Probability Theory and Related Fields (2023) 187:519–522
https://doi.org/10.1007/s00440-023-01215-7

CORRECTION

Correction: CLT in functional linear regression models

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Published online: 23 June 2023

Correction to:
Probab. Theory Relat. Fields (2007) 138:325–361
https://doi.org/10.1007/s00440-006-0025-2

A problem appears in the end of Lemma 8 page 355. The first part of the Lemma remains true but the second weak convergence result for the random predictor is not exact in general. In fact $Z_{i,n}$ is not a martingale difference sequence with respect to the filtration generated by $(X_1, \varepsilon_1, \ldots, X_n, \varepsilon_n)$ and the CLT mentioned in the end of page 355 cannot be invoked.

A new version of the second part of Lemma 8 is given below. An additional assumption denoted (1) below is also required for Theorem 2 to hold. It is satisfied in a wide range of examples and applications.

Lemma 1 *Let $X_i = \sum_{l=1}^{+\infty} \sqrt{\lambda_l}\xi_{l,i} e_l$ be the Karhunen-Loeve expansion of X_i given at page 334. Assume that the sequence of the squared principal component satisfies the weak law of large numbers: when L tends to infinity,*

The original article can be found online at https://doi.org/10.1007/s00440-006-0025-2.

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$$s_L = \frac{1}{L}\sum_{l=1}^{L}\xi_l^2 \overset{\mathbb{P}}{\to} 1 \tag{1}$$

then the second part of Lemma 8 holds namely:

$$\sqrt{\frac{n}{s_n}}\langle R_n, X_{n+1}\rangle \overset{w}{\to} N\left(0, \sigma_\varepsilon^2\right)$$

Remark 1 Assumption (1) holds for Gaussian X and more generally when the principal components ξ_l's are independent.

Proof In order to clarifiy we set below $X_0 = X_{n+1}$, $s_{L,i} = \frac{1}{L}\sum_{l=1}^{L}\xi_{l,i}^2$ and denote $\mathbb{E}_i$ the expectation w.r.t. the couple (X_i, ε_i). The derivation relies on proving classical pointwise convergence for the characteristic function of $S_n = \frac{1}{\sqrt{ns_n}}\sum_{i=1}^{n} Z_{i,n}$ with $Z_{i,n} = \left\langle \Gamma^\dagger X_i, X_0\right\rangle \varepsilon_i$.

We prove the result above in the specific case of PCA-spectral cut then $f_n(x) = 1/x$ for $x \geq \lambda_{k_n}$ (then $s_n = k_n$). The reader will check that it does not alter the generality of the statement.

Then with $\varphi_{S_n}(t) = \mathbb{E}(\exp(itS_n))$

$$\varphi_{S_n}(t) = \mathbb{E}_0\left\{\prod_{j=1}^{n}\mathbb{E}_j \exp\left(\frac{it}{\sqrt{nk_n}}\left\langle\Gamma^\dagger X_j, X_0\right\rangle\varepsilon_j\right)\right\} = \mathbb{E}_0\left\{\mathbb{E}_1\exp\left(\frac{it}{\sqrt{nk_n}}Z_{1,n}\right)\right\}^n.$$

With the above notations on Karhunen-Loeve expansion for X

$$Z_{1,n} = \varepsilon_1\left\langle\Gamma^\dagger X_1, X_0\right\rangle = \varepsilon_1\sum_{l=1}^{k_n}\xi_{l,1}\xi_{l,0},$$

where $\left(\xi_{l,1}\right)_{1\leq l\leq k_n}$ is independent from $\left(\xi_{l,0}\right)_{1\leq l\leq k_n}$. Simple computations give, $\mathbb{E}_1\left[Z_{1,n}\right] = 0$, $\mathbb{E}_1\left[Z_{1,n}^2\right] = \sigma_\varepsilon^2\sum_{l=1}^{k_n}\xi_{l,0}^2$, and Cauchy-Schwarz inequality yields

$$\mathbb{E}_1\left|\left\langle\Gamma^\dagger X_1, X_0\right\rangle\right|^3 \leq \left(\sum_{l=1}^{k_n}\xi_{l,0}^2\right)^{3/2}\mathbb{E}_1\left(\sum_{l=1}^{k_n}\xi_{l,1}^2\right)^{3/2}.$$

Taken from Jensen's inequality, the bound

$$\left(\frac{1}{k_n}\sum_{l=1}^{k_n}\xi_{l,1}^2\right)^{3/2} \leq \frac{1}{k_n}\sum_{l=1}^{k_n}\left|\xi_{l,1}\right|^3$$

leads to $\mathbb{E}_1 \left(\sum_{l=1}^{k_n} \xi_{l,1}^2\right)^{3/2} \leq k_n^{1/2} \sum_{l=1}^{k_n} \mathbb{E}\left|\xi_{l,1}\right|^3 \leq M^{3/4} k_n^{3/2}$ where M appears in assumption $(A.3)$ page 334 and finally to

$$\mathbb{E}_1 \left|\left\langle \Gamma^{\dagger} X_1, X_0 \right\rangle\right|^3 \leq M^{3/4} k_n^{3/2} \left(\sum_{l=1}^{k_n} \xi_{l,0}^2\right)^{3/2}. \tag{2}$$

