Probab. Theory Relat. Fields **113**, 273–285 (1999)

Probability Theory and Related Fields

Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously

Olle Häggström[1,⋆], Yuval Peres[2,3,⋆⋆]

[1] Mathematical Statistics, Chalmers University of Technology, S-412 96 Göteborg, Sweden (e-mail: olleh@math.chalmers.se)
[2] Department of Statistics, University of California, 367 Evans Hall, Berkeley, CA 94720-3860 USA (e-mail: peres@stat.berkeley.edu)
[3] Department of Mathematics, Hebrew University, Jerusalem, Israel

Received: 22 December 1997 / Revised version: 1 July 1998

Abstract. Consider site or bond percolation with retention parameter p on an infinite Cayley graph. In response to questions raised by Grimmett and Newman (1990) and Benjamini and Schramm (1996), we show that the property of having (almost surely) a unique infinite open cluster is increasing in p. Moreover, in the standard coupling of the percolation models for all parameters, a.s. for all $p_2 > p_1 > p_c$, each infinite p_2-cluster contains an infinite p_1-cluster; this yields an extension of Alexander's (1995) "simultaneous uniqueness" theorem. As a corollary, we obtain that the probability $\theta_v(p)$ that a given vertex v belongs to an infinite cluster is depends continuously on p throughout the supercritical phase $p > p_c$. All our results extend to quasi-transitive infinite graphs with a unimodular automorphism group.

Mathematics Subject Classification (1991): 60K35

1. Introduction and statement of results

This paper is concerned with percolation on an infinite, locally finite, connected graph $G = (V, E)$. In **bond percolation** with retention parameter $p \in [0, 1]$, each edge $e \in E$ is independently assigned value 1 (open) with

---

⋆Research supported by a grant from the Swedish Natural Science Research Council
⋆⋆Research partially supported by NSF grant No. DMS-9404391

probability p and value 0 (closed) with probability $1 - p$. In **site percolation**, it is the vertices rather than the edges that are independently assigned values 1 and 0 with respective probabilities p and $1 - p$. We write $\mathbf{P}_p^{G,\text{bond}}$ and $\mathbf{P}_p^{G,\text{site}}$ (sometimes dropping the superscripts) for the resulting probability measures on $\{0, 1\}^E$ and $\{0, 1\}^V$. An infinite connected component of open edges (or vertices, in site percolation) is called an **infinite cluster**. It is elementary that there exists a critical value $p_c = p_c(G, \text{bond}) \in [0, 1]$ such that

$$\mathbf{P}_p^{G,\text{bond}}(\exists \text{ an infinite cluster}) = \begin{cases} 0 \text{ if } p < p_c \\ 1 \text{ if } p > p_c \end{cases} \tag{1}$$

and similarly for site percolation. Further definitions, precise statements and background are given below, but first we indicate our main results, which apply when G is a Cayley graph, or more generally, a unimodular quasi-transitive graph:

Monotonicity of uniqueness. There is a parameter p_u (sometimes different from p_c) such that $\mathbf{P}_p$-a.s. there is a unique infinite cluster for $p \in (p_u, 1]$, and infinitely many for $p \in (p_c, p_u)$ (see Theorems 1.1 and 1.2).

Simultaneous uniqueness and multiplicity. In the standard coupling of percolation models for all $p \in [0, 1]$, almost surely, uniqueness holds simultaneously for all $p > p_u$ and there are infinitely many infinite clusters simultaneously for all $p \in (p_c, p_u)$ (Theorem 1.2).

Continuity above p_c. The probability $\theta_v(p)$ that a vertex v is in an infinite cluster depends continuously on p for $p > p_c$ (Corollary 1.3).

For $G = \mathbf{Z}^d$, it was shown by Aizenman, Kesten and Newman [2] that whenever an infinite cluster exists, it is a.s. unique. For other proofs see [12] or [10]. In contrast, for percolation on the $(n+1)$-regular tree $\mathbf{T}_n$ with parameter $p < 1$, it is easy to see that if an infinite cluster exists, then a.s. there are infinitely many infinite clusters. More exotic is the behaviour of an example studied by Grimmett and Newman [14]. They considered the product graph $\mathbf{T}_n \times \mathbf{Z}$, defined as the graph whose vertex set is $V(\mathbf{T}_n \times \mathbf{Z}) = V(\mathbf{T}_n) \times \mathbf{Z}$, and in which two vertices (a, x) and (b, y) are linked by an edge if and only if either $a = b$ and $|x - y| = 1$, or $x = y$ and $\langle a, b \rangle$ is an edge of $\mathbf{T}_n$. For bond or site percolation on this graph, with n sufficiently large, they showed that if $p > p_c$ is close enough to p_c, then there are a.s. infinitely many infinite clusters, whereas if p is sufficiently close to 1, then there is a.s. a unique infinite cluster. Grimmett and Newman made the implicit conjecture that the a.s. uniqueness of the infinite cluster should be increasing in p, and our first main result (Theorem 1.1) confirms this in a more general setting. It follows (Theorem 1.2) that there is a parameter p_u

such that there is a unique infinite cluster for $p \in (p_u, 1]$, and infinitely many for $p \in (p_c, p_u)$.

