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Two renewal theorems for general random walks tending to infinity

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Summary. Necessary and sufficient conditions for the existence of moments of the first passage time of a random walk S_n into $[x, \infty)$ for fixed $x \geqq 0$, and the last exit time of the walk from $(-\infty, x]$, are given under the condition that $S_n \to \infty$ a.s. The methods, which are quite different from those applied in the previously studied case of a positive mean for the increments of S_n, are further developed to obtain the "order of magnitude" as $x \to \infty$ of the moments of the first passage and last exit times, when these are finite.

A number of other conditions of interest in renewal theory are also discussed, and some results for the first time for which the random walk remains above the level x on K consecutive occasions, which has applications in option pricing, are given.

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1 Introduction

There is a well known trichotomy for a random walk S_n whose increments X_i are not degenerate at 0. One of the following must hold:

$$S_n \to \infty \text{ a.s.}, \tag{1.1}$$

$$S_n \to -\infty \text{ a.s.} \tag{1.2}$$

or

$$-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty \text{ a.s.} \tag{1.3}$$

(Here 'a.s.' means almost surely, and '$n \to \infty$' will be understood in relations like (1.1)–(1.2)). Necessary and sufficient conditions for (1.1)–(1.3) are available from Spitzer (1956, Theorem 5.1, p. 331) in the following form:

$$(1.1)\ \text{holds} \Leftrightarrow \sum_{n \geqq 1} n^{-1} P\{S_n \leqq 0\} < \infty\,, \tag{1.4}$$

$$(1.2)\ \text{holds} \Leftrightarrow \sum_{n \geqq 1} n^{-1} P\{S_n > 0\} < \infty\,, \tag{1.5}$$

$$(1.3)\ \text{holds} \Leftrightarrow \sum_{n \geqq 1} n^{-1} P\{S_n \leqq 0\} = \infty = \sum_{n \geqq 1} n^{-1} P\{S_n > 0\}\,. \tag{1.6}$$

Quite different criteria for (1.1)–(1.3), expressed in terms of integral conditions on the tail of the distribution of the X_i, are implicit in Erickson's (1973) conditions for the a.s. divergence of S_n/n. These are listed in Lemma 1.1 below. Note in particular that (1.1) can hold even when the mean of the positive part of the X_i, i.e., EX_i^+, and the mean of the negative part, EX_i^-, are both infinite.

Both types of conditions are of great importance in understanding the growth of S_n, and in renewal theory. The series in (1.4)–(1.5), or, more generally, the "harmonic renewal series" $\sum_{n \geqq 1} n^{-1} P\{S_n \leqq x\}$ $(x \in \mathbb{R})$, is crucial to the deep theory of random walks developed by Spitzer and others (see Spitzer, 1976, Chap. IV; also Feller, 1971; Chaps. XII, XVIII), Woodroofe (1982), and Siegmund (1985), for discussion and applications). This theory connects the distributions of $\max_{1 \leqq j \leqq n} S_j$ and of various first passage times, to the distribution of S_n, via certain identities which are of great utility in applications. The Erickson criteria, on the other hand, provide simple tests for (1.1)–(1.3) (and for a similar trichotomy relating to S_n/n) in terms of easily calculated integrals such as that of J_- (see (1.14) below) or of $E(X_i^+)$, for example. Our aim here is to study the passage times into $[x, \infty)$ for a random walk which drifts to infinity a.s., i.e., which satisfies (1.1).

To formulate the behaviour we are interested in, some notation and assumptions are needed. The random walk is

$$S_n = X_1 + X_2 + \cdots + X_n \quad (n \geqq 1)\,, \tag{1.7}$$

where the X_i are independent and identically distributed (i.i.d.) random variables with distribution function F *not degenerate at* 0. Let X be any other random variable with distribution F. In Sect. 2 we prove two renewal-type theorems which apply when $S_n \to \infty$ a.s. These extend previous results proved under the restriction $0 < EX \leqq E|X| < \infty$.

To describe the results, we use the colorful language of the gambling casino or stock market. S_n represents our fortune at time n, and if S_n drifts to ∞ we make a lot of money, eventually. But how long does it take? (Of course all of our results have duals for the case when S_n drifts to $-\infty$; we will not state them separately). Define the *first passage time above* $x \geqq 0$ by

$$T(x) = \min\{n \geqq 1\colon S_n > x\} \tag{1.8}$$

(with $T(x) = \infty$ if $S_n \leqq x$ for all n), and let

$$L(x) = \max\{n \geqq 0\colon S_n \leqq x\} \tag{1.9}$$

(with $S_0 = 0$ and with $L(x) = \infty$ if $S_n \leqq x$ for infinitely many n) be the *last exit time from* $(-\infty, x]$. These measure, in some sense, how quickly or slowly S_n grows. Two aspects are considered here: firstly, we seek conditions for moments, of some order, of $T(x)$ and $L(x)$ to be finite; secondly, we can ask how quickly these moments, when finite, increase as x increases. These questions, which are intimately related to the convergence of series such as $\sum_{n\geqq 1} n^{\alpha-1}P\{S_n \leqq x\}$ and $\sum_{n\geqq 1} n^{\alpha}P\{\max_{1\leqq j\leqq n} S_j \leqq x\}$, have been well studied in the case $0 < EX \leqq E|X| < \infty$. Under this restriction, contributions have been made by Heyde (1964, 1966, 1967), Gut (1974), Janson (1986), and many others, but we will only assume (1.1). For a good introduction to the literature, see Gut (1988, Chap. III), and his references.

Our results are set out in the next section and proved in Sect. 3. The discussion in Sect. 4 relates them to previously known ones, and gives some subsidiary results. For the remainder of this section we review Erickson's (1973) criteria for the divergence of S_n/n, extend them to S_n, and list some other facts which will be useful.

Erickson's conditions refer to another trichotomy for random walks due to Kesten (1970). If $E|X| = \infty$, one of the following must hold:

$$\frac{S_n}{n} \to \infty \text{ a.s.}, \tag{1.10}$$

$$\frac{S_n}{n} \to -\infty \text{ a.s.} \tag{1.11}$$

or

$$-\infty = \liminf_{n\to\infty} \frac{S_n}{n} < \limsup_{n\to\infty} \frac{S_n}{n} = \infty \text{ a.s.} \tag{1.12}$$

To state Erickson's criteria for these, define for $y > 0$

$$A_+(y) = \int_0^y (1 - F(u))\, du \quad \text{and} \quad A_-(y) = \int_0^y F(-u)\, du\,, \tag{1.13}$$

and define

$$J_+ = \int_{[0,\infty)} \left\{\frac{y}{A_-(y)}\right\} dF(y) \quad \text{and} \quad J_- = \int_{[0,\infty)} \left\{\frac{y}{A_+(y)}\right\} |dF(-y)|\,. \tag{1.14}$$

[In (1.14), let $J_+ = EX^+ = 0$ and $J_- = EX^-$ if $F(0) = 1$, and let $J_- = EX^- = 0$ and $J_+ = EX^+$ if $F(0-) = 0$. When $F(0) > 0$, define $y/A_-(y)$ at $y = 0$ by its limit as $y \to 0+$, and similarly for $y/A_+(y)$ when $F(0-) < 1$]. Erickson's (1973, Theorem 2) criteria for (1.10)–(1.12) can then be stated as:

$$(1.10) \quad \text{holds} \Leftrightarrow J_- < \infty = EX^+\,, \tag{1.15}$$

$$(1.11) \quad \text{holds} \Leftrightarrow J_+ < \infty = EX^-\,, \tag{1.16}$$

$$(1.12) \quad \text{holds} \Leftrightarrow J_- = \infty = J_+\,. \tag{1.17}$$

The next lemma is included for reference in what follows. It uses (1.15)–(1.17) to give criteria for (1.1)–(1.3).

Lemma 1.1

$$(1.1)\quad holds \;\Leftrightarrow 0 < EX \leqq E|X| < \infty \quad or \quad J_- < \infty = EX^+ , \tag{1.18}$$

$$(1.2)\quad holds \;\Leftrightarrow 0 < -EX \leqq E|X| < \infty \quad or \quad J_+ < \infty = EX^- , \tag{1.19}$$

$$(1.3)\quad holds \;\Leftrightarrow 0 = EX \leqq E|X| < \infty \quad or \quad J_- = \infty = J_+ . \tag{1.20}$$

Lemma 1.1 is simply based on (1.15)–(1.17) and on the Chung–Fuchs (1951) result, that S_n is recurrent if $0 = EX \leqq E|X| < \infty$ (see also Theorem 2.2 of Kesten and Maller (1995)). In fact (1.18)–(1.20) are immediate consequences of the following, whose brief proof is given at the beginning of Sect. 3:

$$\limsup_{n\to\infty} S_n = \infty \text{ a.s. } \Leftrightarrow 0 \leqq EX \leqq E|X| < \infty \quad \text{or} \quad J_+ = \infty , \tag{1.21}$$

$$\liminf_{n\to\infty} S_n = -\infty \text{ a.s. } \Leftrightarrow 0 \leqq -EX \leqq E|X| < \infty \quad \text{or} \quad J_- = \infty . \tag{1.22}$$

We conclude this section with some results known about $T(x)$ and $L(x)$. By, e.g., Chow and Teicher (1988, p. 144, Theorem 1), $T(x) < \infty$ a.s. for all $x \geqq 0$ if and only if $T(0) < \infty$ a.s., and

$$T(0) < \infty \text{ a.s. } \Leftrightarrow \limsup_{n\to\infty} S_n = \infty \text{ a.s.} \tag{1.23}$$

Consequently, $T(0) < \infty$ a.s. if and only if the integral criteria in (1.21) hold. By, e.g., Chow and Teicher (1988, pp. 146–148), $E(T(x)) < \infty$ for all $x \geqq 0$ if and only if $E(T(0)) < \infty$, equivalently, if $S_n \to \infty$ a.s. Thus $E(T(0)) < \infty$ if and only if the integral criteria in (1.18) hold.

The last exit time $L(x)$ is finite a.s. for all $x \geqq 0$ if and only if $L(0) < \infty$ a.s., and this is equivalent to $S_n \to \infty$ a.s. and hence to (1.18) [see Doney (1989) for other results on the distribution of $L(x)$]. As Gut (1988, p. 74) points out, $T(x)$ and $L(x)$ may be of quite different orders of magnitude in general.

2 Results

An object of major interest in renewal theory is the *renewal function*

$$\sum_{n\geqq 1} P\{S_n \leqq x\}, \quad x \in \mathbb{R} . \tag{2.1}$$

This is the expected number of renewal epochs occuring before x, in renewal theory, or it could be thought of as the expected number of times the random walk "dips below x", in general. Another quantity of interest is

$$\sum_{n\geqq 1} P\left\{\max_{1\leqq j\leqq n} S_j \leqq x\right\}, \quad x \in \mathbb{R} , \tag{2.2}$$

this equals $E(T(x)) - 1$. Thus the convergence or divergence of (2.2) tells us how large $T(x)$ is, in some sense, and hence, how long we must wait, on average, till our fortune exceeds x. By the discussion in Sect. 1, the series in (2.2) is finite if and only if $S_n \to \infty$ a.s., but it seems that the convergence of the series in (2.1) has been investigated only in the case $0 < EX \leqq E|X| < \infty$. Our first theorem gives necessary and sufficient conditions for this convergence, in fact for the convergence of the more general series in (2.4) below which includes the harmonic renewal series and others of importance in investigating the moments of $T(x)$ and $L(x)$. Apart from the extra generality achieved, the new methods required for our proofs have other applications (see, for example, Sect. 4.4 below), and in addition the general approach separates out some different kinds of behaviour of S_n; see the discussion in Sect. 4.2 below.

To state our first result we will need the function

$$A(y) = A_+(y) - A_-(y), \quad \text{for } y \geqq 0, \tag{2.3}$$

where A_+ and A_- are defined in (1.13). Elementary properties of these functions are that $A_+(y)/y \downarrow 0$, $A_-(y)/y \downarrow 0$, and $A(y)/y \to 0$, as $y \to \infty$.

Theorem 2.1 *Fix $\alpha \geqq 0$. The following are equivalent*:

$$\sum_{n\geqq 1} n^{\alpha-1} P\{S_n \leqq x\} < \infty \quad \textit{for some (hence every) } x \geqq 0, \tag{2.4}$$

there is a $y_0 > 0$ such that

$$A(y) > 0 \quad \textit{for } y \geqq y_0 \quad \textit{and} \int_{[y_0,\infty)} \left\{\frac{y}{A(y)}\right\}^{1+\alpha} |dF(-y)| < \infty, \tag{2.5}$$

$$\sum_{n\geqq 1} n^{\alpha} P\left\{\max_{1\leqq j\leqq n} S_j \leqq x\right\} < \infty \quad \textit{for some (hence every) } x \geqq 0, \tag{2.6}$$

$$E(T(x))^{1+\alpha} < \infty \quad \textit{for some (hence every) } x \geqq 0 \tag{2.7}$$

and

$$\begin{cases} E(L(x))^{\alpha} < \infty & \textit{for some (hence every) } x \geqq 0, \textit{ if } \alpha > 0, \\ P\{L(x) < \infty\} = 1 & \textit{for some (hence every) } x \geqq 0, \textit{ if } \alpha = 0. \end{cases} \tag{2.8}$$

Remarks. (i) When F is bounded on the left, so that $F(-y_1) = 0$ for some $y_1 > 0$, we will make the convention that (2.5) holds if and only if $A(y) > 0$ for all large y. In particular, when $F(0-) = 0$ (but F is not degenerate at 0), so that $X \geqq 0$ a.s., (2.5) holds trivially, and the series in (2.4) and (2.6) converge for any x such that $F(x) < 1$, because $P\{S_n \leqq x\} \leqq F^n(x)$ when $X_i \geqq 0$ a.s. (Of course by the theorem they continue to converge even for x with $F(x) = 1$.) In this case, also, $T(x) - 1 = L(x)$, and since (2.6) and (2.7) are always equivalent, (2.7)–(2.8) are trivial too.
(ii) For general X_i, (2.4) holding for any $\alpha \geqq 0$ and $x \geqq 0$ implies (2.4) for $\alpha = 0$, $x = 0$, and hence $S_n \to \infty$ a.s., by virtue of (1.4). Conversely, (1.1) implies (2.4) with $\alpha = 0$ by (1.4). Thus the properties in Theorem 2.1 (and Theorem 2.2 below) can only hold for random walks which drift (strongly) to ∞. This is implicit in (2.5) also, because we show in Lemma 3.1 below that (2.5) is equivalent to the validity of one of the conditions (2.9) or (2.10)

below:

$$0 < EX \leqq E|X| < \infty \quad \text{and} \quad E(X^-)^{1+\alpha} < \infty\,, \tag{2.9}$$

$$\int_{[0,\infty)} \left\{ \frac{y}{A_+(y)} \right\}^{1+\alpha} |dF(-y)| < \infty = EX^+\,. \tag{2.10}$$

[We take the integral in (2.10) as ∞ if $A_+(y) \equiv 0$, that is when $F(0) = 1$.] Since $A_+(y) \leqq y$, the integral in (2.10) is larger than J_-, so (2.10) implies $J_- < \infty = EX^+$. Referring to (1.18), we thus see from (2.9)–(2.10) that (2.5) implies $S_n \to \infty$ a.s. For $\alpha = 0$, (2.9)–(2.10) show that (2.5) is equivalent to the right hand side of (1.18). Thus the integral in (2.5) is one kind of generalisation of Erickson's integral J_-.

