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One-sided local large deviation and renewal theorems in the case of infinite mean

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Summary. If $\{S_n, n \geqq 0\}$ is an integer-valued random walk such that S_n/a_n converges in distribution to a stable law of index $\alpha \in (0, 1)$ as $n \to \infty$, then Gnedenko's local limit theorem provides a useful estimate for $P\{S_n = r\}$ for values of r such that r/a_n is bounded. The main point of this paper is to show that, under certain circumstances, there is another estimate which is valid when $r/a_n \to +\infty$, in other words to establish a large deviation local limit theorem. We also give an asymptotic bound for $P\{S_n = r\}$ which is valid under weaker assumptions. This last result is then used in establishing some local versions of generalized renewal theorems.

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1 Introduction

This paper contains results about the asymptotic behaviour of a random walk $S = \{S_n,\ n \geqq 0\}$ which has $E\{|S_1|\} = \infty$ in two distinct, but related areas.

The large deviation result, in its simplest form, applies when S is in the domain of attraction of a stable law which has index $0 < \alpha < 1$ and positivity parameter $0 < \rho \leqq 1$ (so is not concentrated on the negative half-line). In this case a result of Tkachuk [10], which is quoted in Nagaev [8], states that uniformly for n such that $x/a_n \to +\infty$, where a_n is a norming sequence for S,

$$P\{S_n > x\} \sim nP\{X > x\} \text{ as } x \to +\infty\ . \tag{1.1}$$

(A proof of this result and some extensions of it can be found in Doney [4].)

Our concern is with a local version of (1.1), and although our methods can be adapted to the non-lattice case, we assume henceforth that S takes values on the integers. We write X for a typical step in S and denote its distribution and mass functions by F and p, respectively. Put $\overline{F}(x) = P\{X > x\} = 1 - F(x)$, and introduce the tail ratio τ defined by

$$\tau(r) = F(-r)/(\overline{F}(r)\} = \sum_{-\infty}^{-r} p_k \Big/ \left\{\sum_{r+1}^{\infty} p_k\right\} . \tag{1.2}$$

Then the mass function p is regularly varying at $+\infty$ and the above hypotheses hold if and only if for some slowly varying L

$$p_r \sim \alpha r^{-(\alpha+1)} L(r) \quad \text{as } r \to +\infty , \tag{1.3}$$

and

$$\lim_{r\to+\infty} \tau(r) = \rho^{-1} - 1, \quad 0 < \rho \leqq 1 . \tag{1.4}$$

Under these assumptions, the following local version of (1.1) holds.

Theorem A. *If* (1.3) *and* (1.4) *hold then, uniformly in n such that* $r/a_n \to +\infty$,

$$P\{S_n = r\} \sim nP\{X = r\} \quad \textit{as } r \to +\infty . \tag{1.5}$$

The simplest form of the renewal theorem relates to the situation where X takes non-negative integer values only, so that S is a discrete renewal process. The subject of study is the asymptotic behaviour as $r \to +\infty$ of the renewal mass function,

$$u_r = \sum_{n=0}^{\infty} P\{S_n = r\} . \tag{1.6}$$

Here it is known (see e.g. Theorem 8.7.3 of Bingham et al. [2]) that for $0 < \alpha < 1$

$$\overline{F}(n) = \sum_{n+1}^{\infty} p_k \sim n^{-\alpha} L(n) \quad \text{as } n \to \infty , \tag{1.7}$$

is equivalent to

$$\sum_{0}^{n} u_k \sim n^{\alpha}\Gamma(1-\alpha)/\{L(n)\Gamma(1+\alpha)\} \quad \text{as } n \to \infty . \tag{1.8}$$

Again it is a local result which we establish.

Theorem B. *If X takes non-negative integer values only*, (1.7) *holds and in addition*

$$\sup_{n\geqq 0} \{np_n/\overline{F}(n)\} < \infty , \tag{1.9}$$

then

$$\lim_{n\to\infty} n^{1-\alpha} L(n) u_n = \Gamma(1-\alpha)/\Gamma(\alpha) . \tag{1.10}$$

Theorems A and B are actually special cases of results which we state and prove in the following sections.

In the section on large deviations, Theorem 1 shows that (1.5) holds under weaker conditions on the left-hand tail than (1.4) (such as the boundedness of the tail ratio τ), and also that when τ is unbounded, (1.5) can still hold for a smaller range of values of n. Theorem 2 shows that if (1.3) is replaced by the weaker assumption (1.7), then provided τ is bounded above and (1.9) holds, there exists a uniform asymptotic upper bound for the ratio

$$(rP\{S_n = r\})/(n\overline{F}(r)).$$

This upper bound may have other applications, but here we show how it can be used in the renewal theory context. Our approach is to use Gnedenko's local limit theorem to estimate the terms in the expression (1.6) for u_r where a_n/r is bounded away from zero. (For other applications of this technique see Doney [3] and the proof of Theorem 8.6.6 in Bingham et al. [2].) The above estimate is then used to deal with the remaining terms.

It should be remarked that Theorem B has several famous antecedents, including Garsia and Lamperti [6], Williamson [11], and Erickson [5]. These authors have shown that (1.10) actually follows from (1.7) alone when $\frac{1}{2} < \alpha < 1$, but not when $0 < \alpha \leqq \frac{1}{2}$. Also (1.10) with lim replaced by lim inf or by lim* (i.e. convergence off a set of relative measure zero) does follow for $0 < \alpha < 1$, as does (1.10) for $\frac{1}{4} < \alpha < 1$ under (1.9) and for $0 < \alpha < 1$ under the stronger assumption that p_n is ultimately monotone. As well as establishing these last two results, Williamson [11] extended the analysis to the case that X takes negative as well as non-negative values. In this context u_r is also known as the Green's function of S, and we need to supplement (1.7) by the assumption (1.4). Williamson showed that all the above results remain valid, except that the constant appearing on the RHS of (1.10) depends on ρ as well as on α. Theorem 3 of Sect. 3 gives an analogous extension of Theorem B, but also applies to generalized Green's functions, which are of the form

$$g_r = \sum_{n=0}^{\infty} b_n P\{S_n = r\}, \tag{1.11}$$

where $\{b_n,\ n \geqq 0\}$ is some sequence of non-negative constants. This result extends Theorem 2 of Anderson and Athreya [1] although, there only the case of non-negative, non-lattice X is considered.

