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Diffusing particles with electrostatic repulsion

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Summary. We study a diffusion model of an interacting particles system with general drift and diffusion coefficients, and electrostatic inter-particles repulsion. More precisely, the finite particle system is shown to be well defined thanks to recent results on multivalued stochastic differential equations (see [2]), and then we consider the behaviour of this system when the number of particles N goes to infinity (through the empirical measure process). In the particular case of affine drift and constant diffusion coefficient, we prove that a limiting measure-valued process exists and is the unique solution of a deterministic PDE. Our treatment of the convergence problem (as $N \uparrow \infty$) is partly similar to that of T. Chan [3] and L.C.G. Rogers - Z. Shi [5], except we consider here a more general case allowing collisions between particles, which leads to a second-order limiting PDE.

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1 Introduction

The aim of this paper is to study the behaviour of an interacting real N-particles system with general drift and diffusion coefficients, *electrostatic inter-particles repulsion*, when the number of particles N tends to infinity. More precisely, we are interested in systems of particles governed by

$$dX_t^{(i)} = b_N(X_t^{(i)})dt + \sigma_N(X_t^{(i)})dW_t^{(i)} + \gamma_N \sum_{1 \leqslant j \neq i \leqslant N} \frac{dt}{X_t^{(i)} - X_t^{(j)}}\,,\ i = 1, 2, \ldots, N\,. \tag{1.1}$$

The behaviour of this system when $N \uparrow \infty$ will be considered through the sequence of empirical measure processes

$$\mu_t^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{(i)}}\,, \tag{1.2}$$

(where δ_x is the Dirac probability at $x \in \mathbb{R}$) by studying the weak convergence of this sequence and identifying the limit.

Systems of interacting diffusing particles governed by

$$\begin{aligned} dX_t^{(i)} &= b_N(X_t^{(i)})dt + \sigma_N(X_t^{(i)})dW_t^{(i)} \\ &\quad - D_i\Psi_N(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(N)})\, dt\,, \; i = 1, 2, \ldots, N\,, \end{aligned}$$

where Ψ_N is an interaction potentiel of the form

$$\Psi_N(x^{(1)}, x^{(2)}, \ldots, x^{(N)}) = \gamma_N \sum_{i \neq j} V(x^{(i)} - x^{(j)})\,,$$

have been studied by many authors: the reader will find in A.S. Sznitman [6] a treatment of such equations and references. According to (1.1), we are particularly interested in the case V has a logarithmic singularity at 0 so that usual results and tools are no more available. This class of SDE's has already been studied by T. Chan [3] and L.C.G. Rogers - Z. Shi [5] as well: our treatment of the convergence problem as $N \uparrow \infty$ is partly similar, but our hypotheses are more general. Indeed, before considering the asymptotic behaviour, we have to answer the question of existence of solutions to finite-dimensional systems. As a consequence of the explosive form of the drift term, we can't use standard results about existence - uniqueness for SDE's if the particles happen to collide. Unfortunately, to prevent collisions,

1. the coefficients b_N and σ_N must be simple enough to make possible the study of the first collision time

$$\tau = \inf\{t > 0 : X_t^{(i)} = X_t^{(j)} \quad \text{for} \quad 1 \leqslant \mathrm{i} \neq \mathrm{j} \leqslant \mathrm{N}\}. \tag{1.3}$$

2. some restrictions have to be imposed to the coefficients in order that $\tau = \infty$ a.s..

To overcome the difficulty due to possible collisions, we have turned to recent results about multivalued stochastic differential equations (see [2] and Sect. 2). These results allow one to construct some diffusions with possibly reflecting boundary conditions and exploding discontinuous drift: we use those results in a special case and then we have to verify this diffusion is solution of (1.1), which is done by using the special features of the problem (particularly the logarithmic potential V). Then, the tightness of the sequence $(\mu^{(N)})_N$ is easily obtained and the limit of any convergent subsequence must verify some equation. To prove the weak convergence of the global sequence, we need the uniqueness of the solution to this limiting equation, and we are in a position to conclude in a particular case. In this situation, we prove the existence of a limiting measure-valued process which is the solution to a deterministic PDE. In constrast with the previous works ([3, 5]), collisions between particles are possible and the limiting PDE is second-order. The reader can find in [3] and [5] some applications of (1.1) to physics and to the study of the eigenvalues of randomly-diffusing matrices. See also the references therein.

2 Prerequisite on multivalued stochastic differential equations

For the reader's convenience, we recall basic facts on maximal monotone operators and multivalued SDE's: the reader can find in [1] more details and references.

2.1 Maximal monotone operators

Definition 2.1. *A set-valued operator on $\mathbb{R}^d$ is a mapping A from $\mathbb{R}^d$ to $\mathscr{P}(\mathbb{R}^d)$, where $\mathscr{P}(\mathbb{R}^d)$ is the set of all subsets of $\mathbb{R}^d$; its domain is*

$$D(A) = \{x \in \mathbb{R}^d \, : \, A(x) \neq \emptyset\}. \tag{2.4}$$

A set-valued operator A is characterized by its graph:

$$\mathrm{Gr}\,(\mathrm{A}) = \{(\mathrm{x},\mathrm{y}) \in \mathbb{R}^{2\mathrm{d}} \, : \, \mathrm{x} \in \mathbb{R}^{\mathrm{d}},\, \mathrm{y} \in \mathrm{A}(\mathrm{x})\}. \tag{2.5}$$

Definition 2.2. *A set-valued operator A on $\mathbb{R}^d$ is said to be monotone if*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0\,, \qquad \forall (x_1,y_1),(x_2,y_2) \in \mathrm{Gr}\,(\mathrm{A}), \tag{2.6}$$

and maximal monotone if

$$(x,y) \in \mathrm{Gr}\,(\mathrm{A}) \quad \Leftrightarrow \quad \Big\{ \langle \mathrm{y} - \mathrm{v}, \mathrm{x} - \mathrm{u} \rangle \geqslant 0\,, \quad \forall (\mathrm{u},\mathrm{v}) \in \mathrm{Gr}\,(\mathrm{A}) \Big\}. \tag{2.7}$$

The following proposition gives the fundamental example of a maximal monotone operator:

Proposition 2.3. *Let $\varphi : \mathbb{R}^d \to] - \infty\,; +\infty]$ be a lower semi-continuous convex function on $\mathbb{R}^d$ such that its domain* $\mathrm{dom}\,(\varphi) = \{\mathrm{x} \in \mathbb{R}^{\mathrm{d}} \, : \, \varphi(\mathrm{x}) < +\infty\}$ *is not empty (we say that φ is proper is this case and strictly proper if Int (dom(φ)) $\neq \emptyset$).*

The subdifferential of φ, denoted by $\partial\varphi$, is the maximal monotone operator on $\mathbb{R}^d$ defined by

$$(x,y) \in \mathrm{Gr}\,(\partial\varphi) \Leftrightarrow \varphi(\mathrm{x}) \leqslant \varphi(\mathrm{z}) + \langle \mathrm{y}, \mathrm{x} - \mathrm{z} \rangle, \qquad \forall \mathrm{z} \in \mathbb{R}^{\mathrm{d}}. \tag{2.8}$$

Proposition 2.4. *There exists a sequence $(A_n)_n$ of operators (Yosida approximation) satisfying:*

(i) *A_n is a simple-valued Lipschitz operator defined on $\mathbb{R}^d$*

(ii) *for all $x \in D(A)$ such that $A(x)$ has exactly one element (which we note $A(x)$ as well),*

$$A_n(x) \xrightarrow[n\to\infty]{} A(x)\,, \tag{2.9}$$

with $|A_n(x)| \uparrow |A(x)|$ when $n \uparrow \infty$.

