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Logarithmic Sobolev inequalities on noncompact Riemannian manifolds

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Summary. This paper presents a dimension-free Harnack type inequality for heat semigroups on manifolds, from which a dimension-free lower bound is obtained for the logarithmic Sobolev constant on compact manifolds and a new criterion is proved for the logarithmic Sobolev inequalities (abbrev. LSI) on noncompact manifolds. As a result, it is shown that LSI may hold even though the curvature of the operator is negative everywhere.

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1. Introduction

Let M be a connected complete Riemannian manifold with boundary either empty or convex. Consider the operator $L = \Delta + \nabla V$ for some $V \in C^2(M)$ with $Z := \int_M e^V \, dx < \infty$, where dx is the Riemannian volume element. Define

$$k(x) = \inf\{(\mathrm{Ric} - \mathrm{Hess}_V)(X,X) : X \in T_xM, |X| = 1\}, \quad x \in M \ .$$

Throughout this paper, we assume that both of Ricc curvature and k are bounded from below.

Next, let $\mu(dx) = Z^{-1}e^V \, dx$, we say the LSI holds for the L-diffusion process, if there exists $\alpha > 0$ such that

$$\int_M f^2 \log f^2 \, d\mu \le \frac{2}{\alpha} \int_M |\nabla f|^2 \, d\mu$$

holds for all $f \in C_b^1(M)$ with $\mu(f^2) := \int_M f^2 d\mu = 1$. The largest possible constant α is called the LS constant, denoted by $\alpha(V)$ or $\alpha(L)$.

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The first problem we are interested in is when does the LSI hold. It is well known from Bakry-Emery criterion [2] that $\alpha(V) > 0$ whenever $\inf_M k > 0$. If the sectional curvature of M is nonpositive and $\mathrm{cut}(p) = \emptyset$ for some $p \in M$, this condition was relaxed as [5]: $\underline{\lim}_{\rho(p,x)\to\infty} k(x) > 0$. On the other hand, it is proved in [13] that (see also [5] and [10] for $M = \mathbf{R}^d$)

$$\alpha(V) \le \frac{1}{2} \overline{\lim_{\rho(p,x)\to\infty}} \frac{|\nabla V(x)|^2}{-V(x)}(x) , \tag{1.1}$$

where ρ is the Riemannian distance. Hence, to ensure $\alpha(V) > 0$, it is reasonable to assume that $-V(x)$ grows at least in the order of $\rho(p,x)^2$. Also, it is proved in [6] and [15] that the spectral gap exists if there exists a sequence of regular convex domains $D_n \uparrow M$ and

$$-\mathrm{Hess}_V \text{ is uniformly positive definite out of a compact set} . \tag{1.2}$$

So, the spectral gap may exist even though k is negative everywhere, see Example 1.3 below.

Let P_t and λ_1 be, respectively, the semigroup and spectral gap of L with Neumann boundary if $\partial M \neq \emptyset$. The following result due to Korzeniowski and Stroock is a key of the study in this paper, which is a combination of Theorem 3 and Theorem 5 in [10].

Theorem 1.0 (Korzeniowski and Stroock). *Suppose that* $\lambda_1 > 0$. *If there exist* $t, N > 0$ *such that* $\|P_t\|_{2\to4} \le N$, *then*

$$\alpha(V) \ge \frac{1}{2}\left[t + \frac{1}{\lambda_1}\max\{\log 4, \log(2N)\}\right]^{-1} .$$

Remark. 1) According to the proof of [10; Theorem 3], the condition $\|P_t\|_{2\to4} \le N$ can be replaced by $\|P_t f\|_4 \le N\|f\|_2$ for all $f \in C_b(M)$ with $\mu(f) = 0$.

2) It has been proved by Aida [1] that $\lambda_1 > 0$ is a consequence of $\|P_t\|_{2\to q} < \infty$ for some $q > 2$. Hence the assumption in the above result can be removed.

By Theorem 1.0 together with an inequality of heat semigroup (cf. Lemma 2.1 below), we obtain the following result.

Theorem 1.1. *We have* $\alpha(V) > 0$ *provided*

$$\overline{\lim_{\rho(p,x)\to\infty}} \frac{V(x)}{\rho(p,x)^2} < 2 \inf_M k . \tag{1.3}$$

Moreover, as did in [5], Theorem 1.1 can be improved as follows.

