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Probability Theory and Related Fields

Exit time moments, boundary value problems, and the geometry of domains in Euclidean space

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Abstract. Let X_t be a diffusion in Euclidean space. We initiate a study of the geometry of smoothly bounded domains in Euclidean space using the moments of the exit time for particles driven by X_t, as functionals on the space of smoothly bounded domains. We provide a characterization of critical points for each functional in terms of an overdetermined boundary value problem. For Brownian motion we prove that, for each functional, the boundary value problem which characterizes critical points admits solutions if and only if the critical point is a ball, and that all critical points are maxima.

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1. Introduction

In this paper we initiate a study of the geometry of smoothly bounded, connected, open domains in $\mathbb{R}^d$ with compact closure, using diffusions in $\mathbb{R}^d$ and properties of their exit times from these domains.

We begin by fixing notation. Let X_t be a diffusion in $\mathbb{R}^d$ with infinitesimal generator L, a uniformly elliptic operator. We will be particularly interested in operators of *divergence form*. Such operators

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act on $C^\infty(\mathbb{R}^d)$ according to the rule $Lf = \operatorname{div}(a_{ij}\nabla f)$ where the coefficient matrix $a_{ij}(x)$ is smooth and symmetric. Let $\mathscr{D}$ be the space of smoothly bounded, connected open domains in $\mathbb{R}^d$ with compact closure. Let $\mathscr{D}_v$ be the subset of $\mathscr{D}$ consisting of those domains $D \in \mathscr{D}$ such that the volume of D, denoted $|D|$, is v. For $x_0 \in \mathbb{R}^d$ fixed, let $\mathscr{D}_{v,x_0}$ be the subset of $\mathscr{D}_v$ consisting of those domains that contain x_0. We note that each of the spaces $\mathscr{D}, \mathscr{D}_v$ and $\mathscr{D}_{v,x_0}$ is a Frechet manifold.

For $D \in \mathscr{D}$, let $\tau = \tau(\omega) = \inf\{t \geq 0 : X_t(\omega) \notin D\}$ be the first exit time of X_t from D.

Given a domain $D \in \mathscr{D}_{v,x_0}$ and a positive integer, k, we associate to D two sequences of positive real numbers as follows. Let $\mathscr{E}_{k,x_0}(D)$ be given by

$$\mathscr{E}_{k,x_0}(D) = E_{x_0}(\tau^k) \tag{1.1}$$

where E_x denotes expectation under the measure P_x satisfying $P_x\{X_0 = x\} = 1$, for all $x \in \mathbb{R}^d$. Similarly, for $D \in \mathscr{D}_v$ and k a positive integer, let $\mathscr{E}_k(D)$ be defined by

$$\mathscr{E}_k(D) = \int_D E_x(\tau^k)\, dx \tag{1.2}$$

where dx is Lebesgue measure on $\mathbb{R}^d$.

To concisely state our main results, we fix a positive integer k and view (1.1) and (1.2) as defining maps $\mathscr{E}_{k,x_0} : \mathscr{D}_{v,x_0} \to \mathbb{R}$ and $\mathscr{E}_k : \mathscr{D}_v \to \mathbb{R}$. These maps are smooth (cf section 2 below) with respect to the natural Frechet space structure on the spaces $\mathscr{D}_{v,x_0}$ and $\mathscr{D}_v$, respectively. Our main result characterizes the critical points of these maps in terms of overdetermined boundary value problems associated to the operator L:

Theorem 1.1. *Let $D \in \mathscr{D}_v$. Suppose that L is a smooth divergence form operator, $L = \operatorname{div}(a_{ij}\nabla)$. For $1 \leq j \leq k$, let u_j be defined inductively by*

$$\begin{aligned} Lu_1 + 1 &= 0 \text{ on } D \\ u_1 &= 0 \text{ on } \partial D \end{aligned}$$

and

$$\begin{aligned} Lu_j + ju_{j-1} &= 0 \text{ on } D \\ u_j &= 0 \text{ on } \partial D\ . \end{aligned}$$

Then D is a critical point of the functional $\mathscr{E}_k$ if and only if there is a solution of the overdetermined boundary value problem consisting of the above equations and

$$\sum_{j=1}^{k} c_j \frac{\partial u_{k+1-j}}{\partial v} \langle u_j, v\rangle_L = C \ on\ \partial D \tag{1.3}$$

where C is a constant, $\langle u_j, v\rangle_L = (\nabla u_j)^{\mathrm{T}} a_{ij} v$ *is the conormal derivative associated to L, and the constant* c_j *is given by*

$$c_j = \frac{k!}{(k+1-j)!j!} \ .$$

Similarly, $D \in \mathscr{D}_{v,x_0}$ *is a critical point of the functional* $\mathscr{E}_{k,x_0}$ *if and only if there is a solution of the overdetermined boundary value problem consisting of the above equations and*

$$\frac{\partial G}{\partial v}(x_0, \cdot)\langle \nabla u_k, v\rangle_L + \sum_{j=1}^{k-1} d_j \frac{\partial u_{k-j}}{\partial v} \langle \nabla m_j, v\rangle_L = C' \ on\ \partial D \tag{1.4}$$

where C' *is a constant,* $\langle u_j, v\rangle_L = (\nabla u_j)^{\mathrm{T}} a_{ij} v$ *is the conormal derivative associated to L, G is the Green's function for L,* m_j *is defined inductively by*

$$\begin{aligned} Lm_1 + G(x_0, \cdot) &= 0 \ on\ D \\ m_1 &= 0 \ on\ \partial D \end{aligned}$$

and

$$\begin{aligned} Lm_j + m_{j-1} &= 0 \ on\ D \\ m_j &= 0 \ on\ \partial D \ , \end{aligned}$$

and the constant d_j *is given by*

$$d_j = \frac{k!}{(k-j)!} \ .$$

The overdetermined boundary value problems occurring in the statement of our theorem are closely related to a well studied collection of problems first investigated by Serrin [S]. Using our theorem and a technique developed in [FM], we obtain the following result:

Proposition 1.1. *Let* X_t *be standard d-dimensional Brownian motion. Let* $\mathscr{E}_k$ *and* $\mathscr{E}_{k,x_0}$ *be defined as above. Then* $D \in \mathscr{D}_v$ *is a critical point of* $\mathscr{E}_k$ *if and only if D is a ball of volume v. Similarly,* $D \in \mathscr{D}_{v,x_0}$ *is a critical point of* $\mathscr{E}_{k,x_0}$ *if and only if D is a ball of volume v centered at* x_0.