(iii) *for all $x \notin D(A)$,*

$$|A_n(x)| \uparrow +\infty \quad \textit{when} \quad n \uparrow \infty\,; \tag{2.10}$$

2.2 Multivalued stochastic differential equations

The following result shows that for the formal multivalued SDE

$$\begin{cases} dX_t + A(X_t)dt \ni b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0 \in \overline{D(A)}, \end{cases} \tag{2.11}$$

existence and uniqueness hold as soon as A is maximal monotone ; the interested reader will find in [2] a proof of this result (and in [1] another proof and further developments).

Theorem 2.5. *For every $d \in \mathbb{N}^*$, A maximal monotone operator on $\mathbb{R}^d$ such that $Int(D(A)) \neq \emptyset$, Lipschitz applications $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $x_0 \in \overline{D(A)}$, there is exactly one strong (continuous) solution (X, K) of:*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t - dK_t \, ; \quad 0 \leqslant t < \infty$$

with K finite variation process, $K_0 = 0$, $X_t \in \overline{D(A)}$ for any $0 \leqslant t < \infty$, $X_0 = x_0$, and for every continuous process (α, β) such that

$$(\alpha_u, \beta_u) \in Gr(A), \qquad \forall u \in [0; +\infty[, \tag{2.12}$$

the measure

$$\langle X_u - \alpha_u, \, dK_u - \beta_u du \rangle \tag{2.13}$$

is (a.s.) nonnegative on $\mathbb{R}^+$.

3 Existence and uniqueness for the finite particle system

We use the theory of multivalued stochastic differential equations (see [1, 2]) in order to show that the finite particle system is well defined as stated in the next theorem.

Theorem 3.1. *For every $N \in \mathbb{N}^* \setminus \{1\}$, $\gamma > 0$, $-\infty < x_0^{(1)} \leqslant x_0^{(2)} \leqslant \cdots \leqslant x_0^{(N)} < \infty$ and $b : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$ Lipschitz, there is a unique $X = (X^{(1)}, X^{(2)}, \dots, X^{(N)})$ which is the strong solution of the following stochastic differential system:*

$$\begin{cases} dX_t^{(1)} = b(X_t^{(1)})dt + \sigma(X_t^{(1)})dW_t^{(1)} + \gamma \displaystyle\sum_{1 \leqslant j \neq 1 \leqslant N} \frac{dt}{X_t^{(1)} - X_t^{(j)}} \\ \dots = \dots \\ dX_t^{(i)} = b(X_t^{(i)})dt + \sigma(X_t^{(i)})dW_t^{(i)} + \gamma \displaystyle\sum_{1 \leqslant j \neq i \leqslant N} \frac{dt}{X_t^{(i)} - X_t^{(j)}} \\ \dots = \dots \\ dX_t^{(N)} = b(X_t^{(N)})dt + \sigma(X_t^{(N)})dW_t^{(N)} + \gamma \displaystyle\sum_{1 \leqslant j \neq N \leqslant N} \frac{dt}{X_t^{(N)} - X_t^{(j)}}, \end{cases} \tag{3.14}$$

under the conditions:

$$X_0 = (x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(N)}) \tag{3.15}$$

$$X_t^{(1)} \leqslant X_t^{(2)} \leqslant \cdots \leqslant X_t^{(N)} , \ 0 \leqslant t < \infty, \ \mathbb{P} - \text{a.s.} . \tag{3.16}$$

Proof: (1) *Existence part*
For all $x \in \mathbb{R}^N$, we write $x = (x^{(1)}, x^{(2)}, \ldots, x^{(N)})$. We consider the strictly proper lower semi-continuous convex function on $\mathbb{R}^N$ defined by:

$$\varphi(x) = \begin{cases} -\gamma \sum_{1 \leqslant i < j \leqslant N} \ln(x^{(j)} - x^{(i)}) & \text{if } x^{(1)} < x^{(2)} < \cdots < x^{(N)} \\ +\infty & \text{if not,} \end{cases} \tag{3.17}$$

where $\gamma > 0$, and the coefficients $\tilde{b} : \mathbb{R}^N \to \mathbb{R}^N$, $\tilde{\sigma} : \mathbb{R}^N \to \mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$(\tilde{b}(x))^{(i)} = b(x^{(i)}), \qquad \forall x \in \mathbb{R}^N, \ 1 \leqslant i \leqslant N, \tag{3.18}$$

and

$$\tilde{\sigma}_{ij}(x) = \delta_{ij} . \sigma(x^{(i)}), \qquad \forall x \in \mathbb{R}^N, \ 1 \leqslant i, j \leqslant N. \tag{3.19}$$

The domain of φ is the following open subset of $\mathbb{R}^N$

$$D = \{x \in \mathbb{R}^N : x^{(1)} < x^{(2)} < \cdots < x^{(N)}\}, \tag{3.20}$$

and the subdifferential $A = \partial\varphi$ of the convex function φ is a simple-valued maximal monotone operator (since φ is regular on D, we have $\partial\varphi(x) = \{\nabla\varphi(x)\}$, for all $x \in D$) on $\mathbb{R}^N$ such that

$$D(A) = \text{dom}(\varphi) = D. \tag{3.21}$$

The following proposition gives the exact form of the multivalued stochastic differential equation associated with the previous data: in particular, it shows that there is no boundary term in this case (multidimensional version of local time).

Proposition 3.2. *For the data* N, A, $\tilde{b}$, $\tilde{\sigma}$, $(x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(N)})$ *described above, the corresponding multivalued stochastic differential equation*

$$dX_t + A(X_t)dt \ni \tilde{b}(X_t)dt + \tilde{\sigma}(X_t)dW_t , \tag{3.22}$$

can be written in terms of coordinates as:

$$dX_t^{(i)} = b(X_t^{(i)})dt + \sigma(X_t^{(i)})dW_t^{(i)} + \gamma \sum_{1 \leqslant j \neq i \leqslant N} \frac{dt}{X_t^{(i)} - X_t^{(j)}} , \ i = 1, 2, \ldots, N . \tag{3.23}$$

Thanks to Theorem 2.5, (3.22) has a unique strong solution: the problem consists in identifying the process K. The proposition will be proven through several lemma.

Lemma 3.3. *For all* $0 < T < \infty$, *we have*

$$\mathbb{E}\left(\int_0^T |A(X_u)|du \right) < \infty . \tag{3.24}$$

As a consequence,

(i) $u \to A(X_u)$ *is* $\mathbb{P} -$ a.s. *locally integrable on* $\mathbb{R}^+$, *that is to say*

$$\int_0^T |A(X_u)|du < \infty, \qquad \forall 0 < T < \infty,\ \mathbb{P} - \text{a.s.}; \tag{3.25}$$

(ii) *the closed set* $\{0 \leqslant u < \infty : X_u \notin D\}$ *has Lebesgue measure zero* $\mathbb{P} -$ a.s..

