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Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps

Masatoshi Fukushima[1], **Matsuyo Tomisaki**[2]

[1] Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Toyonaka, Osaka, Japan
(e-mail: fuku@sigmath.es.osaka-u.ac.jp)
[2] Department of Mathematics, Faculty of Education, Yamaguchi University, Yamaguchi, Japan
(e-mail: tomisaki@po.yb.cc. yamaguchi-u.ac.jp)

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Summary. We consider a d-dimensional Euclidean domain D whose boundary is Lipschitz continuous but admits locally finite number of outward or inward Hölder cusp points. Using a method of Stampacchia and Moser for PDE, we first construct a conservative diffusion process on the Euclidean closure of D possessing a strong Feller resolvent and associated with a second order uniformly elliptic differential operator of divergence form with measurable coefficients a_{ij}. The sample path of the constructed diffusion can be uniquely decomposed as a sum of a martingale additive functional and an additive functional locally of zero energy. The second additive functional will be proved to be of bounded variation with a Skorohod type expression whenever a_{ij} is weakly differentiable and the Hölder exponent at each outward cusp boundary point is greater than $1/2$ regardless the dimension d.

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1 Introduction

Let D be a domain in the d-dimensional Euclidean space R^d and $\overline{D} = D \cup \partial D \subset R^d$ be its closure. The d-dimensional Lebesgue measure is denoted by $m = m(dx)$ or simply by dx. Given measurable functions $a_{ij}(x),\ 1 \le i,j \le d$, on D such that

$$a_{ij} = a_{ji}, \quad \Lambda^{-1}|\xi|^2 \le \sum_{1 \le i,j \le d} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2,\ x \in D,\ \xi \in R^d, \tag{1.1}$$

for some constant $\Lambda \ge 1$, we consider a Dirichlet form $\mathscr{E}$ on $L^2(D) = L^2(D;m)$ defined by

$$\mathscr{D}[\mathscr{E}] = H^1(D), \quad \mathscr{E}(u,v) = \int_D \sum_{1\le i,j\le d} a_{ij}(x)\partial_i u(x)\partial_j v(x)dx, \ u,v \in H^1(D), \tag{1.2}$$

where $H^1(D) = \{u \in L^2(D) : \partial_i u \in L^2(D), 1 \le i \le d\}$ the Sobolev space of order 1. Let $\{T_t, t > 0\}$ be the strongly continuous semigroup of Markovian symmetric operators on $L^2(D)$ associated with the Dirichlet form $\mathscr{E}$.

We denote by $C_0(D)$ [resp. $B_0(D)$] the space of continuous functions [resp. bounded measurable functions] on D with compact support [resp. vanishing outside a bounded set]. We further denote by $C(\overline{D})$ [resp. $C_0(\overline{D})$] the space of bounded continuous functions on $\overline{D}$ [resp. the restrictions to $\overline{D}$ of functions in $C_0(R^d)$]. Suppose that the Dirichlet form $\mathscr{E}$ is *regular* on $L^2(\overline{D})$ rather than on $L^2(D)$ in the sense that $H^1(D) \cap C_0(\overline{D})$ is dense in the space $H^1(D)$. This is the case for instance when the domain D is of class C in the sense that ∂D is locally expressible as a graph of a continuous function of $d-1$ variables ([16]). According to general theorems ([13]), there exists then a conservative diffusion process $\mathbf{M} = (X_t, P_x)$ on $\overline{D}$ *associated with the Dirichlet form* $\mathscr{E}$ in the sense that the transition probability $p_t(x,E) = P_x(X_t \in E)$ of $\mathbf{M}$ satisfies that

$$p_t f \text{ is a version of } T_t f \text{ for any } f \in B_0(D). \tag{1.3}$$

However we are now concerned with a highly non-trivial problem of constructing the process $\mathbf{M}$ on $\overline{D}$ with a strong Feller resolvent:

$$G_\lambda(B_0(D)) \subset C(\overline{D}), \tag{1.4}$$

which particularly implies the absolute continuity of the transition probability:

$$p_t(x,\cdot) \prec m \quad \text{for any } t > 0 \text{ and } x \in \overline{D}. \tag{1.5}$$

If both the conditions (1.3) and (1.5) are fulfilled, then we can invoke a general decomposition theorem in [13] of additive functionals (AF's in abbreviation) in the strict sense to conclude that the sample path $X_t = (X_t^1, \cdots, X_t^d)$ of $\mathbf{M}$ admits the unique decomposition

$$X_t^i - X_0^i = M_t^i + N_t^i, \quad 1 \le i \le d, \quad P_x\text{-a.s. for any } x \in \overline{D}, \tag{1.6}$$

where M_t^i are martingale additive functionals (MAF's in abbreviation) in the strict sense with covariations

$$\langle M^i, M^j\rangle_t = 2\int_0^t a_{ij}(X_s)ds, \quad 1 \le i,j \le d, \quad P_x\text{-a.s. for any } x \in \overline{D}, \tag{1.7}$$

and N_t^i are continuous additive functionals (CAF's in abbreviation) in the strict sense locally of zero energy. N_t^i are not necessarily of bounded variation (on each finite time interval) but locally of zero quadratic variation in a certain sense ([13]).

Natural questions arise:

(I) Under what condition on the domain D, there exists a conservative diffusion process $\mathbf{M} = (X_t, P_x)$ on $\overline{D}$ satisfying (1.3) and (1.5) ?
(II) Under what additional conditions on the domain D and coefficients a_{ij}, the second terms N_t^i of X_t^i are of bounded variation P_x-a.s. for any $x \in \overline{D}$?

When D is a general bounded Lipschitz domain and $a_{ij} = \frac{1}{2}\delta_{ij}, 1 \le i,j \le d$, Bass and Hsu gave affirmative answer to the both questions (I) and (II) in [2] and [3] respectively. Actually [2] and its refinement [13; Example 5.2.2] gave an explicit expression of N_t^i as

$$N_t^i = \frac{1}{2}\int_0^t n_i(X_s)dL_s, \quad 1 \le i \le d, \quad P_x-\text{a.s. for any } x \in \overline{D}, \tag{1.8}$$

where $\mathbf{n} = (n_1, \cdots, n_d)$ is the inward unit normal vector at the boundary ∂D and L_t is a positive continuous additive functional (a PCAF in abbreviation) in the strict sense associated with the surface measure on ∂D; the local time of X_t on the boundary. In this case, the diffusion $\mathbf{M}$ is called the (normally) *reflecting Brownian motion on* $\overline{D}$ and the decomposition (1.6) with (1.8) is called its *Skorohod representation*. The first process $(M_t^1, \cdots, M_t^d)$ appearing in (1.6) is the standard d-dimensional Brownian motion starting at the origin in this case.

On the other hand, by extending a work of S.R.S.Varadhan and R.J.Williams on an infinite two-dimensional wedge [22], DeBlassie and Toby [7] have formulated under a submartingale problem a normally reflecting Brownian motion on a two-dimensional standard outward cusp domain

$$C = \left\{(x,y) \in R^2 : y \ge |x|^\gamma\right\}, \quad 0 < \gamma < 1,$$

and constructed it from the normally reflecting Brownian motion on the upper half plane by means of a conformal map and a random time change. They have also shown in [8] that the constructed process admits the Skorohod representation if $\gamma > \frac{1}{2}$ but otherwise the process starting at the origin fails to be a semimartingale. By thinking of the direct product of the DeBlassie-Toby reflecting Brownian motion on C with the standard $d-2$-dimensional Brownian motion, we see that $\frac{1}{2}$ is still the critical value of the Hölder exponent for the semi-martingale property of the reflecting Brownian motion on the special Hölder domain $C \times R^{d-2} \subset R^d$.

It is therefore tempting to consider the problem (I) for a general Hölder domain D and further look for a critical value of the Hölder exponent γ with regard to the question (II). In this paper, we do not deal with a most general Hölder domain. However we assume that D is a general (not necessarily bounded) Lipschitz domain allowing locally finite number of outward or inward cusp boundary points with Hölder exponents uniformly bounded away from zero. Our first aim is to give an affirmative answer to the problem (I) (Theorem 2.1 and Theorem 2.2) by employing the PDE methods of Stampacchia and Moser. We then assume that

$$\partial_j a_{ij} \in L^\infty_{loc}(D), \quad 1 \le i,j \le d, \tag{1.9}$$

and give an affirmative answer to the question (II) under the condition that the Hölder exponent at each outward cusp boundary point is greater than $\frac{1}{2}$ regardless the dimension d. Actually an explicit expression of N_t^i using the boundary local time L_t will be derived in this case by invoking an extended version of a general theorem in [13] to characterize N_t^i and by combining the Sobolev inequalities obtained in Sect. 3 with the upper bounds of transition functions due to Carlen-Kusuoka-Stroock [5] (Theorem 2.3).

Furthermore, we shall see that the diffusion process constructed in Theorem 2.2 can be, under the condition that $\partial_j a_{ij} \in L^\infty(D)$, related to a submartingale problem (Theorem 2.4), and accordingly, identified in law with Varadhan-Williams's [resp. DeBlassie-Toby's] normally reflecting Brownian motion when $a_{ij}(x) = \frac{1}{2}\delta_{ij}$ and $\overline{D}$ is a wedge [resp. a cusp C] in R^2.

The present paper is an essential improvement of the previous one [14] where we gave affirmative answers to questions (I) and (II) only under the restriction that the Hölder exponents at cusps are uniformly greater than $\frac{d-1}{d}$, which was technically required in getting a modified Sobolev inequality of Moser's type - a key inequality in our construction of a strong Feller resolvent. This requirement now turns out to be unnecessary thanks to a specific transformation of a standard cusp domain onto a rectangular set exhibited in the last section.

In the next section, we shall formulate a precise condition on the domain D and state main theorems answering the questions (I) and (II). Their proof will be carried out in the subsequent sections.

2 Statement of main theorems

Let F be a real valued function defined on a set E $\left(\subset R^k\right)$ including the origin such that $F(x) = \alpha|x|^\gamma + f(x)$, where $0 < \gamma < 1,\ \alpha \in R$, and f is a k-dimensional Lipschitz continuous function vanishing at the origin. Here $|\cdot|$ denotes the Euclidean norm. In this paper we call such F *a Hölder function* and we denote its Hölder exponent, Hölder constant and Lipschitz constant respectively by

$$\mathrm{Exp}(F) = \gamma, \qquad \mathrm{Höl}(F) = \alpha,$$
$$\mathrm{Lip}(F) = \mathrm{Lip}(f) = \min\left\{K > 0 : |f(x) - f(y)| \le K|x - y|,\ x, y \in E\right\}.$$

For $x = (x_1, \cdots, x_d) \in R^d$, we let $x' = (x_1, \cdots, x_{d-1})$ so that $x = (x', x_d)$.

Let us now consider the following condition (H) on a domain $D \subset R^d$ with $d \ge 2$:

(H) There are four constants $\gamma \in (0,1),\ \delta > 0,\ A \ge 1,\ M > 0$ and a locally finite open covering $\{U_j\}_{j\in J}$ of ∂D satisfying the following properties:

(i) For each $j \in J$, there are a Hölder function F_j of $d-1$ variables and a constant $r_j > \delta$ such that
   F_j is defined on the $d-1$-dimensional ball centered at the origin with radius r_j,
   $\mathrm{Exp}(F_j) \ge \gamma$,

$\text{Höl}(F_j) = 0$, or $1/A \le \text{Höl}(F_j) \le A$, or $-A \le \text{Höl}(F_j) \le -1/A$,
$\text{Lip}(F_j) \le M$,
$U_j \cap D = \{\zeta = (\zeta', \zeta_d) : |\zeta| < r_j,\ F_j(\zeta') < \zeta_d\}$, for some Cartesian coordinate system $\zeta = (\zeta', \zeta_d)$.

(ii) $\partial D \subset \bigcup_{j \in J} \widetilde{U_{j,\delta}}$, where $\widetilde{U_{j,\delta}} = \{x \in U_j : \text{dist}(x, \partial U_j) > \delta\}$.

When D is bounded, condition (H) reduces to a simple one that every point x of ∂D has a neighbourhood U_x such that $\partial D \cap U_x$ is the graph of a Hölder function of $d-1$ variables.

For later convenience, we let

$$\begin{aligned} J_+ &= \{j \in J : \text{Höl}(F_j) > 0\}, \\ J_0 &= \{j \in J : \text{Höl}(F_j) = 0\}, \\ J_- &= \{j \in J : \text{Höl}(F_j) < 0\}. \end{aligned}$$

For $j \in J$, denote by a_j $(\in \partial D)$ the origin of U_j with respect to the coordinate system ζ. a_j is called an *outward* [resp. *inward*] *cusp boundary point* of D if $j \in J_+$ [resp. $j \in J_-$].

In what follows, we work with the Dirichlet form $\left(\mathscr{E}, H^1(D)\right)$ on $L^2(D)$ given by (1.1) and (1.2). Let $\{G_\lambda, \lambda > 0\}$ be the associated resolvent on $L^2(D)$. It is then Markovian in the sense that $0 \le \lambda G_\lambda f \le 1$ whenever $0 \le f \le 1$ and it is well defined as a bounded linear operator on $L^p(D)$ for any $p \in [1, \infty]$. Denote by $C_\infty(\overline{D})$ the space of those functions in $C(\overline{D})$ vanishing at infinity.

Theorem 2.1 *Assume that a domain $D \subset R^d$ satisfies condition (H). Then G_λ enjoys the following properties :*

(i) $G_\lambda\left(L^2(D) \cap L^p(D)\right) \subset C\left(\overline{D}\right), \qquad p > 1 + (d-1)/\gamma.$

(ii) $G_\lambda\left(C_\infty\left(\overline{D}\right)\right)$ *is a dense subspace of* $C_\infty\left(\overline{D}\right)$.

(iii) *There is a function $G_\lambda(x, y)$ continuous on $\overline{D} \times \overline{D}$ off diagonal such that*

$$G_\lambda f(x) = \int_{\overline{D}} G_\lambda(x, y) f(y)\, dy, \qquad x \in \overline{D},\ f \in C_\infty\left(\overline{D}\right). \tag{2.1}$$

As will be seen in Sect. 4, Theorem 2.1 is still valid under condition (A) stated in Sect. 3. Condition (A) is weaker but less concrete than (H) so that we employ (H) in formulating main theorems.

Theorem 2.1 (i) means that G_λ has a strong Feller property. By virtue of Theorem 2.1 (ii) and the Hille-Yosida theorem, there exists a strongly continuous Markovian semigroup $\{T_t, t > 0\}$ on $C_\infty\left(\overline{D}\right)$ such that $G_\lambda f = \int_0^\infty e^{-\lambda t} T_t f\, dt,\ f \in C_\infty\left(\overline{D}\right)$. We have then a Feller transition function by $T_t f(x) = \int_{\overline{D}} p_t(x, dy) f(y)$, which gives rise to a Hunt process (cf. [13; Theorem A.2.2]) $\mathbf{M} = (X_t, P_x)$ on $\overline{D}$ such that

$$P_x\left(X_t \in A\right) = p_t(x, A), \qquad t > 0,\ x \in \overline{D},\ A \in \mathscr{B}\left(\overline{D}\right).$$

$\mathbf{M}$ is associated with the Dirichlet form $\left(\mathscr{E}, H^1(D)\right)$ of (1.2) since the resolvent G_λ is. Since $G_\lambda\left(C_0\left(\overline{D}\right)\right)$ is dense in the Dirichlet space, Theorem 2.1

(ii) implies that the Dirichlet form $\mathscr{E}$ is regular. Therefore we can apply general theorems in [13] to the associated pair $\mathscr{E}$ and $\mathbf{M}$. In particular, $p_t(x,\cdot)$ is absolutely continuous because $G_\lambda(x,\cdot)$ is ([13; Theorem 4.2.4]). Since $\mathscr{E}$ has the strong local property and a_{ij} are uniformly bounded, we can invoke [13; Theorem 4.5.4]) and [13; Theorem 5.7.2, Example 5.7.1] to conclude that $\mathbf{M}$ is a conservative diffusion process on $\overline{D}$. Summing up what has been mentioned, we get

Theorem 2.2 *Under condition (H), there exists a conservative diffusion process* $\mathbf{M} = (X_t, P_x)$ *on* $\overline{D}$ *with resolvent* G_λ *of Theorem 2.1.* $\mathbf{M}$ *is associated with the Dirichlet form given by (1.1) and (1.2), and the transition function* $p_t(x,\cdot)$ *of* $\mathbf{M}$ *satisfies (1.3) and (1.5).*

We next formulate a decomposition of the sample path of $\mathbf{M}$ and its Skorohod representation.