Using symmetric rearrangement, Aizenman-Simon (cf [AS]) give a short proof that the ball centered at x_0 is a global maximum for the map $\mathscr{E}_{k,x_0}$. More precisely, they prove: *Among Lebesgue measurable* *domains of a fixed volume which contain x_0 in their interiors, the ball centered at x_0 maximizes $\mathscr{E}_{k,x_0}(D)$ for each value of k*. Their argument can be modified to establish the same result for the sequence $\mathscr{E}_k(D)$. We have chosen to work in the context of smoothly bounded domains. Our techniques require the existence of a C^1-unit normal vector and depend on results of Serrin for C^2-domains. In this sense, the results of [AS] are more general than those we prove in Proposition 1.1. On the other hand, our results give more detailed information concerning the behavior of exit time moments from less general sets.

Our interest in the sequences defined by (1.1) and (1.2) is largely motivated by the now classical work concerning the extent to which the Dirichlet spectrum of a Euclidean domain determines the geometry of the domain. An early result in this direction is the theorem of Faber-Krahn: *Among smooth domains of a fixed volume, the ball minimizes the principal Dirichlet eigenvalue*. The theorem of Aizenman and Simon cited above can be regarded as an analog of the result of Faber-Krahn for the sequence $\mathscr{E}_{k,x_0}(D)$. This suggests a natural problem: Given a smoothly bounded Euclidean domain, what geometric parameters can be recovered from the sequences (1.1) and (1.2)?

Before proceeding, a few additional words concerning related work are in order. In [KM1], Kinateder and McDonald prove Theorem 1.1 for the special case $k = 1$. Similar results are established for the variance of the exit time and the average variance of the exit time in [KM2].

For $D \subset \mathbb{R}^2$, the quantity $\mathscr{E}_1(D)$ is perhaps better known as the torsional rigidity associated to a beam of uniform cross section D. That $\mathscr{E}_1(D)$ is maximized among domains of a fixed area by a disk is the content of the St. Venant torsion problem, an old problem with extensive associated literature (cf [PS]). In particular, there are a number of results providing bounds for $\mathscr{E}_1(D)$ in terms of various geometric quantities associated to a Euclidean domain. Among these results are a number of estimates which bound $\mathscr{E}_1(D)$ in terms of the *inner radius*, τ_D, associated to D (the supremum of the radii over all disks contained in the domain D) (cf [B]). There are similar estimates for the functional $\mathscr{E}_{1,x_0}(D)$. For example (cf [BC]), *There is a universal constant, b, such that if D is a simply connected planar domain, then*

$$\sup_{x \in D} E_x[\tau] \leq b r_D^2 \ .$$

Inequalities of this form allow one to use the expected lifetime of Brownian motion in D to estimate a number of geometric invariants associated to simply connected planar domains. This approach is introduced in [BC] (cf also [BCH]) where the authors use Brownian motion and conformal mapping (including the schlicht Bloch-Landau constant) to estimate the best constant, a, in Hayman's inequality: *There is a universal constant a such that if D is a simply connected planar domain with principal Dirichlet eigenvalue λ_D, then*

$$\lambda_D \le \frac{a}{r_D^2} \quad .$$

Among the many geometric properties one might choose to study using the sequences defined by (1.1) and (1.2), it would be interesting to know to what extent it is possible to improve the estimates of [BC].

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2. Basic results and definitions

Let $(\Omega, \mathscr{B})$ be a measurable space and $\{P_x\}_{x \in \mathbb{R}^d}$ a family of probability measures on $(\Omega, \mathscr{B})$. Let $\{X_t\}_{t \ge 0}$ denote a d-dimensional diffusion with generator L, a uniformly elliptic operator in divergence form, $Lf = \text{div}\,(a_{ij}\nabla f)$ where the coefficient matrix $a_{ij}(x)$ is smooth and symmetric, and for which $P_x\{X_0 = x\} = 1$, for $x \in \mathbb{R}^d$.

Let D be a smoothly bounded domain in Euclidean space. As in the introduction, we define the first exit time for a particle driven by X_t from D by $\tau = \tau(\omega) = \inf\{t : X_t(\omega) \notin D\}$. For each $x \in \mathbb{R}^d$, we will denote the expected value of a random variable Y under the probability measure P_x by $E_x(Y)$.

There is a useful relationship between the solution of a certain Poisson problem on the domain $D \in \mathscr{D}$ and the expected value of the kth power of the first exit time of a particle driven by X_t from D starting at $x \in D$. Suppose u_k solves the problem

$$\begin{aligned} L^k u_k + (-1)^{k-1} k! &= 0 \text{ on } D \\ u_k = Lu_k = \cdots = L^{k-1} u_k &= 0 \text{ on } \partial D \quad . \end{aligned} \tag{2.1}$$

Note that u_k can be defined inductively by

$$\begin{aligned} Lu_1 + 1 &= 0 \text{ on } D \\ u_1 &= 0 \text{ on } \partial D \end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
Lu_k + ku_{k-1} &= 0 \text{ on } D \\
u_k &= 0 \text{ on } \partial D \ .
\end{aligned}
\tag{2.3}
$$

Using the generalized Dynkin formula [Ha] (cf also [AK] and [P]) we have

$$
\begin{aligned}
E_x[u_k(X_0)] - E_x[u_k(X_\tau)] = &\sum_{j=1}^{k-1} \frac{(-1)^j}{j!} E_x[\tau^j L^j u_k(X_\tau)] \\
&+ \frac{(-1)^k}{(k-1)!} E_x\left[\int_0^\tau s^{k-1} L^k u_k(X_x)\ ds\right] \ .
\end{aligned}
$$

