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Sanov results for Glauber spin-glass dynamics

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Summary. In this paper we prove a Sanov result, i.e. a Large Deviation Principle (**LDP**) for the distribution of the empirical measure, for the annealed Glauber dynamics of the Sherrington-Kirkpatrick spin-glass. Without restrictions on time or temperature we prove a full LDP for the asymmetric dynamics and the crucial upper large deviations bound for the symmetric dynamics. In the symmetric model a new order-parameter arises corresponding to the response function in [SoZi83]. In the asymmetric case we show that the corresponding rate function has a unique minimum, given as the solution of a self-consistent equation. The key argument used in the proofs is a general result for mixing of what is known as Large Deviation Systems (**LDS**) with measures obeying an independent LDP.

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1. Introduction

Sherrington and Kirkpatrick [ShKi75] introduced their model as a "simple" mean-field version of the Edwards-Anderson spin-glass – but the model turned out not to be simple at all – only a few results are proved in a mathematically rigorous way (see for example [AiLeRu87], [CoNe95], [Gui95] and citations therein). In [SoZi83] a dynamical diffusion approach was proposed by Sompolinsky and Zippelius which they used to derive properties of the (statical) model in the limit as $t \to \infty$. This approach was put on firm mathematical ground in a joint work of G. Ben Arous and A. Guionnet which is published in the PhD-thesis of A. Guionnet [Gui95] and a series of papers [BeGu95]-[BeGu94a], which are also part of the PhD-thesis. Based on their large deviations results, they develop a fairly complete picture of the asymmetric model. But in the symmetric case, corresponding to the physical model, they still have a (technical) restriction in their results. If β is the inverse temperature and T the length of the time interval they prove their results under the condition that $\beta^2 T$ is smaller than some fixed constant.

In the Sherrington-Kirkpatrick model (SK-model) it is more natural to use jump processes instead of diffusions. This ansatz was introduced in [Som87]. Some rigorous results for the asymmetric dynamics were proved in [Gru92]. There is a strong advantage in the use of Glauber dynamics. The Girsanov exponent, used to describe the interacting model, is fairly well-behaved, which permits to work without restiction on time and temperature. The price you have to pay for this is the loss of Gaussian techniques.

Let's be more explicit. The SK-model is defined by a random Hamiltonian

$$\mathrm{U}_N^J(\boldsymbol{\sigma}) := -\frac{1}{2\sqrt{N}} \sum_{i \neq j=1}^{N} J^{i,j} \cdot \sigma_i \sigma_j, \quad \boldsymbol{\sigma} \in \mathscr{E}^N$$

on the state space $\mathscr{E}^N := \{-1, 1\}^N$, where for $i > j$ the $J^{i,j}$ are chosen independently according to a centered Gaussian distribution with variance one, and $J^{j,i} := J^{i,j}$ – the coupling-matrix J is supposed to be symmetric. Another way to describe the coupling-matrix is to say that the pairs $(J^{i,j}, J^{j,i})$ are i.i.d. centered Gaussian for $i > j$ with covariance

$$\mathbf{E}_J(J^{i,j})^2 = \mathbf{E}_J(J^{j,i})^2 = 1, \quad \mathbf{E}_J(J^{i,j} \cdot J^{j,i}) = \alpha \tag{1}$$

with $\alpha := 1$. For fixed coupling J the Gibbs measures at temperature β are defined by

$$\mu^J_{\beta,N}(\{\boldsymbol{\sigma}\}) := \frac{1}{Z^J_{\beta,N}} \exp[-\beta \cdot \mathrm{U}_N^J(\boldsymbol{\sigma})], \quad \boldsymbol{\sigma} \in \mathscr{E}^N$$

with the partition-function $Z^J_{\beta,N} := \sum_{\boldsymbol{\sigma} \in \mathscr{E}^N} \exp[-\beta \cdot \mathrm{U}_N^J(\boldsymbol{\sigma})]$ as the normalization. In the dynamical approach a Glauber dynamics for these measures is introduced, i.e. a Markov process with state space $\mathscr{E}^N$ for which the $\mu^J_{\beta,N}$ are reversible invariant measures. Define the field at site i to be

$$\mathrm{h}_i^J(\cdot) : \mathscr{E}^N \to \mathbb{R}, \quad \boldsymbol{\sigma} \mapsto \mathrm{h}_i^J(\boldsymbol{\sigma}) := \frac{1}{\sqrt{N}} \sum_{j=1, j \neq i}^{N} J^{i,j} \cdot \sigma_j .$$

Then the simplest possible choice for the process is a single spin-flip jump process with state space $\mathscr{E}^N$, where the i-th spin changes its state $\sigma_i \to -\sigma_i$ with the transition rate

$$c_i(\boldsymbol{\sigma}) := \frac{1}{1 + \exp[2\beta\sigma_i \mathrm{h}_i^J(\boldsymbol{\sigma})]} .$$

We denote by $\boldsymbol{\sigma}^i := (\sigma_1, \cdots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \cdots)$ the configuration with the i-th coordinate being flipped. Then the transition semi-group $\mathrm{P}_{\boldsymbol{\sigma}', \boldsymbol{\sigma}}(t)$ belonging to the process defined by the rates c_i satisfies the forward equation

$$\frac{d}{dt} \mathrm{P}_{\boldsymbol{\sigma}', \boldsymbol{\sigma}}(t) = \sum_{i=1}^{N} \left(c_i(\boldsymbol{\sigma}^i) \mathrm{P}_{\boldsymbol{\sigma}', \boldsymbol{\sigma}^i}(t) - c_i(\boldsymbol{\sigma}) \mathrm{P}_{\boldsymbol{\sigma}', \boldsymbol{\sigma}}(t) \right)$$

with the initial condition $\mathrm{P}_{\sigma',\sigma}(0) = \delta_{\sigma',\sigma}$. This process can be constructed as a measure on $(\mathscr{D}_{\mathscr{E}}[0,T])^N$, where $\mathscr{D}_{\mathscr{E}}[0,T] =: X$ is the Skorohod space of càdlàg-functions $u : [0,T] \to \mathscr{E}$. For each $x \in \mathscr{D}_{\mathscr{E}}[0,T]^N$

$$t \mapsto \mathrm{h}_i^J(x(t)), \quad x(t) = (x_1(t), \cdots, x_N(t)) \in \mathscr{E}^N,$$

defines a function in the Skorohod space $\mathscr{D}_{\mathbb{R}}[0,T] =: Y$ of real-valued functions on $[0,T]$. We are considering the law of the combined Markov process

$$((\mathrm{h}_1(x), x_1), \ldots, (\mathrm{h}_N(x), x_N)) \in (Y \times X)^N =: Z^N$$

as a measure on the product of the space of paths with the space of the fields, i.e. as a measure on $(Y \times X)^N$. Let $q \in \mathscr{M}(\mathscr{D}_{\mathscr{E}}[0,T])$ be the Markov process starting with some distribution on $\mathscr{E}$ and jumping with rate one between both states. Then we can write the interacting process on the time interval $[0,T]$ via Girsanov formula as

$$p^{N,\mathrm{G},J}(B) := \int_{X^N} \mathbf{1}_B(\mathrm{h}^J(x), x) \cdot \exp\left[\sum_{i=1}^N \mathrm{G}(\mathrm{h}_i^J(x), x_i)\right] q^{\otimes N}(dx) \qquad (2)$$

for some measurable set $B \subset (Y \times X)^N =: Z^N$, where the Girsanov exponent has the simple form

$$\begin{aligned} \mathrm{G}(y,x) \quad := \quad & \sum_{s:x(s)\neq x(s-)} \ln \frac{1}{1+\exp[2\beta y(s-)x(s-)]} \\ & + \int_0^T \frac{\exp[2\beta y(s)x(s)]}{1+\exp[2\beta y(s)x(s)]}\, ds \end{aligned} \qquad (3)$$

for $(y,x) \in Y \times X$. The coupling-matrix J appears in the Eq. (2) only through the field h^J. The idea of Sompolinsky and Zippelius is to average these measures with respect to the distribution of the J's. The physical reasoning behind this is that the "individual dynamics" still evolves with fixed coupling and therefore some information of the distribution of the quenched system can be gained in the limit as $T \to \infty$.

At this point different dynamical models can be defined by choosing different distributions for the coupling-matrix J. The physical SK-model corresponds to the symmetric choice $\alpha = 1$ for the couplings J as in Eq. (1). But every choice $\alpha \in [-1,1]$ is allowed which means that the coupling-matrix J is becoming less and less symmetric up to a complete anti-symmetric matrix for $\alpha = -1$. The most important other choice is the asymmetric case $\alpha = 0$, what implies that the Gaussian variables $J^{i,j}$ and $J^{j,i}$ are independent. We include all possible models $\alpha \in [-1,1]$ in our work to study the influence of the couplings (see [RSZ89] for results in this direction). In the asymmetric model the proofs are much simpler (Sect. 4.1). The effect of the dependence in the coupling-matrix J for $\alpha \neq 0$ can be seen in Sect. 5.

Let's now continue to define the annealed spin-glass dynamics. Following the ideas of [SoZi83] we average the measures $p^{N,\mathrm{G},J}$ with respect to the distribution of the J defined in Eq. (1) for some fixed $\alpha \in [-1, 1]$. Interchanging the order of integration leads to

$$p_\alpha^{N,\mathrm{G}}(B) := \int_{X^N} \int_{Y^N} \mathbf{1}_B(y;x) \cdot \exp\left[\sum_{i=1}^N \mathrm{G}(y_i, x_i)\right] \nu_{\alpha,x}^N(dy)\, q^{\otimes N}(dx), \quad (4)$$

where $\nu_{\alpha,x}^N$ stands for the conditional distribution of the fields $\mathrm{h}^J(x) \in Y^N$ under J for given $x \in X^N$. Since the J^{ij} are centered Gaussian, $\nu_{\alpha,x}^N$ is a centered Gaussian measure on Y^N with covariance

$$\int_{Y^N} y_i(s) y_i(t)\, \nu_{\alpha,x}^N(dy) = \frac{1}{N} \sum_{j=1, j\neq i}^N x_j(s) x_j(t) \quad \text{for} \quad i \in \{1, \cdots, N\}, \quad (5)$$

for the diagonal elements, and

$$\int_{Y^N} y_i(s) y_j(t)\, \nu_{\alpha,x}^N(dy) = \frac{\alpha}{N} x_j(s) x_i(t) \quad \text{for} \quad i, j \in \{1, \cdots, N\}, i \neq j$$

for the off-diagonal elements. Recall that $Z = Y \times X$ and denote by

$$\Theta_N^Z : Z^N \longrightarrow \mathcal{M}_1(\mathrm{Z}), \quad z \longmapsto \frac{1}{N} \sum_{i=1}^N \delta_{z_i}$$

the empirical measure. The measures studied in this paper are the distributions of the empirical measures under $p_\alpha^{N,\mathrm{G}}$, i.e. the measures

$$P_\alpha^{N,\mathrm{G}} := \Theta_N^Z(p_\alpha^{N,\mathrm{G}}), \quad (6)$$

on $\mathcal{M}_1(Z)$, where $\mathrm{f}(\nu) = \nu \circ \mathrm{f}^{-1}$ denotes the image of a measure ν under a map f. $\mathcal{M}_1(Z)$ is the space of Borel probability measures on Z.

In this paper we examine the large deviations properties of the annealed spin-glass dynamics, i.e. of the measures $P_\alpha^{N,\mathrm{G}}$, in the thermodynamic limit as $N \to \infty$. We prove a full LDP for the asymmetric case $\alpha = 0$ in Sect. 4.1 and the upper large deviations bound for the symmetric model and every model $\alpha \neq 0$ in Sect. 6. In the asymmetric case, we prove that the corresponding rate function $\mathrm{S}_0^{\mathrm{G}}$ has a unique minimum $\nu^* \in \mathcal{M}_1(Y \times X)$ given as the solution of the self-consistent equation

$$\nu^* = \exp[\mathrm{G}].\mathcal{N}_{\pi_X(\nu^*)} \otimes q,$$

where $\mathcal{N}_{\pi_X(\nu^*)}$ is a centered Gaussian measure on $\mathcal{D}_{\mathbb{R}}[0, T]$ with the covariance constructed from the X-marginal $\pi_X(\nu^*)$ of the measure ν^*. Via an Borel-Cantelli argument (see [BeGu95], Theorem 2.7) this large deviations result implies the following quenched weak convergence result for the asymmetric spin-glass dynamics.

Corollary 1.1. *Let the J^{ij} be choosen independently. Then for almost all J*

$$P^{N,\mathrm{G},J} \overset{\mathrm{w}}{\Longrightarrow} \delta_{\nu^*} \text{ in } \mathcal{M}_1(Z),$$

where δ_{ν^*} is the Dirac-measure at ν^*.

Also propagation of chaos results (as in [Gui95]) can be derived. The main motivation behind the search for a LDP of the $P_\alpha^{N,\mathrm{G}}$ is to prove weak convergence results and laws of large numbers by analyzing the corresponding rate functions. For such results the upper large deviations bound is sufficient.

We believe that most of the other results for the asymmetric Langevin spin-glass dynamics found in [BeGu95], [Gui95] can be transferred to the Glauber case.

However, we are not going to study the rate function in the symmetric case, which would be the most interesting part from the physical point of view. Since our results are valid for all times and temperatures the longtime behavior as $T \to \infty$ can also be approached. We hope to be able to tackle this problem in a later work.

1.1. Outline of the paper

– Most of the proofs in this paper are based on a general technical result, which is the contents of Sect. 2. Starting point is the fact, that we have (from the construction of the measures) a good representation of the conditional distribution of $P_\alpha^{N,\mathrm{G}}$ given the X-marginal. We will now state the idea behind the technical result: Assume we have a sequence of measures P^N on a Polish space $\mathscr{Z}$ and a continuous surjection

$$\pi : \mathscr{Z} \longrightarrow \mathscr{X}$$

into another Polish space $\mathscr{X}$ such that
1. $Q^N := \pi(P^N)$ satisfies a full LDP with rate $\mathrm{J}_{\mathscr{X}}$,
2. for all fixed $x \in \mathscr{X}$ the conditional distribution P_x^N of P^N given that $\pi = x$, fulfills a LDP with rate $\mathrm{I}(x;\cdot)$ on $\mathscr{Z}$.

Setting $\mathrm{I}(x;A) := \inf_{z\in A} \mathrm{I}(x;z)$ for $A \subset \mathscr{Z}$, we have

$$P^N(A) = \int_{\mathscr{X}} P_x^N(A)\, Q^N(dx) \approx \int_{\mathscr{X}} \exp[-N \cdot \mathrm{I}(x;A)]\, Q^N(dx)$$

and Varadhan's Theorem suggests that the P^N satisfy a LDP with rate

$$\mathrm{S}(z) = \mathrm{I}(\pi(z);z) + \mathrm{J}_{\mathscr{X}}(\pi(z)).$$

In Sect. 2 we prove a result along this line.

– In Sect. 3 we give some general definitions and describe how the measures $P_\alpha^{N,\mathrm{G}}$ of the annealed spin-glass dynamics fit into the concept of Sect. 2.

– For the asymmetric spin-glass dynamics corresponding to the measures $P_0^{N,\mathrm{G}}$ we prove a full LDP in Sect. 4.1. This is possible since the fields having covariance Eq. (5) are independent for $\alpha = 0$. In Sect. 4.2 we prove by a fixed point argument that the rate function $\mathrm{S}_0^{\mathrm{G}}$ governing the LDP of the measures $P_0^{N,\mathrm{G}}$ has a unique minimum, given as the solution of the self-consistent equation Eq. (29).

- The difficult part in the proof of the LDP for $P_\alpha^{N,\mathrm{G}}$, $\alpha \neq 0$, is the weak dependence in the Gaussian variables distributed according to $\nu_{\alpha,x}^N$. This dependence leads to the necessity for the introduction of a new order parameter (see Remark 6.4), which we believe corresponds to the response function in [SoZi83]. The physicist's technique in this situation is the so called Gaussian decoupling or Hubbard-Stratonovich transformation. In this technique complex integration is used. In this paper we give a pure probabilistic proof for this technique, which is contained in Sect. 5. We consider first a simple homogenous model in Sect. 5.1 in order to show the influence of terms of order $\frac{1}{N}$ in the off-diagonal part of a covariance like Eq. (5). In Sect. 5.2 we state the Gaussian decoupling result (in a finite dimensional setting), which is the major step in the proof of the LDP for the "free" spin-glass dynamics P_α^N – defined as $P_\alpha^{N,\mathrm{G}}$ but with $\mathrm{G} \equiv 0$ – i.e. the measures without the physical interaction given by the Girsanov exponent G.
- In Sect. 6 we use the finite dimensional result of Sect. 5 to prove a full LDP for the "free" measures P_α^N and conclude from this principle the full upper bound for the $P_\alpha^{N,\mathrm{G}}$ via Varadhan's Theorem, since $P_\alpha^{N,\mathrm{G}}$ can be written as

$$P_\alpha^{N,\mathrm{G}}(B) = \int_{\mathscr{M}_1(Y\times X)} \mathbf{1}_B(\nu) \exp\left[N \int \mathrm{G}\, d\nu\right] P_\alpha^N(d\nu),$$

for a measurable $B \subset \mathscr{M}_1(Y \times X)$.

2. LDP for mixtures of Large Deviation Systems (LDS)

In this chapter we state the technical result which we need for the proof of a LDP for the situation described in the introduction. Mixtures of LDS were used in the proof of a Sanov result for exchangeable random variables in [DiZa92]. In a mean-field setting the integrating measures – corresponding to the distribution of an order-parameter in physics – will satisfy a LDP.

Let $\mathscr{Z}$ and $\mathscr{X}$ denote Polish spaces and let $\{\mathscr{X}_N\}_{N\in\mathbb{N}}$ be a sequence of measurable subsets of $\mathscr{X}$ with the property that for every point x in a measurable set $\mathscr{X}_\infty \subset \mathscr{X}$ there exists a sequence (x_N), $x_N \in \mathscr{X}_N$, converging to x; we call such a sequence a $(\mathscr{X}_N)$-sequence. We assume that we have a continuous surjective map $\pi : \mathscr{Z} \to \mathscr{X}$ and a family $\Pi = \{P_x^N : x \in \mathscr{X}_N, N \in \mathbb{N}\}$ of finite measures on the Borel σ-field $\mathscr{B}(\mathscr{Z})$ such that

$$P_x^N(\pi^{-1}\{x\}^c) = 0 \quad \forall x \in \mathscr{X}_N,\ N \in \mathbb{N}. \tag{7}$$

We assume further that $P_x^N(\mathscr{Z}) \leq \exp[-N\kappa]$ for some fixed constant $\kappa \in \mathbb{R}$. Let $\mathrm{I} : \mathscr{X}_\infty \times \mathscr{Z} \to [\kappa, \infty]$ be some function and define $\mathrm{I}(x; A) := \inf_{z\in A} \mathrm{I}(x; z)$ for a set $A \subset \mathscr{Z}$. We will use the following (slightly modified) concept due to [DaGä94] (Definition 1.1 and Definition 1.2):

Definition 2.1. *We call Π a Large Deviation System with rate function* I *if the following conditions are satisfied:*

(i) *Compactness of the level sets: For each $x \in \mathscr{X}^{\cdot}_{\infty}$ and each $\rho \geq \kappa$ the set*

$$\Phi(x;\rho) := \{z \in \mathscr{Z} : \mathrm{I}(x;z) \leq \rho\} \tag{8}$$

is compact.

(ii) *Lower large deviations bound: We have the inequality*

$$\liminf_{N\to\infty} \frac{1}{N} \ln P^N_{x_N}(G) \geq -\mathrm{I}(x;G) \tag{9}$$

for each open set G in $\mathscr{Z}$, each $x \in \mathscr{X}^{\cdot}_{\infty}$ and each $(\mathscr{X}^{\cdot}_N)$-sequence converging to x.

(iii) *Upper large deviations bound: We have the inequality*

$$\limsup_{N\to\infty} \frac{1}{N} \ln P^N_{x_N}(F) \leq -\mathrm{I}(x;F) \tag{10}$$

for each closed set F in $\mathscr{Z}$, each $x \in \mathscr{X}^{\cdot}_{\infty}$ and each $(\mathscr{X}^{\cdot}_N)$-sequence converging to x.

