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Singular HJB equations with applications to KPZ on the real line

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Abstract

This paper is devoted to studying Hamilton-Jacobi-Bellman equations with distribution-valued coefficients, which are not well-defined in the classical sense and are understood by using the paracontrolled distribution method introduced in (Gubinelli et al. in Forum Math Pi 3(6):1, 2015). By a new characterization of weighted Hölder spaces and Zvonkin's transformation we prove some new a priori estimates, and therefore establish the global well-posedness for singular HJB equations. As applications, we obtain global well-posedness in polynomial weighted Hölder spaces for KPZ type equations on the real line, as well as modified KPZ equations for which the Cole–Hopf transformation is not applicable.

Keywords Singular SPDEs · HJB equations · KPZ equations · Paracontrolled distributions · Global well-posedness · Zvonkin's transformation

Mathematics Subject Classification 60H15 · 35R60

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1 Introduction

Recall that the classical Kardar-Parisi-Zhang equation is given as follows:

$$\mathscr{L}h := (\partial_t - \Delta)\, h = (\partial_x h)^2 + \xi, \quad h(0) = h_0, \tag{1.1}$$

where ξ is a Gaussian space-time white noise. This equation was introduced in [36] as a model for the growth of interfaces represented by a height function h. In [36] the authors predicted that under a 1–2–3 scaling the height function must converge to a scale invariant random field called the KPZ fixed point (see [8, 41, 47] and references therein). It is conjectured that the large scale behaviour of a large class of interface growth models is described by the KPZ fixed point. These models are said to belong to the KPZ universality class and this is referred to as the strong KPZ universality conjecture. A weaker form of universality which is now called the weak universality conjecture states that the KPZ equation is itself a universal description of the fluctuations of weakly asymmetric growth models (see e.g. [3, 32, 34] and references therein). The main difficulty in solving Eq. (1.1) comes from the singularity of space-time white noise and the nonlinearity, since $(\partial_x h)^2$ cannot be understood in the classical sense because $\partial_x h$ is not a function. This problem can be avoided by using the Cole–Hopf transformation (see [2, 3, 36] and also [7, 28]). In fact, letting

$w := \mathrm{e}^h$ and formally using Itô's formula, one sees that

$$\mathscr{L}w = w\xi, \quad w(0) = \mathrm{e}^{h_0}, \tag{1.2}$$

which can be understood in Itô's sense ([52]). In [2, 3] the solutions to (1.1) are defined by $\log w$, where w is a positive solution to (1.2), now known as the Cole–Hopf solution. It remained an open problem to clarify in what sense the Cole–Hopf solution genuinely solves the original KPZ equation.

A revolutionary step was made by Hairer [26] using methods from rough path theory. He was able to solve the classical KPZ equation on the torus. Later, Hairer introduced the theory of regularity structures in [27] and Gubinelli, Imkeller and Perkowski proposed the paracontrolled distribution method in [20, 23], which made it possible to study a large class of PDEs driven by singular noises. The key idea of these theories is to use the structure of the solution to give a meaning to the terms which are not classically well-defined. These terms are well-defined with the help of renormalization for the "enhanced noise", i.e. the noise and the higher order terms appearing in the decomposition of the equations. More precisely, $(\partial_x h)^2$ can be formally understood as a subtraction of an infinite correction term: $(\partial_x h)^2 - \infty$.

After this breakthrough, an avalanche of excitement and intriguing results followed, proving local/global existence and uniqueness of solutions to a large class of singular SPDEs, including the generalized parabolic Anderson model, the dynamical Φ^4 model and other interesting examples ([9, 29, 30, 55] and references therein). Very recently, geometric stochastic heat equations with values in a Riemannian manifold M were studied in [4] via regularity structures theory and in [11, 49] by Dirichlet forms, which can be written in local coordinates as generalized coupled KPZ equations (see Sect. 6.1 for more details).