Using the definition of u_k and τ, this gives

$$
u_k(x) = E_x[\tau^k] \ .
$$

We now express $\mathscr{E}_{k,x_0}$ and $\mathscr{E}_k$ in terms of u_k:

$$
\mathscr{E}_{k,x_0}(D) = u_k(x_0) \tag{2.4}
$$

$$
\mathscr{E}_k(D) = \int_D u_k(x)\ dx \ . \tag{2.5}
$$

The space $\mathscr{D}$ carries the structure of a Frechet manifold (see [H] for a thorough survey of the geometry of Frechet manifolds). Each element of $D \in \mathscr{D}$ can be naturally identified with its boundary ∂D. As ∂D is a compact hypersurface, there is a diffeomorphism between a tubular neighborhood of ∂D in $\mathbb{R}^d$ and a neighborhood of the zero section of the normal bundle $N\partial D$ of ∂D. A neighborhood $D \in \mathscr{D}$ is identified with a neighborhood of the zero section in the Frechet space $C^\infty(\partial D, N\partial D)$. This identification provides us with the required Frechet structure.

Recall, a map F: $\mathscr{D} \to \mathbb{R}$ is smooth if for any point $D \in \mathscr{D}$ we can find charts around D in $\mathscr{D}$ such that the local representation of F with respect to these charts is a smooth map of Frechet spaces. The derivative of F at a point $D \in \mathscr{D}$ is the induced linear map on the tangent space at $D \in \mathscr{D}$, $\delta_D F$: $T_D\mathscr{D} \to T_{F(D)}\mathbb{R}$.

To see that (2.4) and (2.5) define smooth maps on $\mathscr{D}$, fix $D \in \mathscr{D}$ and consider a neighborhood $U \subset \mathscr{D}$ identified with a neighborhood of the zero section of the normal bundle $C^\infty(\partial D, N\partial D)$. An element $h \in U$ can be smoothly extended to a vector field on D in such a fashion that the extension vanishes outside a collar neighborhood of ∂D in D. This allows one to define a (less than canonical) family of diffeomorphisms $D \to D_t$ given by flowing in the direction of the extension of h. For the case $k = 1$, we can pull back the boundary value problems (2.1) on D_t to boundary value problems for a second order differential operator on D whose coefficients depend smoothly on the diffeomorphism and the choice of h. Hence, for $k = 1$ the maps defined by (2.4) and (2.5) are smooth. Clearly, the argument extends to general k and we see that the maps defined by (2.4) and (2.5) are smooth. A variant of this argument, first given by Hilbert in his study of the dependence of Dirichlet eigenvalues on the underlying domain, was used to examine a wide range of problems in the foundational paper of Garabedian and Schiffer [GS]. A modern version of many of their results appears in [EM].

Throughout this discussion we have focused on the space $\mathscr{D}$. Similar remarks apply to the space $\mathscr{D}_v$ (as well as for $\mathscr{D}_{v,x_0}$, which is an open set in $\mathscr{D}_v$). The space $\mathscr{D}_v$ is a Frechet manifold. Near a point $D \in \mathscr{D}_v$ we have coordinate neighborhoods given by the Frechet space $C_0^\infty(\partial D, N\partial D)$ where

$$C_0^\infty(\partial D, N\partial D) = \left\{ f \in C_0^\infty(\partial D, N\partial D) : \int_{\partial D} f \, d\sigma = 0 \right\} . \tag{2.6}$$

There is a natural identification $T_D\mathscr{D}_v \simeq C_0^\infty(\partial D, N\partial D)$. In the sequel we will compute the derivatives of the maps defined by (2.4) and (2.5) by computing the derivatives in the space $\mathscr{D}$ and restricting to tangent vectors given by (2.6).

3. First variation

In this section we compute the Frechet derivative of each of the maps (1.1) and (1.2) using the natural charts described in the previous section.

Proposition 3.1. *Suppose that L is a divergence form operator and that $\mathscr{E}_k : \mathscr{D}_v \to \mathbb{R}$ is as defined in* (1.2). *Let $D \in \mathscr{D}_v$ and suppose that $\delta_D\mathscr{E}_k : T_D\mathscr{D}_v \to \mathbb{R}$ is the derivative of $\mathscr{E}_k$ at D. Then*

$$\delta_D\mathscr{E}_k(f) = \sum_{j=1}^{k} c_j \int_{\partial D} f \frac{\partial u_{k+1-j}}{\partial v} \langle \nabla u_j, v \rangle_L \, d\sigma \tag{3.1}$$

where $f \in C^\infty\ (\partial D)$ is a tangent vector, $d\sigma$ is surface measure on ∂D, v is the outward pointing unit normal vector, $u_j(x)$ solves (2.3), *$\langle \nabla u_j, v \rangle_L = (\nabla u_j)^T (a_{ij}) v$ is the conormal derivative associated to L, and the constant c_j is given by*

$$c_j = \frac{k!}{(k+1-j)!j!} \quad . \tag{3.2}$$

Proof. Let $D \in \mathscr{D}$ and write $S = \partial D$. Suppose $f \in C^\infty(\partial D)$ represents an infinitesimal variation of D. Let Φ_t be a one-parameter family of diffeomorphisms associated to f, let $S_t = \Phi_t(S)$ and let D_t be the corresponding one-parameter family of domains. Let u_k be the solution of (2.3) and let u_k^t be the solution of

$$\begin{aligned} Lu_k^t + ku_{k-1}^t &= 0 \text{ on } D_t \\ u_k^t &= 0 \text{ on } \partial D_t = S_t \ . \end{aligned} \tag{3.3}$$

Then

$$\begin{aligned} \delta_D \mathscr{E}_k(f) &= \frac{d}{dt}\bigg|_{t=0} \left[\int_{D_t} u_k^t - \int_D u_k \right] \\ &= \frac{d}{dt}\bigg|_{t=0} \left[\int_{D_t \cap D} (u_k^t - u_k) + \int_{D_t \setminus D} u_k^t - \int_{D \setminus D_t} u_k \right] \ . \end{aligned}$$

