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Limit theorems for bivariate Appell polynomials. Part II: Non-central limit theorems*

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Summary. Let $(X_t, t \in \mathbf{Z})$ be a linear sequence with non-Gaussian innovations and a spectral density which varies regularly at low frequencies. This includes situations, known as strong (or long-range) dependence, where the spectral density diverges at the origin. We study quadratic forms of bivariate Appell polynomials of the sequence (X_t) and provide general conditions for these quadratic forms, adequately normalized, to converge to a non-Gaussian distribution. We consider, in particular, circumstances where strong and weak dependence interact. The limit is expressed in terms of multiple Wiener-Itô integrals involving correlated Gaussian measures.

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1 Introduction

We pursue the study, started in Giraitis and Taqqu [12], of the asymptotic distribution of quadratic forms

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$$Q_N := \sum_{t,s=1}^{N} b(t-s)P_{m,n}(X_t, X_s) \tag{1.1}$$

in bivariate Appell polynomials $P_{m,n}$, as $N \to \infty$. Here

$$X_t := \sum_{u \in \mathbf{Z}} a(t-u)\xi_u, \quad t \in \mathbf{Z} \tag{1.2}$$

is a linear sequence. The weights $a(u)$ of the linear sequence satisfy $\sum_u a^2(u) < \infty$ and the innovations ξ_u's are i.i.d. random variables with zero mean, variance 1 and finite $2(m+n)$-th moment so as to ensure that $EP^2_{m,n}(X_t, X_s) < \infty$.

The case $m = n = 1$ which corresponds to $P_{m,n}(X_t, X_s) = X_tX_s - EX_tX_s$, was studied by Fox and Taqqu [3] when $\{X_t\}$ is Gaussian and by Giraitis and Surgailis [8] when $\{X_t\}$ is a linear sequence of the form (1.2). The corresponding limits of Q_N are either Gaussian or non-Gaussian depending on the behavior at the origin of both the spectral density $f(x) = 2\pi|\hat{a}(x)|^2$, $-\pi < x < \pi$ of $\{X_t\}$ and the Fourier transform $\hat{b}(x)$ of the kernel $\{b(t)\}$. Specifically, if $\hat{a}(x) = [|x|^{-\alpha}L(1/|x|)]^{1/2}$ and $\hat{b}(x) = |x|^{-\beta}L_1(1/|x|)$, where L and L_1 are slowly-varying functions at infinity, then Q_N converges to a normal distribution if $\alpha + \beta < 1/2$ and to a non-normal distribution if $\alpha + \beta > 1/2$.

These results have been used to derive the asymptotic behavior of the Whittle estimator of α (Fox and Taqqu [4], Giraitis and Surgailis [8]). The parameter α is of interest because it measures the "intensity" of long-memory in the sequence $\{X_t\}$. We shall say that $\{X_t\}$ is *strongly dependent* if $\alpha > 0$ and *weakly dependent* if $\alpha \leq 0$. In the case of strong dependence the spectral density $f(x)$ of $\{X_t\}$ blows-up at the origin.

Terrin and Taqqu [18] started the investigation of the non-linear case (1.1), where $P_{m,n}$, $m + n > 2$, is an Appell polynomial of the linear sequence X_t. Appell polynomials are natural in this context because they are adapted to the distribution of X_t. In this paper, we weaken the assumptions of Terrin and Taqqu (replacing the assumption on $\hat{a}$ by one on f) and deal with all possible cases where the limit is non-Gaussian, including the particularly delicate situations where strong and weak dependence interact (see Case (A_2) in Section 2). We also include the cases $m = 0$ or $n = 0$. The limits are expressible as multiple Wiener-Itô integrals with *correlated* Gaussian measures. This paper, together with [12, 10] which focus on the central limit theorem, thus provides a complete description of the asymptotic behavior of the quadratic form Q_N as $N \to \infty$.

Extensions to functional limit theorems appear in Giraitis and Taqqu [11]. For applications to Whittle estimation, see Giraitis and Taqqu [9]. Related papers include Taqqu [16], Dobrushin and Major [2], Major [15], Avram and Taqqu [1], Giraitis and Surgailis [6, 7], Terrin and Taqqu [17], Ho and Hsing [13], Koul and Surgailis [14].

The paper is structured as follows. Section 2 contains the definition of the Appell polynomials, the precise assumptions on the spectral density $f(x)$ of X_t and on the kernel $\{b(t)\}$ in (1.1), as well as the statements of the main theorems. Theorem 2.1 deals with the case $m, n > 1$ and Theorem 2.2 with the case where either m or n is zero. Theorem 2.3 provides a multivariate generalization. The proof of the theorems are given in Section 3. These proofs use a number of propositions presented in Section 4 that reduce the problem to the weak convergence of step functions approximations. The weak convergence of these approximations is established in Section 5.

2 Assumptions and results

We start with a definition of the multivariate *Appell polynomials*

$$P_{n_1,\dots,n_k}(X_{t_1},\dots,X_{t_k}) \equiv : \underbrace{X_{t_1},\dots,X_{t_1}}_{n_1},\dots,\underbrace{X_{t_k},\dots,X_{t_k}}_{n_k} :, \quad n_1,\dots,n_k = 0,1,\dots$$

The alternate notation ": . . . :", called *Wick product*, is convenient and we shall use it as well (the indices in P correspond to the number of times that the variables in ":   :" are repeated). The Appell polynomials $P_{n_1,\dots,n_k}$ can be defined by $P_{0,\dots,0} \equiv 1$ and the recurrence relations

$$\frac{\partial}{\partial x_j} P_{n_1,\dots,n_k}(x_1,\dots,x_k) = n_j P_{n_1,\dots,n_j-1,\dots,n_k}(x_1,\dots,x_k) \ ,$$

$$EP_{n_1,\dots,n_k}(X_{t_1},\dots,X_{t_k}) = 0 \ .$$

The first relation indicates that these polynomials behave like power functions. The second relation provides the constants of integration and relates the polynomial to the joint distribution of the X_t's. The multivariate Appell polynomials can also be defined by the generating function

$$\sum_{n_1,\dots,n_k=0}^{\infty} \frac{z_1^{n_1}}{n_1!} \cdots \frac{z_k^{n_k}}{n_k!} P_{n_1,\dots,n_k}(x_1,\dots,x_k) = \frac{\exp(\sum_1^k z_j x_j)}{E\exp(\sum_1^k z_j X_{t_j})}$$

(see [1], [6] for more details). The univariate Appell polynomials

$$P_m(X_t) = : \underbrace{X_t,\dots,X_t}_{m} : \equiv : X_t^{(m)} : \tag{2.1}$$

were considered by Avram and Taqqu [1], Giraitis and Surgailis [6]. If X_t is Gaussian, then $P_n(X_t)$ is the Hermite polynomial. Among all Appell polynomials, the Hermite polynomials are the only ones that are orthogonal. In spite of their lack of orthogonality, the multivariate Appell polynomials possess nice probabilistic properties such as the diagram formula for moments and cumulants. We are interested here in the bivariate Appell polynomials

$$P_{m,n}(X_t, X_s) \equiv : \underbrace{X_t,\dots,X_t}_{m}, \underbrace{X_s,\dots,X_s}_{n} : \ . \tag{2.2}$$

In Giraitis and Taqqu [12] we found conditions for Q_N to satisfy the central limit theorem (CLT). We focus, in this paper, on non-central limit theorems (NCLT) for Q_N. We assume that[1]

$$f(x) \equiv 2\pi|\hat{a}(x)|^2 = |x|^{-\alpha}L(1/|x|), \quad \text{as} \quad x \to 0 \quad (-\infty < \alpha < 1) \ , \tag{2.3}$$

$$\hat{b}(x) = |x|^{-\beta}L_1(1/|x|), \quad \text{as} \quad x \to 0 \quad (-\infty < \beta < 1) \tag{2.4}$$

where $f(x)$, $x \in [-\pi, \pi]$ is spectral density of the sequence (X_i); $\hat{b}(x)$, $x \in [-\pi, \pi]$ is a real even function defined by the relation

$$b(t) = \int_{[-\pi,\pi]} e^{itx}\hat{b}(x)dx, \quad t \in \mathbf{Z}$$

and L, L_1 are slowly varying functions at infinity, i.e. $L(tx)/L(t) \to 1, t \to \infty$ for any fixed $x > 0$. We assume that $f(x)$ and $\hat{b}(x)$ are bounded in $\pi > |x| > \varepsilon$ for any $\varepsilon > 0$, that is, if they diverge, they can do so only at the origin. (Terrin and Taqqu [18] make the stronger assumption $\hat{a}(x) = |x|^{-\alpha/2}L^{1/2}(1/|x|)$).

Let

$$d_m(\alpha) := 1 - m(1 - \alpha), \quad (m \geq 1) \ . \tag{2.5}$$

The number $d_m(\alpha)$ in (2.5) characterizes the dependence structure of the sequence $(: X_t^{(m)} :)_{t \in \mathbf{Z}}$. It is easy to prove that if $d_m(\alpha) < 0$ then $: X_t^{(m)} :$ has a continuous spectral density $\psi_m(x)$, $x \in [-\pi, \pi]$, while if $d_m(\alpha) > 0$, then $\psi_m(x) = |x|^{-d_m(\alpha)}L^{(m)}(1/|x|)$ as $x \to 0$, where $L^{(m)}$ is a slowly varying function at infinity, that is, the spectral density $\varphi_m(x)$ diverges at the origin. Thus, the cases $d_m(\alpha) < 0$ and $d_m(\alpha) > 0$ correspond to the weak and strong dependence of $: X_t^{(m)} :$.

Set now

$$d_m^+(\alpha) := \begin{cases} \alpha, & \text{if} \quad m = 1 \\ \max(d_m(\alpha), 0), & \text{if} \quad m \neq 1 \ , \end{cases} \tag{2.6}$$

and

$$\gamma := d_m^+(\alpha) + d_n^+(\alpha) + 2\beta \ . \tag{2.7}$$

Giraitis and Taqqu [12] studied situations where Q_N satisfies the Central Limit Theorem (CLT) with $\gamma < 1$ (Theorem 2.4). Our goal is to analyze what happens when $\gamma > 1$. But we also treat the case $\gamma > 0$ in Theorem 2.2.

We assume for convenience that $1 \leq m \leq n$ and $d_m(\alpha) \neq 0, d_n(\alpha) \neq 0$ when $m, n \geq 2$. We have to distinguish between "components" $: X_t^{(m)}:$ that are linear $(m = 1, : X_t^{(m)} := X_t)$ and non-linear $(m \geq 2)$ and also between components that are weakly dependent $(d_m(\alpha) < 0)$ and strongly dependent

[1] We use the notation:

$$a(t) = \int_{[-\pi,\pi]} e^{itx}\hat{a}(x)dx, \quad t \in \mathbf{Z}$$

($d_m(\alpha) > 0$). We will consider the boundary case $d_m(\alpha) = 0$ only when $m = 1$. (Then $0 = d_1(\alpha) = \alpha$, that is, $f(x) = L(1/|x|)$ in (2.3).)

We have to distinguish between three situations:

$$
\begin{array}{lll}
(\mathbf{A_1}) & \text{either} & m = n = 1; \\
 & \text{or} & 1 = m < n \text{ and } d_n(\alpha) > 0; \\
 & \text{or} & 2 \le m \le n, \text{ and } d_m(\alpha) > 0,\ d_n(\alpha) > 0; \\
 & & \\
(\mathbf{A_2}) & \text{either} & 1 = m < n \text{ and } d_n(\alpha) < 0 \text{ (case } (\mathbf{A'_2})); \\
 & \text{or} & 2 \le m \le n \text{ and } d_m(\alpha) > 0, d_n(\alpha) < 0; \\
 & & \\
(\mathbf{A_3}) & & 2 \le m \le n \text{ and } d_m(\alpha) < 0, d_n(\alpha) < 0 \ .
\end{array}
$$

In the case (A_1), each component $: X_t^{(m)} :$ and $: X_t^{(n)} :$ is either linear or it is non-linear and strongly dependent. In case (A_2), $: X_t^{(m)} :$ satisfies the same condition as in (A_1) but $: X_t^{(n)} :$ is non-linear and weakly dependent. In case (A_3), both components are non-linear and weakly dependent.

To characterize the limit as $N \to \infty$, introduce a vector

$$\left(Z^{(1)}(dx), Z^{(m)}(dx), Z^{(n)}(dx)\right) \tag{2.8}$$

whose components are complex-valued Gaussian measures $Z^{(1)}(dx)$, $Z^{(m)}(dx)$ and $Z^{(n)}(dx)$ with zero mean such that

$$
\begin{aligned}
&EZ^{(l)}(dx)\overline{Z^{(l')}(dy)} = 0 \quad \text{if} \quad x \neq y; \quad l, l' = 1, m, n \\
&EZ^{(l)}(dx)\overline{Z^{(l')}(dx)} = v(l, l')dx \ ;
\end{aligned}
$$

where

$$
\begin{aligned}
&v(1,1) = 1; \\
&v(1,l) = \mathrm{Cov}(\xi_0, : X_0^{(l)} :) \quad \text{if} \quad 2 \le l, \ l = m, n; \\
&v(l,l') = \psi_{l,l'}(0) \quad \text{if} \quad l, l' \ge 2 \quad \text{and either equal to } m \text{ or } n \ .
\end{aligned}
$$

The Gaussian measures $Z^{(m)}(dx)$ and $Z^{(n)}(dx)$ will be used in the weakly dependent cases $d_m(\alpha) < 0$ and $d_n(\alpha) < 0$ respectively. The cross spectral density $\psi_{m,n}(x)$ of the vector $(: X^{(m)}(t) :, : X^{(n)}(t) :)$ is then continuous at the origin. Its expression is given in (5.20). In the special case

$$\hat{a}(x) \sim [f(x)/2\pi]^{1/2} \sim [|x|^{-\alpha/2} L(1/|x|)]^{1/2} \quad (x \to 0) \tag{2.9}$$

considered in Terrin and Taqqu [18], one can show that

$$\psi_{m,n}(0) = (1/2\pi) \sum_t \mathrm{Cov}(: X_t^{(m)} :, : X_0^{(n)} :) \ . \tag{2.10}$$

The joint vector (2.8) is now well defined. We will need

$$
\begin{aligned}
&Z^{(1)} \text{ in case } (A_1); \ (Z^{(1)}, Z^{(n)}), \ n \ge 2 \text{ in case } (A_2); \\
&\qquad\qquad (Z^{(m)}, Z^{(n)}), \ 2 \le m \le n \text{ in case } (A_3) \ .
\end{aligned}
$$

Note that $Z^{(1)}$ has the standard covariance

$$EZ^{(1)}(A)\overline{Z^{(1)}(B)} = \int_{A\cap B} dx \ ,$$

but $Z^{(n)}, \ n \geq 2$, has covariance

$$EZ^{(n)}(A)\overline{Z^{(n)}(B)} = \psi_{n,n}(0) \int_{A\cap B} dx \ .$$

Moreover, in case (A_2), the joint covariance of $(Z^{(1)}, Z^{(n)}), \ n \geq 2$ is

$$EZ^{(1)}(A)\overline{Z^{(n)}(B)} = \mathrm{Cov}(\xi_0, : X_0^{(n)} :) \int_{A\cap B} dx \ ,$$

and in case (A_3), for $(Z^{(m)}, Z^{(n)})$, $2 \leq m \leq n$,

$$EZ^{(m)}(A)\overline{Z^{(n)}(B)} = \psi_{m,n}(0) \int_{A\cap B} dx \ .$$

Finally, define

$$L^*(N) = \begin{cases} L(N)^{m+n} L_1^2(N), & \text{in case } \ A_1, \\ L(N)^m L_1^2(N), & \text{in case } \ A_2, \\ L_1^2(N), & \text{in case } \ A_3 \ . \end{cases} \tag{2.11}$$

Theorem 2.1 (Non-central limit theorem). *Suppose* $m \geq 1$, $n \geq 1$ *and*

$$\gamma = d_m^+(\alpha) + d_n^+(\alpha) + 2\beta > 1 \ . \tag{2.12}$$

(*In the case* (A_2'), *suppose also that* (2.9) *holds.*) *Then, in the cases* (A_j), $j = 1, 2, 3$, *as* $N \to \infty$,

$$\frac{1}{\sqrt{N^\gamma L^*(N)}} Q_N \overset{d}{\Rightarrow} Z_{m,n}^{(j)} \ , \tag{2.13}$$

where

$$Z_{m,n}^{(1)} = \int_{\mathbf{R}^{m+n}} \Phi_0\left(\sum_{i=1}^m x_i, \sum_{i=m+1}^{m+n} x_i\right) |x_1|^{-\alpha/2} \ldots |x_{m+n}|^{-\alpha/2} Z^{(1)}(dx_1) \ldots Z^{(1)}(dx_{m+n}),$$

$$Z_{m,n}^{(2)} = \int_{\mathbf{R}^{m+1}} \Phi_0\left(\sum_{i=1}^m x_i, x_{m+1}\right) |x_1|^{-\alpha/2} \ldots |x_m|^{-\alpha/2} Z^{(1)}(dx_1) \ldots Z^{(1)}(dx_m) Z^{(n)}(dx_{m+1}),$$

$$Z_{m,n}^{(3)} = \int_{\mathbf{R}^2} \Phi_0(x_1, x_2) Z^{(m)}(dx_1) Z^{(n)}(dx_2) \ .$$

Here

$$\Phi_0(x_1, x_2) := \int_{\mathbf{R}} K_0(x_1 + u) K_0(x_2 - u) |u|^{-\beta} \, du \ , \tag{2.14}$$

and

$$K_0(x) := \frac{e^{ix} - 1}{ix} . \tag{2.15}$$

Observe that the function Φ_0 which appears in the integrand of $Z_{m,n}^{(j)}$ is always bounded. The limits $Z_{m,n}^{(j)}$ are represented as multiple Wiener-Itô integrals (see, for example, Major [15] for a definition).