Finally, Theorem 4 contains the simple observation that, in the special case that S is the renewal process of ladder epochs in some random walk, then again (1.7) and (1.10) are equivalent; this is because, in this case, u is monotone.

2 Local limit theorems

Notice that our basic assumption (1.3) implies the tail estimate

$$\overline{F}(x) \sim x^{-\alpha}L(x) \quad x \to +\infty\ , \tag{2.1}$$

and, whenever (2.1) holds we can, and will, assume, with no loss of generality, that $A(x) := x^{\alpha}/L(x)$ is a continuous monotone increasing function with inverse a. If we now define a sequence $a_n = a(n)$, then under (1.3) and (1.4) it holds that S_n/a_n has a limiting stable distribution of index α. In particular, it follows that $P\{|S_n| > x\} \to 0$ as $n \to \infty$ whenever $x/a_n \to +\infty$. It is therefore clear that Theorem A is a special case of the following result, which also implies that the conclusion of Theorem A is still valid if (1.4) is replaced by the weaker assumption that τ is bounded.

Theorem 1. *Let S be any random walk which takes values on the integers and for which* (1.3) *holds. Suppose also there exists b_n such that*

$$\liminf_{n\to\infty}(b_n/a_n) > 0 \ \text{ and } \ P\{|S_n| > x\} \to 0 \quad \text{whenever } x/b_n \to +\infty\ ; \tag{2.2}$$

then (1.5) *holds uniformly in n such that*

$$r/b_n \to +\infty\ . \tag{2.3}$$

A corollary of this result shows how the uniformity in (1.5) has to be weakened if we know less about the left-hand tail of F:

Corollary 1. *Let S be any random walk which takes values on the integers and for which* (1.3) *holds. Suppose also that*

$$F(-x) = O(R(x)) \quad \text{as } x \to +\infty\ , \tag{2.4}$$

where R is a monotone decreasing function which is regularly varying of index $-\alpha^*$ at ∞, where $0 < \alpha^* < \alpha$. Let a_n^* satisfy $R(a_n^*) \sim 1/n$ as $n \to \infty$; then (1.5) *holds uniformly in n such that $r/a_n^* \to +\infty$.*

Our final result of this section is the only one where we drop the assumption (1.3), and of course our conclusion is also weaker. However this is the result which we use in the next section.

Theorem 2. *Let S be any random walk which takes values on the integers and for which* (2.1) *holds. Suppose also that*

$$F(-x) = O(\overline{F}(x)) \quad \text{as } x \to \infty\ , \tag{2.5}$$

and there are constants m_0 and B such that

$$mp(m) \leqq B\overline{F}(m) \quad \text{for all } m \geqq m_0\ . \tag{2.6}$$

Then, uniformly for n such that $r/a_n \to +\infty$

$$\limsup_{r\to\infty}\{rP\{S_n = r\}/n\overline{F}(r)\} < B/\alpha\ . \tag{2.7}$$

We start the proofs with a simple result, in which c denotes a positive constant whose exact value is immaterial and which can be different on each appearance. (This convention will be used throughout.)

Lemma 1. *Assume* (2.1); *then for any $\delta > 0$, all $y \geqq 1$ and all sufficiently large n*

$$cy^{\alpha-\delta} \leqq n^{-1}A(a_n y) \leqq cy^{\alpha+\delta}\ . \tag{2.8}$$

Proof. This is a consequence of the fact that A is regularly varying of index α and the Potter bounds; see e.g. Theorem 1.5.6 of Bingham et al. [2]. □

However, the key fact on which Theorem 1 depends is

Lemma 2. *Assume that* $F(0) = 0$ *and* (2.1) *holds. Then for all* $n \geqq 1, z$ *large enough and* $x \geqq z$

$$P\{S_n \geqq x,\ M_n \leqq z\} \leqq \{cz/x\}^{x/z}\ , \tag{2.9}$$

where $M_n = \max\{X_1, X_2, \ldots, X_n\}$ *and* $S_n = \sum_1^n X_i$.

Proof. This is an immediate consequence of Corollary 1.5 in Nagaev [7]. □

Proof of Theorem 1. It is convenient to rewrite our basic assumption (1.3), with $p(m) = P\{X = m\}$, as

$$p(m) \sim \alpha/(mA(m)) \quad \text{as } m \to +\infty\ . \tag{2.10}$$

We show first that, uniformly in n such that (2.3) holds,

$$\liminf\,[P\{S_n = r\}/np(r)] \geqq 1\ . \tag{2.11}$$

With $\varepsilon \in (0, \frac{1}{2})$ we note that

$$\begin{aligned} &P\{S_n = r\} \\ &\quad \geqq P\left\{\bigcup_{i=1}^{n} \bigcup_{|m| \leqq \varepsilon r} \{X_i = r - m,\ S_n - X_i = m, X_j \leqq \varepsilon r,\ j \neq i, j \leqq n\right\} \\ &\quad \geqq n \sum_{|m| \leqq \varepsilon r} P\{X_1 = r - m\} P\{M_{n-1} \leqq \varepsilon r,\ S_{n-1} = m\} \\ &\quad \geqq n \inf_{|m| \leqq \varepsilon r} p(r - m) \cdot P\{M_{n-1} \leqq \varepsilon r,\ |S_{n-1}| \leqq \varepsilon r\}\ . \end{aligned}$$

Since $\varepsilon x/a_n \to +\infty$ and $P\{M_{n-1} \leqq \varepsilon x\} \geqq 1 - n\overline{F}(\varepsilon x) \sim 1 - n/A(\varepsilon x)$, it follows from Lemma 1 that $P\{M_{n-1} \leqq \varepsilon x\} \to 1$. By assumption (2.2)

$$P\{|S_{n-1}| \leqq \varepsilon x\} \to 1,$$

so we deduce that the LHS of (2.11) is at least $(1 + \varepsilon)^{-\alpha}$; since ε is arbitrary, (2.11) follows.