Observe that we are not dealing with probability measures - that turns out to be useful to treat interacting systems. The lower bound Eq. (9) and Eq. (7) imply that $\mathrm{I}(x;z) = \infty$ for $z \notin \pi^{-1}(\{x\})$, so we can define the function

$$\begin{aligned} \mathrm{J} : \mathscr{Z} &\longrightarrow [\kappa,\infty] \\ z &\longmapsto \mathrm{J}(z) := \begin{cases} \mathrm{I}(\pi(z);z) & \text{if} \quad z \in \pi^{-1}(\mathscr{X}^{\cdot}_{\infty}) \\ \infty & \text{otherwise} \end{cases} . \end{aligned} \tag{11}$$

Remark 2.2. Since $\mathscr{X}^{\cdot}$ is a metric space, each point $x \in \mathscr{X}^{\cdot}_{\infty}$ has a countable base for the neighborhood in $\mathscr{X}^{\cdot}$ and we have the following observations [DiZa92]:

1. For each closed set $F \subset \mathscr{Z}$ and each $x \in \mathscr{X}^{\cdot}_{\infty}$ such that $\mathrm{I}(x;F) < \infty$ there exists for each $\delta > 0$ a neighborhood U of x in $\mathscr{X}^{\cdot}$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \ln \left[\sup_{x' \in \mathscr{X}^{\cdot}_N \cap U} P^N_{x'}(F) \right] \leq -\mathrm{I}(x;F) + \delta. \tag{12}$$

   If $\mathrm{I}(x;F) = \infty$ there exists for each $L \in \mathbb{R}$ a neighborhood U of x such that the l.h.s. of Eq. (12) is smaller than $-L$.

2. For each open set $G \subset \mathscr{Z}$, each $\delta > 0$ and each $x \in \mathscr{X}^{\cdot}_{\infty}$ there is a neighborhood U of x such that

$$\liminf_{N\to\infty} \frac{1}{N} \ln \left[\inf_{x' \in \mathscr{X}^{\cdot}_N \cap U} P^N_{x'}(G) \right] \geq -\mathrm{I}(x;G) - \delta. \tag{13}$$

We will call the LDS Π measurable if the maps $P_{\cdot}^{N} : \mathscr{X}_N \to \mathscr{M}(\mathscr{Z})$ are $\mathscr{B}(\mathscr{X}) \cap \mathscr{X}_N - \mathscr{B}(\mathscr{M}(\mathscr{Z}))$-measurable, where the space $\mathscr{M}(\mathscr{Z})$ of finite measures on $\mathscr{Z}$ is equipped with the weak topology and $\mathscr{B}(\mathscr{X}) \cap \mathscr{X}_N$ is the trace $\boldsymbol{\sigma}$-field on $\mathscr{X}_N$. Let $\left(Q^N \in \mathscr{M}_1(\mathscr{X})\right)$ be a sequence of probability measures on $\mathscr{X}$, which satisfies a full LDP with a (good) rate function $\mathrm{J}_{\mathscr{X}}$ (the definition can be looked up in [DeZe93], Chap. 1.2). We make the following assumptions:

1. $Q^N(\mathscr{X}_N) = 1 \quad \forall N \in \mathbb{N}$,
2. $\Phi_{\rho}^{\mathscr{X}} := \{\mathrm{J}_{\mathscr{X}} \leq \rho\} \subset \mathscr{X}_{\infty}$.

We set for $B \in \mathscr{B}(\mathscr{Z})$

$$P^N(B) := \int_{\mathscr{X}_N} P_x^N(B)\, Q^N(dx). \tag{14}$$

By the monotone-class theorem and monotone convergence this defines a sequence of finite measures on $\mathscr{Z}$, which maybe none-normalized. Analogous to Varadhan's Theorem we have the

Theorem 2.3. *For the sequence of measures P^N a full LDP with good rate function*

$$\mathrm{S}(z) := \mathrm{J}(z) + \mathrm{J}_{\mathscr{X}}(\pi(z)) \tag{15}$$

holds.

Proof. S is lower semi-continuous: Let $z \in \mathscr{Z}$ be such that $\mathrm{S}(z) < \infty$. We have to show that for $\delta > 0$ we can find a neighborhood $V \subset \mathscr{Z}$ of z such that

$$\mathrm{S}(V) \geq \mathrm{S}(z) - \delta. \tag{16}$$

For a fixed $x \in \mathscr{X}_{\infty}$ the function $\mathrm{I}(x;\cdot)$ is lower semi-continuous and we can find an $\epsilon > 0$ such that $\mathrm{I}(\pi(z); \overline{B_{\epsilon}(z)}) \geq \mathrm{I}(\pi(z), z) - \frac{\delta}{3}$, where $B_{\epsilon}(z)$ is the closed ϵ-ball around z. Because of Remark 2.2 we find an open set $U \subset \mathscr{X}$, $\pi(z) \in U$ such that Eq. (12) holds with $F := \overline{B_{\epsilon}(z)}$, $x := \pi(z)$ for a constant $\frac{\delta}{3}$. This implies together with the upper LD-bound Eq. (10) that

$$\mathrm{I}(\pi(z); z) - \frac{2\delta}{3} \leq \mathrm{I}(\pi(z); \overline{B_{\epsilon}(z)}) - \frac{\delta}{3} \leq \mathrm{I}(x; \overline{B_{\epsilon}(z)})$$

for all $x \in U \cap \mathscr{X}_{\infty}$. Because $\mathrm{J}_{\mathscr{X}}$ is lower semi-continuous there is an open set U', $\pi(z) \in U' \subset \mathscr{X}$ such that $\mathrm{J}_{\mathscr{X}}(U') \geq \mathrm{J}_{\mathscr{X}}(\pi(z)) - \frac{\delta}{3}$. Then for $V := B_{\epsilon}(z) \cap \pi^{-1}(U \cap U')$ Eq. (16) is obtained.

Upper bound: We fix some closed set $F \subset \mathscr{Z}$ and choose $\epsilon > 0$ and $L \geq 0$. We denote by $\Phi_L^{\mathscr{X}} \subset \mathscr{X}_{\infty}$ the compact level set for L of $\mathrm{J}_{\mathscr{X}}$. Because of Eq. (12) and the lower semi-continuity of $\mathrm{J}_{\mathscr{X}}$ we find an open cover $\Phi_L^{\mathscr{X}} \subset \bigcup_{i=1}^k U_{x_i} =: C_L$, $x_i \in \Phi_L^{\mathscr{X}}$, $x_i \in U_{x_i}$, such that

$$\limsup_{N\to\infty} \frac{1}{N} \ln \left[\sup_{x \in \mathscr{X}_N \cap \overline{U}_{x_i}} P_x^N(F) \right] \leq -K_{x_i} + \frac{\epsilon}{3}$$

and $\inf_{x\in\overline{U}_{x_i}} \mathrm{J}_{\mathscr{X}} =: \mathrm{J}_{\mathscr{X}}(\overline{U}_{x_i}) \geq \mathrm{J}_{\mathscr{X}}(x_i) - \frac{\epsilon}{3}$, where

$$K_{x_i} := \begin{cases} \mathrm{I}(x_i; F) & \text{if } \mathrm{I}(x_i; F) < \infty \\ L - \kappa + \epsilon & \text{otherwise} \end{cases}$$

Thus, there exists some N_0 such that for all $N \geq N_0$ we have the inequality

$$\begin{aligned} P^N(F) &= \int_{\mathscr{X}_N} P_x^N(F)\, Q^N(dx) \leq Q^N(C_L^c) + \sum_{i=1}^k \int_{\overline{U}_{x_i} \cap \mathscr{X}_N} P_x^N(F)\, Q^N(dx) \\ &\leq Q^N(C_L^c) + \sum_{i=1}^k \exp\left[-N \left(K_{x_i} - \frac{2\epsilon}{3} \right) \right] \cdot Q^N(\overline{U}_{x_i}). \end{aligned}$$

Taking logarithm, using the upper LD-bound for Q^N and [DeZe93], Lemma 1.2.15 leads to

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N} \ln P^N(F) &\leq -\min_{i\in\{1,\cdots,k\}} \{K_{x_i} + \mathrm{J}_{\mathscr{X}}(x_i) - \epsilon\} \wedge L \\ &\leq -\inf\{\mathrm{I}(x;F) + \mathrm{J}_{\mathscr{X}}(x) - \epsilon : x \in \pi(F)\} \wedge L \\ &= -(\inf_{z\in F} \mathrm{S}(z) - \epsilon) \wedge L. \end{aligned}$$

Since ϵ can be chosen arbitrarily, we obtain the upper bound by letting $L \to \infty$.

Lower bound: Let $G \subset \mathscr{Z}$ be open and fix $z \in G$ and $\epsilon > 0$. Since $\mathrm{J}(z) = \infty$ for $z \notin \pi^{-1}(\mathscr{X}_\infty)$, we assume that $z \in \pi^{-1}(\mathscr{X}_\infty)$. Because of (13) we have an open U, $\pi(z) \in U$ such that

$$\liminf_{N\to\infty} \frac{1}{N} \ln \left[\inf_{x\in\mathscr{X}_N \cap U} P_x^N(G) \right] \geq -\mathrm{I}(\pi(z); G) - \frac{\epsilon}{2}.$$

Then there exists some N_0 such that for all $N \geq N_0$ we have

$$P^N(G) \geq \int_{U\cap\mathscr{X}_N} P_x^N(G)\, Q^N(dx) \geq \exp[-N(\mathrm{I}(\pi(z); G) + \epsilon)] \cdot Q^N(U).$$

Taking the limit as $N \to \infty$ leads to

$$\liminf_{N\to\infty} \frac{1}{N} \ln P^N(G) \geq -\mathrm{I}(\pi(z); G) - \mathrm{J}_{\mathscr{X}}(U) - \epsilon \geq -\mathrm{I}(\pi(z); z) - \mathrm{J}_{\mathscr{X}}(\pi(z)) - \epsilon.$$

This gives the lower bound since ϵ was arbitrary.

Compactness of $\Phi_\rho := \{z : \mathrm{S}(z) \leq \rho\}$: (see [DiZa92]) Let's assume that Φ_ρ is not compact for some $\rho \geq \kappa$. Then we have a sequence of points $z_i \in \Phi_\rho$

such that (z_i) does not have a convergent subsequence. Because of Eq. (15) and the bound $\mathrm{J} \geq \kappa$, $x_i := \pi(z_i)$ is a sequence in the level set $\Phi^{\mathscr{X}}_{\rho-\kappa} := \{\mathrm{J}_{\mathscr{X}} \leq \rho-\kappa\} \subset \mathscr{X}_\infty$ and has a convergent subsequence $x_{i_k} \to x$ since $\Phi^{\mathscr{X}}_{\rho-\kappa}$ is compact. Let x_N be an $(\mathscr{X}_N)$-sequence converging to x and $K_x \subset \mathscr{Z}$ be a compact subset with $\limsup_{N\to\infty} \frac{1}{N} \ln P^N_{x_N}(K_x^c) \leq -3\rho$; such a set K_x exists, i.e. $P^N_{x_N}$ is exponentially tight, because of Definition 2.1 and Theorem (P), [Puk91] (see also [DeZe93], Example 4.1.10 for an outline of the proof). Because z_i does not have a convergent subsequence there is a k_0 such that $z_{i_k} \notin K_x$ for all $k \geq k_0$. Let $D := \{z_{i_k} : k \geq k_0\}$. D is closed because z_{i_k} cannot have any accumulation point. Since $\mathscr{Z}$ is metric we can find disjoint open sets U_D, U_K with $K_x \subset U_K$ and $D \subset U_D$. Because of Eq. (12) we can find some open set V, $x \in V$ such that:

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \ln \left[P^N(U_K^c \cap \pi^{-1}(\overline{V}))\right] &\leq \limsup_{N\to\infty} \frac{1}{N} \ln \left[\sup_{x' \in \overline{V} \cap \mathscr{X}_N} P^N_{x'}(U_K^c)\right] \\
&\leq \limsup_{N\to\infty} \frac{1}{N} \ln P^N_{x_N}(U_K^c) + \rho \qquad (17) \\
&\leq \limsup_{N\to\infty} \frac{1}{N} \ln P^N_{x_N}(K_x^c) + \rho \leq -2\rho
\end{aligned}$$

Applying the lower bound to $U_D \cap \pi^{-1}(V)$ and observing that $z_{i_k} \in \pi^{-1}(V)$ for k large enough we finally obtain

$$\liminf_{N\to\infty} \frac{1}{N} \ln P^N(U_D \cap \pi^{-1}(V)) \geq -\inf_{z \in U_D \cap \pi^{-1}(V)} \mathrm{S}(z) \geq -\rho. \qquad (18)$$

But Eq. (17) and Eq. (18) cannot hold simultaneously and therefore Φ_ρ must be compact for every $\rho \geq \kappa$. □

3. General definitions

In this section we give some general definitions and describe how the annealed spin-glass dynamics fits into the concept of Sect. 2.

Let $X := \mathscr{D}_{\mathscr{E}}[0,T]$ and $Y := \mathscr{D}_{\mathbb{R}}[0,T]$ be the Skorohod spaces of functions on the interval $[0,T]$ with values in $\mathscr{E} := \{-1,+1\}$ and in $\mathbb{R}$ respectively, and define $Z := Y \times X$. $q \in \mathscr{M}(X)$ is the measure on X corresponding to the process jumping between the two states with rate 1 and starting with some fixed distribution on $\mathscr{E}$. Denote by

$$\#_{[0,T]}(x) := \#\{s \in [0,T] : x(s) \neq x(s-)\}$$

the number of jumps of x on the time interval $[0,T]$. $\#_{[0,T]}$ is a continuous map since $\mathscr{E}$ is a descrete topological space and the peculiarity of the Skorohod topology at the endpoint T. For our Sanov result we need a stronger topology on some subset of measures, such that we can easily integrate over the unbounded function $\#_{[0,T]}$. Define

$$\mathscr{X} := \left\{ \nu \in \mathscr{M}_1(X) : \int \#_{[0,T]} \, d\nu < \infty \right\}$$

and

$$\mathscr{Z} := \left\{ \mu \in \mathscr{M}_1(Z) : \int \#_{[0,T]}(x) \, \mu(dy, dx) < \infty \right\},$$

respectively, and equip both spaces with the weakest topology, such that the maps $\nu \mapsto \int f(x) \, \nu(dx)$ and $\mu \mapsto \int g(y, x) \, \mu(dy, dx)$ are continuous for all continuous functions

$$f : X \to \mathbb{R}, \qquad \frac{|f|}{\#_{[0,T]} + 1} \in \mathscr{C}_b(X),$$

and

$$g : Z \to \mathbb{R}, \qquad \frac{|g|}{\#_{[0,T]} + 1} \in \mathscr{C}_b(Z),$$

respectively. With these topologies both spaces are Polish spaces [Léo87], where the Borel $\boldsymbol{\sigma}$-field on $\mathscr{Z}$ and $\mathscr{X}$ are the same as the trace $\boldsymbol{\sigma}$-fields, i.e. $\mathscr{B}(\mathscr{Z}) = \mathscr{B}(\mathscr{M}_1(Z)) \cap \mathscr{Z}$ and $\mathscr{B}(\mathscr{X}) = \mathscr{B}(\mathscr{M}_1(X)) \cap \mathscr{X}$. The Sanov result holds for the measures $Q^N := \Theta_N^X(q^{\otimes N})$ on $\mathscr{X}$ since the condition (Eq. (0.3) in [Léo87])

$$\int \exp[\alpha(\#_{[0,T]} + 1)] \, dq = \exp[\alpha + T(\mathrm{e}^{\alpha} - 1)] < \infty$$

is satisfied for all $\alpha \in \mathbb{R}$. The rate function for the LDP of the Q^N is the relative entropy $\mathrm{H}(\cdot|q)$.

We define the surjection

$$\pi_X : \mathscr{Z} \to \mathscr{X} \quad \nu \mapsto \pi_X(\nu)$$

to be the projection of a measure on its X-marginal. Let

$$\Theta_N^Z : Z^N \to \mathscr{Z}, \qquad z \mapsto \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i}$$

be the continuous map on the empirical measure and define Θ_N^X similar. We set

$$\mathscr{X}_N := \Theta_N^X(X^N) \subset \mathscr{X}$$

and $\mathscr{X}_\infty := \{\nu \in \mathscr{X} : \mathrm{H}(\nu|q) < \infty\}$. If we denote by

$$p_{\alpha,x}^{N,\mathrm{G}}(dy, dx) := \exp\left[\sum_{i=1}^{N} \mathrm{G}(y_i, x_i) \right] \nu_{\alpha,x}^N(dy) \cdot \delta_x(dx) \tag{19}$$

the (none-normalized) conditional distribution of the measure $p_\alpha^{N,\mathrm{G}}$ defined in Eq. (4) for given X-coordinates $x \in X^N$, we can write

$$p_\alpha^{N,\mathrm{G}}(B) = \int_{X^N} p_{\alpha,x}^{N,\mathrm{G}}(B) \, q^{\otimes N}(dx)$$

for a measurable set $B \subset Z^N$. The measures $p^{N,\mathrm{G}}_{\alpha,x}$ are invariant under permutations, i.e. for each permutation ι of $\{1,\dots,N\}$ we have $p^{N,\mathrm{G}}_{\alpha,x} = \iota(p^{N,\mathrm{G}}_{\alpha,\iota(x)})$. Hence, we can write the conditional distribution of the measures $P^{N,\mathrm{G}}_{\alpha}$ defined in Eq. (6) for the given X-marginal $\rho_N \in \mathscr{X}_N$ as

$$P^{N,\mathrm{G}}_{\alpha,\rho_N} := \Theta^Z_N(p^{N,\mathrm{G}}_{\alpha,x^{(N)}}), \tag{20}$$

where $x^{(N)} \in X^N$ is such that

$$\Theta^X_N(x^{(N)}) = \rho_N .$$

The map $P^{N,\mathrm{G}}_{\alpha,(\cdot)} : \mathscr{X}_N \to \mathscr{M}(\mathscr{Z})$ is measurable. Therefore we have the representation

$$P^{N,\mathrm{G}}_{\alpha}(A) = \int_{\mathscr{X}_N} P^{N,\mathrm{G}}_{\alpha,\rho}(A)\, Q^N(d\rho)$$

for $A \subset \mathscr{Z}$ measurable. Because of Theorem 2.3 we can prove a LDP of the measures $P^{N,\mathrm{G}}_{\alpha}$ by showing, that

$$\Pi = \left\{P^{N,\mathrm{G}}_{\alpha,\rho_N} : \rho_N \in \mathscr{X}_N\right\}$$

is a LDS (see Definition 2.1), i.e. for every $\mathscr{X}_N$-sequence $\rho_N \to \rho \in \mathscr{X}_\infty$ the measures $P^{N,\mathrm{G}}_{\alpha,\rho_N}$ satisfy a LDP with a good rate function depending only on the limit $\rho \in \mathscr{X}_\infty$.

For $\mathrm{G} \equiv 0$ we define analogously $P^N_{\alpha,\rho_N} := P^{N,\mathrm{G}\equiv 0}_{\alpha,\rho_N}$ and $P^N_{\alpha} := P^{N,\mathrm{G}\equiv 0}_{\alpha}$, corresponding to a "free" model without the interaction given by the Girsanov exponent G.