(iii) As mentioned in Sect. 1, most renewal theorems for random walks which drift to ∞ have so far been proved under the restriction

$$0 < EX \leqq E|X| < \infty\,. \tag{2.11}$$

In this case, Heyde (1964, Theorem A), (1966, Theorem 1), and (1967) showed that the three conditions (2.4), (2.6), and $E(X^-)^{1+\alpha} < \infty$ are equivalent. Heyde required $\alpha \geqq 1$ to be integral; Gut (1974) extended this to general $\alpha \geqq 1$. These results are contained in Theorem 2.1 (and extended to values of $\alpha \geqq 0$) by (2.9). Heyde's proof is an application of a theorem of Katz (1963) concerning convergence rates in the law of large numbers, while Gut (1974) relies on martingale inequalities for stopped sums. Our proof of the corresponding part of Theorem 2.1 is quite different, relying essentially on techniques developed in Kesten and Maller (1995) to deal with questions relating to the dominance of S_n over its large increments. Having observed this, however, we can reverse the argument and obtain a very general formulation of a convergence rate result related to theorems of Katz (1963), Baum and Katz (1965), and Feller (1946) (see Sect. 4.4 below). An extension of Theorem 2.1, still under (2.11) though, which allows regularly varying functions instead of n^α in (2.4), is in Alsmeyer (1992). Bertoin and Doney (1994) give detailed estimates for some of the random variables we consider, under assumptions which include S_n drifting to ∞ and X_1^- having an exponential moment (when translated into our situation).

Janson (1986, Theorem 1) gave economical proofs of the equivalence of (2.4)–(2.8) in the case $0 < EX \leqq E|X| < \infty$, and included a number of further equivalences (to a total of ten) relating to first passage times and moments of the stopped sum, which had been considered by previous writers in renewal theory. We have listed these further conditions in Sect. 4.2 below, where we point out that some of Janson's conditions [namely, his (iii), (iv) and (vi)] remain equivalent to (2.5) in our more general setup, but his conditions (v), (vii), (viii) and (ix) do not. So one side effect of our general formulation is to separate out this difference.

Moments of ladder times and ladder positions of a random walk which is in case (1.3) are investigated and play a prominent role in Klass and Zhang (1994); see also Chow (1986).

Our next theorem assumes the convergences in Theorem 2.1 and investigates the rates of growth of the various quantities as $x \to \infty$. Some of these were known, at least for some values of the parameter α, in the case $0 < EX \leqq E|X| < \infty$, in which case they are classic "renewal theorems". But in the general case, the clue to the behaviour of a series like $\sum_{n\geqq 1} P\{S_n \leqq x\}$ as $x \to \infty$ comes from Erickson (1973, Lemma 1, p. 377) (see also Chow and Teicher, 1988, p. 153), who showed that *when the X_i are non-negative a.s.*, it is always true that

$$\sum_{n\geqq 1} P\{S_n \leqq x\} \asymp \frac{x}{A(x)} \quad (\text{as } x \to \infty)\,. \tag{2.12}$$

(The relation "$f(x) \asymp g(x)$" is an abbreviation for

$$0 < \liminf_{x\to\infty} f(x)/g(x) \leqq \limsup_{x\to\infty} f(x)/g(x) < \infty\,,$$

when $f(x)$ and $g(x)$ are two positive functions defined on $[0,\infty)$. Also, $f(x) \sim g(x)$ will mean that $f(x)/g(x) \to 1$ as $x \to \infty$.) The next theorem generalises Erickson's result to random variables of arbitrary sign, and adds equivalences for the behaviour as $x \to \infty$ of moments of the first passage time above x, and the last exit time from below x. As explained in Remark (ii) the properties in the next theorem still imply $S_n \to \infty$ a.s.

Theorem 2.2 *For any value of $\alpha > 0$ for which the series in* (2.13) *converges*,

$$\sum_{n\geqq 1} n^{\alpha-1} P\{S_n \leqq x\} \asymp \left(\frac{x}{A(x)}\right)^{\alpha} \quad \text{as } x \to \infty\,, \tag{2.13}$$

$$E(T(x))^{\alpha} \asymp \sum_{n\geqq 1} n^{\alpha-1} P\left\{\max_{1\leqq j\leqq n} S_j \leqq x\right\} \asymp \left(\frac{x}{A(x)}\right)^{\alpha} \quad \text{as } x \to \infty \tag{2.14}$$

and

$$E(L(x))^{\alpha} \asymp \left(\frac{x}{A(x)}\right)^{\alpha} \quad \text{as } x \to \infty\,. \tag{2.15}$$

Assuming only that $S_n \to \infty$ a.s., we have

$$\sum_{n\geqq 1} n^{-1} P\{S_n \leqq x\} \sim \log\left(\frac{x}{A(x)}\right) \quad \text{as } x \to \infty \tag{2.16}$$

and

$$\sum_{n\geqq 1} n^{-1} P\left\{\max_{1\leqq j\leqq n} S_j \leqq x\right\} \sim \log\left(\frac{x}{A(x)}\right) \quad \text{as } x \to \infty\,. \tag{2.17}$$

Remarks. (iv) When $0 < EX \leqq E|X| < \infty$, we have $A(x) \to EX > 0$ as $x \to \infty$, and (2.13)–(2.17) assert that the various series and expectations are of order of magnitude x^{α} if $\alpha > 0$, or $\log x$, if $\alpha = 0$, as $x \to \infty$. In this case (2.13) and (2.16) follow from Theorem 1 of Heyde (1966) (who allows a more general weighting sequence than $n^{\alpha-1}$ in his series, see also his Theorems 4 and 5). Heyde deduces (2.13) and (2.16) with '$\sim$' replacing '$\asymp$', i.e., "asymptotic equivalence" rather than "order of magnitude" holds when $0 < EX \leqq E|X| < \infty$. This extra precision is probably not possible in general, even for the Erickson case when $X_i \geqq 0$ a.s.; see for example the discussion in Bingham et al. (1987, p. 365).

Greenwood, Omey and Teugels (1982, p. 408) sharpen Heyde's result corresponding to (2.16). Alsmeyer (1992) extends it in a different direction.
(v) Note that the powers of n differ by 1 in (2.4) and (2.6), but not in (2.13) and (2.14). The criterion for convergence of $\sum_{n \geqq 1} n^{\alpha-1} P\{\max_{1 \leqq j \leqq n} S_j \leqq x\}$ is less stringent than that for $\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x\}$, but nevertheless when the latter series converges, both series are of the same order of magnitude as $x \to \infty$ (in the sense of (2.13)–(2.14)). Despite Theorem 2.2, we do not know of a criterion for the convergence of $\sum_{n \geqq 1} n^{\alpha-1}\{\max_{1 \leqq j \leqq n} S_j \leqq x\}$ when $0 < \alpha < 1$, or, to put it another way, we do not have a necessary and sufficient condition for the finiteness of $E(T(x))^\gamma$, when $0 < \gamma < 1$. However, (2.5) (with $1+\alpha$ replaced by α) is certainly not the answer in this case. Suppose for example that S_n is in the domain of attraction of a stable law of index $\delta \in (0,1)$ and skewness parameter β, in the terminology of Bingham et al. (1987, p. 347). This means that, as $x \to \infty$, $1 - F(x) + F(-x)$ is regularly varying with index $-\delta$, and

$$\frac{F(-x)}{1 - F(x) + F(-x)} \to p \quad \text{and} \quad \frac{1 - F(x)}{1 - F(x) + F(-x)} \to q\,, \tag{2.18}$$

where $p - q = \beta$. Then by Bingham et al. (1987, pp. 380–381), we have $P\{S_n \leqq 0\} \to 1 - \rho$, where

$$\rho = \frac{1}{2} + \frac{1}{\pi\delta}\arctan\left(\beta \tan \frac{1}{2}\pi\delta\right) \in (0,1)$$

(a result due to Zolotarev, 1957), and $T(0)$ is finite a.s. and satisfies

$$P\{T(0) > n\} \sim \frac{L(n)}{n^\rho} \quad (n \to \infty)$$

for some slowly varying function $L(n) > 0$. Thus $E(T(0))^\gamma < \infty$ for $\gamma < \rho$, while $E(T(0))^\gamma = \infty$ for $\gamma > \rho$. Supposing $0 < p < q < 1$, for example, we have as $y \to \infty$ that

$$A(y) = \int_0^y (1 - F(u))\,du - \int_0^y F(-u)\,du \sim \left(\frac{1}{1-\delta}\right)\left(\frac{q-p}{p}\right) yF(-y),$$

so the integral in (2.5) (with $1+\alpha$ replaced by α) converges, regardless of the value of ρ, for $0 < \alpha < 1$. In other words, (2.5) cannot test for the finiteness or otherwise of $E(T(x))^\gamma$ in this case. Note, however, that S_n here does not satisfy (1.1). In another setting, Klass and Zhang (1994, Theorem 6.3) give some conditions for the convergence or divergence of $E(T(0))^\gamma$, $0 < \gamma < 1$, when $0 = EX \leqq E|X| < \infty$.
(vi) One can add to the equivalences in Theorems 2.1–2.2 another regarding a generalised first passage time which is of some interest in applications. Let $T^{(K)}(x)$ be the first time at which (the first of) K consecutive values of S_n exceeds $x \geqq 0$ (with $T^{(K)}(x) = \infty$ if this never occurs). Clearly $T(x) \leqq T^{(K)}(x) \leqq L(x) + 1$, but we might guess the behaviour of $T^{(K)}$ to be closer to that of $T(x)$ than to that of $L(x)$. This is so, and is proved in Sect. 4.3 below (see also the Acknowledgements).

3 Proofs

In addition to the notation introduced in Sect. 1, we need the following. Define, for $y \geqq 0$,

$$X_i^y = (X_i \vee -y) \wedge y \quad \text{and} \quad X^y = (X \vee -y) \wedge y \,. \tag{3.1}$$

This kind of truncation turns out to be the most convenient for our purposes. Also let

$$S_n^y = \sum_{i=1}^{n} X_i^y \tag{3.2}$$

be the truncated sum, and notice that $E(X^y) = A(y)$, where $A(y)$ is given by (2.3). Also define, for $y \geqq 0$,

$$U(y) = E(X^y)^2 = 2\int_0^y u(1 - F(u) + F(-u-))\,du \,, \tag{3.3}$$

and write $U(y) = U_+(y) + U_-(y)$, where

$$U_+(y) = 2\int_0^y u(1 - F(u))\,du \quad \text{and} \quad U_-(y) = 2\int_0^y uF(-u)\,du \,. \tag{3.4}$$

Proof of (1.21)–(1.22). It suffices to prove (1.21) since (1.22) can then be obtained by replacing X_i with $-X_i$. Suppose $\limsup_{n\to\infty} S_n = \infty$ a.s. First let $E|X| = \infty$ and $J_+ < \infty$. If $EX^- < \infty$, then we see from the definition (1.14) of J_+ that $EX^+ < \infty$, so $E|X| < \infty$, which is impossible. Thus $EX^- = \infty$. But then (1.16) shows that $S_n \to -\infty$ a.s., which is impossible. Consequently, $J_+ = \infty$, and the implication from left to right in (1.21) holds in case $E|X| = \infty$. Alternatively, $E|X| < \infty$. Then $EX \geqq 0$ since $S_n \to -\infty$ if $EX < 0$ by the strong law of large numbers, so the implication from left to right in (1.21) holds in this case too.

Conversely, let $J_+ = \infty$. Then $EX^+ = \infty$, so by (1.15) and (1.17), $\limsup_{n\to\infty} S_n = \infty$ a.s. Alternatively, if $0 \leqq EX \leqq E|X| < \infty$, then $S_n \to \infty$ a.s. if $EX > 0$ by the strong law of large numbers, while $\limsup_{n\to\infty} S_n = \infty$ a.s. if $EX = 0$, by the Chung–Fuchs (1951) condition for recurrence. In either case, $\limsup_{n\to\infty} S_n = \infty$ a.s. □

For the proof of Theorem 2.1 the following lemmas are needed.

Lemma 3.1 (2.5) *holds for a given* $\alpha \geqq 0$ *if and only if* (2.9) *or* (2.10) *holds. If this is the case, then* $S_n \to \infty$ *a.s.*

Proof of Lemma 3.1. [This is an extension of the dual, when X_i is replaced by $-X_i$, of Lemma 4.5 of Kesten and Maller (1995)]. Suppose $A(y) > 0$ for $y \geqq y_0$ and the integral in (2.5) converges. First let $E|X| < \infty$ and $F(-y) > 0$ for all y. Then $EX = \lim_{y\to\infty} A(y) \geqq 0$. If $EX = 0$ we have

$$\begin{aligned} A(y) &= \int_0^y (1 - F(u) - F(-u))\,du = -\int_y^\infty (1 - F(u) - F(-u))\,du \\ &\leqq \int_y^\infty F(-u)\,du \leqq \int_{(y,\infty)} u|\,dF(-u)| =: m(y), \text{ say} \,. \end{aligned}$$

Since $m(y) \downarrow 0$, we have $y/m(y) \uparrow \infty$ as $y \to \infty$, and so (2.5) implies that

$$\infty > \int_{(y_0,\infty)} \left\{ \frac{y}{m(y)} \right\}^{1+\alpha} |dF(-y)| \geqq \left\{ \frac{y_0}{m(y_0)} \right\}^{\alpha} \int_{(y_0,\infty)} \frac{|dm(y)|}{m(y)} .$$

Just as in Lemma 4.5 of Kesten and Maller (1995), this is impossible by the Abel–Dini Theorem. Thus $EX > 0$ and $A(y) \to EX > 0$ as $y \to \infty$, so

$$E(X^-)^{1+\alpha} = \int_{[0,\infty)} y^{1+\alpha} |dF(-y)|$$

$$= O\left(1 + \int_{(y_0,\infty)} \left\{ \frac{y}{A(y)} \right\}^{1+\alpha} |dF(-y)| \right) < \infty .$$

Thus (2.9) holds. Next let $E|X| < \infty$ and $F(-y_1) = 0$ for some $y_1 \geqq 0$. Then fix a $y_2 \geqq y_0 \vee y_1$. For $y \geqq y_2$, $A(y) \geqq A(y_2) > 0$, hence $0 < EX = \lim_{y\to\infty} A(y) \in [A(y_2), \infty)$. (2.9) still holds.