Recall, u_k vanishes on S and u_k^t vanishes on S_t. Hence, $\frac{d}{dt}\big|_{t=0}\left[\int_{D_t\setminus D} u_k^t - \int_{D\setminus D_t} u_k\right] = 0$. We conclude that

$$\delta_D \mathscr{E}_k(f) = \frac{d}{dt}\bigg|_{t=0} \left[\int_{D_t \cap D} (u_k^t - u_k) \right] \ . \tag{3.4}$$

On $D_t \cap D$, let

$$P_j = u_j L(u_{k+1-j}^t - u_{k+1-j}) - (u_{k+1-j}^t - u_{k+1-j}) L u_j \ . \tag{3.5}$$

A straight-forward computation yields

$$(u_k^t - u_k) = \sum_{j=1}^{k} c_j P_j \tag{3.6}$$

where the c_j are the constants given in (3.2). By the divergence theorem,

$$\begin{aligned} \int_{D_t \cap D} P_j = \int_{\partial(D_t \cap D)} &[u_j \langle \nabla(u_{k+1-j}^t - u_{k+1-j}), v\rangle_L \\ &- (u_{k+1-j}^t - u_{k+1-j}) \langle \nabla u_j, v\rangle_L] \ dz \end{aligned}$$

where v is the outward normal vector to $D_t \cap D$, $\langle \nabla u_j, v\rangle_L$ is the conormal derivative associated to L, and dz is surface measure. We conclude that

$$\frac{d}{dt}\bigg|_{t=0} \int_{D_t \cap D} P_j = \frac{d}{dt}\bigg|_{t=0} \int_{\partial(D_t \cap D)} (u_{k+1-j} - u_{k+1-j}^t) \langle \nabla u_j, v\rangle_L \ dz \ .$$

The boundary $\partial(D_t \cap D)$ can be partitioned as $\partial(D_t \cap D) = S^+ \cup S_t^-$, where

$$S^+ = \{\sigma \in S \ : f(\sigma) \geq 0\}$$
$$S^- = \{\sigma \in S \ : f(\sigma) < 0\}$$
$$S_t^- = \{y = \sigma + tf(\sigma)v(\sigma) \in S_t \ : \sigma \in S_-\} \ .$$

We write

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \int_{\partial(D_t \cap D)} (u_{k+1-j} - u^t_{k+1-j}) \langle \nabla u_j, v \rangle_L \, dz \\
&\quad = \left.\frac{d}{dt}\right|_{t=0} \left[\int_{S^+} (u_{k+1-j} - u^t_{k+1-j}) \langle \nabla u_j, v \rangle_L \, d\sigma \right. \\
&\qquad \left. + \int_{S_t^-} (u_{k+1-j} - u^t_{k+1-j}) \langle \nabla u_j, v_t \rangle_L \, dz \right] \ ,
\end{aligned}$$

where v_t is the outward normal to S_t and $d\sigma$ is surface measure. Expanding u_{k+1-j} near $t = 0$

$$\begin{aligned}
u_{k+1-j}(\Phi_t) - u_{k+1-j}(\Phi_o) &= t \left\langle \nabla u_{k+1-j}, \left.\frac{d}{dt}\right|_{t=0} \Phi_t \right\rangle + o(t) \\
&= tf \frac{\partial u_{k+1-j}}{\partial v} + o(t) \ .
\end{aligned} \tag{3.7}$$

Let J_t be the Jacobian of the map Φ_t and note that $J_t(\sigma) = 1 + O(t)$. Then,

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \int_{S_t^-} (u_{k+1-j} - u^t_{k+1-j}) \langle \nabla u_j, v \rangle_L \, dz \\
&\quad = \left.\frac{d}{dt}\right|_{t=0} \int_{S_-} u_{k+1-j}(\Phi_t(\sigma)) \langle \nabla u_j(\Phi_t(\sigma)), v(\Phi_t(\sigma)) \rangle_L J_t(\sigma) \, d\sigma
\end{aligned}$$

and, using (3.7), we conclude that

$$\left.\frac{d}{dt}\right|_{t=0} \int_{S_t^-} (u_{k+1-j} - u^t_{k+1-j}) \langle \nabla u_j, v \rangle_L \, dz = \int_{S_-} f \frac{\partial u_{k+1-j}}{\partial v} \langle \nabla u_j, v \rangle_L \, d\sigma \ .$$

Similarly,

$$\left.\frac{d}{dt}\right|_{t=0} \int_{S^+} (u_{k+1-j} - u^t_{k+1-j}) \langle \nabla u_j, v \rangle_L \, d\sigma = \int_{S_+} f \frac{\partial u_{k+1-j}}{\partial v} \langle \nabla u_j, v \rangle_L \, d\sigma \ .$$

Hence,

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \int_{\partial(D_t \cap D)} (u_{k+1-j} - u^t_{k+1-j}) \langle \nabla u_j, v \rangle_L \, dz \\
&\quad = \int_S f \frac{\partial u_{k+1-j}}{\partial v} \langle \nabla u_j, v \rangle_L \, d\sigma \ .
\end{aligned} \tag{3.8}$$

Combining (3.4)–(3.6) and (3.8) completes the proof of the proposition.

Proposition 3.2. *Suppose that L is a divergence form operator and that $\mathscr{E}_{k,x_0} : \mathscr{D}_v \to \mathbb{R}$ is as defined in* (1.1). *Let $D \in \mathscr{D}_v$ and suppose that $\delta_D \mathscr{E}_{k,x_0} : T_D \mathscr{D}_v \longrightarrow \mathbb{R}$ is the derivative of $\mathscr{E}_{k,x_0}$ at D. Then*

$$\begin{aligned}\delta_D \mathscr{E}_{k,x_0}(f) = \int_{\partial D} f \Bigg[& \frac{\partial G}{\partial v}(x_0, \cdot)\langle \nabla u_k, v\rangle_L \\ & + \sum_{j=1}^{k-1} d_j \frac{\partial u_{k-j}}{\partial v}\langle \nabla m_j, v\rangle_L \Bigg] \, d\sigma\end{aligned} \tag{3.9}$$