Up to now, most of the well-known works in the field of singular SPDEs are considered with the finite volume case. Since the large scale behavior of the KPZ equation is related to the important KPZ fixed point (see [41] and below), it is natural to consider the KPZ equation on the real line. In fact, new phenomena may occur in the infinite volume setting. For example, in [11] it was shown that solutions to geometric stochastic heat equations exhibit different long-time behavior compared to the finite volume setting (see [49]). In general, space-time white noise in infinite volume stays in weighted Besov spaces, as does the solution. Since these spaces are typically not preserved by the nonlinearity, it obstructs the use of simple fixed point arguments for constructing local solutions. The first attempt to overcome this difficulty was due to Hairer and Labbé [29, 30] for the rough linear heat equation by introducing an exponential weight depending on time. For nonlinear equations, suitable a priori estimates in weighted spaces have been established for the dynamical Φ^4_d model by Mourrat and Weber [43, 44] and Gubinelli and Hofmanová [18], which rely on the damping term $-\phi^3$. In [45] a priori estimates and paracontrolled solutions to the KPZ equation on the real line were obtained by using the Cole–Hopf transformation. Moreover, using the probabilistic notion of energy solutions [21, 22, 24] or studying the associated infinitesimal generator and Kolmogorov equation [25] it is possible to give a meaning to the KPZ equation on $\mathbb{R}$, but this is restricted to initial data which is absolutely continuous w.r.t. the stationary measure. We mention that in [11] martingale solutions were constructed for geometric stochastic heat equations in $\mathbb{R}$ by using the Dirichlet form approach, which relies on an integration by parts formula for the invariant measure.

In the present paper we are concerned with the following KPZ type SPDEs on the real line:

$$\mathscr{L}h = \text{“}(\partial_x h)^2\text{”} + g(h) + \xi, \quad h(0) = h_0, \tag{1.3}$$

$$\mathscr{L}h = G(x)\text{“}(\partial_x h)^2\text{”} + K(x)\partial_x h + \xi, \quad h(0) = h_0, \tag{1.4}$$

where g, G, K are bounded Lipschitz functions, and ξ is a Gaussian space-time white noise on $\mathbb{R}^+ \times \mathbb{R}$. Equations (1.3) and (1.4) are typical examples of singular SPDEs and can be viewed as a simplified version of the generalized KPZ equations and geometric stochastic heat equations in [4, 31]. The emphasis of this article is on deriving an a priori estimate by PDE arguments and complements the local solution theory by ruling out the possibility of finite time blow-up. As directly obtaining global well-posedness to geometric stochastic heat equation by PDE arguments is still an open problem, we study the simplified version (1.3) and (1.4). Note that neither equations can be linearized by the Cole–Hopf transformation.

As mentioned above, suitable a priori estimates and global well-posedness have been established for the dynamical Φ^4_d model by using the strong damping term $-\phi^3$ (see [18, 43, 44]) and for the KPZ equation by the Cole–Hopf transformation (see [25]). The main aim of this paper is to obtain global well-posedness of singular SPDEs on the whole space when the strong damping is not at hand and the Cole–Hopf transformation is not applicable. We obtain global well-posedness of Eqs. (1.3) and (1.4) by suitable a priori estimates. By a renormalization and decomposition procedure, one can reduce KPZ type SPDEs (1.3), (1.4) to the following singular Hamilton-Jacobi-Bellman equation in $\mathbb{R}^d$ (abbreviated as HJB) together with some linear equations (see Sect. 6 for more details):

$$\mathscr{L}u := (\partial_t - \Delta)\, u = H(\nabla u) + b \cdot \nabla u + f, \quad u(0) = \varphi, \tag{1.5}$$

where $H : \mathbb{R}^d \to \mathbb{R}$ is a locally Lipschitz function of at most quadratic growth, and for some $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and $\kappa \in (0, 1)$,

$$b \in L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa), \quad f \in L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa).$$

Here $\rho_\kappa(x) := \langle x\rangle^{-\kappa} := (1 + |x|^2)^{-\kappa/2}$ and $\mathbf{C}^{-\alpha}(\rho_\kappa)$ stands for the weighted Hölder (or Besov) space (see Sect. 2.1). We will first derive global well-posedness of Eq. (1.5) under general assumptions on H (see Sect. 5) and then apply it to Eqs. (1.3) and (1.4) in Sect. 6.