Theorem 2.3 *Consider a domain* $D \subset R^d$ *satisfying condition (H).*
(i) *The sample path* $X_t = (X_t^1, \cdots, X_t^d)$ *of the conservative diffusion process* $\mathbf{M}$ *on* $\overline{D}$ *constructed in Theorem 2.2 admits a unique decomposition (1.6) with MAF's* M_t^i *in the strict sense satisfying (1.7) and CAF's* N_t^i *in the strict sense locally of zero energy.*
(ii) *Assume condition (1.9) for* a_{ij}. *We also require the condition that*

$$\operatorname{Exp}(F_j) > \frac{1}{2}, \qquad j \in J_+, \tag{2.2}$$

for the domain D. *Then* N_t^i *has the following representation :*

$$N_t^i = \sum_{j=1}^d \int_0^t \left(\partial_j a_{ij}\right)(X_s)\, ds + \sum_{j=1}^d \int_0^t a_{ij}\,(X_s)\, n_j\,(X_s)\; dL_s, \tag{2.3}$$
$$1 \le i \le d,\ t \ge 0,\ P_x - a.s.\ \text{for any } x \in \overline{D},$$

where L_t *is a unique PCAF in the strict sense with Revuz measure being the surface measure on* ∂D.

Note that (2.3) reduces to (1.8) when $a_{ij} = \frac{1}{2}\delta_{ij}$. The above three theorems extend those results of R.F.Bass and P.Hsu in [2] and [3] formulated for a general bounded Lipschitz domain D and for $a_{ij} = \frac{1}{2}\delta_{ij}$.

Let us denote by Ξ_+ the set of all outward cusp boundary points. $C_b^2\left(\overline{D}\right)$ will stand for the set of twice continuously differentiable functions on R^d that are together with their first and second partial derivatives bounded on $\overline{D}$.

Theorem 2.4 *Under condition (H) for the domain and the assumption that*

$$\partial_i a_{ij} \in L^\infty(D), \qquad 1 \le i, j \le d, \tag{2.4}$$

the conservative diffusion process $\mathbf{M} = (X_t, P_x)$ *of Theorem 2.2 enjoys the following properties : for each* $x \in \overline{D}$, *1.* $P_x\,(X_0 = x) = 1$,

2. $f(X_t) - \int_0^t \sum_{i,j=1}^d \left[\partial_i \left(a_{ij}\, \partial_j f\right)\right](X_s)\, ds$ *is a* P_x*-submartingale,*

whenever $f \in C_b^2\left(\overline{D}\right)$, f *is constant in a neighbourhood of* Ξ_+ *and*

$$\sum_{i,j=1}^d \partial_i f(x)\, a_{ij}(x)\, n_j(x) \geq 0 \qquad \sigma\text{-a.e. on } \partial D, \tag{2.5}$$

3. $E_x\left[\int_0^\infty I_{\Xi_+}(X_s)\, ds\right] = 0.$

When $d = 2$, $a_{ij}(x) = \frac{1}{2}\delta_{ij}$ and $\overline{D} = C$ the standard outward cusp domain, R.D. DeBlassie and E.H. Toby [7] have shown the existence and the uniqueness of the corresponding submartingale problem for a probability measure P_x on $\Omega = \{\omega : \omega$ is a continuous function from $[0,\infty)$ into $C\}$: for each fixed $x \in C$,

1. $P_x\,(\omega(0) = x) = 1$,
2. $f(\omega(t)) - \dfrac{1}{2}\int_0^t \Delta f\left(\omega(s)\right) ds$ is a P_x-submartingale
   whenever $f \in C_b^2(C)$, f is constant in a neighbourhood of the origin
   and $\nabla f(x) \cdot \mathbf{n}(x) \geq 0$ on ∂C,
3. $E_x\left[\int_0^\infty I_0\left(\omega(s)\right) ds\right] = 0.$

Hence, by virtue of Theorem 2.4, the diffusion process of Theorem 2.2 coincides in law with DeBlassie-Toby's one in [7] in this special case.

In exactly the same way, we see that, when $d = 2$, $a_{ij}(x) = \frac{1}{2}\delta_{ij}$ and $\overline{D}$ is a wedge $\{\theta : 0 \leq \theta \leq \xi\} \subset R^2$ for a fixed $\xi \in (0, 2\pi)$, the diffusion process of Theorem 2.2 is identical in law with Varadhan-R.Williams's normally reflecting Brownian motion [22].

3 L^p-estimate, local estimate and Harnack inequality

In this and the next sections, we shall work under another condition (A) on a domain $D \subset R^d$ which will be seen to be more general than (H) (Proposition 4.1). In this section, we derive some estimates for harmonic solutions of equations associated with $\left(\mathcal{E}, H^1(D)\right)$ under condition (A).

In order to state condition (A), we employ the following notations:

$$\begin{aligned}
B(a,\rho) &= \{x \in R^d : |x - a| < \rho\},\\
B(\rho) &= B(0,\rho),\\
B_+(\rho) &= \{(x', x_d) \in B(\rho) : x_d > 0\},\\
C_\gamma(\rho) &= \{(x', x_d) \in B(\rho) : |x'|^\gamma < x_d\},\\
Q_\gamma(\rho) &= \{(x', x_d) \in B(\rho) : -|x'|^\gamma < x_d\},
\end{aligned}$$

for $a \in R^d,\ \rho > 0,\ \gamma \in (0,1)$. For a Lipschitz mapping Φ from a set $E \subset R^k$ into R^l such that $|\Phi(x) - \Phi(y)| \le K|x-y|,\ x, y \in E$, for some constant $K > 0$, we also denote by $\mathrm{Lip}(\Phi)$ the smallest constant K of this property.

We now state condition (A) on a domain $D \subset R^d$:

(A) The following properties hold for an at most countable index set I, a constant $\gamma^* \in (0,1)$ and positive constants $\rho^*,\ r^*,\ M^*$:

(i) There are a point $a_k \in \partial D$ and its neighbourhood V_k associated with each $k \in I$ such that

(i-1) $D \cap V_k \cap V_l = \emptyset, \quad k,\ l \in I,\ k \neq l$;

(i-2) there are a constant $\gamma_k \in [\gamma^*, 1)$ and a one to one mapping Φ_k from $B(\rho^*)$ onto V_k with $\Phi_k(0) = a_k,\ V_k \cap D$ equals either $\Phi_k\left(C_{\gamma_k}(\rho^*)\right)$ or $\Phi_k\left(Q_{\gamma_k}(\rho^*)\right)$, $\mathrm{Lip}(\Phi_k) \le M^*$, $\mathrm{Lip}(\Phi_k^{-1}) \le M^*$.

(ii) For any $a \in \partial D \setminus \bigcup_{k\in I} V_k$, there are its neighbourhood W_a and a one to one mapping Ψ_a from $B(r^*)$ onto W_a such that $\Psi_a(0) = a,\ \Psi_a(B_+(r^*)) = W_a \cap D,\ \mathrm{Lip}(\Psi_a) \le M^*,\ \mathrm{Lip}(\Psi_a^{-1}) \le M^*$.

In our previous paper [14], we considered the same condition as above, but we assumed $\gamma^* > (d-1)/d$. Further we did not consider the case $V_k \cap D = \Phi_k\left(Q_{\gamma_k}(\rho^*)\right)$, namely, we assumed in [14] that every a_k is an outward cusp boundary point but not an inward one. Under those assumptions, we got the same estimates as in this section following the PDE argument due to Stampacchia [18] and Moser [17]. The PDE argument is based on a Sobolev inequality of Moser's type formulated in Proposition 3.1 below. As will be proved in the last section, we need not the previous assumption $\gamma^* > (d-1)/d$ for the validity of Proposition 3.1. Once it is established, we can follow the PDE argument developed in [14] without any change so that we shall state the results of this section omitting the proof and only referring to the corresponding results in [14].

In the rest of this section, we assume (A).

First of all, we note the following easily verifiable observation : if ψ is a one to one mapping from an open set $U \subset R^d$ onto an open set in R^d with $\mathrm{Lip}\left(\psi^{-1}\right) \le M$ and if $B(\widetilde{a}, r) \subset U$ and $\psi(\widetilde{a}) = a$, then

$$B(a, r/M) \subset \psi\left(B(\widetilde{a}, r)\right). \tag{3.1}$$

This observation particularly leads us to the following property of the domain D (see [14; Lemma 3.1]). We set

$$\begin{aligned} I_C &= \left\{k \in I : V_k \cap D = \Phi_k\left(C_{\gamma_k}(\rho^*)\right)\right\}, \\ I_Q &= \left\{k \in I : V_k \cap D = \Phi_k\left(Q_{\gamma_k}(\rho^*)\right)\right\}. \end{aligned}$$

a_k for $k \in I_C$ [resp. I_Q] may be called an outward [resp. inward] cusp boundary point. A collection of open sets is said to have a finite intersection property if there exists an integer M such that any subcollection of cardinality greater than M has an empty intersection.

Lemma 3.1 *For any $\rho \in (0, \rho^*]$, there exist positive constants r_ρ and m_ρ for which D satisfies the following : For every $a \in \partial D^\#_\rho \equiv \partial D \setminus \bigcup_{k \in I} \Phi_k(B(\rho))$, there are a neighbourhood $V_{\rho,a}$ of a and a one to one mapping $\psi_{\rho,a}$ from $B(r_\rho)$ onto $V_{\rho,a}$ such that $\psi_{\rho,a}(0) = a$, $\psi_{\rho,a}(B_+(r_\rho)) = V_{\rho,a} \cap D$, $\mathrm{Lip}(\psi_{\rho,a}) \le m_\rho$, $\mathrm{Lip}(\psi^{-1}_{\rho,a}) \le m_\rho$.*

Furthermore, for any $r \in (0, r_\rho]$ we have a subset $A \subset \partial D^\#_\rho$ and a positive constant $\eta = \eta(r)$ such that $\{\psi_{\rho,a}(B_+(r))\}_{a \in A}$ has a finite intersection property, and for every $b \in \partial D$, the set $B(b, \eta) \cap D$ is contained in one of the following sets: $\Phi_k(C_{\gamma_k}(\rho))$ for $k \in I_C$, $\Phi_k(Q_{\gamma_k}(\rho))$ for $k \in I_Q$, $\psi_{\rho,a}(B_+(r))$ for $a \in A$.

In the following we set

$$\begin{aligned} C^*_k(\rho) &= \Phi_k\left(C_{\gamma_k}(\rho)\right), & k &\in I_C, \\ Q^*_k(\rho) &= \Phi_k\left(Q_{\gamma_k}(\rho)\right), & k &\in I_Q, \\ B^*_a(r) &= \psi_{\rho,a}(B_+(r)), & a &\in \partial D^\#_\rho, \end{aligned}$$

for $0 < \rho \le \rho^*$, $0 < r \le r_\rho$. Since a Sobolev inequality of Moser's type formulated in Lemma 2 in [17] is valid for $u \in H^1(B_+(r))$, we immediately obtain by means of the map $\psi_{\rho,a}$ in Lemma 3.1 that for any $\rho \in (0, \rho^*]$, $\kappa \in (0, 1]$, $q \in [2, 2d/(d-2)]$ ($q \in [2, \infty)$ if $d = 2$) there is a positive constant $C_1 = C_1(\rho, \kappa, q)$ such that

$$\left(\int_{B^*_a(r)} |u|^q dx\right)^{1/q} \le C_1\, r^{d\left(\frac{1}{q} - \frac{1}{2}\right)} \left\{\int_N |u|^2 dx + r^2 \sum_{i=1}^d \int_{B^*_a(r)} |\partial_i u|^2 dx\right\}^{1/2}, \tag{3.2}$$

for $u \in H^1\left(B^*_a(r)\right)$, $N \subset B^*_a(r)$ with $|N| \ge \kappa |B^*_a(r)|$, $0 < r \le r_\rho$, and $a \in \partial D^\#_\rho$. Here $|E|$ denotes the Lebesgue measure for measurable sets E. (C_1 and the other constants C_2, C_3 etc. below also depend on d, ρ^*, r^*, M^* and in some cases on γ^* and Λ. However we omit indicating them.)

Actually (3.2) also holds for $u \in H^1\left(C^*_k(\rho)\right)$ and $u \in H^1\left(Q^*_k(\rho)\right)$:

Proposition 3.1 (i) *For any $\kappa \in (0, 1]$ there is a positive constant $C_2 = C_2(\kappa)$ such that*

$$\left(\int_{C^*_k(\rho)} |u|^q\, dx\right)^{1/q} \le C_2 \rho^{\frac{d-1+\gamma_k}{\gamma_k}\left(\frac{1}{q} - \frac{1}{2}\right)} \times \left\{\int_N |u|^2\, dx + \rho^2 \sum_{i=1}^d \int_{C^*_k(\rho)} |\partial_i u|^2\, dx\right\}^{1/2}, \tag{3.3}$$

for $u \in H^1\left(C^*_k(\rho)\right)$, $N \subset C^*_k(\rho)$ with $|N| \ge \kappa |C^*_k(\rho)|$, $0 < \rho \le \rho^*$, $2 \le q \le 2(d - 1 + \gamma_k)/(d - 1 - \gamma_k)$, and $k \in I_C$.

(ii) *Let $2 \le q < \infty$ in case $d = 2$ or $2 \le q \le 2d/(d-2)$ in case $d \ge 3$. Then for any $\kappa \in (0, 1]$ there is a positive constant $C_3 = C_3(\kappa, q)$ such that*

$$\left(\int_{Q_k^*(\rho)} |u|^q \, dx\right)^{1/q} \le C_3 \rho^{d\left(\frac{1}{q}-\frac{1}{2}\right)} \left\{\int_N |u|^2 \, dx + \rho^2 \sum_{i=1}^d \int_{Q_k^*(\rho)} |\partial_i u|^2 \, dx\right\}^{1/2}, \tag{3.4}$$

for $u \in H^1\left(Q_k^*(\rho)\right)$, $N \subset Q_k^*(\rho)$ *with* $|N| \ge \kappa |Q_k^*(\rho)|$, $0 < \rho \le \rho^*$ *and* $k \in I_Q$.

The proof of Proposition 3.1 will be carried out in the last section by employing a specific transformation of a standard cusp domain onto a rectangular set.

The following Sobolev inequality in an ordinary sense follows from (3.2), (3.3), (3.4) and Lemma 3.1 ([14; Proposition 3.2 (ii)]).

Proposition 3.2 (i) *There is a positive constant* C_4 *such that*

$$\left(\int_D |u|^q dx\right)^{1/q} \le C_4 \left\{\int_D |u|^2 dx + \sum_{i=1}^d \int_D |\partial_i u|^2 dx\right\}^{1/2}, \tag{3.5}$$

for $u \in H^1(D)$, $2 \le q \le 2(d-1+\gamma^*)/(d-1-\gamma^*)$.

(ii) *Assume the absence of outward cusp boundary point* : $I_C = \emptyset$. *Then the above statement is valid for* $2 \le q \le 2d/(d-2)$ *in case* $d \ge 3$ *and for* $2 \le q < \infty$ *in case* $d = 2$.

We denote the norm of the Sobolev space $H^1(E)$ by $\|\cdot\|_{H^1(E)}$. For an open set $E \subset D$, let us consider the following spaces :

$$\widehat{C}(E) = \left\{u \in C^1(E) : \|u\|_{H^1(E)} < \infty, \ u = 0 \text{ on } \partial E \cap D\right\}, \tag{3.6}$$

$$\widehat{H}(E) = \text{the completion of } \widehat{C}(E) \text{ with respect to the norm } \|\cdot\|_{H^1(E)}. \tag{3.7}$$

Note that $\widehat{H}(E)$ coincides with $H_0^1(E)$ if $\overline{E} \subset D$. When $E = C_k^*(\rho)$, $Q_k^*(\rho)$ or $B_a^*(r)$, we can derive the following Sobolev inequalities from Proposition 3.1 ([14; Proposition 3.3]).

Proposition 3.3 (i) *For any* $\delta \in (0,1)$, *there is a positive constant* $C_5 = C_5(\delta)$ *such that*

$$\left(\int_{C_k^*(\rho)} |u|^q \, dx\right)^{1/q} \le C_5 \left(\sum_{i=1}^d \int_{C_k^*(\rho)} |\partial_i u|^2 \, dx\right)^{1/2}, \tag{3.8}$$

for $u \in \widehat{H}(C_k^*(\rho))$, $0 < \rho \le \delta\rho^*$, $2 \le q \le 2(d-1+\gamma_k)/(d-1-\gamma_k)$, $k \in I_C$.