To establish that, uniformly as $r/b_n \to +\infty$,

$$\limsup[P\{S_n = r\}/np(r)] \leqq 1\ , \tag{2.12}$$

we want to be able to assume that $X \geqq 0$ and $b_n \equiv a_n$. So suppose for the moment that we have established (2.12) in this case, and consider the general case with $p^+ = P(X \geqq 0) \in (0, 1)$.

Then $S_n = S_n^{(1)} - S_n^{(2)}$, where $S_n^{(1)} = \sum_1^n X_m^+$ and $S_n^{(2)} = \sum_1^n X_m^-$. Note that we can write $S_n^{(1)} \stackrel{d}{=} \sum_1^{N_n^+} Y_j$, where $N_n^+ = \#\{m \leqq n\colon X_m \geqq 0\}$ has a $B(n, p^+)$ distribution and the Y's are independent of N_n^+ with $P\{Y_j = m\} = p(m)/p^+$, $m = 0, 1, \ldots$. Then

$$P\{S_n = r\}/np(r) = E\left\{\frac{P\{\sum_1^{N_n^+} Y_j = r + S_n^{(2)}\}}{N_n^+ P\{Y_1 = r + S_n^{(2)}\}} \cdot \frac{p(r + S_n^{(2)})}{p(r)} \cdot \frac{N_n^+}{np^+}\right\}.$$

Using the feature that $\bar{p}(m) := \sup_{j \geqq m} p(j) \sim p(m)$ as $m \to +\infty$, it follows, from dominated convergence and our assumption that (2.12) holds with S_n replaced by $\sum_1^n Y_j$, that (2.12) holds in the general case, uniformly for n such that r/a_n (and hence $r/b_n) \to +\infty$. So, from now on we assume that $p(m) = 0$, $m < 0$ and $b_n \equiv a_n$. We will write, for $0 < \gamma < 1$ to be fixed later,

$$w = r/a_n, \qquad z = r^\gamma a_n^{1-\gamma} , \tag{2.13}$$

so that $w \to +\infty$ and

$$z = a_n y \quad \text{where } y = w^\gamma . \tag{2.14}$$

We use the following decomposition;

$$P\{S_n = r\} = \sum_{i=0}^{3} P\{(S_n = r) \cap A_i\} := \sum_{i=0}^{3} q_i ,$$

where, with $0 < \eta < 1$ and $N_n = \#\{m \leqq n \colon X_m > z\}$ we have $A_i = \{M_n \leqq \eta r,\ N_n = i\}$ for $i = 0, 1$, $A_2 = \{M_n \leqq \eta r,\ N_n \geqq 2\}$ and $A_3 = \{M_n > \eta r\}$. Thus $q_3 = P\{S_n = r$ and $M_n > \eta r\} \leqq n \sum_{\ell > \eta r} p(\ell) P\{S_{n-1} = r - \ell\} \leqq n\bar{p}(\eta r)$, so that

$$\lim_{\eta \uparrow 1} \limsup_{r \to \infty} \{q_3/np(r)\} \leqq 1 .$$

Similarly

$$\begin{aligned} q_2 &\leqq \frac{1}{2} n^2 P\{S_n = r,\ X_{n-1} > z,\ X_n > z\} \\ &= \frac{1}{2} n^2 \sum_m P\{X_1 > z,\ X_2 > z,\ X_1 + X_2 = m\} P\{S_{n-2} = r - m\} \\ &\leqq \frac{1}{2} n^2 \sup_m P\{X_1 > z,\ X_2 > z,\ X_1 + X_2 = m\} \\ &\leqq \frac{1}{2} n^2 \overline{F}(z) \bar{p}(z) \sim cn^2/z(A(z))^2 = c \frac{n}{rA(r)} \cdot \frac{r}{z} \cdot \frac{A(r)A(a_n)}{(A(z))^2} . \end{aligned}$$

Using Lemma 1, with $\gamma > (1 + \alpha)/(1 + 2\alpha)$, it follows that $q_2 = o(np(r))$. Also

$$\begin{aligned} q_1 &= n \sum_{z < \ell < \eta r} p(\ell) P\{M_{n-1} \leqq z,\ S_{n-1} = r - \ell\} \\ &\leqq n\bar{p}(z) P\{M_{n-1} \leqq z,\ S_{n-1} > (1 - \eta) r\} , \end{aligned}$$

and it follows from Lemma 2 that $q_1 = o(np(r))$. Thus the theorem will hold if we can show that $q_0 = o(np(r))$; or equivalently in view of the fact that $P\{M_n \leqq z\} \to 1$,

$$P\{\hat{S}_n = r\} = o(np(r)) \quad \text{as } r \to \infty , \tag{2.15}$$

uniformly in n such that $r/a_n \to \infty$, where $\hat{S}_n = \sum_1^n \hat{X}_i$, and

$$P\{\hat{X}_i = m\} = p(m) \Big/ \sum_0^z p(m) := \hat{p}(m), \quad 0 \leqq m \leqq z . \tag{2.16}$$