The difficulties that arrive in solving (1.3) and (1.4) also arise in a slightly different form for (1.5). Concerning (1.5), since $b, f \in L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)$ and $\alpha \in (\frac{1}{2}, \frac{2}{3})$, the best regularity space for u is $L^\infty_T \mathbf{C}^{2-\alpha}$ by Schauder's estimate. Compared to (1.3) and (1.4) there is no difficulty defining $H(\nabla u)$ for (1.5) since f is more regular than space-time white noise. However, the transport term $b \cdot \nabla u$ is not well-defined in the classical sense.

We need to use regularity structures theory or paracontrolled distributions to give a meaning to Eq. (1.5). In this paper we use PDE arguments and paracontrolled distributions to obtain the global well-posedness of (1.5). Notice that for general H, we cannot use the Cole–Hopf transformation to transform (1.5) into a linear equation. In that sense our new approach is much more robust than the previous one.

Finally we also mention that the HJB equation appears originally in optimal control theory, whose solution represents the value function of a stochastic optimal control problem (see [17, 37, 53]). More precisely, consider the following stochastic optimal control problem:

$$V(t,x) := \inf_{\gamma} \mathbb{E}\left[\int_t^T L(s, X_s^\gamma(x), \gamma(s))\mathrm{d}s + \psi(X_T^\gamma(x))\right], \tag{1.6}$$

where the infimum is taken over all controls γ in some class of adapted processes, L is the cost function, ψ is the final bequest value, and $X_t^\gamma(x) = X_t^\gamma$ is the state process which solves the following SDE:

$$\mathrm{d}X_t^\gamma = (b(t, X_t^\gamma) + \gamma_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t,\ X_0^\gamma = x,$$

where W is a d-dimensional standard Brownian motion. Let

$$H(t,x,Q) := \inf_{v\in\mathbb{R}^d}(v\cdot Q + L(t,x,v)).$$

By the dynamical programming principle, V solves the following backward HJB equation:

$$\partial_t V + \Delta V + b\cdot\nabla_x V + H(\nabla V) = 0,\ V(T,x) = \psi(x).$$

Moreover, by the verification theorem, the optimal control γ is then given by $\nabla V(t, X_t^*)$, where X_t^* solves the following SDE:

$$\mathrm{d}X_t^* = (b(t, X_t^*) + \nabla V(t, X_t^*))\mathrm{d}t + \sqrt{2}\mathrm{d}W_t,\ X_0^* = x.$$

In particular, the study of singular HJB equations provides a possibility to study the singular stochastic control problem. By singular, we mean that b may be a distribution. Recently, there is some interest in studying the control problem with rough drift b (see [42] and the reference therein). Notice that our conditions on b are automatically satisfied for $b \in L_T^\infty \mathbf{C}^{-\beta}(\rho_\kappa)$ with $\beta \in (0, \frac{1}{2})$. Thus our main results can be applied to the SDEs in [42], which may give applications to the stochastic control problem considered in [42] and the references therein. For more singular $b \in L_T^\infty \mathbf{C}^{-\alpha}(\rho_\kappa)$ with $\alpha \in (\frac{1}{2}, \frac{2}{3})$, it could be viewed as a random environment and our condition allows for spatial white noise in one dimension, which may be derived from averages of a sequence of i.i.d random variables (see [46, Remark 2.2]). We also mention that the solution to the classical KPZ equation can be viewed as a stochastic control problem with singular $b \in L_T^\infty \mathbf{C}^{-\alpha}(\rho_\kappa)$, $\alpha \in (\frac{1}{2}, \frac{2}{3})$ (see [23, 45]), where the solution is interpreted as a value function defined as in (1.6).

1.1 Main results

As mentioned above, we concentrate on (1.5) first and to define $b \cdot \nabla u$ in (1.5) we need to perform renormalizations by probabilistic calculations. It is not the aim of this paper to discuss the renormalization terms as this has been done extensively (see e.g. [23, 26, 45]). For the main result, we suppose that $b \circ \nabla \mathscr{I} b \in L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_{2\kappa})$ and $b \circ \nabla \mathscr{I} f \in L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_{2\kappa})$ are well defined, where $\mathscr{I} := \mathscr{L}^{-1}$, i.e. $(b, f) \in \mathbb{B}^\alpha_T(\rho_\kappa)$ (see Sect. 2.3 and Sect. 2.4), which in general can be realized by a probabilistic calculation (see Sec. 6 for examples). In the following, we are mainly concerned with the analysis of the deterministic system under the above assumptions.