In order to state our results carefully, we first need to recall some graph terminology. Let $\mathrm{Aut}(G)$ be the group of graph automorphisms of a graph $G = (V, E)$. A subgroup Γ of $\mathrm{Aut}(G)$ is **transitive** if for every $u, v \in V$, there exists a $\gamma \in \Gamma$ which maps u to v. If such a group exists then the graph G is called transitive. Similarly, Γ and G are called **quasi-transitive** if V can be partitioned into a finite number of sets (orbits) $V_1, \dots, V_k$ so that for any $u \in V_i$, $v \in V_j$ there exists a graph automorphism in Γ mapping u on v iff $i = j$.

Given a finite symmetric set of generators $S = \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ for a countable group Γ_0, the corresponding (right) **Cayley graph** of Γ_0 is the graph $G(\Gamma_0, S) := (V, E)$ with $V := \Gamma_0$ and $[g, h] \in E$ iff $g^{-1}h \in S$. Clearly, the group Γ_0 can be identified with a transitive group of automorphisms of its Cayley graph, acting by left multiplication. The most familiar transitive graphs ($\mathbf{Z}^d$, regular trees and their products) are all Cayley graphs.

For a subgroup Γ of $\mathrm{Aut}(G)$ and a vertex $x \in V$, define the **stabilizer**

$$S_\Gamma(x) = \{\pi \in \Gamma : \pi(x) = x\} \ ,$$

and for $y \in V$, define $S_\Gamma(x)y = \{z \in V : \exists \pi \in S_\Gamma(x) \text{ such that } \pi(y) = z\}$. We say that Γ is **unimodular** if for any x, y in the same Γ-orbit we have the symmetry

$$|S_\Gamma(x)y| = |S_\Gamma(y)x|$$

where $|\cdot|$ denotes cardinality; this is equivalent to the standard definition requiring the left and right Haar measures on Γ to coincide, see Trofimov [26]. We call the graph G **unimodular** if $\mathrm{Aut}(G)$ has a quasi-transitive unimodular subgroup; in this case, $\mathrm{Aut}(G)$ itself is unimodular, see Benjamini, Lyons, Peres and Schramm [4], Corollary 6.2.

All examples discussed above, including Cayley graphs, are transitive and unimodular. A transitive graph which is not unimodular can be obtained by fixing an end ξ in the tree $\mathbf{T}_2$, and for each vertex x adding an edge between x and its ξ-grandparent (see [26] or [4]).

The following result provides a partial answer to a question of Benjamini and Schramm [7].

Theorem 1.1 (Monotonicity of uniqueness). *Consider bond percolation on a connected, locally finite, unimodular quasi-transitive graph G, and let $p_1 < p_2$.*

$$\text{If } \mathbf{P}_{p_1}^{G,bond}(\exists \text{ a unique infinite cluster}) = 1$$
$$\text{then } \mathbf{P}_{p_2}^{G,bond}(\exists \text{ a unique infinite cluster}) = 1 \ .$$

The analogous statement for site percolation also holds.

Lalley [21] studied site percolation on Cayley graphs of certain Fuchsian groups. Among many other results, he proved Theorem 1.1 in this special case, making essential use of planarity.

If the edges of G are labeled by i.i.d. random variables $\{U(e)\}_{e\in E}$, uniformly distributed on $[0, 1]$, then the subgraph spanned by $\{e \in E : U(e) \leq p\}$ has the same law as the set of open edges under $\mathbf{P}_p^{G,\text{bond}}$. Thus the following result is a "simultaneous" sharpening of Theorem 1.1.

Theorem 1.2 (Simultaneous uniqueness and multiplicity). *Let $G = (V, E)$ be a connected, locally finite, unimodular quasi-transitive graph. Let $\{U(e)\}_{e\in E}$ be i.i.d. random variables, uniformly distributed in* $[0, 1]$, *so their joint distribution $\mathscr{L}^E$ is a product measure on* $[0, 1]^E$. *For $p \in [0, 1]$, denote $E_p := \{e \in E : U(e) \leq p\}$ and let $G_p = (V, E_p)$. Write $N(p)$ for the number of infinite clusters in G_p. There exist constants $0 < p_c \leq p_u \leq 1$ such that $\mathscr{L}^E$-almost surely,*

$$N(p) = \begin{cases} 0 & \text{for } p \in [0, p_c) \\ \infty & \text{for } p \in (p_c, p_u) \\ 1 & \text{for } p \in (p_u, 1] \end{cases} .$$

More generally, $\mathscr{L}^E$-almost surely, for all $p_1 < p_2$ in the interval $(p_c, 1]$, any infinite cluster of G_{p_2} contains at least one infinite cluster of G_{p_1}. The analogous statement for site percolation also holds.