Theorem 1.2. *If* cut $(p) = \emptyset$ *and the sectional curvature of* M *is nonpositive, then* $\alpha(V) > 0$ *provided*

$$\overline{\lim_{\rho(p,x)\to\infty}} \frac{V(x)}{\rho(p,x)^2} < 2 \underline{\lim_{\rho(p,x)\to\infty}} k(x) . \tag{1.4}$$

The following example shows that $\alpha(V)$ may be positive even though k is negative everywhere.

Example 1.3. Take $M = \mathbf{H}^d$ (the d-dimensional hyperbolic space), $V(x) = -\varepsilon\rho(p,x)^2$ for some $p \in M$ and $\varepsilon > 0$. Then $k(x) = 2\varepsilon + 1 - d, \mathrm{Hess}_V \leq -2\varepsilon$. We have $\lambda_1 > 0$ since (1.2) holds. By Theorem 1.1, $\alpha(V) > 0$ if $\varepsilon > \frac{2(d-1)}{5}$. But $\sup_M k < 0$ for $\varepsilon \in (\frac{2(d-1)}{5}, \frac{d-1}{2})$.

If M is compact, then $\alpha(V) > 0$ (see [14] and references therein for detailed estimates). It is well known that $\lambda_1 \geq \alpha(V)$. Now another problem arises: could we find a simple increasing positive function Λ such that $\alpha(V) \geq \Lambda(\lambda_1)$? Such functions have been presented in [8] and [11] which provide sharp estimates for nonnegative curvature case. But the functions are no longer positive if the lower bound of Ricci curvature is a little negative. Basing on [10; Theorem 5], we obtain the following result.

Theorem 1.4. *Suppose that M is compact. Let D be the diameter of M. If $k \geq -K$ for some $K \in \mathbf{R}$, then*

$$\alpha(V) \geq \frac{1}{2}\sup_{t>0}\left[t + \frac{1}{\lambda_1}\max\left\{\log 4, \log 2 - \frac{1}{2}\lambda_1 t + \frac{K^2 t D^2}{8(1-\mathrm{e}^{-Kt})^2}\right\}\right]^{-1}$$

$$\geq \frac{\lambda_1}{2}\min\left\{\frac{1}{\lambda_1 + \log 4}, \frac{4(1-\mathrm{e}^{-K})^2}{4(1-\mathrm{e}^{-K})^2\log 2 + K^2 D^2}\right\}.$$

Here and in what follows, when $K = 0$, the fraction means its limit as $K \to 0$.

Noting that λ_1 can be estimated from below dimension-freely (see e.g. [6]), by Theorem 1.4 one obtains dimension-free lower bounds for $\alpha(V)$. This disproves a view of Chung-Yau [7] which says that for the unit ball in $\mathbf{R}^d$, $\alpha(0)$ goes to zero as $d \to \infty$.

Finally, we consider diffusion processes on $\mathbf{R}^d$. Let

$$L = \sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i},$$

where $a(x) = (a_{ij}(x))$ is positive definite, $a_{ij} \in C^2(\mathbf{R}^d)$ and

$$b_i(x) = \sum_{j=1}^d a_{ij}(x)\frac{\partial}{\partial x_j}V(x) + \sum_{j=1}^d \frac{\partial}{\partial x_j}a_{ij}(x)$$

for some $V \in C^2(\mathbf{R}^d)$ with $Z := \int \mathrm{e}^V dx < \infty$. Then the L-diffusion process is reversible with respect to $\mathrm{d}\mu = Z^{-1}\mathrm{e}^V dx$ (see [5]). Denote by $\alpha(L)$ the LS constant for the present L, then the LSI becomes

$$\int_{\mathbf{R}^d} f^2 \log f^2 \, d\mu \leq \frac{2}{\alpha(L)}\int_{\mathbf{R}^d}\langle a\nabla f, \nabla f\rangle \, d\mu \tag{1.5}$$

for all $f \in C_b^1(\mathbf{R}^d)$ with $\mu(f^2) = 1$. See [5]. From Theorem 1.2 and (1.5), we obtain the following result.

Corollary 1.5. *Suppose that $a \geq \lambda I$ for some $\lambda > 0$. Let $L' = \Delta + \nabla V$, and let $k(x)$ be the minimal eigenvalue of $(-\frac{\partial^2}{\partial x_i \partial x_j} V(x))$. If*

$$\overline{\lim_{|x|\to\infty}} \frac{V(x)}{|x|^2} < 2 \varliminf_{|x|\to\infty} k(x) \ ,$$

then $\alpha(L) > 0$.

Proof. By (1.5) we have $\alpha(L) \geq \lambda\alpha(L')$, then the corollary follows from Theorem 1.2. □

2. Proofs

To prove Theorem 1.1 and Theorem 1.2, we need the following lemma which gives a dimension-free Harnack type inequality for heat semigroups.