Proof. Let $X^{[n]}$ be the unique strong solution of

$$dX_t^{[n]} + A_n(X_t^{[n]})dt = b(X_t^{[n]})dt + \sigma(X_t^{[n]})dW_t . \tag{3.26}$$

It is shown in ([2]) that there exists $C < \infty$ such that:

$$\mathbb{E}\left(\int_0^T |A_n(X_u^{[n]})|du \right) \leqslant C, \qquad \forall n \in \mathbb{N}^*. \tag{3.27}$$

From the monotonicity of $(|A_n(x)|)_n$, it follows that

$$\mathbb{E}\left(\int_0^T |A_n(X_u^{[p]})|du \right) \leqslant C, \qquad \forall n, p \in \mathbb{N}^*,\ n < p. \tag{3.28}$$

Using the convergence in law of $X^{[n]}$ to X (see [2]) and the continuity of the mapping $x \to \int_0^T |A_n(x(u))|du$ (since A_n is Lipschitz: see Proposition 2.4), we have for any $R < \infty$

$$\mathbb{E}\left(R \wedge \int_0^T |A_n(X_u)|du \right) \leqslant C, \qquad \forall n \in \mathbb{N}^*; \tag{3.29}$$

then Fatou's lemma shows that

$$\mathbb{E}\left(\int_0^T |A_n(X_u)|du \right) \leqslant C, \qquad \forall n \in \mathbb{N}^*; \tag{3.30}$$

and a second application of Fatou's lemma together with Proposition 2.4 proves that

$$\mathbb{E}\left(\int_0^T |A(X_u)|du \right) \leqslant C < \infty . \tag{3.31}$$

■

Lemma 3.4. *The process* K *is* $\mathbb{P} -$ a.s. *absolutely continuous on* $\{0 \leqslant u < \infty : X_u \in D\}$ *with density* $A(X_u) = \nabla\varphi(X_u)$*:*

$$\mathbf{1}_{\{X_u \in D\}}.dK_u \;=\; \nabla\varphi(X_u).du\,. \tag{3.32}$$

Proof. We have to prove that for all $0 \leqslant s < t < \infty$

$$\int_s^t \mathbf{1}_{\{X_u \in D\}}.dK_u \;=\; \int_s^t \nabla\varphi(X_u).du\,. \tag{3.33}$$

Since $\{0 \leqslant u < \infty \;:\; X_u \notin D\}$ has Lebesgue measure zero, its complementary set is dense in $\mathbb{R}^+$, so we may assume $X_s, X_t \in D$ in the proof of (3.33). We define:

$$B \;=\; \{s \leqslant u \leqslant t \;:\; X_u \notin D\} \;=\; \{s < u < t \;:\; X_u \notin D\}, \tag{3.34}$$

$$U \;=\;]s;t[\backslash B, \tag{3.35}$$

so that U is an open subset of $\mathbb{R}$, $U \cup B \;=\;]s;t[$, and thus U can be written

$$U \;=\; \bigcup_l]a_l;b_l[, \tag{3.36}$$

where for every $l \in \mathbb{N}^*$, $s \leqslant a_l < b_l \leqslant t$. Therefore, we have

$$\begin{aligned}\int_s^t \mathbf{1}_{\{X_u \in D\}}.dK_u \;&=\; \int_{\cup_l]a_l;b_l[} .dK_u \\ &=\; \sum_l \int_{a_l}^{b_l} dK_u\,,\end{aligned}$$

and so, it is enough to prove that for all $l \in \mathbb{N}^*$

$$\int_{a_l}^{b_l} dK_u \;=\; \int_{a_l}^{b_l} \nabla\varphi(X_u).du\,. \tag{3.37}$$

Let us write a for a_l and b for b_l. We have to prove

$$\int_a^b dK_u \;=\; \int_a^b \nabla\varphi(X_u).du\,, \tag{3.38}$$

assuming that

$$X_u \;\in\; D, \qquad\qquad \forall u \;\in]a;b[. \tag{3.39}$$

It suffices to show (3.38) under the stronger assumption

$$X_u \;\in\; D, \qquad\qquad \forall u \;\in\; [a;b]. \tag{3.40}$$

If we define

$$\mathscr{K} \;=\; \{X_u \;:\; u \in [a;b]\}, \tag{3.41}$$

then $\mathscr{K}$ is a compact subset of the open set D, which implies the existence of $\delta > 0$ such that

$$\mathscr{K}_\delta \;=\; \{x \in \mathbb{R}^d \;:\; \mathrm{dist}(x,K) \leqslant \delta\} \;\subset\; D. \tag{3.42}$$

Let $e \in \mathbb{R}^N$, $|e| = 1$. Recall that the measure

$$\langle X_u - \alpha_u \,,\, dK_u - \beta_u du \rangle \tag{3.43}$$

is nonnegative on $[a;b]$ for (α,β) such that

$$(\alpha_u,\beta_u) \in \mathrm{Gr}(A), \qquad \forall u \in [a;b]\,. \tag{3.44}$$

Taking $(\alpha_u,\beta_u) = (X_u - \varepsilon e, \nabla\varphi(X_u - \varepsilon e))$, $0 < \varepsilon < \delta$, we obtain:

$$\langle e, \int_a^b dK_u - \nabla\varphi(X_u - \varepsilon e)du \,\rangle \;\geqslant\; 0, \tag{3.45}$$

and then we let ε tend to 0 to get

$$\langle e, \int_a^b dK_u - \nabla\varphi(X_u)du \,\rangle \;\geqslant\; 0, \tag{3.46}$$

thanks to (3.42), the smoothness of φ on D and Lebesgue's convergence theorem. Replacing ε by $-\varepsilon$, we finally obtain

$$\langle e, \int_a^b dK_u - \nabla\varphi(X_u)du \,\rangle \;=\; 0, \quad \forall e \in \mathbb{R}^N,\ |e| = 1, \tag{3.47}$$

and consequently,

$$\int_a^b dK_u \;=\; \int_a^b \nabla\varphi(X_u)du\,. \tag{3.48}$$

■

From the previous lemma, we assert that the measure dK can be written

$$dK_u \;=\; \nabla\varphi(X_u)du \;+\; dG_u\,, \tag{3.49}$$

where G is a continuous boundary process, that is to say:

$$G_t \;=\; \int_0^t \mathbf{1}_{\{X_u\in\partial D\}} dG_u\,. \tag{3.50}$$

Lemma 3.5. *For all* $0\leqslant t < \infty$, $1\leqslant i < j \leqslant N$, *we have*

$$\int_0^t \frac{du}{X_u^{(j)} - X_u^{(i)}} \;<\; \infty. \tag{3.51}$$

Proof. Let $i = 1$. For all $2\leqslant j \leqslant N$, since K, G are finite variation processes and all the terms in the absolute value have the same sign,

$$\int_0^t \frac{du}{X_u^{(j)} - X_u^{(1)}} \leqslant \int_0^t \left| \sum_{l=2}^{l=N} \frac{1}{X_u^{(l)} - X_u^{(1)}} \right| du \;<\; \infty. \tag{3.52}$$

Let $i = 2$. According to the case $i = 1$, we know that

$$\int_0^t \frac{du}{X_u^{(2)} - X_u^{(1)}} \ < \ \infty, \tag{3.53}$$

and consequently for all $3 \leqslant j \leqslant N$

$$\int_0^t \frac{du}{X_u^{(j)} - X_u^{(2)}} \leqslant \int_0^t \left| \sum_{l=3}^{l=N} \frac{1}{X_u^{(l)} - X_u^{(2)}} \right| du \ < \ \infty\,. \tag{3.54}$$

Let $i = 3$. According to the cases $i = 1$ et $i = 2$, we know that

$$\int_0^t \frac{du}{X_u^{(3)} - X_u^{(1)}} \ < \ \infty, \tag{3.55}$$

$$\int_0^t \frac{du}{X_u^{(3)} - X_u^{(2)}} \ < \ \infty, \tag{3.56}$$

and consequently for all $4 \leqslant j \leqslant N$

$$\int_0^t \frac{du}{X_u^{(j)} - X_u^{(3)}} \leqslant \int_0^t \left| \sum_{l=4}^{l=N} \frac{1}{X_u^{(l)} - X_u^{(3)}} \right| du \ < \ \infty\,. \tag{3.57}$$

By iteration, we obtain (3.51). ■

The following result displays in which direction the boundary term G acts to keep X inside $\overline{D}$:

Lemma 3.6. *For all* $0 \leqslant t < \infty$, *we have*

$$G_t \ = \ \int_0^t n_s \, d|G|_s, \tag{3.58}$$

where n_s *belongs* $d|G|$ – a.e. *to the set of unitary outward normals of* D *at* X_s.