(ii) *For any* $\delta \in (0,1)$ *and for any* $2 \le q \le 2d/(d-2)$ $(2 \le q < \infty$ *if* $d = 2)$, *there is a positive constant* $C_6 = C_6(\delta, q)$ *such that*

$$\left(\int_{Q_k^*(\rho)} |u|^q \, dx\right)^{1/q} \le C_6 \left(\sum_{i=1}^d \int_{Q_k^*(\rho)} |\partial_i u|^2 \, dx\right)^{1/2}, \tag{3.9}$$

for $u \in \widehat{H}(Q_k^*(\rho))$, $0 < \rho \le \delta\rho^*$, $k \in I_Q$.

(iii) *Let* $0 < \rho \le \rho^*$, $0 < \delta < 1$ *and* $2 \le q \le 2d/(d-2)$ $(2 \le q < \infty$ *if* $d = 2)$. *Then there is a positive constant* $C_7 = C_7(\rho, \delta, q)$ *satisfying*

$$\left(\int_{B_a^*(r)} |u|^q\,dx\right)^{1/q} \le C_7 \left(\sum_{i=1}^d \int_{B_a^*(r)} |\partial_i u|^2\,dx\right)^{1/2}, \tag{3.10}$$

for $u \in \widehat{H}(B_a^*(r)),\ 0 < r \le \delta r_\rho,\ a \in \partial D_\rho^\#$.

We now turn to the Dirichlet form $\left(\mathscr{E}, H^1(D)\right)$ given by (1.1) and (1.2). We also consider the following form $\left(\mathscr{E}_E, \widehat{H}(E)\right)$ for an open set $E \subset D$.

$$\mathscr{E}_E(u,v) = \sum_{i,j=1}^d \int_E \partial_i u(x)\partial_j v(x) a_{ij}(x)\,dx, \qquad u,\ v \in \widehat{H}(E), \tag{3.11}$$

with $a_{ij},\ 1 \le i,j \le d$ satisfying (1.1). Since $\left(\mathscr{E}_E, \widehat{H}(E)\right)$ is a Dirichlet form on $L^2(E)$, we have the associated Markovian resolvent $\{G_{E,\lambda}, \lambda > 0\}$ on $L^2(E)$.

Let T be a functional defined by

$$\langle T, \varphi\rangle = \int_D f_0 \varphi\,dx + \sum_{i=1}^d \int_D f_i \partial_i \varphi\,dx, \qquad \varphi \in H^1(D), \tag{3.12}$$

for $f_i \in L^2(D),\ i = 0, 1, \cdots, d$. Since T is a continuous linear functional on $H^1(D)$, there is for each $\lambda > 0$ a unique element $u \in H^1(D)$ such that

$$\mathscr{E}_\lambda(u,\varphi) = \langle T, \varphi\rangle, \qquad \varphi \in H^1(D). \tag{3.13}$$

Here $\mathscr{E}_\lambda(\ ,\ ) = \mathscr{E}(\ ,\ ) + \lambda(\ ,\ )_{L^2(D)}$. We denote this function u by $G_\lambda T$. If T is defined by (3.12) with D and $H^1(D)$ replaced by E and $\widehat{H}(E)$ respectively and if every f_i belongs to $L^2(E)$, then we have for each $\lambda > 0$ a unique $u \in \widehat{H}(E)$ denoted by $G_{E,\lambda}T$ such that

$$\mathscr{E}_{E,\lambda}(u,\varphi) = \langle T, \varphi\rangle, \qquad \varphi \in \widehat{H}(E), \tag{3.14}$$

where $\mathscr{E}_{E,\lambda}(\ ,\ ) = \mathscr{E}_E(\ ,\ ) + \lambda(\ ,\ )_{L^2(E)}$.

Obviously $G_\lambda T$ [resp. $G_{E,\lambda}T$] coincides with $G_\lambda f$ [resp. $G_{E,\lambda} f$] in the case where $\langle T, \varphi\rangle = (f, \varphi)$ for $f \in L^2(D)$ [resp. $f \in L^2(E)$]. If $E = C_k^*(\rho),\ Q_k^*(\rho)$ or $B_a^*(r)$, then the norm $\|\cdot\|_{H^1(E)}$ is equivalent to $\|\cdot\|_{\widehat{H}(E)} \equiv \mathscr{E}_E(\cdot,\cdot)^{1/2}$ in view of Proposition 3.3. Therefore there exists $G_{E,0}T \in \widehat{H}(E)$ satisfying (3.14) with $\lambda = 0$.

Using Sobolev inequalities (3.5), (3.8), (3.9), (3.10) and following a standard argument as in [18; Theorem 4.1] (see also [10]), we can get the following L^p-estimates.

Theorem 3.1 (i) *Let* $p > (d-1)/\gamma^* + 1$ *and* $\lambda > 0$. *Then it holds that, for some* $C_8 = C_8(p,\lambda) > 0$,

$$\|G_\lambda T\|_{L^\infty(D)} \le C_8 \sum_{i=0}^d \left(\|f_i\|_{L^2(D)} + \|f_i\|_{L^p(D)}\right), \tag{3.15}$$

where T is given by (3.12) with $f_i \in L^2(D) \cap L^p(D),\ i = 0, 1, \cdots, d$.
(ii) *Let $k \in I_C,\ p > (d-1)/\gamma_k + 1$ and $0 < \delta < 1$. Then there is a positive constant $C_9 = C_9(p, \delta)$ such that*

$$\|G_{E,\lambda} T\|_{L^\infty(E)} \leq C_9 \rho^{\frac{1}{p}\left(p-1-\frac{d-1}{\gamma_k}\right)} \sum_{i=0}^{d} \|f_i\|_{L^p(E)}, \tag{3.16}$$

where $E = C_k^*(\rho),\ 0 < \rho \leq \delta\rho^*,\ \lambda \geq 0$, and T is a continuous linear functional given by (3.12) with $f_i \in L^p(E)$ and $\varphi \in \widehat{H}(E)$.
(iii) *Let $p > d$ and $0 < \delta < 1$. Then there is a positive constant $C_{10} = C_{10}(p, \delta)$ such that*

$$\|G_{E,\lambda} T\|_{L^\infty(E)} \leq C_{10} \rho^{(p-d)/p} \sum_{i=0}^{d} \|f_i\|_{L^p(E)}, \tag{3.17}$$

where $E = Q_k^*(\rho),\ k \in I_Q,\ 0 < \rho \leq \delta\rho^*,\ \lambda \geq 0$, and T is a continuous linear functional given by (3.12) with $f_i \in L^p(E)$ and $\varphi \in \widehat{H}(E)$.
(iv) *Let $0 < \rho \leq \rho^*,\ p > d$ and $0 < \delta < 1$. Then there is a positive constant $C_{11} = C_{11}(p, \delta)$ such that*

$$\|G_{E,\lambda} T\|_{L^\infty(E)} \leq C_{11} r^{(p-d)/p} \sum_{i=0}^{d} \|f_i\|_{L^p(E)}, \tag{3.18}$$

for $\lambda \geq 0,\ E = B_a^*(r),\ 0 < r \leq \delta r_\rho,\ a \in \partial D_\rho^\#$, and for T defined by (3.12) with $E,\ \widehat{H}(E),\ f_i \in L^p(E)$ instead of $D,\ H^1(D),\ f_i \in L^p(D)$ respectively.

We are next concerned with local estimates for subsolutions of the equations associated with $\mathscr{E}_E$. A function $u \in H^1(E)$ is called a subsolution if

$$\mathscr{E}_E(u, \varphi) \leq 0, \qquad \varphi \geq 0,\ \varphi \in \widehat{H}(E). \tag{3.19}$$

In the same way as in [18; Theorem 5.1] or in [17; Theorem 1], we obtain the following local estimates from Proposition 3.3 or (3.2) ([14; Theorem 3.2]).

Theorem 3.2 (i) *Let $0 < \rho \leq \delta\rho^*$ for some $\delta \in (0,1)$ and $E = C_k^*(\rho)$ with $k \in I_C$. Then every nonnegative subsolution $u \in H^1(E)$ of (3.19) satisfies*

$$\operatorname*{ess\,sup}_{C_k^*(s)} u \leq C_{12}(\rho - s)^{-\frac{d-1+\gamma_k}{2\gamma_k}} \left(\int_{C_k^*(\rho)} u^2 \, dx \right)^{1/2}, \quad 0 < s < \rho, \tag{3.20}$$

for some $C_{12} = C_{12}(\delta) > 0$.
(ii) *Let $0 < \rho \leq \delta\rho^*$ for some $\delta \in (0,1)$ and $E = Q_k^*(\rho)$ with $k \in I_Q$. Then every nonnegative subsolution $u \in H^1(E)$ of (3.19) satisfies*

$$\operatorname*{ess\,sup}_{Q_k^*(s)} u \leq C_{13}(\rho - s)^{-d/2} \left(\int_{Q_k^*(\rho)} u^2 \, dx \right)^{1/2}, \quad 0 < s < \rho, \tag{3.21}$$

for some $C_{13} = C_{13}(\delta) > 0$.

(iii) *Let* $0 < \rho \le \rho^*,\ 0 < \delta < 1,\ 0 < r \le \delta r_\rho,\ a \in \partial D^\#_\rho$ *and* $E = B^*_a(r)$. *Then every nonnegative subsolution* $u \in H^1(E)$ *of (3.19) satisfies*

$$\operatorname*{ess\,sup}_{B^*_a(s)} u \le C_{14}(r-s)^{-d/2}\left(\int_{B^*_a(r)} u^2\,dx\right)^{1/2}, \quad 0 < s < r, \tag{3.22}$$

for some $C_{14} = C_{14}(\rho, \delta) > 0$.

If $u \in H^1(E)$ satisfies

$$\mathscr{E}_{E,\lambda}(u, \varphi) = 0, \qquad \varphi \in \widehat{H}(E), \tag{3.23}$$

for some $\lambda \ge 0$, then $u \vee 0$ and $(-u) \vee 0$ are both nonnegative subsolutions of (3.19). Therefore as an immediate consequence of Theorem 3.2 we get the following result.

Corollary 3.1 (i) *Let* $0 < \rho \le \delta\rho^*$ *for some* $\delta \in (0, 1)$ *and* $E = C^*_k(\rho)$ *with* $k \in I_C$ $\big[$ *resp.* $Q^*_k(\rho)$ *with* $k \in I_Q$ $\big]$. *Then every solution* $u \in H^1(E)$ *of (3.23) satisfies (3.20)* [*resp. (3.21)*] *with* u *being replaced by* $|u|$.

(ii) *Let* $0 < \rho \le \rho^*,\ 0 < \delta < 1,\ 0 < r \le \delta r_\rho,\ a \in \partial D^\#_\rho$ *and* $E = B^*_a(r)$. *Then every solution* $u \in H^1(E)$ *of (3.23) satisfies (3.22) with* u *being replaced by* $|u|$.

Finally, by means of Proposition 3.1 and Theorem 3.2, we can get the following Harnack inequality for solutions $u \in H^1(E)$ of the equation (3.23) with $\lambda = 0$ ([14; Theorem 3.3]).

Theorem 3.3 (i) *Let* $k \in I_C$ $\big[$ *resp.* $k \in I_Q$ $\big]$, $0 < \rho \le \rho^*,\ 0 < \kappa < 1$ *and* $E = C^*_k(\rho)$ $\big[$ *resp.* $Q^*_k(\rho)$ $\big]$. *If* $u \in H^1(E)$ *is a nonnegative solution of (3.23) with* $\lambda = 0$ *and satisfies* $|\{x : u(x) \ge 1\} \cap C^*_k(\rho/2)| \ge \kappa\,|C^*_k(\rho/2)|$ $\big[$*resp.* $|\{x : u(x) \ge 1\} \cap Q^*_k(\rho/2)| \ge \kappa\,|Q^*_k(\rho/2)|$ $\big]$, *then there is a positive constant* $C_{15} = C_{15}(\kappa)$ *such that*

$$\operatorname*{ess\,inf}_{C^*_k(\rho/4)} u \ge C_{15} \quad \left[\ \textit{resp.}\quad \operatorname*{ess\,inf}_{Q^*_k(\rho/4)} u \ge C_{15}\ \right]. \tag{3.24}$$

(ii) *Let* $0 < \rho \le \rho^*,\ 0 < r \le r_\rho,\ a \in \partial D^\#_\rho,\ 0 < \kappa < 1$ *and set* $E = B^*_a(r)$. *If* $u \in H^1(E)$ *is a nonnegative solution of (3.23) with* $\lambda = 0$ *and satisfies* $|\{x : u(x) \ge 1\} \cap B^*_a(r/2)| \ge \kappa\,|B^*_a(r)|$, *then there is a positive constant* $C_{16} = C_{16}(\rho, \kappa)$ *such that*

$$\operatorname*{ess\,inf}_{B^*_a(r/4)} u \ge C_{16}. \tag{3.25}$$

4 Strong Feller resolvent

In this section, we will show that the resolvent $\{G_\lambda\}$ associated with the Dirichlet form $\left(\mathscr{E}, H^1(D)\right)$ has the same properties as those of Theorem 2.1 under condition (A). At the end of this section, we will show that condition (H) reduces to (A) and hence Theorem 2.1 follows.

Theorem 4.1 *Under condition (A), G_λ satisfies the same properties as in Theorem 2.1. Namely,*
(i) $G_\lambda\left(L^2(D)\cap L^p(D)\right)\subset C\left(\overline{D}\right), \qquad p>1+(d-1)/\gamma^*.$
(ii) $G_\lambda\left(C_\infty\left(\overline{D}\right)\right)$ *is a dense subspace of* $C_\infty\left(\overline{D}\right)$.
(iii) *There is a function $G_\lambda(x,y)$ continuous on $\overline{D}\times\overline{D}$ off diagonal such that*

$$G_\lambda f(x)=\int_{\overline{D}} G_\lambda(x,y)f(y)\,dy, \qquad x\in\overline{D},\ f\in C_\infty\left(\overline{D}\right). \tag{4.1}$$

Theorem 4.1 is obtained essentially by the same argument as in [14; Sect. 4] but we give the proof here for completeness.

Theorem 4.1 (i) is an immediate consequence of the following theorem.

Theorem 4.2 *Assume condition (A). Let $p>(d-1)/\gamma^*+1$, T be a functional given by (3.12) with $f_i\in L^2(D)\cap L^p(D)$, $i=0,1,2,\cdots,d$, and $\lambda>0$. Then $G_\lambda T$ is uniformly continuous in D and accordingly $G_\lambda T$ can be extended to a continuous function on $\overline{D}$.*

Proof Put $u=G_\lambda T$. Fix a $k\in I_C$ and an $s\in(0,\rho^*/2]$ arbitrarily. Set $E=C_k^*(s)$. Let $v\equiv G_{E,0}(T-\lambda u)\in\widehat{H}(E)$ be the solution of the equation (3.14) with $\lambda=0$ and $T=T-\lambda u$. We see by means of Theorem 3.1 (i), (ii),

$$\begin{aligned}\|v\|_{L^\infty(E)} &\le C_9(p,1/2)s^{\frac{1}{p}\left(p-1-\frac{d-1}{\gamma_k}\right)}\left\{\|f_0-\lambda u\|_{L^p(E)}+\sum_{i=1}^{d}\|f_i\|_{L^p(E)}\right\}\\ &\le c_1 s^{\frac{1}{p}\left(p-1-\frac{d-1}{\gamma^*}\right)}\sum_{i=0}^{d}\left\{\|f_i\|_{L^2(D)}+\|f_i\|_{L^p(D)}\right\},\end{aligned} \tag{4.2}$$

for some positive c_1 independent of s and k. Since $w\equiv u-v$ belongs to $H^1(E)$ and satisfies (3.23) with $\lambda=0$, following the same argument as in [17] we get by means of Theorem 3.3 (i)

$$\begin{aligned}\operatorname{Osc}\left(w;C_k^*(s/4)\right) &\le \left(1-\frac{1}{2}C_{15}(1/2)\right)\operatorname{Osc}\left(w;C_k^*(s)\right)\\ &\le c_2\operatorname{Osc}\left(w;C_k^*(s)\right),\end{aligned}$$

for $c_2\in(0,1)$ independent of s and k. Here $\operatorname{Osc}(g;F)$ denotes the oscillation of a function g over a set F : $\operatorname{Osc}(g;F)=\operatorname{ess\,sup}_F g-\operatorname{ess\,inf}_F g$. Hence

$$\begin{aligned}&\operatorname{Osc}\left(u;C_k^*(s/4)\right)\le\operatorname{Osc}\left(v;C_k^*(s/4)\right)+\operatorname{Osc}\left(w;C_k^*(s/4)\right)\\ &\le 2\|v\|_{L^\infty(E)}+c_2\operatorname{Osc}\left(w;C_k^*(s)\right)\le 4\|v\|_{L^\infty(E)}+c_2\operatorname{Osc}\left(u;C_k^*(s)\right).\end{aligned}$$

Combining this with (4.2) and using [18; Lemma 7.3], we get

$$\operatorname{Osc}\left(u; C_k^*(s)\right) \le c_3 s^{\xi_1}, \qquad 0 < s \le \rho^*/4,\ k \in I_C, \tag{4.3}$$

for some constants $c_3 > 0$ and $\xi_1 \in (0,1)$. In the same way we also get

$$\operatorname{Osc}\left(u; Q_k^*(s)\right) \le c_4 s^{\xi_2}, \qquad 0 < s \le \rho^*/4,\ k \in I_Q, \tag{4.4}$$

for some constants $c_4 > 0$ and $\xi_2 \in (0,1)$. Recall r_ρ and $\partial D_\rho^\#$ appearing in Lemma 3.1. Similarly, for any $\rho \in (0, \rho^*/2]$, there then exist a $c_5 > 0$ and a $\xi_3 \in (0,1)$ such that

$$\operatorname{Osc}\left(u; B_a^*(s)\right) \le c_5 s^{\xi_3}, \qquad 0 < s \le r_\rho/4,\ a \in \partial D_\rho^\#. \tag{4.5}$$

The estimate for oscillations on open balls with closures contained in D, which is due to Stampacchia [18], asserts that

$$\operatorname{Osc}(u; B(a,s)) \le c_6 s^{\xi_4}, \qquad 0 < s \le \eta/4,\ a \in D \setminus D_\eta. \tag{4.6}$$

Here η is a positive number fixed arbitrarily, $D_\eta = \{x \in D : \operatorname{dist}(x, \partial D) < \eta\}$, and constants $c_6 > 0$ and $\xi_4 \in (0,1)$ depend on η but are independent of $a \in D \setminus D_\eta$.