Alternatively, $E|X| = \infty$. Since $A(y) > 0$ for y large, $EX^- \leqq EX^+$. Thus $EX^+ = \infty$, otherwise $E|X| < \infty$. Also $A(y) \leqq A_+(y)$, so by (2.5), the integral in (2.10) converges.

Conversely, if (2.9) holds then $A(y) \to EX \in (0,\infty)$ as $y \to \infty$, so (2.5) holds as a consequence of $E(X^-)^{1+\alpha} < \infty$. Alternatively, if (2.10) holds, then, since $y/A_+(y) \uparrow \infty$, we have

$$J_- = \int_{[0,\infty)} \frac{y|dF(-y)|}{A_+(y)} < \infty ,$$

and, since $EX^+ = \infty$, this further implies $A_-(y)/A_+(y) \to 0$ as $y \to \infty$. The latter follows by first observing that $y_2F(-y_2)/A_+(y_2) \to 0$ as $y_2 \to \infty$ and then by letting $y_2 \to \infty$ and then $y_1 \to \infty$ in

$$\frac{A_-(y_2) - A_-(y_1) + y_1F(-y_1-) - y_2F(-y_2)}{A_+(y_2)}$$

$$= \frac{\int_{[y_1,y_2]} y|dF(-y)|}{A_+(y_2)} \leqq \int_{[y_1,\infty)} \frac{y|dF(-y)|}{A_+(y)} .$$

Consequently $A(y) \sim A_+(y)$ as $y \to \infty$, and (2.5) holds again.

Finally, if (2.9) or (2.10) holds then $S_n \to \infty$ a.s. by Lemma 1.1. □

Our main concern is with random walks such that $S_n \to \infty$ a.s., but a number of subsidiary results to follow will be proved under the condition that S_n drifts *weakly* to ∞, in the sense that $S_n \xrightarrow{P} \infty$. This kind of divergence was studied in Kesten and Maller (1994), and found to depend strongly on the behaviour for large y of the function $A(y)$, which is a kind of truncated mean of X. In general $A(y)$ may oscillate in sign, but when $S_n \xrightarrow{P} \infty$, $A(y) > 0$ for y large enough (see Lemma 3.2 below), and this is certainly true when $S_n \to \infty$ a.s. In fact this stronger condition is equivalent to the existence of a finite positive mean for X (i.e., to $0 < EX \leqq E|X| < \infty$), in which case $A(y) \to EX > 0$ as $y \to \infty$, or to the integral criterion $J_- < \infty = EX^+$ as we saw in Lemma 1.1, and in the latter case $A(y) \sim A_+(y)$ as $y \to \infty$, as was shown in the proof of Lemma 3.1.

The next lemma collects some useful consequences of the divergence $S_n \xrightarrow{P} \infty$.

Lemma 3.2 *Suppose* $S_n \xrightarrow{P} \infty$. *Then* $A(y) > 0$ *for* y *large enough*, *and*

$$\lim_{y\to\infty} \frac{U_-(y)}{yA(y)} = 0 \,, \tag{3.5}$$

$$\limsup_{y\to\infty} \frac{U(y)}{yA(y)} \leqq 2 \,. \tag{3.6}$$

Moreover, *there exists a* y_1 *such that for* $y_1 \leqq y < u$ *and* $c < \frac{1}{3}$,

$$\frac{A(y)}{y} > \frac{cA(u)}{u} \,. \tag{3.7}$$

Proof of Lemma 3.2. Suppose $S_n \xrightarrow{P} \infty$. Then when $EX^2 = \infty$, the facts that $A(y) > 0$ for y large enough, and that (3.5) and (3.6) hold, are immediate consequences of Lemma 4.3 and Theorem 2.1 of Kesten and Maller (1994). (Note that (3.5) holds even if $F(-y_1) = 0$ for some $y_1 = 0$, since $U_-(y)$ is bounded then, while $yA(y) \to \infty$ as $y \to \infty$, as is shown in Lemma 4.3 of Kesten and Maller (1994).) When $EX^2 < \infty$, $S_n \xrightarrow{P} \infty$ forces $EX > 0$ (Theorem 2.1 of Kesten and Maller, 1994), and since $A(y) \to EX$ as $y \to \infty$, we have $A(y) > 0$ for y large enough, and (3.5) and (3.6) hold again.

(3.7) is the dual of (4.35) of Kesten and Maller (1995), under interchange of $+$ and $-$, but we prove it here for convenience. Suppose $y_0 > 0$ is so large that $A(y) > 0$ and $U(y) \leqq 3yA(y)$ for $y \geqq y_0$, as is possible by (3.6). Thus $y[1 - F(y)] \leqq U(y)/y \leqq 3A(y)$ for $y \geqq y_0$, and if $y_0 < y < u$ we have

$$\begin{aligned} uA(y) - cyA(u) &= uA(y) - cyA(y) - cy\int_y^u (1 - F(z) - F(-z))\,dz \\ &\geqq yA(y) + (u-y)A(y) - cyA(y) - cy(u-y)(1 - F(y)) \\ &= (1-c)yA(y) + (u-y)(A(y) - cy(1 - F(y))) \,. \end{aligned} \tag{3.8}$$

If $c < \frac{1}{3}$ this is positive, since $y(1 - F(y)) \leqq 3A(y)$. □

Now fix $\delta > 0$ and suppose $A(y) > 0$ for $y \geqq y_0 > 0$. We will need the sequence $B_n = B_n(\delta)$ defined by

$$B_n = \sup\left\{ y \geqq y_0 \colon \frac{A(y)}{y} \geqq \frac{\delta}{n} \right\}, \quad n \geqq 1 \,. \tag{3.9}$$

Then $y_0 \leqq B_n < \infty$, $B_n \uparrow \infty$ since $A(y)/y \to 0$ as $y \to \infty$, and, by the continuity of $A(y)$,

$$\frac{nA(B_n)}{B_n} = \delta \tag{3.10}$$

for large n, say for $n \geqq n_0(\delta)$.

Lemma 3.3 *Suppose* (2.5) *holds for some* $\alpha \geqq 0$ *and suppose* B_n *is defined by* (3.9) *with arbitrary* $\delta > 0$. *Then*

$$\sum_{n \geqq 1} n^\alpha F(-B_n) < \infty . \tag{3.11}$$

Conversely, *if* (3.11) *holds for some* $\alpha \geqq 0$ *and* B_n *is defined by* (3.9) *for some* $\delta > 0$, *and if also* $S_n \xrightarrow{P} \infty$, *then* (2.5) *holds.*

Proof of Lemma 3.3. Suppose $A(y) > 0$ for $y \geqq y_0$ and the integral in (2.5) converges for some $\alpha \geqq 0$. We have

$$\sum_{n \geqq 1} n^\alpha F(-B_n) = \sum_{n \geqq 1} n^\alpha \sum_{j \geqq n} \int_{[B_j, B_{j+1})} |dF(-y)| \asymp \sum_{j \geqq 1} j^{1+\alpha} \int_{[B_j, B_{j+1})} |dF(-y)| . \tag{3.12}$$

When $y \geqq B_j$, (3.9) implies that $A(y)/y \leqq \delta/j$. Thus (3.12) gives, for some $c > 0$,

$$\sum_{n \geqq 1} n^\alpha F(-B_n) \leqq c \sum_{j \geqq 1} \int_{[B_j, B_{j+1})} \left\{ \frac{y}{A(y)} \right\}^{1+\alpha} |dF(-y)|$$

and this is finite by (2.5).

Conversely, let (3.11) hold and suppose $S_n \xrightarrow{P} \infty$. If $F(-y_1) = 0$ for some $y_1 > 0$ then, by our convention, the integral in (2.5) will be finite if we can show that $A(y) > 0$ for $y \geqq y_0$ for some $y_0 > 0$. But this was already proven in Lemma 3.2. If, on the other hand, $F(-y) > 0$ for all $y > 0$, then by (3.7) we can choose y_1 so that $y_1 \leqq y < B_j$ implies

$$\frac{A(y)}{y} > \frac{A(B_j)}{4B_j} = \frac{\delta}{4j} . \tag{3.13}$$

Thus (3.12) gives, for some $c > 0$,

$$\begin{aligned} \sum_{n \geqq 1} n^\alpha F(-B_n) &\asymp \sum_{j \geqq 2} j^{1+\alpha} \int_{[B_{j-1}, B_j)} |dF(-y)| \\ &\geqq c \sum_{j \geqq 2} \int_{[B_{j-1}, B_j)} \left\{ \frac{y}{A(y)} \right\}^{1+\alpha} |dF(-y)| , \end{aligned}$$

and so (2.5) must hold. □

Proof of Theorem 2.1. This proof is partitioned into various stages.

(2.4) (for some $x \geqq 0$) ⇒ (2.5).

Suppose (2.4) holds for some $\alpha \geqq 0$ and some $x \geqq 0$. Then (2.4) holds for $x = 0$, so by Spitzer's criterion (1.4), $S_n \to \infty$ a.s. (In fact, we will only need $S_n \xrightarrow{P} \infty$ for this part of the proof). By Lemma 3.2, $S_n \xrightarrow{P} \infty$ implies $A(y) > 0$ for $y \geqq y_0$, say, so if $F(-y_1) = 0$ for some $y_1 > 0$ then (2.5) holds by convention. So we can assume that $F(-y) > 0$ for all $y > 0$.

Fix $\delta \leqq 1/12$ and let B_n be the sequence defined in (3.9). Recall that $B_n \uparrow \infty$. An argument originally due to Erdös (1949) gives

$$P\{S_n \leqq 0\} \geqq P\left\{\bigcup_{j=1}^{n}\left\{\sum_{1 \leqq i \leqq n, i \neq j} X_i \leqq B_{n-1},\ X_j \leqq -B_{n-1}\right\}\right\}$$

$$\geqq nP\{X \leqq -B_{n-1}\}\left(P\{S_{n-1} \leqq B_{n-1}\} - nP\{X \leqq -B_{n-1}\}\right) . \tag{3.14}$$

From (3.5) we have that $U_{-}(B_n) = o(B_n A(B_n))$ as $n \to \infty$. This together with the inequality $B_n^2 F(-B_n) \leqq U_{-}(B_n)$ and (3.10) shows that

$$nP\{X \leqq -B_{n-1}\} = nF(-B_{n-1}) = o\left(\frac{nA(B_{n-1})}{B_{n-1}}\right) = o(1) . \tag{3.15}$$

Next we estimate the probability $P\{S_{n-1} \leqq B_{n-1}\}$. Write $X_i^n = (X_i \vee -B_n) \wedge B_n$, and $S_n^n = \sum_{i=1}^n X_i^n$, and note that $E(X_i^n) = A(B_n)$ and $E(X_i^n)^2 = U(B_n)$. Thus

$$P\{S_n \leqq B_n\} \geqq P\left\{\sum_{i=1}^{n}(X_i \vee -B_n) \leqq B_n\right\}$$

$$\geqq P\left\{\sum_{i=1}^{n}(X_i \vee -B_n) \wedge B_n \leqq B_n, \max_{1 \leqq i \leqq n} X_i \leqq B_n\right\}$$

$$\geqq P\{S_n^n \leqq B_n\} - P\left\{\max_{1 \leqq i \leqq n} X_i > B_n\right\}$$

$$\geqq P\{S_n^n - nA(B_n) \leqq B_n - nA(B_n)\} - n[1 - F(B_n)] . \tag{3.16}$$

We defined B_n so that $nA(B_n) = \delta B_n$ eventually. Also $B_n^2[1 - F(B_n)] \leqq U(B_n)$. So we have from (3.16) and the fact that $\delta \leqq 1/12$, that for large n

$$P\{S_n \leqq B_n\} \geqq P\{S_n^n - nA(B_n) \leqq (1-\delta)B_n\} - nU(B_n)/B_n^2$$

$$= 1 - P\{S_n^n - E(S_n^n) > (1-\delta)B_n\} - nU(B_n)/B_n^2$$

$$\geqq 1 - \frac{nU(B_n)}{(1-\delta)^2 B_n^2} - \frac{nU(B_n)}{B_n^2} \geqq 1 - \frac{3nU(B_n)}{B_n^2} . \tag{3.17}$$

Now by (3.6), $U(B_n) \leqq 3B_n A(B_n)$ for n large enough. Consequently, recalling that $\delta \leqq 1/12$, we have

$$P\{S_n \leqq B_n\} \geqq 1 - \frac{9nA(B_n)}{B_n} = 1 - 9\delta \geqq \tfrac{1}{4} . \tag{3.18}$$

By (3.15) we can assume that n is so large that $nP\{X \leqq -B_{n-1}\} \leqq 1/8$. Substituting these bounds in (3.14) gives

$$P\{S_n \leqq 0\} \geqq \tfrac{1}{8} nF(-B_{n-1}) , \tag{3.19}$$

provided n is large enough. From the convergence in (2.4) we now deduce the convergence of $\sum_{n \geqq 1} n^{\alpha} F(-B_n)$, and by Lemma 3.3, this implies the convergence of the integral in (2.5).

(2.5) $\Longrightarrow$ (2.4) (for all $x \geqq 0$).

Now let (2.5) hold. By Lemma 3.1, this implies $S_n \to \infty$ a.s. Assume first that $F(-y) > 0$ for all $y > 0$. By (3.6) we can assume that

$$\frac{U(y)}{yA(y)} \leqq 3 \tag{3.20}$$

provided y is large enough, say $y \geqq y_2$. Without loss of generality we take $y_2 \geqq y_0 \vee y_1$, where y_0 is as in (2.5) and y_1 as in (3.7). Now keep x fixed at a value greater than y_2; it suffices to prove (2.4) for each such x.