The fact that, for each fixed p, $N(p)$ is an a.s. constant that equals either 0, 1 or ∞ follows from the arguments of Newman and Schulman [24], as noted e.g. in [7] and in [21]. The constants p_c and p_u depend on G and on whether bond or site percolation is considered. Benjamini and Schramm [7] conjectured that $N(p_c) = 0$ for quasi-transitive graphs as long as $p_c < 1$. For the case $\mathbf{Z}^d$, $d \geq 2$, this is a classical open problem which has been solved only for $d = 2$ [20] and for d sufficiently large [19]. If $p_c < p_u$, then $N(p_c) = 0$ a.s., see [4] and [5].

Theorem 1.2 separates the parameter space $[0, 1]$ into three qualitatively different intervals (phases), rather than just the two phases indicated in (1). The bottom phase ("no infinite clusters") is always a nondegenerate interval. The middle phase ("multiple infinite clusters") cannot consist of a single point (see [5]). For Euclidean lattices we have that $p_c = p_u$ and that the middle phase is empty; in this case Theorem 1.2 reduces to the "simultaneous uniqueness" theorem of Alexander [3]. For the tree $\mathbf{T}_n$ we have $p_u = 1$, so the uniqueness phase consists of the single point $\{1\}$. In the Grimmett–Newman example, as well as in the Cayley graphs considered in [21], all three phases are intervals of positive length.

Next, we define, for $v \in V$, the percolation function $\theta_v : [0, 1] \to [0, 1]$ by letting

$$\theta_v(p) = \mathbf{P}_p(v \text{ is in an infinite cluster})$$

(for G transitive, this quantity is of course independent of v). The following corollary is an immediate consequence of Theorem 1.2 in conjunction with the results of Van den Berg and Keane [8].

Corollary 1.3 (Continuity above p_c). *For bond percolation on a connected, locally finite, unimodular quasi-transitive graph G, and any vertex $v \in V(G)$, the percolation function $\theta_v(p)$ is continuous on $(p_c, 1]$. The analogous statement holds for site percolation.*

The key tool in our proofs is the **mass-transport method**, which was first used in the percolation setting by Häggström [15], and was fully developed by Benjamini et al. [4]. (Precursors of this method were applied earlier by Adams [1] and by Van den Berg and Meester [9].) The method is briefly explained in the next section. Section 3 is devoted to the proof of Theorem 1.1, and Section 4 to an alternative proof using random walk. Section 5 contains the proof of Theorem 1.2. Finally, in Section 6, we discuss various extensions.

2. The mass-transport method

Let G be a locally finite graph, and suppose that $\Gamma \subset \mathrm{Aut}(G)$ is quasi-transitive and unimodular. Suppose that the elements of Γ can be identified with measure preserving transformations of a probability space $(\Omega, \mathbf{Q})$. (For example, $\mathbf{Q}$ could be the percolation measure $\mathbf{P}_p$ on the space $\{0, 1\}^E$: Clearly $\mathbf{P}_p$ is invariant under any graph automorphism of G.)

Let $m(x, y, \omega)$ be a nonnegative function of three variables: two vertices x, y in the same orbit of Γ, and $\omega \in \Omega$. Intuitively, $m(x, y, \omega)$ represents the mass transported from x to y given the "configuration" ω. We suppose that $m(\cdot, \cdot, \cdot)$ is invariant under the diagonal action of Γ, i.e., $m(x, y, \omega) = m(\gamma x, \gamma y, \gamma \omega)$ for all x, y, ω and $\gamma \in \Gamma$.

Theorem 2.1 (The Mass-Transport Principle). *Let $\Gamma \subset \mathrm{Aut}(G)$ be unimodular and quasi-transitive. Given $m(\cdot, \cdot, \cdot)$ as above, let $M(x, y) := \int_\Omega m(x, y, \omega)\, d\mathbf{Q}(\omega)$. Then the expected total mass transported out of any vertex x equals the expected total mass transported into x, i.e.,*

$$\forall x \in V \quad \sum_{y \in \Gamma x} M(x, y) = \sum_{y \in \Gamma x} M(y, x) \ . \tag{2}$$

Proof. On a Cayley graph, it is straightforward to verify (2): Since $M(\cdot,\cdot)$ is also invariant under the diagonal action of Γ,

$$\sum_{y\in\Gamma x} M(x,y) = \sum_{\gamma\in\Gamma} M(x,\gamma x) = \sum_{\gamma\in\Gamma} M(\gamma^{-1}x,x) = \sum_{y\in\Gamma x} M(y,x) \ .$$

See [4] for the general case. □

We remark that (2) fails in the nonunimodular case; see [4] again. The key step in applying the mass-transport method is to make a suitable choice of the transport function $m(\cdot,\cdot,\cdot)$; examples can be found in [15], [4], [5] and in the following sections.