To shed some light on Theorem 2.1, recall that the limiting distribution of properly normalized sums $\sum_{t=1}^{N} : X_t^m :$ may be represented by a 1- or m-tuple Wiener-Itô integral (see Giraitis [5], Giraitis and Surgailis [7]), respectively. The limiting distribution of Q_N has a much more complicated structure. As Theorem 2.1 indicates, it is represented by a *multiple* Wiener-Itô integral whose order depends on whether $: X_t^{(m)} :$ and $: X_t^{(n)} :$ are strongly dependent. A strongly dependent component $: X_t^{(m)} :$ contributes an m-fold integral to the limit, and a weakly dependent component $: X_t^{(m)} :$ contributes a single integral.

In the next theorem we study the asymptotic behavior of the quadratic form Q_N (1.1) when $m \geq 1$ and $n = 0$, that is, when

$$Q_N := \sum_{t,s=1}^{N} b(t-s) P_m(X_t) \ . \tag{2.16}$$

In this case, $P_{m,0}(X_t, X_t) = P_m(X_t) =: X_t^{(m)} :$ is the univariate Appell polynomial, and the parameter γ in (2.7) becomes $\gamma = d_m^+(\alpha) + 1 + 2\beta$.

The following theorem addresses the cases:

$$\begin{array}{ll} (\mathbf{A_4}) & m \geq 2,\ n = 0,\ d_m(\alpha) < 0; \\ (\mathbf{A_5}) & \text{either } \ m = 1, n = 0; \\ & \text{or } m \geq 2,\ n = 0,\ d_m(\alpha) > 0. \end{array}$$

We set $L^*(N) = L_1^2(N)$ in the case (A_4) and $L^*(N) = L^m(N) L_1^2(N)$ in the case (A_5).

Theorem 2.2 *Assume* $m \geq 1$, $n = 0$, *and*

$$0 < \gamma = d_m^+(\alpha) + 1 + 2\beta \ . \tag{2.17}$$

Then as $N \to \infty$,

$$[N^\gamma L^*(N)]^{-1/2} Q_N \overset{d}{\Rightarrow} Z_{m,0}^{(j)} \tag{2.18}$$

in the cases (A_j), $j = 4, 5$. *The limit* $Z_{m,0}^{(4)}$ *is Gaussian*:

$$Z_{m,0}^{(4)} = \psi_m(0) \int_{\mathbf{R}} \Phi_0(x, 0) Z(dx) \ , \tag{2.19}$$

but the limit $Z_{m,0}^{(5)}$ *is non-Gaussian*:

$$Z_{m,0}^{(5)} = \int_{\mathbf{R}^m} \Phi_0(x_1 + \cdots + x_m, 0) |x_1|^{-\alpha/2} \ldots |x_m|^{-\alpha/2} Z(dx_1) \ldots Z(dx_m) \ . \tag{2.20}$$

Here $\psi_m(0)$ *is the spectral density of* $: X_t^{(m)} :$ *at the origin and* $Z(dx)$ *is a complex Gaussian random measure with* $E|Z(dx)|^2 = dx$.

Using the notation of Theorem 2.1, $Z_m^{(4)} = \int \Phi_0(x, 0) Z^{(m)}(dx)$ and, in $Z_{m,0}^{(5)}, Z(dx) = Z^{(1)}(dx)$.

Theorem 2.2 implies that both the CLT and non-CLT can be valid with the normalizing factor $N^{-\gamma'}$, $0 < \gamma' < 1$. Whether the limit $Z_{m,0}$ is Gaussian or not is determined by the dependence type of the variables $P_m(X_t) =: X_t^{(m)}$:, $t \in \mathbf{Z}$. It is Gaussian if $P_m(X_t), t \in \mathbf{Z}$ has weak dependence ($d(\alpha) < 0$) and non-Gaussian (given by the stochastic Wiener-Itô intergral) if $P_m(X_t), t \in \mathbf{Z}$ is strongly dependent ($d(\alpha) > 0$).

The following example provides a "non-CLT" with the classical normalization $\sqrt{N}$.

Example 2.1 Suppose that all slowly varying functions equal 1, $m \geq 2$, $1 - 1/m < \alpha < 1$, so that $d_m(\alpha) = 1 - m(1-\alpha) > 0$. Choose $\beta = -d_m(\alpha)/2$. Then $\gamma = 1$ and

$$\frac{1}{\sqrt{N}} Q_N \overset{d}{\Rightarrow} Z_{m,0}^{(5)}$$

where $Z_{m,0}^{(5)}$ is a non-Gaussian random variables given by (2.20).

Remarks

1.1. Theorem 2.1 in the case (A_1) was proved by Taqqu and Terrin ([18], [17]), for $d_m(\alpha) > 0$ and $d_n(\alpha) > 0$ and with condition (2.3) on the spectral density $f(x) = 2\pi|\hat{a}(x)|^2$ replaced by the stronger assumption $\hat{a}(x) = |x|^{-\alpha/2} L(1/|x|)$ $(x \to 0)$. This stronger condition was not really needed because the proof in [18] uses only conditions on the behavior of the spectral density $f(x) = 2\pi|\hat{a}(x)|^2$, around the origin.
1.2. Case (A_4) involves $d_m(\alpha) < 0$ and excludes $m = 1$. It is instructive, however, to try to apply it to the case $m = 1$. In order to do so, one has to use the definition of $d_m^+(\alpha)$ for $m > 1$, hence redefine $d_1^+(\alpha)$ as $\max(d_1(\alpha), 0) = \max(\alpha, 0) = 0$. Then the γ in (2.12) becomes $\gamma^* = 0 + 1 + 2\beta$. This γ^* is greater than the $\gamma = \alpha + 1 + 2\beta$ which is used in (A_5) with the standard definition (2.6) of $d_1^+(\alpha)$. The corresponding normalization $N^{\gamma^*/2}$ is thus too high and the limit in (2.13) ought to be 0. This is indeed the case since $\psi_1(0) = 0$ implies a limit of $Z_{1,0}^{(4)} = 0$. By using the correct case, namely (A_5), one would normalize by $N^{\gamma/2}$ and obtain the non-degenerate Gaussian limit $Z_{1,0}^{(5)}$.
1.3. We do not consider the cases $\gamma > 1$ when either $m \geq 2$, $d_m(\alpha) = 0$ or $n \geq 2, d_n(\alpha) = 0$. These are boundary cases. We conjecture, that in these cases, (2.13) holds with the norming factor $N^{-\gamma/2}L'(N)$, where $L'(N)$ is a slowly varying function.
1.4. The boundary case $\gamma = 1$ was considered in Theorem 2.2 when $m \geq 1$ and $n = 0$. We conjecture that if $\gamma = 1$ and $m, n \geq 1$, then the CLT holds.
1.5. Theorems 2.1 and 2.2 involve Wiener-Itô integrals $Z_{m,n}^{(j)}$, $j = 1, \ldots, 5$. Let $F^{(j)}$ denote the (non-random) integrand of $Z_{m,n}^{(j)}$. The condition $\|F^{(j)}\|_{L^2} < \infty$, which ensures the existence of the Wiener-Itô integral, is proved in the Appendix.
1.6. Under the assumptions of the Theorems 2.1 and 2.2 the finite-dimensional distributions of $\{(N^\gamma L^*(N))^{-1/2} Q_{[Nt]},\ t \geq 0\}$ converge to those of

$\{Z^{(j)}_{m,n}(t),\ t \geq 0\}$ in the cases $j = 1, \ldots, 5$, respectively. The limits $(Z^{(j)}_{m,n}(t))$ are obtained by replacing, in the definition of the limits $Z^{(j)}_{m,n} = Z^{(j)}_{m,n}(1)$, the kernel 2.14 $\Phi_0(x_1, x_2)$ by the kernel

$$\Phi_t(x_1, x_2) := \int_{\mathbf{R}} K_t(x_1 + u) K_t(x_2 - u) |u|^{-\beta}\, du \ ,$$

where

$$K_t(x) := \frac{e^{itx} - 1}{ix} \ .$$

1.7. For functional limit theorems, see Giraitis and Taqqu [11].

Multivariate extension

We now provide a multivariate generalization of the Theorems 2.1 and 2.2

Suppose we have $k \geq 1$ quadratic forms

$$Q^{(l)}_N := \sum_{t,s=1}^{N} b_l(t-s) P_{m_l,n_l}(X_t, X_s), \quad (m_l + n_l \geq 1), \quad l = 1, \ldots, k$$

where (X_j) is the linear sequence (1.2) whose spectral density has the asymptotic behavior (2.3) and where the weights $b_l(.)$ are Fourier coefficients of the functions $\hat{b}_l(x), |x| \leq \pi$,

$$\widehat{b_l}(x) = |x|^{-\beta_l} L_l(1/|x|),\ |x| \leq \pi, \quad (|\beta_l| < 1) \quad l = 1, \ldots, k \ ,$$

where $L_l, l = 1, \ldots, k$ are slowly varying functions.

Assume $E|\xi_0|^{2(m_l+n_l)} < \infty$, in order to ensure that $Q^{(l)}_N$ has finite second moments for all l. Let

$$\gamma_l = d^+_{m_l}(\alpha) + d^+_{n_l}(\alpha) + 2\beta_l, \quad l = 1, \ldots, k \tag{2.21}$$

and $L^*_l(N)$, $l = 1, \ldots, k$ be defined as $L^*(N)$ in (2.11), but replacing $L_1(N)$ by $L_l(N)$ respectively. Under the assumptions of Theorems 2.1 and 2.2 each of the quadratic forms $A^{-1/2}_{N,l} Q_{[Nt]}$, $l = 1, \ldots, k$, normalized by $A_{N,l} = N^{\gamma_l} L^*_l(N)$ converges to the limit $(I^{(l)}_t)_{t \geq 0} = (Z^{(j)}_{m_l,n_l}))_{t \geq 0}$ in cases $j = 1, \ldots, 5$. The indexes l and t in $I^{(l)}_t$ indicate that the parameter β in the definition of the integral $Z^{(j)}_{m,n}$ is replaced by β_l and the kernel $K_0(x)$ by the kernel $K_t(x)$.

Theorem 2.3 *Suppose that each of the quadratic form* $Q^{(l)}_N$, $l = 1, \ldots, k$ *satisfies the assumption of the Theorem* 2.1 *or* 2.2. *Then, for any* $t_1, \ldots, t_k \geq 0$,

$$\left(A^{-1/2}_{N,1} Q^{(1)}_{[Nt_1]}, \ldots, A^{-1/2}_{N,k} Q^{(k)}_{[Nt_k]} \right) \Rightarrow \left(I^{(1)}_{t_1}, \ldots, I^{(k)}_{t_k} \right) \quad (N \to \infty) \ . \tag{2.22}$$

The proof of Theorem 2.3 can be derived in the same way as that of Theorems 2.1 and 2.2.

3 Proof of the theorems

Since Theorem 2.2 involves the particular case $n = 0$, and since its proof is related to that of Theorem 2.1, we combine the proofs of these two theorems. Set

$$\Phi_N(u_1, u_2) := \int_{[-\pi,\pi]} D_N(u_1 + y) D_N(u_2 - y) \hat{b}(y) dy \ , \tag{3.1}$$

where

$$D_N(x) := \left(\sum_{t=1}^{N} e^{itx} \right) e^{-i(N+1)x/2} = \frac{\sin(Nx/2)}{\sin(x/2)}$$

is the Dirichlet kernel. Applying the multilinearity of $P_{m,n}(X_t, X_s)$ we can write (see Giraitis and Taqqu [12], Section 4 or Giraitis and Taqqu [10], Section 3):

$$Q_N = \sum_{u_1,\ldots,u_{m+n} \in \mathbf{Z}} d_N(u_1, \ldots, u_{m+n}) : \xi_{u_1}, \ldots, \xi_{u_{m+n}} : \tag{3.2}$$

where

$$\begin{aligned} d_N(u_1, \ldots, u_{m+n}) &:= \sum_{t,s=1}^{N} b(t-s) a(t-u_1) \cdots a(t-u_m) \\ &\quad \times a(s-u_{m+1}) \cdots a(s-u_{m+n}) \\ &= \int_{[-\pi,\pi]^{m+n}} e^{i(u_1 x_1 + \cdots + u_{m+n} x_{m+n})} \hat{d}_N(x_1, \ldots, x_{m+n}) d^{m+n} x \end{aligned}$$

and

$$\begin{aligned} \hat{d}_N(x_1, \ldots, x_{m+n}) &:= \hat{a}(x_1) \ldots \hat{a}(x_{m+n}) e^{i\frac{(N+1)}{2}(x_1 + \cdots + x_{m+n})} \\ &\quad \times \Phi_N(x_1 + \cdots + x_m, x_{m+1} + \cdots + x_{m+n}) \ , \end{aligned} \tag{3.3}$$

where Φ_N is defined in (3.1).

If $n = 0$, we interpret $\Phi_N(x_1 + \cdots + x_m, x_{m+1} + \cdots + x_{m+n})$ as $\Phi_N(x_1 + \cdots + x_m, 0)$, and set

$$\hat{d}_N(x_1, \ldots, x_m) := \hat{a}(x_1) \ldots \hat{a}(x_m) e^{i\frac{(N+1)}{2}(x_1 + \cdots + x_m)} \Phi_N(x_1 + \cdots + x_m, 0) \ . \tag{3.4}$$

In order to make explicit the dependence in (3.2) of the quadratic form Q_N on the weights d_N or $\widehat{d_N}$, we shall often write $Q_N(d_N)$ or $Q_N(\widehat{d_N})$.

Our goal is to find the asymptotic behavior of $Q_N(\widehat{d_N})$, adequately normalized.

3.1. Approximation by step functions

We first approximate Φ_N in (3.3) by "step functions".

If $d_m(\alpha) > 0$ and $m \geq 2$ (or $d_n(\alpha) > 0$, $n \geq 2$), that is, if $: X_t^m :$ (or $: X_t^n :$) in (2.2) are strongly dependent, we shall focus on the components $\hat{a}(x_i)$ of $\hat{d}_N$ with x_i close to the 0. Here, $1 \leq i \leq m$ (or $m + 1 \leq i \leq m + n$).

In the case $d_m(\alpha) < 0$, $m \geq 2$, which corresponds to weak dependence of $:X_t^{(m)}:$, the local behaviour of $\hat{a}(x_i)$ does not play any role and we shall discard a small neighborhood of $x_i = 0$. The function $\hat{a}(x_i)$ will be bounded outside that interval.