Proof. Since $(X, K)$is the solution of (3.22), it follows that for all $(\alpha, \beta) \in \mathrm{Gr}(A)$

$$\langle X_u - \alpha, dK_u - \beta du \rangle \geqslant 0, \tag{3.59}$$

and considering the product with $\mathbf{1}_{\{X_u \in \partial D\}}$which is du-a.e. zero (see Lemma 3.3) and writing $dG_u = n_u d|G|_u$ where $|n_u| = 1$ (corollary of Radon-Nikodym's theorem)

$$\langle X_u - \alpha, n_u d|G|_u \rangle \geqslant 0, \tag{3.60}$$

thus, for $d|G|$-almost all u, we have

$$\langle X_u - \alpha, n_u \rangle \geqslant 0, \qquad\qquad \forall \alpha \in D, \tag{3.61}$$

and so for all $\alpha \in \overline{D}$ as well, which characterizes the fact that n_u belongs to the outward normal cone of D at X_u. ■

Using the decomposition (3.49) of dK, we write differently the system of equations satisfied by X:

$$\begin{aligned} dX_t^{(i)} \ &= \ b(X_t^{(i)})dt + \sigma(X_t^{(i)})dW_t^{(i)} \\ &\quad + \gamma \sum_{1 \leqslant j \neq i \leqslant N} \frac{dt}{X_t^{(i)} - X_t^{(j)}} \ - \ dG_t^{(i)}\,, \ i = 1, 2, \ldots, N\,. \end{aligned}$$

Lemma 3.7. *Let* $x \in \partial D$*,* $x = (x^{(1)} \leqslant x^{(2)} \leqslant \cdots \leqslant x^{(j-1)} < x^{(j)} = x^{(j+1)} = \cdots = x^{(k-1)} = x^{(k)} < x^{(k+1)} \leqslant \cdots \leqslant x^{(N)})$ *and* $n = (n^{(1)}, n^{(2)}, \ldots, n^{(N)})$ *belong to the outward normal cone of* D *at* x*. Then* $n^{(j)} + n^{(j+1)} + \cdots + n^{(k-1)} + n^{(k)} = 0$ *(with natural modifications for* $j = 1$*,* $k = N$*,* $j = k$*).*

Proof. By definition of the outward normal cone of D at x, we have:

$$\langle x - \alpha, n \rangle \geqslant 0\,, \quad \forall \alpha \in \overline{D} = \{y : y^{(1)} \leqslant y^{(2)} \leqslant \cdots \leqslant y^{(N)}\}. \tag{3.62}$$

Let $\varepsilon > 0$ such that $x^{(j-1)} < x^{(j)} - \varepsilon < x^{(k)} + \varepsilon < x^{(k+1)}$. The inequality (3.62) for $\alpha = (x^{(1)}, x^{(2)}, \cdots, x^{(j-1)}, x^{(j)} - \varepsilon, x^{(j+1)} - \varepsilon, \ldots, x^{(k-1)} - \varepsilon, x^{(k)} - \varepsilon, x^{(k+1)}, \ldots, x^{(N)}) \in \overline{D}$ gives $\varepsilon(n^{(j)} + n^{(j+1)} + \cdots + n^{(k-1)} + n^{(k)}) \geqslant 0$, and with $\alpha = (x^{(1)}, x^{(2)}, \cdots, x^{(j-1)}, x^{(j)} + \varepsilon, x^{(j+1)} + \varepsilon, \ldots, x^{(k-1)} + \varepsilon, x^{(k)} + \varepsilon, x^{(k+1)}, \ldots, x^{(N)}) \in \overline{D}$, we obtain $\varepsilon(n^{(j)} + n^{(j+1)} + \cdots + n^{(k-1)} + n^{(k)}) \leqslant 0$. Consequently, we have shown that $n^{(j)} + n^{(j+1)} + \cdots + n^{(k-1)} + n^{(k)} = 0$. ∎

In the following lemma, it is shown the process G is zero in fact and consequently there is no boundary term: thus the proof of Proposition 3.2 will be complete.

Lemma 3.8. $G = 0$.

Proof. We deduce from Lemma 3.6 and Lemma 3.7 (with $j = k$) that the measure $dG^{(i)}$, $1 \leqslant i \leqslant d$, is supported by

$$\{0 \leqslant u < \infty \,:\, X_u^{(i)} = X_u^{(i+1)}\} \,\cup\, \{0 \leqslant u < \infty \,:\, X_u^{(i)} = X_u^{(i-1)}\}, \tag{3.63}$$

with natural modifications for $i = 1$, $i = d$. Therefore, in order to prove Lemma 3.8, it suffices to show that for all $1 \leqslant i \leqslant N - 1$

$$\mathbf{1}_{\{X_u^{(i)} = X_u^{(i+1)}\}} dG_u^{(i)} = \mathbf{1}_{\{X_u^{(i)} = X_u^{(i+1)}\}} dG_u^{(i+1)} = 0, \tag{3.64}$$

or for all $1 \leqslant i \leqslant d - 1$, $1 \leqslant j \leqslant i < k \leqslant N$,

$$\begin{aligned} & \mathbf{1}_{\{X_u^{(1)} \leqslant X_u^{(2)} \leqslant \cdots \leqslant X_u^{(j-1)} < X_u^{(j)} = \cdots = X_u^{(i)} = X_u^{(i+1)} = \cdots = X_u^{(k)} < X_u^{(k+1)} \leqslant \cdots \leqslant X_u^{(N)}\}} dG_u^{(i)} \\ = \; & \mathbf{1}_{\{X_u^{(1)} \leqslant X_u^{(2)} \leqslant \cdots \leqslant X_u^{(j-1)} < X_u^{(j)} = \cdots = X_u^{(i)} = X_u^{(i+1)} = \cdots = X_u^{(k)} < X_u^{(k+1)} \leqslant \cdots \leqslant X_u^{(N)}\}} dG_u^{(i+1)} \\ = \; & 0. \end{aligned} \tag{3.65}$$

Let $j \leqslant l < m \leqslant k$. We are going to prove

$$\begin{aligned} & \mathbf{1}_{\{X_u^{(1)} \leqslant X_u^{(2)} \leqslant \cdots \leqslant X_u^{(j-1)} < X_u^{(j)} = \cdots = X_u^{(i)} = X_u^{(i+1)} = \cdots = X_u^{(k)} < X_u^{(k+1)} \leqslant \cdots \leqslant X_u^{(N)}\}} dG_u^{(l)} \\ = \; & \mathbf{1}_{\{X_u^{(1)} \leqslant X_u^{(2)} \leqslant \cdots \leqslant X_u^{(j-1)} < X_u^{(j)} = \cdots = X_u^{(i)} = X_u^{(i+1)} = \cdots = X_u^{(k)} < X_u^{(k+1)} \leqslant \cdots \leqslant X_u^{(N)}\}} dG_u^{(m)} \\ = \; & 0. \end{aligned} \tag{3.66}$$