For an $\varepsilon > 0$ fixed arbitrarily, we see by virtue of (4.3) and (4.4) that there exists an $s_1 = s_1(\varepsilon) \in (0, \rho^*/4]$ such that

$$\operatorname{Osc}\left(u; C_k^*(s_1)\right) < \varepsilon, \qquad k \in I_C, \tag{4.7}$$

$$\operatorname{Osc}\left(u; Q_k^*(s_1)\right) < \varepsilon, \qquad k \in I_Q. \tag{4.8}$$

By means of (4.5), we further find an $s_2 = s_2(\varepsilon, s_1) \in (0, r_{s_1}/4]$ such that

$$\operatorname{Osc}\left(u; B_a^*(s_2)\right) < \varepsilon, \qquad a \in \partial D_{s_1}^\#. \tag{4.9}$$

In view of Lemma 3.1, we can find an $\eta_o > 0$ such that

$$\begin{aligned}
&\text{every pair } x,\ y \in D_{\eta_o} \text{ with } |x-y| < \eta_o \text{ is} \\
&\text{simultaneously contained in one of sets} \\
&C_k^*(s_1) \text{ with some } k \in I_C,\ Q_k^*(s_1) \text{ with some } k \in I_Q, \\
&B_a^*(s_2) \text{ with some } a \in \partial D_{s_1}^\#.
\end{aligned} \tag{4.10}$$

(4.6) with this η_o leads us to

$$\operatorname{Osc}(u; B(a, s_3)) < \varepsilon, \qquad a \in D \setminus D_{\eta_o/2}, \tag{4.11}$$

for some $s_3 = s_3(\varepsilon, \eta_o) \in (0, \eta_o/8]$.

We now set $\delta = (\eta_o/2) \wedge s_3$. Let $x,\ y \in D$ with $|x-y| < \delta$. If x or y belongs to $D_{\eta_o/2}$, then $|u(x) - u(y)| < \varepsilon$ by (4.10), (4.7), (4.8), (4.9). Otherwise, $|u(x) - u(y)| < \varepsilon$ by (4.11). □

Employing Corollary 3.1 in place of Theorem 3.1 (i) in getting (4.2), we obtain the following in the same way as above:

Theorem 4.3 *Let W be an open set of R^d and $E = W \cap D$. Every solution $u \in H^1(E)$ of (3.23) for some $\lambda \geq 0$ is uniformly continuous in $W_1 \cap D$ for every open set W_1 satisfying $\overline{W_1} \subset W$.*

We next give

Proof of Theorem 4.1 (ii) We first follow an argument in [20; Proposition 5.1] to show

$$G_\lambda \left(C_\infty(\overline{D}) \right) \subset C_\infty(\overline{D}). \tag{4.12}$$

Since $C_0(\overline{D})$ is dense in $C_\infty(\overline{D})$, it suffices to show that

$$G_\lambda \left(C_0(\overline{D}) \right) \subset C_\infty(\overline{D}) \tag{4.13}$$

in the case where D is unbounded.

Let $g \in C_0(\overline{D})$ and $\varepsilon > 0$. Choose an $R_1 > 0$ such that

$$\mathrm{Supp}[g] \subset B(R_1) \cap \overline{D}, \qquad c_1 \| G_\lambda g \|_{L^2(D \setminus B(R_1))} < \varepsilon, \tag{4.14}$$

c_1 being a positive constant specified later. We next take an $R_2 > R_1$ satisfying the following :

$$\begin{array}{ll} C_k^*(\rho^*) \subset D \setminus B(R_1) & \text{for } k \in I_C \text{ with } a_k \in \partial D \setminus B(R_2), \\ Q_k^*(\rho^*) \subset D \setminus B(R_1) & \text{for } k \in I_Q \text{ with } a_k \in \partial D \setminus B(R_2), \\ B_a^*(r^*) \subset D \setminus B(R_1) & \text{for } a \in \partial D^\#_{\rho^*} \setminus B(R_2). \end{array}$$

We set $J_C = \{k \in I_C : a_k \in \partial D \setminus B(R_2)\}$, $J_Q = \{k \in I_Q : a_k \in \partial D \setminus B(R_2)\}$, and $A = \partial D^\#_{\rho^*/2} \setminus B(R_2)$. Then, on account of (3.1), there is a constant $\eta \in (0, R_2 - R_1)$ depending on ρ^* but not on R_1, R_2 such that

$$D_{2\eta} \setminus B(R_2) \subset \bigcup_{k \in J_C} C_k^* \left(\frac{\rho^*}{2} \right) \cup \bigcup_{k \in J_Q} Q_k^* \left(\frac{\rho^*}{2} \right) \cup \bigcup_{a \in A} B_a^* \left(\frac{r_{\rho^*/2}}{2} \right).$$

We consider the set $K = \left[\bigcup_{k \in J_C \cup J_Q} \{a_k\} \right] \cup A \cup \left[D \setminus D_{2\eta} \setminus B(R_2) \right]$ and, for each $a \in K$, we define a constant s and a set $E_a(s)$ as follows :

$$E_a(s) = \begin{cases} C_k^*(s), & s = \rho^*, \quad \text{if } a = a_k,\ k \in J_C, \\ Q_k^*(s), & s = \rho^*, \quad \text{if } a = a_k,\ k \in J_Q, \\ B_a^*(s), & s = r_{\rho^*/2}, \quad \text{if } a \in A, \\ B(a, s), & s = \eta, \quad \text{if } a \in D \setminus D_{2\eta} \setminus B(R_2). \end{cases}$$

Note that

$$D \setminus B(R_2) \subset \bigcup_{a \in K} E_a(s/2) \subset \bigcup_{a \in K} E_a(s) \subset D \setminus B(R_1), \tag{4.15}$$

and

$$\mathscr{E}_{E_a(s), \lambda}(G_\lambda g, \varphi) = (g, \varphi) = 0, \qquad \varphi \in \widehat{H}\,(E_a(s)).$$

By virtue of Corollary 3.1, we then have

$$\|G_\lambda g\|_{L^\infty\left(E_a(s/2)\right)} \le c_1 \|G_\lambda g\|_{L^2(E_a(s))}, \tag{4.16}$$

where we used a local estimate due to Stampacchia [18] or Moser [17] in the case that $a \in D \setminus D_{2\eta} \setminus B(R_2)$. It should be noted that c_1 is a positive constant independent of a, R_1 and R_2. By (4.14), (4.15) and (4.16), we find that

$$\|G_\lambda g\|_{L^\infty\left(D\setminus B(R_2)\right)} \le c_1 \|G_\lambda g\|_{L^2\left(D\setminus B(R_1)\right)} < \varepsilon, \tag{4.17}$$

which along with Theorem 4.2 proves (4.13).

We next adopt Kunita's argument [15]. Let denote by $C_0^\infty\left(\overline{D}\right)$ the space of the restrictions to $\overline{D}$ of all infinitely continuously differentiable functions on R^d with compact support. For each $u \in C_0^\infty\left(\overline{D}\right)$, we define a functional Lu by

$$\langle Lu, \varphi\rangle = -\sum_{j=1}^d \int_D \left(\sum_{i=1}^d a_{ij}\, \partial_i u\right) \partial_j \varphi\, dx, \quad \varphi \in H^1(D).$$

Then $T = \lambda u - Lu$ satisfies the condition of Theorem 3.1 (i) and $u = G_\lambda T$ for each $\lambda > 0$. By virtue of Theorem 3.1 (i), there is for any $\varepsilon > 0$ a $g \in C_0^\infty\left(\overline{D}\right)$ such that

$$\|u - G_\lambda g\|_{L^\infty(D)} < \varepsilon.$$

Since $C_0^\infty\left(\overline{D}\right)$ is dense in $C_\infty\left(\overline{D}\right)$, we thus obtain the denseness of $G_\lambda\left(C_\infty\left(\overline{D}\right)\right)$ in $C_\infty\left(\overline{D}\right)$. □

Proof of Theorem 4.1 (iii) Since G_λ is Markovian, there exists a function $G_\lambda(x, y)$ satisfying (4.1) by virtue of Theorem 3.1 (i) and Theorem 4.2. Hence it is enough to show that

$$G_\lambda(x, \cdot) \text{ belongs to } H^1(U) \text{ and is continuous on } \overline{U}, \tag{4.18}$$

for any open set U with $\overline{U} \subset \overline{D} \setminus \{x\}$, where $x \in \overline{D}$ and $\lambda > 0$.

Let us denote the dual space of $H^1(E)$ by $\left(H^1(E)\right)'$. There exists for each $\lambda > 0$ and $T \in \left(H^1(D)\right)'$ a unique element $u \in H^1(D)$ such that

$$\mathscr{E}_\lambda(u, \varphi) = \langle T, \varphi\rangle, \qquad \varphi \in H^1(D).$$

We denote this function u by $G_\lambda T$. (We already used this notation for T given by (3.12) which is actually a general expression of $T \in \left(H^1(D)\right)'$ (cf. [16; 1.1.14]).)

For a while we fix an $x \in \overline{D}$ arbitrarily. We define a set $E_x(s)$ according as three different cases.

(*Case 1*) x is a cusp point, that is, $x = a_k$ for some $k \in I$. In this case we take an $s \in (0, \rho^*]$.

(*Case 2*) x is a boundary point but not a cusp point, that is, $x \in \partial D \setminus \bigcup_{k\in I}\{a_k\}$. Choose $\rho \in (0, \rho^*]$ such that $x \in \partial D \setminus \bigcup_{k\in I} \Phi_k\,(B(\rho))$. Then for an r_ρ given in Lemma 3.1 we take an $s \in (0, r_\rho]$.

(*Case 3*) x is an interior point of D. In this case we take an $s \in (0, d_x/2]$, where $d_x = \operatorname{dist}(x, \partial D)$.

Let us put

$$E_x(s) = \begin{cases} C_k^*(s) & \text{if } k \in I_C, \text{ in Case 1,} \\ Q_k^*(s) & \text{if } k \in I_Q, \text{ in Case 1,} \\ B_x^*(s) & \text{in Case 2,} \\ B(x,s) & \text{in Case 3.} \end{cases}$$

Then there exists a unique element $g_s^{x,\lambda} \in H^1(D)$ such that

$$\mathscr{E}_\lambda \left(g_s^{x,\lambda}, \varphi\right) = \frac{1}{|E_x(s)|} \int_{E_x(s)} \varphi(y)\, dy, \quad \varphi \in H^1(D). \tag{4.19}$$

Here we note the following lemma which is obtained by the same method as in [14; Lemma 4.3].

Lemma 4.1 *Let U be an open set such that $\overline{U} \subset \overline{D} \setminus \{x\}$. Then $G_\lambda(x,\cdot)|_U \in H^1(U)$ and $g_s^{x,\lambda}|_U$ converges to $G_\lambda(x,\cdot)|_U$ weakly in $H^1(U)$ as $s \downarrow 0$.*

Take an open set V of R^d such that $\overline{U} \subset V$ and $x \notin \overline{V}$ and set $E = V \cap D$. On account of Lemma 4.1, $G_\lambda(x,\cdot)|_E \in H^1(E)$ and $g_s^{x,\lambda}|_E \to G_\lambda(x,\cdot)|_E$ weakly in $H^1(E)$ as $s \downarrow 0$. Take any $\varphi \in \widehat{C}(E)$ and extend it to D by putting $\varphi = 0$ on $D \setminus E$. Then

$$\begin{aligned} \mathscr{E}_{E,\lambda}(G_\lambda(x,\cdot)|_E, \varphi) &= \lim_{s\downarrow 0} \mathscr{E}_{E,\lambda}\left(g_s^{x,\lambda}|_E, \varphi\right) \\ &= \lim_{s\downarrow 0} \mathscr{E}_\lambda \left(g_s^{x,\lambda}, \varphi\right) = \lim_{s\downarrow 0} \frac{1}{|E_x(s)|} \int_{E_x(s)} \varphi(y)\, dy = 0. \end{aligned}$$

This implies that $G_\lambda(x,\cdot)|_E \in H^1(E)$ is a solution of (3.23) and hence, in view of Theorem 4.3, $G_\lambda(x,\cdot)$ is continuous in $\overline{U}$. □

We finally note the following proposition which along with Theorem 4.1 implies Theorem 2.1.

Proposition 4.1 *Condition (H) reduces to condition (A).*

Proof For each $j \in J$, a Hölder function F_j in (H) (i) is given by

$$F_j(x') = \alpha_j |x'|^{\gamma_j} + f_j(x'),$$

where $\gamma \le \gamma_j < 1$, $\alpha_j = 0$ or $1/A \le \alpha_j \le A$ or $-A \le \alpha_j \le -1/A$ according to $j \in J_0$ or $j \in J_+$ or $j \in J_-$, and f_j is a Lipschitz continuous function defined on the $d-1$-dimensional closed ball $\left\{x' \in R^{d-1} : |x'| \le r_j\right\}$ with $f_j(0) = 0$ and $\mathrm{Lip}(f_j) \le M$. Then

$$U_j \cap D = \left\{ \left(\zeta^{(j)\prime}, \zeta_d^{(j)}\right) \in B(r_j) : F_j\left(\zeta^{(j)\prime}\right) < \zeta_d^{(j)} \right\},$$

for some Cartesian coordinate system $\zeta^{(j)} = \left(\zeta^{(j)\prime}, \zeta_d^{(j)}\right) = \left(\zeta_1^{(j)}, \zeta_2^{(j)}, \cdots, \zeta_d^{(j)}\right)$.

Let us put $I = J_+ \cup J_-$. For $k \in I$, a_k is the point of ∂D corresponding to the origin in $\zeta^{(k)}$-coordinate system. $\Xi \equiv \{a_k : k \in I\}$ is then the totality of cusp boundary points. For each $k \in I$, the neighbourhood U_k of a_k contains no cusp boundary point other than a_k, and hence $|a_k - a_l| \geq \delta,\ k \neq l,\ k, l \in I$.

For each $k \in I$, we define a mapping Φ_k from $E_k \equiv \{(x', x_d) : |x'| < r_k,\ x_d \in R\}$ into $\zeta^{(k)}$-space by

$$\Phi_k(x', x_d) = \left(\zeta^{(k)\prime}, \zeta_d^{(k)}\right),$$
$$\zeta^{(k)\prime} = x', \quad \zeta_d^{(k)} = |\alpha_k| x_d + f_k(x').$$

We then have $\mathrm{Lip}\big(\Phi_k\big) \leq 1 + A + M$ and $\mathrm{Lip}\big(\Phi_k^{-1}\big) \leq 1 + A + AM$. Put $\rho^* = \delta/2(1 + A + M)$, $V_k = \Phi_k\,(B(\rho^*))$ and $M_1^* = (1 + A)(1 + M)$. Then we see from (3.1) that $\{V_k\}_{k\in I}$ satisfies (A) (i) with $\gamma^* = \gamma$ and $M^* = M_1^*$. In particular, $V_k \cap D = \Phi_k\left(C_{\gamma_k}(\rho^*)\right)$ if $\alpha_k > 0$, $= \Phi_k\left(Q_{\gamma_k}(\rho^*)\right)$ if $\alpha_k < 0$.