4. Asymmetric spin-glass dynamics

4.1. LDP for the asymmetric dynamics

Annealed asymmetric spin-glass dynamics means that we are interested in a LDP of the distribution of the empirical measure under (see Eq. (4))

$$p^{N,\mathrm{G}}_0(B) := \int_{X^N}\int_{Y^N} \mathbf{1}_B(y,x)\cdot \exp\left[\sum_{i=1}^N \mathrm{G}(y_i,x_i)\right]\ \nu^N_{0,x}(dy)\, q^{\otimes N}(dx),$$

i.e. under the average of the dynamic in the case of the completely independent coupling matrix J ($\alpha = 0$). This situation is considerably simple, since for fixed $x \in X^N$ the distribution of the fields

$$\mathrm{h}^J_i(x(\cdot)) \in Y, \quad i = 1,\dots,N,$$

are i.i.d. centered Gaussian with covariance (compare Eq. (5))

$$\mathbf{E}_J\left(\mathrm{h}^J_i(x(t))\mathrm{h}^J_i(x(s))\right) = \frac{1}{N}\sum_{j=1}^N x_j(t)x_j(s)$$

for $s,t \in [0,T]$ – for convenience we add the missing diagonal terms ($i = j$) which has no influence on a large deviations scale. The covariance is only a function of the empirical measure. We define

$$\mathrm{D}_\rho(s,t) := \int x(s)x(t)\,\rho(dx) \tag{21}$$

for $\rho \in \mathscr{X}$ and denote with $\mathscr{N}_\rho \in \mathscr{M}_1(Y)$ the Gaussian measure on Y with covariance D_ρ (see Proposition 4.4). Because of the independence of the fields y_i under $\nu^N_{0,x}$ the none-normalized conditional distribution $p^{N,\mathrm{G}}_{\alpha=0,x}$ in Eq. (19) has the simple form

$$p^{N,\mathrm{G}}_{0,x}(dy,dx) := \exp\left[\sum_i \mathrm{G}(y_i,x_i)\right]\left((\mathscr{N}_{\Theta^X_{N,x}} \otimes \delta_{x_1}) \otimes \cdots (\mathscr{N}_{\Theta^X_{N,x}} \otimes \delta_{x_N})\right).$$

As in Sect. 3. we denote by $P^{N,\mathrm{G}}_{0,\rho} := \Theta^Z_N(p^{N,\mathrm{G}}_{0,x})$ for $x \in X^N$ such that $\Theta^X_N(x) = \rho_N \in \mathscr{X}_N$. We will prove the following results for the asymmetric spin-glass dynamics.

Lemma 4.1. *The system*

$$\Pi := \left\{P^{N,\mathrm{G}}_{0,\rho_N} : \rho_N \in \mathscr{X}_N\right\}$$

is a LDS with rate

$$\mathrm{I}^{\mathrm{G}}_0(\rho;\nu) := \begin{cases} \mathrm{H}(\nu|\mathscr{N}_\rho \otimes \rho) - \int \mathrm{G}\,d\nu & \text{if } \pi_X(\nu) = \rho \\ \infty & \text{otherwise} \end{cases}$$

and the

Corollary 4.2. *The probability measures $P^{N,\mathrm{G}}_0 \in \mathscr{M}_1(\mathscr{Z})$ obey a full LDP with rate*

$$\mathrm{S}^{\mathrm{G}}_0(\nu) := \mathrm{H}(\nu|\mathscr{N}_{\pi_X(\nu)} \otimes q) - \int \mathrm{G}\,d\nu = \mathrm{H}(\nu|\exp[\mathrm{G}]\mathscr{N}_{\pi_X(\nu)} \otimes q).$$

Before we prove these results we state some properties of G and the map $\rho \mapsto \mathscr{N}_\rho$. An immediate consequence of the definition of G in (3) and the Skorohod topology ([Bill68], Chapter 3) is:

Remark 4.3. $\mathrm{G} : Z \to \mathbb{R}$ is measurable. G is continuous at $(y,x) \in Z$ when y is continuous at the (finite) jumps $s \in [0,T], x(s) \neq x(s-)$ of x. $\mathrm{G}(z) \le T$ for all $z \in Z$. We have a very crude estimate

$$\|\mathrm{G}(y,x) - \mathrm{G}(y',x)\| \le 2\beta(\#_{[0,T]}(x) + T)\|y - y'\|_\infty, \tag{22}$$

where $\|y\|_\infty := \sup_{s\in[0,T} |y(s)|$ is the ∞-norm.

Because of the discontinuity of G we have to use at some places in the proofs a approximate version of G, which we define now. For a function $y \in Y$ we define for $\delta > 0$ a smoothed version by

$$y^{\delta}(t) := \frac{1}{\delta}\left[\int_{(t-\delta)\wedge 0}^{t} y(s)\,ds + ((\delta - t)\vee 0)\,y(0)\right]. \tag{23}$$

We define a lower cutoff for the function G as:

$$\begin{aligned} \mathrm{G}^L(y,x) \quad := \quad & \sum_{s:x(s)\neq x(s-)} \ln \frac{1}{1+\exp[2\beta y(s-)x(s-)]\wedge L} \\ & + \int_0^T \frac{\exp[2\beta y(s)x(s)]\wedge L}{1+\exp[2\beta y(s)x(s)]\wedge L}\,ds. \end{aligned} \tag{24}$$

We obtain for G^L the monotony

$$\mathrm{G} \le \mathrm{G}^L + \frac{T}{1+L} \tag{25}$$

and the bound

$$T > \mathrm{G}^L(y,x) \ge -\ln(1+L)\#_{[0,T]}(x). \tag{26}$$

The functions $\mathrm{G}^L((\cdot)^\delta,\cdot) =: \mathrm{G}^L_\delta$ and $\mathrm{G}((\cdot)^\delta,\cdot) =: \mathrm{G}_\delta$ are continuous everywhere and converge point-wise to G^L, G respectively, for $\delta \to 0$.

Proposition 4.4. *For every measure $\rho \in \mathscr{X}$ there exists a centered Gaussian measure $\mathscr{N}_\rho \in \mathscr{M}_1(Y)$ with covariance Eq. (21). If $\rho_N \Rightarrow \rho \in \mathscr{X}_\infty$ in the stronger topology on $\mathscr{X}$ then*

$$\mathscr{N}_{\rho_N} \overset{w}{\Longrightarrow} \mathscr{N}_\rho$$

in $\mathscr{M}_1(Y)$, i.e. weakly, and $\mathscr{N}_\rho(\mathscr{C}_{\mathbb{R}}[0,T]) = 1$, i.e. the fields y are $\mathscr{N}_\rho$-a.s. continuous.

Proof. Because of the definition, D_ρ is a covariance and there exists a centered Gaussian process $X_t, t \in [0,T]$ with covariance D_ρ. For $\rho \in \mathscr{X}$

$$F_\rho(t) := \int \#_{[0,t]}(x)\,\rho(dx), \quad t \in [0,T]$$

is a well-defined, nondecreasing, right-continuous function on $t \in [0,T]$. We calculate:

$$\begin{aligned} &\mathbf{E}(X_{t_2}-X_t)^2(X_t-X_{t_1})^2 \le \sqrt{\mathbf{E}(X_{t_2}-X_t)^4\mathbf{E}(X_t-X_{t_1})^4} \\ &\quad = \quad 3(D_\rho(t_2,t_2)+D_\rho(t,t)-2D_\rho(t_2,t))(D_\rho(t_1,t_1)+D_\rho(t,t)-2D_\rho(t_1,t)) \\ &\quad = \quad 3\int (x(t_2)-x(t))^2\,\rho(dx)\int (x(t)-x(t_1))^2\,\rho(dx) \\ &\quad \le \quad 3\cdot 2^4\left[F_\rho(t_2)-F_\rho(t)\right]\left[F_\rho(t)-F_\rho(t_1)\right] \end{aligned} \tag{27}$$

for $t_2 \geq t \geq t_1$ and therefore [Bill68], p.133-134, ensures the existence of the measure $\mathscr{N}_\rho^{\cdot}$. Let now $\rho \in \mathscr{X}_\infty$, i.e. $\mathrm{H}(\rho|q) < \infty$ implying $\rho \ll q$. Then

$$F_\rho(t) - F_\rho(t-) = \rho(\{x(t) \neq x(t-)\}) = 0,$$

i.e. F_ρ is continuous, since the distribution of the jumps under q is the same as under a Poisson process. Then $\#_{[0,t]}(\cdot)$ is ρ a.s. continuous which implies $F_{\rho_N}(t) \to F_\rho(t)$ for all $t \in [0,T]$ and since F_ρ is continuous and monotone the convergence is uniform. Calculating as in Eq. (27) shows

$$\int (x(t_2) - x(t))^2 (x(t) - x(t_1))^2 \, \rho_N(dx) \leq 3 \cdot 2^4 \left[F_{\rho_N}(t_2) - F_{\rho_N}(t_1)\right]^2,$$

and a slight modification of [Bill68], Theorem 15.6, proves that $\mathscr{N}_{\rho_N}^{\cdot} \overset{\mathrm{w}}{\Longrightarrow} \mathscr{N}_\rho^{\cdot}$. Since

$$\int (x(t_2) - x(t_1))^4 \, \rho(dx) \leq 3 \cdot 2^2 \left[F_\rho(t_2) - F_\rho(t_1)\right]^2$$

[Bill68], Theorem 12.4, shows $\mathscr{N}_\rho^{\cdot}(\mathscr{C}_{\mathbb{R}}[0,T]) = 1$. □

Remark 4.5. Instead of the usual Skorohod topology $\mathscr{T}$, the space $\mathscr{D}_{\mathbb{R}}[0,T]$ can be equipped with the topology generated by the uniform norm $\|\cdot\|_\infty$. Denote this topology by $\mathscr{T}_\infty$ and the corresponding Borel σ-field by $\mathscr{U} := \boldsymbol{\sigma}(\mathscr{T}_\infty)$. Because $\mathscr{T} \subset \mathscr{T}_\infty$,

$$\boldsymbol{\sigma}(\mathscr{T}) \subset \boldsymbol{\sigma}(\mathscr{T}_\infty) = \mathscr{U}.$$

If $\rho_N \Rightarrow \rho \in \mathscr{X}_\infty$ is a $\mathscr{X}_N$-sequence the measures $\mathscr{N}_{\rho_N}^{\cdot}, \mathscr{N}_\rho^{\cdot}$ can be extended to the σ-field $\mathscr{U}$. This is true for $\mathscr{N}_\rho^{\cdot}$ since $\mathscr{N}_\rho^{\cdot}(\mathscr{C}_{\mathbb{R}}[0,T]) = 1$ and for $\mathscr{N}_{\rho_N}^{\cdot}$ because for fixed $x \in X^N$, $\Theta_N^X(x) = \rho_N$,

$$\phi : \mathbb{R}^N \to \mathscr{D}_{\mathbb{R}}[0,T], \quad u \mapsto \phi(u) := \frac{1}{\sqrt{N}} \sum_{i=1}^N u_i x_i$$

is continuous for the uniform topology and $\mathscr{N}_{\rho_N}^{\cdot} = \phi(\mathscr{N}_1^{\cdot \otimes N})$ for $\mathscr{N}_1^{\cdot}$ the standard Gaussian measure on $\mathbb{R}$. This implies ([Bill68], p.150-151), that the extended measures $\mathscr{N}_{\rho_N}^{\cdot}, \mathscr{N}_\rho^{\cdot}$ converge weakly as Borel measures on the metric space $(\mathscr{D}_{\mathbb{R}}[0,T], \mathscr{T}_\infty)$.

Proof of Lemma 4.1. The advantage in the asymmetric case is that the rate function $\mathrm{I}_0^{\mathrm{G}}$ is convex and we can apply a general result due to [DaGä87] for the proof, which we state as Theorem A.1 in the appendix. Normalizing the measures $P_{0,\rho_N}^{N,\mathrm{G}}$ shows that the result can be applied as long as $\lim_{N\to\infty} \frac{1}{N} \ln P_{0,\rho_N}^{N,\mathrm{G}}(\mathscr{Z})$ converges and this is true as the calculations below will show. In our situation

$$W = \left\{ \mathrm{f} \in \mathscr{C}(\mathrm{Z}) : \frac{|\mathrm{f}|}{\#_{[0,T]} + 1} \in \mathscr{C}_b(\mathrm{Z}) \right\}$$

is the set of continuous functions bounded by $\#_{[0,T]}$ and $\mathscr{Z}$ as before. We fix a $(\mathscr{X}_N)$ sequence $\rho_N \to \rho$ and let $x^{(N)} \in X^N$ be such that $\Theta_N^X(x^{(N)}) = \rho_N$. In order to show condition (i) and (ii) of Theorem A.1 we will prove for $\mathrm{f} \in W$:

$$\begin{aligned}
\Lambda_N^{\mathrm{G}}(\mathrm{f}) &= \frac{1}{N}\ln\int_{\mathscr{Z}}\exp\left[N\int \mathrm{f}\,d\mu\right]P_{0,\rho_N}^{N,\mathrm{G}}(d\mu)\\
&= \frac{1}{N}\ln\prod_{i=1}^{N}\int_Y \exp[\mathrm{f}(y,x_i^{(N)})+\mathrm{G}(y,x_i^{(N)})]\,\mathscr{N}_{\rho_N}^{\cdot}(dy)\\
&= \int_X \ln\left(\int_Y \exp[\mathrm{f}(y,x)+\mathrm{G}(y,x)]\,\mathscr{N}_{\rho_N}^{\cdot}(dy)\right)\rho_N(dx)\\
&\overset{N\to\infty}{\longrightarrow} \int_X \ln\left(\int_Y \exp[\mathrm{f}(y,x)+\mathrm{G}(y,x)]\,\mathscr{N}_{\rho}^{\cdot}(dy)\right)\rho(dx) =: \Lambda^{\mathrm{G}}(\mathrm{f}).
\end{aligned}$$

We can split this estimate into:

$$\begin{aligned}
&\left|\int \ln\int \exp[\mathrm{f}+\mathrm{G}]\,d\mathscr{N}_{\rho}^{\cdot}\,d\rho - \int \ln\int \exp[\mathrm{f}+\mathrm{G}]\,d\mathscr{N}_{\rho_N}^{\cdot}\,d\rho_N\right|\\
&\leq \underbrace{\left|\int \ln\int \exp[\mathrm{f}+\mathrm{G}]\,d\mathscr{N}_{\rho}^{\cdot}\,d\rho - \int \ln\int \exp[\mathrm{f}+\mathrm{G}]\,d\mathscr{N}_{\rho}^{\cdot}\,d\rho_N\right|}_{=:A_N}\\
&\quad + \underbrace{\left|\int \ln\int \exp[\mathrm{f}+\mathrm{G}]\,d\mathscr{N}_{\rho}^{\cdot}\,d\rho_N - \int \ln\int \exp[\mathrm{f}+\mathrm{G}]\,d\mathscr{N}_{\rho_N}^{\cdot}\,d\rho_N\right|}_{=:B_N}.
\end{aligned}$$

$A_N \overset{N\to\infty}{\longrightarrow} 0$: Because of the definition of W, $|\mathrm{f}| \leq c_{\mathrm{f}}(\#_{[0,T]}+1)$. Since $\mathscr{N}_{\rho}^{\cdot}(\mathscr{C}[0,T]) = 1$ and Remark 4.3 $\int \exp[\mathrm{f}(y,\cdot)+\mathrm{G}(y,\cdot)]\,\mathscr{N}_{\rho}^{\cdot}(dy)$ is a continuous function on X. We have to show that

$$\mathrm{g}_{\rho}(\cdot) := \ln\int \exp[\mathrm{f}(y,\cdot)+\mathrm{G}(y,\cdot)]\,\mathscr{N}_{\rho}^{\cdot}(dy) \in W.$$

Then $A_N \to 0$ as $N\to\infty$ due to the definition of the topology on $\mathscr{X}$. We will show the bounds for g_{ρ_N} defined similar for later use. We get the upper bound

$$\mathrm{g}_{\rho_N} \leq (c_{\mathrm{f}}+T)(\#_{[0,T]}+1).$$

We fix some compact set $K_1 \subset Y$ such that

$$\mathscr{N}_{\rho_N}^{\cdot}(K_1) \geq \frac{1}{2} \quad \text{for all} \quad N \geq N_1$$

and some N_1. Compactness in $\mathscr{D}_{\mathbb{R}}[0,T]$ implies ([Bill68], Theorem 14.3) that

$$\sup_{y\in K_1}\|y\|_\infty \leq L_1 < \infty.$$

This leads to

$$\begin{aligned}
&\int \exp[\mathrm{f}(y,x)+\mathrm{G}(y,x)]\,\mathscr{N}_{\rho_N}(dy) \\
&\quad\geq \quad \mathrm{e}^{-c_{\mathrm{f}}(\#_{[0,T]}(x)+1)} \int \left(\frac{1}{1+\exp[2\beta\|y\|_\infty]}\right)^{(\#_{[0,T]}(x))} \mathscr{N}_{\rho_N}(dy) \\
&\quad\geq \quad \mathrm{e}^{-c_{\mathrm{f}}(\#_{[0,T]}(x)+1)} \left(\int \left(\frac{1}{1+\exp[2\beta\|y\|_\infty]}\right) \mathscr{N}_{\rho_N}(dy)\right)^{\#_{[0,T]}(x)} \\
&\quad\geq \quad \exp\left[-\left(c_{\mathrm{f}} - \ln\left(\frac{1}{2(1+\exp[2\beta L_1])}\right)\right)(\#_{[0,T]}(x)+1)\right]
\end{aligned}$$

and therefore $|\mathrm{g}_{\rho_N}| \leq c_1(\#_{[0,T]}+1)$ for some constant c_1 and $N \geq N_1$.
$B_N \overset{N\to\infty}{\longrightarrow} 0$: Further we approximate

$$\begin{aligned}
B_N \quad &\leq \quad \left|\int_{\{\#_{[0,T]}\geq L_2\}} \mathrm{g}_{\rho}\, d\rho_N\right| + \left|\int_{\{\#_{[0,T]}\geq L_2\}} \mathrm{g}_{\rho_N}\, d\rho_N\right| \\
&\quad + \underbrace{\left|\int_{\{\#_{[0,T]}< L_2\}} \mathrm{g}_{\rho}\, d\rho_N - \int_{\{\#_{[0,T]}< L_2\}} \mathrm{g}_{\rho_N}\, d\rho_N\right|}_{=:\,C_N} \\
&\leq \quad 2c_1 \int \mathbf{1}_{\{\#_{[0,T]}\geq L_2\}}(\#_{[0,T]}+1)\, d\rho_N + C_N\,.
\end{aligned}$$

Since $\mathrm{h}_{L_2} := \mathbf{1}_{\{\#_{[0,T]}\geq L_2\}}(\#_{[0,T]}+1) \in W$ the first term tends to $\int \mathrm{h}_{L_2}\, d\rho$ as $N \to \infty$ and this can be made small for L_2 large enough. Define

$$\mathrm{g}_{\nu}^{\delta} := \ln \int \exp[\mathrm{f}(y,\cdot)+\mathrm{G}(y^{\delta},\cdot)]\mathscr{N}_{\nu}(dy)$$

for $\nu \in \{\rho_1, \cdots\} \cup \{\rho\}$. Then

$$\begin{aligned}
C_N \leq\ & \underbrace{\left|\int_{\{\#_{[0,T]}< L_2\}} \mathrm{g}_{\rho}\, d\rho_N - \int_{\{\#_{[0,T]}< L_2\}} \mathrm{g}_{\rho}^{\delta}\, d\rho_N\right|}_{=:\,C_N^1} \\
&+ \underbrace{\left|\int_{\{\#_{[0,T]}< L_2\}} \mathrm{g}_{\rho}^{\delta}\, d\rho_N^{\delta} - \int_{\{\#_{[0,T]}< L_2\}} \mathrm{g}_{\rho_N}^{\delta}\, d\rho_N\right|}_{=:\,C_N^2} + \underbrace{\left|\int_{\{\#_{[0,T]}< L_2\}} \mathrm{g}_{\rho_N}^{\delta}\, d\rho_N - \int_{\{\#_{[0,T]}< L_2\}} \mathrm{g}_{\rho_N}\, d\rho_N\right|}_{=:\,C_N^3}.
\end{aligned}$$

First we study C_N^3 and C_N^1. Let $K_2 \subset \mathscr{C}_{\mathbb{R}}[0,T]$ be a compact set such that $\mathscr{N}_{\rho}(K_2) \geq 1-\epsilon_2$ and denote by $K_2^{\epsilon_3}$ the ϵ_3-ball around K_2 in the uniform metric. Because of Remark 4.5

$$\liminf_{N\to\infty} \mathscr{N}^{\cdot}_{\rho_N}(K_2^{\epsilon_3}) \geq \mathscr{N}_{\rho}(K_2^{\epsilon_3})$$

for the measures extended to the σ-field $\mathscr{U}$. Then because of Eq. (22)

$$\begin{aligned} C_N^3 \quad &\leq \quad \exp[c_1(L_2+1)] \sup_{x\in\{\#_{[0,T]}<L_2\}} \left| \int \exp[\mathrm{f}(y,x)+\mathrm{G}(y,x)]\mathscr{N}^{\cdot}_{\rho_N}(dy) \right. \\ &\qquad \left. - \int \exp[\mathrm{f}(y,x)+\mathrm{G}(y^{\delta},x)]\mathscr{N}^{\cdot}_{\rho_N}(dy) \right| \\ &\leq \quad \mathrm{e}^{[(c_{\mathrm{f}}+c_1)(L_2+1)+T]} \left((L_2+1)2\beta \int_{K_2^{\epsilon_3}} \|y-y^{\delta}\|_{\infty} + \mathscr{N}^{\cdot}_{\rho_N}(K_2^{\epsilon_3}) \right). \end{aligned}$$

For $y \in K_2^{\epsilon_3}$ there is a $\tilde{y} \in K_2$ with $\|y-\tilde{y}\|_{\infty} \leq \epsilon_3$ and therefore

$$\begin{aligned} |y(s)-y^{\delta}(s)| \quad &\leq \quad |y(s)-\tilde{y}(s)| + |\tilde{y}(s)-\tilde{y}^{\delta}(s)| + |y^{\delta}(s)-\tilde{y}^{\delta}(s)| \\ &\leq \quad 2\epsilon_3 + w_{\tilde{y}}(\delta), \end{aligned} \tag{28}$$

where $w_{\tilde{y}}(\delta)$ is the modulus of continuity of $\tilde{y}$ ([Bill68], Eq. (8.1)). Because of the theorem of Arzelà-Ascoli

$$\sup_{\tilde{y}\in K_2} w_{\tilde{y}}(\delta) \stackrel{\delta\to 0}{\longrightarrow} 0$$

for the compact set K_2 and we are done with C_N^1 and C_N^3 since a appropriate choice of K_2, ϵ_3, δ will ensure that they are small for N large enough.