We divide now the interval $T_K := [-K, -1/K) \cup (1/K, K]$, $K > 0$ into l_K smaller intervals. Let $t_0 = 1/K < t_1 < \cdots < t_{l_K} = K$, $t_l := t_0 + hl$, $l = 1, \ldots, l_K$. Set $\Delta_l = (t_{l-1}, t_l], \Delta_{-l} = [-t_l, -t_{l-1}),\ l = 1, \ldots, l_K$. We assume that the mesh size

$$h \equiv h_K := \frac{K - K^{-1}}{l_K} \to 0 \quad \text{as} \quad K \to \infty \ . \tag{3.5}$$

Let $D_K^{(1)} = \{\Delta_l, l = 1, \ldots, l_K, -1, \ldots, -l_K\}$ denote the set of intervals Δ_l whose union is T_K. When $d \geq 2$, let $D_K^{(c)} = \{\Delta_{l_1} \times \cdots \times \Delta_{l_c} : \Delta_{l_i} \in D_K^{(1)}$, $l_i \neq \pm l_{i'}$ if $i \neq i'\}$, denotes the collection of the mini-cubes $\{\Delta\} = \Delta_{l_1} \times \cdots \times \Delta_{l_c}$ of the c-dimensional cube $T_K^c = T_K \times \ldots \times T_K$ where "diagonals" $\Delta_{l_i} = \pm\Delta_{l_{i'}}$ have been excluded.

For each mini-cube $\{\Delta\}$, we introduce the constants $C_\Delta^{(j)}$ for each of the cases $(A_j),\ j = 1, \ldots, 5$:

$$\begin{aligned}
C_\Delta^{(1)} &\equiv C^{(1)}_{\Delta_{l_1} \times \ldots \times \Delta_{l_{m+n}}} = \Phi_0\left(\sum_{i=1}^{m} t_{l_i}, \sum_{i=m+1}^{m+n} t_{l_i}\right) \ , \\
C_\Delta^{(2)} &\equiv C^{(2)}_{\Delta_{l_1} \times \ldots \times \Delta_{l_m} \times \Delta_{l_{m+1}}} = \Phi_0\left(\sum_{i=1}^{m} t_{l_i}, t_{l_{m+1}}\right) \ , \\
C_\Delta^{(3)} &\equiv C^{(3)}_{\Delta_{l_1} \times \Delta_{l_2}} = \Phi_0(t_{l_1}, t_{l_2}), \\
C_\Delta^{(4)} &\equiv C^{(5)}_{\Delta_{l_1}} = \Phi_0(t_{l_1}, 0), \\
C_\Delta^{(5)} &\equiv C^{(4)}_{\Delta_{l_1} \times \ldots \times \Delta_{l_m}} = \Phi_0\left(\sum_{i=1}^{m} t_{l_i}, 0\right) \ ,
\end{aligned}$$

where Φ_0 is defined by (2.14).

Finally, we introduce the renormalized step functions $\Phi_{N,\Delta}^{(j)}$ where $j = 1, 2, 3$ denote the case (A_1), (A_2), (A_3) of Theorem 2, respectively. Set

$$\Phi_{N,\Delta}^{(1)}(x_1, \ldots, x_{m+n}) = \sum_{\Delta \in D_K^{(m+n)}} C_\Delta^{(1)} \prod_{k=1}^{m+n} \mathbb{1}(N x_k \in \Delta_{l_k}) \ , \tag{3.6}$$

$$\begin{aligned}
\Phi_{N,\Delta}^{(2)}(x_1, \ldots, x_{m+n}) = &\sum_{\{\Delta\} \in D_K^{(m+1)}} C_\Delta^{(2)} \prod_{k=1}^{m} \mathbb{1}(N x_k \in \Delta_{l_k}) \\
&\cdot \mathbb{1}\left(N\left[\left(\sum_{i=m+1}^{m+n} x_i\right) \bmod 2\pi\right] \in \Delta_{l_{m+1}}\right) \\
&\times \prod_{i=m+1}^{m+n} \mathbb{1}(|x_i| > 1/\ln N),
\end{aligned} \tag{3.7}$$

$$\Phi^{(3)}_{N,\Delta}(x_1,\dots,x_{m+n}) = \sum_{\{\Delta\}\in D_K^{(2)}} C^{(3)}_{\{\Delta\}} \cdot \mathbb{1}\Bigg(N\Bigg[\Bigg(\sum_{i=1}^{m} x_i\Bigg)\bmod 2\pi\Bigg] \in \Delta_{l_1}, \\ N\Bigg[\Bigg(\sum_{i=m+1}^{m+n} x_i\Bigg)\bmod 2\pi\Bigg] \in \Delta_{l_2}\Bigg)\prod_{i=1}^{m+n}\mathbb{1}(|x_i| > 1/\ln N)\ , \tag{3.8}$$

where Φ_0 is defined by (2.14).

If the assumptions of Theorem 2 are satisfied, i.e. $m \geq 1, n = 0$, we use the following step functions instead:

In the case (A_4), that is $d_m(\alpha) < 0$, $m \geq 2$, we set

$$\Phi^{(4)}_{N,\Delta}(x_1,\dots,x_m) := \sum_{\Delta_j\in D_K^{(1)}} C^{(4)}_{\Delta}\mathbb{1}\Bigg(N\Bigg[\Bigg(\sum_{i=1}^{m} x_i\Bigg)\bmod 2\pi\Bigg] \in \Delta_j\Bigg) \\ \times \prod_{j=1}^{m}\mathbb{1}(|x_j| > 1/\ln N) \tag{3.9}$$

and denote by $Z^{(4)}_{m,0}$ the Gaussian limit $Z_{m,0}$ in (2.19).

In the case (A_5), that is $d_m(\alpha) > 0$, we set

$$\Phi^{(5)}_{N,\Delta}(x_1,\dots,x_m) := \sum_{\{\Delta\}\in D_K^{(m)}} C^{(5)}_{\{\Delta\}}\prod_{k=1}^{m}\mathbb{1}(Nx_k \in \Delta_{l_k})\ , \tag{3.10}$$

and denote by $Z^{(5)}_{m,0}$ the non-Gaussian limit $Z_{m,0}$ in (2.20).

3.2. Convergence

Now, we divide the function

$$\hat{d}_N = \hat{d}^{(j)}_{N,\Delta} + \hat{r}^{(j)}_{N,\Delta} \quad j = 1,2,3,4,5 \tag{3.11}$$

defined in (3.3) into a main part

$$\hat{d}^{(j)}_{N,\Delta} := \hat{a}(x_1)\cdots\hat{a}(x_{m+n})N^{1+\beta}L_1(N)\mathrm{e}^{\mathrm{i}\frac{(N+1)}{2}(x_1+\cdots+x_{m+n})}\Phi^{(j)}_{N,\Delta}(x_1,\dots,x_{m+n}) \tag{3.12}$$

and a part corresponding to the remainder term

$$\hat{r}^{(j)}_{N,\Delta} := \hat{a}(x_1)\cdots\hat{a}(x_{m+n})\mathrm{e}^{\mathrm{i}\frac{(N+1)}{2}(x_1+\cdots+x_{m+n})}U^{(j)}_{N,\Delta}(x_1,\dots,x_{m+n})\ , \tag{3.13}$$

where

$$U^{(j)}_{N,\Delta}(x_1,\dots,x_{m+n}) = \Bigg(\Phi_N\Bigg(\sum_{i=1}^{m} x_i, \sum_{i=m+1}^{m+n} x_i\Bigg) \\ - N^{1+\beta}L_1(N)\,\Phi^{(j)}_{N,\Delta}(x_1,\dots,x_{m+n})\Bigg)\ ,$$

$j = 1, 2, 3, 4, 5$. Thus,

$$Q_N(\hat{d}_N) = Q_N(\hat{d}^{(j)}_{N,\Delta}) + Q_N(\widehat{r}^{(j)}_{N,\Delta}) \ . \tag{3.14}$$

From Proposition 4.1, 4.2 and 4.3 in Section 4 it follows that $\forall \varepsilon > 0$, there exists $K > 0$ and a set of partitions $\{\Delta\}$ which define the step functions $\Phi^{(j)}_{N,\Delta}$, such that

$$[L^*(N)N^\gamma]^{-1} \mathrm{Var}\ Q_N\left(\widehat{r}^{(j)}_{N,\Delta}\right) < \varepsilon \ , \tag{3.15}$$

the limit

$$[L^*(N)N^\gamma]^{-1/2} Q_N\left(\hat{d}^{(j)}_{N,\Delta}\right) \overset{d}{\Rightarrow} q^{(j)}_\Delta \tag{3.16}$$

exists and

$$\mathrm{Var}\left(q^{(j)}_\Delta - Z^{(j)}_{m,n}\right) < \varepsilon \tag{3.17}$$

as $N \to \infty$ for $j = 1, 2, 3, 4, 5$. The statements (2.13) of Theorem 2.1 and (2.18) of Theorem 2.2 follow from (3.14)–(3.17) and (3.11). This concludes the proof of the theorems.

4 Propositions

The following propositions have been used.

Proposition 4.1 *For any* $\varepsilon > 0$, *there exists* $K > 0$ *and a set of partitions* $\{\Delta\}$, *such that* (3.15) *holds.*

Proof of Proposition 4.1. Since (see Giraitis and Taqqu [12], (4.8))

$$\mathrm{Var} Q(d_N) \le C(\xi) \sum_{t_1,\ldots,t_{m+n} \in \mathbf{Z}} d_N^2(t_1, \ldots, t_{m+n}) = C(\xi) \left\|\hat{d}_N^2\right\|^2_{L^2} \ , \tag{4.1}$$

we get

$$\mathrm{Var}\ Q_N\left(\widehat{r}^{(j)}_{N,\Delta}\right) \le C(\xi) \left\|\widehat{r}^{(j)}_{N,\Delta}\right\|^2_{L^2} \ . \tag{4.2}$$

From (4.2) and the definition (3.13) of $\widehat{r}^{(j)}_{N,\Delta}$, it follows that the norm

$$\begin{aligned} r_N := \left\|\widehat{r}^{(j)}_{N,\Delta}\right\|^2_{L^2} &\le C \int_{[-\pi,\pi]^{m+n}} f(x_1) \ldots f(x_{m+n}) |U^{(j)}_{N,\Delta}(x_1, \ldots, x_{m+n})|^2\, d^{m+n}x \\ &= C\left(\int_{[-\pi,\pi]^{m+n} \cap A^{(m)}_{K,N} \times A^{(n)}_{K,N}} [\ldots] + \int_{[-\pi,\pi]^{m+n} \setminus A^{(m)}_{K,N} \times A^{(n)}_{K,N}} [\ldots]\right) =: C(r_{N,1} + r_{n,2}) \ . \end{aligned} \tag{4.3}$$

The set $A^{(m)}_{K,N}$ in (4.3) denotes $\{(x_1, \ldots, x_m) \in \mathbf{R} : 1/K \le |Nx_i| \le K,\ i = 1, \ldots, m\}$ if $d_m(\alpha) > 0$; and the set $\{(x_1, \ldots, x_m) \in \mathbf{R} : 1/K \le N|(x_1 + \ldots + x_m) \mathrm{mod}\ 2\pi| \le K,\ |x_i| \ge 1/\log N,\ i = 1, \ldots, m\}$ if $d_m(\alpha) < 0$, respectively. (The definition of $A^{(n)}_{K,N}$ is similar). Note the asymmetry in the definition of $A_{K,N}^{(m)}$ between the strongly and weakly dependent cases. In the case of strong dependence, we require all the x_i's to be "outside" the origin. In the case of weak dependence it is only their sum that can is required to lie outside the origin. Note also that, whereas $r_{N,1}$ involves integration over $A_{K,N}^{(m)} \times A_{K,N}^{(n)}$, $r_{N,2}$ involves integration over the complement of that set.

Clearly,

$$\Phi_{N,\Delta}^{(j)}(x_1, \dots, x_{m+n}) = 0 \quad \text{for} \quad (x_1, \dots, x_{m+n}) \in [-\pi, \pi] \setminus A_{K,N}^{(m)} \times A_{K,N}^{(n)} \ . \tag{4.4}$$

It remains to estimate $r_{N,1}$ and $r_{N,2}$.

We estimate first $r_{N,1}$. Consider, for example, the case $j = 2$ for which $d_m(\alpha) > 0$ and $d_n(\alpha) < 0$. Note that in this case, $\gamma = d_m^+(\alpha) + d_n^+(\alpha) + 2\beta = 1 - m + m\alpha + 2\beta$. After the change of variables $(x'_1, \dots, x'_m) = N(x_1, \dots, x_m)$, $x'_{m+1} = N[(x_{m+1} + \cdots + x_{m+n}) \text{mod } 2\pi]$, we get

$$\begin{aligned}[N^\gamma L^*(N)]^{-1} r_{N,1} \le C(K) N^{-m\alpha} L^{-m}(N) \int_{\substack{1/K \le |x'_i| \le K \\ i=1,\dots,m+1}} & f(x'_1/N) \dots f(x'_m/N) \\ & \times f^{(*n)}(x'_{m+1}/N) \cdot |h_N(x'_1, \dots, x'_m; x'_{m+1})|^2 d^{m+1} x \\ & =: R_{N,K}(h) \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} h_N(x'_1, \dots, x'_m; x'_{m+1}) = N^{-1-\beta} L_1^{-1}(N) \Phi_N \left(\sum_{i=1}^m x'_i/N, x'_{m+1}/N \right) \\ - h_\Delta(x'_1, \dots, x'_m; x'_{m+1}) \end{aligned}$$

and

$$h_\Delta(x'_1, \dots, x'_m; x'_{m+1}) = \sum_{\{\Delta\} \in D_K^{(m+1)}} C_\Delta^{(2)} \prod_{k=1}^m \mathbb{1}(x'_k \in \Delta_{l_k}) \mathbb{1}(x'_{m+1} \in \Delta_{l_{m+1}}) \ .$$

The function

$$f^{(*n)}(u) = \int_{[-\pi,\pi]^{n-1}} f(u - x_1 - \cdots - x_n) f(x_1) \dots f(x_{n-1}) \, dx_1 \dots dx_n \tag{4.6}$$

denotes the n^{th} convolution of f (periodically extended in $\mathbf{R}$). Under the assumption $d_n(\alpha) < 0$, the convolution $f^{(*n)}(u)$ is a bounded function, and under the assumption $d_m(\alpha) > 0$ we can also bound

$$\prod_{i=1}^m f(x'_i/N) \le C(K) N^{m\alpha} L^m(N)$$

in (4.5), using the following well-known property of the slowly varying functions:

$$N^{-\alpha} L^{-1}(N) f(x/N) \le C|x|^{-(\alpha+\epsilon)} \text{ uniformly in } N \ge 1 \ \text{ and } |x| \le N\pi \tag{4.7}$$

(for any $\epsilon > 0$ fixed), where $C = C(\epsilon)$. Thus,

$$R_{N,K}(h) \leq C(K) \int_{|x_i'| \leq K: i=1,\ldots,m+1} |h_N(x_1', \ldots, x_m'; x_{m+1}')|^2 d^{m+1}x' \quad . \tag{4.8}$$

Using the estimate

$$\left|D_N(x)\right| \leq \text{ const } N(1 + |Nx|)^{-1}, \; |x| \leq 3\pi/2 \tag{4.9}$$

it follows that the function

$$p_N(u_1, u_2) := |N^{-1-\beta} L_1^{-1}(N) \Phi_N(u_1/N, u_2/N)|$$

satisfies

$$\begin{aligned} p_N(u_1, u_2) &\leq p(u_1, u_2) \\ &:= \text{const} \int_{\mathbf{R}} (1 + |u_1 + y|)^{-1} (1 + |u_2 - y|)^{-1} |y|^{-\beta - \epsilon'} dy < \infty \end{aligned} \tag{4.10}$$

uniformly in $(u_1, u_2) \in [-\frac{\pi}{2}N, \frac{\pi}{2}N]^2$, and hence uniformly in $|u_1|, |u_2| \leq K$. Here $\epsilon' > 0$ is a fixed small number. Since $N^{-1}D_N(u/N) \to K_0(u)$ as $N \to \infty$, applying the dominated convergence theorem, we get

$$N^{-1-\beta} L_1^{-1}(N) \Phi_N(u_1/N, u_2/N) \to \Phi_0(u_1, u_2) \quad (N \to \infty) \ ,$$

for $|u_1|, |u_2| \leq K$, where the limit Φ_0 is defined by (2.14). We now want to examine the integral defining $R_{N,K}(h)$ in (4.5).