Using the occupation times formula, we can claim

$$\int_0^\infty \frac{L_t^a(X^{(m)} - X^{(l)})}{a} da = \int_0^t \frac{\sigma^2(X_u^{(m)}) + \sigma^2(X_u^{(l)})}{X_u^{(m)} - X_u^{(l)}} du, \tag{3.67}$$

where $L^a(X^{(m)}-X^{(l)})$ is the local time at a of the real continuous semi-martingale $X^{(m)}-X^{(l)}$. From the continuity of X and the Lipschitz continuity of σ, we deduce

$$\sup_{0\leqslant u\leqslant t}\left(\sigma^2(X_u^{(m)})+\sigma^2(X_u^{(l)})\right)\leqslant C.\left(1+\sup_{0\leqslant u\leqslant t}|X_u|^2\right)\leqslant C, \tag{3.68}$$

so, using Equation (3.67) and Lemma 3.5

$$\int_0^\infty \frac{L_t^a(X^{(m)}-X^{(l)})}{a}da < \infty, \tag{3.69}$$

which implies $L_t^0(X^{(m)}-X^{(l)})=0$ thanks to the right continuity of local time. From the identity

$$X_u^{(m)}-X_u^{(l)} = (X_u^{(m)}-X_u^{(l)})^+, \tag{3.70}$$

calculating $(X_u^{(m)}-X_u^{(l)})^+$ with Tanaka's formula and since $L^0(X^{(m)}-X^{(l)})\equiv 0$, we have

$$\int_0^t (dG_u^{(m)}-dG_u^{(l)}) = \int_0^t \mathbf{1}_{\{X_u^{(m)}>X_u^{(l)}\}}(dG_u^{(m)}-dG_u^{(l)}), \tag{3.71}$$

where we have used Lemma 3.3 to assert that $\mathbf{1}_{\{X_u^{(m)}>X_u^{(l)}\}} = 1$ a.e. with respect to the Lebesgue measure in order to identify the others terms in the semimartingale decompositions of $X_u^{(m)}-X_u^{(l)}$ and $(X_u^{(m)}-X_u^{(l)})^+$. The last equality can be written

$$\int_0^t \mathbf{1}_{\{X_u^{(l)}=X_u^{(m)}\}}dG_u^{(m)} = \int_0^t \mathbf{1}_{\{X_u^{(l)}=X_u^{(m)}\}}dG_u^{(l)}, \tag{3.72}$$

hence also

$$\begin{aligned}&\mathbf{1}_{\{X_u^{(1)}\leqslant X_u^{(2)}\leqslant\cdots\leqslant X_u^{(j-1)}<X_u^{(j)}=\cdots=X_u^{(i)}=X_u^{(i+1)}=\cdots=X_u^{(k)}<X_u^{(k+1)}\leqslant\cdots\leqslant X_u^{(N)}\}}dG_u^{(l)}\\ =\ &\mathbf{1}_{\{X_u^{(1)}\leqslant X_u^{(2)}\leqslant\cdots\leqslant X_u^{(j-1)}<X_u^{(j)}=\cdots=X_u^{(i)}=X_u^{(i+1)}=\cdots=X_u^{(k)}<X_u^{(k+1)}\leqslant\cdots\leqslant X_u^{(N)}\}}dG_u^{(m)}.\end{aligned} \tag{3.73}$$

Let $v = e_j + e_{j+1} + \ldots + e_k$ where $(e_1, e_2, \ldots, e_N)$ is the canonical basis of $\mathbb{R}^N$. From Lemma 3.7, v is orthogonal to every normal vector at any point of $\{x^{(1)}\leqslant x^{(2)}\leqslant\cdots\leqslant x^{(j-1)} < x^{(j)} = \cdots = x^{(i)} = x^{(i+1)} = \cdots = x^{(k)} < x^{(k+1)}\leqslant\cdots\leqslant x^{(N)}\}$. So, using Lemma 3.6, we have

$$\mathbf{1}_{\{X_u^{(1)}\leqslant X_u^{(2)}\leqslant\cdots\leqslant X_u^{(j-1)}<X_u^{(j)}=\cdots=X_u^{(i)}=X_u^{(i+1)}=\cdots=X_u^{(k)}<X_u^{(k+1)}\leqslant\cdots\leqslant X_u^{(N)}\}}\langle v, dG_u\rangle = 0, \tag{3.74}$$

and from the definition of v

$$\sum_{p=j}^{p=k}\mathbf{1}_{\{X_u^{(1)}\leqslant X_u^{(2)}\leqslant\cdots\leqslant X_u^{(j-1)}<X_u^{(j)}=\cdots=X_u^{(i)}=X_u^{(i+1)}=\cdots=X_u^{(k)}<X_u^{(k+1)}\leqslant\cdots\leqslant X_u^{(N)}\}}dG_u^{(p)} = 0\,. \tag{3.75}$$

Now, using (3.73) and (3.75), we get (3.64). The proof of Lemma 3.8 is complete and, consequently, that of Proposition 3.2 as well. ■

Proof: (2) *Uniqueness part*
For the sake of completeness, we prove uniqueness for the problem (3.14) but it

is clear this proof is just a particular case of the general result of uniqueness for multivalued stochastic differential equations (see [2]). In particular, uniqueness is a direct consequence of the Lipschitz continuity of b, σ and the monotonicity of $\partial\varphi$.

Let X and Y be solutions of (3.14). We define:

$$S_p = \inf\{t \geqslant 0 : |X_t| + |Y_t| \geqslant p\}, \qquad p \in \mathbb{N}^*. \tag{3.76}$$

Using Ito's formula, we have:

$$\begin{aligned} & |X_{t\wedge S_p} - Y_{t\wedge S_p}|^2 \\ = \; & 2\int_0^{t\wedge S_p} \langle b(X_s) - b(Y_s), X_s - Y_s\rangle ds \\ & + 2\int_0^{t\wedge S_p} \langle \sigma(X_s) - \sigma(Y_s), X_s - Y_s\rangle dW_s \\ & - 2\gamma \int_0^{t\wedge S_p} \sum_{1\leqslant i\leqslant N} \sum_{1\leqslant j\neq i\leqslant N} (X_s^{(i)} - Y_s^{(i)}) \left(\frac{1}{X_s^{(i)} - X_s^{(j)}} - \frac{1}{Y_s^{(i)} - Y_s^{(j)}} \right) ds \\ & + \int_0^{t\wedge S_p} \operatorname{tr}\left[\{\sigma(X_s) - \sigma(Y_s)\} \{\sigma(X_s) - \sigma(Y_s)\}^* \right] ds \,, \end{aligned}$$

but for all $x, y \in D$, we can claim

$$\begin{aligned} & \textstyle\sum_{1\leqslant i\leqslant N} \sum_{1\leqslant j\neq i\leqslant N} (x^{(i)} - y^{(i)}) \left(\frac{1}{x^{(i)} - x^{(j)}} - \frac{1}{y^{(i)} - y^{(j)}} \right) \\ = \; & \textstyle\sum_{1\leqslant i\leqslant N} \sum_{1\leqslant j< i\leqslant N} \left(\frac{1}{x^{(i)} - x^{(j)}} - \frac{1}{y^{(i)} - y^{(j)}} \right) ((x^{(i)} - y^{(i)}) - (x^{(j)} - y^{(j)})) \\ = \; & \textstyle\sum_{1\leqslant i\leqslant N} \sum_{1\leqslant j< i\leqslant N} \left(\frac{1}{x^{(i)} - x^{(j)}} - \frac{1}{y^{(i)} - y^{(j)}} \right) ((x^{(i)} - x^{(j)}) - (y^{(i)} - y^{(j)})) \\ \leqslant \; & 0 \,, \end{aligned}$$

which implies that

$$\begin{aligned} |X_{t\wedge S_p} - Y_{t\wedge S_p}|^2 \quad \leqslant \quad & 2\int_0^{t\wedge S_p} \langle b(X_s) - b(Y_s), X_s - Y_s\rangle ds \\ & + 2\int_0^{t\wedge S_p} \langle \sigma(X_s) - \sigma(Y_s), X_s - Y_s\rangle dW_s \\ & + \int_0^{t\wedge S_p} \operatorname{tr}\left[\{\sigma(X_s) - \sigma(Y_s)\} \{\sigma(X_s) - \sigma(Y_s)\}^* \right] ds \,, \end{aligned}$$

and then from the assumption on b, σ:

$$\begin{aligned} \mathbb{E}|X_{t\wedge S_p} - Y_{t\wedge S_p}|^2 \quad & \leqslant \quad C\, \mathbb{E} \int_0^{t\wedge S_p} |X_s - Y_s|^2 ds \\ & \leqslant \quad C \int_0^t \mathbb{E}|X_{s\wedge S_p} - Y_{s\wedge S_p}|^2 ds \,. \end{aligned}$$