C_N^2 remains to be studied. Since $\rho_N \stackrel{\mathrm{w}}{\Longrightarrow} \rho$ for every ϵ_4 there is a compact set $K_4 \subset X$ such that

$$\rho_N(K_4^c) \leq \epsilon_4$$

for N large enough. Observe that

$$\mathscr{F} := \left\{ \exp[\mathrm{f}(\cdot,x)+\mathrm{G}((\cdot)^{\delta},x)] : x \in K_4 \right\}.$$

is a family of functions on Y that is uniformly bounded and uniformly continuous on compact sets. Therefore the last term

$$C_N^2 \leq \exp[c_1(L_2+1)] \left(\rho_N(K_4^c) + \sup_{\{\mathrm{h}\in\mathscr{F}\}} \left| \int \mathrm{h}\, d\mathscr{N}^{\cdot}_{\rho_N} - \int \mathrm{h}\, d\mathscr{N}_{\rho} \right| \right)$$

vanishes because of [DeZe93], Theorem D.11.

The function Λ^{G} is Gâteaux differentiable and therefore (i) and (ii) of Theorem A.1 are satisfied.

A slight modification of [Sep93], Lemma 2.19 proves that $\Lambda^{\mathrm{G}*}(\nu) = \mathrm{I}_0^{\mathrm{G}}(\rho;\nu)$ on $\mathscr{Z}$.

This concludes the proof of Lemma 4.1 because condition (iii) of Theorem A.1 holds as for the standard Sanov result. □

Proof of Corollary 4.2. Since $P_{0,\rho_N}^{N,\mathrm{G}}$ is a LDS and (Q^N) satisfies a full LDP with rate $\mathrm{H}(\cdot|q)$ in view of Theorem 2.3 all that remains is to observe that

$$\begin{aligned}
\mathrm{S}_0^{\mathrm{G}}(\nu) &= \mathrm{I}_0^{\mathrm{G}}(\pi_X(\nu)) + \mathrm{H}(\pi_X(\nu)|q) \\
&= \int_X \mathrm{H}(\nu_x|\mathscr{N}^{\cdot}_{\pi_X(\nu)} \otimes \delta_x)(\pi_X(\nu))(dx) + \mathrm{H}(\pi_X(\nu)|q) - \int \mathrm{G}\, d\nu \\
&= \mathrm{H}(\nu|\mathscr{N}^{\cdot}_{\pi_X(\nu)} \otimes q) - \int \mathrm{G}\, d\nu,
\end{aligned}$$

because of Appendix A, Eq. (65). $\mathrm{S}_0^{\mathrm{G}}(\nu) = \mathrm{H}(\nu|\exp[\mathrm{G}].\mathscr{N}^{\cdot}_{\pi_X(\nu)} \otimes q)$ since $\exp[\mathrm{G}].\mathscr{N}^{\cdot}_{\pi_X(\nu)} \otimes q$ is a probability measure and [DeSt89], (3.2.14). □

4.2. Minimum of the rate $\mathrm{S}_0^{\mathrm{G}}$

We can draw two immediate conclusions from our large deviations result Corollary 4.2. Since the rate function $\mathrm{S}_0^{\mathrm{G}}$ has compact level sets the set

$$M := \{\nu \in \mathscr{Z} : \mathrm{S}_0^{\mathrm{G}}(\nu) = 0\} \neq \emptyset$$

of minima of $\mathrm{S}_0^{\mathrm{G}}$ is compact and non-empty. From the form of the rate $\mathrm{S}_0^{\mathrm{G}}(\nu) = \mathrm{H}(\nu|\mathscr{N}^{\cdot}_{\pi_X(\nu)} \otimes q)$ we know how the minima must look like. Since $\mathrm{H}(\nu|\mu) = 0 \iff \nu = \mu$ we get that an element $\nu \in M$ must be a solution of the self-consistent equation

$$\nu = \exp[\mathrm{G}].\mathscr{N}^{\cdot}_{\pi_X(\nu)} \otimes q\,. \tag{29}$$

The process given by Eq. (29) describes a particle jumping in a centered Gaussian field y with rate $c(x(s-)y(s-)) := \frac{1}{1+\exp[2\beta x(s-)y(s-)]}$, where the covariance of the Gaussian field is the same as the covariance of the X-marginal of the process. Since the r.h.s. of Eq. (29) is completely determined by the X-marginal of ν we will look on the self-consistent equation of the X-marginal, i.e.

$$\frac{d\rho}{dq}(x) = \int \exp[\mathrm{G}(y,x)]\,\mathscr{N}_\rho(dy) \tag{30}$$

for a measure $\rho \in \mathscr{X}^{\cdot}$. We will prove

Lemma 4.6. *Equation (30) has exactly one solution* ρ^* *and therefore Eq. (29) has a unique solution* $\nu^* := \exp[\mathrm{G}].\mathscr{N}^{\cdot}_{\rho^*} \otimes q$.

So far we have nowhere used the fact that we are dealing with stochastic processes. Of course we have to use this fact to show that Eq. (30) has only one solution. We will start with a description of exp[G] as a solution of a stochastic differential equation. The process $t \mapsto \#_{[0,t]}(x)$ is a Poisson process under q and $M_t^x := \#_{[0,t]}(x) - t$, the compensated Poisson process, is a martingale. Let $y \in \mathscr{D}_{\mathbb{R}}[0,T]$ be some fixed function and

$$\mathrm{r} : \mathbb{R} \times \mathscr{E} \longrightarrow \mathbb{R}$$

be a continuous bounded function. Then the unique solution of the stochastic differential equation

$$Z_t^x = 1 - \int_0^t Z_{s-}^x \,\mathrm{r}\,(y(s-), x(s-))\, dM_s^x$$

is the martingale (see [Pro90], Theorem 36)

$$Z_t^x := \exp\left[\int_0^t \mathrm{r}\,(y(s), x(s))\, ds\right] \prod_{s \le t: x(s-) \ne x(s)} (1 - \mathrm{r}\,(y(s-), x(s-)))\,.$$

For $\mathrm{r}\,(h, \sigma) := \frac{\exp[2\beta h\sigma]}{1+\exp[2\beta h\sigma]}$ we get $Z_t^x =: \exp[\mathrm{G}_t(y, x)]$ and $Z_T^x = \exp[\mathrm{G}(y, x)]$ with the definition of G as in Eq. (3).

For the proof of Lemma 4.6 we will use the same ideas as in [Gui95] and [BeGu95] for the proof of Theorem 5.5. For some measure $\nu \in \mathscr{X}$ we define the probability measure $L_\nu \in \mathscr{X}$ by

$$\frac{dL_\nu}{dq}(x) := \int \exp[\mathrm{G}(y, x)]\, \mathscr{N}_\nu(dy).$$

We will use a fixed point argument for the map $\nu \mapsto L_\nu$ to show the uniqueness. Since the measures L_ν are absolutely continuous with respect to q a very useful metric on $\mathscr{X}$ is the variational distance. We will need the variational distance with respect to a filtration on $\mathscr{D}_{\mathscr{E}}[0, T]$. Denote by

$$\mathscr{F}_t := \boldsymbol{\sigma}\left(\{x(s) : s \le t\}\right)$$

the standard filtration generated by the evaluation maps $x \mapsto x(s)$ up to time t. Then for some measures $\nu, \mu \in \mathscr{M}_1(X)$ the variational distance is

$$D_t(\nu, \mu) := \sup\left\{\left|\int \mathrm{f}\; d\nu - \int \mathrm{f}\; d\mu\right|\right\}, \tag{31}$$

where the supremum is taken over all $\mathscr{F}_t$ measurable functions f bounded by one. If $\mu \ll q$ and $\nu \ll q$ then

$$D_t(\nu, \mu) = \int \left|\mathbf{E}_q\left[\frac{d\nu}{dq}\middle|\mathscr{F}_t\right] - \mathbf{E}_q\left[\frac{d\mu}{dq}\middle|\mathscr{F}_t\right]\right| dq\,,$$

where $\mathbf{E}_q\left[\frac{d\nu}{dq}\middle|\mathscr{F}_t\right]$ denotes the conditional distribution of $\frac{d\nu}{dq}$ given $\mathscr{F}_t$. We have finished the proof of Lemma 4.6, if we show that $\rho^* = L_{\rho^*}$ has only one solution, which is true by Gronwall's Lemma when we have shown the following

Proposition 4.7. *There is a constant C such that for all $t \le T$*

$$D_t(L_\nu, L_\mu) \le C \int_0^t D_s(\nu, \mu)\, ds.$$

Proof. Since $\exp[\mathrm{G}_t(y,\cdot)]$ is a martingale

$$\mathbf{E}_q\left[\frac{dL_\nu}{dq}\middle|\mathscr{F}_t\right]=\int\exp[\mathrm{G}_t(y,\cdot)]\mathscr{N}_\nu(dy).$$

If we denote by

$$Z_s^\nu(x):=\int\mathrm{r}\,(y(s),x(s))\exp[\mathrm{G}_s(x,y)]\mathscr{N}_\nu(dy)$$

then the stochastic integral definition of $\exp[\mathrm{G}_t]$ leads to

$$\begin{aligned}D_t(L_\nu,L_\mu)&=\int\left|\int_0^t\left(Z_{s-}^\nu(x)-Z_{s-}^\mu(x)\right)\,dM_s^x\right|\,q(dx)\\&\leq\int\int_0^t\left|Z_{s-}^\nu(x)-Z_{s-}^\mu(x)\right|\,dN_s^x\,q(dx),\end{aligned}$$

where $N_t^x:=\#_{[0,t]}(x)+t$ is a strictly increasing process. We will first prove the crucial step

$$\left|Z_{s-}^\nu(x)-Z_{s-}^\mu(x)\right|\leq C_1\left(\#_{[0,s[}(x)+1+s\right)D_s(\nu,\mu).$$

Denote by W the Hilbert space $\mathscr{L}_2([0,s],\lambda^x)$ where λ^x is the measure $\lambda^x:=dN_\cdot^x+\delta_s$ and denote by $<f,g>:=\int_{[0,s[}f(u)g(u)\,dN_u^x+f(s)g(s)$ the inner product on W. Let $\mathscr{N}_{\nu,\mu}$ be any measure on $W\times W$ such that the first and the second marginal are centered Gaussian measures with covariance

$$\int_{W\times W}<f,g_1>^2\ \mathscr{N}_{\nu,\mu}(dg_1,dg_2)=\int_X<f,x((\cdot)-)>^2\ \nu(dx)$$

and

$$\int_{W\times W}<f,g_2>^2\ \mathscr{N}_{\nu,\mu}(dg_1,dg_2)=\int_X<f,x((\cdot)-)>^2\ \mu(dx)$$

respectively. Then a telescopic-product argument and

$$0\leq\mathrm{r}\leq1,\quad|\mathrm{r}\,(h,\sigma)-\mathrm{r}\,(h',\sigma)|\leq2\beta|h-h'|$$

leads to the inequality

$$\begin{aligned}\left|Z_{s-}^\nu(x)-Z_{s-}^\mu(x)\right|&\leq2\beta\mathrm{e}^s\int_{W\times W}<|g-g'|,1>\mathscr{N}_{\nu,\mu}(dg,dg')\\&\leq\ 2\beta\mathrm{e}^s\sqrt{\left(\#_{[0,s[}(x)+1+s\right)\int_{W\times W}\|g-g'\|_w^2\,\mathscr{N}_{\nu,\mu}(dg,dg')},\end{aligned}\tag{32}$$

where we have used the Cauchy-Schwarz inequality in the last step. Denote by $\pi_W:\mathscr{D}_{\mathscr{E}}[0,T]\to W$ the map $\pi_W(x)(u):=x(u-)$ for $u\in[0,s]$ and by $\Xi_{\nu,\mu}$ any measure on $W\times W$ having marginals $\pi_W(\nu)$ and $\pi_W(\mu)$ respectively. Now take $\mathscr{N}_{\nu,\mu}$ to be the centered Gaussian on $W\times W$ having covariance

$$\int \left(< f, g >^2 + < f', g' >^2\right) \mathscr{N}^{\cdot}_{\nu,\mu}(dg, dg')$$
$$= \int \left(< f, g >^2 + < f', g' >^2\right) \Xi_{\nu\,\mu}(dg, dg'). \tag{33}$$

Because

$$\int_{W\times W} \|g - g'\|^2 \mathscr{N}^{\cdot}_{\nu,\mu}(dg, dg') = \int_{W\times W} \|g - g'\|^2\, \Xi_{\nu,\mu}(dg, dg')$$
$$\leq 4\left(\#_{[0,s[}(x) + 1 + s\right) (\int_{W\times W} \left(\|g - g'\|^2 \wedge 1\right) \Xi_{\nu,\mu}(dg, dg')$$

by taking the infimum over all measures $\Xi_{\nu,\mu}$ on $W \times W$ having the required marginals, we obtain the estimate

$$\begin{aligned} \left|Z^{\nu}_{s-}(x) - Z^{\mu}_{s-}(x)\right| &\leq 4\beta \mathrm{e}^s \left(\#_{[0,s[}(x) + 1 + s\right) d^W(\pi_W(\nu), \pi_W(\mu)) \\ &\leq 4\beta \mathrm{e}^s \left(\#_{[0,s[}(x) + 1 + s\right) D^W(\pi_W(\nu), \pi_W(\mu)), \end{aligned} \tag{34}$$

where d^W is the Vaserstein-metric on $\mathscr{M}_1(W)$ and D^W the corresponding variational-metric. The last inequality in (34) is always true since the Vaserstein-metric is smaller than the Prohorov-metric ([EtKu85], Chapter 3, Eq. (1.1) and Theorem 1.2). Since for every bounded, measurable function f on W, $f \circ \pi_W$ is bounded and $\mathscr{F}_{s-}$-measurable on $\mathscr{D}_{\mathscr{E}}[0, T]$, we have

$$D^W(\pi_W(\nu), \pi_W(\mu)) \leq D_s(\nu, \mu),$$

which proves our interim result.

$D_s(\nu, \mu)$ is monotone in s and therefore measurable. Since for every measurable function f on $[0, T]$

$$\int_X \int_0^t (\#_{[0,s[}(x) + 1 + s) f(s)\, dN^x_s\, q(dx) = \int_0^t 2(2s + 1) f(s)\, ds$$

we have shown Proposition 4.7 by choosing

$$C := 8\beta \mathrm{e}^T(2T + 1).$$

□

5. Gaussian decoupling

The main difficulty in generalizing the results of Sect. 4.1 is the weak dependence of the Gaussian variables given by the covariance Eq. (5). To clearify the influence of this weak dependence we study a simple homogenous “toy”-model in Sect. 5.1. In Sect. 5.2 we will use the idea behind the “toy”-model to prove a finite dimensional Gaussian decoupling result, which will be the key step for the proof of the LDP for the symmetric spin-glass dynamics in Sect. 6.

5.1. A "toy"-model

As an application of Theorem 2.3 we prove a Sanov result for a sequence of weakly dependent Gaussian variables.

Let Y be some separable Banach-space and let p^N be a centered Gaussian measure on Y^N with a covariance of the form

$$\begin{aligned}\int_{Y^N} < z, y >< z', y > \; p^N(dy) &= \sum_{i=1}^{N} A(z_i, z_i') + \frac{1}{N} \sum_{i,j=1}^{N} B(z_i, z_j') \\ &= \sum_{i=1}^{N} A(z_i, z_i') + N \cdot B(\bar{z}, \bar{z}') \qquad (35)\end{aligned}$$

for $z, z' \in (Y^*)^N$, the dual space to Y^N, and $\bar{z} = \frac{1}{N} \sum_i z_i$. A and B are some symmetric bilinearforms on $Y^* \times Y^*$, where necessarily A has to be positive and B has to be such that $C := A + B$ is positive. An example could be $B = \alpha A$ for $\alpha \in [-1, 1]$ if a Gaussian measure with covariance A exists on Y. The dependence between the Gaussian variables y_i distributed according to p^N is getting weaker and weaker as $N \to \infty$ but the variables "feel" a dependence on the mean-value $\frac{1}{N} \sum_{i=1}^N y_i =: \mathrm{m}(y)$, which is of the same order for all N as can be seen in the covariance Eq. (35). Denote by ν a centered, Gaussian measure on Y with covariance A, e.g. the first marginal of p^N. The mean-value is an important order-parameter since the conditional distribution p_x^N of p^N for given mean-value $\mathrm{m} = x$ is the same as the conditional distribution of the Gaussian measure $\nu^{\otimes N}$ – corresponding to the distribution of independent variables – for given $\mathrm{m} = x$. We define the map

$$\begin{aligned}\phi_x : Y^N &\longrightarrow Y^N \\ y &\longmapsto ((y_1 - \mathrm{m}(y) + x), \cdots, (y_N - \mathrm{m}(y) + x)).\end{aligned}$$

By Gaussian calculus can be proved, that $\phi_x(p^N) = p_x^N$ and that indeed $\phi_x(p^N) = \phi_x(\nu^{\otimes N})$. By Q^N we denote the distribution of the mean-value under p^N. Using Eq. (35) we evaluate the characteristic function of Q^N and find for $z \in Y^*$ that

$$\int \exp\left[\mathrm{i}\frac{1}{N} < z, \sum_i y_i >\right] p^N(dy) = \exp\left[-\frac{1}{2 \cdot N}(A(z,z) + B(z,z))\right].$$

Hence, $\{Q^N\}_{N \in \mathbb{N}}$ is a sequence of centered Gaussian measures with covariance $\frac{1}{N} C$ which satisfies a full LDP with rate function

$$\Lambda^*(y) := \sup_{z \in Y^*} \left\{< z, y > -\frac{1}{2} C(z,z)\right\}, \quad y \in Y,$$

due to Schilder's Theorem ([DeSt89], Theorem 3.4.5).

Let $P^N := \Theta_N^Y(p^N)$ be the law of the empirical measure under p^N and $\mathrm{m}(\mu)$ be the mean-value of a measure $\mu \in \mathscr{M}_1(Y)$, which is well-defined whenever $\int \|y\| \, \mu(dy) < \infty$ (see App. B). We have the following Sanov result:

Lemma 5.1. *The sequence* (P^N) *of measures obeys a full LDP on* $\mathscr{M}_1(Y)$ *with rate function*

$$\mathrm{S}(\mu) = \inf_{y \in Y} \mathrm{H}(\mu|\varepsilon_y * \nu) + \Lambda^*(\mathrm{m}(\mu)),$$

where $\varepsilon_y * \nu$ *is the convolution of* ν *with the Dirac-measure at* y, *i.e. the shift of* ν *by a fixed* $y \in Y$. $\mathrm{S}(\mu) = \infty$ *if* $\mathrm{m}(\mu)$ *is not defined.*

Remark 5.2. It can be easily seen that $\mathrm{S}(\mu) = 0 \iff \mu = \nu$ and therefore $P^N \stackrel{\mathrm{w}}{\Longrightarrow} \varepsilon_\nu$ as in the case of independent Gaussian variables; the effect of the weak dependence is visible only on an exponential scale. But on a large deviations scale the influence is fairly strong. If for example $B := -A$ then the rate $\mathrm{S}(\mu)$ is finite only when $\mathrm{m}(\mu) = 0$.