The function $h_\Delta(x_1', \ldots, x_m'; x_{m+1}')$ involves an indicator function of a set concentrated around the diagonals whose measure tends to 0 with the mesh size h. Since the integral in the estimate (4.8) of $R_{N,K}(h)$ is over a compact set, since h_N is bounded, and in view of the definition of the functions $\Phi_{N,\Delta}^{(j)}$ and h_Δ it follows that

$$\limsup_N R_{N,K}(h) \to 0 \quad h \to 0 \tag{4.11}$$

for any fixed $K > 0$.

We deal now with $r_{N,2}$ in (4.3) and consider again the case $j = 2$. Using (4.4), we get

$$r_{N,2} = \int_{[-\pi,\pi] \backslash A_{K,N}^{(m)} \times A_{K,N}^{(n)}} f(x_1) \ldots f(x_{m+n}) \left| \Phi_N \left(\sum_{i=1}^{n} x_i, \sum_{i=m+1}^{m+n} x_i \right) \right|^2 d^{m+n}x \ .$$

Write

$$r_{N,2} = r_{N,3} + r_{N,4} \tag{4.12}$$

where $r_{N,3}$ and $r_{N,4}$ are obtained by replacing $\Phi_N(u_1, u_2)$, defined in (3.1), by

$$\Phi_{N,k}(u_1, u_2) = \int_{B_k} D_N(u_1 + y) D_N(u_2 - y) \hat{b}(y) dy$$

$k = 3, 4$ respectively and where

$$\begin{aligned} B_3 &= \left\{ y \in [-\pi, \pi] : |u_1 + y| > \tfrac{3}{2}\pi \;\text{ or } |u_2 + y| > \tfrac{3}{2}\pi \right\} \ , \\ B_4 &= \left\{ y \in [-\pi, \pi] : |u_1 + y| \leq \tfrac{3}{2}\pi, |u_2 + y| \leq \tfrac{3}{2}\pi \right\} \ . \end{aligned}$$

Consider first $r_{N,3}$. We have

$$r_{N,3} \le \int_{[-\pi,\pi]^2} f^{(*m)}(u_1) f^{(*n)}(u_2) \cdot \left(\int_{|y|\le\pi, |u_1+y|\ge 3/2\pi \text{ or } |u_2+y|\ge 3/2\pi} |D_N(u_1+y) D_N(u_2-y)\hat{b}(y)| dy \right)^2 du_1\, du_2 \tag{4.13}$$

where $f^{(*m)}(u)$ is the m^{th} convolution defined in (4.6). Note that $f^{(*m)}(u)$, $m \ge 1$, is bounded for $|u| \ge \epsilon$ (since $f(x)$ is bounded for $|x| \ge \epsilon$, one can always bound one of the integrants in (4.6) and hence separate the variables). Suppose first that $m, n \ge 1$. If $|u_1 + y| \ge \frac{3}{2}\pi$, then $|u_1| \ge \frac{\pi}{2}$ and $|y| \ge \frac{\pi}{2}$, and hence both functions $f^{(*m)}(u_1)$ and $\hat{b}(y)$ are bounded. Thus,

$$\begin{aligned} r_{N,3} &\le C \int_{[-\pi,\pi]^2} |f^{(*n)}(u_1) + f^{(*m)}(u_2)| \left(\int_{-\pi}^{\pi} |D_N(u_1 - y) D_N(u_2 - y)| dy \right)^2 du_1 du_2 \\ &\le CN = o(N^{\gamma} L^*(N)) \end{aligned}$$

(see e.g. Giraitis and Taqqu [12], Relation (5.9)).

Suppose now $m \ge 1, n = 0$. Then $f^{(*n)}(u_2) \equiv 1$ and

$$r_{N,3} = \int_{-\pi}^{\pi} f^{(*m)}(u_1) \left(\int_{|y|\le\pi, |u_1+y|\ge 3/2\pi} |D_N(u_1+y) D_N(-y)\hat{b}(y)| dy \right)^2 du_1 \ .$$

Since $f^{(*m)}(u_1)$, $|D_N(y)|$ and $\hat{b}(y)|$ are bounded for $|u_1| \ge \frac{\pi}{2}$ and $\frac{\pi}{2} \le |y| \le \pi$, we get

$$\begin{aligned} r_{N,3} &\le C \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |D_N(u_1 + y)| dy \right)^2 du_1 \\ &\le C \int_{-\pi}^{\pi} \ln^2 N\, du_1 = O(\ln^2 N) = o(N^{\gamma} L^*(N)) \ . \end{aligned} \tag{4.14}$$

We finally consider $r_{N,4}$ defined in (4.12). Relations (4.4) and (4.9) lead to the bound:

$$[N^{\gamma} L^*(N)]^{-1} r_{N,4} \le \text{ const} \int_{[-\pi N, \pi N]^2} f_K(u_1, u_2) p^2(u_1, u_2) du_1\, du_2 \ ,$$

where $p(u_1, u_2)$ is defined in (4.10), and where $f_K(u_1, u_2)$ will be defined below. To motivate the form of $f_K(u_1, u_2)$, note that we want to bound the renormalized $\prod_{i=1}^{m} f(x_i) \prod_{i=m+1}^{m+n} f(x_i)$. We shall use the bound (4.7) if $d_m(\alpha) > 0$, and (2.3) if $d_m(\alpha) < 0$ and make a change of variables as above. For example, if $d_m(\alpha) > 0$, we set $N^{-1}x'_i = x_i$, $i = 1, \ldots, m$; $u_1 = \sum_{i=1}^{m} x'_i$, and we get

$$\prod_{i=1}^{m} f(Nx_i) \le a_N \prod_{i=1}^{m} |x'_i|^{-\alpha'} \le a_N |u_1 - x'_2 - \cdots - x'_m|^{-\alpha'} \prod_{i=2}^{m} |x'_i|^{-\alpha'}$$

where a_N is some function of N and $\alpha' = \alpha + \epsilon$. Taking advantage of the symmetry in the variables of integration, we can choose for N sufficiently large,

$$f_K(u_1,u_2) := \tilde{f}_K^{(*m)}(u_1)\tilde{f}_0^{(*n)}(u_2) + \tilde{f}_0^{(*m)}(u_1)\tilde{f}_K^{(*n)}(u_2) + \tilde{f}_0^{(*m)}(u_1)\tilde{f}_0^{(*n)}(u_2)\big(\mathbb{1}\big(|u_1| < 1/K \text{ or } |u_1| > K\big) + (\mathbb{1}\big(|u_2| < 1/K \text{ or } |u_2| > K\big)\big) \ .$$

Here $\tilde{f}_K^{(*m)}(u) :=$

$$\begin{aligned}
&|u|^{-\alpha'}\mathbb{1}\big(|u| \le 1/K \text{ or } |u| \ge K\big),\\
&\text{if} \quad m = 1;\\
&\int_{\mathbf{R}^{m-1}} \big|(u - x_2' - \cdots - x_m')x_2' \ldots x_m'\big|^{-\alpha'}\mathbb{1}\big(|x_m'| \le 1/K \text{ or } |x_m'| \ge K\big)d^{m-1}x',\\
&\text{if} \quad m \ge 2, d_m(\alpha) > 0;\\
&\sup_{u\in[-\pi,\pi]} \int_{[-\pi,\pi]^{m-1}} f(u - x_2' - \cdots - x_m')f(x_2')\ldots f(x_m')\mathbb{1}\big(|x_m'| \le 1/K\big)d^{m-1}x',\\
&\text{if} \quad m \ge 2, d_m(\alpha) < 0 \ .
\end{aligned}$$

The function $\tilde{f}_0^{(*m)}(u)$ is $\tilde{f}_K^{(*m)}(u)$ with $K = 0$ and $1/K = \infty$. The function $\tilde{f}_K^{(*n)}(u)$ is defined in a similar way. In the case $m \ge 2$, $d_m(\alpha) < 0$, we used the inequality $\mathbb{1}(|x_m'| \le 1/\ln N) \le \mathbb{1}(|x_m'| \le 1/K)$, valid for N sufficiently large.

In order to apply the dominated convergence theorem, we shall derive an upper bound for the function f_K.

Taking into account (4.7), it is easy to show (as in Giraitis and Taqqu [12], Lemma 5.2) that $\tilde{f}_K(u) \le \text{const}|u|^{-d_m^+(\alpha)-\epsilon'}$ uniformly in $K \ge 0$ and $u \in \mathbf{R}$ (for small $\epsilon' > 0$). (In fact, if $d_m(\alpha) < 0$, then $\tilde{f}_K(u)$ is bounded.) Thus,

$$f_K(u_1,u_2) \le \text{const}|u_1|^{-d_m^+(\alpha)-\epsilon'}|u_2|^{-d_n^+(\alpha)-\epsilon'} =: f(u_1,u_2) \ , \tag{4.15}$$

uniformly in $u_1, u_2 \in [-\pi N, \pi N]$, with $\epsilon' > 0$ small. Note also, that for any fixed u_1, u_2,

$$f_K(u_1,u_2) \to 0 \quad \text{as} \quad K \to \infty \ . \tag{4.16}$$

Therefore, from the assumption $\gamma > 1$ in (2.12) and Lemma 4.1 below, it follows that $f(\cdot,\cdot)p^2(\cdot,\cdot) \in L(\mathbf{R}^2)$. Thus, using relations (4.15) and (4.16), we get

$$\limsup_N [N^\gamma L^*(N)]^{-1} r_{N,4} \le \text{const}\int_{\mathbf{R}^2} f_K(u_1,u_2)p^2(u_1,u_2)du_1\,du_2 \to 0 \quad (K \to \infty) \ .$$

This, together with (4.3), (4.12) and (4.14), implies the statement (3.15) of Proposition 4.1. □

Lemma 4.1 *Let* $p(u_1,u_2)$ *be defined as in* (4.10) *and set* $-1 < \alpha_1, \alpha_2, \beta < 1$. *If* $\alpha_1 + \alpha_2 + 2\beta > 1$, *then*

$$I := \int_{\mathbf{R}^2} |u_1|^{-\alpha_1}|u_2|^{-\alpha_2}|p(u_1,u_2)|^2 du_1\,du_2 < \infty \ .$$

If $\alpha_1 + 2\beta + 1 > 0$, *then*

$$I := \int_{\mathbf{R}} |u_1|^{-\alpha_1} |p(u_1, 0)|^2 \, du_1 < \infty \ .$$

This lemma is proved in the Appendix.

Proposition 4.2 *Relation* (3.16) *holds for any* $K > 0$ *and partition* $\{\Delta\}$.

Proof. We start with the proof of (3.16). Let $\Delta = (a, b]$ be an interval from $D_K^{(1)}$ (see Subsection 3.1). Set

$$\hat{a}_{N,\Delta}(x) := N^{-\alpha/2} L^{-1/2}(N) \hat{a}(x) \mathbb{1}(x \in \Delta/N) \mathrm{e}^{\mathrm{i}xN/2}, \quad |x| \le \pi \ . \tag{4.17}$$

For $m' \ge 2$, put

$$\hat{a}_{N,\Delta}(x_1, \ldots, x_{m'}) := \hat{a}(x_1) \ldots \hat{a}(x_{m'}) \mathrm{e}^{\mathrm{i}N(x_1 + \cdots + x_{m'})/2} \prod_{k=1}^{m'} \mathbb{1}(|x_k| > 1/\ln N)$$
$$\cdot \mathbb{1}\big((x_1 + \cdots + x_{m'}) \mathrm{mod} 2\pi \in \Delta/N\big), \quad |x_1| \le \pi, \ldots, |x_{m'}| \le \pi \ . \tag{4.18}$$

(Below $m' \ge 2$ will be used to denote either $m' = m \ge 2$ or $m' = n \ge 2$.)

From (4.17)–(4.18) and the assumption (2.3) it follows that

$$\sup_{x_1, \ldots, x_{m'}} \Big| (\hat{a}_{N,\Delta}(x_1, \ldots, x_{m'}) \Big| \le C(K) (\ln N)^p < \infty \quad (m' \ge 2) \tag{4.19}$$

for large $p > 0$, and $|\hat{a}_{N,\Delta}(x)| \le C(K)$, for $m' = 1$ uniformly in the intervals $\Delta \in D_K^{(1)}$. Introduce the Fourier coefficients

$$a_{N,\Delta}(t_1, \ldots, t_{m'}) := \int_{[-\pi,\pi]^{m'}} \mathrm{e}^{\mathrm{i}(t_1 x_1 + \cdots + t_{m'} x_{m'})} \hat{a}_{N,\Delta}(x_1, \ldots, x_{m'}) d^{m'} x \quad (m' \ge 1) \ . \tag{4.20}$$

Then from Definition (3.12) of $\hat{d}_{N,\Delta}$ it follows that

$$N^{-\gamma/2} Q_N(d_{N,\Delta}^{(j)}) \equiv \sum_{\{\Delta\}}^{(j)} C_\Delta^{(j)} \sum_{t_1, \ldots, t_{m+n} \in \mathbf{Z}} A_{N,\Delta}^{(j)}(t_1, \ldots, t_{m+n}) : \xi_{t_1}, \ldots, \xi_{t_{m+n}} : \ ,$$

where $j = 1, 2, 3, 4, 5$ refers to the Cases (A_1)–(A_5), and where $\sum_{\{\Delta\}}^{(j)}$ denotes the sum over all partitions $\{\Delta\} = \Delta_{l_1} \times \ldots \times \Delta_{l_{c(j)}}$ with $l_i \ne l_{i'}$ for $i \ne i'$ and $c(1) = m + n$, $c(2) = m + 1$, $c(3) = 2$, $c(4) = 1$, $c(5) = m$;

$$A_{N,\Delta}^{(j)}(t_1, \ldots, t_{m+n})$$
$$= \begin{cases} \sqrt{N} a_{N,\Delta_{l_1}}(t_1) \ldots \sqrt{N} a_{N,\Delta_{l_{m+n}}}(t_{m+n}) & \text{if } \ j = 1, \\ \sqrt{N} a_{N,\Delta_{l_1}}(t_1) \ldots \sqrt{N} a_{N,\Delta_{l_m}}(t_m) & \\ \quad \times \sqrt{N} a_{N,\Delta_{l_{m+1}}}(t_{m+1}, \ldots, t_{m+n}) & \text{if } \ j = 2, \\ \sqrt{N} a_{N,\Delta_{l_1}}(t_1, \ldots, t_m) \sqrt{N} a_{N,\Delta_{l_2}}(t_{m+1}, \ldots, t_{m+n}) & \text{if } \ j = 3, \\ \sqrt{N} a_{N,\Delta_{l_1}}(t_1, \ldots, t_m) & \text{if } \ j = 4, \\ \sqrt{N} a_{N,\Delta_{l_1}}(t_1) \ldots \sqrt{N} a_{N,\Delta_{l_m}}(t_m) & \text{if } \ j = 5 \ . \end{cases} \tag{4.21}$$

Setting

$$Z_{N,\Delta}^{(m')} := \sum_{t_1,\dots,t_{m'}\in\mathbf{Z}} \sqrt{N} a_{N,\Delta}(t_1,\dots,t_{m'}) : \xi_{t_1},\dots,\xi_{t_{m'}} :, \quad (m' \geq 1) \tag{4.22}$$

we can rewrite

$$N^{-\gamma/2} Q_N(\hat{d}_{N,\Delta}) = Q_{N,\Delta}^{(j)} + R_N^{(j)} \tag{4.23}$$

as a sum of a main term

$$Q_{N,\Delta}^{(j)} := \begin{cases} \sum\limits^{(1)}_{\{\Delta'_1\times\ldots\times\Delta'_{m+n}\}} C_\Delta^{(1)} Z_{N,\Delta'_1}^{(1)} \cdots Z_{N,\Delta'_{m+n}}^{(1)}, & \text{if} \quad j=1, \\ \sum\limits^{(2)}_{\{\Delta'_1\times\ldots\times\Delta'_m\times\Delta'_{m+1}\}} C_\Delta^{(2)} Z_{N,\Delta'_1}^{(1)} \cdots Z_{N,\Delta'_m}^{(1)} Z_{N,\Delta'_{m+1}}^{(n)}, & \text{if} \quad j=2, \\ \sum\limits^{(3)}_{\{\Delta'_1\times\Delta'_2\}} C_\Delta^{(3)} Z_{N,\Delta'_1}^{(m)} Z_{N,\Delta'_2}^{(n)}, & \text{if} \quad j=3, \\ \sum\limits^{(4)}_{\{\Delta'_1\}} C_\Delta^{(4)} Z_{N,\Delta'_1}^{(m)}, & \text{if} \quad j=4, \\ \sum\limits^{(5)}_{\{\Delta'_1\times\ldots\times\Delta'_m\}} C_\Delta^{(5)} Z_{N,\Delta'_1}^{(1)} \cdots Z_{N,\Delta'_m}^{(1)}, & \text{if} \quad j=5 \;, \end{cases} \tag{4.24}$$

and a remainder term

$$R_N^{(j)} = \sum^{(j)}_{\{\Delta\}} C_\Delta^{(j)} \sum_{(t_1,\dots,t_{m+n})\in I_j} A_{N,\Delta}^{(j)}(t_1,\dots,t_{m+n}) \Big(: \xi_{t_1},\dots,\xi_{t_{m+n}} : - \xi^{(j)}(t_1,\dots,t_{m+n}) \Big) \;, \tag{4.25}$$

$j = 1,2,3,5 (R_N^{(4)} = 0)$, where, to simplify notation, we now write $\Delta'_1, \Delta'_2, \ldots$, instead of $\Delta_{l_1}, \Delta_{l_2}, \ldots$, and where I_j and $\xi^{(j)}$ are given below.