Then a Taylor expansion for the characteristic function of $\left\langle \Gamma^{\dagger} X_1, X_0 \right\rangle \varepsilon_1$ is

$$\mathbb{E}_1 \exp\left(\frac{it}{\sqrt{nk_n}} Z_{1,n}\right) = 1 - \frac{t^2 \sigma_\varepsilon^2}{2nk_n} \sum_{l=1}^{k_n} \xi_{l,0}^2 - i \frac{t^3}{6\left(nk_n\right)^{3/2}} H_n(t),$$

where $H_n(t) = \mathbb{E}_1\left[\varepsilon_1^3 \left\langle \Gamma^{\dagger} X_1, X_0 \right\rangle^3 \exp\left(i\tau_t \left\langle \Gamma^{\dagger} X_1, X_0 \right\rangle \varepsilon_1\right)\right]$ for some $\tau_t \in \left(0, t/\sqrt{nk_n}\right)$ is a remainder term in the Taylor's expansion of $\varphi_{Z_{1,n}}$. Hence, we get from (2),

$$\left|H_n(t)\right| \leq \mathbb{E}\left|\varepsilon_1\right|^3 \mathbb{E}_1 \left|\left\langle \Gamma^{\dagger} X_1, X_0 \right\rangle\right|^3 \leq \mathbb{E}\left|\varepsilon_1\right|^3 M^{3/4} k_n^{3/2} \left(\sum_{l=1}^{k_n} \xi_{l,0}^2\right)^{3/2}.$$

Remind that $s_{k_n,0} = \frac{1}{k_n} \sum_{l=1}^{k_n} \xi_{l,0}^2$. From the equations above we can write,

$$\mathbb{E}_1 \exp\left(\frac{it}{\sqrt{nk_n}} Z_{1,n}\right) = 1 - \frac{t^2 \sigma_\varepsilon^2}{2n} s_{k_n,0} \left(1 + t k_n^{3/2} \sqrt{\frac{s_{k_n,0}}{n}} \widetilde{H}_n(t)\right), \tag{3}$$

where this time: $\sup_{(n,t)} \left|\widetilde{H}_n(t)\right| \leq M^{3/4} \mathbb{E}\left|\varepsilon_1\right|^3$. Then, with assumption (7) page 334 in Theorem 2, we have that k_n^3/n tends to zero when n tends to infinity so that,

$$\mathbb{E}_1 \exp\left(\frac{it}{\sqrt{nk_n}} \left\langle \Gamma^{\dagger} X_1, X_0 \right\rangle \varepsilon_1\right) = 1 - \frac{t^2 \sigma_\varepsilon^2}{2n} \left(\frac{1}{k_n} \sum_{l=1}^{k_n} \xi_{l,0}^2\right) \left(1 + o_{\mathbb{P}}(1)\right),$$

when $\frac{1}{k_n} \sum_{l=1}^{k_n} \xi_{l,0}^2$ is an $O_{\mathbb{P}}(1)$. In order to conclude we have to take expectation with respect to X_0 and integrate to the limit.

First of all it is plain from (3), the assumption on $\frac{1}{k_n} \sum_{l=1}^{k_n} \xi_{l,0}^2$ and the continuous mapping theorem that

$$\left[\mathbb{E}_1 \exp\left(\frac{it}{\sqrt{nk_n}} \left\langle \Gamma^{\dagger} X_1, X_0 \right\rangle \varepsilon_1\right)\right]^n \underset{n \to +\infty}{\overset{\mathbb{P}}{\to}} \exp\left(-\frac{t^2}{2} \sigma_\varepsilon^2\right).$$

Besides $\left[\mathbb{E}_1 \exp\left(\frac{it}{\sqrt{nk_n}} Z_{1,n}\right)\right]^n \leq 1$ almost surely and is uniformly integrable with respect to $\mathbb{E}_0$. We can conclude that

$$\mathbb{E}_0 \left\{ \mathbb{E}_1 \exp\left(\frac{it}{\sqrt{nk_n}} \left\langle \Gamma^{\dagger} X_1, X_0 \right\rangle \varepsilon_1 \right) \right\}^n \underset{n \to +\infty}{\to} \exp\left(-\frac{t^2}{2} \sigma_{\varepsilon}^2 \right),$$

which concludes the proof of the Lemma. □

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