Proof. As shown in Appendix B we see that a Sanov result holds for the measures $\Theta_N^Y(\nu^{\otimes N})$ on

$$\mathscr{Y} := \left\{ \mu \in \mathscr{M}_1(Y) : \int \|y\|^{1+\delta} \mu(dy) \right\}$$

equipped with the stronger topology such that the map on the mean value $\mu \mapsto \mathrm{m}(\mu)$ is continuous. We define the continuous map

$$\begin{aligned} \Phi_x : \mathscr{Y} &\longrightarrow \mathscr{Y} \\ \mu &\longmapsto \int \varepsilon_{(y - \mathrm{m}(\mu) + x)} \mu(dy) = \varepsilon_{(x - \mathrm{m}(\mu))} * \mu \end{aligned}$$

and set $P_x^N := \Phi_x\left(\Theta_N^Y(\nu^{\otimes N})\right)$. By applying the contraction principle and an approximation argument can be seen that $\Pi = \{P_x^N : x \in Y, N \in \mathbb{N}\}$ is a LDS on $\mathscr{Y}$ with rate

$$\mathrm{I}(x;\mu) := \begin{cases} \inf_{y \in Y} \mathrm{H}(\mu|\varepsilon_y * \nu) & \text{if} \quad \mathrm{m}(\mu) = x \\ \infty & \text{otherwise} \end{cases}.$$

Since $P^N(\cdot) = \int_Y P_x^N(\cdot)\, Q^N(dx)$ we conclude the proof of Lemma 5.1 by applying Theorem 2.3. □

5.2. Gaussian decoupling for the spin-glass dynamics

Let X be some Polish space and W be some finite dimensional Hilbert-space, i.e. some vector space with an inner product $< \cdot, \cdot >$. Assume we have a continuous, bounded map

$$\tau : X \to W, \qquad x \mapsto \tau(x) = {}^\tau x, \qquad \|{}^\tau x\| \le \kappa,$$

for some constant κ. We define for some measure $\rho \in \mathscr{M}_1(X)$ the covariance operator

$$\mathrm{D}_\rho \cdot w := \int_X {}^\tau x < {}^\tau x, w > \rho(dx)$$

on W. Since W is finite dimensional there is a centered Gaussian measure $\mathcal{N}_\rho^{\cdot}$ on W having covariance D_ρ and the map $\rho \mapsto \mathcal{N}_\rho^{\cdot}$ is continuous. For $x \in X^N$ we define the centered Gaussian measure $\nu_{\alpha,x}^N$ on W^N through the covariance

$$\mathbf{D}_{\alpha,x}^N(w,w') := \sum_{i=1}^N < w_i, \mathrm{D}_{\Theta_N^X(x)} \cdot w_i' > + \frac{\alpha}{N} \sum_{i,j=1}^N < w_i, {}^\tau x_j >< w_j, {}^\tau x_i > \quad (36)$$

for $w, w' \in W^N$ and a parameter $\alpha \in [-1,1]$. This is the analogous construction as in the fields for the annealed spin-glass dynamics in Eq. (5), with the slight difference, that for (36) an additional independent Gaussian coupling $J_{i,i}$, $i = 1, \cdots, N$ appears; an easy argument using Eq. (43) and [DeZe93], Theorem 4.2.13, shows that the LDP is unaltered. We define the measure $p_{\alpha,x}^N$ on $W^N \times X^N \simeq (W \times X)^N =: Z^N$ by

$$p_{\alpha,x}^N := \nu_{\alpha,x}^N \otimes \delta_x .$$

Analogous to the definitions in Sect. 3 we set $P_{\alpha,\rho_N}^N := \Theta_N^Z(p_{\alpha,x}^N)$ for some $\rho_N = \Theta_N^X(x)$ and $x \in X^N$. We denote by $\mathscr{X}_N^{\cdot} := \Theta_N^X(X^N)$, $\mathscr{X}_\infty^{\cdot} = \mathscr{X}^{\cdot} := \mathscr{M}_1(X)$.

Before we state the LDP for $P_{\alpha,\rho}^N$ we need some additional definitions. We denote by $\mathscr{B}_2(W)$ the space of linear operators on W. With the inner product

$$(A,B) := \mathrm{Tr}\, A^* B$$

$\mathscr{B}_2(W)$ is a finite dimensional Hilbert-space. We define

$$\mathscr{Z}_W := \left\{ \nu \in \mathscr{M}_1(Z) : \int \|w\|^{1+\delta} \nu(dw, dx) \right\}$$

equipped with the stronger topology as in App. B. In this topology the map

$$\begin{aligned} \mathrm{C} : \mathscr{Z} \quad &\longrightarrow \quad \mathscr{B}_2(W) \\ \nu \quad &\longmapsto \quad \mathrm{C}(\nu) := \int w < {}^\tau x, \cdot > \nu(dw, dx) \end{aligned} \quad (37)$$

is well-defined and continuous. $\mathrm{C}(\nu)$ is just the mean-value of the measure $\varphi(\nu)$, where

$$\varphi : Z \to \mathscr{B}_2(W) \qquad z = (w,x) \mapsto w < {}^\tau x, \cdot > .$$

Observe that $\|\varphi(z)\|^2 = \sum_{e_k} < w, w >< {}^\tau x, e_k >^2 = \|w\|^2 \cdot \|{}^\tau x\|^2 \le \|w\|^2 \kappa^2$ since $\|\tau(x)\| \le \kappa$ for all $x \in X$. We will also use the notation

$$w < w', \cdot > =: w \hat{\otimes} w'$$

indicating that $\mathscr{B}_2(W) \simeq W \hat{\otimes} W$ - the tensor product of $W \times W$. For $A \in \mathscr{B}_2(W)$ we denote by $A^s := \frac{1}{2}(A + A^*)$, $A^a := \frac{1}{2}(A - A^*)$ respectively the symmetric- and antisymmetric part of A. We define the positive symmetric operator

$$\begin{aligned} \mathbf{C}_{\alpha,\rho}(A) \quad &:= \quad \mathrm{D}_\rho \cdot A \cdot \mathrm{D}_\rho + \alpha \mathrm{D}_\rho \cdot A^* \cdot \mathrm{D}_\rho \\ &= \quad (1+\alpha)\mathrm{D}_\rho \cdot A^s \cdot \mathrm{D}_\rho + (1-\alpha)\mathrm{D}_\rho \cdot A^a \cdot \mathrm{D}_\rho \end{aligned} \quad (38)$$

on $\mathscr{B}_2(W)$ and define

$$\Gamma^*_{\alpha,\rho}(A) := \sup_{B\in\mathscr{B}_2(W)} \left\{ (A,B) - \frac{1}{2}(B, \mathbf{C}_{\alpha,\rho}(B)) \right\}. \tag{39}$$

We are now in the position to state the main result of this section

Theorem 5.3. *For each $(\mathscr{X}_N)$-sequence $\rho_N \overset{w}{\Longrightarrow} \rho \in \mathscr{M}_1(X)$ the measures P^N_{α,ρ_N} obey a full LDP on $\mathscr{M}_1(Z)$ with good rate function*

$$\mathrm{I}_\alpha(\rho;\mu) := \begin{cases} \inf\limits_{B\in\mathscr{B}_2(W)} \mathrm{H}(\mu|\mathscr{N}_\rho \otimes_B \rho) + \Gamma^*_{\alpha,\rho}(\mathrm{C}(\mu)) & \text{if } \pi_X(\mu) = \rho \text{ and} \\ & \int \|w\|\, \mu(dw,dx) < \infty \\ \infty & \text{otherwise} \end{cases}, \tag{40}$$

where $\mathscr{N}_\rho$ is the Gaussian measure with covariance D_ρ and $\mathscr{N}_\rho \otimes_B \rho$ is given by

$$\frac{d(\mathscr{N}_\rho \otimes_B \rho)}{d(\mathscr{N}_\rho \otimes \rho)}(x,w) := \exp\left[< w, B\cdot x > - \frac{1}{2} < B\cdot x, \mathrm{D}_\rho \cdot B\cdot x > \right].$$

Remark 5.4. $\mathscr{N}_\rho \otimes_B \rho$ can be characterized by the conditional distribution for the given second coordinate:

$$(\mathscr{N}_\rho \otimes_B \rho)_x = \mathscr{N}_{(\mathrm{D}_\rho, \mathrm{D}_\rho\cdot B\cdot {}^\tau x)} \otimes \delta_x,$$

where $\mathscr{N}_{(\mathrm{D}_\rho, \mathrm{D}_\rho\cdot B\cdot {}^\tau x)}$ is the Gaussian measure with covariance D_ρ and mean $\mathrm{D}_\rho \cdot B \cdot {}^\tau x$.

If $\alpha = 0$ in Eq. (36) the Gaussian variables w_i distributed according to $\nu^N_{0,x}(dw)$ are independent. We get analogous to Lemma 4.1, with $\mathrm{G} \equiv 0$, the following result, where we have lifted the LDP to the stronger topology as in App. B.

Lemma 5.5. *For every $\mathscr{X}_N$-sequence $\rho_N \overset{w}{\Longrightarrow} \rho \in \mathscr{M}_1(X)$ the sequence of measures P^N_{0,ρ_N} on $\mathscr{Z}_W$ satisfies a full LDP with rate*

$$\mathrm{I}(\rho;\nu) = \begin{cases} \mathrm{H}(\nu|\mathscr{N}_\rho \otimes \rho) & \pi_X(\nu) = \rho \\ \infty & \text{otherwise} \end{cases},$$

where $\mathscr{N}_\rho$ is the centered Gaussian measure on W with covariance D_ρ.

For the remainder of this section we fix a $(\mathscr{X}_N)$-sequence $\rho_N \to \rho$ and $x^N \in X^N$ such that $\Theta^X_N(x^N) = \rho_N$. We set $\mathrm{D}_N := \mathrm{D}_{\rho_N}$. $\Gamma^*_{\alpha,\rho}$ will be the rate function for the LDP of the order parameter C. Denote by $R^N_{\alpha,\rho_N} := \mathrm{C}(P^N_{\alpha,\rho_N})$ the distribution of C on $\mathscr{B}_2(W)$. Then we get the

Proposition 5.6. *R^N_{α,ρ_N} has a full LDP on $\mathscr{B}_2(W)$ with good rate function $\Gamma^*_{\alpha,\rho}$.*

Proof. We evaluate the characteristic function of R^N_{α,ρ_N}:

$$\begin{aligned}\int_{\mathscr{B}_2(W)} \exp[\mathrm{i}(A, W)]\, R^N_{\alpha,\rho_N}(dW) &= \int \exp\left[\mathrm{i}\frac{1}{N}\sum_{i=1}^{N}(A, w_i \hat{\otimes}\,{}^\tau x_i^N)\right] \nu^N_{\alpha,x^N}(dw) \\ &= \int \left[\frac{\mathrm{i}}{N}\sum_i < \underbrace{A\cdot {}^\tau x_i^N}_{=:\, z_i}, w_i >\right] \nu^N_{\alpha,x^N}(dw) \\ &= \exp\left[-\frac{1}{2N^2}\mathbf{D}^N_{\alpha,N}(z,z)\right]\end{aligned}$$

and

$$\begin{aligned}\frac{1}{N^2}\mathbf{D}^N_{\alpha,N}(z,z) &= \frac{1}{N^2}\sum_i \mathrm{Tr}\,(A^*\cdot \mathrm{D}_N \cdot A \cdot {}^\tau x_i \hat{\otimes}\,{}^\tau x_i) \\ &\quad + \frac{\alpha}{N^3}\sum_{i,j}\mathrm{Tr}\,({}^\tau x_i \hat{\otimes}\,{}^\tau x_i \cdot A \cdot {}^\tau x_j \hat{\otimes}\,{}^\tau x_j \cdot A) \\ &= \frac{1}{N}\left(\mathrm{Tr}\,(A^*\cdot \mathrm{D}_N \cdot A\cdot \mathrm{D}_N) + \alpha \mathrm{Tr}\,(A\cdot \mathrm{D}_N \cdot A \cdot \mathrm{D}_N)\right) \\ &= \frac{1}{N}(A, \mathbf{C}_{\alpha,\rho_N}(A))\end{aligned}$$

R^N_{α,ρ_N} is a centered Gaussian measure with covariance $\frac{1}{N}\mathbf{C}_{\alpha,\rho_N}$ and therefore we have a Schilders-type result which can be proved by applying Theorem A.1. □

Before we start with the proof of Theorem 5.3, we give a useful representation for the coupling-matrix J given by Eq. (1).

Remark 5.7. For every $\alpha \in [-1,1]$ there are constants $\kappa_1, \kappa_2 \in [-1,1]$ such that

$$\kappa_1{}^2 + \kappa_2{}^2 = 1 \quad \text{and} \quad 2\kappa_1\cdot\kappa_2 = \alpha, \quad |\kappa_1| \ge |\kappa_2|$$

(For example $\kappa_1 = \kappa_2 = \frac{1}{\sqrt{2}}$ will do in the symmetric case $\alpha = 1$) and the coupling-matrix J can be chosen as

$$J^{ij} := \kappa_1 E^{ij} + \kappa_2 E^{ji}, \tag{41}$$

where the E^{ij} for $(i,j)\in\{1,\dots,N\}^2$ are i.i.d. standard centered Gaussian.

Proof of Theorem 5.3. We split the proof in two parts.
Step 1: Assume for the moment that $\mathrm{D} = \mathrm{D}_\rho$ is invertible on W. Since $\mathrm{D}_N \to \mathrm{D}$ in $\mathscr{B}_2(W)$ we have that D_N^{-1} exists for $N \ge N_0$ and $\mathrm{D}_N^{-1} \to D^{-1}$ in $\mathscr{B}_2(W)$.

We define the linear subspace $U_N \subset W^N$ by

$$U_N := \left\{ w \in W^N : \frac{1}{N}\sum_i w_i \hat{\otimes}\,{}^\tau x_i = \mathrm{C}\,(\Theta^Z_N(w, x^N)) = 0 \right\}$$

and denote by $U_N^\perp$ the orthogonal complement to U_N. The projections on U_N, $U_N^\perp$ are $\mathbf{P}_N$ and $\mathbf{P}_N^\perp$ respectively. The space $U_N^\perp$ is just

$$U_N^\perp = \left\{ w \in W^N : \exists B \in \mathscr{B}_2(W),\ w_i = B \cdot {}^\tau x_i^N, \forall i = 1, \dots, N \right\}.$$

For $N \geq N_0$ we can express the projection $\mathbf{P}_N$ as

$$(\mathbf{P}_N \cdot w)_i = w_i - \mathrm{C}\,(\Theta_N^Z(w, x^N)) \cdot \mathrm{D}_N^{-1} \cdot {}^\tau x_i^N.$$

The subspaces U_N and $U_N^\perp$ have the property that the projections $\mathbf{P}_N$ and $\mathbf{P}_N^\perp$ of a Gaussian variable distributed according to ν_{α,ρ_N}^N are independent. To show this we set for $w \in W^N$, $\bar{w} := \mathbf{P}_N \cdot w$ and $\tilde{w} := \mathbf{P}_N^\perp \cdot w$. Then we have

$$\begin{aligned}
\mathbf{D}_{\alpha,x^N}^N(w,w) &= \sum_i < \bar{w}_i + \tilde{w}_i, \mathrm{D}_N \cdot (\bar{w}_i + \tilde{w}_i) > \\
&\quad + \frac{\alpha}{N} \mathrm{Tr} \left(\sum_i ((\bar{w}_i + \tilde{w}_i) \hat{\otimes} {}^\tau x_i^N) \cdot \sum_j ((\bar{w}_j + \tilde{w}_j) \hat{\otimes} {}^\tau x_j^N) \right) \\
&= \mathbf{D}_{0,x^N}^N(\bar{w}, \bar{w}) + \mathbf{D}_{\alpha,x^N}^N(\tilde{w}, \tilde{w}).
\end{aligned} \tag{42}$$

Because of this independence we have the following nice representation which can be proved by Gaussian-calculus:

Lemma 5.8. *Let $N \geq N_0$ and define for $M \in \mathscr{B}_2(W)$*

$$\begin{aligned}
\phi_M : Z^N &\longrightarrow Z^N \\
z &\longmapsto \Big(\big(w_1 - \big(\mathrm{C}\,(\Theta_N^Z(z)) - M\big) \cdot \mathrm{D}_N^{-1} \cdot {}^\tau x_1, x_1 \big), \cdots \\
&\qquad \cdots, \big(w_N - \big(\mathrm{C}\,(\Theta_N^Z(z)) - M\big) \cdot \mathrm{D}_N^{-1} \cdot {}^\tau x_N, x_N \big) \Big)
\end{aligned}$$

Then $p_{M,x^N}^N := \phi_M(p_{\alpha,x^N}^N) = \phi_M(p_{0,x^N}^N)$ is a regular conditional probability for p_{α,x^N}^N given $\mathrm{C}\,(\Theta_{N,x^N}^Z) = M$, i.e.

$$p_{\alpha,x^N}^N = \int_{\mathscr{B}_2(W)} p_{M,x^N}^N \, R_{\alpha,\rho_N}^N(dM).$$

We define $P_{M,x^N}^N := \Theta_N^Z(p_{M,x^N}^N)$. Because of Theorem 2.3 and Proposition 5.6 we have completed the proof of Step 1 if we show the

Lemma 5.9. *For $N \geq N_0$*

$$\Pi := \left\{ P_{M,x^N}^N : M \in \mathscr{B}_2(W) \right\}$$

is a LDS on $\mathscr{Z}_W$ with rate function

$$\mathrm{J}\,(M;\nu) := \begin{cases} \inf\limits_{B \in \mathscr{B}_2(W)} \mathrm{H}\,(\mu | \mathscr{N}_{\mathrm{D}} \otimes_B \rho) & \text{if } \mathrm{C}\,(\nu) = M \text{ and } \pi_X(\nu) = \rho \\ \infty & \text{otherwise} \end{cases}.$$

Proof. We will prove the result on $\mathcal{M}_1(Z)$ first. Define for $M \in \mathcal{B}_2(W)$

$$\begin{aligned} \Phi_{M,\mathrm{D}} : \mathcal{Z}_W &\to \mathcal{M}_1(Z) \\ \nu &\mapsto \int \delta_{(y-(\mathrm{C}(\nu)-M)\cdot \mathrm{D}^{-1}\cdot {}^{\tau}x,x)}\, \nu(dw,dx). \end{aligned}$$

$\Phi_{M,\mathrm{D}}$ is continuous since $\|{}^{\tau}x\|$ is bounded. Therefore for fixed M,D we have a LDP for $\Phi_{M,\mathrm{D}}(P^N_{0,\rho})$ due to the contraction principle and Lemma 5.5 with the rate

$$\mathrm{J}(M;\nu) = \inf\{\mathrm{H}(\mu|\mathcal{N}_\rho \otimes \rho) : \Phi_{M,\mathrm{D}}(\mu) = \nu, \pi_X(\mu) = \rho\}$$

if $\pi_X(\nu) = \rho$ and ∞ otherwise. We assume that $\pi_X(\nu) = \rho$. Since $\mathrm{C}(\Phi_{M,\mathrm{D}}(\mu)) = M$ for all $\pi_X(\mu) = \rho$ we have $\mathrm{J}(M;\nu) = \infty$ for $\mathrm{C}(\nu) \neq M$. If $\Phi_{M,\mathrm{D}}(\mu) = \nu$ and $\mathrm{C}(\mu) = M'$ then $\mu = \Phi_{M',\mathrm{D}}(\nu)$ and therefore the infimum is taken over the measures $\varphi_B(\nu) := \int \delta_{(w-B\cdot {}^{\tau}x,x)}\, \nu(dw,dx)$ with $B \in \mathcal{B}_2(W)$. Again [DeSt89], (3.2.13) shows

$$\mathrm{H}(\varphi_B(\nu)|\mathcal{N}_\rho \otimes \rho) = \mathrm{H}(\nu|\varphi_{-B}(\mathcal{N}_\rho \otimes \rho)).$$

But $\varphi_B(\mathcal{N}_\mathrm{D} \otimes \rho) = \mathcal{N}_\mathrm{D} \otimes_{(\mathrm{D}^{-1}\cdot B)} \rho$ and therefore the result holds.