First we note that without loss of generality we can assume that as $r \to +\infty$

$$n \to \infty, \quad r/na_n \to 0\,. \tag{2.17}$$

The first holds because $\hat{S}_n \leqq nz$, so that the LHS of (2.15) is zero unless $n \geqq r/z$, and we know $r/z \to \infty$. Secondly, if $r \geqq cna_n$ it is easy to see that $np(r) \geqq c(r/z)^{-c}$ and it follows from Lemma 2 that $P\{\hat{S}_n \geqq r\} = o(np(r))$. Next, for $\lambda \geqq 0$ we introduce the probability measure $P^{(\lambda)}$ such that

$$P^{(\lambda)}(\hat{X}_1 = m) = e^{m\lambda}\,\hat{p}(m)/M(\lambda) \tag{2.18}$$

where $M(\lambda) = E(e^{\lambda\hat{X}_1}) = \sum e^{m\lambda}\,\hat{p}(m)$. For each fixed r and n, $\mu(\lambda) := E^{(\lambda)}(\hat{X}_1) = M'(\lambda)/M(\lambda)$ is a strictly increasing function on $[0,\infty)$ with $\mu(\infty) = z$ and $\mu(0) = \sum m\hat{p}(m)$. As $r \to \infty$, $\mu(0) \sim \sum_0^z mp(m) \sim \alpha z/(1-\alpha)A(z)$, so that if $\theta = r/n\mu(0)$ we have $\theta \sim crA(z)/zA(a_n) = cwA(a_n y)/yA(a_n)$. Using the bounds in (2.8) and writing $\beta = 1 + (\alpha - 1)\gamma$ we see that for any $\delta > 0$ and all sufficiently large r

$$cw^{\beta-\delta} \leqq \theta \leqq cw^{\beta+\delta}\,. \tag{2.19}$$

Thus $\theta \to \infty$, and since $r/n < z$ there exists a unique $\rho \in (0,\infty)$ with

$$\mu(\rho) = r/n\,, \tag{2.20}$$

and we will prove (2.15) by working with $P^{(\rho)}$. To do so, we need information about the behaviour of ρ as $r \to \infty$, and we claim that, for any $\delta > 0$ and sufficiently large r,

$$cw^{\beta-\delta} \leqq e^{\rho z} \leqq cw^{\beta+\delta}\,. \tag{2.21}$$

The lower bound follows from the observation that

$$\begin{aligned} e^{\rho z}\mu(0) &= e^{\rho z}\sum_0^z m\hat{p}(m) \geqq \sum_0^z m\hat{p}(m)e^{m\rho} \\ &= M'(\rho) = \frac{r}{n}M(\rho) \geqq \frac{r}{n}\,, \end{aligned}$$

so that $\theta \leqq e^{\rho z}$, and (2.19). For the upper bound, write $\rho^* = z^{-1}k\log w$, where $k > 0$ will be fixed later. Note that $a_n\rho^* = y^{-1}k\log w \to 0$, so that

$$M(\rho^*) \leqq \sum_0^{a_n-1} e^{m\rho^*}\,\hat{p}(m) + \sum_{a_n}^z e^{m\rho^*}\,\hat{p}(m) := \Sigma_1 + \Sigma_2\,,$$

where $\Sigma_1 \to 1$ as $r \to \infty$. Suppose first that $\Sigma_2 \geqq \Sigma_1$; then for all sufficiently large r,

$$\mu(\rho^*) \geqq \sum_{a_n}^z me^{m\rho^*}\,\hat{p}(m)/2\Sigma_2 \geqq \frac{1}{2}a_n \geqq \frac{r}{n} = \mu(\rho)\,, \tag{2.22}$$

where we have used (2.17).

But if $\Sigma_2 < \Sigma_1$ then, for sufficiently large r, $M(\rho^*) \leqq c$ and for any $\varepsilon \in (0,1)$

$$\mu(\rho^*) \geqq c\sum_{\varepsilon z}^z me^{m\rho^*}\,\hat{p}(m) \geqq ce^{\varepsilon z\rho^*}\sum_{\varepsilon z}^z m\hat{p}(m) \sim cw^{k\varepsilon}\mu(0)\,. \tag{2.23}$$

Using (2.17) and the previous estimate for $\mu(0)$, it follows that provided $k > \beta$ and ε is chosen suitably, we have $\mu(\rho^*) \geqq r/n = \mu(\rho)$ and hence in all cases $\rho^* \geqq \rho$ for all sufficiently large r. Since $P \geqq \rho$ is equivalent to $e^{\rho z} \leqq w^k$, this establishes the upper bound in (2.21).

Now for any $\lambda > 0$, $P^{(\lambda)}(\hat{S}_n = r) = \{M(\lambda)\}^{-n} e^{r\lambda} P\{\hat{S}_n = r\}$, so that

$$P\{\hat{S}_n = r\} = e^{-nf(\lambda)} P^{(\lambda)}(\hat{S}_n = r)\,, \tag{2.24}$$

where $f(\lambda) = n^{-1}r\lambda - \log M(\lambda)$. The next step is to find a good lower bound for $f(\rho) = \rho\mu(\rho) - \int_0^\rho \mu(\lambda)\,d\lambda = \int_0^\rho (\mu(\rho) - \mu(\lambda))\,d\lambda$, and for this we will use the inequality

$$f(\rho) \geqq \frac{1}{2}\eta^2\rho^2 \inf_{\tilde{\rho} \leqq \lambda \leqq \rho} \mu'(\lambda)\,, \tag{2.25}$$

where $0 < \eta < 1$ and $\tilde{\rho} = (1-\eta)\rho$. Note that on $[\tilde{\rho}, \rho]$ we have

$$\mu'(\lambda) = E^{(\lambda)}(\hat{X}_1^2) - (\mu(\lambda))^2 \geqq E^{(\lambda)}(\hat{X}_1^2) - (\mu(\rho))^2\,. \tag{2.26}$$

Furthermore, on this range we have

$$\begin{aligned} M(\lambda)E^{(\lambda)}(\hat{X}_1^2) &= \sum_0^z m^2 e^{m\lambda}\,\hat{p}(m) \geqq e^{-\eta z\rho} \sum_{(1/3)z}^{z} m^2 e^{m\rho}\,\hat{p}(m) \\ &\geqq \frac{1}{3} z e^{-\eta z\rho}\Big(M'(\rho) - \sum_{m<(1/3)z} m e^{m\rho}\,\hat{p}(m)\Big)\,. \end{aligned}$$