3. Monotonicity of uniqueness

In order to prove Theorem 1.1, we will need to define a number of random processes which will live on a common probability space $(\Omega, \mathscr{F}, \mathbf{P})$; we write $\mathbf{E}$ for expectation with respect to $\mathbf{P}$.

Consider the following standard coupling (ω_1,ω_2) of the bond percolation process on G at two different parameter values $0 \le p_1 \le p_2 \le 1$: For each $e \in E$ independently, let

$$(\omega_1(e),\omega_2(e)) = \begin{cases} (1,1) \text{ w.p. } p_1 \\ (0,1) \text{ w.p. } p_2 - p_1 \\ (0,0) \text{ w.p. } 1-p_2 \ . \end{cases} \tag{3}$$

It is clear that ω_1 and ω_2 have distributions $\mathbf{P}^{G,\text{bond}}_{p_1}$ and $\mathbf{P}^{G,\text{bond}}_{p_2}$, and that the set of open edges in ω_2 contains the set of open edges in ω_1. Moreover, the joint distribution of (ω_1,ω_2) is the same as that of (G_{p_1}, G_{p_2}), defined as in Theorem 1.2.

Next, let **dist** denote graph-theoretical distance in G, and for each $v \in V$, define

$$D_1(v) := \inf\{\mathbf{dist}(v,w) : \ w \text{ is in an infinite cluster of } \omega_1\} \tag{4}$$

where $\mathbf{Z}_+ = \{0,1,2,\dots\}$, and as usual, the infimum of the empty set is taken to be ∞.

The following proposition implies Theorem 1.1.

Proposition 3.1. *Let G be an infinite locally finite connected unimodular quasi-transitive graph and pick $0 < p_1 < p_2 \le 1$ so that*

$$\mathbf{P}^{G,\text{bond}}_{p_1}(\exists \textit{ an infinite open cluster}) = 1 \ . \tag{5}$$

Pick ω_1 and ω_2 according to the coupling defined in (3). *Then, a.s., each infinite cluster of ω_2 contains some infinite cluster of ω_1. The analogous statement for site percolation also holds.*

Proof. We give the proof for bond percolation; the site percolation case is completely analogous. Assume first that G is transitive. Pick p_1 and p_2 satisfying the assumptions of the proposition, and denote by $\mathscr{C}(v, \omega_i)$ the connected component of $v \in V$ in the open subgraph determined by ω_i. Also, let $\mathscr{C}(\infty, \omega_i)$ be the union of all infinite open clusters in ω_i. For $u \in V$, let $A(u)$ be the event that

$$D_1(u) = \min_{v \in \mathscr{C}(u, \omega_2)} D_1(v) > 0 \ .$$

By considering the closest vertices to $\mathscr{C}(\infty, \omega_1)$ in each ω_2 cluster, we see that to establish the proposition, it suffices to prove that

$$\mathbf{P}[\{|\mathscr{C}(u, \omega_2)| = \infty\} \cap A(u)] = 0 \quad \forall u \in V \ . \tag{6}$$

Consider the mass transport m where $m(x, y, \omega) = 1$ if y is the *unique* vertex in $\mathscr{C}(x, \omega_2)$ which is closest to $\mathscr{C}(\infty, \omega_1)$, and $m(x, y, \omega) = 0$ otherwise. By the mass-transport principle, the expected incoming mass to any vertex u is finite, so $\mathbf{P}[\{|\mathscr{C}(u, \omega_2)| = \infty\} \cap A^*(u)] = 0$, where $A^*(u)$ is the event that u is the unique minimizer of D_1 in $\mathscr{C}(u, \omega_2)$. Consequently

$$\mathbf{P}[|\mathscr{C}(u, \omega_2)| = \infty \cap A(u) \cap \{D_1(u) > 1\}] = 0 \tag{7}$$

since opening (in ω_2 only) an edge connecting u to a vertex w with $D_1(w) = D_1(u) - 1$, and closing all other edges adjacent to w, changes probabilities by a bounded factor and maps the event in (7) to $\{|\mathscr{C}(u, \omega_2)| = \infty\} \cap A^*(w)$.