We next show (A) (ii). Let ξ_o be the positive solution of the equation $\xi^{2\gamma} + \xi^2 = \big(\rho_o\big)^2$ for $\rho_o = \rho^* \wedge 1$. We then take a constant $R > 1$ satisfying

$$\left\{1 + M + A\left(\frac{R-1}{R}\xi_o\right)^{\gamma-1}\right\}\frac{\xi_o}{R} < \delta,$$

and put $r^* = \xi_o/R$. This r^* will play the role of r^* in (A) (ii).

Let us fix a $p \in \partial D \setminus \bigcup_{k\in I} V_k$ arbitrarily. By means of (H) (ii), there is a $j \in J$ such that $p \in \widetilde{U_{j,\delta}}$. Denote the $\zeta^{(j)}$-coordinate of p by $(p^{(j)\prime}, p_d^{(j)})$. We shall define a mapping Ψ_p and a neighbourhood W_p in two cases $j \in I$ and $j \notin I$ separately.

In the case that $j \in I$, $(p^{(j)\prime}, p_d^{(j)})$ belongs to $\Phi_j(E_j)$. Putting $(\widetilde{p}', \widetilde{p}_d) = \Phi_j^{-1}\left(p^{(j)\prime}, p_d^{(j)}\right)$, we have that $(\widetilde{p}', \widetilde{p}_d) \in E_j \setminus B(\rho^*)$ and $\widetilde{p}_d = \pm|\widetilde{p}'|^{\gamma_j}$ in accordance to the sign of α_j. We then define a mapping Ψ_p from the set $G_p \equiv \{(x', x_d) : |x' + \widetilde{p}'| < r_j,\ x_d \in R\}$ into $\zeta^{(j)}$-space as follows:

$$\Psi_p(x', x_d) = \left(\zeta^{(j)\prime}, \zeta_d^{(j)}\right),$$
$$\zeta^{(j)\prime} = x' + \widetilde{p}',$$
$$\zeta_d^{(j)} = \alpha_j\,|x' + \widetilde{p}'|^{\gamma_j} + x_d + f_j\left(x' + \widetilde{p}'\right).$$

Notice that, on the region $\{x \in G_p : |x'| < \xi_o/S\}$ for $S > 1$, $\mathrm{Lip}\big(\Psi_p\big) \leq 1 + M + A\left(\frac{S-1}{S}\xi_o\right)^{\gamma_j - 1}$. Since the distance of $p = \Psi_p(0)$ from $\partial\left(\Psi_p(G_p)\right)$ is greater than δ, we can conclude from (3.1) that $B(r^*) \subset G_p$ for the above chosen $r^* = \xi_o/R$.

In the case that $j \notin I$, we define a mapping Ψ_p from the set $G_p \equiv \{(x', x_d) : |x' + p^{(j)\prime}| < r_j,\ x_d \in R\}$ into $\zeta^{(j)}$-space by

$$\Psi_p(x', x_d) = \left(\zeta^{(j)\prime}, \zeta_d^{(j)}\right),$$
$$\zeta^{(j)\prime} = x' + p^{(j)\prime}, \quad \zeta_d^{(j)} = x_d + f_j\left(x' + p^{(j)\prime}\right).$$

In both cases, $B(r^*) \subset G_p$. Accordingly we put $W_p = \Psi_p(B(r^*))$. It is easy to see that Ψ_p is one-to-one, $\Psi_p(0) = p$, $\Psi_p(B_+(r^*)) = W_p \cap D$, and $\mathrm{Lip}(\Psi_p) \leq M_2^*$, $\mathrm{Lip}(\Psi_p^{-1}) \leq M_2^*$ where $M_2^* = 1 + M + A\left(\frac{R-1}{R}\xi_o\right)^{\gamma-1}$.

Thus (H) reduces to (A) with $I,\ \gamma^*,\ \rho^*,\ r^*$ as above and $M^* = M_1^* \vee M_2^*$.
□

5 Decomposition of the sample path and additive functionals

This section is devoted to the proof of Theorem 2.3 and Theorem 2.4. To this end, we first prepare an extended version of a general theorem [13; Theorem 5.5.5] to characterize the second term in the decomposition (1.6).

Let X be a locally compact separable metric space, m be a positive Radon measure on X with full support and $(\mathscr{E}, \mathscr{F})$ be a strongly local regular Dirichlet form on $L^2(X; m)$. We assume that there exists a conservative diffusion process $\mathbf{M} = (X_t, P_x)$ on X associated with the form $\mathscr{E}$ whose transition function $p_t(x, \cdot)$ is absolutely continuous with respect to m for any $t > 0$ and $x \in X$.

Then the resolvent of $\mathbf{M}$ admits a symmetric density $G_\lambda(x,y)$ with respect to m which is λ-excessive in two variables $x,\ y$. The potential of a measure μ is denoted by $G_\lambda\mu(x) = \int_X G_\lambda(x,y)\,\mu(dy)$. The integral of a function f against a measure μ is denoted by $\langle \mu, f\rangle$ or $\langle f, \mu\rangle$. A positive Radon measure μ on X is said to be of finite energy integral if there exists a constant C_{17} such that

$$\int_X |v(x)|\mu(dx) \le C_{17}\sqrt{\mathscr{E}_1(v,v)}, \quad v \in \mathscr{C}, \tag{5.1}$$

for some special standard core $\mathscr{C}$ of $\mathscr{E}$. The totality of such measures is denoted by S_0. It is known that $\mu \in S_0$ if and only if $\langle \mu, G_\lambda\mu\rangle$ is finite and that, in this case, $G_\lambda\mu$ is a λ-excessive and quasi-continuous version of the potential $U_\lambda\mu$ considered in [13; Sect. 2.2]. We further introduce two classes of positive Radon measures on X by

$$S_{00} = \{\mu : \mu(X) < \infty,\ \ \sup_{x\in X} G_\lambda\mu(x) < \infty\}$$

$$S_{01} = \{\mu : \mu \in S_0,\ \ G_\lambda\mu(x) < \infty\ \forall x \in X\}.$$

Obviously $S_{00} \subset S_{01} \subset S_0$. In our later application, the family S_{01} turns out to be more useful than S_{00}.

An increasing sequence $\{E_\ell\}$ of finely open sets is said to be an exhaustive sequence if $\bigcup_{\ell=1}^\infty E_\ell = X$. A positive Borel measure μ on X is called smooth in the strict sense if there exists an exhaustive sequence $\{E_\ell\}$ of finely open sets such that $I_{E_\ell}\cdot\mu \in S_{00},\ \ \ell = 1, 2, \cdots$. Let S_1 be the totality of smooth measures in the strict sense. S_1 is known to be in one to one correspondence with the (equivalence classes of) positive continuous additive functionals (PCAF's in abbreviation) in the strict sense of $\mathbf{M}$ under the Revuz correspondence ([13; Theorem 5.1.7]).

Lemma 5.1 *$\mu \in S_1$ if and only if there exists an exhaustive sequence $\{E_\ell\}$ of finely open sets such that $I_{E_\ell}\cdot\mu \in S_{01},\ \ell = 1, 2, \cdots$.*

Proof It suffices to show that any $\mu \in S_{01}$ admits an exhaustive sequence $\{E_\ell\}$ of finely open sets such that $I_{E_\ell} \cdot \mu \in S_{00}$, $\ell = 1, 2, \cdots$. We may choose E_ℓ as follows:

$$E_\ell = \{x \in O_\ell : G_\lambda \mu(x) < \ell\}, \quad \ell = 1, 2, \cdots,$$

where $\{O_\ell\}$ is an exhaustive sequence of relatively compact open sets. Then, $(I_{E_\ell} \cdot \mu)(X) = \mu(E_\ell)$ is finite, and further $G_\lambda(I_{E_\ell} \cdot \mu)(x) \leq \ell$ for m-a.e. $x \in X$ by the maximum principle ([13; Lemma 2.2.4]) and hence for every $x \in X$ by the absolute continuity of the transition function. □

We denote by $\mu_{\langle u \rangle}$ the energy measure of $u \in \mathscr{F}_{loc}$. The Dirichlet form $\mathscr{E}$ is expressible as

$$\mathscr{E}(u, v) = \frac{1}{2} \mu_{\langle u,v \rangle}(X), \quad u, v \in \mathscr{F}$$

by using the co-energy measure $\mu_{\langle u,v \rangle}$. The second assertion of the next proposition replaces S_{00} in [13; Theorem 5.5.5] by S_{01}.

Proposition 5.1 (i) *Suppose that a function u satisfies the following conditions:*

1. *u is finite valued, finely continuous and $u \in \mathscr{F}_{loc}$.*
2. *$I_G \cdot \mu_{\langle u \rangle} \in S_{00}$ for any relatively compact open set G.*

Then we have the unique decomposition

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad \forall t \geq 0, \quad P_x - \text{a.e. } \forall x \in X, \tag{5.2}$$

where $M^{[u]}$ is a CAF in the strict sense such that, for any relatively compact open set G,

$$E_x\left(M_{t \wedge \tau_G}^{[u]}\right) = 0, \quad E_x\left(\left(M_{t \wedge \tau_G}^{[u]}\right)^2\right) = E_x\left(B_{t \wedge \tau_G}\right), \quad \forall x \in G, \tag{5.3}$$

B being the PCAF in the strict sense with Revuz measure $\mu_{\langle u \rangle}$ and τ_G being the first leaving time from G. $N_t^{[u]}$ is a CAF in the strict sense locally of zero energy.
(ii) *Assume further the following property of u:*

$\exists \nu = \nu^{(1)} - \nu^{(2)}$ with $I_G \cdot \nu^{(1)}$, $I_G \cdot \nu^{(2)} \in S_{01}$ for any relatively compact open set G and

$$\mathscr{E}(u, v) = \langle \nu, v \rangle, \quad \forall v \in \mathscr{C}, \tag{5.4}$$

for some special standard core $\mathscr{C}$ of $\mathscr{E}$.

Then

$$N^{[u]} = -A^{(1)} + A^{(2)}, \quad P_x - \text{a.s. } \forall x \in X, \tag{5.5}$$

where $A^{(1)}$ and $A^{(2)}$ are PCAF's in the strict sense with Revuz measures $\nu^{(1)}$ and $\nu^{(2)}$ respectively.

Proof The first assertion is a consequence of [11; Theorem 2]. Since $\nu^{(1)}$, $\nu^{(2)}$ in (ii) are in the class S_1 by the preceding lemma, we can see the validity of identity (5.5) on account of [12; Theorem 3.3, Corollary 3.1]. □

We are in a position to prove Theorem 2.3. We fix a domain $D \subset R^d$ possessing the property (H) and a data a_{ij} satisfying (1.1). We now apply the general theory prepared above to the specific Dirichlet form (1.2) on $L^2(\overline{D}; m)$, the resolvent G_λ of Theorem 2.1 and the conservative diffusion $\mathbf{M}$ of Theorem 2.2. Here m denotes the $d-$dimensional Lebesgue measure. $\mathbf{M}$ will be called the reflecting diffusion on $\overline{D}$ (associated with a_{ij}). Accordingly G_λ will be called the resolvent of the reflecting diffusion on $\overline{D}$.

Notice that, under the condition (H) for the domain D, the surface measure σ on ∂D is well defined with a local expression

$$\sigma(E) = \int_{E_*} \sqrt{1 + |\nabla F_j(\zeta')|^2} d\zeta', \quad E \subset U_j \cap \partial D, \tag{5.6}$$

where $E_* = \{\zeta' : (\zeta', F_j(\zeta')) \in E\}$. Further, with the unit inward normal vector $\mathbf{n}(\zeta) = (n_1(\zeta), \cdots, n_d(\zeta))$ making sense σ-a.e. on ∂D according as

$$\mathbf{n}(\zeta) = (-\nabla F_j(\zeta'), 1) \Big/ \sqrt{1 + |\nabla F_j(\zeta')|^2}\,, \quad \zeta \in U_j \cap \partial D,$$

we have the divergence theorem

$$\int_D \frac{\partial w}{\partial x_i} dm = -\int_{\partial D} w n_i \, d\sigma, \qquad 1 \le i \le d, \; w \in C_0^\infty(\overline{D}).$$

This formula extends to a wider class of functions w and in particular the condition (1.9) for a_{ij} guarantees the identity

$$\int_D \partial_j (a_{ij} \cdot v) \, dm = -\int_{\partial D} v \, a_{ij} \, n_j \, d\sigma, \quad v \in C_0^\infty(\overline{D}). \tag{5.7}$$

Denote by ϕ_i the coordinate functions: $\phi_i(x) = x_i$, $1 \le i \le d$. Then, $\phi_i \in H^1_{loc}(D)$, and the co-energy measures $\mu_{\langle \phi_i, \phi_j \rangle}$ with respect to the Dirichlet form (1.2) are given by (cf. [13; Example 5.2.1])

$$\mu_{\langle \phi_i, \phi_j \rangle} = 2 a_{ij} \cdot m, \quad 1 \le i, j \le d. \tag{5.8}$$

Let us denote by B an arbitrary ball in R^d. Since a_{ij} are bounded, we have

$$I_{B \cap \overline{D}} \cdot \mu_{\langle \phi_i \rangle} \in S_{00}.$$

The PCAF in the strict sense with Revuz measure m is just a constant functional t. Therefore Proposition 5.1 (i) implies the decomposition (1.6) where $M^i, 1 \le i \le d$, are CAF's in the strict sense satisfying (1.7) with t being replaced by $t \wedge \tau_{B \cap \overline{D}}$. Owing to the boundedness of a_{ij} however, we can let $B \uparrow R^d$ to get (1.7), proving Theorem 2.3 (i).

Turning to the proof of Theorem 2.3 (ii), we have under the condition (1.9)

$$\mathscr{E}(\phi_i, v) = \int_{\overline{D}} v(x) \, \nu(dx) \quad \text{with} \quad \nu = -\sum_{j=1}^d (\partial_j a_{ij}) \cdot m - \sum_{j=1}^d a_{ij} \, n_j \cdot \sigma, \tag{5.9}$$

holding for any $v \in C_0^\infty(\overline{D})$, because

$$\begin{aligned}\mathscr{E}(\phi_i, v) &= \sum_{j=1}^{d} \int_D a_{ij}(x)\partial_j v(x)\, m(dx) \\ &= -\sum_{j=1}^{d} \int_D \partial_j a_{ij} \cdot v\, dm + \sum_{j=1}^{d} \int_D \partial_j (a_{ij} \cdot v)\, dm,\end{aligned}$$

which equals to the right hand side of (5.9) by virtue of (5.7).

Suppose that the surface measure σ satisfies

$$I_{B\cap\partial D} \cdot \sigma \in S_{01} \quad \text{for any ball} \quad B \subset R^d. \tag{5.10}$$

We then have from the boundedness of a_{ij} and condition (1.9)

$$I_{B\cap\overline{D}} \cdot |\nu| \in S_{01}.$$

Hence (5.9) and Proposition 5.1 (ii) lead us to the expression (2.3) in terms of a PCAF L in the strict sense with Revuz measure being the surface measure σ, proving Theorem 2.3 (ii).

It only remains to show (5.10) for the proof of Theorem 2.3.

Theorem 5.1 *If (2.2) holds, namely, each outward cusp boundary point is of Hölder exponent greater than $\frac{1}{2}$, then the surface measure σ on ∂D satisfies condition (5.10).*

For any ball B, the compact set $\overline{B} \cap \partial D$ can be covered by finite number of open sets $\widetilde{U_{j,\delta}}$ appearing in the condition (H) (ii) for the domain D. Besides $G_\lambda(x, y)$ is jointly continuous off diagonal by Theorem 2.1 (iii). For the proof of (5.10), it is therefore sufficient to show

$$I_\Gamma \cdot \sigma \in S_0 \quad \text{and} \quad G_\lambda I_\Gamma \cdot \sigma(x) < \infty, \quad x \in \Gamma, \tag{5.11}$$

where

$$\Gamma = \{\zeta = (\zeta', \zeta_d) : |\zeta| < \rho,\ \zeta_d = F_j(\zeta')\} \subset U_j \cap \partial D$$

for each fixed $j \in J$ and $\rho < r_j$. Exp(F_j) will be denoted by γ_j. $c_1, c_2, \cdots$ will denote some positive constants. We further let

$$\Gamma_* = \{\zeta' : (\zeta', F_j(\zeta')) \in \Gamma\} (\subset \{\zeta' : |\zeta'| < \rho\}).$$

Lemma 5.2 *Let $j \in J_+ \cup J_-$.*

(i) $I_\Gamma \cdot \sigma \in S_0$ *whenever* $d \geq 3$. *When* $d = 2$, *this is true if* $\gamma_j > \frac{1}{2}$.