We will show that for each sequence $M_N \to M$ in $\mathcal{B}_2(W)$ $P^N_{M_N,\rho_N} = \Phi_{M_N,\mathrm{D}_N}$ is exponentially approximated by $\Phi_{M,\mathrm{D}}(P^N_{0,\rho})$. To this end we introduce a version of the Vaserstein-metric

$$d_1(\nu,\nu')^2 := \inf\left\{\int_{W\times W} ((\|w-w'\|^2 + d(x,x')^2)\wedge 1)\,\chi(dz,dz') : \chi \in \mathcal{M}_{\nu,\nu'}\right\}, \tag{43}$$

where the infimum is taken over all measures χ on $Z \times Z$ having first respectively second marginals $\pi_1(\chi) = \nu$, $\pi_2(\chi) = \nu'$. It is a result due to [Dob70], Theorem 2, that d_1 is a complete metric compatible with the weak topology. d is some metric on X. We define $Z_N(\nu) := (\Phi_{M,\mathrm{D}}(\nu), \Phi_{M_N,\mathrm{D}_N}(\nu)) \in \mathcal{M}_1(Z)^2$ and set

$$\Gamma_L := \left\{\nu \in \mathcal{M}_1(Z) : \int \|w\|^2\,\nu(dw,dx) \le L\right\}.$$

For $\nu \in \Gamma_L$ we have:

$$\begin{aligned} &d_1(\Phi_{M,\mathrm{D}}(\nu), \Phi_{M_N,\mathrm{D}_N}(\nu))^2 \\ &\quad\le \|\mathrm{D}_N^{-1} - \mathrm{D}^{-1}\|_2^2 \int \|\mathrm{C}(\nu) - M_N\|_2^2 \cdot \|{}^{\tau}x\|^2\,\nu(dw,dx) \\ &\quad\le \|\mathrm{D}_N^{-1} - \mathrm{D}^{-1}\|_2^2 \cdot \beta^2 \left(2\|M_N\|_2^2 + 2\beta^2 \int \|w\|^2 \nu(dw,dx)\right) \end{aligned} \tag{44}$$

where we have used that the operator norm is smaller than the norm on $\mathcal{B}_2(W)$. For large N the last expression in Eq. (44) is uniformly small on Γ_L and therefore

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \ln P^N_{0,\rho_N} \underbrace{\{\nu : d_1(\Phi_{M,\mathrm{D}}(\nu), \Phi_{M_N,\mathrm{D}_N}(\nu)) > \delta\}}_{=:\,\Lambda_\delta} \\ &\quad\le \limsup_{N\to\infty} \frac{1}{N} \ln(P^N_{0,\rho_N}(\Gamma_L^c) + P^N_{0,\rho_N}(\Gamma_L \cap \Lambda_\delta)) \le -L', \end{aligned}$$

where we have used the notation of App. B. L' tends to ∞ as $L \to \infty$ and therefore Z_N defines an exponential approximation [DeZe93], Theorem 4.2.13 and $P^N_{M_N,D_N}$ has a LDP with the same rate J.

To strengthen the LDP to $\mathscr{Z}_W$ observe that

$$\sum_{i=1}^{N} \|w_i - (\mathrm{C}(\Theta^Z_{N,Z}) - M_N) \cdot {}^\tau x_i, x_i)\|^2 \leq K \left(\sum_{i=1}^{N} \|w_i\|^2 + 1 \right)$$

for some constant K and then apply App. B. □

Step 2: For the proof of the general case we denote by

$$V := \{w \in W : D \cdot w = 0\}^\perp$$

the subspace on which the restriction $D|_V$ is invertible and by P the orthogonal projection on V. We define the map $\phi_V : \mathscr{M}_1(Z) \to \mathscr{M}_1(Z)$ by

$$\nu \longmapsto \phi_V(\nu) := \int_Z \delta_{(\mathrm{P}\cdot w, x)}\, \nu(dw, dx).$$

Because of **Step 1** and the continuity of the injection $\iota : V \to W$ we have a LDP for the measures $P^{N,V}_{\alpha,\rho_N} := \phi_V(P^N_{\alpha,\rho_N})$. The rate function is actually the same rate function defined in (40) since

$$\mathscr{N}_\rho \otimes \rho\left(\{V \times \tau^{-1}(V)\}\right) = 1.$$

Therefore it remains to show that $P^{N,V}_{\alpha,\rho_N}$ is an exponential approximation of P^N_{α,ρ_N} (see [DeZe93], Definition 4.2.10). We will use again the Vaserstein-metric d_1 Eq. (43) and have to show, that

$$\lim_{N\to\infty} \frac{1}{N} \ln P^N_{\alpha,\rho_N}(\underbrace{\{\nu : d_1(\nu, \phi_V(\nu)) \geq \delta\}}_{=:\,\Gamma_\delta}) = -\infty$$

for every $\delta > 0$. However, because of Eq. (43) and using Eq. (41)

$$\begin{aligned} P^N_{\alpha,\rho_N}(\Gamma_\delta) &\leq \nu^N_{\alpha,x^{(N)}}\left(\left\{ w \in W^N : \frac{1}{N}\sum_{i=1}^{N} \|\mathrm{P}^\perp w_i\|^2 \geq \delta \right\}\right) \\ &\leq \mathrm{e}^{-N\alpha\delta} \int_{W^N} \exp\left[\alpha \sum_{i=1}^{N} \|\mathrm{P}^\perp w_i\|^2\right] \nu^N_{\alpha,x^{(N)}}(dw) \\ &\leq \mathrm{e}^{-N\alpha\delta} \mathbf{E}_E \exp\left[2\alpha \sum_{i=1}^{N} \kappa_1^2 \left\| \frac{\mathrm{P}^\perp}{\sqrt{N}} \sum_{j=1}^{N} E^{ij} x_j \right\|^2 + \kappa_2^2 \left\| \frac{\mathrm{P}^\perp}{\sqrt{N}} \sum_{j=1}^{N} E^{ji} x_j \right\|^2 \right] \\ &\leq \mathrm{e}^{-N\alpha\delta} \mathbf{E}_E \exp\left[4\alpha \sum_{i=1}^{N} \kappa_1^2 \left\| \mathrm{P}^\perp \frac{1}{\sqrt{N}} \sum_{j=1}^{N} E^{ij} x_j \right\|^2 \right] \\ &= \mathrm{e}^{-N\alpha\delta} \left(\int \exp\left[4\alpha \|\mathrm{P}^\perp w\|^2\right] \mathscr{N}_{\rho_N}(dw) \right)^N, \end{aligned} \tag{45}$$

by Cauchy-Schwarz and the fact that $\kappa_1^2 \geq \kappa_2^2$. $\mathrm{P}^\perp w$ under $\mathcal{N}^{\cdot}_{\rho_N}$ are centered, Gaussian variables with covariance $\mathrm{P}^\perp \cdot \mathrm{D}_N \cdot \mathrm{P}^\perp = \mathrm{P}^\perp \cdot (\mathrm{D}_N - \mathrm{D}) \cdot \mathrm{P}^\perp$. Since the operator-norm of $(\mathrm{D}_N - \mathrm{D})$ is smaller than the norm $\|D_N - D\|_2$ the variational characterization of the largest eigenvalue λ_N of $\mathrm{P}^\perp \cdot \mathrm{D}_N \cdot \mathrm{P}^\perp$

$$\lambda_N = \sup\{\|\mathrm{P}^\perp \cdot \mathrm{D}_N \cdot \mathrm{P}^\perp \cdot w\| : w \in W, \|w\| = 1\} \leq \|\mathrm{D}_N - \mathrm{D}\|_2$$

implies that $\lambda_N \to 0$ for $N \to \infty$. Because of Eq. (69) and $\mathrm{Tr}\, \mathrm{P}^\perp \cdot \mathrm{D}_N \cdot \mathrm{P}^\perp \to 0$ the integral in the last expression of Eq. (45) converges to 1 for all $\delta > 0$ and all α. This completes the proof. $\square$

6. Symmetric spin-glass dynamics

6.1. "Free"-case

In the proof of the LDP for the asymmetric spin-glass dynamics we were able to include the interaction given by the Girsanov exponent G due to the independence of the Gaussian fields. In the symmetric case, or more general in the cases when

$$\mathbf{E}_J J^{ij} J^{ji} = \alpha \neq 0,$$

we have to show first a LDP for the underlying measures

$$P^N_\alpha := \int_{\mathscr{X}_N} P^N_{\alpha,\rho}\, Q^N(d\rho),$$

where for $x \in X^N$, $\rho = \Theta^X_{N,x} \in \mathscr{X}^{\cdot}_N$ (see Sect. 3 for the definitions),

$$P^N_{\alpha,\rho} := \Theta^Z_N \left(\nu^N_{\alpha,x} \otimes \delta_x\right).$$

In Sect. 6.2 we then get the upper bound in the interacting model via Varadhan's Theorem. We split the proof again into first establishing that the $P^N_{\alpha,\rho}$ define a LDS and then averaging with Q^N.

We need some additional definitions to state the results. Let

$$\|y\|_2 := \sqrt{\int_0^T |y(s)|^2\, ds}, \quad y \in \mathscr{L}_2$$

be the usual norm on $\mathscr{L}_2 := \mathscr{L}_2([0,T],\lambda)$ - λ Lebesgue-measure on $[0,T]$ - and we denote the inner-product by $< y, y' >:= \int_0^T y(s)y'(s)\, ds$. For a measure $\nu \in \mathscr{Z}$ such that

$$\int \|y\|_2\, \nu(dy, dx) < \infty$$

we define a function $\mathrm{C}_\nu \in \mathscr{L}_2([0,T]^2, \lambda \otimes \lambda)$ by

$$\mathrm{C}_\nu(s,t) := \int y(s)x(t)\, \nu(dy, dx). \tag{46}$$

We are thinking of a $B \in \mathscr{L}_2([0,T]^2, \lambda \otimes \lambda)$ as an operator acting on $\mathscr{L}_2$ by

$$(B \cdot y)(s) := \int_0^T B(s,t)y(t)\,dt.$$

$\mathscr{L}_2([0,T]^2, \lambda \otimes \lambda)$ is isometric to the Hilbert-space of Hilbert-Schmidt operators $\mathscr{B}_2(\mathscr{L}_2)$ on $\mathscr{L}_2$. On $\mathscr{B}_2(\mathscr{L}_2)$ we have the inner product

$$(A, B) := \int_0^T \int_0^T A(t,s)B(t,s)\,ds\,dt.$$

For $A \in \mathscr{B}_2(\mathscr{L}_2)$ we define $\mathbf{C}_{\alpha,\rho}(A)$ as in Eq. (38) and $\Gamma^*_{\alpha,\rho}(A)$ similar to Eq. (39).

Remark 6.1. In the case $\alpha = 0$, contraction principle, Lemma 4.1 (with $\mathrm{G} \equiv 0$) and App. B show that we have the representation

$$\Gamma^*_{0,\rho}(\mathrm{C}_\nu) = \inf\{\mathrm{H}(\mu|\mathscr{N}_\rho \otimes \rho) : \pi_X(\mu) = \rho, \mathrm{C}_\mu = \mathrm{C}_\nu\} \tag{47}$$

for the rate function governing the LDP of the $R^N_{0,\rho_N} = \mathrm{C}(P^N_{\alpha,\rho_N})$.

We are now in the position to state our main

Theorem 6.2. *For each ($\mathscr{X}_N$)-sequence $\rho_N \to \rho \in \mathscr{X}_\infty$ the measures P^N_{α,ρ_N} obey a full LDP on $\mathscr{Z}$ with good rate function*

$$\mathrm{I}_\alpha(\rho;\mu) := \begin{cases} \inf_{B \in \mathscr{B}_2(\mathscr{L}_2)} \mathrm{H}(\mu|\mathscr{N}_\rho \otimes_B \rho) + \Gamma^*_{\alpha,\rho}(\mathrm{C}_\mu) & \text{if } \pi_X(\mu) = \rho \text{ and} \\ & \int \|y\|\mu(dy) < \infty \\ \infty & \text{otherwise} \end{cases}, \tag{48}$$

where $\mathscr{N}_\rho$ is the Gaussian measure with covariance D_ρ as in Proposition 4.4 and the measure $\mathscr{N}_\rho \otimes_B \rho$ is defined by

$$\frac{d(\mathscr{N}_{\mathrm{D}_\rho} \otimes_B \rho)}{d(\mathscr{N}_{\mathrm{D}_\rho} \otimes \rho)}(x,y) := \exp\left[< y, B \cdot x > -\frac{1}{2} < B \cdot x, \mathrm{D}_\rho \cdot B \cdot x >\right].$$

As an immediate corollary we get

Corollary 6.3. *The sequence of measures P^N_α on $\mathscr{Z}$ has a full LDP with rate function:*

$$\mathrm{S}_\alpha(\nu) := \inf_{B \in \mathscr{B}_2(\mathscr{L}_2)} \mathrm{H}(\nu|\mathscr{N}_{\pi_X(\nu)} \otimes_B q) + \Gamma^*_{\alpha,\pi_X(\nu)}(\mathrm{C}_\nu), \tag{49}$$

where $\mathrm{S}_\alpha(\nu) = \infty$ for $\int \|y\|\nu(dy) = \infty$.

Remark 6.4. The measures $\mathscr{N}_{\pi_X(\nu)} \otimes_B q$ can be characterized by their conditional distribution as in Remark 5.4. The effect of the order parameter C_μ in an interacting model (with an "energy" exponent G), is to produce an x-depending shift in the Y-marginal. This is the influence of the response function in the symmetric model, as can be seen in [RSZ89], or in [Gru92] for a simple Markov-chain model. This effect will probably become more transparent in a later work, when we deal with the interpretation of the rate $\mathrm{S}^{\mathrm{G}}_\alpha$ of the interacting model.

Proof Theorem 6.2. Most of the work has actually been done in Sect. 5. We have to apply an appropriate approximation scheme to the measures P^N_{α,ρ_N} and we must have an exponential control over the error we make.

For the proof we fix a $\dot{\mathscr{X}}_N$-sequence $\rho_N \to \rho \in \dot{\mathscr{X}}_\infty$ and $x^{(N)} \in X^N$ such that $\Theta^X_{N,x^{(N)}} = \rho_N$.

For some function $y \in \mathscr{D}_{\mathbb{R}}[0,T]$ we define a approximate version by

$$y^k(s) := \frac{2^k}{T} \int_{i\frac{T}{2^k}}^{(i+1)\frac{T}{2^k}} y(t)\,dt, \qquad \text{for} \qquad s \in \left[i\frac{T}{2^k}, (i+1)\frac{T}{2^k} \right[=: \mathrm{I}_{k,i}, \tag{50}$$

for $s \in [0,T[$ and set $y^k(T) = \lim_{s\uparrow T} y^k(s)$. We define the map

$$\begin{aligned} \phi_k : \mathscr{M}_1(\mathrm{Z}) &\longrightarrow \mathscr{M}_1(\mathrm{Z}) \\ \nu &\longmapsto \phi_k(\nu) := \int \delta_{(y^k,x)}\, \nu(dy,dx). \end{aligned} \tag{51}$$

The approximation scheme we will use is

$$P^{k,N}_{\alpha,\rho_N} := \phi^k \left(P^N_{\alpha,\rho_N} \right). \tag{52}$$

Using the results of Sect. 5 we get the following LDP result for the $P^{k,N}_{\alpha,\rho_N}$.

Corollary 6.5. *The sequence of measures $P^{k,N}_{\alpha,\rho_N}$ has a full LDP on $\mathscr{M}_1(Z)$ with good rate function*

$$\mathrm{I}^k_\alpha(\rho;\mu) := \begin{cases} \inf\limits_{B\in \mathscr{B}^k_2(\mathscr{L}_2)} \mathrm{H}\left(\mu|\phi_K(\dot{\mathscr{N}}_\rho \otimes_B \rho)\right) + \Gamma^{*,k}_{\alpha,\rho}(\mathrm{C}_\mu) & \pi_X(\mu) = \rho, \\ & \int \|y^k\|\mu(dy) < \infty \\ \infty & \text{otherwise} \end{cases}, \tag{53}$$

where

$$\mathscr{B}^k_2(\mathscr{L}_2) := \{ B \in \mathscr{B}_2(\mathscr{L}_2) : B(s,t) = B(s',t') \text{ for } s,s' \in I_{k,i}, t,t' \in I_{k,j} \}, \tag{54}$$

and $\Gamma^{*,k}_{\alpha,\rho}(\mathrm{C}_\mu)$ is defined as in Definition (39) but with the supremum running over $\mathscr{B}^k_2(\mathscr{L}_2)$ instead of $\mathscr{B}_2(\mathscr{L}_2)$.

Proof. We have to translate into the notation in Sect. 5. Let $W := (\mathscr{D}_{\mathbb{R}}[0,T])^k$ be the 2^k-dimensional space of piecewise constant functions equipped with the $\mathscr{L}_2$ inner-product $< \cdot,\cdot >$. Identify $\tau(x) := x^k$ for every function $x \in \mathscr{D}_{\mathscr{E}}[0,T]$. The last changes come from the fact that $y \to y^k$ acts as a orthogonal projection on W in the space $\mathscr{L}_2$. $\square$

We will apply the general approach of [DeZe93], Chap. 4.2.2, and have to check

Proposition 6.6. *$P^{k,N}_{\alpha,\rho_N}$ is an exponentially good approximation of P^N_{α,ρ_N}.*

Proof. Let d_X and d_Y be the usual Skohorod-metric on $\mathscr{D}_{\mathscr{E}}[0,T]$ and on $\mathscr{D}_{\mathbb{R}}[0,T]$ respectively (observe that $d_Y(y,y') \leq \|y-y'\|_\infty$ is always true). Then we define as in Eq. (43) the Vaserstein metric

$$d_{\mathscr{Z}}(\nu,\nu') := \inf\left\{\int_{Z\times Z}((d_Y(y,y')+d_X(x,x'))\wedge 1)\chi(dz,dz') : \chi \in \mathscr{M}_{\nu,\nu'}\right\} \tag{55}$$

on $\mathscr{M}_1(Z)$. We have to check that

$$\lim_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N}\ln P^N_{\alpha,\rho_N}\left(\underbrace{\{\nu : d_{\mathscr{Z}}(\nu,\phi^k(\nu)) > \delta\}}_{=:\,\Lambda_\delta}\right) = -\infty$$

for all $\delta > 0$ ([DeZe93], Definition 4.2.14). We define (see Eq. (41))

$$z^N_{1,i}(E) := \frac{1}{\sqrt{N}}\sum_{j=1}^N E^{ij}x^{(N)}_j, \quad \text{and} \quad z^N_{2,i}(E) := \frac{1}{\sqrt{N}}\sum_{j=1}^N E^{ji}x^{(N)}_j, \tag{56}$$

for $i,j \in \{1,\dots,N\}$. Then, like in Eq. (45), we obtain

$$\begin{aligned}
P^N_{\alpha,\rho_N}(\Lambda_\delta) &\leq \nu^N_{\alpha,x^{(N)}}\left(\left\{y\in Y^N : \left(\frac{1}{N}\sum_{i=1}^N d_Y(y_i,y_i^k)\right)\wedge 1 \geq \delta\right\}\right)\\
&\leq \nu^N_{\alpha,x^{(N)}}\left(\left\{y\in Y^N : \frac{1}{N}\sum_{i=1}^N \|y_i - y_i^k\|_\infty \wedge 1 \geq \delta\right\}\right)\\
&\leq \exp[-N\alpha\delta]\int_{Y^N}\exp\left[\alpha\sum_{i=1}^N\|y_i-y_i^k\|\wedge 1\right]\,\nu^N_{\alpha,x^{(N)}}(dy)\\
&\leq \exp[-N\alpha\delta]\mathbf{E}_E\exp\left[\alpha\left(|\kappa_1|\sum_{i=1}^N\|z^N_{1,i}(E)-z^N_{1,i}(E)^k\|\wedge 1\right.\right.\\
&\qquad\left.\left.+|\kappa_2|\sum_{i=1}^N\|z^N_{2,i}(E)-z^N_{1,i}(E)^k\|\wedge 1\right)\right]\\
&\leq \exp[-N\alpha\delta]\left(\int\exp[2\alpha|\kappa_1|\cdot\|y-y^k\|\wedge 1]\mathscr{N}^{\cdot}_{\rho_N}(dy)\right)^N.
\end{aligned} \tag{57}$$

Because of Remark 4.5 there is a compact set $K \subset \mathscr{C}_{\mathbb{R}}[0,T]$ such that

$$\mathscr{N}^{\cdot}_{\rho}(K^c) \leq \frac{1}{2}\exp[-2\alpha|\kappa_1|],$$

and because of the weak convergence in the $\|\cdot\|_\infty$-norm, for every $\epsilon > 0$ there is a number N_0 such that

$$\mathscr{N}^{\cdot}_{\rho_N}((K^\epsilon)^c) \leq \exp[-2\alpha|\kappa_1|]$$

for $N \geq N_0$, where K^ϵ is the ϵ-ball around K in $\|\cdot\|_\infty$. Lets take $\epsilon := \frac{1}{4\alpha|\kappa_1|}$. Then

$$\int \exp[2\alpha|\kappa_1| \cdot \|y - y^k\| \wedge 1] \mathcal{N}_{\rho_N}^{\cdot}(dy) \leq 1 + \exp[2\alpha|\kappa_1| \sup_{y \in K^\epsilon} \|y - y^k\|].$$

But

$$\sup_{y \in K^\epsilon} \|y - y^k\| \leq 2\epsilon + \sup_{y \in K^\epsilon} w_y \left(\frac{T}{2^k}\right),$$

and since $\sup_{y \in K^\epsilon} w_y \left(\frac{T}{2^k}\right) \to 0$ for $k \to \infty$, we get

$$\lim_{k \to \infty} \limsup_{N \to \infty} \frac{1}{N} \ln P^N_{\alpha,\rho_N}(\Lambda_\delta) \leq -\alpha\delta + \ln(1 + \mathrm{e})$$

for arbitrarily large α. □

As a direct consequence of [DeZe93], Theorem 4.2.16, P^N_{α,ρ_N} has a weak LDP with rate

$$\tilde{\mathrm{I}}_\alpha(\rho; \nu) := \sup_{\delta > 0} \liminf_{k \to \infty} \inf_{\mu \in B_\delta(\nu)} \mathrm{I}^k_\alpha(\rho; \mu), \tag{58}$$

where $B_\delta(\nu)$ is the δ-ball around ν. The function $\tilde{\mathrm{I}}_\alpha$ is lower semi-continuous by construction.