where $f \in C^\infty(\partial D)$ is a tangent vector, G is the Green's function for the operator L on the domain D, $d\sigma$ is surface measure on ∂D, v is the outward pointing unit normal vector, $u_j(x)$ solves (2.3), *$\langle \nabla u_j, v\rangle_L = (\nabla u_j)^T (a_{ij}) v$ is the conormal derivative associated to L, the function m_j are defined inductively as the solutions of the boundary value problems given by*

$$\begin{aligned} Lm_1 + G(x_0, \cdot) &= 0 \text{ on } D \\ m_1 &= 0 \text{ on } \partial D \end{aligned}$$

and

$$\begin{aligned} Lm_j + m_{j-1} &= 0 \text{ on } D \\ m_j &= 0 \text{ on } \partial D \end{aligned} \tag{3.10}$$

and the constant d_j is given by

$$d_j = \frac{k!}{(k-j)!} \ . \tag{3.11}$$

Proof. The proof is similar to the proof of the previous proposition. Let $D \in \mathscr{D}$ and write $S = \partial D$. Suppose $f \in C^\infty(S)$ represents an infinitesimal variation of D. Let Φ_t be a one-parameter family of diffeomorphisms associated to f, let $S_t = \Phi_t(S)$ and let D_t be the corresponding one-parameter family of domains. Let u_k be the solution of (2.3) and let u_k^t be the solution of (3.3). Let $G = G^0 = G(x_0, \cdot)$ be the Green's function for the operator L on the domain D and let $G^t = G^t(x_0, \cdot)$ be the Green's function for the operator L on the domain D^t. Then

$$\begin{aligned}\delta_D \mathscr{E}_{k,x_0}(f) &= \frac{d}{dt}\bigg|_{t=0}\left[k\int_{D_t} u^t_{k-1}G^t - k\int_D u_{k-1}G\right]\\ &= \frac{d}{dt}\bigg|_{t=0}\left[k\int_{D_t\cap D} u^t_{k-1}(G^t - G) + (u^t_{k-1} - u_{k-1})G\right]\\ &\quad + \frac{d}{dt}\bigg|_{t=0}\left[k\int_{D_t\setminus D} u^t_{k-1}G^t - k\int_{D\setminus D_t} u_{k-1}G\right] .\end{aligned}$$

Recall, u_{k-1} and $G^t(x_0,\cdot)$ vanish on S and u^t_{k-1} and $G(x_0,\cdot)$ vanish on S_t. Hence,

$$\frac{d}{dt}\bigg|_{t=0}\left[\int_{D_t\setminus D} u^t_{k-1}G^t - \int_{D\setminus D_t} u_{k-1}G\right] = 0 .$$

We conclude that

$$\delta_D \mathscr{E}_{k,x_0}(f) = \frac{d}{dt}\bigg|_{t=0}\left[k\int_{D_t\cap D} u^t_{k-1}(G^t - G) + (u^t_{k-1} - u_{k-1})G\right] . \tag{3.12}$$

Let m_j be given as in (3.10) and suppose m^t_j is defined inductively by

$$\begin{aligned} Lm^t_1 + G^t(x_0,\cdot) &= 0 \text{ on } D_t\\ m^t_1 &= 0 \text{ on } \partial D_t\end{aligned}$$

and

$$\begin{aligned} Lm^t_j + m^t_{j-1} &= 0 \text{ on } D_t\\ m^t_j &= 0 \text{ on } \partial D_t .\end{aligned}$$

For $x \in D_t \cap D$, note that $G^t(x_0,\cdot) - G(x_0,\cdot)$ satisfies $L(G^t - G) = 0$. In particular,

$$u^t_{k-1}(G^t - G) = \frac{1}{k}[u^t_k L(G^t - G) - (G^t - G)Lu^t_k] . \tag{3.13}$$

Let

$$R_j = m_j L(u^t_{k-j} - u_{k-j}) - (u^t_{k-j} - u_{k-j})Lm_j . \tag{3.14}$$

Note that

$$k(u^t_{k-1} - u_{k-1})G = \sum_{j=1}^{k-1} d_j R_j \tag{3.15}$$

where d_j are the constants given in (3.11). By the divergence theorem,

$$\left.\frac{d}{dt}\right|_{t=0}\int_{D_t\cap D} u_{k-1}^t(G^t-G) = \left.\frac{d}{dt}\right|_{t=0}\frac{1}{k}\int_{\partial(D_t\cap D)} u_k^t\langle\nabla(G^t-G),v\rangle_L \\ -(G^t-G)\langle\nabla u_k^t,v\rangle_L\,dz \\ = \left.\frac{d}{dt}\right|_{t=0} -\frac{1}{k}\int_{\partial(D_t\cap D)}(G^t-G)\langle\nabla u_k^t,v\rangle_L\,dz \tag{3.16}$$

and

$$\left.\frac{d}{dt}\right|_{t=0}\int_{D_t\cap D} R_j = \left.\frac{d}{dt}\right|_{t=0}\int_{\partial(D_t\cap D)} m_j\langle\nabla(u_{k-j}^t-u_{k-j}),v\rangle_L \\ -(u_{k-j}^t-u_{k-j})\langle\nabla m_j,v\rangle_L\,dz \\ = \left.\frac{d}{dt}\right|_{t=0} -\int_{\partial(D_t\cap D)}(u_{k-j}^t-u_{k-j})\langle\nabla m_j,v\rangle_L\,dz \tag{3.17}$$

where v is the outward normal vector to $D_t\cap D$ and $\langle\nabla v_j,v\rangle_L$ is the conormal derivative associated to L.