(ii) $I_{\Gamma_\delta} \cdot \sigma \in S_0$ *for any* $\delta > 0$ *where* $\Gamma_\delta = \{\zeta : \delta < |\zeta|\}$.

Proof (i) In view of (1.1) and (5.1), it suffices to prove the inequality

$$\int_{\Gamma_*} |u(\zeta', F_j(\zeta'))| \sigma(d\zeta') \leq c_1 \sqrt{\mathbf{D}(u,u) + (u,u)_{L^2(D)}}, \quad u \in C_0^\infty(\overline{D}), \tag{5.12}$$

where $\mathbf{D}(u,u)$ denotes the Dirichlet integral of u on D. On account of (5.6), the surface measure σ has a density $\sigma(\zeta')$ with respect to $d\zeta'$ satisfying

$$\sigma(\zeta') \leq c_2 |\zeta'|^{\gamma_j - 1}. \tag{5.13}$$

Hence the square of the left hand side of (5.12) is dominated by

$$c_2^2 \int_{\Gamma_*} u(\zeta', F_j(\zeta'))^2 d\zeta' \cdot \int_{|\zeta'|<\rho} |\zeta'|^{2\gamma_j - 2} d\zeta'. \tag{5.14}$$

The second factor equals $\int_0^\rho r^{2\gamma_j + d - 4} dr$, which is finite under the stated condition. Consider a function $\psi \in C_0^\infty(U)$ taking value 1 on the set Γ. Then from the expression

$$u(\zeta', F_j(\zeta')) = -\int_{F_j(\zeta')}^{\sqrt{r^2 - |\zeta'|^2}} \frac{\partial}{\partial \zeta_d} \{\psi(\zeta', \zeta_d) u(\zeta', \zeta_d)\} d\zeta_d, \quad \zeta' \in \Gamma_*,$$

we see that the first factor of (5.14) is dominated by

$$c_3 \int_{U \cap D} (u^2 + |\nabla u|^2) d\zeta$$

arriving at (5.12).
(ii) Since $\sigma(\zeta')$ is bounded on $\Gamma_{\delta,*} = \{\zeta' : (\zeta', F_j(\zeta')) \in \Gamma_\delta\}$, (5.12) with Γ_* being replaced by $\Gamma_{\delta,*}$ holds for any $\delta > 0$. □

In order to complete the proof of (5.11), we prepare a lemma on a comparison of resolvent densities.

Lemma 5.3 *Let K be a compact subset of $\overline{D}$ and U be a bounded domain containing K such that the domain $D_1 = D \cap U$ possesses the property (H). Denote by $G_\lambda^1(x,y)$, $x, y \in \overline{D}_1$, the resolvent density of the reflecting diffusion on $\overline{D}_1$. Then,*

$$G_\lambda(x,y) \leq G_\lambda^1(x,y) + C_{18}, \qquad x, y \in K, \ x \neq y, \tag{5.15}$$

for some positive constant C_{18} depending on the set K.

Proof Consider the set $F = \overline{D} \cap U$ and the resolvent density $G_\lambda^0(x,y)$, $x, y \in F$, of the part $\mathbf{M}_F$ of $\mathbf{M}$ on the set F. $\mathbf{M}_F$ is obtained from $\mathbf{M}$ by killing the sample paths upon leaving the set F. Then by Dynkin's formula

$$G_\lambda(x,y) = G_\lambda^0(x,y) + E_x\left(e^{-\lambda\tau} G_\lambda(X_\tau, y)\right), \ x, y \in F,$$

where τ denotes the leaving time from the set F. Take an open set W such that $K \subset W \subset \overline{W} \subset U$. The second term of the right side of the above identity with y being restricted to $\overline{D} \cap W$ is dominated by

$$C_{18} = \sup_{x \in D \cap \partial U,\ y \in \overline{D} \cap W} G_\lambda(x, y)$$

which is finite owing to the off diagonal continuity Theorem 2.1 (iii).

Let $\mathbf{M}_1$ be the reflecting diffusion on $\overline{D}_1$ and $\mathbf{M}_F^1$ be its part on the set $F(\subset \overline{D}_1)$. On account of [13; Theorem 4.4.3], $\mathbf{M}_F$ and $\mathbf{M}_F^1$ share a common Dirichlet form $\mathscr{E}_F$ on $L^2(F)$ given by

$$\begin{aligned} \mathscr{E}_F(u, v) &= \mathscr{E}(u, v), \qquad u, v \in \mathscr{D}[\mathscr{E}_F] \\ \mathscr{D}[\mathscr{E}_F] &= \widehat{H}(D_1), \end{aligned}$$

where $\widehat{H}(D_1)$ is defined by (3.7). Therefore we have the inequality

$$G_\lambda^0(x, y) \le G_\lambda^1(x, y)$$

holding for $m \times m$-a.e. $(x, y) \in F \times F$. In view of the continuity of G_λ and G_λ^1, we get (5.15) for every $x \in F$ and every $y \in \overline{D} \cap W$. □

We return to the set $\Gamma \subset U_j \cap \partial D$ specified before Lemma 5.2.

Lemma 5.4 *Following inequalities hold for $x,\ y \in \Gamma,\ x \neq y$ and a positive constant C_{19} depending on the set $U_j \cap D$:*

(i) *If $j \in J_+$ and $d \ge 2$, then*

$$G_\lambda(x, y) \le C_{19} |x - y|^{-\frac{d-1-\gamma_j}{\gamma_j}}. \tag{5.16}$$

(ii) *If $j \in J_0 \cup J_-$ and $d \ge 3$, then*

$$G_\lambda(x, y) \le C_{19} |x - y|^{-d+2}. \tag{5.17}$$

(iii) *If $j \in J_0 \cup J_-$ and $d = 2$, then*

$$G_\lambda(x, y) \le C_{19} |x - y|^{-\varepsilon} \qquad \textit{for any} \quad \varepsilon > 0. \tag{5.18}$$

Proof (i) Notice that the Sobolev inequality in the statement of Proposition 3.2 (i) holds with D and γ^* being replaced by $D_j = U_j \cap D$ and γ_j respectively. Since the domain D_j is bounded, we can invoke Carlen-Kusuoka-Stroock [5] to conclude in the same way as in [3; Sect. 2] that the resolvent density $G_\lambda^1(x, y)$ of the reflecting diffusion on $\overline{D_j}$ admits the estimate

$$G_\lambda^1(x, y) \le c_1 |x - y|^{-\beta}, \qquad x,\ y \in \overline{D_j}, \tag{5.19}$$

for $\beta = 4/(q-2)$. In particular, by taking $q = 2(d-1+\gamma_j)/(d-1-\gamma_j)$ we see that (5.19) is valid for $\beta = (d-1-\gamma_j)/\gamma_j$. We can then use Lemma 5.3 to get (5.16).

(ii), (iii) In these cases, Proposition 3.2 (ii) is applicable to the domain $D_j = U_j \cap D$ and we see the validity of (5.19) for $\beta = d-2$ [resp. $\beta = \varepsilon > 0$] by taking $q = 2d/(d-2)$ [resp. $q = 4/\varepsilon + 2$]. We again use Lemma 5.3 to get (5.17) [resp. (5.18)]. □

Lemma 5.5 *Let $j \in J_+ \cup J_-$. Assume that $\gamma_j > \frac{1}{2}$ in case $j \in J_+$. Then $G_\lambda I_\Gamma \cdot \sigma(\zeta) < \infty,\ \zeta \in \Gamma$.*

Proof Keeping the expression

$$G_\lambda I_\Gamma \cdot \sigma(\zeta) = \int_{\Gamma_*} G_\lambda(\zeta, (\eta', F(\eta'))\sigma(\eta')d\eta'$$

and the bound (5.13) of σ in mind, we first prove the finiteness of the potential for $\zeta = 0$ in case that $j \in J_+$ and $\gamma_j > \frac{1}{2}$. From (5.16), we have the bound

$$G_\lambda I_\Gamma \cdot \sigma(0) \le c_1 \int_{\Gamma_*} \left\{ \left(|\eta'|^2 + |F(\eta')|^2\right)^{\frac{d-1-\gamma_j}{2\gamma_j}} |\eta'|^{1-\gamma_j} \right\}^{-1} d\eta'. \tag{5.20}$$

Since

$$\left(|\eta'|^2 + |F(\eta')|^2\right)^{\frac{d-1-\gamma_j}{2\gamma_j}} \ge \left(\frac{\alpha_j}{2}\right)^{\frac{d-1-\gamma_j}{\gamma_j}} |\eta'|^{d-1-\gamma_j}, \qquad |\eta'| < \delta,$$

for some $\delta > 0$, where $\alpha_j = \text{Höl}(F_j)$, we obtain

$$G_\lambda I_\Gamma \cdot \sigma(0) \le c_2 \int_0^\delta r^{d+\gamma_j-3-(d-1-\gamma_j)}\, dr + c_3 \delta^{-\frac{d-1-\gamma_j}{\gamma_j}} \int_\delta^\rho r^{d+\gamma_j-3}\, dr < \infty.$$

In the case that $j \in J_-$, we get the finiteness of $G_\lambda I_\Gamma \cdot \sigma(0)$ from (5.17) and (5.18) in a similar manner to the above.

Next take a $\zeta \in \Gamma,\ \zeta \ne 0$. We can choose a neighbourhood V of ζ such that $0 \notin V,\ V \subset U_j$ and $D_1 = V \cap D$ is a Lipschitz domain. Let $\tilde{\Gamma} = \Gamma \cap V$. Then the same reasoning as the proof of Lemma 5.3 works to see that $G_\lambda(\zeta, \eta),\ \eta \in \tilde{\Gamma}$, is dominated by $c_4|\zeta - \eta|^{-d+2}$ in case that $d \ge 3$ and by $c_5|\zeta - \eta|^{-\varepsilon},\ \varepsilon > 0$, in case that $d = 2$. Since $\sigma(\eta)$ is bounded on $\tilde{\Gamma}$, we see the finiteness of $G_\lambda I_{\tilde{\Gamma}} \cdot \sigma(\zeta)$ and hence of $G_\lambda I_\Gamma \cdot \sigma(\zeta)$. □

Proof of Theorem 5.1 We divide the situation into four cases:

(I) $j \in J_+,\ \gamma_j > 1/2$ (II) $j \in J_-,\ d \ge 3$

(III) $j \in J_-,\ d = 2$ (IV) $j \in J_0$

In view of Lemma 5.2 and Lemma 5.5, we see that (5.11) holds in cases (I) and (II). Hence it remains to prove (5.11) in cases (III) and (IV). We can instead prove a stronger property

$$\sup_{x \in \Gamma} G_\lambda I_\Gamma \cdot \sigma(x) < \infty \tag{5.21}$$

in these cases.

Indeed, when $j \in J_-$ and $d = 2$, we have the bound (5.18) of $G_\lambda(x, y)$ for any $\varepsilon > 0$, and we can proceed in a similar manner to the proof of Theorem 6.1 in our preceding paper [14] in getting (5.21) by choosing ε smaller than γ_j. When

$j \in J_0$, then we have the bound (5.17) or (5.18) of $G_\lambda(x, y)$ which, together with the uniform boundedness on Γ of the density function σ of the surface measure, readily leads us to (5.21). □

Proof of Theorem 2.4 Take any function f as is stated in the theorem and denote by W a neighbourhood of Ξ_+ on which f is constant. Then

$$I_{B\cap(\partial D\setminus W)} \cdot \sigma \in S_{01} \quad \text{for any ball } B \subset R^d. \tag{5.22}$$

To see this, it suffices to show

$$I_{\Gamma\setminus W} \cdot \sigma \in S_0 \quad \text{and} \quad G_\lambda I_{\Gamma\setminus W} \cdot \sigma(x) < \infty,\ x \in \Gamma, \tag{5.23}$$

for the set $\Gamma \subset U_j \cap \partial D$ appearing in (5.11) and exclusively for $j \in J_+$. Since $\Gamma \setminus W \subset \Gamma_\delta$ for some $\delta > 0$, the first assertion in (5.23) follows from Lemma 5.2 (ii). The second one for $x = 0$ [resp. for $x \neq 0$] is immediate from the continuity of $G_\lambda(0, y)$ [resp. from Lemma 5.5].

Now just as computations made in (5.8) and (5.9), we have

$$\mu_{\langle f,f\rangle} = 2\left(\sum_{i,j=1}^{d} a_{ij} \cdot \partial_i f \cdot \partial_j f\right) \cdot m \tag{5.24}$$

and

$$\mathscr{E}(f, v) = \int_{\overline{D}} v(x)\, \nu(dx), \qquad v \in C_0^\infty\left(\overline{D}\right),$$

with

$$\nu = -\sum_{i,j=1}^{d} \partial_i \left(a_{ij}\, \partial_j f\right) \cdot m - \sum_{i,j=1}^{d} \partial_i f \cdot a_{ij}\, n_j\, I_{\partial D\setminus W} \cdot \sigma. \tag{5.25}$$

$I_{\partial D\setminus W}$ can be inserted in the last expression because $\partial_i f$ vanishes on W.

In view of (2.4), (5.22), (5.24) and (5.25), Proposition 5.1 applies and we get

$$f(X_t) - f(X_0) = M_t^{[f]} + N_t^{[f]}, \qquad P_x\text{-a.s.},\ x \in \overline{D}, \tag{5.26}$$

where $M^{[f]}$ is a MAF in the strict sense with

$$\langle M^{[f]}\rangle_t = 2\int_0^t \left(\sum_{i,j=1}^{d} a_{ij}\, \partial_i f \cdot \partial_j f\right)(X_s)\, ds \tag{5.27}$$

and

$$N_t^{[f]} = \int_0^t \left(\sum_{i,j=1}^{d} \partial_i \left(a_{ij}\, \partial_j f\right)\right)(X_s)\, ds + \int_0^t \left(\sum_{i,j=1}^{d} \partial_i f \cdot a_{ij}\, n_j\right)(X_s)\, d\widetilde{L}_s. \tag{5.28}$$

Here $\widetilde{L}_t$ is a PCAF in the strict sense with Revuz measure $I_{\partial D\setminus W} \cdot \sigma$. (5.27) and (5.28) are valid P_x-a.e. for every $x \in \overline{D}$. Under the condition (2.5) for f, the second functional in the right hand side of (5.28) is a PCAF in the strict sense. Therefore the desired conclusion follows from (5.26) and (5.28). □

6 Sobolev inequality of Moser type

In this section we will show Proposition 3.1. Throughout this section we assume condition (A).

Proposition 3.1 is immediate from the following two propositions.

Proposition 6.1 *For any $\kappa \in (0,1]$ there is a positive constant $C_{20} = C_{20}(\kappa, \gamma^*, \rho^*, M^*, d)$ such that*

$$\int_{E_k^*(\rho)} |u|^2\,dx \le C_{20}\left\{\int_N |u|^2\,dx + \rho^2 \sum_{i=1}^d \int_{E_k^*(\rho)} |\partial_i u|^2\,dx\right\}, \tag{6.1}$$

for $E_k^*(\rho) = C_k^*(\rho)$ $\left[\text{ resp. } Q_k^*(\rho)\ \right]$, $u \in H^1\left(E_k^*(\rho)\right)$, N being a Borel subset of $E_k^*(\rho)$ with $|N| \ge \kappa |E_k^*(\rho)|$, $0 < \rho \le \rho^*$ and $k \in I_C$ $\left[\text{ resp. } k \in I_Q\ \right]$.

Proposition 6.2 (i) *Let $k \in I_C$ and $1 \le p \le (d-1+\gamma_k)/\gamma_k$. Take a q satisfying $p \le q \le p\left(d-1+\gamma_k\right)/(d-1+\gamma_k-\gamma_k p)$ if $p < (d-1+\gamma_k)/\gamma_k$, or $p \le q < \infty$ if $p = (d-1+\gamma_k)/\gamma_k$. Then there is a positive constant $C_{21} = C_{21}(p, q, \gamma^*, \rho^*, M^*, d)$ such that*

$$\begin{aligned}\left(\int_{C_k^*(\rho)} |u|^q\,dx\right)^{1/q} \ &\le\ C_{21}\rho^{\frac{d-1+\gamma_k}{\gamma_k}\left(\frac{1}{q}-\frac{1}{p}\right)}\\ &\quad\times\ \left\{\int_{C_k^*(\rho)} |u|^p\,dx + \rho^p \sum_{i=1}^d \int_{C_k^*(\rho)} |\partial_i u|^p\,dx\right\}^{1/p},\end{aligned}$$

for $u \in H^1\left(C_k^*(\rho)\right)$, $0 < \rho \le \rho^*$.