We complete the proof of Theorem 5.3 with the next proposition, since lifting the result to $\mathscr{Z}$ is possible because $P^N_{\alpha,\rho_N}(\{\pi_X^{-1}(\rho_N)\}) = 1$ and the rate function I_α is finite only on the fixed point ρ.

Proposition 6.7. $\tilde{\mathrm{I}}_\alpha = \mathrm{I}_\alpha$. I_α *is a good rate function. The condition*

$$\inf_F \mathrm{I}_\alpha \leq \limsup_{k \to \infty} \inf_F \mathrm{I}^k_\alpha \tag{59}$$

([DeZe93], (4.2.18)) *is satisfied for every closed set $F \subset \mathscr{M}_1(Z)$ and therefore P^N_{α,ρ_N} has a full LDP.*

Proof. For the proof we will restrict the rate functions I and H to the closed set $\{\mu : \pi_X(\mu) = \rho\}$, since $\mathrm{I} = \infty$ on the complement, anyway. Observe that because of App. B $\mathrm{C}(\nu)$ is well-defined for $\nu \in \{\mu : \mathrm{H}_\rho(\mu|\mathcal{N}_\rho^{\cdot} \otimes \rho) < \infty\} =: \Phi_{\mathrm{H}_\rho}$, where H_ρ is the standard relative entropy H defined to be $\mathrm{H}_\rho(\mu|\mathcal{N}_\rho^{\cdot} \otimes q) = \infty$ for $\pi_X(\mu) \neq \rho$.

We will establish some properties of the rate functions. If $\mathrm{H}_\rho(\nu|\mathcal{N}_\rho^{\cdot} \otimes \rho) = \infty$ then $\mathrm{H}_\rho(\nu|\mathcal{N}_\rho^{\cdot} \otimes_B \rho) = \infty$ for all $B \in \mathscr{B}_2(\mathscr{L}_2)$ since $\mathcal{N}_\rho^{\cdot} \otimes \rho$ and $\mathcal{N}_\rho^{\cdot} \otimes_B \rho$ are mutually absolutely continuous and

$$\begin{aligned} \mathrm{H}_\rho(\nu|\mathcal{N}_\rho^{\cdot} \otimes_B \rho) &= \mathrm{H}_\rho(\nu|\mathcal{N}_\rho^{\cdot} \otimes \rho) \\ &\quad - \int \left[< y, B \cdot x > - \frac{1}{2} < B \cdot x, \mathrm{D} \cdot B \cdot x > \right] \nu(dy, dx) \\ &= \mathrm{H}_\rho(\nu|\mathcal{N}_\rho^{\cdot} \otimes \rho) - \left[(B, \mathrm{C}(\nu)) - \frac{1}{2}(B, \mathbf{C}_{0,\rho}(B)) \right]. \end{aligned} \tag{60}$$

In addition Eq. (60) and Eq. (39) show that

$$\inf_{B\in\mathscr{B}_2(\mathscr{L}_2)} \mathrm{H}_{\rho}(\nu|\mathscr{N}_{\rho}^{\cdot}\otimes_B \rho) = \mathrm{H}_{\rho}(\nu|\mathscr{N}_{\rho}^{\cdot}\otimes \rho) - \Gamma_{0,\rho}(\mathrm{C}_{\nu}) \tag{61}$$

for $\nu \in \Phi_{\mathrm{H}_{\rho}}$. For $B \in \mathscr{B}_2(\mathscr{L}_2)$

$$(B, \mathbf{C}_{\alpha,\rho}(B)) = (1+\alpha)(B^s, \mathbf{C}_{0,\rho}(B^s)) + (1-\alpha)(B^a, \mathbf{C}_{0,\rho}(B^a)) \le 2\cdot (B, \mathbf{C}_{0,\rho}(B))$$

and therefore

$$\Gamma^*_{\alpha,\rho}(A) \ge \frac{1}{2}\Gamma^*_{0,\rho}(A)$$

for all $\alpha \in [-1,1]$. Combining Eq. (60) and Eq. (47) leads to

$$\mathrm{I}_{\alpha}(\rho;\nu) \ge \mathrm{H}_{\rho}(\nu|\mathscr{N}_{\rho}^{\cdot}\otimes\rho) - \frac{1}{2}\Gamma^*_{0,\rho}(\mathrm{C}_{\nu}) \ge \frac{1}{2}\mathrm{H}_{\rho}(\nu|\mathscr{N}_{\rho}^{\cdot}\otimes\rho). \tag{62}$$

The inequality Eq. (62) is actually valid for all $\nu \in \mathscr{M}_1(Z)$ and therefore the level set

$$\Phi^L_{\mathrm{I}_{\alpha}} := \{\mathrm{I}_{\alpha} \le L\} \subset \{\mathrm{H}_{\rho} \le 2L\}$$

is precompact. Similar results hold for I^k_{α}.

We are now going to prove the upper inequality $\mathrm{I}_{\alpha} \ge \tilde{\mathrm{I}}_{\alpha}$ for $\nu \in \Phi_{\mathrm{H}_{\rho}}$. Since $y^k \to y$ in $\mathscr{D}_{\mathbb{R}}[0,T]$ for every continuous function y

$$\phi^k(\nu) \overset{\mathrm{w}}{\Longrightarrow} \nu \quad \text{for} \quad k\to\infty,$$

since $\mathscr{N}_{\rho}(\mathscr{C}_{\mathbb{R}}[0,T] = 1$ (Proposition 4.4) and $\nu \ll \mathscr{N}_{\rho}^{\cdot}\otimes\rho$. Hence, for all $\delta > 0$, $\phi^k(\nu) \in B_{\delta}(\nu)$ for k large enough. The characterization ([DeZe93], Lemma. 6.2.13) shows that

$$\mathrm{H}_{\rho}(\phi_k(\cdot)|\phi_k(\cdot)) = \mathrm{H}^{\mathscr{F}^k}_{\rho}(\cdot|\cdot),$$

where $\mathrm{H}^{\mathscr{F}^k}_{\rho}(\cdot|\cdot)$ is the relative entropy restricted to the σ-field $\mathscr{F}^k := \boldsymbol{\sigma}(\varphi_k)$ generated by $\varphi_k(y,x) := (y^k, x)$. Since $\mathscr{F}^k \subset \mathscr{F}^{k'}$ for $k \le k'$ and $\boldsymbol{\sigma}\left(\bigcup_{k>0}\mathscr{F}^k\right) = \mathscr{B}(Z)$ is the Borel-σ-field on Z [Geo85], Proposition 15.6, leads to the monotone convergence

$$\mathrm{H}_{\rho}(\phi^k(\mu)|\phi^k(\mathscr{N}_{\rho}^{\cdot}\otimes\rho)) \nearrow \mathrm{H}_{\rho}(\mu|\mathscr{N}_{\rho}^{\cdot}\otimes\rho)$$

for all measures $\mu \in \mathscr{M}_1(Z)$. We define the projection P^k on $\mathscr{L}_2$ by $(\mathrm{P}^k\cdot y) := y^k$ and set $B^k := \mathrm{P}^k\cdot B\cdot \mathrm{P}^k$ for some operator $B \in \mathscr{B}_2(\mathscr{L}_2)$. With the representation Eq. (60) for I^k_{α} we get the estimate

$$\begin{aligned}
\tilde{\mathrm{I}}_{\alpha}(\rho;\nu) &\le \liminf_{k\to\infty} \mathrm{I}^k_{\alpha}(\rho;\phi^k(\nu)) \\
&\le \lim_{k\to\infty}\Big(\mathrm{H}_{\rho}(\phi^k(\nu)|\phi^k(\mathscr{N}_{\rho}^{\cdot}\otimes\rho)) - \Big[(B^k, \mathrm{C}_{\nu}) - \frac{1}{2}(B^k, \mathbf{C}_{0,\rho}(B^k))\Big] \\
&\quad + \Gamma^{*;k}_{\alpha,\rho}(\mathrm{C}_{\nu})\Big) \\
&= \mathrm{H}_{\rho}(\nu|\mathscr{N}_{\rho}^{\cdot}\otimes\rho) - \Big[(B, \mathrm{C}_{\nu}) - \frac{1}{2}(B, \mathbf{C}_{0,\rho}(B))\Big] + \Gamma^*_{\alpha,\rho}(\mathrm{C}_{\nu})
\end{aligned}$$

for all $B \in \mathscr{B}_2(\mathscr{L}_2)$ since the last two expressions in the second line converge due to the convergence $\|B - B^k\|_2 \to 0$ in $\mathscr{B}_2(\mathscr{L}_2)$. Since B was arbitrary, taking the infimum proves the inequality.

Lower estimate $\tilde{\mathrm{I}}_\alpha \geq \mathrm{I}_\alpha$: To simplify the proof of the lower bound we will switch to the stronger topology on a subset $\mathscr{W} \subset \mathscr{M}_1(Z)$ (as in App. B) such that the map $\nu \to \mathrm{C}_\nu$ is continuous and $\Gamma^*_{\alpha,\rho}(\mathrm{C}_\nu)$ is lower semi-continuous. Because of Theorem 5.3 the result is still valid. The condition for the lower bound is only getting stronger.

Since

$$\mathrm{I}^k_\alpha(\rho;\nu) \geq \frac{1}{2}\mathrm{H}_\rho(\nu|\phi^k(\dot{\mathscr{N}_\rho} \otimes \rho)) \geq \frac{1}{2}\mathrm{H}_\rho(\phi^k(\nu)|\phi^k(\dot{\mathscr{N}_\rho} \otimes \rho)),$$

and therefore $\tilde{\mathrm{I}}_\alpha(\rho;\nu) = \infty$ for ν such that $\mathrm{H}_\rho(\nu|\dot{\mathscr{N}_\rho} \otimes \rho) = \infty$, we assume that $\mathrm{H}_\rho(\nu|\dot{\mathscr{N}_\rho} \otimes \rho) =: L < \infty$. $\mathrm{I}^k_\alpha(\rho;\nu)$ is finite only when $\phi^k(\nu) = \nu$, i.e. $\nu(\varphi_k(Y \times X)) = 1$. We therefore have the relation $\mathrm{I}^k_\alpha(\rho;\nu) \geq \mathrm{I}^k_\alpha(\rho;\phi^k(\nu))$ for all $\nu \in \mathscr{W}$. Then

$$\begin{aligned}\inf_{B\in\mathscr{B}_2(\mathscr{L}_2)} \mathrm{H}_\rho(\nu|\dot{\mathscr{N}_\rho}\otimes_B \rho) &= \inf_{\left\{B\in\mathscr{B}_2(\mathscr{L}_2):\frac{1}{2}(B,\mathbf{C}_{0,\rho}(B))\leq 2L\right\}} \mathrm{H}_\rho(\nu|\dot{\mathscr{N}_\rho}\otimes_B \rho)\\ &= \inf_{\mu\in S_L} \mathrm{H}_\rho(\nu|\mu),\end{aligned}$$

where

$$S_L := \left\{ \dot{\mathscr{N}_\rho}\otimes_B \rho \,\middle|\, B \in \mathscr{B}_2(\mathscr{L}_2) : \frac{1}{2}(B, \mathbf{C}_{0,\rho}(B)) \leq 2L \right\} \subset \mathscr{W}.$$

S_L is precompact since $\mathrm{H}_\rho(\dot{\mathscr{N}_\rho}\otimes_B \rho|\dot{\mathscr{N}_\rho}\otimes \rho) = \frac{1}{2}(B, \mathbf{C}_{0,\rho}(B))$. Because $\phi^K(\nu) \in B_\delta(\nu)$ for $k \geq k_0$ we have

$$\begin{aligned}&\inf_{\xi\in B_\delta(\nu)} \mathrm{I}^k_\rho(\rho;\xi)\\ &\geq \inf_{\xi\in B_\delta(\nu)} \left[\inf_{\left\{B\in\mathscr{B}^k_2(\mathscr{L}_2):\frac{1}{2}(B,\mathbf{C}_{0,\rho}(B))\leq 2L\right\}} \mathrm{H}_\rho(\phi^k(\xi)|\phi^k(\dot{\mathscr{N}_\rho}\otimes_B \rho)) + \Gamma^{*,k}_{\alpha,\rho}(\mathrm{C}_\xi)\right]\\ &\geq \inf_{\xi\in B_\delta(\nu)} \left[\inf_{\mu\in S_L} \mathrm{H}_\rho(\phi^k(\xi)|\phi^k(\mu)) + \Gamma^{*,k}_{\alpha,\rho}(\mathrm{C}_\xi)\right] =: \inf_{\xi\in B_\delta(\nu)} \mathrm{I}^{k\,\prime}_\alpha(\rho;\xi).\end{aligned}$$

Because $\mathrm{H}_\rho(\nu|\mu)$ is lower semi-continuous in (ν,μ) and S_L is precompact, $\mathrm{I}^{k\,\prime}_\alpha(\rho;\cdot)$ are lower semi-continuous functions converging monotone at ν,

$$\mathrm{I}^{k\,\prime}_\alpha(\rho;\nu) \nearrow \mathrm{I}_\alpha(\rho;\nu).$$

We get the lower bound since

$$\begin{aligned}\tilde{\mathrm{I}}_\alpha(\rho;\nu) &= \sup_\delta \liminf_{k\to\infty} \inf_{\xi\in B_\delta(\nu)} \mathrm{I}^k_\alpha(\rho;\xi)\\ &\geq \sup_\delta \lim_{k\to\infty} \inf_{\xi\in B_\delta(\nu)} \mathrm{I}^{k\,\prime}_\alpha(\rho;\xi) = \mathrm{I}_\alpha(\rho;\nu).\end{aligned}$$

The last missing step is to establish the condition Eq. (59) for all closed sets $F \subset \mathscr{M}_1(Z)$. From the explicit construction of the compact set for the exponential tightness of the Sanov result ([DeZe93], Lemma 6.2.6) and the convergence

$$\phi^k(\mathscr{N}_\rho \otimes \rho) \stackrel{w}{\Longrightarrow} \mathscr{N}_\rho \otimes \rho$$

we have a compact set $K_L \subset \mathscr{M}_1(Z)$ such that

$$\mathrm{I}^{\,k}_{\alpha}(\rho;\nu) \geq \frac{1}{2}\mathrm{H}_{\,\rho}(\phi^k(\nu)|\phi^k(\mathscr{N}_\rho \otimes \rho)) \geq L$$

for all $k = 1, 2, \cdots$ and $\nu \notin K_L$. In the case $\limsup_{k\to\infty} \inf_{\nu\in F} \mathrm{I}^{\,k}_{\alpha}(\rho;\nu) = \infty$ we do not have to prove anything. We assume therefore that

$$\limsup_{k\to\infty} \inf_{\nu\in F} \mathrm{I}^{\,k}_{\alpha}(\rho;\nu) = \kappa < \infty,$$

i.e. $\inf_{\nu\in F} \mathrm{I}^{\,k}_{\alpha}(\rho;\nu) \leq 2\kappa$ for k large enough. Since the $\mathrm{I}^{\,k}_{\alpha}$ are good rate functions the infimum is attained at some point

$$\nu_k \in K_{2\kappa} \cap F, \quad \inf_{\nu\in F} \mathrm{I}^{\,k}_{\alpha}(\rho;\nu) = \mathrm{I}^{\,k}_{\alpha}(\rho;\nu_k).$$

We will find a convergent subsequence $\nu_{k_n} \stackrel{w}{\Longrightarrow} \nu \in F \cap K_{2\kappa}$. From the definition Eq. (58) we get

$$\begin{aligned} \inf_{\mu\in F} \mathrm{I}_{\,\alpha}(\rho;\mu) \leq \mathrm{I}_{\,\alpha}(\rho;\nu) &\leq \liminf_{k\to\infty} \inf_{\mu\in F} \mathrm{I}^{\,k}_{\alpha}(\rho;\mu) \\ &\leq \liminf_{n\to\infty} \mathrm{I}^{\,k_n}_{\alpha}(\rho;\nu_{k_n}) \leq \kappa, \end{aligned}$$

which shows the result.

□

□

6.2. "Interacting"-case

It is now a simple task to show large deviations results for interacting models, i.e. models with present Girsanov exponent G, corresponding to the annealed symmetric spin-glass.

We first prove a full LDP in an approximate situation. Fix some $L > 0$. Then we have the

Lemma 6.8. *The sequence of probability-measures* $P^{N,\mathrm{G}^L}_{\alpha}(d\nu) := \exp[N \int \mathrm{G}^L\, d\nu] P^N_{\alpha}(d\nu)$ *on* $\mathscr{Z}$ *defined with the Girsanov exponent* G^L *as in Eq. (24) stisfy a full LDP with rate* $\mathrm{S}^{\,\mathrm{G}^L}_{\alpha}(\nu) = \mathrm{S}_{\,\alpha}(\nu) - \int \mathrm{G}^L\, d\nu$.