To prove (6), it only remains to show that

$$\mathbf{P}[|\mathscr{C}(u, \omega_2)| = \infty \cap A(u) \cap \{D_1(u) = 1\}] = 0 \ . \tag{8}$$

For $w \in \mathscr{C}(\infty, \omega_1)$ let $S(w)$ consist of the vertices v in $\mathscr{C}(w, \omega_2)$ such that w is the unique vertex in $\mathscr{C}(\infty, \omega_1)$ which is closest to v in the metric of $\mathscr{C}(w, \omega_2)$. By considering the mass transport where each vertex in $S(w)$ sends unit mass to w, we see that $S(w)$ must be finite a.s. Now observe that, on the event in (8), we have that opening (in ω_2 only) an edge between u and $w \in \mathscr{C}(\infty, \omega_1)$ takes us into the event $\{|S(w)| = \infty\}$. Hence (8) holds, and the transitive case is taken care of.

Minor modifications handle the quasi-transitive case. For instance, to show that $S(w)$ above is finite, consider the mass transport that sends unit mass from each vertex x to all the vertices in its orbit that are closest to w.

□

Remark. The proof above actually yields the quantitative bound

$$\mathbf{E}[|\mathscr{C}(u, \omega_2)| \, \mathbf{1}_{A(u)}] < \infty \quad \forall u \in V \ . \tag{9}$$

4. A random walk approach

Our first proof of Theorem 1.1 used random walk on the percolation cluster. A complete proof using this approach is longer than the mass transport argument we gave above. Nevertheless, we sketch the main ideas here, since we believe they can be useful in other percolation problems. Indeed, very recently Lyons and Schramm [23] used this approach to prove a remarkable indistinguishability property of percolation clusters (see Section 6.4 below). In that application, it seems the random walk approach cannot be replaced by direct mass transport.

Let $p_c < p_1 < p_2$, and consider the coupling (ω_1, ω_2) from (3). Define the V-valued random process $Z_v = \{Z_v(j)\}_{j=0}^{\infty}$ which depends on ω_2, but whose distribution given ω_2 is taken to be conditionally independent of ω_1, as follows. Take $Z_v(0) = v$. Given $Z_v(0), \ldots, Z_v(j)$, pick a random vertex $W_v(j)$ uniformly from the set of nearest neighbours of $Z_v(j)$, and let

$$Z_v(j+1) = \begin{cases} W_v(j) \text{ if } \omega_2(\langle Z_v(j), W_v(j)\rangle) = 1 \\ Z_v(j) \text{ otherwise} \end{cases} .$$

Recall the definition of D_1 in (4). The random walk approach requires the following lemma, which is easy to prove in the Cayley graph case, and which can be proved in the transitive unimodular case using the mass-transport principle.

Lemma 4.1. *If G is transitive and unimodular, then $(D_1(Z_v(0)), D_1(Z_v(1)), \ldots)$ is a stationary sequence.*

Due to space restrictions, we omit the proof; a more general statement can be found in [23]. Using the lemma, Proposition 3.1. can be proved as follows, and Theorem 1.1 is a consequence.

Alternative proof of Proposition 3.1. for bond percolation on transitive graphs

Suppose that G is transitive, and pick p_1 and p_2 as in the proposition. Consider the sequence $(D_1(Z_v(0)), D_1(Z_v(1)), \ldots)$ which is stationary by the lemma, and define the random variable $M = \min_{j=0}^{\infty} D_1(Z_v(j))$. By the assumption (5), we have a.s. that $M < \infty$. Moreover, the set of times j such that $D_1(Z_v(j)) = M$ has a strictly positive (but possibly random) density in $\mathbf{Z}_+$, i.e. the limit

$$\lim_{n\to\infty} \frac{|\{j \in \{0, \ldots, n-1\} : D_1(Z_v(j)) = M\}|}{n} \tag{10}$$

a.s. exists and is strictly positive; this follows from the ergodic theorem. We are done if we can show that if v is in an infinite ω_2-cluster, then $M = 0$ a.s. For $k \in \mathbf{Z}_+$, define the event $A_k = \{v \text{ is in an infinite } \omega_2\text{-cluster}, M = k\}$, and suppose for contradiction that $\mathbf{P}(A_k) > 0$ for some $k > 0$. Simple random walk on any infinite locally finite connected graph is either transient or null recurrent; hence, we have a.s. on A_k that the limiting density of times $\{j : Z_v(j) = w\}$ is 0 for any $w \in V$. In conjunction with the strictly positive limit in (10), this implies that Z_v has to visit infinitely many vertices $w \in V$ satisfying $D_1(w) = k$. Writing B_k for the event that there exist infinitely many times $j \in \mathbf{Z}_+$ such that