Thus, in particular, $x > 0$. Define $(X^-)_n^{(1)} = \max_{1 \leqq i \leqq n}(X_i^-)$, and note that

$$\begin{aligned}
&\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ (X^-)_n^{(1)} \leqq x\} \\
&\quad \leqq \sum_{n \geqq 1} n^{\alpha-1} P\{X_i \geqq -x,\ 1 \leqq i \leqq n\} \\
&\quad = \sum_{n \geqq 1} n^{\alpha-1}[1-F(-x-)]^n \asymp \begin{cases} \frac{1}{(F(-x-))^\alpha} & \text{for } \alpha > 0\,, \\ -\log(F(-x-)) & \text{for } \alpha = 0\,, \end{cases}
\end{aligned} \tag{3.21}$$

and these are finite because $F(-x) > 0$. Next, Lemma 3.3 tells us that the series $\sum n^\alpha F(-B_n)$ converges, so we have

$$\begin{aligned}
\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ (X^-)_n^{(1)} > B_n\} &\leqq \sum_{n \geqq 1} n^{\alpha-1} P\{X_i \leqq -B_n \quad \text{for some } i \leqq n\} \\
&\leqq \sum_{n \geqq 1} n^\alpha F(-B_n) < \infty\,.
\end{aligned}$$

It remains to prove the convergence of

$$\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ x < (X^-)_n^{(1)} \leqq B_n\}\,. \tag{3.22}$$

Let X_{i_n} be the first X_i such that $X_{i_n} = (X^-)_n^{(1)}$. Then

$$\begin{aligned}
&\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ x < (X^-)_n^{(1)} \leqq B_n\} \\
&\quad = \sum_{n \geqq 1} n^{\alpha-1} \sum_{j=1}^{n} P\left\{\sum_{1 \leqq i \leqq n, i \neq j} X_i + X_j \leqq x,\ x < (-X_j) \leqq B_n,\ i_n = j\right\} \\
&\quad \leqq \sum_{n \geqq 1} n^\alpha \int_{(x,B_n]} P\left\{\sum_{i=1}^{n-1} X_i + (-y) \leqq x,\ \max_{1 \leqq i \leqq n-1} X_i^- \leqq y\right\} |dF(-y)| \\
&\quad \leqq \sum_{n \geqq 1} n^\alpha \int_{(x,B_n]} P\{S_{n-1} \leqq 2y,\ (X^-)_{n-1}^{(1)} \leqq y\} |dF(-y)|\,.
\end{aligned} \tag{3.23}$$

To find an upper bound for the probability in (3.23) we proceed as in the proof of Theorem 2.6 of Kesten and Maller (1995), except that here we have $S_n \stackrel{P}{\to} \infty$ rather than $S_n \stackrel{P}{\to} -\infty$. (Compare (3.23) with (4.38) of Kesten and

Maller, 1995). We again use the sequence $B_n = B_n(\delta)$ defined by (3.9), but this time we take $\delta = 12$. Keep $y_1 < y \leqq B_n$. Then by (3.7)

$$\frac{nA(y)}{y} \geqq \frac{nA(B_n)}{3B_n} = \frac{\delta}{3} = 4 ,$$

so $y \leqq nA(y)/4$. Define X_i^y and S_n^y by (3.1)–(3.2), and recall that $E(X_i^y) = A(y)$ and $E(X_i^y)^2 = U(y)$. The last probability in (3.23) satisfies

$$\begin{aligned} &P\{S_n \leqq 2y,\ (X^-)_n^{(1)} \leqq y\} \\ &\quad \leqq P\left\{\sum_{i=1}^n (X_i \vee -y) \leqq 2y\right\} \leqq P\left\{\sum_{i=1}^n (X_i \vee -y) \wedge y \quad \leqq 2y\right\} \\ &\quad = P\{S_n^y - nA(y) \leqq 2y - nA(y)\} \leqq P\{S_n^y - nA(y) \leqq -nA(y)/2\} \end{aligned} \tag{3.24}$$

because $y \leqq nA(y)/4$. Now we apply Bernstein's inequality (e.g., Chow and Teicher, 1988, Exercise 4.3.14). Let $t = nA(y)/2$. Then since

$$|X_i^y - EX_i^y| \leqq 2y \quad \text{a.s.} ,$$

we have

$$\begin{aligned} P\{S_n^y - nA(y) \leqq -nA(y)/2\} &= P\left\{\sum_{i=1}^n (EX_i^y - X_i^y) > t\right\} \\ &\leqq \exp\left\{\frac{-t^2}{2(s_n^2 + 2yt)}\right\} , \end{aligned} \tag{3.25}$$

where

$$s_n^2 = n \ \mathrm{Var}(X_i^y) \leqq nE(X_i^y)^2 = nU(y) .$$

By (3.20)

$$s_n^2 \leqq nU(y) \leqq 3nyA(y) ,$$

so that

$$s_n^2 + 2yt \leqq 3nyA(y) + (2y)nA(y)/2 = 4nyA(y) .$$

Inequality (3.25) then gives

$$P\{S_n^y - nA(y) \leqq -nA(y)/2\} \leqq \exp\left\{-\frac{n^2A^2(y)}{32nyA(y)}\right\} = \rho^n(y), \quad \text{say} , \tag{3.26}$$

where

$$\rho(y) = \exp\left(-\frac{A(y)}{32y}\right) < 1 . \tag{3.27}$$

Returning to (3.23) we see via (3.24) and (3.26) that, for $\alpha \geqq 0$,

$$\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ x < (X^-)_n^{(1)} \leqq B_n\}$$
$$\leqq \sum_{n \geqq 1} n^{\alpha} \int_{(x,\infty)} \rho^{n-1}(y)\ |dF(-y)|$$
$$\asymp \int_{(x,\infty)} \left\{\frac{1}{1-\rho(y)}\right\}^{1+\alpha} |dF(-y)| \asymp \int_{(x,\infty)} \left\{\frac{y}{A(y)}\right\}^{1+\alpha} |dF(-y)|\,, \tag{3.28}$$

since $1-\rho(y) \asymp A(y)/y$ as $y \to \infty$ (from (3.27)). This proves that (2.4) holds for any $x \geqq 0$, when $F(-y) > 0$ for all $y > 0$.

Now suppose (2.5) holds and there is a $y_3 > 0$ with $F(-y_3) = 0$. Then, by convention, $A(y) > 0$ for all $y \geqq y_0$, and $(X^-)_n^{(1)} \leqq y_3$ a.s. We still have $S_n \xrightarrow{P} \infty$ by Lemma 3.1. Take $x > y_0 \vee y_3$ and, without loss of generality, also such that $U(x) \leqq 3xA(x)$ (see (3.6)). For $n_0 \geqq 1$, we can write

$$\sum_{n \geqq n_0} n^{\alpha-1} P\{S_n \leqq x\} = \sum_{n \geqq n_0} n^{\alpha-1} P\left\{\sum_{i=1}^{n} (X_i \vee -x) \leqq x\right\}$$
$$\leqq \sum_{n \geqq n_0} n^{\alpha-1} P\left\{\sum_{i=1}^{n} (X_i \vee -x) \wedge x \leqq x\right\}$$
$$= \sum_{n \geqq n_0} n^{\alpha-1} P\{S_n^x - nA(x) \leqq x - nA(x)\}\,. \tag{3.29}$$

Since $A(x) > 0$, we can choose $n_0 = n_0(x)$ so large that $x \leqq nA(x)/2$ for $n \geqq n_0$. The same argument as used for (3.26) then gives

$$P\{S_n^x - nA(x) \leqq x - nA(x)\} \leqq P\{S_n^x - nA(x) \leqq -nA(x)/2\} \leqq \rho^n(x)\,, \tag{3.30}$$

with $\rho(\cdot)$ again defined by (3.27). Thus

$$\sum_{n \geqq n_0} n^{\alpha-1} P\{S_n \leqq x\} \leqq \sum_{n \geqq 1} n^{\alpha-1} \rho^n(x)$$
$$\asymp \begin{cases} \frac{1}{(1-\rho(x))^{\alpha}} & \text{for } \alpha > 0 \\ -\log(1-\rho(x)) & \text{for } \alpha = 0\,. \end{cases} \tag{3.31}$$

Since $\rho(x) < 1$ these are finite, and so we see that (2.4) holds for any $x \geqq 0$ in this case also.

(2.4) $\Longleftrightarrow$ (2.6).

Next we prove the equivalence of (2.4) and (2.6). We showed above that the series in (2.4) converges for all x if it converges for some x, and the same is true of the series in (2.6) by Gut (1988, p. 81). So it will suffice to consider the case $x = 0$ in both (2.4) and (2.6). For this we will use the following identity of Spitzer (1956, Corollary 2)

$$1 + \sum_{n \geqq 1} s^n P\left\{\max_{1 \leqq j \leqq n} S_j \leqq 0\right\} = \exp\left[\sum_{n \geqq 1} \frac{s^n}{n} P\{S_n \leqq 0\}\right] \tag{3.32}$$

for any $s \in [0,1)$. This will be combined with a general lemma on a pair of related positive random variables.

Lemma 3.4 *Let $\Delta(z) = \sum_{n=1}^{\infty} d_n z^n$ be a power series with radius of convergence $r > 0, d_1 > 0$ (and no zeroth power present). Let $X^{(i)}, i = 1,2$ be positive random variables with*

$$\Phi^{(i)}(\lambda) = Ee^{-\lambda X^{(i)}}, \quad i = 1,2, \ \lambda \geqq 0 .$$

Assume that

$$1 - \Phi^{(1)}(\lambda) = \Delta(1 - \Phi^{(2)}(\lambda)) \quad \text{for } 0 \leqq \lambda < \delta, \text{ for some } \delta > 0 . \tag{3.33}$$

For any constants $\mu_j^{(i)}, 1 \leqq j \leqq k$, define $R_k^{(i)}$ as

$$R_k^{(i)}(\lambda; \mu^{(i)}) = 1 - \Phi^{(i)}(\lambda) + \sum_{j=1}^{k} \mu_j^{(i)} \frac{(-\lambda)^j}{j!}, \quad k \geqq 1 ,$$

and $R_0^{(i)} = 1 - \Phi^{(i)}(\lambda)$. Then for each fixed integer $k \geqq 0$,

$$\text{there exist constants } \mu_j^{(1)}, 1 \leqq j \leqq k, \quad \text{such that } R_k^{(1)}(\lambda; \mu^{(1)}) = o(\lambda^k) \text{ as } \lambda \downarrow 0 \tag{3.34}$$

if and only if

$$\text{there exist constants } \mu_j^{(2)}, 1 \leqq j \leqq k, \quad \text{such that } R_k^{(2)}(\lambda; \mu^{(2)}) = o(\lambda^k) \text{ as } \lambda \downarrow 0. \tag{3.35}$$

Moreover, if (3.34) *and* (3.35) *hold for certain $\mu_j^{(i)}$, then for these $\mu_j^{(i)}$*

$$R_k^{(1)}(\lambda; \mu^{(1)}) = d_1(1 + o(1)) R_k^{(2)}(\lambda; \mu^{(2)}) + O(\lambda^{k+1}) . \tag{3.36}$$

Proof of Lemma 3.4. Substitution of

$$-\sum_{j=1}^{k} \mu_j^{(2)} \frac{(-\lambda)^j}{j!} + R_k^{(2)}(\lambda)$$

for $1 - \Phi^{(2)}$ into the right hand side of (3.33) makes it obvious that (3.35) implies (3.34).

This same substitution can be used to give a proof by induction on k that (3.34) implies (3.35). Indeed, both (3.34) and (3.35) hold for $k = 0$. Assume now that their equivalence has been proven for a given k and that now (3.34) holds also with k replaced by $k + 1$. Then we know that for a suitable choice of $\mu^{(2)}$, $R_k^{(2)}(\lambda; \mu^{(2)}) = o(\lambda^k)$. The above substitution then shows that the right hand side of (3.33) equals

$$\text{a polynomial in } \lambda \text{ of degree } (k+1) + d_1(1 + o(1)) R_k^{(2)}(\lambda; \mu^{(2)}) + O(\lambda^{k+2}) . \tag{3.37}$$

By assumption the left hand side of (3.33) equals

$$-\sum_{j=1}^{k+1} \mu_j^{(1)} \frac{(-\lambda)^j}{j!} + R_{k+1}^{(1)}(\lambda; \mu^{(1)}) \tag{3.38}$$

with $R^{(1)}_{k+1} = o(\lambda^{k+1})$. From the equality of (3.37) and (3.38) we obtain that for a suitable choice of the constant $\mu^{(2)}_{k+1}$

$$R^{(2)}_k(\lambda;\mu^{(2)}) = -\frac{\mu^{(2)}_{k+1}}{(k+1)!}(-\lambda)^{k+1} + o(\lambda^{k+1}) .$$

Therefore (3.35) holds with k replaced by $k+1$ and

$$R^{(2)}_{k+1}(\lambda;\mu^{(2)}) = R^{(2)}_k(\lambda;\mu^{(2)}) + \frac{\mu^{(2)}_{k+1}}{(k+1)!}(-\lambda)^{k+1} .$$

Now the equality of (3.38) and the analogue of (3.37) with k replaced by $k+1$ gives (3.36) with k replaced by $k+1$. □

An immediate corollary is that under the conditions of the lemma, $X^{(1)}$ has a finite moment of order $\alpha \geqq 0$ if and only if the same holds for $X^{(2)}$. This follows from (3.36) and Theorems 8.1.6 and 8.1.8 of Bingham et al. (1987). Indeed, if $\alpha = k \in \{1,2,\ldots\}$, then a positive random variable X has a moment of order α if and only if there exist constants $\mu_1,\ldots,\mu_{k-1}$ such that

$$R_{k-1}(\lambda;\mu) := 1 - Ee^{-\lambda X} + \sum_{j=1}^{k-1} \mu_j \frac{(-\lambda)^j}{j!} = O(\lambda^k) . \tag{3.39}$$

If $\alpha \geqq 0$ is not an integer, then X has an αth moment if and only if for $k = \lfloor \alpha \rfloor$ and for some $\mu_1,\ldots,\mu_k$

$$\int_0^1 |R_k(\lambda;\mu)|\,\lambda^{-1-\alpha}\,d\lambda < \infty . \tag{3.40}$$

Note that if (3.40) holds for some μ_j, then it is necessarily the case that $\mu_j = EX^j, 1 \leqq j \leqq k = \lfloor\alpha\rfloor$, and if (3.39) holds then $\mu_j = EX^j$ for $1 \leqq j \leqq k-1$.

Now (3.32) shows that the two sums

$$\sum_{n=1}^{\infty} P\left\{\max_{1\leqq j\leqq n} S_j \leqq 0\right\} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leqq 0\}$$

converge or diverge together. If they both diverge, then neither (2.4) nor (2.6) hold for any $\alpha \geqq 0$. If they converge, let $X^{(i)}$, $i = 1,2$, be two random variables with distributions

$$P\{X^{(1)} = n\} = C^{(1)} P\left\{\max_{1\leqq j\leqq n} S_j \leqq 0\right\}, \quad n \geqq 1,$$

and

$$P\{X^{(2)} = n\} = \frac{C^{(2)}}{n} P\{S_n \leqq 0\}, \quad n \geqq 1 ,$$

where the normalizing constants $C^{(i)}$ are chosen so that these are indeed probability distributions. (3.32) can be rewritten as

$$\begin{aligned}
&C^{(1)} \sum_{n \geqq 1} (1 - s^n) P\left\{ \max_{1 \leqq j \leqq n} S_j \leqq 0 \right\} \\
&\quad = 1 - E s^{X^{(1)}} \\
&\quad = C^{(1)} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leqq 0\} \right] \left[1 - \exp\left[\frac{C^{(2)}}{C^{(2)}} \sum_{n=1}^{\infty} \frac{s^n - 1}{n} P\{S_n \leqq 0\} \right] \right] \\
&\quad = C^{(1)} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leqq 0\} \right] \left[1 - \exp\left[-\frac{1}{C^{(2)}} (1 - E s^{X^{(2)}}) \right] \right] .
\end{aligned}$$

This is of the form (3.33) (with s written for $e^{-\lambda}$) and we therefore conclude that $X^{(1)}$ has a finite moment of order $\alpha \geqq 0$ if and only if this holds for $X^{(2)}$. For our choice of the distributions of $X^{(i)}$, this is just the equivalence of (2.4) and (2.6) with $x = 0$.