However

$$\begin{aligned} \sum_{m<(1/3)z} m e^{m\rho}\,\hat{p}(m) &\leqq e^{(1/3)z\rho} \sum_{m<(1/3)z} m\hat{p}(m) \sim c e^{(1/3)z\rho} \\ &\qquad \times \sum_{(2/3)z}^{z} m\hat{p}(m) \leqq c e^{-(1/3)z\rho} M'(\rho)\,, \end{aligned}$$

so we conclude that for all large enough r we have

$$E^{(\lambda)}(\hat{X}_1^2) \geqq cze^{-\eta\rho z}\mu(\rho) = cn^{-1}rze^{-\eta z\rho}\,, \tag{2.27}$$

for all $\lambda \in [\tilde{\rho}, \rho]$. Since $n(\mu(\rho))^2/rz = r/nz = (r/na_n)\cdot(a_n/z) = o(w^{-\gamma})$ we see from (2.21) that provided $\beta\eta < \gamma$ we have $(\mu(\rho))^2 = o(E^{(\lambda)}(\hat{X}_1^2))$. Hence, using (2.21), (2.26) and (2.27) in (2.25) we see that for all sufficiently large r

$$\begin{aligned} nf(\rho) \geqq c\rho^2 rze^{-\eta\rho z} &\geqq c(\log w)^2 rz^{-1} w^{-(\beta+\delta)\eta} \\ &= c(\log w)^2 w^{1-\gamma-\eta(\beta+\delta)} \geqq cw^{(1/2)(1-\gamma)}\,, \end{aligned}$$

by suitable choice of η. Thus, putting $\lambda = \rho$ in (2.24) and noting that $np(r) = A(a_n)/\{a_n w A(a_n w)\} \geqq (a_n w^{1+\alpha+\varepsilon})^{-1}$ for any $\varepsilon > 0$ and all sufficiently large r, it is clear that (2.15) will follow if we can show that

$$a_n P^{(\rho)}(\hat{S}_n = r) = 0(w^c) \quad \text{as } r \to \infty\,. \tag{2.28}$$

To achieve this, we need the following:

Lemma 3. *Let $W_1, W_2, \ldots, W_n$ be independent and identically distributed random variables taking values on the integer lattice $\mathscr{L}$ and having span* 1, *and write $Z_n = \sum_1^n W_i$. Suppose that $\mu = E(W_1)$, $\sigma^2 = \mathrm{Var}(W_1)$ and $v = E(|W_1 - \mu|^3)$ are finite, and write $L = v/(n\sigma^4)$. Then $\exists$ an absolute constant A such that*

$$\sup_{v:\, v+n\mu\in\mathscr{L}} |P\{Z_n = n\mu + v\} - (2\pi n\sigma^2)^{-1/2} \exp -\{v^2/2n\sigma^2\}| \leqq AL + d \,, \tag{2.29}$$

where, with $\phi(t) = E(e^{itW_1})$ and $\ell = (4Ln\sigma^2)^{-1} = \sigma^2/4v$,

$$d = 2\int_\ell^\pi e^{-n(1-|\phi(t)|)}\, dt \,. \tag{2.30}$$

Proof of Lemma 3. With $\Psi(t) = E(e^{it(Z_n - n\mu)/\sqrt{n}\sigma}) = \{\phi(t/\sigma\sqrt{n})\}^n e^{-it\sigma\sqrt{n}/\mu}$, it follows from the inversion theorem that the LHS of (2.29) is bounded above by $I_1 + I_2 + I_3$, where

$$\sigma\sqrt{n} I_1 = \int_{|t|<\sqrt{n}\sigma\ell} |\Psi(t) - e^{-(1/2)t^2}|\, dt,$$

$$\sigma\sqrt{n} I_2 = \int_{|t|\geqq\sqrt{n}\sigma\ell} e^{-(1/2)t^2}\, dt \text{ and } I_3 = 2\int_\ell^\pi |\phi(t)|^n\, dt \,.$$

It is clear that $I_2/L = 4\sigma\sqrt{n} I_2 \cdot (\sqrt{n}\sigma\ell)$ is bounded by an absolute constant and it follows from Lemma 1, p. 109 of Petrov [9] that the same is true of I_1/L. Finally the bound for I_3 follows from the simple estimate $|\phi(t)| \leqq e^{-(1-|\phi(t)|)}$. $\square$

Returning to the proof of the theorem, we will now apply the lemma, with W_1 having the $P^{(\rho)}$ distribution of $\hat{X}_1$, so that $\mu = r/n$, to establish (2.28). We use (2.29) with $v = 0$, and note that, from the argument leading to (2.27) and an obvious upper bound, it follows that $n\sigma^2/rz$ is bounded away from 0 and ∞ for all large enough r. A similar calculation yields the same conclusion for the quantity nv/rz^2, and it then follows that $a_n(\sigma\sqrt{n})^{-1}$ is $0(w^{-(1/2)(1+\gamma)})$ and $a_n L$ is $0(w^{-1})$, so (2.28) will follow if we can show that

$$a_n d = 0(w^c) \,. \tag{2.31}$$

Now $\phi(t) = E^{(\rho)}(e^{it\hat{X}_1}) = E(e^{(\rho+it)\hat{X}_1})/M(\rho)$, so that for $t > 0$

$$\begin{aligned} M(\rho)(1 - |\phi(t)|) &\geqq M(\rho)\,\mathrm{Re}(1 - \phi(t)) \geqq E\{e^{\rho\hat{X}_1}(1 - \cos t\hat{X}_1)\} \\ &\geqq ct^2 E\{\hat{X}_1^2;\ 0 \leqq \hat{X}_1 \leqq t^{-1}\} \geqq ct^2 \overline{F}(1/t^*)/(t^*)^2 \end{aligned}$$