As in Proposition 3.1, the boundary $\partial(D_t\cap D)$ can be partitioned as $\partial(D_t\cap D)=S^+\cup S_t^-$, where

$$S^+ = \{\sigma\in S\ : f(\sigma)\geq 0\}$$
$$S^- = \{\sigma\in S\ : f(\sigma)<0\}$$
$$S_t^- = \{y=\sigma+tf(\sigma)v(\sigma)\in S_t\ :\sigma\in S_-\}\ .$$

Beginning with (3.16) and proceeding as in Proposition 3.1 we obtain

$$\left.\frac{d}{dt}\right|_{t=0} k\int_{D_t\cap D} u_{k-1}^t(G^t-G) = \int_S f\frac{\partial G}{\partial v}(x_0,\cdot)\langle\nabla u_k,v\rangle_L\,d\sigma \tag{3.18}$$

where v is the outward pointing unit normal vector to S. Similarly, beginning with (3.17) and proceeding as in Proposition 3.1, we obtain

$$\left.\frac{d}{dt}\right|_{t=0}\int_{D_t\cap D} R_j = \int_S f\frac{\partial u_{k-j}}{\partial v}\langle\nabla m_j,v\rangle_L\,d\sigma\ . \tag{3.19}$$

Combining (3.12), (3.13), (3.15), (3.18) and (3.19) concludes the proof of the proposition.

Proof of Theorem 1.1. Theorem 1.1 follows immediately from Proposition 3.1, Proposition 3.2, and the characterization of the tangent space given in (2.6).

4. Brownian motion

Proof of Proposition 1.1. We begin by noting that for $L = \frac{1}{2}\Delta$, where Δ is the Laplace operator, Theorem 1.1 implies that D is a critical point for $\mathscr{E}_k$ if and only if there is a solution to the heirarchy of boundary value problems

$$\begin{aligned} \frac{1}{2}\Delta u_1 + 1 &= 0 \text{ on } D \\ u_1 &= 0 \text{ on } \partial D \ , \\ \frac{1}{2}\Delta u_j + j u_{j-1} &= 0 \text{ on } D \\ u_j &= 0 \text{ on } \partial D, \quad \text{for } j = 2, \ldots, k \\ \sum_{j=1}^{k} c_j \frac{\partial u_{k+1-j}}{\partial \nu} \frac{\partial u_j}{\partial \nu} &= C \text{ on } \partial D \ . \end{aligned} \tag{4.1}$$

Similarly, Theorem 1.1 implies that D is a critical point for $\mathscr{E}_{k,x_0}$ if and only if there is a solution to the heirarchy of boundary value problems

$$\begin{aligned} \frac{1}{2}\Delta u_1 + 1 &= 0 \text{ on } D \\ u_1 &= 0 \text{ on } \partial D \ , \\ \frac{1}{2}\Delta u_j + j u_{j-1} &= 0 \text{ on } D \\ u_j &= 0 \text{ on } \partial D, \quad \text{for } j = 2, \ldots, k \\ \frac{\partial G}{\partial \nu}(x_0, \cdot)\frac{\partial u_k}{\partial \nu} + \sum_{j=1}^{k-1} d_j \frac{\partial u_{k-j}}{\partial \nu} \frac{\partial m_j}{\partial \nu} &= C' \ . \end{aligned} \tag{4.2}$$

where G and m_j are as in the statement of Theorem 1.1.

Suppose that D is a ball of volume v and radius r. Suppose that $k = 1$. Then the solution to (4.1) is given by

$$u_1(x) = \frac{r^2 - \sum x_i^2}{d}$$

where the x_i are Euclidean coordinates with origin at the center of the ball. We can express u_k in terms of u_{k-1} and the Green's function for D:

$$u_k(x) = -k \int_D u_{k-1}(y) G(x, y)\, dy \ .$$

In particular, if u_{k-1} is radial with respect to the center of D, then u_k is radial with respect to the center of D. This proves that a ball of volume v is a critical point for each of the functional $\mathscr{E}_k$ and $\mathscr{E}_{k,x_0}$ for every k.

The converse statement is much involved. The proof we give follows the ideas developed in [FM] which in turn is inspired by the beautiful paper of Serrin, [S]. The argument is by moving planes.

Let D be a critical point for the function $\mathscr{E}_k$. By Theorem 1.1, D is a domain for which we have a solution, u_k, of the overdetermined boundary value problem (4.1).

In [S] Serrin uses the method of moving planes to prove the following fundamental result:

Theorem S. *Suppose that $D \subset \mathbb{R}^d$ is a smoothly bounded, connected domain with compact closure and that Δ is the Laplace operator. Suppose that u solves*

$$\begin{aligned} \Delta u + 1 &= 0 \text{ on } D \\ u &= 0 \text{ on } \partial D \\ \frac{\partial u}{\partial \nu} &= k' \text{on } \partial D \ . \end{aligned} \tag{4.3}$$

Then D is a ball and u is a radially symmetric function.

This establishes that D is a critical point for the functional $\mathscr{E}_k$ if and only if D is a ball of volume v in the case $k = 1$. In [FM] the authors use a moving planes argument to establish that D is a critical point for the functional $\mathscr{E}_k$ if and only if D is a ball of volume v for the case $k = 2$. It is also shown in [FM] that D is a critical point for the functional $\mathscr{E}_{k,x_0}$ if only if D is a ball of volume v centered at x_0 for the case $k = 1$. In the sequel we will modify the proofs given in [FM] to complete the proof of Proposition 1.1 for all k.

Fix a hyperplane $H_0 \subset \mathbb{R}^d$ not intersecting D. Move the hyperplane parallel to itself toward D. The hyperplane will eventually cut off from D an open cap, $\Sigma(H)$, where H is parallel to H_0 and intersects D. (By "cap" we do not preclude the possibility that the region being cut off from D is disconnected, which may occur if D is not convex.) The reflection of $\Sigma(H)$ in H, denoted $\Sigma'(H)$, will initially lie in D. This will remain true as H continues to move into D, until one of the following two conditions occurs:

(i) $\Sigma'(H)$ is internally tangent to ∂D at a point P not on H, or
(ii) H is orthogonal to ∂D at some point Q.