Lemma 2.1. *Let $f \in C_b(M)$. For any $\alpha > 1, t \geq 0$ and positive $g \in C[0,t]$, we have*

$$|P_t f(x)|^\alpha \leq P_t|f|^\alpha(y) \exp\left[\frac{\alpha\rho(x,y)^2 \int_0^t g(s)^2\, ds}{4(\alpha-1)(\int_0^t g(s)e^{-Ks}\, ds)^2}\right], \quad x, y \in M \ .$$

Proof. We may assume that $f > 0$ since $|P_t f| \leq P_t|f|$. Next, since $P_t f$ is positive and smooth for $t > 0$, we have $|P_t f|^\alpha \in C^2(M)$. Given $x \neq y$ and $t > 0$, let $x_s : [0,t] \to M$ be the geodesic from x to y with length $\rho(x,y)$. Let $v_s = dx_s/ds$, we have $|v_s| = \rho(x,y)/t$. Set

$$h(s) = \frac{t}{\int_0^t g(s)e^{-Ks}\, ds} \int_0^s g(u)e^{-Ku}\, du, \quad s \in [0,t] \ .$$

Then $h(0) = 0, h(t) = t$. Let $y_s = x_{h(s)}$. Define

$$\phi(s) = \log P_s(P_{t-s}f)^\alpha(y_s), \quad s \in [0,t] \ .$$

Noting that (see [3] and [13])

$$|\nabla P_s F| \leq P_s|\nabla F|e^{Ks}, \quad s \geq 0, \ F \in C_b^1(M) \ ,$$

we have

$$\begin{aligned}
\frac{d\phi}{ds} &= \frac{1}{P_s(P_{t-s}f)^\alpha}\Big\{\alpha(\alpha-1)P_s(P_{t-s}f)^{\alpha-2}|\nabla P_{t-s}f|^2 + h'(s)\langle\nabla P_s(P_{t-s}f)^\alpha, v_s\rangle\Big\} \\
&\geq \frac{\alpha}{P_s(P_{t-s}f)^\alpha} P_s\Big\{(\alpha-1)(P_{t-s}f)^{\alpha-2}|\nabla P_{t-s}f|^2 \\
&\qquad - t^{-1}\rho e^{Ks}h'(s)(P_{t-s}f)^{\alpha-1}|\nabla P_{t-s}f|\Big\} \\
&= \frac{\alpha}{P_s(P_{t-s}f)^\alpha} P_s\big\{(P_{t-s}f)^\alpha\big((\alpha-1)x^2 - t^{-1}\rho h'(s)e^{Ks}x\big)\big\} \\
&\geq -\frac{\alpha\rho^2 g(s)^2}{4(\alpha-1)\big(\int_0^t g(s)e^{-Ks}\, ds\big)^2} \ ,
\end{aligned}$$

where $x = |\nabla P_{t-s}f|/P_{t-s}f$. By integrating over s from 0 to t, we complete the proof. □

We remark that when $|\nabla V|$ is bounded, the same type of estimate given in Lemma 2.1 can be deduced from the extended Li-Yau's Harnack inequality for heat semigroups, see [12]. But we are not in the case since according to (1.1), $|\nabla V|$ is necessarily unbounded in order for $\alpha(V) > 0$ when M is unbounded.

Proof of Theorem 1.1. a) Since $\alpha(V) \geq \inf_M k$, we need only to prove for the case that $\inf k \leq 0$. If (1.3) holds, then there exists $K > -\inf k$ (hence $K > 0$) and $\delta, m > 0$ such that

$$V(x) \leq m - 2(1+\delta)K\rho(p,x)^2 \ . \tag{2.1}$$

Choose $\varepsilon \in (0,1)$ such that

$$\frac{r(r-\varepsilon')}{(r-2\varepsilon')(r-\varepsilon'-1)} \leq 2+\delta, \quad r \in [2,4], \ \varepsilon' \in (0,\varepsilon] \ . \tag{2.2}$$