(ii) *Let $1 \le p \le d$ and take a q satisfying $p \le q \le pd/(d-p)$ if $p < d$, or $p \le q < \infty$ if $p = d$. Then there is a positive constant $C_{22} = C_{22}(p, q, \gamma^*, \rho^*, M^*, d)$ such that*

$$\begin{aligned}\left(\int_{Q_k^*(\rho)} |u|^q\,dx\right)^{1/q} \ &\le\ C_{22}\rho^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\\ &\quad\times\ \left\{\int_{Q_k^*(\rho)} |u|^p\,dx + \rho^p \sum_{i=1}^d \int_{Q_k^*(\rho)} |\partial_i u|^p\,dx\right\}^{1/p},\end{aligned}$$

for $u \in H^1\left(Q_k^*(\rho)\right)$, $0 < \rho \le \rho^*$, $k \in I_Q$.

The part (i) of Proposition 6.2 is obtained by the same method as in [1; pp.128–135] if we employ a transformation $\Psi_\gamma(x)$ defined below in place of the transformation $r_k(x)$ in [1; p.130]. The part (ii) is also obtained following argument in [1; pp.103–104]. So we omit the proof of Proposition 6.2.

In order to show Proposition 6.1 with $E_k^*(\rho) = C_k^*(\rho)$, we make use of a mapping $\Psi_\gamma(x)$ from a cusp $C_\gamma(\rho)$ onto a direct product set $\Sigma(\rho)$. We begin with the definition of Ψ_γ. Let $R_+^d = \{(x', x_d) \in R^d : x_d > 0\}$ and Ξ_d be the following product space:

$$\varXi_d = \begin{cases} \{(r,t) : 0 < r < \infty,\ -\infty < t < \infty\} & \text{if } d = 2, \\ \{(r,t,\theta) : 0 < r < \infty,\ 0 \le t < \infty,\ \theta \in \varTheta_{d-2}\} & \text{if } d \ge 3, \end{cases}$$

where $\varTheta_{d-2} = \{(\theta_1, \theta_2, \cdots, \theta_{d-2}) : 0 \le \theta_j \le \pi\ (j = 1, \cdots, d-3),\ 0 \le \theta_{d-2} < 2\pi\}$. Given $\gamma \in (0,1)$, we define the mapping $\varPsi_\gamma : R^d_+ \longrightarrow \varXi_d$ as follows: When $d = 2$, we set for $x = (x_1, x_2) \in R^2_+$

$$\varPsi_\gamma(x) = (r,t), \quad r = |x|\ (> 0), \quad t = x_1 x_2^{-1/\gamma}\ (\in R);$$

When $d \ge 3$, we set for $x = (x', x_d) \in R^d_+$

$$\begin{aligned} &\varPsi_\gamma(x) = (r,t,\theta), \\ &r = |x|\ (> 0), \quad t = |x'| x_d^{-1/\gamma}\ (\ge 0), \\ &\theta = \begin{cases} (0, \cdots, 0)\ (\in \varTheta_{d-2}) & \text{if } |x'| = 0, \\ \theta\ (\in \varTheta_{d-2}) \text{ satisfying } \big(\phi_1(\theta), \cdots, \phi_{d-1}(\theta)\big) = x'/|x'| & \text{if } |x'| > 0. \end{cases} \end{aligned}$$

Here $\big(\phi_1(\theta), \cdots, \phi_{d-1}(\theta)\big)$ is the spherical polar coordinate of S^{d-2}, that is, for $\theta = (\theta_1, \cdots, \theta_{d-2}) \in \varTheta_{d-2}$,

$$\begin{aligned} \phi_1(\theta) &= \cos\theta_1, \\ \phi_2(\theta) &= \sin\theta_1 \cos\theta_2, \\ &\ \vdots \\ \phi_{d-2}(\theta) &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-3} \cos\theta_{d-2}, \\ \phi_{d-1}(\theta) &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{d-3} \sin\theta_{d-2}. \end{aligned}$$

For each $r > 0$, we identify the points $(r, 0, \theta)$, $\theta \in \varTheta_{d-2}$ by regarding them as a same point. Under this identification, $\varPsi_\gamma$ is one to one from R^d_+ onto $\varXi_d$ and the inverse mapping $\varPsi_\gamma^{-1}$ is as follows: For $(r,t) \in \varXi_2$, let $\xi = \xi(r,t)$ be the (unique) positive solution of the equation

$$t^2 \xi^{2/\gamma} + \xi^2 = r^2. \tag{6.2}$$

Then

$$\varPsi_\gamma^{-1}(r,t) = (x_1, x_2), \quad x_1 = t\xi^{1/\gamma}\ (\in R), \quad x_2 = \xi\ (> 0). \tag{6.3}$$

For $(r,t,\theta) \in \varXi_d$ with $d \ge 3$, let $\xi = \xi(r,t)$ be the positive solution of the equation (6.2). Then

$$\begin{aligned} &\varPsi_\gamma^{-1}(r,t,\theta) = (x_1, x_2, \cdots, x_d), \\ &x_i = t\xi^{1/\gamma}\phi_i(\theta)\ (\in R), \quad i = 1, 2, \cdots, d-1, \quad x_d = \xi\ (> 0). \end{aligned} \tag{6.4}$$

We next note that

$$\varPsi_\gamma : C_\gamma(\rho) \longrightarrow \varSigma(\rho) \quad \text{one to one, onto}$$

where $\varSigma(\rho)$ is a subset of $\varXi_d$ given by

$$\varSigma(\rho) = \begin{cases} \{(r,t) : 0 < r < \rho,\ -1 < t < 1\} & \text{if } d = 2, \\ \{(r,t,\theta) : 0 < r < \rho,\ 0 \le t < 1,\ \theta \in \varTheta_{d-2}\} & \text{if } d \ge 3. \end{cases} \tag{6.5}$$

For the sake of convenience, we write $(r,t,\theta) \in \Sigma(\rho)$ in case $d = 2$ too, and use (6.4) with the convention that $\phi_1(\theta) \equiv 1$. Since $\xi = \xi(r,t)$ is the solution of the equation (6.2), $(x_1, \cdots, x_d) = \Psi_\gamma^{-1}(r,t,\theta)$ satisfies the following relations:

$$\frac{\partial x_d}{\partial r} = \frac{r}{\xi + (1/\gamma)t^2\xi^{2/\gamma-1}}, \tag{6.6}$$

$$\frac{\partial x_d}{\partial t} = \frac{-t\xi^{2/\gamma}}{\xi + (1/\gamma)t^2\xi^{2/\gamma-1}}, \tag{6.7}$$

$$\frac{\partial x_d}{\partial \theta_j} = 0 \quad (j = 1, 2, \cdots, d-2), \tag{6.8}$$

and for $i = 1, 2, \cdots, d-1$,

$$\frac{\partial x_i}{\partial r} = \frac{1}{\gamma} t \xi^{1/\gamma-1} \frac{\partial x_d}{\partial r} \phi_i(\theta), \tag{6.9}$$

$$\frac{\partial x_i}{\partial t} = \left(\xi^{1/\gamma} + \frac{1}{\gamma} t \xi^{1/\gamma-1} \frac{\partial x_d}{\partial t} \right) \phi_i(\theta), \tag{6.10}$$

$$\frac{\partial x_i}{\partial \theta_j} = t\xi^{1/\gamma} \frac{\partial \phi_i(\theta)}{\partial \theta_j} \quad (j = 1, 2, \cdots, d-2). \tag{6.11}$$

In the following, $A_1, A_2, \cdots$ denote positive constants depending only on γ^*, ρ^* and d. Let $\gamma^* \le \gamma < 1$, $0 < \rho \le \rho^*$ and $(x_1, x_2, \cdots, x_d) = \Psi_\gamma^{-1}(r,t,\theta)$, $(r,t,\theta) \in \Sigma(\rho)$. Then we get the following estimates by means of (6.2), (6.3), (6.4), (6.6), (6.7):

$$A_1 r \le x_d \le r, \tag{6.12}$$

$$A_2 \le \frac{\partial x_d}{\partial r} \le A_3, \tag{6.13}$$

$$\left| \frac{\partial x_d}{\partial t} \right| \le A_4 r. \tag{6.14}$$

Further we see that the Jacobian determinant is given by

$$J(r,t,\theta) = \frac{\partial(x_1, \cdots, x_d)}{\partial(r,t,\theta_1, \cdots, \theta_{d-2})} = (-1)^{d+1} x_d^{(d-1)/\gamma} \frac{\partial x_d}{\partial r} t^{d-2} S_d(\theta), \tag{6.15}$$

where

$$S_d(\theta) = \begin{cases} 1 & \text{if } d = 2, \\ \sin^{d-3}\theta_1 \sin^{d-4}\theta_2 \cdots \sin\theta_{d-3} & \text{if } d \ge 3, \end{cases} \tag{6.16}$$

for $\theta = (\theta_1, \cdots, \theta_{d-2}) \in \Theta_{d-2}$. By means of (6.12)–(6.16), we readily get

$$|J(r,t,\theta)| \le A_3 r^{(d-1)/\gamma}, \tag{6.17}$$

$$A_5 \rho^{(d-1)/\gamma+1} \le |C_\gamma(\rho)| = \int_{\Sigma(\rho)} |J(r,t,\theta)| \, dr \, dt \, d\theta \le A_6 \rho^{(d-1)/\gamma+1}. \tag{6.18}$$

We next note the following fact:

Lemma 6.1 *Let $\gamma^* \leq \gamma < 1,\ 0 < \rho \leq \rho^*$ and $x^{(i)} \in C_\gamma(\rho),\ i = 0, 1,\ x^{(0)} \neq x^{(1)}$. Set $\left(r^{(i)}, t^{(i)}, \theta^{(i)}\right) = \Psi_\gamma\left(x^{(i)}\right),\ \theta^{(i)} = \left(\theta_1^{(i)}, \cdots, \theta_{d-2}^{(i)}\right),\ i = 0, 1$. For $0 \leq s \leq 1$, put*

$$
\begin{aligned}
r^{(s)} &= r^{(0)} + s\left(r^{(1)} - r^{(0)}\right),\\
t^{(s)} &= t^{(0)} + s\left(t^{(1)} - t^{(0)}\right),\\
\theta^{(s)} &= \left(\theta_1^{(s)}, \cdots, \theta_{d-2}^{(s)}\right),\\
\theta_j^{(s)} &= \theta_j^{(0)} + s\left(\theta_j^{(1)} - \theta_j^{(0)}\right), \quad j = 1, 2, \cdots, d-2.
\end{aligned}
\tag{6.19}
$$

Then $\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)$ belongs to $\Sigma(\rho)$ for $0 \leq s \leq 1$. Moreover the following estimate holds :

$$
\left| \frac{J\left(r^{(0)}, t^{(0)}, \theta^{(0)}\right) J\left(r^{(1)}, t^{(1)}, \theta^{(1)}\right)}{J\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)} \right| \leq A_7\, \rho^{(d-1)/\gamma}, \tag{6.20}
$$

$$
\textit{for} \quad 0 \leq s \leq 1 \quad \textit{if} \quad d = 2, \ \textit{and}
$$

$$
\textit{for} \quad 0 < s < 1 \quad \textit{if} \quad d = 3 \quad \textit{and} \quad \left(t^{(0)} + t^{(1)}\right) \prod_{j=1}^{d-3} \left(\sin\theta_j^{(0)} + \sin\theta_j^{(1)}\right) \neq 0.
$$

Proof In view of (6.5) it is obvious that $\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right) \in \Sigma(\rho),\ 0 \leq s \leq 1$.We now assume that $\left(t^{(0)} + t^{(1)}\right) \prod_{j=1}^{d-3} \left(\sin\theta_j^{(0)} + \sin\theta_j^{(1)}\right) \neq 0$ in case $d \geq 3$. Let $0 \leq s \leq 1$ in case $d = 2$, and $0 < s < 1$ in case $d \geq 3$. Then $\left|J\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)\right| > 0$ by virtue of (6.12), (6.13), (6.15) and (6.16). Moreover

$$
\begin{aligned}
&\left| \frac{J\left(r^{(0)}, t^{(0)}, \theta^{(0)}\right) J\left(r^{(1)}, t^{(1)}, \theta^{(1)}\right)}{J\left(r^{(s)}, t^{(s)}, \theta^{(s)}\right)} \right| \\
&\leq \left(\frac{r^{(0)} r^{(1)}}{A_1\, r^{(s)}} \right)^{(d-1)/\gamma} \frac{A_3^2}{A_2} \left(\frac{t^{(0)} t^{(1)}}{t^{(s)}} \right)^{d-2} \frac{S_d\left(\theta^{(0)}\right) S_d\left(\theta^{(1)}\right)}{S_d\left(\theta^{(s)}\right)}.
\end{aligned}
$$

Note that

$$
r^{(s)} \geq r^{(0)} \wedge r^{(1)}, \quad \text{and hence} \quad r^{(0)} r^{(1)} / r^{(s)} \leq \rho,
$$

and if $d \geq 3$, then

$$
t^{(s)} \geq t^{(0)} \wedge t^{(1)}, \quad \text{and hence} \quad t^{(0)} t^{(1)} / t^{(s)} \leq 1,
$$

and for $j = 1, 2, \cdots, d-3$,

$$
\sin\theta_j^{(s)} \geq \begin{cases} \sin\theta_j^{(0)} & \text{if } \left|\theta_j^{(0)} - \pi/2\right| \geq \left|\theta_j^{(1)} - \pi/2\right|, \\ \sin\theta_j^{(1)} & \text{if } \left|\theta_j^{(0)} - \pi/2\right| \leq \left|\theta_j^{(1)} - \pi/2\right|, \end{cases}
$$

consequently

$$
0 \leq \frac{S_d\left(\theta^{(0)}\right) S_d\left(\theta^{(1)}\right)}{S_d\left(\theta^{(s)}\right)} \leq 1.
$$

We thus get (6.20). □

For an open set $E \subset R^d$, denote by $C^1\left(\overline{E}\right)$ the restrictions to $\overline{E}$ of all continuously differentiable functions on R^d. $c_1,\ c_2$, etc. appearing in what follows denote positive constants depending only on $\gamma^*,\ \rho^*$ and d.

Lemma 6.2 *For any $\kappa \in (0,1]$, there is a positive constant $C_{23} = C_{23}(\kappa, \rho^*, \gamma^*, d)$ such that*

$$\int_{C_\gamma(\rho)} |u|^2\, dx \le C_{23} \left\{ \int_N |u|^2\, dx + \rho^2 \sum_{i=1}^{d} \int_{C_\gamma(\rho)} |\partial_i u|^2\, dx \right\}, \tag{6.21}$$

for $u \in C^1\left(\overline{C_\gamma(\rho)}\right)$, a Borel subset $N \subset C_\gamma(\rho)$ with $|N| \ge \kappa |C_\gamma(\rho)|,\ \gamma^* \le \gamma < 1$ and $0 < \rho \le \rho^*$.