Proof. We will again use the LDS-approach. Therefore we fix some $\mathscr{X}_N$-sequence $\rho_N \to \rho \in \mathscr{X}_\infty$. We have to show that $P^{N,\mathrm{G}^L}_{\alpha,\rho_N}$ is a LDS with rate

$$\mathrm{I}_\alpha(\rho;\cdot) - \int \mathrm{G}^L\, d(\cdot) =: \mathrm{I}^{\mathrm{G}^L}_\alpha(\rho;\cdot).$$

The level sets $\{\mathrm{I}^{\mathrm{G}^L}_\alpha(\rho;\cdot) \le M\}$ are precompact since $\mathrm{I}^{\mathrm{G}^L}_\alpha(\rho;\cdot) \ge \mathrm{I}_\alpha(\rho;\cdot) - T$. Since all measures $\nu \in \{\mathrm{I}^{\mathrm{G}^L}_\alpha(\rho;\cdot) \le M\}$ satisfy $\nu(\mathscr{C}[0,T]\times X) = 1$ the function $\mathrm{I}^{\mathrm{G}^L}_\alpha(\rho;\cdot)$ is lower semi-continuous. The function G^L_δ (see Eq. (24) below), is continuous and bounded by Eq. (26). Since the Theorem 6.2 is valid in $\mathscr{Z}$ the measures $P^{N,\mathrm{G}^L_\delta}_{\alpha,\rho_N}$ are a LDS with rate $\mathrm{I}_\alpha(\rho;\cdot) - \int \mathrm{G}^L_\delta\, d(\cdot) =: \mathrm{I}^{\mathrm{G}^L_\delta}_\alpha(\rho;\cdot)$. We will show: For each $M > 0$ and each $\epsilon > 0$ there is a set $B_M \subset \mathscr{Z}$, $\delta_0 > 0$ and a N_0 such that

$$\limsup_{N\to\infty} \frac{1}{N} \ln P^N_{\alpha,\rho_N}(B^c_M) \le -M$$

and $|\mathrm{G}^L - \mathrm{G}^L_\delta| \le \epsilon$ for $\delta \le \delta_0$, $\nu \in B_M$, and $\pi_X(\nu) = \rho_N$ for $N \ge N_0$ or $\pi_X(\nu) = \rho$. Let $K \subset \mathscr{C}_{\mathbb{R}}[0,T]$ (Remark 4.5) be a compact set such that $\mathscr{N}_\rho(K^c) \le \frac{1}{2}\epsilon_1$. We assume that K is star-shaped, i.e. for $y \in K$, $\lambda \in [-1,1]$ $\lambda y \in K$. Such sets exist due to Arzelà-Ascoli's Theorem. Set $K' := K + K = \{y + y' : y, y' \in K\}$ and let $K'_{2\delta_1}$ be the $2\delta_1$ ball around K' in the Skohorod metric d_Y and K_{δ_1} be the δ_1 ball for K in the $\|\cdot\|_\infty$-norm (remember $d_Y(y,y') \le \|y - y'\|_\infty$). We define

$$B_M := \left\{\nu \in \mathscr{Z} : \nu\left((K'_{2\delta_1} \times X)^c\right) \le \delta_2\right\},$$

which is a closed set in $\mathscr{Z}$ due to the Portmanteau Theorem, and obtain the estimate (see Eq. (56))

$$\begin{aligned}
P^N_{\alpha,\rho}(B^c_M) &= \nu^N_{\alpha,x^{(N)}}\left(\left\{y \in Y^N : \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{K'_{2\delta_1}{}^c}(y_i) \ge \delta_2\right\}\right)\\
&\le \exp[-N\alpha\delta_2] \int_{Y^N} \exp\left[\alpha_2 \sum_{i=1}^N \mathbf{1}_{K'_{2\delta_1}{}^c}(y_i)\right] \nu^N_{\alpha,x^{(N)}}(dy)\\
&\le \exp[-N\alpha\delta_2]\mathbf{E}_E \exp\left[\alpha_2 \sum_{i=1}^N \left(\mathbf{1}_{K_{\delta_1}{}^c}(\kappa_1 z_{1,i}(E)) + \mathbf{1}_{K_{\delta_1}{}^c}(\kappa_1 z_{2,i}(E))\right)\right]\\
&\le \exp[-N\alpha\delta_2]\left(\int \exp\left[2\alpha \cdot \mathbf{1}_{K_{\delta_1}{}^c}(y)\right] \mathscr{N}_{\rho_N}(dy)\right)^N. \qquad (63)
\end{aligned}$$

Because of Remark 4.5 we conclude

$$\limsup_{N\to\infty} \frac{1}{N} \ln P^N_{\alpha,\rho_N}(B^c_M) \le -\alpha\delta_2 + \ln(\epsilon_1 \cdot \mathrm{e}^{2\alpha} + 1).$$

Assuming that $\nu \in B_M$ and $\pi_X(\nu) = \rho_N$ we have for all L_2 (see Eq. (22)) and Eq. (28)):

$$\left|\int (\mathrm{G}^L - \mathrm{G}^L_\delta)\, d\nu\right| \le 2\ln(1+L) \int_{\{\#_{[0,T]}\ge L_2\}} (\#_{[0,T]}(x)+1)\,\rho_N(dx)$$
$$+\,\delta_2(L_2+1)\ln(1+L) + \beta(L_2+1)\left(4\delta_1 + \sup_{y\in K} w_y(\delta)\right).$$

Now we are almost done. We have to choose (in this order) L_2, N_0, δ_2, δ_1, ϵ_1, K and then δ_0 appropriatly to ensure the existence of B_M, δ_0 and N_0 with the required properties.

The rest is now fairly standard. Let $F \subset \mathscr{Z}$ be a closed set. Then for $N \ge N_0$, $\delta \le \delta_0$

$$P^{N,\mathrm{G}^L}_{\alpha,\rho_N}(F) \le \mathrm{e}^{NT}\cdot P^N_{\alpha,\rho_N}(B^c_M) + \mathrm{e}^{N\epsilon}\cdot P^{N,\mathrm{G}^L_\delta}_{\alpha,\rho_N}(F\cap B_M).$$

Making use of the inequality $\mathrm{I}^{\mathrm{G}^L}_\alpha - \epsilon \le \mathrm{I}^{\mathrm{G}^L_\delta}_\alpha$ on B_M we get in the limit:

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}P^{N,\mathrm{G}^L}_{\alpha,\rho_N}(F) &\le \max\left\{2\epsilon - \inf_{\nu\in F\cap B_M}\mathrm{I}^{\mathrm{G}^L}_\alpha(\rho;\nu), -M+T\right\}\\ &\le \max\left\{2\epsilon - \inf_{\nu\in F}\mathrm{I}^{\mathrm{G}^L}_\alpha(\rho;\nu), -M+T\right\}.\end{aligned}$$

Since ϵ and M were arbitrary we are done with the upper bound.

$\mathrm{G}^L_\delta(y,x) \to \mathrm{G}^L(y,x)$ for all (y,x) and therefore by dominated convergence

$$\int \mathrm{G}^L_\delta\, d\nu \to \int \mathrm{G}^L\, d\nu$$

as $\delta \to 0$ for all $\nu \in \mathscr{Z}$. We fix some open set $U \subset \mathscr{Z}$. For the lower bound we start with the general inequality

$$\liminf_{\delta\to 0}\inf_{\nu\in U}\mathrm{I}^{\mathrm{G}^L_\delta}_\alpha(\rho;\nu) \le \inf_{\nu\in U}\mathrm{I}^{\mathrm{G}^L}_\alpha(\rho;\nu)$$

and complete the proof since

$$\begin{aligned}&-\inf_{\nu\in U}\mathrm{I}^{\mathrm{G}^L_\delta}_\alpha(\rho;\nu) \le \liminf_{N\to\infty}\frac{1}{N}\ln P^{N,\mathrm{G}^L_\delta}_{\alpha,\rho_N}(U)\\ &\le \max\left\{\epsilon + \liminf_{N\to\infty}\frac{1}{N}\ln P^{N,\mathrm{G}^L}_{\alpha,\rho_N}(U); T + \limsup_{N\to\infty}\frac{1}{N}\ln P^N_{\alpha,\rho_N}(B_M)\right\}.\end{aligned}$$

Mixing the measures $P^{N,\mathrm{G}^L}_{\alpha,(\cdot)}$ by Q^N and Theorem 2.3 proves the lemma. □

Now we come to the most interesting result from a physicist's point of view, which for a mathematician is just a

Corollary 6.9. *The measures* $\left(P^{N,\mathrm{G}}_\alpha\right)$ *of the annealed symmetric spin-glass dynamics obey a full Large Deviation upper bound on* $\mathscr{Z}$ *with good rate function* $\mathrm{S}^{\mathrm{G}}_\alpha(\nu) := \mathrm{S}_\alpha(\nu) - \int \mathrm{G}\, d\nu$.

Proof. First we will show that $\mathrm{S}_{\alpha}^{\mathrm{G}}$ is a good rate function. Since $\mathrm{S}_{\alpha}^{\mathrm{G}} \geq \mathrm{S}_{\alpha} - T$ the set

$$\Phi_L := \{\mathrm{S}_{\alpha}^{\mathrm{G}} \leq L\} \subset \{\mathrm{S}_{\alpha} \leq L + T\}$$

is precompact. We have to show, that Φ_L is closed. Fix a sequence $\nu_n \to \nu$, $\nu_n \in \Phi_L$. By the Definition of S_{α}, $\mathrm{H}(\pi_X(\nu_n)|q) \leq L + T$ and Proposition 4.4 shows that $\nu_n(\mathscr{C}_{\mathbb{R}}[0,T] \times X) = 1$ for all $n \in \mathbb{N}$. Because of the lower semi-continuity of S_{α} and [DeZe93], Theorem D.12.

$$L \geq \liminf_{n\to\infty} \mathrm{S}_{\alpha}^{\mathrm{G}}(\nu_n) \geq \liminf_{n\to\infty} \mathrm{S}_{\alpha}(\nu_n) + \liminf_{n\to\infty}(-\int \mathrm{G}\, d\nu_n) \geq \mathrm{S}_{\alpha}(\nu) - \int \mathrm{G}\, d\nu,$$

which proves $\nu \in \Phi_L$. Since we do not have the lower bound, we have to prove that $\mathrm{S}_{\alpha}^{\mathrm{G}} \geq 0$. Since Φ_L is compact, there is a ν such that $\mathrm{S}_{\alpha}^{\mathrm{G}}(\nu) = \inf_{\mu\in\mathscr{Z}} \mathrm{S}_{\alpha}^{\mathrm{G}}(\mu)$. Then $-\int \mathrm{G}\, d\nu < \infty$ and by dominated convergence

$$\mathrm{S}_{\alpha}^{\mathrm{G}}(\nu) = \lim_{L\to\infty} \mathrm{S}_{\alpha}^{\mathrm{G}^L}(\nu) \geq 0,$$

since the $\mathrm{S}_{\alpha}^{\mathrm{G}^L}$ are non-negative.

The measures $P_{\alpha}^{N,\mathrm{G}}$ are exponentially tight since the P_{α}^{N} are exponentially tight ([Puk91], Theorem (P)) and $\mathrm{G} \leq T$. Therefore it is enough to prove a weak upper LD-bound ([DeZe93], Lemma 1.2.18). We fix some compact set $F \subset \mathscr{Z}$. Then Eq. (25) shows that for $L \geq L_0$

$$\limsup_{N\to\infty} \frac{1}{N} \ln P_{\alpha}^{N,\mathrm{G}}(F) \leq \frac{T}{1+L_0} + \limsup_{N\to\infty} \frac{1}{N} \ln P_{\alpha}^{N,\mathrm{G}^L}(F) \leq \frac{T}{1+L_0} - \inf_F \mathrm{S}_{\alpha}^{\mathrm{G}^L}.$$

The only missing step is to show

$$\lim_{L\to\infty} \inf_F \mathrm{S}_{\alpha}^{\mathrm{G}^L} = \inf_F \mathrm{S}_{\alpha}^{\mathrm{G}}.$$

Define $\bar{\mathrm{G}}^L$ as in Eq. (24) omitting the $\wedge L$-parts in the second term. Then $\mathrm{S}_{\alpha}^{\bar{\mathrm{G}}^L}(\nu) := \mathrm{S}_{\alpha}(\nu) - \int \bar{\mathrm{G}}^L\, d\nu$ are lower semi-continuous functions converging monotone to $\mathrm{S}_{\alpha}^{\mathrm{G}}$,

$$\mathrm{S}_{\alpha}^{\bar{\mathrm{G}}^L} \nearrow \mathrm{S}_{\alpha}^{\mathrm{G}}$$

as $L \to \infty$ by Fatou's Lemma. We have the inequality

$$\mathrm{S}_{\alpha}^{\mathrm{G}^L} \geq \mathrm{S}_{\alpha}^{\bar{\mathrm{G}}^L} - \frac{T}{1+L_0} \text{ for } L \geq L_0.$$

Since F is compact we end the proof by

$$\begin{aligned} \lim_{L\to\infty} \inf_F \mathrm{S}_{\alpha}^{\mathrm{G}^L} &\geq \lim_{L\to\infty} \inf_F \mathrm{S}_{\alpha}^{\bar{\mathrm{G}}^L} - \frac{T}{1+L_0} \\ &= \inf_F \mathrm{S}_{\alpha}^{\mathrm{G}} - \frac{T}{1+L_0} \end{aligned}$$

since L_0 can be chosen arbitrarily large. □

A. Some technical results

For the proof of Lemma 4.1 we will use a quite general result due to [DaGä87] (Theorem 3.4), that we will state for the sake of completeness.

Let W be a real vector space and $\mathscr{Z}$ a subset of its algebraic dual W^*. We equip W^* with the weak* topology $\sigma(W^*, W)$ induced by W and equip $\mathscr{Z}$ with the relative topology. Let (μ_N) be a sequence of probability measures on $\mathscr{Z}$. In this setting we have the

Theorem A.1 ([DaGä87]). *Suppose the following conditions are satisfied:*

1. *for each $w \in W$, the limit*

$$\Lambda(w) = \lim_{N\to\infty} \Lambda_N(w) = \lim_{N\to\infty} \frac{1}{N} \ln \int_{\mathscr{Z}} \exp[N \cdot < z, w >] \, \mu_N(dz)$$

   exists and is finite;
2. *Λ is Gâteaux differentiable, i.e. the real function $t \mapsto \Lambda(w + tw')$ is differentiable for every $w, w' \in W$. Define*

$$\Lambda^*(y) = \sup_{w\in W} [< y, w > - \Lambda(w)], \quad y \in W^*, \tag{64}$$

   and suppose further that
3. *$\{y \in W^* : \Lambda^*(y) < \infty\} \subset \mathscr{Z}$.*

Then the sequence (μ_N) satisfies a full LDP with Λ^* restricted to $\mathscr{Z}$ as rate function.

We will state a second result that we will need to identify the rate function.

Let Z, X be Polish spaces, $\pi : Z \to X$ a measurable map and μ a measure on Z. A regular conditional probability distribution (RCPD) $\mu.$ will be a measurable map $\mu. : X \to \mathscr{M}_1(Z)$ with the following properties:

1. $\mu_x(\pi^{-1}\{x\}) = 1$
2. $\mu(B) = \int_X \mu_x(B) \, \pi(\mu)(dx), \quad B \in \mathscr{B}(Z)$.

Such a RCPD $\mu.$ exists and is $\pi(\mu)$-almost everywhere unique (see [DeZe93], Theorem D.3). For this situation the result [DeSt89], Lemma. 4.4.7, holds:

Lemma A.2. *Let μ and ν be measures on Z and denote by $\mu.$ and $\nu.$ the RCPD of the measures. Then $x \mapsto \mathrm{H}(\nu_x|\mu_x)$ is measurable and*

$$\mathrm{H}(\nu|\mu) = \mathrm{H}(\pi(\nu)|\pi(\mu)) + \int_X \mathrm{H}(\nu_x|\mu_x) \, \pi(\nu)(dx) \tag{65}$$

holds.

B. Sanov result for a stronger topology

In the Sect. 5 and Sect. 6 respectively we need the map C to be well-defined and continuous. We will therefore outline the arguments for strengthening a Sanov result to a stronger topology. The argument is adapted for the case of (not necessarily independent) Gaussian variables on a Banach-space.

Let Z be some separable Banach-space and P^N a sequence of measures on $\mathscr{M}_1(Z)$ having a full LDP. Let

$$\mathscr{Z} := \left\{ \nu \in \mathscr{M}_1(Z) : \int \|z\|^{1+\delta} \right\} \subset \mathscr{M}_1(Z) \tag{66}$$

for some fixed $\delta \in]0,1[$ and let

$$\mathscr{C}^\delta := \left\{ f \in \mathscr{C}(Z) : \sup_{z\in Z} \frac{|f(z)|}{\|z\|^{1+\delta} \wedge 1} < \infty \right\}. \tag{67}$$

As in [Léo87] we will equip $\mathscr{Z}$ with the weakest topology such that all maps

$$\nu \longmapsto \int_Z f \, d\nu$$

for $f \in \mathscr{C}^\delta$ are continuous. If we want to emphasize the topology we use on $\mathscr{Z}$, we will denote by $\mathscr{Z}_w$, $\mathscr{Z}$ equipped with the relative topology and with $\mathscr{Z}_s$ the stronger topology defined above. $\mathscr{Z}_s$ is a Polish space [Léo87]. The Borel σ-field $\mathscr{B}(\mathscr{Z}_s)$ on $\mathscr{Z}_s$ is just the trace σ-field induced by $\mathscr{B}(\mathscr{M}_1(Z))$. Define the mean-value of a measure $\nu \in \mathscr{Z}$ to be the unique element $m(\nu) \in Z$ such that

$$\int_Z < y, z > \nu(dz) = < y, m(\nu) >$$

holds for all $y \in Z^*$, the dual space of Z. On $\mathscr{Z}_s$ the map $m(\cdot)$ is continuous as an argument outlined in ([DeZe93], Example 6.2.21) shows. Then we have the

Proposition B.1. *If the sequence P^N has the properties that*

$$\int_{\mathscr{Z}} \exp[\alpha \cdot N \int \|z\|^2 \nu(dz)] P^N(d\nu) \leq \kappa^N < \infty \tag{68}$$

for some $\alpha > 0$ and that $P^N(\mathscr{Z}) = 1$ then the restrictions of the measures P^N have a full LDP on $\mathscr{Z}_s$.

Proof. In the I.I.D.-case this would be (a even weaker) result as in [Léo87], but we need the result in a dependent version. Define for $L \geq 0$ the set

$$\Gamma_L := \left\{ \nu \in \mathscr{M}_1(Z) : \int \|z\|^2 \nu(dz) \leq L \right\}.$$

Γ_L is closed in $\mathscr{M}_1(Z)$ because of ([DeZe93], D.12). Chebycheff's inequality and Eq. (68) show that

$$P^N(\Gamma_L^c) \leq \exp[-\alpha N L] \cdot \kappa^N$$

This inequality has two consequences.

1. The lower LD bound of P^N applied to the open set Γ_L^c and $P^N(\mathscr{Z}) = 1$ establish the conditions of [DeZe93], Lemma 4.1.5. Therefore P^N restricted to $\mathscr{Z}_w$ has a full LDP.
2. On Γ_L equipped with the relative topology induced by $\mathscr{M}_1(Z)$ the maps $\nu \mapsto \int \mathrm{f}\, d\nu$ are continuous for all $\mathrm{f} \in \mathscr{C}^\delta$ and therefore the topology is the same as the one induced by $\mathscr{Z}_s$. Because P^N has a full LDP we have for $L' = \alpha L - \ln \kappa$ a compact set $K_{L'} \subset \mathscr{Z}_w$ such that:
$$\limsup_{N\to\infty} \frac{1}{N} \ln P^N(K_{L'}^c) \le -L'$$
Then $C_{L'} := K_{L'} \cap \Gamma_L$ is a compact set in $\mathscr{Z}_s$ because both topologies coincide on Γ_L, and
$$\limsup_{N\to\infty} \frac{1}{N} \ln P^N(C_{L'}^c) \le -L'$$
holds, i.e. the family P^N is exponentially tight on $\mathscr{Z}_s$.

An application of the "inverse contraction principle" [DeZe93], Theorem 4.2.4, shows the proposition. □

C. Hilbert-space variables

Let Y be some separable Hilbert-space and $\mathscr{N}_{\mathrm{D}}$ a centered Gaussian measure on Y defined by the covariance $\mathrm{D} \in \mathscr{B}_1(Y)$ - the space of trace-class operators on Y.

Proposition C.1. *Let $\lambda = \lambda_1 \ge \lambda_2 \cdots$ be the eigenvalues of* D *then for all α such that*
$$2\alpha\lambda \le \delta < 1$$
we have the inequality
$$\int [\alpha\|y\|^2]\, \mathscr{N}_{\mathrm{D}} \le \exp[2\mu_\delta \alpha \mathrm{Tr}\, \mathrm{D}] \tag{69}$$
with $\mu_\delta = \frac{-\ln(1-\delta)}{2\delta}$.

Proof. [GiSk74], p. 351, establishes the equation
$$\int \exp[\alpha\|y\|^2]\, \mathscr{N}_{\mathrm{D}} = \prod_i \frac{1}{\sqrt{1-2\alpha\lambda_i}}.$$
For $z \le \delta < 1$ we obtain the bound
$$\sqrt{1-z} \ge \exp[-\mu_\delta z] \Leftrightarrow (1-z) \ge \exp[-2\mu_\delta z]$$
for $\mu_\delta = \frac{-\ln(1-\delta)}{2\delta}$ since $\exp[-z]$ is convex and therefore
$$\prod_i \frac{1}{\sqrt{1-2\alpha\lambda_i}} \le \exp\left[2\mu_\delta\alpha \sum_i \lambda_i\right] = \exp[2\mu_\delta \alpha \mathrm{Tr}\, \mathrm{D}].$$
□

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