Assume that H satisfies either (i) or (ii) above. Define a function U_k in $\Sigma' = \Sigma'(H)$ by $U_k(x) = u_k(\sigma(x))$, where σ is the reflection map about H. Note that $\Delta U_k(x) = -kU_{k-1}$ on Σ' and that U_k satisfies the following boundary conditions in Σ':

$$\begin{aligned} &U_k = u_k \text{ on } H \cap \partial\Sigma' \\ &U_k = 0 \text{ on } H^c \cap \partial\Sigma' \\ &\sum_{j=1}^{k} c_j \frac{\partial U_{k+1-j}}{\partial v} \frac{\partial U_j}{\partial v} = C \end{aligned} \tag{4.4}$$

where the c_j and C are as in (4.1), above.

To show that D is a sphere, if suffices to show that D is symmetric about H for arbitrary initial data H_0. To show that D is symmetric about H, it sufficies to show that $u_j = U_j$ for some value of $j \le k$.

Assume that $u_j \ne U_j$ for all $j \le k$. We require the following lemma:

Lemma 4.1. *Suppose that u_k satisfies* (4.1). *Suppose that $\Sigma' = \Sigma'(H)$ and U_k are defined as above and that $u_j \ne U_j$ for all $j \le k$. Then $u_k > U_k$ on the interior of Σ'.*

Proof. Suppose that $k = 1$. Then $\Delta(u_k - U_k) = 0$. From (4.4) $U_k = 0$ on $H^c \cap \partial\Sigma'$ and, by the maximum principle, $u_k \ge 0$ on Σ'. Hence, by the maximum principle and the assumption that $u_k \ne U_k$, $u_k > U_k$ on Σ' when $k = 1$. Suppose that $u_{k-1} > U_{k-1}$ on Σ'. Then $\Delta(u_k - U_k) = -k(u_{k-1} - U_{k-1}) < 0$ on Σ'. Since $u_k - U_k \ge 0$ on $\partial\Sigma'$, we get that $u_k > U_k$ on the interior of Σ' for all k by the maximum principle.

Suppose now that H satisfies condition (i). Then by the boundary point maximum principle (cf [PW]),

$$\frac{\partial}{\partial v}(u_j - U_j) > 0 \text{ at } P$$

for each value of $j \le k$. This contradicts the fact that at P,

$$\sum_{j=1}^{k} c_j \frac{\partial u_{k+1-j}}{\partial v} \frac{\partial u_j}{\partial v} = \sum_{j=1}^{k} c_j \frac{\partial U_{k+1-j}}{\partial v} \frac{\partial U_j}{\partial v} \quad ,$$

where the c_j are the positive constants occuring in Proposition 3.1.

Next, suppose that H satisfies condition (ii). We can not apply the standard version of the Hopf boundary point maximum principle (as we did above) as the exterior sphere condition fails at Q. We will use a sharpening of the theorem due to Serrin.

Choose coordinates in $\mathbb{R}^d$ such that the origin is at Q, the positive x_n-axis has the same direction as the inward normal to ∂D at Q, and the positive x_1-axis is normal to H, pointing away from Σ'. Let s be a vector at Q which is nontangential with respect to $\partial\Sigma'$, and let $\frac{\partial}{\partial s}$ be the corresponding directional derivative. We will apply the following lemma (essentially Lemma 1 of [S], cf also [FM]):

Lemma 4.2 *Suppose that* $w \in C^2(\Sigma')$, $\Delta w \leq 0$ it and $w \geq 0$ at Q. *Then if* $w \neq 0$, *either*

$$\frac{\partial w}{\partial s} > 0 \ or \ \frac{\partial^2 w}{\partial s^2} > 0 \ at \ Q \quad .$$

As in [FM], we apply this lemma to $u_j - U_j$, $j \leq k$, with $\frac{\partial}{\partial s} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_n}$. Note that near Q,

$$U_j(x_1, x_2, \ldots . x_n) = u_j(-x_1, x_2, \ldots, x_n), \quad \text{for all } j \leq k \ . \tag{4.5}$$

From this it follows that at Q, $\frac{\partial u_j}{\partial x_n} = \frac{\partial U_j}{\partial x_n}$, $j \leq k$. Since u_j is zero on ∂D and x_1 is a tangential direction at Q, $\frac{\partial u_j}{\partial x_1} = 0$, $j \leq k$, at the point Q. Hence, $\frac{\partial u_j}{\partial x_1} = \frac{\partial U_j}{\partial x_1}$, $j \leq k$, and we obtain $\frac{\partial(u_j - U_j)}{\partial s} = 0$ at Q. Using Lemma 4.2, we conclude that

$$\frac{\partial^2}{\partial s^2}(u_j - U_j) > 0 \text{ at } Q \text{ for all } j \leq k \ . \tag{4.6}$$

Now we compute:

$$\frac{\partial^2}{\partial s^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_n^2} - 2\frac{\partial^2}{\partial x_1 \partial x_n} \quad .$$

From (4.5) we conclude

$$\frac{\partial^2 u_j}{\partial x_1^2} = \frac{\partial^2 U_j}{\partial x_1^2} \text{ for all } j \leq k$$

$$\frac{\partial^2 u_j}{\partial x_n^2} = \frac{\partial^2 U_j}{\partial x_n^2} \text{ for all } j \leq k \ .$$

Since u_j, $1 \leq j \leq k$, vanishes on ∂D, ∇u_j is normal to ∂D and we can write the boundary condition given in (4.1) as

$$\sum_{i=1}^{n} \sum_{j=1}^{k} c_j \frac{\partial u_j}{\partial x_i} \ \frac{\partial u_{k+1-j}}{\partial x_i} = C \ .$$

If we differentiate this in the tangential direction x_1 we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{k} c_j\left(\frac{\partial^2 u_j}{\partial x_i \partial x_1}\ \frac{\partial u_{k+1-j}}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\ \frac{\partial^2 u_{k+1-j}}{\partial x_i \partial x_1}\right) = 0 \ . \tag{4.7}$$

But, at Q, $\frac{\partial u_j}{\partial x_i} = 0$ for $j \leq k, 1 \leq i \leq n-1$. Hence, (4.7) becomes

$$\sum_{j=1}^{k} c_j\left(\frac{\partial^2 u_j}{\partial x_n \partial x_1}\ \frac{\partial u_{k+1-j}}{\partial x_n} + \frac{\partial u_j}{\partial x_n}\ \frac{\partial^2 u_{k+1-j}}{\partial x_n \partial x_1}\right) = 0 \text{ at } Q \ .$$