Next, choose $2 = r_0 < r_1 < \cdots < r_n = 4$ such that $\max_{1\leq i\leq n}(r_i - r_{i-1}) \leq \varepsilon$. Let

$$G(t) = \frac{1}{1-\mathrm{e}^{-2Kt}}, \quad t > 0 \ .$$

By taking $g(s) = \mathrm{e}^{-Ks}$ in Lemma 2.1, we obtain

$$\begin{aligned}
\int_M |P_t f|^{r_i}\, d\mu &= \int_M |P_t f|^{r_i/2}(|P_t f|^{r_{i-1}})^{r_i/(2r_{i-1})}\, d\mu \\
&\leq \int_{M\times M} \mu(dx)\mu(dy)|P_t f(x)|^{r_i/2}(P_t|f|^{r_{i-1}}(y))^{r_i/(2r_{i-1})}\exp\left[\frac{r_i K\rho(x,y)^2}{4(r_{i-1}-1)}G(t)\right] \\
&\leq \left[\int_{M\times M} |P_t f(x)|^{r_{i-1}} P_t|f|^{r_{i-1}}(y)\mu(dx)\mu(dy)\right]^{r_i/(2r_{i-1})} R_i(t)^{(2r_{i-1}-r_i)/(2r_i)} \\
&\leq \|f\|_{r_{i-1}}^{r_i} R_i(t)^{(2r_{i-1}-r_i)/(2r_i)} \ ,
\end{aligned}$$

where

$$R_i(t) = \int_{M\times M} \exp\left[\frac{r_i r_{i-1} K\rho(x,y)^2 G(t)}{2(2r_{i-1}-r_i)(r_{i-1}-1)}\right] \mu(dx)\,\mu(dy) \ .$$

b) Since Ricci curvature is bounded from below, there exists $c_1 > 0$ such that

$$\text{volume}\{x : \rho(p,x) \leq r\} \leq \mathrm{e}^{c_1 r}, \quad r \geq 0 \ .$$

Hence

$$\int_M \exp\left[-r\rho(p,x)^2\right] dx < \infty, \quad r > 0 \ . \tag{2.3}$$

By (2.2) we have

$$\frac{r_i r_{i-1}}{(2r_{i-1}-r_i)(r_{i-1}-1)} \leq 2+\delta \ .$$

Noting that $\rho(x,y)^2 \le 2\rho(p,x)^2 + 2\rho(p,y)^2$, by (2.1) we obtain

$$\begin{aligned} R_i(t) &\le Z^{-2}\int_{M\times M}\exp\left[\frac{1}{2}(2+\delta)KG(t)\rho(x,y)^2 + V(x) + V(y)\right]dx\,dy \\ &\le Z^{-2}\int_{M\times M}\exp\Big[2m + ((2+\delta)KG(t) \\ &\qquad - 2(1+\delta)K)\Big(\rho(p,x)^2 + \rho(p,y)^2\Big)\Big]\,dx\,dy\ . \end{aligned}$$

Note that $\lim_{t\to\infty} G(t) = 1$, then there exists $t_0 > 0$ such that

$$R_i(t_0) \le Z^{-2}\int_{M\times M}\exp\left[2m - \frac{1}{2}\delta K\Big(\rho(p,x)^2 + \rho(p,y)^2\Big)\right]dx\,dy\ .$$

By (2.3) we have $R_i(t_0) < \infty$. Hence, there exists $N_i > 0$ such that $\|P_{t_0}f\|_{r_i} \le N_i\|f\|_{r_{i-1}}$. Therefore

$$\|P_{nt_0}f\|_4 \le N_n\big\|P_{(n-1)t_0}f\big\|_{r_{n-1}} \le \cdots \le N_1N_2\cdots N_n\|f\|_2\ .$$

Now the proof is completed by Theorem 1.0. □

Just as pointed out by Aida [1], the proof of Theorem 1.1 implies the following result.

Theorem 2.1. *Assume that there exists* $K \ge 0$ *and* $\varepsilon > 0$ *such that* $\inf_M k \ge -K$ *and* $\mu(\exp[2(K+\varepsilon)\rho(p,x)^2]) := N < \infty$. *Then*

$$\|P_{t_\theta}\|_{2\to 2\theta} \le N^{\frac{2-\theta}{2\theta}}\ ,$$

where $t_\theta > 0$ *and* $\theta \in (1,2)$ *satisfy* $\frac{\theta K}{(2-\theta)(1-e^{-2Kt_\theta})} \le K + \varepsilon$.