(2.6) $\Longleftrightarrow$ (2.7).

(2.6) and (2.7) are equivalent because $\{\max_{1 \leqq j \leqq n} S_j \leqq x\} = \{T(x) > n\}$, so it remains only to prove the equivalence of (2.4)–(2.7) with (2.8).

(2.8) (for some $x \geqq 0$) $\Longrightarrow$ (2.4) (for all $x \geqq 0$)

Since the case $\alpha = 0$ is contained in (1.4) we keep $\alpha > 0$ for the rest of the proof. Now suppose (2.8) holds for some $x \geqq 0$, so $E(L(x))^\alpha < \infty$ for some $x \geqq 0$. Since $L(0) \leqq L(x)$ a.s., we have $E(L(0))^\alpha < \infty$. Also, since $\{S_n \leqq 0\} \subseteq \{L(0) \geqq n\}$, we have

$$\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq 0\} \leqq \sum_{n \geqq 1} n^{\alpha-1} P\{L(0) \geqq n\} < \infty . \tag{3.41}$$

Hence (2.4) holds for $x = 0$ and hence for all $x \geqq 0$.

(2.4) (for some $x \geqq 0$) $\Longrightarrow$ (2.8) (for all $x \geqq 0$).

Conversely, let (2.4) hold for some $x \geqq 0$. Then (2.4) holds with $x = 0$; denote this by $(2.4)_0$. Notice that, for $x \geqq 0$,

$$T(x) - 1 \leqq L(x) \leqq T(x) + \tilde{L}(x) , \tag{3.42}$$

where

$$\tilde{L}(x) = \max\{n \geqq T(x) \colon S_n \leqq S_{T(x)}\} - T(x) . \tag{3.43}$$

The random variable $\tilde{L}(x)$ is independent of $T(x)$ and has the same distribution as $L(0)$. Thus $E(\tilde{L}(x))^\alpha < \infty$ if and only if $E(L(0))^\alpha < \infty$. Now $(2.4)_0$ implies $E(T(x))^\alpha < \infty$ for all $x \geqq 0$ (in fact, $E(T(x))^{1+\alpha} < \infty$, by (2.7)), so from (3.42), $E(L(x))^\alpha < \infty$ whenever $E(L(0))^\alpha < \infty$. So it suffices to show that $(2.4)_0$ implies $E(L(0))^\alpha < \infty$.

To do this, introduce the random variables

$$v_0 = \min\{n \geqq 1 \colon S_n \leqq 0\} \tag{3.44}$$

(with $\nu_0 = \infty$ if $S_n > 0$ for all n), and

$$N_+ = \min\{n > \nu_0 \colon S_n > 0\} - \nu_0 \quad (\text{on } \{\nu_0 < \infty\}) \tag{3.45}$$

(with $N_+ = \infty$ if $S_n \leqq 0$ for all $n > \nu_0$). When $(2.4)_0$ holds we have $S_n \to \infty$ a.s., so

$$\pi := P\left\{\min_{n \geqq 1} S_n > 0\right\} = P\{\nu_0 = \infty\} > 0 \tag{3.46}$$

(because $P\{\nu_0 < \infty\} = 1$ if and only if $\liminf_{n\to\infty} S_n = -\infty$ a.s.; see the discussion in Sect. 1). The idea of the proof now is as follows. In order for the walk to arrive at $L(0)$, the first time after which it remains forever positive, it may make a number of excursions which consist of a piece in $(-\infty, 0]$ and a piece in $(0, \infty)$. The number of such excursions is a geometric random variable with success probability π, and the times spent in these excursions are independently distributed as $\nu_0 + N_+$. We can model this process as successive independent tosses of a coin with success probability π. The first success in the cointossing game corresponds to the end of the excursions which enter $(-\infty, 0]$; after that the random walk stays in $(0, \infty)$, so that $L(0)$ has been reached.

This idea is formalised as follows. Let $\nu_0^{(1)}$ and $N_+^{(1)}$ be the ν_0 and N_+, respectively, which were just introduced, and let $K_1 = \nu_0^{(1)} + N_+^{(1)}$. Now for $i > 1$ define on the event $\{K_i < \infty\}$

$$\nu_0^{(i+1)} = \inf\{n > K_i \colon S_n \leqq S_{K_i}\} - K_i \,. \tag{3.47}$$

If $\nu_0^{(i+1)} < \infty$ set further

$$N_+^{(i+1)} = \inf\{n > K_i + \nu_0^{(i+1)} \colon S_n > S_{K_i}\} - K_i - \nu_0^{(i+1)},$$
$$K_{i+1} = K_i + \nu_0^{(i+1)} + N_+^{(i+1)} \,.$$

When $K_i = \infty$ we also take $\nu_0^{(i+1)} = N_+^{(i+1)} = K_{i+1} = \infty$, and when $K_i + \nu_0^{(i+1)} = \infty$, we take $N_+^{(i+1)} = \infty$ and $K_{i+1} = \infty$. Note that $K_{i+1} < \infty$ a.e. on $\{K_i + \nu_0^{(i+1)} < \infty\}$ because $S_n \to \infty$ a.s. Finally we define

$$\tau = \min\{i \geqq 1 \colon K_i = \infty\} \,. \tag{3.48}$$

Our first observation is that on $\{K_i < \infty\}$ we have

$$S_n > S_{K_i} \quad \text{for } K_i < n < K_i + \nu_0^{(i+1)} \,. \tag{3.49}$$

If in addition $\{K_{i+1} < \infty\}$, then also

$$S_{K_{i+1}} > S_{K_i} \,.$$

In particular

$$S_{K_i} > 0 \quad \text{for } 1 \leqq i < \tau$$

and

$$S_n > S_{K_{\tau-1}} > 0 \quad \text{for all } n \geqq K_{\tau-1} \,.$$

Thus

$$L(0) \leqq K_{\tau-1} = \sum_{i=1}^{\tau-1} (v_0^{(i)} + N_+^{(i)}) \,. \tag{3.50}$$

Next we note that the condition $\{\tau = r\}$ only tells us that $v_0^{(i)} < \infty$ for $i < r$ and $v_0^{(r)} = \infty$. In particular, given this condition, the random variables $v_0^i, N_+^{(j)}, 1 \leqq i, j \leqq r-1$ are still independent with the distribution of v_0, conditioned on $\{v_0 < \infty\}$ and of N_+, respectively. Moreover these distributions are independent of r. Also, given that $K_i < \infty$, the conditional probability of $\{K_{i+1} = \infty\} = \{v_0^{(i+1)} = \infty\}$ equals π. Thus τ has a geometric distribution with success probability π.

From these observations we obtain (as a crude bound)

$$\begin{aligned} E(L(0))^\alpha &\leqq E\left(\sum_{i=1}^{\tau-1} (v_0^{(i)} + N_+^{(i)})\right)^\alpha \\ &\leqq 2^\alpha E\left\{(\tau-1)^\alpha \max_{1 \leqq i \leqq \tau-1} \left((v_0^{(i)})^\alpha \vee (N_+^{(i)})^\alpha\right)\right\} \\ &\leqq 2^\alpha E\left\{(\tau-1)^\alpha \sum_{i=1}^{\tau-1} \left((v_0^{(i)})^\alpha + (N_+^{(i)})^\alpha\right)\right\} \\ &\leqq 2^\alpha E\left((\tau-1)^{\alpha+1}\right) \left(E(v_0^\alpha |\ v_0 < \infty) + E(N_+^\alpha |\ v_0 < \infty)\right) \\ &= c(\alpha, p)\left(E(v_0^\alpha;\ v_0 < \infty) + E(N_+^\alpha;\ v_0 < \infty)\right) \end{aligned} \tag{3.51}$$

for some $c(\alpha, p) < \infty$ when $\pi < 1$ [or see Gut (1988, Theorem I.5.2) for a better bound]. If $\pi = 1$, the left hand side of (3.51) equals 0 so we take $\pi < 1$ from now on. (3.51) shows that

$$E(v_0^\alpha;\ v_0 < \infty) + E(N_+^\alpha;\ v_0 < \infty) < \infty \tag{3.52}$$

is sufficient for $E(L(0))^\alpha < \infty$.

It is quite easy to demonstrate that $E(N_+^\alpha;\ v_0 < \infty) < \infty$ under $(2.4)_0$. For $x \geqq 0$ define

$$\Lambda(x) = \sum_{k \geqq 1} \leqq\{S_k \leqq x\} \tag{3.53}$$

Then

$$N_+ \leqq 1 + \#(k \geqq 1\colon S_k \leqq 0) = 1 + \Lambda(0)$$

(where # denotes the number of elements of a set), so $E(N_+)^\alpha \leqq E(1 + \Lambda(0))^\alpha$. We now show that $(2.4)_0$ implies $E(\Lambda(0))^\alpha < \infty$. To see this, define $\Lambda_n = \sum_{k=1}^n \leqq\{S_k \leqq 0\}$ and use formula (3.3) in Sparre–Andersen (1954) applied to $n - \Lambda_n = \#\{1 \leqq k \leqq n\colon S_k > 0\}$ to write, for $0 \leqq k \leqq n$,

$$P\{n - \Lambda_n = k\} = P\{k - \Lambda_k = k\}\ P\{(n-k) - \Lambda_{n-k} = 0\} \,,$$

or equivalently, for $0 \leqq k \leqq n$,

$$P\{\Lambda_n = k\} = P\{\Lambda_{n-k} = 0\}\ P\{\Lambda_k = k\} \,.$$

As $n \to \infty$, $\Lambda_n \uparrow \Lambda(0)$ a.s., so we get

$$P\{\Lambda(0) = k\} = P\left\{\min_{n \geqq 1} S_n > 0\right\} P\{\Lambda_k = k\} = \pi P\{T(0) > k\}\,, \qquad (3.54)$$

because $P\{\Lambda_k = k\} = P\{T(0) > k\}$, and we defined $P\{\min_{n \geqq 1} S_n > 0\} = \pi$ (see (3.46)). Now $S_n \to \infty$ a.s., or equivalently $P\{\Lambda(0) < \infty\} = 1$, implies $\pi > 0$. Thus $(2.4)_0$, which is equivalent to $E(T(0))^{1+\alpha} < \infty$, is also equivalent to $E(\Lambda(0))^{\alpha} < \infty$.

It remains only to prove that, under $(2.4)_0$,

$$E(v_0^{\alpha};\ v_0 < \infty) < \infty\,. \qquad (3.55)$$

From Feller (1971, Theorem XII.7.1) (or after some manipulations from Spitzer (1976, P17.5(d)) it follows that

$$Es^{v_0} = 1 - \exp\left[-\sum_{n \geqq 1} \frac{s^n}{n} P\{S_n \leqq 0\}\right].$$

We now obtain (3.55) from another application of Lemma 3.4, or rather its corollary, with

$$P\{X^{(1)} = n\} = P\{v_0 = n | v_0 < \infty\}$$

and

$$P\{X^{(2)} = n\} = \frac{C^{(2)}}{n} P\{S_n \leqq 0\}$$

(again with $C^{(2)}$ chosen so that the above is a probability distribution). In fact under the condition that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leqq 0\} < \infty\,,$$

or equivalently $S_n \to \infty$ a.s. (see (1.4)), (3.55) is equivalent to $(2.4)_0$. We shall need this last step in Sect. 4.2. □.

Proof of Theorem 2.2. Suppose the series in (2.13) converges for some $\alpha > 0$. Then (2.4)–(2.8) hold for that value of α. We first show that the series in (2.13) and (2.16) are no larger than stated in these formulae. Let B_n again be defined by (3.9) for $\delta = 12$. Clearly

$$\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ (X^-)_n^{(1)} > B_n\} \leqq \sum_{n \geqq 1} n^{\alpha} F(-B_n)\,,$$

and by Lemma 3.3 the last series has value c_1, say, where $c_1 < \infty$. By (2.5), we have $A(y) > 0$ for $y \geqq y_0 > 0$, and, in any case, $A(y)/y \to 0$ as $y \to \infty$. In what follows we keep $x > y_0$, so $A(x) > 0$. Thus $c_1 = O(x/A(x))^{\alpha}$ or $c_1 = o(\log(x/A(x)))$ $(x \to \infty)$, as required for (2.13) or (2.16).

So we consider

$$\sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ (X^-)_n^{(1)} \leqq B_n\}$$
$$= \sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ (X^-)_n^{(1)} \leqq x\}$$
$$+ \sum_{n \geqq 1} n^{\alpha-1} P\{S_n \leqq x,\ x < (X^-)_n^{(1)} \leqq B_n\}\ . \quad (3.56)$$

We showed in the proof of Theorem 2.1 (see (3.28)) that the second series on the right hand side of (3.56) does not exceed

$$c_2 \int_{(x,\infty)} \left\{ \frac{y}{A(y)} \right\}^{1+\alpha} |dF(-y)|\ , \quad (3.57)$$

and (3.57) is certainly $O(x/A(x))^\alpha$ or $o(\log(x/A(x))$, since the integral converges when $F(-y) > 0$ for all y, while (3.57) vanishes for large x if $F(-y) = 0$ for some y.

It remains to deal with the first series on the right hand side of (3.56), which we write as

$$\left(\sum_{n \leqq d_\alpha x/A(x)} + \sum_{n > d_\alpha x/A(x)} \right) n^{\alpha-1} P\{S_n \leqq x,\ (X^-)_n^{(1)} \leqq x\}\ . \quad (3.58)$$

Here d_α is a constant which we take as 2 when $\alpha > 0$, and $d_0 \geqq 2$. The first sum in (3.58) is bounded by $O(x/A(x))^\alpha$ if $\alpha > 0$, and by $\log^+(d_0 x/A(x)) + 1$ if $\alpha = 0$. In the second sum, $x < nA(x)/d_\alpha \leqq nA(x)/2$, and we can write, for large x, just as in (3.29)–(3.30),

$$\sum_{n > d_\alpha x/A(x)} n^{\alpha-1} P\{S_n \leqq x,\ (X^-)_n^{(1)} \leqq x\}$$
$$\leqq \sum_{n > d_\alpha x/A(x)} n^{\alpha-1} P\{S_n^x - nA(x) \leqq -nA(x)/2\}$$
$$\leqq \sum_{n > d_\alpha x/A(x)} n^{\alpha-1} \rho^n(x)\ . \quad (3.59)$$

Since $1 - \rho(x) \asymp A(x)/x$ as $x \to \infty$ (from (3.27)), the right hand side of (3.59) is $O(x/A(x))^\alpha$ when $\alpha > 0$. So for $\alpha > 0$ we proved that the series in the left hand side of (2.13) is $O(x/A(x))^\alpha$ as $x \to \infty$.