To motivate the definition of I_j and $\xi^{(j)}$, $j = 1,2,3,5$, recall that, since the sequence (ξ_i) is i.i.d, it follows from the well-known properties of Wick products (see [3]) that

$$: \xi_{t_1},\dots,\xi_{t_k},\xi_{t_{k+1}},\dots,\xi_{t_{k'}} :=: \xi_{t_1},\dots,\xi_{t_k} :: \xi_{t_{k+1}},\dots,\xi_{t_{k'}} : \tag{4.26}$$

if $\{t_1,\dots,t_k\} \cap \{t_{k+1},\dots,t_{k'}\} = \emptyset$. Thus $I_j \subset \mathbf{Z}^{m+n}$ will be the set, where the difference $: \xi_{t_1},\dots,\xi_{t_{m+n}} : - \xi^{(j)}(t_1,\dots,t_m) \neq 0$, with $\xi^{(1)}(t_1,\dots,t_{m+n}) =: \xi_{t_1} : \ldots : \xi_{t_{m+n}} :$, $\xi^{(2)}(t_1,\dots,t_{m+n}) =: \xi_{t_1} : \ldots : \xi_{t_m} :: \xi_{t_{m+1}},\dots,\xi_{t_{m+n}} :$, $\xi^{(3)}(t_1,\dots,t_{m+n}) =: \xi_{t_1},\dots,\xi_{t_m} :: \xi_{t_{m+1}},\dots,\xi_{t_{m+n}} :$, $\xi^{(5)}(t_1,\dots,t_m) =: \xi_{t_1} : \ldots : \xi_{t_m} :$. Equivalently, then, the sets $I_j \subset \mathbf{Z}^{m+n}$, $j = 1,2,3,5$, are subsets of $\mathbf{Z}^{m+n}$ defined as follows: The vector $(t_1,\dots,t_{m+n}) \in I_1$ if $t_s = t_{s'}$ for some $1 \leq s < s' \leq m+n$. The vector is in I_2 if, in addition, $s \leq m$, and it is in I_3 if, in addition to the previous two criteria, $t_s = t_{s'}$ for some $m < s'$. The vector is in I_5 if $t_s = t_{s'}$ for some $1 \leq s < s' \leq m$.

From Lemma 5.1 below it follows that

$$Q_{N,\Delta}^{(j)} \overset{d}{\Rightarrow} q_\Delta^{(j)} \quad (N \to \infty) \quad j = 1,2,3,4,5 \;, \tag{4.27}$$

where the limit $q_\Delta^{(j)}$ is defined by the right hand side of (4.24), after replacing the $Z_{N,\Delta}^{(m')}$ with $m' = 1, m$ or n by $Z_\Delta^{(m')}$ defined in (5.1).

From Lemma 5.2 below it follows that the remaining term $R_M^{(j)}$ in (4.23) is negligible:

$$R_N^{(j)} \overset{d}{\Rightarrow} 0 \quad (N \to \infty) \quad j = 1, 2, 3, 4, 5 \ . \tag{4.28}$$

Hence, the relations (4.23), (4.27) and (4.28) imply the convergence (existence of the limit) (3.16).

Proposition 4.3 *The number* $K > 0$ *and the partition* $\{\Delta\}$ *in Proposition* 4.2 *can be chosen so that Relation* (3.17) *holds.*

Proof. We prove (3.17) only in the case (A_2) (or $j = 2$) (the proof in the cases $j = 1, 3, 4, 5$ is similar and based on the Lemma 4.1).

Let $(Z^{(1)}, Z^{(m)}, Z^{(n)})$ be defined in (2.8). Then, $q_\Delta^{(2)}$ defined in (4.27) can be written in the form

$$q_\Delta^{(2)} = \int_{\mathbf{R}^{m+1}} F_\Delta^{(2)}(x_1, \ldots, x_{m+1}) Z^{(1)}(dx_1) \ldots Z^{(1)}(dx_m) Z^{(n)}(dx_{m+1}) \ ,$$

where

$$F_\Delta^{(2)}(x_1, \ldots, x_{m+1}) := \Phi_\Delta^{(2)}(x_1, \ldots, x_{m+1}) |x_1 \ldots x_m|^{-\alpha/2}$$

and

$$\Phi_\Delta^{(2)}(x_1, \ldots, x_{m+1}) = \sum_{\{\Delta\} \in D_K^{(m+1)}} \Phi_0\left(\sum_{i=1}^m t_{l_i}, t_{l_{m+1}}\right) \prod_{k=1}^{m+1} \mathbb{1}(x_k \in \Delta_{l_k}) \ .$$

On the other hand, we have that

$$Z_{m,n}^{(2)} = \int_{\mathbf{R}^{m+1}} F^{(2)}(x_1, \ldots, x_{m+1}) Z^{(1)}(dx_1) \ldots Z^{(1)}(dx_m) Z^{(n)}(dx_{m+1})$$

where

$$F^{(2)}(x_1, \ldots, x_{m+1}) = \Phi_0\left(\sum_{i=1}^m x_i, x_{m+1}\right) |x_1 \ldots x_m|^{-\alpha/2} \ .$$

Therefore, by the well-known properties of the stochastic Wiener-Itô integrals,

$$\begin{aligned}
\mathrm{Var}\big(q_\Delta^{(2)} - Z_{m,n}^{(2)}\big) &\le \mathrm{const} \big\| F_\Delta^{(2)} - F^{(2)} \big\|_{L^2}^2 \\
&= C\bigg(\int_{[-K,K]^{m+1}} \big|\Phi_\Delta(x_1, \ldots, x_{m+1}) \\
&\quad - \Phi_0(\sum_{i=1}^m x_i, x_{m+1})\big|^2 |x_1 \ldots x_m|^{-\alpha} dx_1 \ldots dx_{m+1} \\
&\quad + \int_{\mathbf{R}^{m+1} \setminus [-K,K]^{m+1}} \big|F^{(2)}(x_1, \ldots, x_{m+1})\big|^2 dx_1 \ldots dx_{m+1} \bigg) \\
&=: j_K + j'_K \ .
\end{aligned} \tag{4.29}$$

The dominant convergence theorem implies

$$j'_K \to 0 \quad (K \to \infty) \tag{4.30}$$

since $F^{(2)} \in L^2(\mathbf{R}^{m+1})$ (see Corollary 6 below) and $\mathbf{R}^{m+1} \backslash [-K,K]^{m+1} \to \emptyset \quad (K \to \infty)$.

Clearly, since the function Φ_0 is continuous, for any fixed $\epsilon > 0$, $K > 0$, the term j_K can be made arbitrary small ($j_K < \epsilon$) by choosing $\{\Delta\}$ with sufficiently small mesh size h.

This, together with (4.29) and (4.30) proves (3.17). □

Using Lemma 4.1, we show in the Appendix (Corollary 6.1) that the limiting processes $Z^{(j)}_{m,n}$, $j = 1, \ldots, 5$ in Theorems 2.1 and 2.2 are well-defined.

5 Convergence of the step function approximations

Using diagram formulas for cumulants (Giraitis and Surgailis [6]), we first prove that the step function approximations converge to the desired limits. This convergence is used in the proof of Proposition 4.2.

Lemma 5.1 *Let* $Z^{(m_1)}_{N,\Delta'_1}, \ldots, Z^{(m_p)}_{N,\Delta'_p}$ $m_i \in \{1, m, n\}$ *be defined by* (4.22) *and* m_i *are such that* $d(m_j) = 1 - m_j(1-\alpha) < 0$ *if* $m_j \geq 2$. *Let* $\Delta'_i \in D^{(1)}_K$, $i = 1, \ldots, p$ *are such that*

$$\Delta'_i \cap \pm\Delta'_j = \emptyset \quad (i \neq j) \ .$$

Then

$$\left(Z^{(m_1)}_{N,\Delta'_1}, \ldots, Z^{(m_p)}_{N,\Delta'_p}\right) \overset{d}{\Rightarrow} \left(Z^{(m_1)}_{\Delta'_1}, \ldots, Z^{(m_p)}_{\Delta'_p}\right)$$

as $N \to \infty$, *where the Gaussian complex valued variables* (*measures*) $Z^{(m_i)}_{\Delta'_i}$, $i = 1, \ldots, p$ *are defined by*

$$Z^{(l)}_{\Delta} := \begin{cases} \int_\Delta |x|^{-\alpha/2} Z^{(1)}(dx), & \text{if} \quad l = 1 \\ \int_\Delta Z^{(l)}(dx), & \text{if} \quad l = m, n \geq 2 \ , \end{cases} \tag{5.1}$$

and the $Z^{(l)}$*'s are defined by* (2.8).

Proof. Let $Y_{N,i} = \text{Re}\, Z^{(m_i)}_{N,\Delta'_i}$, $Y_{N,p+i} = \text{Im}\, Z^{(m_i)}_{N,\Delta'_i}$, $Y_i = \text{Re}\, Z^{(m_i)}_{\Delta'_i}$, $Y_{p+i} = \text{Im}\, Z^{(m_i)}_{\Delta'_i}$, $i = 1, \ldots, p$. It is sufficient to prove that for any real numbers $u_1, \ldots, u_p$ the convergence

$$S_N := \sum_{i=1}^{2p} u_i Y_{N,i} \overset{d}{\Rightarrow} S := \sum_{i=1}^{2p} u_i Y_i \tag{5.2}$$

holds as $N \to \infty$. Because the limit S in (5.2) is Gaussian, it is sufficient to check that

$$\text{Var}\ S_N \to \text{Var}\ S \quad (N \to \infty) \tag{5.3}$$

and

$$\text{cum}_k(S_N) \to 0 \quad (N \to \infty) \quad \text{for } k \geq 3 \ , \tag{5.4}$$

where $\text{cum}_k(S_N)$ denotes the k-th cumulant of S_N.

Without loss of generality we assume below that all moments of ξ_s exist (otherwise ξ_s can be approximated in (4.22) by $\xi_s \mathbb{1}(|\xi_s| \leq K')$, $K' > 0$).

Let us prove (5.4) first. Because of the multilinearity properties of the cumulant, it is sufficient to prove that

$$\chi_k := \mathrm{cum}\big(Y_{N,i_1}, \ldots, Y_{N,i_k}\big) \to 0 \quad (N \to \infty) \ \forall\, k \geq 3 \tag{5.5}$$

for any $(i_1, \ldots i_k) \in (1, \ldots, 2p)$.

From the definition (4.22) of $Z_{N,\Delta}^{(m_i)}$ we get

$$Y_{N,i} = \sum_{t_1,\ldots,t_{m_i} \in \mathbf{Z}} \sqrt{N}\theta_{N,i}(t_1, \ldots, t_{m_i}) : \xi_{t_1}, \ldots, \xi_{t_{m_i}} :$$

$i = 1, \ldots, 2p$ after setting

$$\theta_{N,i} = \ \mathrm{Re}\ a_{N,\Delta_i'}(t_1, \ldots, t_{m_i}), \quad \theta_{N,p+i} = \mathrm{Im}\ a_{N,\Delta_i'}(t_1, \ldots, t_{m_{p+i}}), \quad i = 1, \ldots, p$$

with $m_{p+i} = m_i$.

Using again the multilinearity of the cumulants, the fact that (ξ_s) is an i.i.d. sequence, and the cumulant formula for $: \xi_{t_1}, \ldots, \xi_{t_l} :$ (see, e.g. [6], Theorem 4 (iv)), we get

$$\chi_k = \sum_{\gamma=(V_1,\ldots,V_r)} d(\gamma) U_N(\gamma) \ , \tag{5.6}$$

where

$$U_N(\gamma) = \sum_{t_1,\ldots,t_r \in \mathbf{Z}} \prod_{j=1}^{k} [\sqrt{N}\theta_{N,i_j}(t_{j,1}, \ldots, t_{j,m_{i_j}})] \ , \tag{5.7}$$

and where $\sum_\gamma$ and $d(\gamma)$ are defined below. $\sum_\gamma$ is taken over all *connected* partitions of the table

$$W = \begin{pmatrix} (1,1), \ldots, (1, m_{i_1}) \\ \ldots \ , \ldots, \quad \ldots \\ (k,1), \ldots, (k, m_{i_k}) \end{pmatrix} \tag{5.8}$$

by subsets $V_1, \ldots, V_r$ such that V_i contains elements from two or more rows of the table W. (Since $\chi_1(\xi_0) = E\xi_0 = 0$, we do not need to consider V's with $|V| = 1$). A partition is *connected* if it does not split the rows of the table W into two or more disjoint subsets. Finally,

$$d(\gamma) := \chi_{|V_1|}(\xi_0) \ldots \chi_{|V_r|}(\xi_0)$$

where $\chi_l(\xi_0)$ is the l-th cumulant of ξ_0 and $t_{ij}, (i,j) \in W$ are defined by

$$t_{i,j} \equiv t_s \quad \text{if } (i,j) \in V_s, \ V_s \in \gamma = (V_1, \ldots V_r) \ .$$

Using (5.6), we get

$$|\chi_k| \leq C(\xi) \sum_{\gamma=(V_1,\ldots,V_r)} |U_N(\gamma)| \ .$$

We shall now apply Parseval's equality in the form

$$\sum_t g_1(t)\dots g_k(t) = 2\pi \int_{[-\pi,\pi]^k} \widehat{g}_1(x_1)\dots\widehat{g}_k(x_k)\mathbb{1}\left(\left(\sum_{i=1}^k x_i\right)\mathrm{mod}\ 2\pi \equiv 0\right)d^k x, \quad g_i \in L^2 \ , \tag{5.9}$$

to $U_N(\gamma)$. If $\widehat{\theta}_{N,i}$ denotes the Fourier transform of $\theta_{N,i}$, we get

$$U_N(\gamma) = (2\pi)^r N^{k/2} \int_{[-\pi,\pi]^{m_0}}^{(\gamma)} \prod_{j=1}^k \widehat{\theta}_{N,i_j}(x_{j,1},\dots,x_{j,m_{i_j}})d^{m_0}x \tag{5.10}$$

where $m_0 := m_{i_1} + \dots + m_{i_k}$ and the integral $\int_{[-\pi,\pi]^{m_0}}^{(\gamma)} \dots d^{m_0}x$ is taken over all hyperplanes

$$\left(\sum_{(i,j)\in V_s} x_{i,j}\right)\mathrm{mod}\ 2\pi \equiv 0, \quad s = 1,\dots,r \tag{5.11}$$

in $[-\pi,\pi]^{m_0}$. Note that

$$|\widehat{\theta}_{N,i}(x_1,\dots,x_{m_i})| \le |\hat{a}_{N,\Delta_i}(x_1,\dots,x_{m_i})| + |\hat{a}_{N,-\Delta_i}(x_1,\dots,x_{m_i})| \ .$$

By (4.17)–(4.19), $|\widehat{\theta}_{N,i}(x_1,\dots,x_l)|$ is bounded by $C(K)$ if $l=1$, and by $C(K)(\log N)^p$ if $l \ge 2$ $(l = m,n)$ for large p. Moreover, since the partition $\gamma = (V_1,\dots,V_r)$ is connected, the variables

$$y_j := (x_{j,1} + \dots + x_{j,m_{i_j}})\mathrm{mod}\ 2\pi,\ j = 1,\dots,k-1$$

are linearly independent. After this (partial) change of variables we integrate over the remaining variables and obtain

$$\begin{aligned} |\chi_k| &\le C(K)N^{k/2}(\ln N)^{kp} \int_{[-\pi,\pi]^{k-1}} \prod_{j=1}^{k-1} \mathbb{1}(y_j \in (\Delta'_{i_j} \cup -\Delta'_{i_j})/N)d^{k-1}y \\ &\le C(K)(\ln N)^{kp}N^{k/2}N^{-(k-1)} \to 0 \quad (N\to\infty) \quad \forall k \ge 3 \ , \end{aligned}$$

i.e. (5.5) holds. This concludes the proof of (5.4).