$$D_1(Z_v(j)) = k, \quad \text{and} \quad Z_v(i) \neq Z_v(j) \text{ for } i = 0, \ldots, j-1 \ , \tag{11}$$

we thus have that

$$\mathbf{P}[B_k \mid A_k] = 1 \ . \tag{12}$$

All vertices in V have the same degree d, by transitivity. We claim that at any time j such that $D_1(Z_v(i)) \geq k$ for $i = 0, \ldots, j-1$ and (11) holds, we have

$$\mathbf{P}[D_1(Z_v(j+1)) = k-1 \mid Z_v(0), \ldots, Z_v(j), \ \omega_1] \geq \frac{p_2 - p_1}{d(1-p_1)} \ . \tag{13}$$

This is because $Z_v(j)$ has some neighbour w such that $D_1(w) = k-1$, and the conditional probability that the edge between $Z_v(j)$ and w is open is at least $(p_2 - p_1)/(1 - p_1)$. By repeated use of (13), we infer that the probability that Z_v visits at least m vertices w with $D_1(w) = k$ before visiting any vertex u with $D_1(u) = k-1$ tends to 0 as $m \to \infty$. Hence, $\mathbf{P}[A_k \mid B_k] = 0$, and using (12) we conclude that $\mathbf{P}[A_k] = \mathbf{P}[A_k \cap B_k] \leq \mathbf{P}[A_k \mid B_k] = 0$. □

5. Simultaneous uniqueness

In this section we prove Theorem 1.2. We restrict attention to bond percolation on transitive graphs, the extensions to site percolation and quasi-transitive graphs being straightforward.

Proof of Theorem 1.2. Let $p_1 < p_2 < p_3$ be rationals bigger than p_c. If for some $p \in (p_2, p_3)$ the vertex x is in an infinite cluster $\mathscr{C}(x, p)$ disjoint from $\mathscr{C}(\infty, p_1)$, then this value of p is uniquely determined by x (since we can directly apply Proposition 3.1. to all rational p simultaneously). We call such an infinite cluster $\mathscr{C}(x, p)$ *exceptional*, and denote it $\mathscr{C}^*(x)$. Note that any two exceptional infinite clusters are either identical or disjoint.

Let A_x be the event that x is in an exceptional infinite cluster, and assume for contradiction that $\mathscr{P}^E(A_x) > 0$. Define

$$p^*(x) = \inf\{p : \ x \text{ is in an infinite cluster of } G_p\}$$

and note that on the event A_x we have $\mathscr{C}(x, p^*(x)) = \mathscr{C}^*(x)$. For non-negative integer k, define the random variable N_x^k to be the number of vertices in $\mathscr{C}(x, p^*(x))$ that are at distance exactly k from $\mathscr{C}(\infty, p_1)$. Furthermore set $K_x = \min\{k : \ N_x^k > 0\}$. By conditioning on $\mathscr{C}(\infty, p_1)$ and on the status of all edges at distance k or more from $\mathscr{C}(\infty, p_1)$, we see that

$$\mathscr{P}^E(A_x, K_x = k, N_x^k = n) \leq \left(\frac{1 - p_2}{1 - p_1}\right)^n$$

for any $k, n \geq 1$. In particular, $\mathscr{P}^E(A_x, K_x = k, N_x^k = \infty) = 0$ for $k \geq 1$. Hence, any exceptional infinite cluster contains a.s. only a finite number of vertices y that minimize the distance from $\mathscr{C}(\infty, p_1)$. Consider the mass transport where $m(x, y, \omega) = 1/n$ if x and y are in the same exceptional infinite cluster and y is one of exactly n such minimizers, and $m(x, y, \omega) = 0$ otherwise. The expected mass sent from a vertex is at most 1, whereas if exceptional infinite clusters exist with positive probability, then any vertex v is, with positive probability, one of the minimizers defined above. Thus the expected mass received at v is infinite, contradicting the mass-transport principle. □

6. Extensions

6.1. *Non-isotropic edge probabilities*

The class of percolation processes on $\mathbf{T}_n \times \mathbf{Z}$ considered by Grimmett and Newman [14] is actually more general than what we indicated in the introduction, in that they allowed non-isotropic retention probabilities. Let us call an edge $e \in E(\mathbf{T}_n \times \mathbf{Z})$ a tree-edge (resp. line-edge) if its two endpoints differ only in their $\mathbf{T}_n$-coordinate (resp. $\mathbf{Z}$-coordinate). In the Grimmett–Newman setup, all edges are independent, but tree-edges have retention probability p_T and line-edges have retention probability p_Z, where $(p_T, p_Z) \in [0, 1]^2$. The proof of Theorem 1.1 can easily be generalized in such a way as to show that if $p_T \leq p'_T$, $p_Z \leq p'_Z$ and a.s. uniqueness of the infinite cluster holds at (p_T, p_Z), then the same thing holds at (p'_T, p'_Z). Similarly, the proof of Corollary 1.3 can be generalized to show that the percolation function $\theta_v(p_T, p_Z)$ is continuous throughout the interior of the supercritical regime $\{(p_T, p_Z) \in [0, 1]^2 : \theta_v(p_T, p_Z) > 0\}$, thus providing a new proof of a continuity result of Zhang [27].