We next complete the bound for (2.16) when $\alpha = 0$. It is true that

$$\sum_{n > d_0 x/A(x)} n^{-1} \rho^n(x) \leqq \rho^{d_0 x/A(x)}\ (-\log(1 - \rho(x))) + 1$$
$$\leqq e^{-d_0/32} (\log(x/A(x)) + c_3\ ,$$

so when $\alpha = 0$ the upper bound for (3.58) is

$$\log^+(d_0 x/A(x)) + 1 + e^{-d_0/32} (\log(x/A(x)) + c_3$$
$$\leqq \left(1 + e^{-d_0/32} + \frac{\log d_0 + c_4}{\log(x/A(x))} \right) \log(x/A(x))\ . \quad (3.60)$$

We can choose d_0 so large that $1 + e^{-d_0/32}$ is arbitrarily close to 1, then let $x \to \infty$ to get an upper bound which is $\sim \log(x/A(x))$, as required for (2.16).

Next note that the series in (2.14) or (2.17) is no bigger than the corresponding one in (2.13) or (2.16). So to complete the proof of Theorem 2.2, we next should find lower bounds for the series in (2.14) and (2.17). This part of the proof again only requires $S_n \xrightarrow{P} \infty$, so we state it as a separate lemma.

Lemma 3.5 *Let* $S_n \xrightarrow{P} \infty$. *Then if* $\alpha > 0$

$$\liminf_{x\to\infty} \left\{ \frac{A(x)}{x} \right\}^{\alpha} \sum_{n \geqq 1} n^{\alpha-1} P \left\{ \max_{1 \leqq j \leqq n} S_j \leqq x \right\} > 0 \,, \tag{3.61}$$

while

$$\liminf_{x\to\infty} \frac{1}{\log(x/A(x))} \sum_{n \geqq 1} n^{-1} P \left\{ \max_{1 \leqq j \leqq n} S_j \leqq x \right\} \geqq 1 \,. \tag{3.62}$$

Proof of Lemma 3.5. Since $S_n \xrightarrow{P} \infty$, $A(x) > 0$ for x large enough by Lemma 3.2. If $x \in \mathbb{R}$ and $\delta > 0$ define

$$M_\delta(x) = \min \left\{ n \geqq 1 \colon P \left\{ \max_{1 \leqq j \leqq n} S_j \leqq x \right\} < 1 - \delta \right\} . \tag{3.63}$$

Then $M_\delta(x) < \infty$ for all x and $\delta > 0$, since $S_n \xrightarrow{P} \infty$, and $M_\delta(x) \uparrow \infty$ as $x \to \infty$. Also

$$n < M_\delta(x) \quad \text{implies} \quad P \left\{ \max_{1 \leqq j \leqq n} S_j \leqq x \right\} \geqq 1 - \delta \,, \tag{3.64}$$

whereas

$$P \left\{ \max_{1 \leqq j \leqq M_\delta(x)} S_j \leqq x \right\} < 1 - \delta \,. \tag{3.65}$$

By (3.64)

$$\begin{aligned} \sum_{n \geqq 1} n^{\alpha-1} P \left\{ \max_{1 \leqq j \leqq n} S_j \leqq x \right\} &\geqq \sum_{n < M_\delta(x)} n^{\alpha-1} P \left\{ \max_{1 \leqq j \leqq n} S_j \leqq x \right\} \\ &\geqq \begin{cases} (1-\delta) c_1 \, (M_\delta(x) - 1)^{\alpha} & \text{for } \alpha > 0 \\ (1-\delta) \log(M_\delta(x)) & \text{for } \alpha = 0 \,, \end{cases} \end{aligned} \tag{3.66}$$

for some constant $c_1 > 0$. Now we show that

$$\liminf_{x\to\infty} \frac{M_\delta(x) A(x)}{x} = c_2 > 0 \,. \tag{3.67}$$

If (3.67) fails we can find a sequence $x_k \uparrow \infty$ such that $M_\delta(x_k) A(x_k)/x_k \to 0$ as $k \to \infty$. Given $\varepsilon \in (0,1)$, choose k so that

$$M_\delta(x_k) A(x_k) \leqq \varepsilon x_k \tag{3.68}$$

and, for brevity, let $M_k = M_\delta(x_k)$. Note that, since $P\{S_n \geqq 0\} \to 1$, we can fix n_0 so that $P\{S_n \geqq 0\} \geqq 1/2$ for $n \geqq n_0$. If $1 \leqq n < n_0$

$$P\{S_n \geqq 0\} \geqq P\{X_i > 0 \quad \text{for } 1 \leqq i \leqq n_0\} = (1 - F(0))^{n_0}$$

so, letting $c = 2 \vee (1 - F(0))^{-n_0}$, we have

$$P\{S_n \geqq 0\} \geqq 1/c \quad \text{for all } n \geqq 1 . \tag{3.69}$$

Now note that

$$\begin{aligned}
&P\left\{\max_{1 \leqq j \leqq M_k} S_j > x_k\right\} \\
&\quad = \frac{P\{S_1 > x_k, \sum_{i=2}^{M_k} X_i \geqq 0\}}{P\{S_{M_k-1} \geqq 0\}} \\
&\quad\quad + \sum_{j=2}^{M_k} \frac{P\{\max_{1 \leqq \ell \leqq j-1} S_\ell \leqq x_k < S_j,\ \sum_{i=j+1}^{M_k} X_i \geqq 0\}}{P\left\{S_{M_k-j} \geqq 0\right\}} \\
&\quad \leqq c\left(P\{S_{M_k} > x_k, S_1 > x_k\} + \sum_{j=2}^{M_k} P\{S_{M_k} > x_k,\ \max_{1 \leqq \ell \leqq j-1} S_\ell \leqq x_k < S_j\}\right) \\
&\quad = cP\{S_{M_k} > x_k\} .
\end{aligned} \tag{3.70}$$

Next, write

$$\begin{aligned}
P\{S_{M_k} > x_k\} &= P\left\{\sum_{i=1}^{M_k} X_i > x_k\right\} \leqq P\left\{\sum_{i=1}^{M_k} (X_i \vee -x_k) > x_k\right\} \\
&\leqq P\left\{\sum_{i=1}^{M_k} ((X_i \vee -x_k) \wedge x_k) > x_k\right\} + M_k(1 - F(x_k)) .
\end{aligned} \tag{3.71}$$

Let $X_i^k = (X_i \vee -x_k) \wedge x_k$, so that $E(X_i^k) = A(x_k)$ and $E(X_i^k)^2 = U(x_k)$. By (3.68), $x_k - M_k A(x_k) \geqq (1 - \varepsilon)x_k$, so from (3.71) and Chebychev's inequality,

$$\begin{aligned}
P\{S_{M_k} > x_k\} &\leqq P\left\{\sum_{i=1}^{M_k} X_i^k - M_k A(x_k) > (1 - \varepsilon)x_k\right\} + M_k(1 - F(x_k)) \\
&\leqq \frac{M_k U(x_k)}{(1 - \varepsilon)^2 x_k^2} + M_k(1 - F(x_k)) \leqq \frac{2M_k U(x_k)}{(1 - \varepsilon)^2 x_k^2} .
\end{aligned} \tag{3.72}$$

(Recall that $x^2(1 - F(x)) \leqq U(x)$). By (3.6), $U(x) \leqq 3xA(x)$ for x large enough, so by (3.70) and (3.72)

$$P\left\{\max_{1 \leqq j \leqq M_k} S_j > x_k\right\} \leqq \frac{6cM_k A(x_k)}{(1 - \varepsilon)^2 x_k} \leqq \frac{6c\varepsilon}{(1 - \varepsilon)^2} \quad \text{(by (3.68))} .$$

For ε so small that $6c\varepsilon/(1 - \varepsilon)^2 < \delta$, this contradicts (3.65). Consequently (3.67) holds, and (3.61) and (3.62) follow from (3.66) (letting $\delta \downarrow 0$ for (3.62)). □

To complete the proof of Theorem 2.2, it remains only to establish (2.15). But this follows from (3.42), (2.14), and the fact that $E(L(0))^\alpha < \infty$ (see the proof that (2.4) $\Longrightarrow$ (2.8)). □

4 Discussion

In this section some additional results of interest are discussed briefly.

4.1 The ladder height process

Let $\{T_j, S_{T_j}\}_{j \geqq 1}$ be the strict ascending ladder point process for S_n. Thus, if we take $T_0 = 0$, $S_0 = 0$, then $T_j = \min\{n > T_{j-1}\colon S_n > S_{T_{j-1}}\}$ for $j \geqq 1$. A classical result is then that $\{T_j - T_{j-1},\ Z_j = S_{T_j} - S_{T_{j-1}}\}_{j \geqq 1}$ are i.i.d. random vectors (see Feller, 1971, pp. 391–392). Define, as in Feller,

$$\begin{aligned}\Psi(y) &= I_{[0,\infty)}(y) + \sum_{n \geqq 1} P\left\{\max_{0 \leqq j \leqq n-1} S_j < S_n \leqq y\right\}\\ &= I_{[0,\infty)}(y) + \sum_{k \geqq 1} P\{S_{T_k} \leqq y\}\\ &= I_{[0,\infty)}(y) + \sum_{k \geqq 1} P\left\{\sum_{j=1}^{k} Z_j \leqq y\right\},\end{aligned} \tag{4.1}$$

the renewal measure for the ladder height process. Define

$$A_Z(y) = \int_0^y P\{Z_1 > u\}\, du\,.$$

Since the Z_j are nonnegative random variables we have by (2.12) and (4.1) that

$$\Psi(y) \asymp \frac{y}{A_Z(y)} \quad \text{as } y \to \infty\,. \tag{4.2}$$

Now suppose $S_n \to \infty$ a.s., so (2.4) and (2.6) hold with $\alpha = 0$. Then we can write for $y \geqq 0$ (and with $\max_{0 \leqq k \leqq -1} S_k = -\infty$),

$$\begin{aligned}&\sum_{n=1}^{\infty} P\left\{\max_{1 \leqq j \leqq n} S_j \leqq y\right\}\\ &\quad= \sum_{n=1}^{\infty} P\left\{\max_{0 \leqq j \leqq n} S_j \leqq y\right\}\\ &\quad= \sum_{n=1}^{\infty}\sum_{j=0}^{n} P\left\{\max_{0 \leqq k \leqq j-1} S_k < S_j \leqq y,\ \max_{j \leqq k \leqq n} S_k \leqq S_j\right\}\\ &\quad= \sum_{j=0}^{\infty} P\left\{\max_{0 \leqq k \leqq j-1} S_k < S_j \leqq y\right\} \sum_{n \geqq j \vee 1} P\left\{\max_{j \leqq k \leqq n} S_k - S_j \leqq 0\right\}\\ &\quad= \sum_{j=0}^{\infty} P\left\{\max_{0 \leqq k \leqq j-1} S_k < S_j \leqq y\right\} \sum_{n \geqq 0} P\left\{\max_{0 \leqq j \leqq n} S_j \leqq 0\right\} - 1\\ &\quad= \Psi(y) \sum_{n \geqq 0} P\left\{\max_{0 \leqq j \leqq n} S_j \leqq 0\right\} - 1\,.\end{aligned} \tag{4.3}$$

From (3.61) with $\alpha = 1$ and (2.6) with $\alpha = 0$ we deduce that $\Psi(y) \geqq c_1 y/A(y)$ for some $c_1 > 0$ and y large enough. Hence, by (4.2), we have

$A_Z(y) \geqq c_2 A(y)$ for some $c_2 > 0$ and y large enough. But $A_Z(y) \geqq A_+(y) \geqq A(y)$, so we have proved under the sole assumption

$$S_n \to \infty \text{ a.s.}\,, \tag{4.4}$$

that

$$A_Z(y) \asymp A_+(y) \asymp A(y) \quad \text{as } y \to \infty\,. \tag{4.5}$$

This relationship will be needed in the next section.

4.2 Janson's conditions

Recall that $\Lambda(x)$ is the number of S_n smaller than or equal to $x \geqq 0$ (see (3.53)). Define also

$$\tau(-x) = \min\{n \geqq 1 \colon S_n < -x\} \quad (\text{for } x \geqq 0) \tag{4.6}$$

(with $\tau(-x) = \infty$ if $S_n \geqq -x$ for all $n \geqq 1$) and

$$K_{\min} = \min\left\{k \geqq 1 \colon S_k = \min_{n \geqq 1} S_n\right\}\,. \tag{4.7}$$

In his Theorem 1, Janson (1986) shows that, in the case when

$$\alpha > 0 \quad \text{and} \quad 0 < EX \leqq E|X| < \infty\,, \tag{4.8}$$

$E(X^-)^{\alpha+1} < \infty$ is also equivalent to any of the following conditions:

$$E(\Lambda(x))^\alpha < \infty \quad \text{for some (hence every) } x \geqq 0\,, \tag{4.9}$$

$$E((\tau(-x))^\alpha;\ \tau(-x) < \infty) < \infty \quad \text{for some (hence every) } x \geqq 0\,, \tag{4.10}$$

and

$$E(K_{\min})^\alpha < \infty\,. \tag{4.11}$$

By Theorem 2.1 and Lemma 3.1, these conditions are thus also equivalent to (2.4)–(2.8) (under the side condition (4.8)).

Even the following more general statement holds in our setup: If $\alpha > 0$ and (4.4) holds, then (2.4)–(2.8) and (4.9)–(4.11) are all equivalent. For proof of this assertion, we only consider the case where $F(0-) > 0$, because everything is rather trivial if $X_i \geqq 0$ a.s. We now observe that

$$K_{\min} \leqq L(X_1)\,,$$

because $S_n > S_1 = X_1$ for all $n > L(X_1)$. However, conditionally on $X_1, L(X_1) - 1$ has the same distribution as $L(0)$. Therefore (2.4)–(2.8) imply (4.11), which in turn implies (4.10) for any $x \geqq 0$ because $\tau(-x) \leqq K_{\min}$ on $\{\tau(-x) < \infty\}$. Furthermore

$$\tau(-x)I[\tau(-x) < \infty] \geqq \tau(0)I[\tau(0) < \infty]\,I\left[\min_{k \geqq \tau(0)} S_k - S_{\tau(0)} < -x\right]\,,$$

so that

$$E\{\tau^\alpha(-x); \tau(-x) < \infty\} \geqq E\{\tau^\alpha(0); \tau(0) < \infty\} P\left\{\min_{k \geqq 0} S_k < -x\right\} .$$

Consequently (4.10) for any $x \geqq 0$ implies (4.10) for $x = 0$. In the same way, (4.10) for any $x \geqq 0$ implies

$$E\{v_0^\alpha; v_0 < \infty\} < \infty . \tag{4.12}$$

(Recall that v_0 is the first $n \geqq 1$ for which $S_n \leqq 0$ if such an n exists; see (3.44)). We saw in the proof of Theorem 2.1 (see the end of the proof of (2.4) $\Longrightarrow$ (2.8)) that (4.12) is equivalent to (2.4) (under (4.4)). Thus (2.4)–(2.8) are equivalent to (4.10)–(4.11). Finally, $\Lambda(0) \leqq \Lambda(x) \leqq L(x)$ for all $x \geqq 0$, and the equivalence of $E(\Lambda(0))^\alpha < \infty$ with (2.7) (see the proof of Theorem 2.1) easily shows the equivalence of (2.4)–(2.8) and (4.9).