Hence $\mathbb{E}|X_{t\wedge S_p} - Y_{t\wedge S_p}|^2 = 0$ from Gronwall's lemma, and then, using Fatou's lemma, $\mathbb{E}|X_t - Y_t|^2 = 0$. The proof of uniqueness for (3.14) is therefore complete, and Theorem 3.1 follows at once. ∎

4 Tightness

For a particular choice of the coefficients b, σ, we are able to study the collision between particles governed by the system (3.14) (see [1] or [5] for a proof).

Proposition 4.1. *We consider the solution X of (3.14) for the data $N \in \mathbb{N}^* \backslash \{1\}$, $\gamma > 0$, $-\infty < x_0^{(1)} < x_0^{(2)} < \cdots < x_0^{(N)} < \infty$ and the coefficients b, σ given by $b(x) = \theta x + \delta$, $\sigma(x) = \sigma$ ($\theta \in \mathbb{R}$, $\delta \in \mathbb{R}$, $0 \leqslant \sigma < \infty$). Let τ be the first collision time between the particles, that is to say*

$$\tau \;=\; \inf\{t > 0 \,:\, X_t^{(i)} \;=\; X_t^{(j)} \quad \text{for} \quad 1 \leqslant \mathrm{i} \neq \mathrm{j} \leqslant \mathrm{N}\}. \tag{4.77}$$

If we suppose

$$2\gamma \geqslant \sigma^2, \tag{4.78}$$

then there is no collision between particles, that is to say

$$\mathbb{P}(\tau \;=\; \infty) \;=\; 1. \tag{4.79}$$

We are interested in the behaviour of the interacting particles system (3.14) when

$$\gamma \;=\; \frac{2\lambda}{N}\,, \tag{4.80}$$

as the number N of particles tends to infinity, which we study through the empirical measure process:

$$\mu_t^{(N)} \quad = \quad \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{(i)}}\,, \tag{4.81}$$

where δ_x is the Dirac probability at x for $x \in \mathbb{R}$. Thus, we would like to establish a kind of “strong law” limiting behaviour as $N \uparrow \infty$ for (3.14).

This interacting SDE, and particularly its behaviour when $N \uparrow \infty$, have recently been studied by L.C.G. Rogers-Z. Shi [5] and T. Chan [3] but they were forced to suppose that there was no collision between particles (otherwise they couldn’t define a solution for the system (3.14) up to ∞), that is to say they studied the limiting behaviour in the particular case

$$b(x) \;=\; -\theta.x, \qquad\qquad \forall\, x \in \mathbb{R}, \tag{4.82}$$

$$\sigma(x) \;\equiv\; \sigma, \qquad\qquad \forall\, x \in \mathbb{R}, \tag{4.83}$$

($0 < \theta < \infty$, $0 < \sigma < \infty$) with $\sigma^2 \leqslant 2\gamma = \dfrac{4\lambda}{N}$. That’s the reason why they had to consider $\sigma = \dfrac{const.}{\sqrt{N}}$, and consequently tending to 0 as N tends to infinity.

Our treatment is partly similar to that of T. Chan [3] and L.C.G. Rogers - Z. Shi [5] except that we consider the most general case for b and σ (thanks to

Theorem 3.1) at least for tightness, but we will consider particular coefficients b and σ when studying convergence (see Sect. 5).

Henceforth, we will assume (to simplify)

$$x_0^{(1)} = x_0^{(2)} = \ldots = x_0^{(N)} = x_0 \in \mathbb{R}, \forall N \in \mathbb{N}^*. \tag{4.84}$$

Theorem 4.2. *With the same assumptions as in Theorem 3.1 except γ is given by (4.80) and the initial data by (4.84), the sequence of measure-valued processes $(\mu^{(N)})_N$ is tight and any limit μ is a continuous probability measure-valued process satisfying:*

$$\begin{aligned} \int f(x) d\mu_t(x) &= f(x_0) + \int_0^t ds \left(\int b(x).f'(x)\mu_s(dx) \right) \\ &+ \frac{1}{2} \int_0^t ds \left(\int \sigma^2(x).f''(x)\mu_s(dx) \right) \\ &+ \lambda \int_0^t ds \left(\int \int \frac{f'(x) - f'(y)}{x - y} \mu_s(dx)\mu_s(dy) \right) \end{aligned} \tag{4.85}$$

for all $f \in C_b^2(\mathbb{R})$ (= $\{f \in C^2(\mathbb{R}) : f, f', f''$ bounded $\}$) with $x^2 f''(x)$ and $xf'(x)$ bounded.

Proof. Let $(f_n)_{n\in\mathbb{N}^*}$ be a dense subsequence of functions in $C_c^2(\mathbb{R})$ and f_0 be a positive function in $C^2(\mathbb{R})$ with $x^2 f_0''(x)$, $xf_0'(x)$ bounded and $f_0(x) \to \infty$ as $|x| \uparrow \infty$. In order to obtain the tightness of $(\mu^{(N)})_N$, it is sufficient (see [3]) to prove that for each $n \in \mathbb{N}$, the sequence of continuous *real* $-$ *valued* processes $(\int f_n(x) d\mu_\cdot^{(N)}(x))_N$ is tight. Consequently, it is enough to prove the tightness of $(\int f(x) d\mu_\cdot^{(N)}(x))_N$ for all $f \in C^2(\mathbb{R})$ with $x^2 f''(x)$ and $xf'(x)$ bounded.

From (3.14), (4.81) and Itô's formula, we have:

$$\begin{aligned}\int f(x)d\mu_t^{(N)}(x) &= \frac{1}{N}\sum_{i=1}^{N} f(X_t^{(i)}) \\ &= f(x_0) + \frac{1}{N}\sum_{i=1}^{N}\int_0^t f'(X_s^{(i)})\sigma(X_s^{(i)})dW_s^{(i)} \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}\int_0^t b(X_s^{(i)}).f'(X_s^{(i)})ds \\ &\quad + \frac{1}{N}\sum_{i=1}^{N}\int_0^t f'(X_s^{(i)})\left(\frac{2\lambda}{N}\sum_{1\leqslant j\neq i\leqslant N}\frac{1}{X_s^{(i)}-X_s^{(j)}}\right)ds \\ &\quad + \frac{1}{2N}\sum_{i=1}^{N}\int_0^t \sigma^2(X_s^{(i)}).f''(X_s^{(i)})ds.\end{aligned} \tag{4.86}$$

Hence, using the definition (4.81) of $(\mu^{(N)})_N$ and the symmetry of the interaction, the equality (4.86) can also be written:

$$\begin{aligned}\int f(x)d\mu_t^{(N)}(x) &= f(x_0) + M_t^{(N)} \\ &\quad + \int_0^t ds\left(\int b(x).f'(x)\mu_s^{(N)}(dx)\right) \\ &\quad + \lambda\int_0^t ds\left(\int\int_{\{x\neq y\}}\frac{f'(x)-f'(y)}{x-y}\mu_s^{(N)}(dx)\mu_s^{(N)}(dy)\right) \\ &\quad + \frac{1}{2}\int_0^t ds\left(\int \sigma^2(x).f''(x)\mu_s^{(N)}(dx)\right),\end{aligned} \tag{4.87}$$

where $M^{(N)}$ is a continuous martingale such that:

$$\langle M^{(N)}\rangle_t = \frac{1}{N^2}\sum_{i=1}^{N}\int_0^t \{\sigma(X_s^{(i)}).f'(X_s^{(i)})\}^2 ds. \tag{4.88}$$