Similar reasoning allows us to conclude

$$\sum_{j=1}^{k} c_j\left(\frac{\partial^2 U_j}{\partial x_n \partial x_1}\ \frac{\partial U_{k+1-j}}{\partial x_n} + \frac{\partial U_j}{\partial x_n}\ \frac{\partial^2 U_{k+1-j}}{\partial x_n \partial x_1}\right) = 0 \text{ at } Q \ .$$

As we saw above, $\frac{\partial U_j}{\partial x_n} = \frac{\partial u_j}{\partial x_n}$ at Q. Subtracting we obtain

$$\begin{aligned}\sum_{j=1}^{k} c_j\Bigg(\left(\frac{\partial^2 u_j}{\partial x_n \partial x_1} - \frac{\partial^2 U_j}{\partial x_n \partial x_1}\right)\ &\frac{\partial u_{k+1-j}}{\partial x_n} \\ &+ \frac{\partial u_j}{\partial x_n}\ \left(\frac{\partial^2 u_{k+1-j}}{\partial x_n \partial x_1} - \frac{\partial^2 U_{k+1-j}}{\partial x_n \partial x_1}\right)\Bigg) = 0 \text{ at } Q \ .\end{aligned}$$

This contradicts (4.6), the fact that the constants c_j are positive, and the fact that $\frac{\partial u_j}{\partial x_n}$ is positive at Q. This concludes the proof that critical points for the functional $\mathscr{E}_k$ must be balls of the appropriate volume.

To treat the functional ξ_{k,x_0}, we assume that D is a critical point for $\mathscr{E}_{k,x_0}$ and that H is a hyperplane satisfying either condition (i) or (ii) above. Let G be the Green's function for the Laplace operator on D,

$$\begin{aligned}\Delta G &= -\delta_{x_0} \text{ on } D \\ G &= \ 0 \text{ on } \partial D\end{aligned}$$

where δ_{x_0} is the delta mass at x_0. For notational convenience we will write $G = m_0$. As above, let $\Sigma = \Sigma(H)$ be the cap defined by H and let $\Sigma' = \Sigma'(H)$ be the reflection of Σ about H. Define a function M_k in $\Sigma' = \Sigma'(H)$ by $M_k(x) = m_k(\sigma(x))$, where σ is the reflection map about H. Note that $\Delta M_k(x) = -M_{k-1}$ on Σ' and that M_k satisfies the following boundary conditions in Σ':

$$\begin{aligned}M_k &= m_k \text{ on } H \cap \partial\Sigma' \\ M_k &= 0 \text{ on } H^c \cap \partial\Sigma'\end{aligned}$$

$$\sum_{j=0}^{k-1} d_j \frac{\partial U_{k-j}}{\partial v}\frac{\partial M_j}{\partial v} = C' \text{on } \partial D \cap \partial\Sigma' \tag{4.8}$$

where the d_j and C' are as in (1.4) and U_{k-j} satisfy (4.4), as above.

We will show that D is symmetric about H for arbitrary initial data H_0 and, in addition, that $x_0 \in H$. To show that D is symmetric about H, it suffices to show that $u_j = U_j$ or that $m_j = M_j$ for some value of $j \leq k$.

To begin, assume that $x_0 \in H$. Then, as in Lemma 4.1, $m_j - M_j$ is superharmonic on Σ' and nonnegative on $H^c \cap \partial\Sigma'$. Hence, applying the proof given above to the pairs m_j, u_{k-j} and M_j, U_{k-j} we conclude that D is symmetric about H.

Next, assume that $x_0 \notin H$. There are three possibilities:

Case 1: $x_0 \in \Sigma'$. Note than $m_j - M_j$ is still superharmonic for all j. Hence, the argument given above leads immediately to a contradiction.

Case 2: $x_0 \notin \Sigma' \cup H \cup \Sigma$. As above, $m_j - M_j$ is superharmonic on Σ'. Hence, by the above argument, D is symmetric about H and $D = \Sigma' \cup (D \cap H) \cup \Sigma$, contradicting the fact that $x_0 \in D$.

Case 3: $x_0 \in \Sigma$. In this case, $m_j - M_j$ is no longer superharmonic. To obtain a contradiction, consider the family of planes parallel to H_0 but starting on the opposite side of D. These planes eventually reach a stopping plane, H^*, which defines a cap $\Sigma(H^*)$ and its reflection, $\Sigma'(H^*)$. Note that the moving planes defining H^* never pass the plane H because $\Sigma'(H) \subset D$ and $\Sigma'(H^*) \subset D$. Since $x_0 \in \Sigma$, it follows that $x_0 \notin \Sigma(H^*)$. Hence, either case 1 or case 2, above, holds for the case where the plane H is replaced by the plane H^* and we obtain a contradiction.

This proves that D is a critical point for the functional ξ_{k,x_0} if and only if D is a ball of volume v centered at x_0.

This concludes the proof of Proposition 1.1.

Remark 4.1: It would be natural to hope that these results could be generalized and extended to the case of non-constant coefficient operators. One natural direction in which to proceed would be to consider those diffusions with associated infinitesimal generator given by the Laplace operator with respect to some Riemannian metric on $\mathbb{R}^n$ (these operators will in general not fall into the class of divergence form operators, as defined in the introduction). In this context, one would hope that the exit time moments and the averaged (with respect to the associated induced volume form) exit time moments would be maximized at geodesic balls and that the critical points of the associated functionals could be characterized by overdetermined boundary value problems which “rigidly” determine the domain. There are a number of obstructions to carrying out such a program. First, the boundedness of the functionals determined by the exit time moments on the set of domains of constant volume must be established. There are a number of restrictions which will assure that this is the case. For example, if the metric satisfies a classic isoperimetric inequality (not automatically satisfied), the functionals will be bounded. Next, one must establish powerful results analogous to Serrin's theorem. While such theorems are not true in the general case, there are contexts (such as constant curvature space forms) for which there is hope that such results hold. This will be treated in another paper.

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