where $t^* = \max(t, 1/z)$. Also the previous estimates show that $\ell z \geqq c$ for all large enough r, and hence for $t \in [\ell, \pi]$

$$nM(\rho)(1 - |\phi(t)|) \geqq cn/A(1/t) = cA(a_n)/A(a_n \cdot (ta_n)^{-1}) \geqq c(ta_n)^{\alpha_0} \,,$$

where $0 < \alpha_0 < \alpha$ and we have again used (2.8). Hence we have the asymptotic bound

$$
\begin{aligned}
a_n d &\leqq a_n \int_{cz^{-1}}^{\pi} \exp(-c(ta_n)^{\alpha_0}/M(\rho))\,dt \\
&\leqq \{M(\rho)\}^{1/\alpha_0} \int_0^{\infty} \exp(-cu^{\alpha_0})\,du \leqq c\,\exp\{\rho z/\alpha_0\}\,.
\end{aligned}
$$

Now (2.31) follows from (2.21), and the proof is finished. □

Proof of Theorem 2. This follows by repeating the argument that establishes (2.12), with minor modifications, to get (2.7). Firstly, the argument that shows we can take $F(0) = 0$ remains valid when we replace $p(m)$ for $m \geqq m_0$ by $m^{-1}B\overline{F}(m) = \tilde{p}(m)$ say. Next we see that

$$
\lim_{\eta\uparrow 1}\limsup_{r\to\infty}\{q_3/n\tilde{p}(r)\} \leqq B/\alpha\,,
$$

and that $q_2 = o(n\tilde{p}(r))$, just as in Theorem 1. Since the assumptions of Lemma 2 are still valid, the argument that gives $q_1 = o(n\tilde{p}(r))$ also goes through, so it remains only to show that $P\{\hat{S}_n = r\} = o(n\tilde{p}(r))$. However one can check that all the estimates that are used in the proof depend only on (2.1), so (2.7) is established. □

3 Renewal theorems

Our main result, from which Theorem B follows by taking $b_n \equiv 1$, is;.

Theorem 3. *Suppose that b is regularly varying at* ∞ *with exponent* $\beta > -2$. *Assume, in the renewal case, that* (1.7) *and* (1.9) *hold, and in the random walk case, that* (1.7), (1.4), (1.9) *hold and that* $\alpha(\beta+1) < 1$. *Then, if* $\gamma(\cdot) = b(A(\cdot))$,

$$
\lim_{n\to\infty} n^{1-\alpha}L(n)g_n/\gamma(n) = k(\alpha,\rho,\beta)\,. \tag{3.1}
$$

Here $0 < k(\alpha,\rho,\beta) = \alpha\int_0^{\infty} x^{-\alpha(\beta+1)}g_{\alpha,\rho}(x)\,dx < \infty$, *where* $g_{\alpha,\rho}$ *is the density of the limiting stable law.*

Our final result applies to the special case that S_n is the time at which the nth increasing ladder epoch occurs in a random walk $\{Z_n,\ n \geqq 0\}$. Thus $Z_0 \equiv 0$, $Z_n = \sum_{j=1}^n Y_j$ for $n \geqq 1$, where $Y_1, Y_2, \ldots$ are independent identically distributed random variables, and

$$
X_m = \min\{k \geqq 1\colon Z_{S_{m-1}+k} > Z_{S_{m-1}}\}, \quad m \geqq 1\,. \tag{3.2}
$$

Referring to this as the ladder case, recall that in this case (1.2) is equivalent to

$$
\frac{1}{n}\sum_1^n P\{Z_i > 0\} \to \alpha\,, \tag{3.3}
$$

which is Spitzer's condition. It is now known that (3.3) is equivalent to

$$P\{Z_n > 0\} \to \alpha\,, \tag{3.4}$$

(see [3]), and our observation gives the following result.

Theorem 4. *In the ladder case for each* $0 < \alpha < 1$ *the conditions* (1.7), (1.8), (1.10), (3.3) *and* (3.4) *are all equivalent.*

Proof of Theorem 3. Suppose first that $\alpha(\beta+1) < 1$ and hence that b_n/a_n is regularly varying of index $-\kappa$ where $\kappa = \alpha^{-1} - \beta > 1$. With $\delta > 0$ we write

$$\frac{rg_r}{A(r)\gamma(r)} = \sum_r^{(1)} + \sum_r^{(2)}\,,$$

where

$$\sum_r^{(2)} = \frac{r}{A(r)\gamma(r)} \cdot \sum_{n:\ a_n > \delta r} b_n P\{S_n = r\}\,.$$

Since $a_n > \delta r$ is equivalent to $n > A(\delta r)$ and

$$\sum_{n > A(\delta r)} b_n/a_n \sim A(\delta r)\gamma(\delta r)/\{(\kappa-1)\delta r\} \sim \delta^{\alpha(\beta+1)-1} A(r)\gamma(r)/\{(\kappa-1)r\}\,,$$

it follows from (1.8) that

$$\sum_r^{(2)} = \frac{r}{A(r)\gamma(r)} \sum_{A(\delta r)}^{\infty} b_n g_{\alpha,\rho}(r/a_n)/a_n + o(1) \quad \text{as } r \to \infty\,. \tag{3.5}$$

In this sum we write $r = a_n x_n^{(r)}$, so that

$$(x_n^{(r)} - x_{n+1}^{(r)})\frac{na_n}{r} = n(1 - a_n/a_{n+1}) \to \alpha^{-1}\,,$$

uniformly for $n > A(\delta r)$, as $r \to \infty$. We may therefore replace r/a_n in (3.5) by $\alpha n(x_n^{(r)} - x_{n+1}^{(r)})$ to see (since $n = A(r/x_n^{(r)})$) that the RHS of (3.5) is a Riemann sum approximating

$$\alpha \int_0^{1/\delta} \frac{A(r/x)\gamma(r/x)}{A(r)\gamma(r)}\, g_{\alpha,\rho}(x)\, dx\,.$$