The triple integral of (4.87) is

$$\begin{aligned}&\lambda\int_0^t ds\left(\int\int\frac{f'(x)-f'(y)}{x-y}\mu_s^{(N)}(dx)\mu_s^{(N)}(dy)\right) \\ &\quad - \frac{\lambda}{N}\int_0^t ds\left(\int f''(x)\mu_s^{(N)}(dx)\right)\end{aligned} \tag{4.89}$$

so that

$$\int f(x)d\mu_t^{(N)}(x) = f(x_0) + M_t^{(N)} + \int_0^t ds\left(\int b(x).f'(x)\mu_s^{(N)}(dx)\right) + \lambda\int_0^t ds\left(\int\int \frac{f'(x)-f'(y)}{x-y}\mu_s^{(N)}(dx)\mu_s^{(N)}(dy)\right) - \frac{\lambda}{N}\int_0^t ds\left(\int f''(x)\mu_s^{(N)}(dx)\right) + \frac{1}{2}\int_0^t ds\left(\int \sigma^2(x).f''(x)\mu_s^{(N)}(dx)\right). \tag{4.90}$$

Now, using (4.90), the assumption that $xf'(x)$, $x^2f''(x)$ are bounded, b and σ are Lipschitz, and the well-known Aldous criterion (see [4]), the sequence of continuous real-valued processes $(\int f(x)d\mu_\cdot^{(N)}(x))_N$ is easily shown to be tight and, consequently, the laws of the processes $(\mu^{(N)}))_N$ are shown to be tight. From the tightness, we have at least the convergence $\mu^{(N)} \Rightarrow \mu$ along a subsequence (N_k). Let k tend to infinity in (4.90) (written for $N = N_k$) for suitable f and use the convergence $\mu^{(N_k)} \Rightarrow \mu$, the boundedness of f, $xf'(x)$, $x^2f''(x)$: thus, we show that any such limit process μ satisfies (4.85). This concludes the proof of Theorem 4.2.

5 Convergence

In this section, we would like to obtain the weak convergence of $(\mu^{(N)})_N$ when $N \uparrow \infty$ to a measure - valued process μ. From Sect. 4, it remains to prove there is only one possible limit for all $(\mu^{(N_k)})_k$. We shall do this for quite particular coefficients, more precisely

$$b(x) = \theta.x + \rho, \qquad \forall x \in \mathbb{R}, \tag{5.91}$$

$$\sigma(x) = \sigma, \qquad \forall x \in \mathbb{R}, \tag{5.92}$$

for some constants $\theta,\ \rho \in \mathbb{R}$, $0 < \sigma < \infty$; so we henceforth consider the system

$$dX_t^{(i)} = \sigma dW_t^{(i)} + \frac{2\lambda}{N}\sum_{1\leqslant j\neq i\leqslant N}\frac{dt}{X_t^{(i)} - X_t^{(j)}} + (\theta X_t^{(i)} + \rho)dt\ ,\ i = 1, 2, \ldots, N\ . \tag{5.93}$$

However, it is worth noticing that we don't assume $\sigma^2 \leqslant 2\gamma$. In particular, this implies:

- in contrast with [3] and [5], the sequence of diffusion coefficients σ_N does not tend to 0 as $N \to \infty$, so that the process "hidden behind" the limit μ of $\mu^{(N)}$ is not deterministic anymore ;
- there might be collisions between the particles governed by the system (5.93).

Theorem 5.1. *With the same assumptions as in Theorem 4.2 (with b, σ given by (5.91), (5.92)), the sequence of measure-valued process $(\mu^{(N)})_N$ is (weakly) convergent and the limit μ is the unique continuous probability measure-valued function satisfying:*

$$\begin{aligned}\int f(x)d\mu_t(x) \quad = \quad & f(x_0) + \int_0^t ds\left(\int(\theta x+\rho).f'(x)\mu_s(dx)\right)\\ & + \frac{\sigma^2}{2}\int_0^t ds\left(\int f''(x)\mu_s(dx)\right)\\ & + \lambda\int_0^t ds\left(\int\int\frac{f'(x)-f'(y)}{x-y}\mu_s(dx)\mu_s(dy)\right)\end{aligned} \tag{5.94}$$

for all $f \in C_b^2(\mathbb{R})$ with $xf'(x)$ bounded.

Proof. Thanks to Theorem 4.2, we have at least that $\mu^{(N)} \Rightarrow \mu$ along a subsequence and any such limit process $\mu = \{\mu_t, t \in \mathbb{R}^+\}$ satisfies

$$\begin{aligned}\int f(x)d\mu_t(x) \quad = \quad & f(x_0) + \int_0^t ds\left(\int(\theta x+\rho).f'(x)\mu_s(dx)\right)\\ & + \frac{\sigma^2}{2}\int_0^t ds\left(\int f''(x)\mu_s(dx)\right)\\ & + \lambda\int_0^t ds\left(\int\int\frac{f'(x)-f'(y)}{x-y}\mu_s(dx)\mu_s(dy)\right)\end{aligned} \tag{5.95}$$

for all $f \in C_b^2(\mathbb{R})$ with $xf'(x)$ bounded (note that σ is constant and so the boundedness of $x^2f''(x)$ isn't necessary anymore). If we can show that (5.95) has a unique solution, we actually prove the convergence of $(\mu^{(N)})_N$, this time not only up to subsequences, and thereby Theorem 5.1. Now, for $z = x + iy \in \mathbb{C}$, $y > 0$, set

$$f(u) = \frac{1}{u-z}, \tag{5.96}$$

and

$$M_t(z) = \int\frac{\mu_t(du)}{u-z}. \tag{5.97}$$

Simple calculations, with (5.95) at their starting point, show that M satisfies the PDE

$$\begin{cases} \dfrac{\partial M_t(z)}{\partial t} = -\theta M_t(z) - (\theta z + \rho)\dfrac{\partial M_t(z)}{\partial z} + \sigma^2 \dfrac{\partial^2 M_t(z)}{\partial z^2} + 2\lambda M_t(z).\dfrac{\partial M_t(z)}{\partial z} \\ M_0(z) = \dfrac{1}{x_0 - z}\,. \end{cases} \tag{5.98}$$

Let us note that $|M_t(z)| \leqslant \frac{1}{y}$ for all $t \geqslant 0$, $x \in \mathbb{R}$, $|M_t(z)| \to 0$ when $|x| \to \infty$ and $M_.(. + iy) \in L^2([0;t] \times \mathbb{R})$ for all $t \geqslant 0$, $y > 0$. Now, let us fix $y > 0$ and set

$$u(t,x) = M_t(x + iy)\,. \tag{5.99}$$

We define the linear operator L by

$$Lu = \frac{\partial u}{\partial t} + \theta.u + (\theta x + i\theta y + \rho)\frac{\partial u}{\partial x} - \sigma^2 \frac{\partial^2 u}{\partial x^2}\,. \tag{5.100}$$

Thanks to (5.98), u is a solution to the PDE

$$\begin{cases} Lu = 2\lambda u.\dfrac{\partial u}{\partial x} \\ u(0,x) = \dfrac{1}{x_0 - x - iy} \\ u \in L^2([0;T] \times \mathbb{R})\,,\ \forall T < \infty\,. \end{cases} \tag{5.101}$$

In order to prove Theorem 5.1, it also suffices to show uniqueness for (5.101). We will prove it thanks to several lemma.