6.2. *More general graph structures*

After seeing our arguments, Schonmann [25] extended our Theorem 1.1 and Corollary 1.3 by showing that the unimodularity assumption can be dropped. One may ask whether the assertions of Theorems 1.1 and Corollary 1.3 even hold for any graph, but the answer is negative in both cases. A simple counterexample to the behaviour in Theorem 1.1 can be obtained by taking a copy of the $\mathbf{Z}^3$ lattice and a copy of the binary tree $\mathbf{T}_2$, and adding a single edge between one vertex in $\mathbf{Z}^3$ and one vertex in $\mathbf{T}_2$. By similarly connecting two copies of $\mathbf{Z}^2$, we see that the number of infinite clusters need not be an a.s. constant, and by replacing $\mathbf{T}_2$ in the above construction by some tree $\mathbf{T}$ which has $p_c(\mathbf{T}) \in (p_c(\mathbf{Z}^3), 1)$ and which percolates at criticality (such trees exist, see [22]), we obtain an example which shows that the assertion of Corollary 1.3 fails for general graphs. In fact, one can even construct trees where $\theta_v(p)$ has infinitely many discontinuities; see [17].

6.3. *Ends of infinite clusters*

Say that two self-avoiding paths ξ_1, ξ_2 in a graph $G = (V, E)$ are *equivalent* if for any finite set $V_0 \subset V$, both paths eventually remain in the same connected component of $V \setminus V_0$. Equivalence classes of self-avoiding paths are called **ends** of G. Benjamini and Schramm [7] conjectured that when G is quasi-transitive and has multiple infinite clusters $\mathbf{P}_p$-a.s., then each infinite cluster must have uncountably many ends a.s. The submitted version of this paper contained a proof of this conjecture in the unimodular case. More precisely, we established the following.

Theorem 6.1 (Uncountably many ends). *Consider bond percolation on a connected, locally finite, unimodular quasi-transitive graph G. Fix p such that the $\mathbf{P}_p$-probability of having infinitely many infinite clusters is* 1*. Then $\mathbf{P}_p$-a.s., every infinite cluster has a set of ends with cardinality of the continuum.*

Our two-page proof of this result used Menger's Theorem and mass transport. Due to lack of space, we had to omit the proof in the final version. Generalizations of Theorem 6.1 were subsequently proved by Lyons and Schramm [23] and by Häggström, Peres and Schonmann [18]; in particular, the reader can find proofs of Theorem 6.1 in those references.

6.4. *Infinite clusters with special properties*

Let A be some G-automorphism invariant property of an infinite cluster. An interesting example of such a property, studied in [13], [6] and [16], is transience, i.e. the property that simple random walk on the cluster under consideration is transient. For G satisfying the usual assumptions, the existence of an infinite cluster with property A has probability 0 or 1 for any p, and our proof of Proposition 3.1 can be modified to prove the following generalization.

Proposition 6.2. *Let G be a locally finite connected unimodular quasi-transitive graph, and pick $0 < p_1 \leq p_2 \leq 1$ such that*

$$P_{p_1}^{G,\mathrm{bond}}(\exists \textit{ an infinite cluster with property } A) = 1 \quad .$$

If we then pick the $\{0, 1\}^V$-valued random objects ω_1 and ω_2 according to the coupling defined in Section 3, then we have a.s. that each infinite cluster of ω_2 contains some infinite cluster of ω_1 with property A.

Write $p_c(G, A) := \inf\{p > 0 \,:\, \mathbf{P}_p[\exists \text{ an infinite cluster with property } A] > 0\}$. Suppose that A is an *increasing* property, i.e., that if an infinite cluster has property A and we add edges or vertices to it, then it still has property A; by Rayleigh's monotonicity principle (see [11]), this holds for the transience property. Under this assumption, Proposition 6.2. implies that the coexistence of infinite clusters with and without property A has $\mathbf{P}_p$-probability 0 for all $p > p_c(G, A)$. Two natural questions are whether this probability must vanish also for $p = p_c(G, A)$, and whether the assumption that A is increasing can be dropped. Very recently, both questions were answered affirmatively by Lyons and Schramm [23].