To prove the convergence (5.3) of the variances, it is sufficient to show that as $N \to \infty$

$$k_N := EZ^{(m_1)}_{N,\Delta'_1}\overline{Z^{(m_2)}_{N,\Delta'_2}} \to \begin{cases} 0, & \text{if } \Delta'_1 \cap \Delta'_2 = \emptyset; \\ EZ^{(m_1)}(\Delta'_1)\overline{Z^{(m_2)}(\Delta'_2)}, & \text{if } \Delta'_1 = \Delta'_2 \ . \end{cases} \tag{5.12}$$

Indeed, from the relations (5.12), $\mathrm{Re}\, Z^{(m_1)}_{N,\Delta} = (Z^{(m_1)}_{N,\Delta} + \overline{Z^{(m_1)}_{N,\Delta}})/2$, $\mathrm{Im}\, Z^{(m_1)}_{N,\Delta} = (Z^{(m_1)}_{N,\Delta} - \overline{Z^{(m_1)}_{N,\Delta}})/2i$, and the equality $\overline{Z^{(m_1)}_{N,\Delta}} \stackrel{d}{=} Z^{(m_1)}_{N,-\Delta}$ (see definitions (4.22), (4.20)), it follows that as $N \to \infty$, for $m_1, m_2 = 1, m, n$,

$$\begin{aligned} &E\ \mathrm{Re}\ Z^{(m_1)}_{N,\Delta_i}\ \mathrm{Im}\ Z^{(m_2)}_{N,\Delta_j} \to 0 = E\ \mathrm{Re}\ Z^{(m_1)}_{\Delta_i}\ \mathrm{Im}\ Z^{(m_2)}_{\Delta_j}, \\ &E\ \mathrm{Re}\ Z^{(m_1)}_{N,\Delta_i}\ \mathrm{Re}\ Z^{(m_2)}_{N,\Delta_j} \to EZ^{(m_1)}(\Delta_i)\overline{Z^{(m_2)}(\Delta_j)}/2 = E\ \ \mathrm{Re}\ Z^{(m_1)}_{\Delta_i}\ \mathrm{Re}\ Z^{(m_2)}_{\Delta_j}, \\ &E\ \ \mathrm{Im}\ Z^{(m_1)}_{N,\Delta_i}\ \mathrm{Im}\ Z^{(m_2)}_{N,\Delta_j} \to EZ^{(m_1)}(\Delta_i)\overline{Z^{(m_2)}(\Delta_j)}/2 = E\ \ \mathrm{Im}\ Z^{(m_1)}_{\Delta_i}\ \mathrm{Im}\ Z^{(m_2)}_{\Delta_j} \ . \end{aligned}$$

This obviously implies (5.3).

It remains thus to prove 5.12. Since

$$k_N := EZ^{(m_1)}_{N,\Delta_1'}\overline{Z^{(m_2)}_{N,\Delta_2'}} = EZ^{(m_1)}_{N,\Delta_1'}Z^{(m_2)}_{N,-\Delta_2'} \ , \tag{5.13}$$

using formula (5.6) for the right side of (5.13), we obtain

$$k_N = \sum_{\gamma} d(\gamma)U_N'(\gamma) \tag{5.14}$$

where $\gamma = (V_1, \dots, V_r)$ denotes the partitions of the table W in (5.8) consisting in this case of two rows, and the numbers $U_N'(\gamma)$ are obtained from (5.7)–(5.10) with $k = 2$:

$$\begin{aligned} U_N'(\gamma) &= \sum_{t_1,\dots,t_r \in \mathbf{Z}} \prod_{j=1}^{2} [\sqrt{N} a_{N,\Delta_j''}(t_{j,1}, \dots, t_{j,m_j})] \\ &= (2\pi)^r N \int_{[-\pi,\pi]^{m_1+m_2}}^{(\gamma)} \prod_{j=1}^{2} \hat{a}_{N,\Delta_j''}(x_{j,1}, \dots, x_{j,m_j}) \\ &\qquad \times \mathbb{1}\big((x_{j,1} + \cdots + x_{j,m_j}) \bmod 2\pi \in \Delta_j''/N\big) d^{m_1+m_2}x \ , \end{aligned} \tag{5.15}$$

where $\Delta_1'' = \Delta_1', \Delta_2'' = -\Delta_2'$. Suppose first $\Delta_1' \cap \Delta_2' = \emptyset$. In (5.15), $(x_{1,1} + \cdots + x_{1,m_1}) \bmod 2\pi \in \Delta_1'/N$, $(x_{2,1} + \cdots, +x_{2,m_2}) \bmod 2\pi \in -\Delta_2'/N$. Since by (5.11),

$$(x_{1,1} + \cdots + x_{1,m_1}) \bmod 2\pi \equiv -(x_{2,1} + \cdots + x_{2,m_2}) \bmod 2\pi \ , \tag{5.16}$$

we are integrating in (5.15) over an empty set. Thus $U_N'(\gamma) = 0$ and $k_N = 0$.

Suppose now $\Delta_1' = \Delta_2'$. To prove the second relation in (5.12), we consider three cases:

1. Let $\Delta_1' = \Delta_2'$ and $m_1 = m_2 = 1$. Then from (5.14) and the definition (4.22) of $Z^{(1)}_{N,\Delta_1'}$, it follows that

$$\begin{aligned} k_N &= N \sum_{t \in \mathbf{Z}} a_{N,\Delta_1'}(t)\overline{a_{N,\Delta_1'}(t)} E\xi_t^2 = N2\pi \int_{[-\pi,\pi]} \hat{a}_{n,\Delta_1'}(x)\overline{\hat{a}_{N,\Delta_1'}(x)} dx \\ &= \int_{[-\pi,\pi]} 2\pi |\hat{a}(x)|^2 N^{-\alpha+1} L^{-1}(N) \mathbb{1}(x \in \Delta_1'/N) dx \sim \int_{\Delta_1'} |x|^{-\alpha} \, dx \quad (N \to \infty) \end{aligned}$$

because of (4.17), (2.3), and $E\xi_0^2 = 1$. In view of (5.1), Relation (5.12) holds.

2. Let $\Delta_1' = \Delta_2'$, $m_1 = 1$, $m_2 \geq 2$. Then $d_{m_2}(\alpha) < 0$ and by (5.6) and (5.13)–(5.15),

$$\begin{aligned} k_N = \chi_{m_2+1}(\xi_0) 2\pi &\int_{[-\pi,\pi]^{m_2+1}}^{(\gamma)} [N^{-\alpha/2} L^{-1/2}(N) \hat{a}(x_{1,1}) \mathbb{1}(x_{1,1} \in \Delta_1'/N)] \\ &\cdot \hat{a}(x_{2,1}) \dots \hat{a}(x_{2,m_2}) \mathbb{1}((x_{2,1} + \cdots + x_{2,m_2}) \bmod \ 2\pi \in -\Delta_2'/N) \\ &\times \prod_{k=1}^{m_2} \mathbb{1}(|x_{2,k}| > 1/\ln N) d^{m_2+1}x \ . \end{aligned} \tag{5.17}$$

In view of (5.11), we have in (5.17) $(x_{1,1} + x_{2,1} + \cdots + x_{2,m_2}) \bmod 2\pi = 0$, and therefore $x_{2,m_2} = (-x_{1,1} - x_{2,1} - \cdots - x_{2,m_2-1}) \bmod 2\pi$. Hence,

$$k_N = \int_{[-\pi,\pi]} dx_{1,1}[N^{-\alpha/2}L^{-1/2}(N)\hat{a}(x_{1,1})\mathbb{1}(x_{1,1} \in \Delta_1'/N)]g_N(x_{1,1}) \ ,$$

where

$$g_N(x_{1,1}) = \chi_{m_2+1}(\xi_0)2\pi \int_{[-\pi,\pi]^{m_2-1}} \hat{a}(x_{2,1}) \dots \hat{a}(x_{2,m_2-1})$$
$$\times \hat{a}(-x_{1,1} - x_{2,1} - \cdots - x_{2,m_2-1}) \prod_{j=1}^{m_2-1} \mathbb{1}(|x_{2,j}| > 1/\ln N)$$
$$\cdot \mathbb{1}(|(x_{1,1} + x_{2,1} + \cdots + x_{2,m_2-1}))\mathrm{mod} 2\,\pi| > 1/\ln N)d^{m_2-1}x$$

and where the function $\hat{a}(x)$ has been periodically extended in $\mathbf{R}$. Since g_N is continuous (see Lemma 5.2 of [12]), $g_N(x_{1,1}) = g(0) + \epsilon_N(x_{1,1})$, where $\epsilon_N(x_{1,1}) \to 0$ uniformly in N on Δ_1'/N and

$$g(0) = \chi_{m_2+1}(\xi_0)2\pi \int_{[-\pi,\pi]^{m_2-1}} \hat{a}(x_{2,1}) \dots \hat{a}(x_{2,m_2-1})$$
$$\times \hat{a}(-x_{2,1} - \cdots - x_{2,m_2-1})d^{m_2-1}x$$
$$= \chi_{m_2+1}(\xi_0) \sum_{j \in \mathbf{Z}} (a(j))^{m_2} \ .$$

Hence (2.9) implies

$$\lim_{N\to\infty} k_N = g(0) \int_{\Delta_1'} |x_{1,1}|^{-\alpha/2} dx_{1,1} \ . \tag{5.18}$$

On the other hand, by (5.1) and (2.8),

$$EZ^{(1)}(\Delta_1')\overline{Z^{(m_2)}(\Delta_1')} = E \int_{\Delta_1'} |x|^{-\alpha/2} Z^{(1)}(dx) \int_{\Delta_1'} \overline{Z^{(m_2)}(dy)}$$
$$= \mathrm{Cov}(\xi_0, : X_0^{(m_2)} :) \int_{\Delta_1'} |x|^{-\alpha/2} dx \ . \tag{5.19}$$

But

$$\mathrm{Cov}(\xi_0, : X_0^{(m_2)} :) = \sum_{s_1,\dots,s_{m_2}} a(-s_1) \dots a(-s_{m_2}) E : \xi_{s_1}, \dots, \xi_{s_{m_2}} : : \xi_0 :$$
$$= \chi_{m_2+1}(\xi_0) \sum_{j \in \mathbf{Z}} (a(j))^{m_2}$$

because $E : \xi_{s_1}, \dots, \xi_{s_{m_2}} :: \xi_0 := \mathrm{cum}(\xi_{s_1}, \dots, \xi_{s_{m_2}}, \xi_0)$ is non-zero only if $s_1 = \dots = s_{m_2} = 0$. By Parserval identity (5.9), $\mathrm{Cov}(\xi_0, : X_0^{(m_2)} :) = g(0)$. Comparing (5.18) and (5.19), we get (5.12).

3. Let $\Delta_1' = \Delta_2'$ and $m_1, m_2 \geq 2$. In this case, we assume in the lemma $d_{m_1}(\alpha) < 0$ and $d_{m_2}(\alpha) < 0$. This ensures that the joint spectral density

$$\psi_{m_1,m_2}(\lambda) := \sum_{\gamma=(V_1,\dots,V_r)} d(\gamma)(2\pi)^r \int_{[-\pi,\pi]^{m_1+m_2}}^{(\gamma)} \prod_{j=1}^{2} \hat{a}(x_{j,1}) \dots \hat{a}(x_{j,m_j})$$
$$\cdot \mathbb{1}\Big((x_{1,1} + \cdots + x_{1,m_1}) \bmod 2\pi = \lambda\Big) d^{m_1+m_2}x \tag{5.20}$$

$\lambda \in [-\pi, \pi]$ of the sequences $(: X^{(m_1)}(t) :)_{t \in \mathbf{Z}}$ and $(: X^{(m_2)}(t) :)_{t \in \mathbf{Z}}$ is continuous at $\lambda = 0$. Then from (5.14), (5.15) and the definition (4.18) of $\hat{a}_{N,\Delta}$, it follows that as $N \to \infty$,

$$k_N \sim \psi_{m_1,m_2}(0) N \int_{-\pi}^{\pi} \mathbb{1}(y \in \Delta'_1/N) dy = |\Delta'_1| \psi_{m_1,m_2}(0) \ .$$

Hence (5.12) also holds in this case. □

The following lemma is used in the proof of Proposition 4.2.

Lemma 5.2 *Let* $R_N^{(j)}$ *be defined by* (4.2). *Then*

$$\operatorname{Var} R_N^{(j)} \to 0 \quad (N \to \infty), \quad j = 1, 2, 3, 5 \ . \tag{5.21}$$

Proof. Recall that $R_N^{(j)}$ involves a sum over repeated indices $t_1, \ldots, t_{m+n}$. To motivate the notation that will be used in this proof, consider for example the case $j = 3$ with $m = 4$ and $n = 2$ and focus on the term $X =: \xi_{t_1}, \xi_{t_2}, \xi_{t_3}, \xi_{t_4} :: \xi_{t_5}, \xi_{t_6} :$ where all the t_i's are different except that $t_1 = t_2 = t_5$ and $t_3 = t_4$. Let $W_1 = \{1, 2, 5\}, V_1 = \{3, 4\}, V_2 = \{6\}$ (In the case $j = 3$ considered here, the letter W is used when the indices appear in different Wick products and the letter V is used when the indices appear in the same Wick product). Note that V_1, V_2, W_1 form a partition of the index set $M = \{1, \ldots, m+n\} = \{1, \ldots, 6\}$. Setting $s_l^* = t_j$ if $j \in W_l$, $t_l^* = t_j$ if $j \in V_l$, and using the notation (2.1) and property (4.26), we get

$$X =: \xi_{s_1^*}, \xi_{s_1^*}, \xi_{t_1^*}, \xi_{t_1^*} :: \xi_{s_1^*}, \xi_{t_2^*} :=: \xi_{s_1^*}^{(2)} :: \xi_{t_1^*}^{(2)} :: \xi_{s_1^*} :: \xi_{t_2^*} :=: \xi_{t_1^*}^{V_1} :: \xi_{t_2^*}^{V_2} :: \xi_{s_1^*}^{(2)} :: \xi_{s_1^*} : \ ,$$

where, in terms of the original t variables, $: \xi_{t_1^*}^{V_1} :=: \xi_{t_3}, \xi_{t_4} :, : \xi_{t_2^*}^{V_2} :=: \xi_{t_6} :$. To deal with the repeated indices s_1^*, we shall use the formula

$$: \xi_s^{(n_1)} : \ldots : \xi_s^{(n_k)} := E : \xi_s^{(n_1)} : \ldots : \xi_s^{(n_k)} : + \sum_{j=1}^{n_1 + \cdots + n_k} c(j) : \xi_s^{(j)} : \tag{5.22}$$

where $c(j)$ are some weights. Thus, we can write $: \xi_{s_1^*}^{(2)} :: \xi_{s_1^*} := c(3) : \xi_{s_1^*}^{(3)} : + c(2) : \xi_{s_1^*}^{(2)} : + c(1) : \xi_{s_1^*} : + c(0) = c(3) : \xi_{s_1^*}^{W'_{1,1}} : + c(2) : \xi_{s_1^*}^{W'_{1,2}} : \ + c(1) : \xi_{s_1^*}^{W'_{1,3}} : + c(0) : \xi_{s_1^*}^{W'_{1,4}} :$ where $W'_{1,1} = \{1, 2, 5\}, W'_{1,2} = \{1, 2\}$ (for example), $W'_{1,3} = \{1\}$ (for example), $W'_{1,4} = \emptyset$, $\xi_{s_1^*}^{W'_{1,1}} =: \xi_{t_1}, \xi_{t_2}, \xi_{t_5} :$, etc. Then $X = c(3) : \xi_{t_1^*}^{V_1} :: \xi_{t_2^*}^{V_2} : \xi_{s_1^*}^{W'_{1,1}} : + \cdots + \quad c(0) : \xi_{t_1^*}^{V_1} :: \xi_{t_2^*}^{V_2} :: \xi_{s_1^*}^{W'_{1,4}} : \quad = c(3) : \xi_{t_1^*}^{V_1}, \xi_{t_2^*}^{V_2} \quad, \xi_{s_1^*}^{W'_{1,1}} :: + \cdots + c(0) : \xi_{t_1^*}^{V_1}, \xi_{t_2^*}^{V_2}, \xi_{s_1^*}^{W'_{1,4}} :$ by using again the property (4.26). In any one of the terms, the important thing is that the indices are the same within V_1, the same within V_2, the same within W_1. Note that we considered here the case $j = 3$. The definition of V and W will be different from case to case (see below).