We note in addition that the inequalities

$$T(x) - 1 \leqq \Lambda(x) \leqq L(x)$$

together with Theorem 2.2, show that, when $\alpha > 0$ and (2.4) holds,

$$E(\Lambda(x))^\alpha \asymp \left\{\frac{x}{A(x)}\right\}^\alpha \quad \text{as } x \to \infty .$$

Janson (1986) also lists in his Theorem 1 some other conditions which he shows, under (4.8), to be equivalent to (4.9)–(4.11) [and thus to (2.4)–(2.8)]. These other conditions involve the *position* S_N at certain random times N, while so far we have only dealt with moments of random *times*. In a similar spirit, we show that the following conditions are equivalent to each other, but in general *not*, as it turns out, to (4.9)–(4.11). We discuss their connection with Janson's conditions in more detail below.

Proposition 4.1 *The following are equivalent when $S_n \to \infty$ a.s. and $\alpha > 0$:*

$$E(|S_{v_0}|^\alpha;\ v_0 < \infty) < \infty , \tag{4.13}$$

$$\int_0^\infty \frac{y^{\alpha+1}}{A_+(y)} |dF(-y)| < \infty , \tag{4.14}$$

$$E\left(\left|\min_{n \geqq 0} S_n\right|^\alpha\right) < \infty , \tag{4.15}$$

$$E(|S_{\tau(-x)}|^\alpha;\ \tau(-x) < \infty) < \infty \quad \textit{for some (hence every) } x \geqq 0 , \tag{4.16}$$

$$E\left(\max_{0 \leqq n \leqq L(x)} |S_n|^\alpha\right) < \infty \quad \textit{for some (hence every) } x \geqq 0 , \tag{4.17}$$

and

$$E\left(\max_{0 \leqq n \leqq \tau(-x)} |S_n|^\alpha;\ \tau(-x) < \infty\right) < \infty \quad \textit{for some (hence every) } x \geqq 0 . \tag{4.18}$$

Proof of Proposition 4.1. Assume (4.4) and $\alpha > 0$. Note that, from Feller (1971, Eq. (XII.3.7a) or Doney (1982, p. 374), for $t \geqq 0$

$$P\{-S_{\nu_0} \geqq t,\ \nu_0 < \infty\} = \int_{[0,\infty)} F(-t-y)\,d\Psi(y) \tag{4.19}$$

where $\Psi(y)$ is the renewal measure for the strict ascending ladder height process defined in Sect. 4.1. We showed in (4.2) and (4.5) that, when $S_n \to \infty$ a.s.,

$$\Psi(y) \asymp \frac{y}{A_Z(y)} \asymp \frac{y}{A_+(y)}, \quad \text{as } y \to \infty\ . \tag{4.20}$$

From (4.19) we obtain, using (4.20) and Fubini's theorem, for $t > 0$,

$$\begin{aligned} &P\{-S_{\nu_0} \geqq t, \nu_0 < \infty\} \\ &\quad = \int_{[0,\infty)} |dF(-z)|\Psi(z-t) \asymp \int_{[t,\infty)} |dF(-z)| \frac{z-t}{A_+(z-t)} \\ &\quad = \int_{[t,\infty)} |dF(-z)| \int_{[0,z-t]} d\left(\frac{y}{A_+(y)}\right)\ . \end{aligned} \tag{4.21}$$

(In the last integral we take $y/A_+(y) = 0$ for $y < 0$ and include the contribution due to the jump of size $1/[1-F(0)]$ at 0 in $y/A_+(y)$.) Now multiply by $t^{\alpha-1}$ and integrate over $t \in [0,\infty)$. We have

$$\begin{aligned} &\int_0^\infty t^{\alpha-1} dt \int_{[t,\infty)} |dF(-z)| \int_{[0,z-t]} d\left(\frac{y}{A_+(y)}\right) \\ &\quad = \frac{1}{\alpha} \int_{[0,\infty)} |dF(-z)| \int_{[0,z]} (z-y)^\alpha d\left(\frac{y}{A_+(y)}\right)\ . \end{aligned} \tag{4.22}$$

On the one hand, the right hand side of (4.22) is at most

$$\frac{1}{\alpha} \int_{[0,\infty)} |dF(-z)| \frac{z^{\alpha+1}}{A_+(z)}\ ,$$

while on the other hand it is at least equal to

$$\begin{aligned} &\frac{1}{\alpha} \int_{[0,\infty)} |dF(-z)| \int_{[0,z/2]} (z-y)^\alpha d\left(\frac{y}{A_+(y)}\right) \\ &\quad \geqq \frac{1}{\alpha} 2^{-\alpha-1} \int_{[0,\infty)} |dF(-z)| \frac{z^{\alpha+1}}{A_+(z/2)} \\ &\quad \geqq \frac{1}{\alpha} 2^{-\alpha-1} \int_{[0,\infty)} |dF(-z)| \frac{z^{\alpha+1}}{A_+(z)}\ . \end{aligned}$$

Thus the integral in the right hand side of (4.22) converges if and only if (4.14) holds. This establishes the equivalence of (4.13) and (4.14).

The equivalence of (4.13), (4.15) and (4.16) follows just as in Janson (1986, p. 871) [compare also the argument above for showing that (4.10) for any $x \geqq 0$ implies (4.10) for $x = 0$].

For the next step, we use the inequality

$$P\left\{\max_{0\leqq n\leqq L(x)}|S_n| > t\right\} \leqq 2P\left\{\left|\min_{n\geqq 0}S_n\right| > t-x\right\}, \quad t > x \geqq 0\,. \tag{4.23}$$

To prove (4.23), fix $t > x \geqq 0$. We work throughout on the event $\{T(t) < \infty, L(x) < \infty\}$, which has probability 1 since $S_n \to \infty$ a.s. Following Janson's (1986) working, we have $\max_{0\leqq n\leqq L(x)} S_n > t \Longleftrightarrow \max_{1\leqq n\leqq L(x)} S_n > t \Longleftrightarrow L(x) \geqq 1$ and $T(t) \leqq L(x) \Longleftrightarrow S_n \leqq x$ for some $n \geqq T(t) \Longleftrightarrow S_{T(t)+n} - S_{T(t)} \leqq x - S_{T(t)}$ for some $n \geqq 0$. Since $S_{T(t)} > t$, the last equivalence implies that $S_{T(t)+n} - S_{T(t)} < x - t$ for some $n \geqq 0$, so

$$P\left\{\max_{0\leqq n\leqq L(x)} S_n > t\right\} \leqq P\left\{\min_{n\geqq 0} S_n < x-t\right\} \leqq P\left\{\left|\min_{n\geqq 0}S_n\right| > t-x\right\}\,.$$

Also

$$P\left\{\min_{0\leqq n\leqq L(x)} S_n < -t\right\} \leqq P\left\{\min_{n\geqq 0} S_n < -t\right\} \leqq P\left\{\left|\min_{n\geqq 0}S_n\right| > t-x\right\}$$

so the inequality in (4.23) follows.

To complete the proof of Proposition 4.1 we use (4.23) to show that (4.15) implies (4.17) (for all $x \geqq 0$). Then since $\tau(-x) \leqq L(0)$ on $\{\tau(-x) < \infty\}$, (4.17) (for some $x \geqq 0$, hence for $x = 0$) implies (4.18) (for all $x \geqq 0$), while clearly (4.18) (for some $x \geqq 0$) implies (4.16) (for that same $x \geqq 0$, hence for all $x \geqq 0$). □

Remarks. (i) The conditions in Proposition 4.1 are similar to Janson's (1986, Theorem 1) remaining conditions (other than (4.9)–(4.11)), but there are some differences. First note that, under (4.4), $A(y) \asymp A_+(y)$ is bounded away from 0 as $y \to \infty$, so (4.14) implies (2.5). However, (2.5) does not imply (4.14) in general, since we may have $A_+(y) \to \infty$ as $y \to \infty$. Thus (4.13)–(4.18) are more stringent conditions than (2.4)–(2.8) and (4.9)–(4.11), in general. However, if (4.8) holds then all these conditions are equivalent to each other, and to $E(X^-)^{\alpha+1} < \infty$ (see also Chow, 1986, Corollary 1).
(ii) Theorem 2.1, conditions (4.9)–(4.11), and Proposition 4.1 provide an analogue of Janson's Theorem 1 which is not quite the same in respect to conditions (4.17)–(4.18); Janson's analogous conditions (viii) and (ix) are written with S_n where we have $|S_n|$. Under (4.8) these modified conditions are equivalent to (4.13)–(4.16), as Janson shows, but in general they are not. The working of Proposition 4.1 shows that (4.15) implies

$$E\left(\max_{0\leqq n\leqq L(x)} S_n\right)^{\alpha} < \infty \quad \text{for some (hence every) } x \geqq 0\,, \tag{4.24}$$

and (4.24) implies

$$E\left(\left(\max_{0\leqq n\leqq \tau(-x)} S_n\right)^{\alpha};\ \tau(-x) < \infty\right) < \infty \quad \text{for some (hence every) } x \geqq 0\,. \tag{4.25}$$

However it can be shown that the distribution F whose tails satisfy

$$1 - F(y) \sim \frac{1}{\log y} \quad \text{and} \quad F(-y) \sim \frac{1}{y^{\alpha}(\log y)^2} \quad (y \to \infty)$$

satisfies (4.25) but not (4.13).

4.3 Consecutive S_n above x

The following problem arises in the theory of option pricing. Presumably as an incentive to productivity, the new executive of a company has the option to buy stock in the company at an agreed price, provided the stock price of the company remains above a pre-assigned level x on at least K consecutive trading days at some time in the future. (One major Australian company is known to follow this practice with $K = 5$). The finer points of "pricing" (determining the value) of such an option have been well studied when $K = 1$, assuming a certain random walk model S_n for the stock price, but not for $K > 1$. Here we will prove that the behaviour of

$$T^{(K)}(x) = \min\left\{n \geqq 1\colon \min_{0 \leqq j \leqq K-1} S_{n+j} > x\right\},$$

the first time at which (the first of) K consecutive values of S_n exceeds x (with $T^{(K)}(x) = \infty$ if this never occurs) is similar to that of $T(x)$ as defined in (1.8), when $S_n \to \infty$ a.s.

To see this, take $K > 1$, and note that just as in (3.42), we have the inequality

$$T(x) \leqq T^{(K)}(x) \leqq T(x) + \tilde{T}^{(K)}(x), \tag{4.26}$$

where

$$\tilde{T}^{(K)}(x) = \min\left\{l \geqq 1\colon \min_{0 \leqq j \leqq K-1} S_{T(x)+l+j} > S_{T(x)}\right\}.$$

Now $\tilde{T}^{(K)}(x)$ has the same distribution as $T^{(K)}(0)$ and is independent of $T(x)$. If $T^{(K)}(x) < \infty$ a.s., or has moments of a certain order, then (4.26) shows that the same is true of $T(x)$. Inequality (4.26) also shows that $T^{(K)}(x)/T(x) \xrightarrow{P} 1$ as $x \to \infty$.

Conversely, suppose $T(x) < \infty$ a.s. As in Sect. 4.1, let T_r be the r-th strict ascending ladder time of the random walk, with $T_1 = T(0) = \min\{n \geqq 1\colon S_n > 0\}$. Since $T(0) < \infty$ a.s., $T_r < \infty$ a.s. Let E_r be the event $\{\min_{1 \leqq j \leqq K-1} S_{T_r+j} > S_{T_r}\}$. When E_r occurs, $T^{(K)}(0)$ has occurred, and $T^{(K)}(0) \leqq T_r$. Let $N = \min\{r\colon E_{rK} \text{ occurs}\}$. Then

$$T^{(K)}(0) \leqq T_{NK}.$$

Now N is a geometric random variable with success probability

$$\pi_K = P\left\{\min_{1 \leqq j \leqq K-1} S_j > 0\right\} \geqq P\left\{\min_{1 \leqq j \leqq K-1} X_j > 0\right\} = (1 - F(0))^{K-1} > 0.$$

It follows that $N < \infty$ a.s. Thus $T^{(K)}(0) < \infty$ a.s. [This incidentally shows that $\limsup_{n\to\infty} S_n = \infty$ a.s. if and only if $\limsup_{n\to\infty}(\min_{1\leqq j\leqq K-1} S_{n+j}) = \infty$ a.s., too.]