Clearly the coefficient of $g_{\alpha,\rho}(x)$ converges pointwise to $x^{-\alpha(\beta+1)}$, and the Potter bounds show that it is dominated by $cx^{-\alpha(\beta+1)-\varepsilon}$ where $\varepsilon > 0$ can be chosen so that $\alpha(\beta+1)+\varepsilon < 1$. So dominated convergence applies to give

$$\lim_{r\to\infty} \sum_r^{(2)} = \alpha \int_0^{1/\delta} x^{-\alpha(\beta+1)} g_{\alpha,\rho}(x)\, dx\,,$$

and hence that

$$\lim_{\delta\downarrow 0} \lim_{r\to\infty} \sum_r^{(2)} = k(\alpha,\rho,\beta)\,.$$

On the other hand it is immediate from Theorem 2 that for sufficiently small δ and large r,

$$\sum_r^{(1)} \leqq c\left\{\frac{(\sum_r^\infty p_k)(\sum_{n\leqq A(\delta r)} nb_n)}{A(r)\gamma(r)}\right\} . \tag{3.6}$$

Since nb_n is regularly varying with index >-1, it is easily seen that the RHS of (3.6) converges to $c\delta^{(2+\beta)\alpha}$ as $r\to\infty$, so that

$$\lim_{\delta\downarrow 0}\limsup_{r\to\infty}\sum_r^{(1)} = 0 ,$$

and the result follows.

In the remaining case $X\geqq 0$ and $\beta\geqq \alpha^{-1}-1$; here b_n/a_n is regularly varying of index $\geqq -1$, and the estimate of the error term in $\sum_r^{(2)}$ breaks down. Note however we still have

$$\lim_{\delta\downarrow 0}\limsup_{r\to\infty}\sum_r^{(1)} = 0 ,$$

as this depends only on $\beta > -2$. It is also easy to see that, for each fixed $0<\delta<\Delta<\infty$

$$\lim_{r\to\infty}\frac{r}{A(r)\gamma(r)}\sum_{A(\delta r)}^{A(\Delta r)} b_n P\{S_n = r\} = \alpha\int_{\Delta^{-1}}^{\delta^{-1}} x^{-\alpha(\beta+1)} g_{\alpha,\rho}(x)\,dx , \tag{3.7}$$

so it remains only to show that

$$\lim_{\Delta\uparrow\infty}\limsup_{r\to\infty}\sum_r^{(3)} = 0 ,$$

where

$$\sum_r^{(3)} = \frac{r}{A(r)\gamma(r)}\sum_{n>A(\Delta r)} b_n P\{S_n = r\} .$$

We now write $\phi(\theta) = E(e^{-\theta X_1})$, and for $\theta\geqq 0$ define probability measures $P^{(\theta)}$ by

$$P^{(\theta)}(X=m) = e^{-m\theta}p_m/\phi(\theta),\quad m\geqq 0 .$$

Note that $\mu(\theta) := E^{(\theta)}(X) = -\phi'(\theta)/\phi(\theta)$ is a non-negative, continuous, strictly decreasing function with $\mu(0+)=\infty,\ \mu(\infty)=0$ so $\exists$ a unique $\rho=\mu^{-1}(r/n)$. Furthermore we can write

$$P\{S_n = r\} = e^{-nh(\rho)}P^{(\rho)}\{S_n = r\} \tag{3.8}$$

where

$$h(\rho) = -\log\phi(\rho) - n^{-1}r\rho = -\log\phi(\rho) - \rho\mu(\rho) . \tag{3.9}$$

It is easily seen that h is a positive, monotone increasing function of ρ on $(0,\infty)$. We then have

Lemma 4. *Suppose that $X\geqq 0$ and* (1.7) *holds. Then* (i) *for all r and n such that $r\leqq nC_1$ we have*

$$P\{S_n = r\}\leqq e^{-nC_2}\quad\text{where } C_2 = h(\mu^{-1}(C_1))\ : \tag{3.10}$$

(ii) *uniformly as* $r/n \to \infty$ *and* $r/a_n \to 0$, *we have* $P\{S_n = r\} \sim q(n,r)$ *and* $h(\rho) \sim r\rho(1-\alpha)$, *where*

$$q(n,r) = \left\{ \frac{\rho}{2\pi r(1-\alpha)} \right\}^{1/2} e^{-nh(\rho)} . \tag{3.11}$$

Proof of Lemma 4. (i) This follows from (3.8) and the monotonicity of h.
(ii) By a tauberian argument, (see e.g. Theorem 1.7.6 in [2]) (1.7) implies that as $\theta \downarrow 0$

$$1 - \phi(\theta) \sim \Gamma(1-\alpha)/A(\theta^{-1}) , \tag{3.12}$$

and

$$\theta^3 v(\theta)/(1-\alpha)(2-\alpha) \sim \theta^2\sigma^2(\theta)/(1-\alpha) \sim \theta\mu(\theta) \sim \alpha(1-\phi(\theta)) , \tag{3.13}$$

where $\sigma^2(\theta) = \mathrm{Var}^{(\theta)}(X_1)$, $v(\theta) = E^{(\theta)}(|X_1 - \mu(\theta)|^3)$. Thus, in particular, $\mu(\theta)$ is regularly varying of index $\alpha - 1$ at zero, so that μ^{-1} is monotone decreasing and regularly varying of index $-(1-\alpha)^{-1}$ at ∞, and $\rho = \mu^{-1}(r/n) \to 0$ as $r/n \to \infty$. We now apply Lemma 3 to see that $n^{1/2}\sigma(\rho)P^{(\rho)}(S_n = r) - 1/\sqrt{2\pi}$ is bounded in absolute value by $A\Lambda + \delta$, where