In general, using (5.22) and the property (4.26) of the Wick products of i.i.d. variables $: \xi_t :$, we can write:

$$R_N^{(j)} - ER_N^{(j)} = \sum_{(V_1,\dots,V_r;W_1,\dots,W_{r'})} \sum_{W_1'\subset W_1,\dots,W_{r'}'\subset W_{r'}} c(V,W,W')$$

$$\cdot \sum_{\substack{t_1^*,\dots,t_r^*;s_1^*,\dots,s_{r'}^*\in\mathbf{Z} \\ t_i^*\neq t_l^*, s_i^*\neq s_l^*(i\neq l); t_i^*\neq s_l^*}} A_{N,\Delta}^{(j)}(t_1,\dots,t_{m+n}) : \xi_{t_1^*}^{V_1},\dots,\xi_{t_r^*}^{V_r}, \xi_{s_1^*}^{W_1'},\dots,\xi_{s_{r'}^*}^{W_{r'}'} : \tag{5.23}$$

where the coefficients $c(V,W,W')$ depend on partitions $(V_1,\dots,V_r;W_1,\dots,W_{r'})$, $(W_1',\dots,W_{r'}')$ defined below but do not depend on $t_1,\dots,t_{m+n}$.

Here the sum $\sum_{V_1,\dots,V_r;W_1,\dots,W_{r'}}$ is taken over all partitions $(V_1,\dots,V_r;W_1,\dots,W_{r'})$, $r+r'>0$ of the index set $M:=\{1,\dots,m+n\}$, and such the V_k and W_k are characterized as follows:

Definition 5.1 The sets V_k and W_k are such that $|V_k|\ge 1, |W_k|\ge 2$, and
1) in the case $j=1$: $|V_k|=1$, $|W_k|\ge 2$;
2) in the case $j=2$: $|V_k|=1$ if $V_k\subset\{1,\dots,m\}$, no restrictions on V_k if $V_k\subset\{m+1,\dots,m+n\}$, W_k contains at least one point of $\{1,\dots,m\}$;
3) in the case $j=3$: $V_k\subset\{1,\dots,m\}$ or $V_k\subset\{m+1,\dots,m+n\}$, W_k contains points from both $\{1,\dots,m\}$ and $\{m+1,\dots,m+n\}$;
5) in the case $j=5$: $|V_k|=1$, $|W_k|\ge 2$.

(These requirements follow from the definition of the sets I_j in (4.25)). The sum $\sum_{W_1',\dots,W_{r'}'}$ is taken over all subsets $W_l'\subset W_l, l=1,\dots,r'$ including the empty set. In the sum $\sum_{t_1^*,\dots,t_r^*;s_1^*,\dots,s_{r'}^*}$, one sets $t_j\equiv t_l^*\in\mathbf{Z}$ if $j\in V_l$, $l=1,\dots,r$ and $t_j\equiv s_l^*\in\mathbf{Z}$ if $j\in W_l$, $l=1,\dots,r'$. Finally for $V=(i_1,\dots,i_l)\subset M$, $:\xi_{t^*}^V:$ denotes the Wick product $:\xi_{t^*}^V:=:\xi_{t_{i_1}},\dots,\xi_{t_{i_l}}:$ where $t^*=t_{i_1}=\dots=t_{i_l}$. ξ_s^W is defined similarly. Note that the term $:\xi_{t_1},\dots,\xi_{t_{m+n}}:$ in (4.25) is included in (5.23); it corresponds to $W_1'=W_1,\dots,W_r'=W_r$.

We can rewrite (5.23) as

$$R_N^{(j)} - ER_N^{(j)} = \sum_{(V_1,\dots,V_r;W_1,\dots,W_{r'})} \sum_{W_1'\subset W_1,\dots,W_{r'}'\subset W_{r'}} c'(V,W,W')$$

$$\cdot \sum_{t_1^*,\dots,t_r^*;s_1^*,\dots,s_{r'}^*\in\mathbf{Z}} A_{N,\Delta}^{(j)}(t_1,\dots,t_{m+n}) : \xi_{t_1^*}^{V_1},\dots,\xi_{t_r^*}^{V_r}, \xi_{s_1^*}^{W_1'},\dots,\xi_{s_{r'}^*}^{W_{r'}'} : \tag{5.24}$$

that is, as a sum in $(t_1^*,\dots,t_r^*,s_1^*,\dots,s_{r'}^*)\in\mathbf{Z}^{r+r'}$ without excluding the diagonals, with some other coefficients $c'(V,W,W')$. (For example, if $V_1=\{1\}, V_2=\{2\}$, $\sum_{t_1^*\neq t_2^*} A_{N,\Delta}^{(j)}(t_1,t_2):\xi_{t_1^*}^{V_1},\xi_{t_2^*}^{V_2}: = \sum_{t_1^*,t_2^*} A_{N,\Delta}^{(j)}(t_1,t_2):\xi_{t_1^*}^{V_1},\xi_{t_2^*}^{V_2}: - \sum_{t_1^*} A_{N,\Delta}^{(j)}(t_1,t_2):\xi_{t_1^*}^{V_1\cup V_2}:$). Then

$$\mathrm{Var}(\mathbf{R}_N^{(j)}) \le C \sum_{(V_1,\dots,V_r;W_1,\dots,W_{r'})} \sum_{W_1',\dots,W_{r'}'} q_N(V,W,W') \tag{5.25}$$

where

$$q_N(V,W,W') = \mathrm{Var}\left(\sum_{t_1^*,\dots,t_r^*;s_1^*,\dots,s_{r'}^* \in \mathbf{Z}} A_{N,\Delta}^{(j)}(t_1,\dots,t_{m+n}) : \xi_{t_1^*}^{V_1},\dots,\xi_{t_r^*}^{V_r},\xi_{s_1^*}^{W_1'},\dots,\xi_{s_{r'}^*}^{W_{r'}'} : \right) .$$

It remains to estimate the terms $q_N(V,W,W')$.

First we consider the case when $W_1' \neq \emptyset, \dots, W_{r'}' \neq \emptyset$. Then: a) there exists a pair (i,l) of indices such that $W_k' := (i,l)$ satisfies the conditions in definition 5.1; b) any index $t_1,\dots,t_{m+n}$ will have a corresponding ξ_{t_i} in the Wick product $: \xi_{t_1^*}^{V_1},\dots,\xi_{t_r^*}^{V_r},\xi_{s_1^*}^{W_1'},\dots,\xi_{s_{r'}^*}^{W_{r'}'} :$. Thus, (4.1) implies the bound

$$q_N(V,W,W') \le C u_N^{(i,l)} \tag{5.26}$$

where

$$u_{N,\Delta}^{(i,l)} := \sum_{t_1,\dots,t_{m+n} \in \mathbf{Z}: t_i = t_l} \left| A_{N,\Delta}^{(j)}(t_1,\dots,t_{m+n}) \right|^2 .$$

Now consider the case where $(W_1',\dots,W_{r'}')$ contains $W_i' = \emptyset$. Set for simplicity that $W_1' = \emptyset,\dots,W_l' = \emptyset$, $W_{l+1}' \neq \emptyset,\dots,W_{r'}' \neq \emptyset$. Then

$$\begin{aligned} q_N(V,W,W') &= \mathrm{Var}\Bigg(\sum_{t_1^*,\dots,t_r^*} \sum_{s_i^*: W_i' \neq \emptyset} \left[\sum_{s_i^*: W_i' = \emptyset} A_{N,\Delta}^{(j)}(t_1,\dots,t_{m+n}) \right] \\ &\qquad \times : \xi_{t_1^*}^{V_1},\dots,\xi_{t_r^*}^{V_r},\xi_{s_1^*}^{W_1'},\dots,\xi_{s_{r'}^*}^{W_{r'}'} : \Bigg) \\ &= \mathrm{Var}\Bigg(\sum_{t_1^*,\dots,t_r^*} \sum_{s_{l+1}^*,\dots,s_{r'}^*} \left[\sum_{s_1^*,\dots,s_l^*} A_{N,\Delta}^{(j)}(t_1,\dots,t_{m+n}) \right] \\ &\qquad \times : \xi_{t_1^*}^{V_1},\dots,\xi_{t_r^*}^{V_r},\xi_{s_1^*}^{W_1'},\dots,\xi_{s_{r'}^*}^{W_{r'}'} : \Bigg) . \end{aligned}$$

Note that in this case the Wick product $: \xi_{t_1^*}^{V_1},\dots,\xi_{t_r^*}^{V_r},\xi_{s_1^*}^{W_1'},\dots,\xi_{s_{r'}^*}^{W_{r'}'} :$ does not depend on the indices $s_1^*,\dots,s_l^*$. Using the same argument as above, we get from (4.1) that

$$q_N(V,W,W') \le C v_{N,\Delta}^{(j)}(W_1,\dots,W_l) \tag{5.27}$$

where

$$v_{N,\Delta}^{(j)}(W_1,\dots,W_l) := \sum_{t_1^*,\dots,t_r^*,s_{l+1}^*,\dots,s_{r'}^* \in \mathbf{Z}} \left(\sum_{s_1^*,\dots,s_l^*}^{(W)} A_{N,\Delta}^{(j)}(t_1,\dots,t_{m+n}) \right)^2 . \tag{5.28}$$

Taking into account (5.25), (5.26) and (5.27), it remains to check, for the proof of (5.21) that

$$u_{N,\Delta}^{(i,l)} \to 0 \quad (N \to \infty) , \tag{5.29}$$

and

$$v^{(j)}_{N,\Delta}(W_1,\dots,W_l)\to 0 \quad (N\to\infty), \quad j=1,2,3,5\ . \tag{5.30}$$

We first prove (5.29)–(5.30) in the most complicated case (A_3), i.e. $j=3$.

We prove (5.29) first. Because of symmetry, it is sufficient to treat the case $(i,l)=(1,m+1)$. By (4.21),

$$A^{(3)}_{N,\Delta}(t_1,\dots,t_{m+n}) = Na_{N,\Delta_1}(t_1,\dots,t_m)a_{N,\Delta_2}(t_{m+1},\dots,t_{m+n})$$

with $\Delta_1\cap\Delta_2=\emptyset$. Since $(i,l)=(1,m+1)$,

$$u^{(1,m+1)}_{N,\Delta} = N^2 \sum_{t_2,\dots,t_m,t_{m+2},\dots,t_{m+n}} \sum_s \left|a_{N,\Delta_1}(s,t_2,\dots,t_m)a_{N,\Delta_2}(s,t_{m+2},\dots,t_{m+n})\right|^2\ . \tag{5.31}$$

From Parseval's identity, using the expression (4.18) for $\hat{a}_{N,\Delta}$ and the corresponding bound (4.19), we get

$$\begin{aligned} u^{(1,m+1)}_{N,\Delta} \le N^2C(K)(\ln N)^{2p} \int_{[-\pi,\pi]^{m+n-2}} \bigg| \int_{[-\pi,\pi]} & \mathbb{1}\Big((x_1+\cdots+x_m)\text{mod } 2\pi\in\Delta_1/N\Big) \\ & \cdot\mathbb{1}\Big((z-x_1)+x_2+\cdots+x_{m+n})\text{mod } 2\pi\in\Delta_2/N\Big)dx_1\bigg|^2 \\ & \times dzdx_2\dots dx_m dx_{m+2}\dots dx_{m+n} \end{aligned}$$

for some $p>0$. Setting $y_1=(x_2+\cdots+x_m)$, and $y_2=(z+x_2+\cdots+x_{m+n})$, we get

$$\begin{aligned} u^{(1,m+1)}_{N,\Delta} &\le N^2C_1(K)(\ln N)^{2p}\int_{[-\pi,\pi]^4} \mathbb{1}((x_1+y_1)\text{mod } 2\pi\in\Delta_1/N) \\ &\qquad \times\mathbb{1}((x_1'+y_1)\text{mod } \ 2\pi\in\Delta_1/N) \\ &\qquad \cdot\mathbb{1}((-x_1+y_2)\text{mod } 2\pi\in\Delta_2/N)dx_1\,dx_1'\,dy_1\,dy_2 \\ &= O(N^{-1}(\ln N)^{2p})\to 0 \quad (N\to\infty)\ , \end{aligned}$$

because y_2 and x_1 are linearly independent. Hence (5.29) holds.

It remains to verify (5.30). Again Parseval's identity and (4.19) imply

$$\begin{aligned} v^{(3)}_{N,\Delta}\big(W_1,\dots,W_l\big) &\le N^2C(K)(\ln N)^{2p} \\ &\times \int_{[-\pi,\pi]^{n'}} \bigg|\int^{(W)}_{[-\pi,\pi]^{n_0}} \mathbb{1}\Big((x_1+\cdots+x_m)\text{mod } 2\pi\in\Delta_1/N\Big) \\ &\cdot\mathbb{1}\Big((x_{m+1}+\cdots+x_{m+n})\text{mod } 2\pi\in\Delta_2/N\Big)d^{n_0}x\bigg|^2 \prod_{i\in W^c} dx_i\ , \end{aligned} \tag{5.32}$$

where $W^c=\{1,\dots,m+n\}\setminus(W_1\cup\ldots\cup W_l)$, $n_0:=|W_1\cup\ldots\cup W_l|$, $n_0':=|W^c|=m+n-n_0$ and the integral $\int^{(W)}\dots d^{n_0}x$ is taken over the hyperplanes

$$\left(\sum_{s\in W_p} x_s\right)\text{mod } 2\pi=0, \quad p=1,\dots,l. \tag{5.33}$$

If $n' = 0$, we have $n_0 = m + n$ and, setting

$$y := (x_1 + \cdots + x_m) \mathrm{mod}\ 2\pi \equiv -(x_{m+1} + \cdots + x_{m+n}) \mathrm{mod}\ 2\pi \ , \tag{5.34}$$

we obtain from (5.32) and (5.34) that

$$\begin{aligned} v_{N,\Delta}^{(3)}(W_1, \ldots, W_l) &\le C(K)N(\ln N)^{2d} \int_{[-\pi,\pi]^{n_0}}^{(W)} [\ldots] d^{m+n}x \\ &\le C(K)N(\ln N)^{2d} \int_{[-\pi,\pi]} \mathbb{1}(y \in \Delta_1/N)\mathbb{1}(-y \in \Delta_2/N)dy = 0 \end{aligned}$$

since $\Delta_1 \cap -\Delta_2 = \emptyset$ by definition of $\{\Delta\}$.

If $n' \neq 0$, proceeding as in the proof of (5.29), one obtains

$$v_{N,\Delta}^{(3)}\Big(W_1, \ldots, W_l\Big) = O(N^{-1}(\ln N)^{2p}) \to 0 \quad (N \to \infty) \ . \tag{5.35}$$

Therefore, (5.30) holds. Lemma 5.2 is now proved for the case $j = 3$.

The proof in the cases $j = 1, 2, 5$ is similar but one must use the following properties of the coefficient $a_{N,\Delta}$:

$$\begin{aligned} \Big|a_{N,\Delta}(t)\Big| &= \Big| \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}tx} \frac{\hat{a}(x)}{N^{\alpha/2}L(N)} \mathbb{1}(x \in \Delta/N)dx \Big| \\ &\le C \int_{-\pi}^{\pi} \mathbb{1}(x \in \Delta/N)dx = C(K)N^{-1} \ , \end{aligned} \tag{5.36}$$

$$\sum_{t\in\mathbf{Z}} \big|a_{N,\Delta}(t)\big|^2 = 2\pi \int_{\Delta} \big|\hat{a}_{N,\Delta}(x)\big|^2 dx \sim N^{-1} \int_{\Delta} |x|^{-\alpha} dx \quad (N \to \infty) \tag{5.37}$$

which follow from (4.17) and (4.20).