Proof of Theorem 1.2. Let $k_0 = \underline{\lim}_{\rho(p,x)\to\infty} k(x)$. If (1.4) holds, then there exists $\delta > 0$ such that

$$\overline{\lim}_{\rho(p,x)\to\infty}\frac{V(x)}{\rho(p,x)^2} \le 2(k_0 - \delta)\ . \tag{2.4}$$

If $\mathrm{cut}(p) = \emptyset$ and the sectional curvature of M is nonpositive, then there exists $U \in C^2(M)$ such that $U = V$ out of a compact set but $\mathrm{Ric} - \mathrm{Hess}_U \ge k_0 - \delta/2$. See [15] or [4]. Let $\lambda_1(U)$ be the spectral gap of $\Delta + \nabla U$, then (see [8] and [15])

$$\lambda_1(U) \ge \exp[\inf(U - V) - \sup(U - V)]\lambda_1 > 0\ .$$

By Theorem 1.1 we have $\alpha(U) > 0$. Hence (see [8])

$$\alpha(V) \ge \exp[\inf(U - V) - \sup(U - V)]\alpha(U) > 0\ . \quad \square$$

Proof of Theorem 1.4. Since M is compact, by taking $g(s) \equiv 1$ in Lemma 2.1, we obtain

$$(P_tf(x))^2 \le P_tf^2(y)\exp\left[\frac{K^2tD^2}{2(1-e^{-Kt})^2}\right]\ .$$

Then

$$\int_M (P_t f)^4 \, d\mu \le \exp\left[\frac{K^2 t D^2}{2(1-e^{-Kt})^2}\right] \int_{M\times M} (P_t f(x))^2 P_t f^2(y) \, \mu(dx) \, \mu(dy)$$
$$= \exp\left[\frac{K^2 t D^2}{2(1-e^{-Kt})^2}\right] \|f\|_2^2 \|P_t f\|_2^2 \ .$$

If $\mu(f) = 0$, then

$$\|P_t f\|_2 \le e^{-\lambda_1 t} \|f\|_2 \ .$$

Hence

$$\|P_t f\|_4 \le \exp\left[-\frac{\lambda_1}{2} t + \frac{K^2 t D^2}{8(1-e^{-Kt})^2}\right] \|f\|_2, \quad \mu(f) = 0 \ .$$

Now the first inequality follows from Theorem 1.0 and its remark, and the second one follows from the choice $t = 1$ and the fact that

$$\max\{\log 2, N\} \le \max\{\log 2, 2N\} \ . \quad \square$$

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References

[1] Aida, S.: Uniform positivity improving property, Sobolev inequality and spectral gaps, preprint (1997)

[2] Bakry, D., Emery, M.: Hypercontractivité de semigroups de diffusion. C. R. Acad. Sci. Paris, Série I, **209**(15), 775–778 (1984)

[3] Bakry, D., Ledoux, M., Lévy-Gromov's isoperimetric inequality for infinite dimensional diffusion generator. Invent. Math. **123**, 259–281 (1996)

[4] Chen, M.F., Wang, F.Y.: Estimation of the first eigenvalue of second order elliptic operators, J. Funct. Anal. **131**, 345–363 (1995)

[5] Chen, M.F., Wang, F.Y.: Estimates of logarithmic Sobolev constant – an improvement of Bakry-Emery criterion. J. Funct. Anal., **144**(2), 287–300 (1997)

[6] Chen, M.F., Wang, F.Y.: General formula for lower bound of the first eigenvalue on Riemannian manifolds. Sci. Sin. (A), **40**(4), 384–394 (1997)

[7] Chung, F.R.K., Yau, S.T.: Logarithmic Harnack inequalities. Math. Research Letters, **3**, 793–812 (1996)

[8] Deuschel, J.D., Stroock, D.W.: Hypercontractivity and spectral gap of symmetric diffusions with applications to the stochastic Ising models. J. Funct. Anal. **92**(1), 30–48 (1990)

[9] Gross, L.: Logarithmic Sobolev inequalities and contractivity properties of semigroups. Lecture Notes in Maths. 1563, 54–82 (1993)

[10] Korzeniowski, A.: On logarithmic Sobolev constant for diffusion semigroups. J. Funct. Anal. **71**, 363–370 (1987)

[11] Rothaus, A.J.: Logarithmic Sobolev inequalities and the spectrum of Schrödinger operator. J. Funct. Anal. **42**, 110–120 (1981)

[12] Setti, A.G.: Gaussian estimates for the heat kernel of the weighted Laplacian and fractal measures. Canad. J. Maths. **44**, 1061–1078 (1992)

[13] Wang, F.Y.: Logarithmic Sobolev inequalities for diffusion processes with application to path spaces. Chinese. J. Appl. Probab. Stat. **12**(3), 255–264 (1996)

[14] Wang, F.Y.: On estimation of logarithmic Sobolev constant and gradient estimates of heat semigroups. Probab. Theory Relat. Fields, **108**, 87–101 (1997)

[15] Wang, F.Y.: Spectral gap of diffusion processes on noncompact manifolds. Chinese Sci. Bull. **40**(4), 1145–1149 (1995)