$$\Lambda = v(\rho)/n^{1/2}\sigma^3(\rho), \quad \ell = \sigma^2(\rho)/4v(\rho), \quad \delta = 2n^{1/2}\sigma(\rho)\int_\ell^\pi e^{-n(1-|\psi(t)|)}\,dt ,$$

and $\psi(t) = E^{(\rho)}(e^{itX_1})$. From (3.13) it is immediate that $\Lambda \sim (2-\alpha)((1-\alpha)r\rho)^{-1/2}$, and it is easy to deduce from $a_n/r \to \infty$ that $r\rho \to \infty$. Thus we need only show that $\delta \to 0$. To this end let $Y \stackrel{d}{=} X_1 - X_2$ (where X_1 and X_2 are independent copies of X) be a symmetrized version of X, and note that $E^{(\rho)}(e^{itY}) = |\psi(t)|^2$. Writing $\tilde{p}_m = P^{(\rho)}(Y = m)$ it follows that

$$\begin{aligned} 2(1 - |\psi(t)|) \;&\geqq\; 1 - |\psi(t)|^2 = 2\sum_1^\infty \tilde{p}_m(1 - \cos mt) \\ &\geqq\; 4\sum_1^\infty \tilde{p}_m \sin^2\left(\frac{1}{2}mt\right) \;\geqq\; t^2 V(t^{-1}) \end{aligned}$$

where $V(x) = \sum_1^{[x]} m^2 \tilde{p}_m$. Now, choosing a fixed k and $p_k > 0$ we see that for all $x \geqq 4/\rho$ and all sufficiently small ρ

$$V(x) \;\geqq\; (\phi(\rho))^{-2}\sum_1^{[x]} m^2 e^{-m\rho}\sum_0^\infty e^{-2\rho j} p_j\, p_{m+j} \;\geqq\; c\sum_1^{[x]} m^2 p_{m+k} \;\geqq\; cx^2/A(x) .$$

Since (3.13) gives $\ell \sim 4(2-\alpha)^{-1}\rho \geqq 4\rho$ we see that for $t \in (\ell, \pi]$ and all sufficiently small ρ we have

$$n(1 - |\psi(t)|) \;\geqq\; cn/A(t^{-1}) \;\geqq\; c(ta_n)^{\alpha_0} ,$$

where $0 < \alpha_0 < \alpha$ and we have used Lemma 1. It follows that

$$\delta \;\leqq\; 2\sqrt{n}\sigma(\rho)\int_\ell^\pi e^{-c(ta_n)^{\alpha_0}}\,dt \;\leqq\; \frac{2\sqrt{n}\sigma(\rho)}{a_n}\int_0^\infty e^{-ct_0^\alpha}\,dt.$$

Since

$$\sqrt{n}\sigma(\rho)/a_n \sim \left(\frac{(1-\alpha)r}{a_n}\right)^{1/2} \cdot (a_n\rho)^{-1/2} \to 0 ,$$

this is clearly $o(1)$, and we have shown that $P\{S_n = r\} \sim q(n,r)$. That $h(\rho) \sim \rho r(1-\alpha)$ follows by noting that $-\log\phi(\rho) \sim 1 - \phi(\rho)$ and using (3.12) and (3.13) in (3.9). □

Thus in all cases we have, for $r \geqq nc$ and r and a_n/r large enough

$$P\{S_n = r\} \leqq c(\rho/r)^{1/2} e^{-cr\rho} . \tag{3.14}$$

Using (3.10) it is easy to see that

$$\frac{r}{A(r)\gamma(r)} \sum_{n>r/c} b_n P\{S_n = r\} \to 0 . \tag{3.15}$$

Finally, using (3.14) we have

$$\frac{r}{A(r)\gamma(r)} \sum_{A(\Delta r)}^{r/c} b_n P\{S_n = r\} \leqq \frac{c}{A(r)\gamma(r)} \sum_{A(\Delta r)}^{\infty} b(n) e^{-cr\rho} (r\rho)^{1/2} . \tag{3.16}$$

Now $\sqrt{x}e^{-x} \leqq ce^{-(1/2)x}$ and $\rho r = \mu^{-1}(r/n)/\mu^{-1}(\mu(1/r))$ so that the Potter bounds give $\rho r \geqq c\{n\mu(1/r)/r\}^{\omega}$, for any $0 < \omega < (1-\alpha)^{-1}$, on $n \geqq A(\Delta r)$ for Δ and r sufficiently large. Since $r/\mu(1/r) \sim A(r)\Gamma(\alpha)$, we see that the RHS of (3.16) is asymptotically bounded by

$$\frac{c}{A(r)\gamma(r)} \sum_{A(\Delta r)}^{\infty} b(n) \exp - \{c(n/A(r))^{\omega}\} .$$

Writing $n = A(r)x_n^{(r)}$, this in turn is asymptotic to

$$\frac{c}{\gamma(r)} \int_{\Delta^{\alpha}}^{\infty} \gamma(rx) e^{-cx^{\omega}}\, dx$$

and since $\gamma(rx)/\gamma(r)$ is bounded by x^c, this converges to $c\int_{\Delta^{\alpha}}^{\infty} x^{-\alpha\beta} e^{-cx^{\omega}}\, dx$. From this and (3.15) we deduce that

$$\lim_{\Delta\to\infty} \limsup_{r\to\infty} \sum_{r}^{(3)} = 0 ,$$

as required. □

Proof of Theorem 4. We need only show that u_n is monotone, since then the equivalence of (1.8) and (1.10) is a consequence of the monotone density argument, the equivalence of (1.7) and (3.3) is well-known (see e.g. Theorem 8.9.12 of [2]), and the equivalence of (3.3) and (3.4) is in [3]. But, by duality,

$$\begin{aligned} u_n &= P\{S_n > 0,\ S_n > S_1,\ \ldots,\ S_n > S_{n-1}\} \\ &= P\{S_1 > 0,\ S_2 > 0,\ \ldots,\ S_n > 0\} , \end{aligned}$$

which is clearly decreasing in n. □

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