For instance, consider the case $j = 2$. To evaluate (5.31), use (4.21). One may get

$$\begin{aligned} u_{N,\Delta}^{(1,m+1)} = &\sum_{t_3,\ldots,t_m,t_{m+1},\ldots,t_{m+n}} \sum_s \big| \sqrt{N} a_{N,\Delta_1}(s) \sqrt{N} a_{N,\Delta_2}(s) \\ &\times \sqrt{N} a_{N,\Delta_3}(t_3) \ldots \sqrt{N} a_{N,\Delta_4}(t_m) \cdot \sqrt{N} a_{N,\Delta_{m+1}}(t_{m+1}, \ldots, t_{m+n}) \big|^2 \\ = &\ O(N^{-1}(\ln N))^{2p}, \end{aligned}$$

since

$$\sum_s \big|\sqrt{N} a_{N,\Delta_1}(s) \sqrt{N} a_{N,\Delta_2}(s)\big|^2 \le \frac{C^2(K)}{N} \sum_s \big|\sqrt{N} a_{N,\Delta_1}(s)\big|^2 = O(1/N)$$

by (5.36), (5.37) and

$$\begin{aligned} &\sum_{t_{m+1},\ldots,t_{m+n}} \big|\sqrt{N} a_{N,\Delta_{m+1}}(t_{m+1}, \ldots, t_{m+n})\big|^2 = CN \int_{[-\pi,\pi]^n} \big|\hat{a}_{N,\Delta_{m+1}}(x_{m+1}, \ldots, x_{m+n})\big|^2 d^n x \\ &\le CN(\ln N)^{2p} \int_{[-\pi,\pi]^n} \mathbb{1}((x_{m+1} + \cdots + x_{m+n}) \mathrm{mod}\ 2\pi \in \Delta_{m+1}/N) d^n x \\ &= O(\ln N)^{2p} \ . \end{aligned}$$

One can also get

$$u_{N,\Delta}^{(1,m+1)} = \sum_{t_2,\dots,t_m,t_{m+2},\dots,t_{m+n}} \sum_s \Big| \sqrt{N} a_{N,\Delta_1}(s) \sqrt{N} a_{N,\Delta_2}(t_2) \dots \sqrt{N} a_{N,\Delta_m}(t_m) \cdot \sqrt{N} a_{N,\Delta_{m+1}}(s, t_{m+2}, \dots, t_{m+n}) \Big|^2$$
$$\le CN^2 \sum_{t_{m+2},\dots,t_{m+n}} \sum_s \big| a_{N,\Delta_1}(s) a_{N,\Delta_{m+1}}(s, t_{m+2}, \dots, t_{m+n}) \big|^2$$

by (5.37). One then proceeds as in the case $j = 3$. □

6 Appendix

Our goal here is to establish Lemma 4.1 and to show that the limiting processes $Z_{m,n}^{(j)},\ j = 1, \dots, 5$ are well-defined. Let

$$p(u_1, u_2) := \int_{\mathbf{R}} \big(1 + |u_1 + y|\big)^{-1} \big(1 + |u_2 - y|\big)^{-1} |y|^{-\beta}\, dy \ , \tag{6.1}$$

be the function introduced in (4.10). Lemma 4.1 is a consequence of the two following lemmas.

Lemma 6.1 *If* $-1 < \alpha_1, \alpha_2, \beta < 1$ *and*

$$\alpha_1 + \alpha_2 + 2\beta > 1 \ , \tag{6.2}$$

then

$$I := \int_{\mathbf{R}^2} |u_1|^{-\alpha_1} |u_2|^{-\alpha_2} |p(u_1, u_2)|^2\, du_1\, du_2 < \infty \ .$$

Proof. i) Suppose first $\beta > 0$. Relation (6.2) implies that either

$$\alpha_1 + \beta > 0 \quad \text{or} \quad \alpha_2 + \beta > 0 \ . \tag{6.3}$$

For simplicity, we assume below that $\alpha_1 + \beta > 0$. If $\alpha_1 + \beta \le 1$, set $\beta_1 = \beta_2 = \beta/2$. If $\alpha_1 + \beta > 1$ (and, therefore, $\alpha_1 > 0$) put $\beta_1 = (1 - \alpha_1 - \epsilon)/2$, $\beta_2 = \beta + (\alpha_1 - 1 + \epsilon)/2$, where $\epsilon > 0$ is small number. Then $0 < \beta_1, \beta_2 < 1$, $\beta_1 + \beta_2 = \beta$. Using in (6.1) the inequalities

$$(1 + |x|)^{-1} (1 + |y|)^{-1} \le (1 + |x + y|)^{-1}, \quad x, y \in \mathbf{R}$$

and

$$\int_{\mathbf{R}} |x|^{-\alpha} (1 + |x + y|)^{-1}\, dx \le C(\varepsilon)(1 + |y|)^{-\alpha+\varepsilon}, \quad (0 < 2\varepsilon < \alpha < 1) \tag{6.4}$$

we obtain

$$\begin{aligned} p(u_1, u_2) &\le (1 + |u_1 - u_2|)^{-1/2} \int_{\mathbf{R}} (1 + |u_1 + y|)^{-1/2} (1 + |u_2 - y|)^{-1/2} |y|^{-\beta}\, dy \\ &\le (1 + |u_1 - u_2|)^{-1/2} \prod_{j=1}^{2} \left(\int_{\mathbf{R}} (1 + |u_j - y|)^{-1} |y|^{-2\beta_j}\, dy \right)^{1/2} \\ &\le C(\varepsilon)(1 + |u_1 - u_2|)^{-1/2} (1 + |u_1|)^{-\beta_1 + \frac{\varepsilon}{2}} (1 + |u_2|)^{-\beta_2 + \frac{\varepsilon}{2}} \ . \end{aligned} \tag{6.5}$$

Using (6.5), we obtain

$$I \le C \int_{\mathbf{R}^2} \prod_{i=1}^{2} |u_i|^{-\alpha_i} (1+|u_i|)^{-2\beta_i+\varepsilon} (1+|u_1-u_2|)^{-1}\, du_1\, du_2 \ . \tag{6.6}$$

Since $0 < \alpha_1 + \beta_1 < 1$, setting $|1+|u_1||^{-\beta_1+\varepsilon} \le |u_1|^{-\beta_1+\varepsilon}$, integrating over u_1 and using (6.4), we get

$$I \le C \int_{\mathbf{R}} |u_2|^{-\alpha_2} (1+|u_2|)^{-\beta_2+\varepsilon} (1+|u_2|)^{-\alpha_1-\beta_1+2\varepsilon}\, du < \infty$$

because of (6.2), $\alpha_1 + \beta_1 + \alpha_2 + \beta_2 = \alpha_1 + \alpha_2 + \beta > 1$.

ii) Suppose now $\beta \le 0$. Then from (6.2) it follows that

$$1 > \alpha_i + \beta > 0, \quad i = 1, 2; \tag{6.7}$$

$$\alpha_1 + \alpha_2 > 1, \quad \alpha_1 > 0,\ \alpha_2 > 0 \tag{6.8}$$

and

$$(1+\beta) > 1/2 \ , \tag{6.9}$$

because if (6.9) is not true, then $\alpha_1 + \alpha_2 + 2\beta = (\alpha_1 + \alpha_2 - 2) + 2(1+\beta) \le 0 + 1 = 1$, i.e. (6.2) does not hold. Let us estimate $p(\cdot)$ in (6.1):

$$p(u_1, u_2) = \int_{\mathbf{R}} [\ldots] dy \le \int_{\mathbf{R}} [\ldots] \Big(\mathbb{1}(|u_1| < |y|/2) + \mathbb{1}(|u_2| < |y|/2) + \mathbb{1}\big(|y| \le 2\sqrt{|u_1 u_2|}\big) \Big) dy =: \sum_{j=1}^{3} p_j(u_1, u_2) \ .$$

Then

$$I \le C \sum_{j=1}^{3} \int_{\mathbf{R}^2} |u_1|^{-\alpha_1} |u_2|^{-\alpha_2} \left| p_j(u_1, u_2) \right|^2 du_1\, du_2 =: C \sum_{j=1}^{3} I_j \ .$$

It remains to prove that

$$I_j < \infty, \quad j = 1, 2, 3 \ . \tag{6.10}$$

Let $j = 1$ (the case $j = 2$ is similar by symmetry). Since here $|u_1| \le |y|/2$, we have $1 + |u_1 + y| \ge 1 + ||y| - |u_1|| \ge 1 + (|y|/2) \ge |y|/2$, and thus, since $\beta \le 0$, we get $(1 + |u+y|)^{-1} \le ((1 + |u_1 + y|)^{1+\beta} |y/2|^{-\beta})^{-1}$. Hence,

$$\begin{aligned} p_1(u_1, u_2) &\le C \int_{\mathbf{R}} (1+|u_1+y|)^{-1-\beta} (1+|u_2-y|)^{-1}\, dy \\ &\le C \int_{\mathbf{R}} (1+|u_1+u_2+y|)^{-1-\beta} (1+|y|)^{-1}\, dy \\ &\le C(1+|u_1+u_2|)^{-1-\beta+\varepsilon} \le C(1+|u_1+u_2|)^{-1/2} \end{aligned}$$

because of (6.4) and (6.9). Then

$$I_1 \le C \int_{\mathbf{R}^2} |u_1|^{-\alpha_1} |u_2|^{-\alpha_2} (1 + |u_1 + u_2|)^{-1} \, d^2u$$
$$\le C \int_{\mathbf{R}^2} |u_2|^{-\alpha_2} (1 + |u_2|)^{-\alpha_1 + \varepsilon} \, du < \infty$$

because of (6.4) and (6.8), when $\varepsilon > 0$ is sufficiently small.

It remains to prove (6.10) in the case $j = 3$. Taking into account that $\beta \le 0$ and using (6.4), we obtain

$$p_3(u_1, u_2) = \int_{|y| \le 2\sqrt{|u_1 u_2|}} (1 + |u_1 + y|)^{-1} (1 + |u_2 - y|)^{-1} |y|^{-\beta} \, dy$$
$$\le C |u_1 u_2|^{-\beta/2} \int_{\mathbf{R}} (1 + |u_1 + y|)^{-1} (1 + |u_2 - y|)^{-1} \, dy$$
$$\le C |u_1 u_2|^{-\beta/2} \int_{\mathbf{R}} |y|^{-1+\epsilon} (1 + |u_1 + u_2 - y|)^{-1} \, dy$$
$$\le C(\varepsilon) |u_1 u_2|^{-\beta/2} (1 + |u_1 + u_2|)^{-1+2\varepsilon} \ .$$

Then

$$I_3 \le C \int_{\mathbf{R}^2} |u_1|^{-\alpha_1 - \beta} |u_2|^{-\alpha_2 - \beta} (1 + |u_1 + u_2|)^{-1} \, du_1 \, du_2$$

and, by (6.4), (6.7) and (6.2) we obtain

$$I_3 \le C(\varepsilon) \int_{\mathbf{R}} |u_2|^{-\alpha_2 - \beta} (1 + |u_2|)^{-\alpha_1 - \beta + \varepsilon} \, du < \infty \ ,$$

where $\varepsilon > 0$ is small enough. □

Lemmma 6.2 *If* $-1 < \alpha, \beta < 1$, *and*

$$1 + 2\beta + \alpha > 0 \ ,$$

then

$$I = \int_{\mathbf{R}} |u|^{-\alpha_1} |p(u)|^2 \, du < \infty \ ,$$

where $p(u) = \int_{\mathbf{R}} (1 + |u + y|)^{-1} (1 + |y|)^{-1} |y|^{-\beta} \, dy$.

Proof. Let $\varepsilon > 0$ be sufficiently small. If $\beta < 0$, then, by (6.4),

$$p(u) \le \int_{\mathbf{R}} (1 + |u + y|)^{-1} |u|^{-1-\beta} \, du \le C(1 + |u|)^{-1-\beta+\varepsilon},$$

and thus

$$I \le \int_{\mathbf{R}} |u_1|^{-\alpha} (1 + |u|)^{-2(1+\beta-\varepsilon)} \, du < \infty \ .$$

If $\beta \ge 0$, then $(1 + |y|)^{-1} \le (1 + |y|)^{-1+\beta+\varepsilon} \le |y|^{-1+\beta+\varepsilon}$,

$$p(u) \le C \int_{\mathbf{R}} (|1 + |u + y|)^{-1} |y|^{-1+\varepsilon} dy \le C(\epsilon)(1 + |u|)^{-1+2\varepsilon} \ ,$$

and

$$I \le C \int_{\mathbf{R}} |u|^{-\alpha_1} (1+|u|)^{-2+4\varepsilon} \, du < \infty \ . \qquad \square$$

Let $F^{(j)}$ be the integrand in the integral defining $Z^{(j)}_{m,n}$ in Theorem 2.1, $j = 1, 2, 3$ and the integrand in the integral defining $Z^{(j)}_{m,0}$ in Theorems 2.2 $j = 4, 5$.

Corollary 6.1 *If the conditions of Theorem* 2.1 *are satisfied, then the functions* $F^{(j)}$, $j = 1, 2, 3$ *satisfy*

$$\|F^{(j)}\|_{L^2} < \infty \tag{6.11}$$

in the cases (A_j), $j = 1, 2, 3$ *respectively. If the conditions of Theorem* 2.2 *are satisfied, then* (6.11) *also holds in the cases* (A_j), $j = 4, 5$.

Proof. The functions $F^{(j)}$ involve Φ_0, defined in (2.14). Since

$$|K_0(x)| = \left| \frac{\mathrm{e}^{\mathrm{i}x} - 1}{\mathrm{i}x} \right| \le C \frac{1}{1+|x|}, \quad x \in \mathbf{R} \ , \tag{6.12}$$

we have $\Phi_0(x_1, x_2) \le Cp(x_1, x_2)$ for all $(x_1, x_2) \in \mathbf{R}^2$. Consider the case $j = 1$ first, where $F^{(1)}(x_1, \ldots, x_{m+n}) = \Phi_0(\sum_{i=1}^m x_i, \sum_{i=m+1}^{m+n} x_i)|x_1|^{-\alpha/2} \ldots |x_{m+n}|^{-\alpha/2}$.

Setting $u_1 = \sum_{i=1}^m x_i, \ u_2 = \sum_{i=m+1}^{m+n} x_i$, we get

$$\begin{aligned} \|F^{(1)}\|^2_{L^2} &= \int_{\mathbf{R}^{m+n}} F^{(1)}(x_1, \ldots, x_{m+n})^2 \, d^{m+n}x \\ &\le \int_{\mathbf{R}^2} p^2(u_1, u_2) g^{(*m)}(u_1) g^{(*n)}(u_2) \, du_1 \, du_2 \end{aligned}$$

where

$$g^{(*m)}(u) = \int_{\mathbf{R}^{m-1}} (|u - x_2 - \cdots - x_m||x_2 \cdots x_m|)^{-\alpha} \, d^{m-1}x \ . \tag{6.13}$$

By using the inequality

$$\int_{\mathbf{R}} |x|^{-\alpha} |x+y|^{-\beta} \, dx \le C|y|^{-\alpha-\beta+1}, \quad \alpha + \beta > 1, \ 0 < \alpha, \ \beta < 1$$

repeatedly, we get

$$g^{(*m)}(u) \le C|u|^{-d_m(u)}, \quad u \in \mathbf{R} \ , \tag{6.14}$$

as in the proof of Lemma 5.2 in [12]. Hence

$$\|F^{(1)}\|^2_{L^2} \le C \int_{\mathbf{R}^2} p^2(u_1, u_2) |u_1|^{-d_m^+(\alpha)} |u_2|^{-d_n^+(\alpha)} \, du_1 \, du_2 < \infty$$

by Condition (2.12) and Lemma 6.1.

The cases $j = 2$ and 3 are treated similarly. When $j = 2$, for example, we set $u_2 = x_{m+n}$ and define $g^{(*n)}$ not through (6.13), but as a bounded function. Since $d_n(\alpha) < 0$ in this case, Relation (6.14) still holds and the rest of the proof applies.

In the cases $j = 4$ and 5, we end up with

$$\|F^{(j)}\|_{L^2} \le C \int_{\mathbf{R}} |u|^{-d_m^+(\alpha)} |p(u)|^2 \, du < \infty \ ,$$

where $p(u)$ is as in Lemma 6.2. Condition (2.17) and this lemma conclude the proof in these cases. □

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