Proof Let $\gamma^* \le \gamma < 1$ and $0 < \rho \le \rho^*$. For $u \in C^1\left(\overline{C_\gamma(\rho)}\right)$, we set

$$\begin{aligned} &x = \Psi_\gamma^{-1}(r,t,\theta) \in C_\gamma(\rho), \\ &\widetilde{u}(r,t,\theta) = u \circ \Psi_\gamma^{-1}(r,t,\theta) = u(x), \\ &\left|\widetilde{\partial u}(r,t,\theta)\right| = \sum_{i=1}^{d} \left| \frac{\partial u}{\partial x_i} \circ \Psi_\gamma^{-1}(r,t,\theta) \right|. \end{aligned}$$

By virtue of (6.8) – (6.11),

$$\begin{aligned} \frac{\partial \widetilde{u}}{\partial r}(r,t,\theta) &= \left\{ \frac{1}{\gamma}\, t\, x_d^{1/\gamma - 1} \sum_{i=1}^{d-1} \frac{\partial u}{\partial x_i}(x)\, \phi_i(\theta) + \frac{\partial u}{\partial x_d}(x) \right\} \frac{\partial x_d}{\partial r}, \\ \frac{\partial \widetilde{u}}{\partial t}(r,t,\theta) &= x_d^{1/\gamma} \sum_{i=1}^{d-1} \frac{\partial u}{\partial x_i}(x)\, \phi_i(\theta) \\ &\quad + \left\{ \frac{1}{\gamma}\, t\, x_d^{1/\gamma - 1} \sum_{i=1}^{d-1} \frac{\partial u}{\partial x_i}(x)\, \phi_i(\theta) + \frac{\partial u}{\partial x_d}(x) \right\} \frac{\partial x_d}{\partial t}, \\ \frac{\partial \widetilde{u}}{\partial \theta_j}(r,t,\theta) &= t\, x_d^{1/\gamma} \sum_{i=j}^{d-1} \frac{\partial u}{\partial x_i}(x)\, \frac{\partial \phi_i}{\partial \theta_j}(\theta), \quad j = 1, 2, \cdots, d-2. \end{aligned}$$

Combining these with (6.12), (6.13), (6.14), we find that

$$\left| \frac{\partial \widetilde{u}}{\partial r}(r,t,\theta) \right| \le A_8 \left| \widetilde{\partial u}(r,t,\theta) \right|, \tag{6.22}$$

$$\left| \frac{\partial \widetilde{u}}{\partial t}(r,t,\theta) \right| \le A_9\, r \left| \widetilde{\partial u}(r,t,\theta) \right|, \tag{6.23}$$

$$\left| \frac{\partial \widetilde{u}}{\partial \theta_j}(r,t,\theta) \right| \le A_{10}\, rt \left| \widetilde{\partial u}(r,t,\theta) \right|, \quad j = 1, 2, \cdots, d-2. \tag{6.24}$$

Let $0 < \kappa \le 1,\ N \subset C_\gamma(\rho)$ with $|N| \ge \kappa |C_\gamma(\rho)|,\ x^{(0)} \in C_\gamma(\rho)$ and $x^{(1)} \in N$. Put $\sigma^{(i)} = \left(r^{(i)}, t^{(i)}, \theta^{(i)}\right) = \Psi_\gamma\left(x^{(i)}\right),\ i = 0, 1$. Then

$$u\left(x^{(0)}\right)-u\left(x^{(1)}\right)=\widetilde{u}\left(\sigma^{(0)}\right)-\widetilde{u}\left(\sigma^{(1)}\right)=-\int_0^1\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)ds,$$

where $\sigma^{(s)}=\left(r^{(s)},t^{(s)},\theta^{(s)}\right)$, and $r^{(s)},\ t^{(s)},\ \theta^{(s)}$ are those given by (6.19). Integrating over $x^{(1)}\in N$, we find that

$$|N|\,\left|u\left(x^{(0)}\right)\right|\leq\int_N|u|\,dx+\int_{\sigma^{(1)}\in\Sigma(\rho)}\left|J\left(\sigma^{(1)}\right)\right|\,d\sigma^{(1)}\int_0^1\left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right|ds.$$

Here $d\sigma^{(i)}$ denotes the product measure $dr^{(i)}dt^{(i)}d\theta^{(i)}$ for each $i=1,2$. From this

$$\begin{aligned}&|N|^2\int_{C_\gamma(\rho)}|u|^2\,dx\\&\leq 2|C_\gamma(\rho)|\left(\int_N|u|\,dx\right)^2\\&\quad+2\int_{\sigma^{(0)}\in\Sigma(\rho)}\left|J\left(\sigma^{(0)}\right)\right|\,d\sigma^{(0)}\left(\int_{\substack{\sigma^{(1)}\in\Sigma(\rho)\\0<s<1}}\left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right|\left|J\left(\sigma^{(1)}\right)\right|\,d\sigma^{(1)}ds\right)^2\\&\equiv 2(I+I\!I).\end{aligned}\tag{6.25}$$

Obviously,

$$I\leq|C_\gamma(\rho)|^2\int_N|u|^2\,dx.\tag{6.26}$$

On account of Lemma 6.1 and (6.17),

$$\begin{aligned}I\!I&\leq\int_{\sigma^{(0)}\in\Sigma(\rho)}d\sigma^{(0)}\int_{\substack{\sigma^{(1)}\in\Sigma(\rho)\\0<s<1}}\left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right|^2\left|J\left(\sigma^{(s)}\right)J\left(\sigma^{(1)}\right)\right|\,d\sigma^{(1)}ds\\&\quad\times\int_{\substack{\sigma^{(1)}\in\Sigma(\rho)\\0<s<1}}\left|\frac{J\left(\sigma^{(0)}\right)J\left(\sigma^{(1)}\right)}{J\left(\sigma^{(s)}\right)}\right|\,d\sigma^{(1)}ds\\&\leq A_3A_7\rho^{2(d-1)/\gamma}\int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\\sigma^{(1)}\in\Sigma(\rho)\\0<s<1}}\left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right|^2\left|J\left(\sigma^{(s)}\right)\right|\,d\sigma^{(0)}d\sigma^{(1)}ds\\&\quad\times\int_{\substack{\sigma^{(1)}\in\Sigma(\rho)\\0<s<1}}d\sigma^{(1)}ds\\&=c_1\rho^{2(d-1)/\gamma+1}\int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\\sigma^{(1)}\in\Sigma(\rho)\\0<s<1}}\left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right|^2\left|J\left(\sigma^{(s)}\right)\right|\,d\sigma^{(0)}d\sigma^{(1)}ds.\end{aligned}\tag{6.27}$$

On the other hand, we get from (6.22)–(6.24)

$$\begin{aligned}\left|\frac{\partial}{\partial s}\widetilde{u}\left(\sigma^{(s)}\right)\right| &\le \left|\frac{\partial\widetilde{u}}{\partial r}\left(\sigma^{(s)}\right)\right|\left|r^{(0)}-r^{(1)}\right| + \left|\frac{\partial\widetilde{u}}{\partial t}\left(\sigma^{(s)}\right)\right|\left|t^{(0)}-t^{(1)}\right| \\ &\quad + \sum_{j=1}^{d-2}\left|\frac{\partial\widetilde{u}}{\partial\theta_j}\left(\sigma^{(s)}\right)\right|\left|\theta_j^{(0)}-\theta_j^{(1)}\right| \\ &\le c_2\rho\sum_{i=1}^{d}\left|\frac{\partial u}{\partial x_i}\circ\Psi_\gamma^{-1}\left(\sigma^{(s)}\right)\right|. \end{aligned} \tag{6.28}$$

Denoting $\dfrac{\partial u}{\partial x_i}\circ\Psi_\gamma^{-1}$ by v_i and substituting (6.28) into (6.27), we arrive at

$$\begin{aligned} I\!I &\le c_3\,\rho^{2(d-1)/\gamma+3}\sum_{i=1}^{d}\int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\ \sigma^{(1)}\in\Sigma(\rho)\\ 0<s<1}} v_i\left(\sigma^{(s)}\right)^2\left|J\left(\sigma^{(s)}\right)\right|\,d\sigma^{(0)}d\sigma^{(1)}ds \\ &\equiv c_3\,\rho^{2(d-1)/\gamma+3}\sum_{i=1}^{d} I\!I_i(\rho). \end{aligned} \tag{6.29}$$

By fixing $\sigma^{(0)} = \left(r^{(0)},t^{(0)},\theta^{(0)}\right) \in \Sigma(\rho)$ and $0 < s < 1$, we make use of the transformation $\sigma^{(1)} = \left(r^{(1)},t^{(1)},\theta^{(1)}\right) \longmapsto \sigma^{(s)} = \left(r^{(s)},t^{(s)},\theta^{(s)}\right)$. Putting $\sigma = (r,t,\theta) = \left(r^{(s)},t^{(s)},\theta^{(s)}\right)$, we find that the Jacobian determinant is given by $\partial\sigma^{(1)}/\partial\sigma = s^{-d}$. Moreover $\sigma = (r,t,s)$ exhausts a set $\Sigma\left(\sigma^{(0)},s\right)$ specified by

$$\begin{aligned} (1-s)r^{(0)} &< r < \rho s + (1-s)r^{(0)}, \\ a_d + (1-s)t^{(0)} &< t < s + (1-s)t^{(0)}, \\ (1-s)\theta_j^{(0)} &< \theta_j < \alpha_j s + (1-s)\theta_j^{(0)}, \quad j = 1,2,\cdots,d-2, \end{aligned}$$

where $a_d = -1$ if $d = 2$, $= 0$ if $d \ge 3$, and $\alpha_j = \pi$ $(j = 1,2,\cdots,d-3)$, $\alpha_{d-2} = 2\pi$. So we get, for each $i = 1,2,\cdots,d$,

$$\begin{aligned} I\!I_i(\rho) &= \int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\ 0<s<1}} d\sigma^{(0)}ds\int_{\sigma^{(1)}\in\Sigma(\rho)} v_i\left(\sigma^{(s)}\right)^2\left|J\left(\sigma^{(s)}\right)\right|\,d\sigma^{(1)} \\ &= \int_{\substack{\sigma^{(0)}\in\Sigma(\rho)\\ 0<s<1}} d\sigma^{(0)}ds\int_{\sigma\in\Sigma\left(\sigma^{(0)},s\right)} v_i(\sigma)^2\left|J(\sigma)\right|s^{-d}\,d\sigma. \end{aligned}$$

By exchanging the order of integration,

$$\begin{aligned} I\!I_i(\rho) &= \int_{\substack{\sigma\in\Sigma(\rho)\\ 0<s<1}} v_i(\sigma)^2|J(\sigma)|s^{-d}\left(\frac{r}{1-s}\wedge\rho - \frac{r-\rho s}{1-s}\vee 0\right) \\ &\quad\times\left(\frac{t}{1-s}\wedge 1 - \frac{t-s}{1-s}\vee a_d\right)\prod_{j=1}^{d-2}\left(\frac{\theta_j}{1-s}\wedge\alpha_j - \frac{\theta_j-\alpha_j s}{1-s}\vee 0\right)d\sigma ds \\ &\le 2^{d+1}\pi^{d-2}\rho\int_{\Sigma(\rho)} v_i(\sigma)^2|J(\sigma)|\,d\sigma. \end{aligned}$$

Combining this with (6.29) and (6.18), we have

$$\begin{aligned} I\!I &\le c_4\rho^{2(d-1)/\gamma+4}\sum_{i=1}^{d}\int_{\Sigma(\rho)} v_i(\sigma)^2|J(\sigma)|\,d\sigma \\ &\le c_5\rho^2|C_\gamma(\rho)|^2\sum_{i=1}^{d}\int_{C_\gamma(\rho)}|\partial_i u|^2\,dx. \end{aligned} \tag{6.30}$$

Since $|N|\ge\kappa|C_\gamma(\rho)|$, (6.25), (6.26) and (6.30) lead us to (6.21). □

Lemma 6.3 *Let $0<\rho\le\rho^*$ and $E(\rho)$ be the following subset of R^d with $d\ge 2$.*

$$E(\rho)=\{(x',x_d)\in B(\rho):x_d>g(x')\},$$

where g is a continuous function on $\{x'\in R^{d-1}:|x'|<\rho^*\}$ such that $g(0)=0$ and $g(x')\le 0$, $|x'|<\rho^*$. Then the statement of Lemma 6.2 with $C_\gamma(\rho)$ replaced by $E(\rho)$ above holds.

Proof Let $u\in C^1\big(\overline{E(\rho)}\big)$ and N $(\subset E(\rho))$ be a Borel subset satisfying $|N|\ge\kappa|E(\rho)|$. For $x=(x',x_d)\in E(\rho)$, we set $\bar x=(x',|x_d|)$. We also use the polar coordinate with center $\bar x:\bar y=\bar x+r\omega,\ r=|\bar y-\bar x|,\ \omega=(\bar y-\bar x)/r\in S^{d-1}$. The following inequality is obvious:

$$\begin{aligned} |N|\,|u(x)| \le& \int_N|u(y)|\,dy+|N|\,\big|u(x)-u(\bar x)\big| \\ &+\int_N\big|u(y)-u(\bar y)\big|\,dy+\int_N\big|u(\bar x)-u(\bar y)\big|\,dy, \end{aligned}$$

from which we obtain

$$\begin{aligned} &|N|^2|u(x)|^2 \\ &\le c_1\Bigg\{\left(\int_N|u|\,dy\right)^2+|N|^2\left(\int_{x_d}^{|x_d|}\partial_d u(x',s)\,ds\right)^2 \\ &\qquad+\left(\int_{y\in E(\rho),y_d<0}dy\int_{y_d}^{|y_d|}|\partial_d u(y',s)|\,ds\right)^2 \\ &\qquad+\sum_{i=1}^{d}\left(\int_{y\in E(\rho),\bar y=\bar x+r\omega}dy\int_0^r\big|\partial_i u(\bar x+s\omega)\big|\,ds\right)^2\Bigg\}, \\ &\le c_1\Bigg\{|E(\rho)|\int_N|u|^2\,dy+2|E(\rho)|^2\rho\int_{x_d}^{|x_d|}\big|\partial_d u(x',s)\big|^2\,ds \\ &\qquad+2|E(\rho)|\rho^2\int_{E(\rho)}|\partial_d u|^2\,dy \\ &\qquad+4|E(\rho)|\rho^{d+1}\sum_{i=1}^{d}\int_{z\in E(\rho),z_d\ge 0}\frac{|\partial_i u(z)|^2}{|\bar x-z|^{d-1}}\,dz\Bigg\}, \end{aligned}$$

where we used the following estimate for the last term.

$$\begin{aligned}
&\left(\int_{y\in E(\rho),\bar{y}=\bar{x}+r\omega} dy \int_0^r |\partial_i u\left(\bar{x}+s\omega\right)|\, ds\right)^2 \\
&\le 4|E(\rho)|\rho \int_{y\in E(\rho),\bar{y}=\bar{x}+r\omega} d\bar{y} \int_0^r |\partial_i u\left(\bar{x}+s\omega\right)|^2\, ds \\
&= 4|E(\rho)|\rho \int_{y\in E(\rho),\bar{y}=\bar{x}+r\omega} r^{d-1}\, dr d\omega \int_0^r |\partial_i u\left(\bar{x}+s\omega\right)|^2\, ds \\
&\le 4|E(\rho)|\rho^{d+1} \int_{z\in E(\rho),z_d\ge 0} \frac{|\partial_i u(z)|^2}{|\bar{x}-z|^{d-1}}\, dz\,.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&|N|^2 \int_{E(\rho)} |u(x)|^2 dx \\
&\le c_2\Bigg\{|E(\rho)|^2 \int_N |u|^2\, dy + |E(\rho)|^2\rho^2 \int_{E(\rho)} |\partial_d u|^2\, dx \\
&\qquad +|E(\rho)|\rho^{d+1} \sum_{i=1}^d \int_{z\in E(\rho),z_d\ge 0} |\partial_i u(z)|^2\, dz \int_{E(\rho)} \frac{dx}{|\bar{x}-z|^{d-1}}\Bigg\} \\
&\le c_3\Bigg\{|E(\rho)|^2 \int_N |u|^2\, dy + |E(\rho)|^2\rho^2 \int_{E(\rho)} |\partial_d u|^2\, dx \\
&\qquad +|E(\rho)|\rho^{d+2} \sum_{i=1}^d \int_{E(\rho)} |\partial_i u|^2\, dx\Bigg\}.
\end{aligned}$$

Noting that $c_4\rho^d \le |E(\rho)| \le c_5\rho^d$, we get the conclusion. □

Proposition 6.1 now follows from Lemmas 6.3 and 6.4.

Proof of Proposition 6.1 Let $k \in I_C$, $0 < \rho \le \rho^*$, and N be a Borel subset of $C_k^*(\rho)$ satisfying $|N| \ge \kappa|C_k^*(\rho)|$. In view of [16; Theorem 1.1.7], $u \circ \Phi_k \in H^1\big(C_{\gamma_k}(\rho)\big)$ provided $u \in H^1\big(C_k^*(\rho)\big)$. By means of [1 : Theorem 3.18], $u \circ \Phi_k$ is approximated by functions belonging to $C^1\left(\overline{C_{\gamma_k}(\rho)}\right)$ in H^1-norm. Noting that $\Phi_k^{-1}(N)$ is a Borel subset of $C_{\gamma_k}(\rho)$ and satisfies $\left|\Phi_k^{-1}(N)\right| \ge \kappa' \left|C_{\gamma_k}(\rho)\right|$ for some $\kappa' \in (0,1]$, we obtain (6.1) with $E_k^*(\rho) = C_k^*(\rho)$ from Lemma 6.3. Similarly (6.1) with $E_k^*(\rho) = Q_k^*(\rho)$ follows from Lemma 6.4. □

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