Now suppose $E(T(0))^{1+\alpha} < \infty$. Let $U_1, U_2, \ldots$ be a sequence of auxiliary random variables such that the distribution of each U_i equals the conditional distribution of T_K, given

$$(E_0)^c = \left\{ \min_{1\leqq j\leqq K-1} S_j \leqq 0 \right\} .$$

Thus U_i also has the conditional distribution of $T_{iK} - T_{(i-1)K}$, given $\left(E_{(i-1)K}\right)^c$. We take the U_i independent of all Xs and define further

$$\Delta_j = \{T_{jK} - T_{(j-1)K}\} I[(E_{(j-1)K})^c] + U_j I[E_{(j-1)K}] ,$$

$$\mathscr{F}_j = \sigma\text{-field generated by } T_{iK}, U_i, I[E_{iK}], i \leqq j .$$

N is an $\{\mathscr{F}_j\}$-stopping time. Note that this requires us to put E_{jK}, which depends on X_i for $i \leqq jK + K - 1$, into $\mathscr{F}_j$. Nevertheless, Δ_{j+1} is independent of $\mathscr{F}_j$, because if E_{jK} fails, then Δ_{j+1} has the conditional distribution of $T_{(j+1)K} - T_{jK}$, given $\left(E_{jK}\right)^c$, and if E_{jK} occurs, then $\Delta_{j+1} = U_{j+1}$ has the same distribution. We can therefore apply Theorem I.5.2 of Gut (1988) to obtain that

$$\begin{aligned} E[\tilde{T}^{(K)}(x)]^{1+\alpha} &\leqq E[T_{NK}]^{1+\alpha} = E\left[\sum_{j=1}^{N} \Delta_j\right]^{1+\alpha} \\ &\leqq d'_\alpha E N^{1+\alpha} E[\Delta_1]^{1+\alpha} \\ &\leqq d_\alpha [P\{(E_K)^c\}]^{-1} E T_K^{1+\alpha} \end{aligned}$$

for some constants d_α, d'_α. But T_K is the sum of K independent copies of $T(0)$. Therefore by Minkowski's inequality

$$E[\tilde{T}^{(K)}(x)]^{1+\alpha} \leqq d_\alpha [P\{(E_K)^c\}]^{-1} K^{1+\alpha} E[T(0)]^{1+\alpha} < \infty .$$

Returning to (4.26), we have by Minkowski's inequality,

$$E(T^{(K)}(x))^{1+\alpha} \leqq \left[(E(T(x))^{1+\alpha})^{(1/(1+\alpha))} + (E(T^{(K)}(0))^{1+\alpha})^{(1/(1+\alpha))} \right]^{1+\alpha} . \tag{4.27}$$

Now for $a \geqq 0,\ b \geqq 0$,

$$\begin{aligned} (a+b)^{1+\alpha} &= a^{1+\alpha} + (1+\alpha) \int_0^b (a+\xi)^\alpha \, d\xi \\ &\leqq a^{1+\alpha} + (1+\alpha)(a+b)^\alpha b \\ &\leqq a^{1+\alpha} + (1+\alpha) 2^\alpha (a^\alpha b + b^{1+\alpha}) \end{aligned}$$

so that, using (4.26) and (4.27),

$$\begin{aligned}E(T(x))^{1+\alpha} &\leqq E(T^{(K)}(x))^{1+\alpha}\\ &\leqq E(T(x))^{1+\alpha} + (1+\alpha)2^{\alpha}E(T^{(K)}(0))^{1+\alpha}\\ &\quad + (1+\alpha)2^{\alpha}\left(E(T(x))^{1+\alpha}\right)^{(\alpha/(1+\alpha))}\left(E(T^{(K)}(0))^{1+\alpha}\right)^{1/(1+\alpha)} .\end{aligned}$$

Consequently

$$|E(T^{(K)}(x))^{1+\alpha} - E(T(x))^{1+\alpha}| = O(E(T(x)^{1+\alpha}))^{(\alpha/(1+\alpha))} = o(E(T(x)^{1+\alpha}) .$$

Thus $E(T^{(K)}(x))^{1+\alpha} \sim E(T(x)^{1+\alpha})$ as $x \to \infty$, and further, by Theorem 2.2 we then deduce that (under (2.7))

$$E(T^{(K)}(x))^{1+\alpha} \asymp \left(\frac{x}{A(x)}\right)^{1+\alpha} \quad \text{as } x \to \infty . \tag{4.28}$$

So we have shown that $T^{(K)}(x)$ is finite a.s. if and only if the same is true of $T(x)$, and furthermore that $T^{(K)}(x)$ and $T(x)$ are asymptotically equivalent as $x \to \infty$. (Of course $T^{(K)}(x)$ could still be considerably larger than $T(x)$ for finite x).

A variant of the procedure mentioned above is to allow the executive's exercising of the option if the *average* of K consecutive values of the stock price remains above x at some stage. Clearly the first time $T_A^{(K)}(x)$ at which this occurs lies between $T(x)$ and $T^{(K)}(x)$, so (4.28) is also true for $T_A^{(K)}(x)$.

4.4 A convergence rate theorem

The method of proof of Theorem 2.1 can be used to obtain the following "convergence rate" theorem. Such theorems are of interest in boundary crossing problems, see for example Kao (1978, Theorem 3.1), and the papers of Katz (1963) and Baum and Katz (1965). Proposition 4.2 also generalises Feller (1946).

Proposition 4.2 *Suppose $b_n > 0$ is a nonstochastic sequence satisfying*

$$\frac{b_n}{n} \to \infty \tag{4.29}$$

and, for some constant $0 < \gamma \leqq 1$,

$$\frac{b_m}{m} \geqq \gamma \frac{b_n}{n} \quad \textit{whenever} \quad m \geqq n \geqq 1 . \tag{4.30}$$

Then for any $\alpha \geqq 0$

$$\sum_{n \geqq 1} n^{\alpha} P\{X > b_n\} < \infty \tag{4.31}$$

implies

$$\sum_{n \geqq 1} n^{\alpha-1} P\{S_n > b_n\} < \infty . \tag{4.32}$$

Proof of Proposition 4.2. Let (4.29)–(4.31) hold. In proving (4.32), we may assume that $X \geqq 0$ a.s., since replacing X by X^+ does not affect (4.31), and $P\{S_n > b_n\} \leqq P\{\sum_{i=1}^n X_i^+ > b_n\}$. In this case the X_i^y and S_n^y defined in (3.1) and (3.2) become $X_i \wedge y$ and $\sum_{i=1}^n X_i \wedge y$. Write

$$\begin{aligned} P\{S_n > b_n\} & \\ &\leqq P\{X_i > \tfrac{1}{2}b_n \quad \text{for some } i \leqq n\} + P\left\{S_n > b_n, \max_{1\leqq i\leqq n} X_i \leqq \tfrac{1}{2}b_n\right\} \\ &\leqq nP\{X > \tfrac{1}{2}b_n\} + \sum_{j=1}^n \int_{\left[0,\frac{1}{2}b_n\right]} P\left\{\sum_{i=1, i\neq j}^n X_i^y > b_n - y\right\} dF(y) \\ &\leqq nP\{X > \tfrac{1}{2}b_n\} + n \int_{\left[0,\frac{1}{2}b_n\right]} P\{S_{n-1}^y > \tfrac{1}{2}b_n\}\, dF(y)\,. \end{aligned} \tag{4.33}$$

Now by (4.30) we have

$$\tfrac{1}{2}b_n \geqq \frac{\gamma}{2}\frac{n}{\lfloor \gamma n/2 \rfloor} b_{\lfloor \gamma n/2 \rfloor} \geqq b_{\lfloor \gamma n/2 \rfloor} \tag{4.34}$$

so that

$$\begin{aligned} \sum_{n\geqq 1} n^{\alpha-1} nP\{X > \tfrac{1}{2}b_n\} &\leqq \sum_{n\geqq 1} n^\alpha P\{X > b_{\lfloor \gamma n/2 \rfloor}\} \\ &\leqq \sum_{k\geqq 1} P\{X > b_k\} \sum_{\frac{2k}{\gamma} \leqq n < \frac{2(k+1)}{\gamma}} n^\alpha \leqq c_1 \sum_{k\geqq 1} k^\alpha P\{X > b_k\} \end{aligned} \tag{4.35}$$

for some $c_1 = c_1(\alpha,\gamma)$. The last series is finite by (4.31).

Next we need to estimate $P\{S_{n-1}^y > \frac{1}{2}b_n\}$. First we need an estimate of $E(X^y)$ for $y \leqq \frac{1}{2}b_n$. We have, for $y \leqq \frac{1}{2}b_n$,

$$\begin{aligned} E(X^y) = A(y) = \int_0^y P\{X > z\}\, dz &\leqq \int_0^{b_n/2} P\{X > z\}\, dz \\ &= \tfrac{1}{2}b_n P\{X > \tfrac{1}{2}b_n\} + \int_{\left[0,\frac{1}{2}b_n\right]} z dF(z)\,. \end{aligned} \tag{4.36}$$

For $\gamma n/4 \leqq i \leqq \gamma n/2$, we have, as in (4.34), that $P\{X > \frac{1}{2}b_n\} \leqq P\{X > b_i\}$, and therefore

$$\begin{aligned} \tfrac{1}{2}b_n P\{X > \tfrac{1}{2}b_n\} &\leqq \tfrac{1}{2}\frac{b_n}{\gamma n/4} \sum_{\gamma n/4 \leqq i \leqq \gamma n/2} P\{X > b_i\} \\ &\leqq c_2 \frac{b_n}{n} \sum_{i \geqq \gamma n/4} P\{X > b_i\} \leqq \frac{\gamma}{32}\frac{b_n}{n} \end{aligned} \tag{4.37}$$

provided we keep $n \geqq n_0$, for some n_0 (in fact, this expression is $o(b_n/n)$.) Also for $n \geqq n_1$, for some n_1, and with $b_0 = 0$,

$$\begin{aligned}
&\frac{n}{b_n} \int_{[0,\frac{1}{2}b_n]} z dF(z) \\
&\leqq \frac{n}{b_n} \sum_{k=1}^{n} b_k P\{b_{k-1} < X \leqq b_k\} \\
&\leqq \frac{n}{b_n} \max_{1 \leqq k \leqq n_1} b_k + \sum_{k=n_1+1}^{n} \left(\frac{n}{b_n}\frac{b_k}{k}\right) kP\{b_{k-1} < X \leqq b_k\} \\
&\leqq \frac{n}{b_n} n_1 \max_{1 \leqq k \leqq n_1} \frac{b_k}{k} + \frac{1}{\gamma} \sum_{k=n_1+1}^{n} kP\{b_{k-1} < X \leqq b_k\} \\
&\leqq \frac{n}{b_n} n_1 \left(\frac{1}{\gamma}\frac{b_{n_1}}{n_1}\right) + \frac{1}{\gamma}\left(\sum_{k=n_1}^{n-1}(k+1)P\{X > b_k\} - \sum_{k=n_1+1}^{n} kP\{X > b_k\}\right) \\
&\leqq \frac{b_{n_1}}{\gamma}\frac{n}{b_n} + \frac{1}{\gamma}\left((n_1+1)P\{X > b_{n_1+1}\} + \sum_{k=n_1}^{n-1} P\{X > b_k\}\right) \\
&\leqq \frac{b_{n_1}}{\gamma}\frac{n}{b_n} + \frac{1}{8} \quad \text{(by (4.31) and the estimates in (4.37))}\,. \qquad (4.38)
\end{aligned}$$

Note that the first inequality here does not require that b_k is increasing. Since $b_n/n \to \infty$ by (4.29), (4.36)–(4.38) show that for $y \leqq \frac{1}{2}b_n$

$$A(y) \leqq \frac{b_n}{4n} \qquad (4.39)$$

provided n is large enough. This means that $E(S_{n-1}^y) \leqq \frac{1}{4}b_n$ when $y \leqq \frac{1}{2}b_n$. By Bernstein's inequality (Chow and Teicher, 1986, Exercise 4.3.13) we then have

$$\begin{aligned}
P\{S_{n-1}^y > \tfrac{1}{2}b_n\} &\leqq P\{S_{n-1}^y - E(S_{n-1}^y) > \tfrac{1}{4}b_n\} \\
&\leqq \exp\left\{\frac{-c_3 b_n^2}{n \operatorname{Var}(X^y) + y b_n/4}\right\}.
\end{aligned}$$

Moreover, by (4.39) we have for large n

$$n \operatorname{Var}(X^y) \leqq 2n \int_0^y zP\{X > z\}\, dz \leqq 2nyA(y) \leqq \tfrac{1}{2} y b_n\,,$$

so, for $y \leqq \frac{1}{2}b_n$ and large n,

$$P\{S_{n-1}^y > \tfrac{1}{2}b_n\} \leqq e^{-c_4 b_n/y}\,.$$

Next, define

$$c(y) = \max\{n \geqq 1 : b_n \leqq y\}\,. \qquad (4.40)$$

Then we have

$$\sum_{n\geqq 1} n^\alpha \int\limits_{\left[0,\frac{1}{2}b_n\right]} P\{S^y_{n-1} > \tfrac{1}{2}b_n\}\; dF(y) \leqq \int\limits_0^\infty \sum_{b_n\geqq 2y} n^\alpha e^{-c_4 b_n/y} dF(y)$$

and

$$\begin{aligned}\sum_{b_n\geqq 2y} n^\alpha e^{-c_4 b_n/y} &\leqq \sum_{n\leqq c(y)} n^\alpha + \sum_{n>c(y)} n^\alpha e^{-c_4 b_n/y}\\ &\leqq c_5(c(y))^{1+\alpha} + \sum_{n>c(y)} n^\alpha \;\exp\left(-c_4\gamma\frac{b_{c(y)+1}}{c(y)+1}\frac{n}{y}\right)\\ &\leqq c_5(c(y))^{1+\alpha} + c_6\left(\frac{b_{c(y)+1}}{c(y)+1}\frac{1}{y}\right)^{-\alpha-1}\\ &\leqq c_7(c(y)+1)^{\alpha+1}\,,\end{aligned}$$

because $b_{c(y)+1} > y$.

Finally this shows

$$\begin{aligned}\sum_{n\geqq 1} n^\alpha \int\limits_{\left[0,\frac{1}{2}b_n\right]} P\{S^y_{n-1} > \tfrac{1}{2}b_n\}dF(y) &\leqq c_7 \int\limits_{[0,\infty)} (c(y)+1)^{\alpha+1}dF(y)\\ &= c_7 E(c(X)+1)^{\alpha+1}\,.\end{aligned}$$

But

$$\{c(X) \geqq k\} \subseteq \{X \geqq b_{\lfloor\gamma k\rfloor}\}\,,$$

because $X < b_{\lfloor\gamma k\rfloor}$ implies

$$c(X) = \max\{n \geqq 1\colon b_n \leqq X\} \leqq \max\{n \geqq 1\colon b_n < b_{\lfloor\gamma k\rfloor}\}$$

and this does not exceed $k-1$, since by (4.30) $n \geqq k$ implies $b_n \geqq \gamma n b_{\lfloor\gamma k\rfloor}/\lfloor\gamma k\rfloor \geqq b_{\lfloor\gamma k\rfloor}$. Thus

$$E(c(X))^{\alpha+1} \leqq c_8 \sum_{k\geqq 0} k^\alpha P\{c(X) \geqq k\} \leqq c_8 \sum_{k\geqq 0} k^\alpha P\{X \geqq b_{\lfloor\gamma k\rfloor}\}\,,$$

and this converges, just as in (4.35). This, together with (4.33) and (4.35), proves (4.32). □

Remarks. (vii) We can drop assumption (4.29), but, under (4.30), this just leads to the finite mean case. In fact if $b_{n_i}/n_i \leqq C < \infty$ for some sequence $n_i \to \infty$, then (4.30) gives $b_k \leqq C\gamma k$ for k large enough. Then (4.31) implies $EX < \infty$. When $b_n = n\varepsilon$ for some $\varepsilon > 0$ and $\alpha \geqq 1$, Proposition 4.2 is contained in Theorem 3.1 of Kao (1978).

(viii) As a special case of Proposition 4.2, let $b_n = n^{r/t}$, where $r \geqq 1, t > 0$, and $r/t > 1$. Then Proposition 4.2 with $\alpha = r-1$ tells us that $\sum_{n\geqq 1} n^{r-1}P\{X > n^{r/t}\} < \infty$ implies the convergence of $\sum_{n\geqq 1} n^{r-2}P\{S_n > n^{r/t}\}$, which is essentially the implication (d) ⇒ (e) in Theorem 3 of Baum and Katz (1965).

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