Acknowledgements. We are grateful to Itai Benjamini, Russ Lyons, Roberto Schonmann, Oded Schramm and Jeff Steif for useful discussions.

References

1. Adams, S.: Trees and amenable equivalence relations. Ergod. Th. Dynam. Sys., **10,** 1–14 (1990)
2. Aizenman, M., Kesten H., Newman, C.M.: Uniqueness of the infinite cluster and continuity of connectivity functions for short- and long-range percolation. Commun. Math. Phys., **111,** 505–532 (1987)
3. Alexander, K.: Simultaneous uniqueness of infinite clusters in stationary random labeled graphs. Commun. Math. Phys., **168,** 39–55 (1995)
4. Benjamini, I., Lyons, R., Peres, Y., Schramm, O.: Group-invariant percolation on graphs. Geom. Funct. Analysis, to appear

5. Benjamini, I., Lyons, R., Peres, Y., Schramm, O.: Critical percolation on any nonamenable group has no infinite clusters. Ann. Probab., to appear (1997b)
6. Benjamini, I., Pemantle, R., Peres, Y.: Unpredictable paths and percolation. Ann. Probab., **26,** 1198–1211 (1998)
7. Benjamini, I., Schramm, O.: Percolation beyond $\mathbf{Z}^d$: many questions and a few answers. Electr. Commun. Probab., **1,** 71–82 (1996)
8. Van den Berg, J., Keane, M.: On the continuity of the percolation probability function. Conference in modern analysis and probability (New Haven, Conn., 1982), R. Beals et al. (editors), 61–65, Amer. Math. Soc., Providence, RI (1984)
9. Van den Berg, J., Meester, R.: Stability properties of a flow process in graphs. Random Structures and Algorithms, **2,** 335–341 (1991)
10. Burton, R., Keane, M.: Density and uniqueness in percolation. Commun. Math. Phys., **121,** 501–505 (1989)
11. Doyle, P., Snell, J.L.: Random Walks and Electric Networks. Mathematical Monograph, **22,** Mathematical Association of America (1984)
12. Gandolfi, A., Grimmett, G.R., Russo, L.: On the uniqueness of the infinite open cluster in the percolation model. Commun. Math. Phys., **114,** 549–552 (1988)
13. Grimmett, G.R., Kesten, H., Zhang, Y.: Random walk on the infinite cluster of the percolation model. Probab. Th. Rel. Fields, **96,** 33–44 (1993)
14. Grimmett, G.R., Newman, C.M.: Percolation in $\infty + 1$ dimensions. Disorder in Physical Systems (G.R. Grimmett and D.J.A. Welsh, editors), 219–240, Clarendon Press, Oxford (1990)
15. Häggström, O.: Infinite clusters in dependent automorphism invariant percolation on trees. Ann. Probab., **25,** 1423–1436 (1996)
16. Häggström, O., Mossel, E.: Nearest-neighbor walks ith low predictability profile and percolation in $2 + \varepsilon$ dimensions. Ann. Probab., **26,** 1212–1231 (1998)
17. Häggström, O., Pemantle, R.: On near-critical and dynamical percolation in the tree case. preprint (1997)
18. Häggström, O., Peres, Y., Schonmann, R.: Percolation on transitive graphs as a coalescent process: relentless merging followed by simultaneous uniqueness. In Perplexing Probability Problems: Papers in Honor of Harry Kesten, Birkhäuser, to appear
19. Hara, T., Slade, G.: Mean field behavior and the lace expansion. Probability and Phase Transition (G.R. Grimmett, ed.), 87–122, Kluwer, Dordrecht (1994)
20. Kesten, H.: The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. Commun. Math. Phys., **74,** 41–59 (1980)
21. Lalley, S.: Percolation on Fuchsian groups. Ann. Inst. H. Poincaré, Probab. Stat., **34,** 151–177 (1998)
22. Lyons, R.: Random walks, capacity, and percolation on trees. Ann. Probab., **20,** 2043–2088 (1992)
23. Lyons, R., Schramm, O.: Indistinguishability of percolation clusters. preprint (1998)
24. Newman, C.M., Schulman, L.S.: Infinite clusters in percolation models. J. Statist. Phys., **26,** 613–628 (1981)
25. Schonmann, R.H.: Stability of infinite clusters in supercritical percolation. In Perplexing Probability Problems: Papers in Honor of Harry Kesten, Birkhäuser, to appear
26. Trofimov, V.I.: Automorphism groups of graphs as topological groups. Math. Notes, **38,** 717–720 (1985)
27. Zhang, Y.: Continuity of percolation probability in $\infty + 1$ dimensions. J. Appl. Probab., **33,** 427–433 (1996)