Lemma 5.2. *Let u be a solution to the PDE*

$$\begin{cases} Lu = 0 \\ u(0,x) = u_0(x)\,,\ u_0 \in L^1 \cap L^\infty \\ u \in L^2([0;T] \times \mathbb{R})\,,\ \forall T < \infty \end{cases} \tag{5.102}$$

Then we have:

$$\begin{aligned} u(t,x) &= \int u_0(e^{-\theta t}(x - v))e^{-\theta t} \\ &\quad \times \exp\left(- \frac{\left(v - \frac{1}{\theta}(\rho + i\theta y)(e^{\theta t} - 1)\right)^2}{2\sigma^2 \frac{(e^{2\theta t} - 1)}{\theta}} \right) \\ &\quad \times \frac{1}{\sqrt{2\pi}\sigma\sqrt{\frac{e^{2\theta t} - 1}{\theta}}}\, dv\,. \end{aligned} \tag{5.103}$$

Proof. This kind of result is very classic and so we don't give a detailed proof. Let us just note that it suffices to consider the Fourier transform of the equation $Lu = 0$ to obtain the new equation

$$\frac{\partial \hat{u}}{\partial t} - \theta\xi.\frac{\partial \hat{u}}{\partial \xi} - (\theta y - i\rho)\xi\hat{u} + \sigma^2\xi^2\hat{u} = 0\,, \tag{5.104}$$

(with the notation $\hat{u}(t,\xi) = \int e^{-ix\xi} u(t,x)\,dx$) and, then, using the well-known method of the characteristics, we get

$$\hat{u}(t,\xi) \;=\; \hat{u}_0(e^{\theta t}\xi)\exp\left(\frac{1}{\theta}(\theta y - i\rho)\xi(e^{\theta t}-1) \;-\; \xi^2\frac{\sigma^2}{2\theta}(e^{2\theta t}-1)\right). \qquad (5.105)$$

The inverse Fourier formula allows us to conclude. ∎

Lemma 5.3. *Let u be a solution of the PDE*

$$\begin{cases} Lu \;=\; \dfrac{\partial g}{\partial x} = h\,, \qquad g \text{ and } h \text{ bounded on } [0;T]\times\mathbb{R} \\ u(0,.) \;=\; 0 \\ u \;\in\; L^2([0;T]\times\mathbb{R})\,,\ \forall T<\infty \end{cases} \qquad (5.106)$$

Then u must be given by

$$u(t,x) \;=\; \int_0^t\int \frac{1}{\sqrt{2\pi}\sigma\sqrt{\frac{e^{2\theta s}-1}{\theta}}}\, e^{\theta s}\frac{\left(ve^{\theta s}-x+\frac{1}{\theta}(\rho+i\theta y)(e^{\theta s}-1)\right)}{\sigma^2\dfrac{(e^{2\theta s}-1)}{\theta}} \times\exp\left(-\frac{\left(x-ve^{\theta s}-\dfrac{1}{\theta}(\rho+i\theta y)(e^{\theta s}-1)\right)^2}{2\sigma^2\frac{(e^{2\theta s}-1)}{\theta}}\right) \times g(t-s,v)\,ds\,dv\,. \qquad (5.107)$$

Proof. Uniqueness follows from Lemma 5.2. We are now looking for $f(t,x,v)$ such that

$$u(t,x) \;=\; \int_0^t\int f(s,x,v)h(t-s,v)dsdv\,, \qquad (5.108)$$

and consequently

$$\hat{u}(t,\xi) \;=\; \int_0^t\int\int e^{-ix\xi}f(s,x,v)h(t-s,v)dsdvdx\,. \qquad (5.109)$$

We want to find

$$\varphi(t,\xi,v) \;=\; \int e^{-ix\xi}f(t,x,v)dx\,, \qquad (5.110)$$

satisfying

$$\hat{u}(t,\xi) \;=\; \int_0^t\int\varphi(s,\xi,v)h(t-s,v)dsdv \;=\; \int_0^t\int\varphi(t-s,\xi,v)h(s,v)dsdv\,, \qquad (5.111)$$

with

$$\int\varphi(0,\xi,v)h(t,v)dv \;=\; \int e^{-iv\xi}h(t,v)dv\,. \qquad (5.112)$$

So we have

$$\frac{\partial\varphi}{\partial t} - \theta\xi.\frac{\partial\varphi}{\partial\xi} - (\theta y - i\rho)\xi\varphi + \sigma^2\xi^2\varphi = 0\,, \qquad (5.113)$$

$$\varphi(0,\xi,v) = e^{-iv\xi}\,. \qquad (5.114)$$

Thanks to the proof of Lemma 5.2 (see (5.105)), we can assert that

$$\varphi(t,\xi,v) = e^{-i\xi e^{\theta t}v}\exp\left(\frac{1}{\theta}(\theta y - i\rho)\xi(e^{\theta t}-1) - \xi^2\frac{\sigma^2}{2\theta}(e^{2\theta t}-1)\right). \qquad (5.115)$$

Using (5.110), (5.115) and the inverse Fourier formula, we obtain $f(t,x,v)$. Then, we deduce from (5.108) the expression of $u(t,x)$ in terms of $h = \frac{\partial g}{\partial x}$ and, finally, an integration by parts gives formula (5.107). ■

Now, we are able to show the uniqueness of the solution to Equation (5.101). Let us also assume that u_1 and u_2 satisfy

$$L\,u = 2\lambda u.\frac{\partial u}{\partial x} = \lambda.\frac{\partial u^2}{\partial x}\,, \qquad (5.116)$$

with $|u_i(t,x)| \leqslant 1/y$, $i = 1,2$. Writing (5.116) for u_1 and u_2 and then making the difference, we obtain

$$L\,(u_1 - u_2) = \lambda.\frac{\partial(u_1^2 - u_2^2)}{\partial x}\,. \qquad (5.117)$$

Hence, there exists a constant $C < \infty$ such that

$$v(t) \leqslant C\int_0^t v(s)\frac{1}{\sqrt{\frac{e^{2\theta(t-s)}-1}{\theta}}}\,ds\,, \qquad (5.118)$$

where

$$v(t) = \sup_x \sup_{s\leqslant t} |u_1(s,x) - u_2(s,x)|\,. \qquad (5.119)$$

Using the elementary fact that we can find a constant $\nu > 0$ such that

$$\sqrt{\frac{e^{2\theta u}-1}{\theta}} \geqslant \nu\sqrt{u}\,, \quad 0\leqslant u\leqslant T\,, \qquad (5.120)$$

we can write

$$v(t) \leqslant C'\int_0^t v(s)\frac{1}{\sqrt{t-s}}\,ds\,, \qquad (5.121)$$

and iterating this inequality

$$\begin{aligned} v(t) &\leqslant C'^2\int_0^t\left(\int_0^s v(u)\frac{1}{\sqrt{s-u}}\,du\right)\frac{1}{\sqrt{t-s}}\,ds \\ &= C'^2\int_0^t v(u)\,du\int_u^t\frac{ds}{\sqrt{s-u}\sqrt{t-s}}\,, \end{aligned} \qquad (5.122)$$

$$v(t) \leqslant C'^2 \int_0^t v(u)\,du \int_0^1 \frac{dr}{\sqrt{r(1-r)}} \leqslant C'' \int_0^t v(u)\,du\,, \tag{5.123}$$

and, finally, thanks to Gronwall's lemma, we can claim that $v = 0$ and, consequently, $u_1 = u_2$. This ends the proof of Theorem 5.1.

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