The following result is a special case of the main Theorem 5.1, where a more general condition on the nonlinear term H is given (see Remark 5.2 for examples of H).

Theorem 1.1 *Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and κ be small enough so that $\delta := 2(\frac{9}{2-3\alpha} + 1)\kappa < 1$, $\bar\alpha := \alpha + \kappa^{1/4} \in (\frac{1}{2}, \frac{2}{3})$. Suppose that for some $c > 0$,*

$$|\partial_Q H(Q)| \leq c(1 + |Q|).$$

If $d \geq 2$, we also suppose H has sub-quadratic growth, i.e., for some $\zeta \in [0, 2)$,

$$|H(Q)| \leq c(|Q|^\zeta + 1).$$

Then for any $(b, f) \in \mathbb{B}^\alpha_T(\rho_\kappa)$ and initial value $\varphi \in \mathbf{C}^{1+\alpha+\varepsilon}(\rho_{\varepsilon\delta})$, where $\varepsilon \in (0, 1)$, there exists a unique paracontrolled solution $u \in \mathbb{S}^{2-\bar\alpha}_T(\rho_\eta)$ to the HJB equation (1.5) in the sense of (5.4) and (5.5) below, where $\eta = \eta(\kappa, \alpha, \zeta) < \frac{1-\alpha}{2}$ converges to zero as $\kappa \to 0$.

As the main application, we obtain global well-posedness of (1.3) and (1.4). The regularity of the space-time white noise ξ is more rough than the coefficient f given in (1.5). To apply Theorem 1.1 we need to introduce some random distributions and use the Da Prato-Debussche trick to reduce (1.3) to (1.5) (see e.g. [13]). This is the usual pathwise approach to the KPZ equation (cf. [23, 26, 45]). Let Y and $Y^{\vee}$, $Y^{\tilde\vee}$ be random distributions defined in Sect. 6.

Theorem 1.2 *Let $g : \mathbb{R} \to \mathbb{R}$ be bounded, Lipschitz continuous, and $\kappa > 0$ be small enough, $\delta := 40\kappa < 1$. For $h_0 = Y(0) + \widetilde{h}(0)$ with $\widetilde{h}(0) \in \mathbf{C}^{\frac{3}{2}+\varepsilon+\gamma}(\rho_{\varepsilon\delta})$, where $\varepsilon \in (0, 1)$ and $0 < \gamma < 1/4$, there exists a unique paracontrolled solution to (1.3) in the sense that $h - Y - Y^{\vee} - Y^{\tilde\vee} := \widetilde{h} \in \mathbb{S}^{\frac{3}{2}-\kappa^{1/4}-\gamma}_T(\rho_\eta)$ is the unique paracontrolled solution to (6.4) with $2(\kappa^{1/4} + 80\kappa) < \eta < 1/4$.*

This result improves the weight for the solution space obtained in [45] for $g \equiv 0$ and is proved in Theorem 6.3. As a further application, we also establish global well-posedness of Eq. (1.4), which is presented in Theorem 6.7.

1.2 Sketch of proofs and structure of the paper

In Sect. 2 we first introduce the basic notations and the spaces used throughout the paper. The regularization effect of heat semigroups and paracontrolled calculus are recalled in Sects. 2.2 and 2.3, respectively. The conditions for the coefficient (b, f) are discussed in Sect. 2.4.

The bulk of our argument is contained in Sections 3-5 and we now proceed to explain the strategy. We decompose (1.5) into the following two equations: $u = u_1 + u_2$

$$(\partial_t - \Delta)\, u_1 = b \cdot \nabla u_1 + f, \tag{1.7}$$

$$(\partial_t - \Delta)\, u_2 = b \cdot \nabla u_2 + H(\nabla u_1 + \nabla u_2). \tag{1.8}$$

In Sect. 3 we first establish Schauder's estimate for (1.7) with sublinear weights (see Theorem 3.7). This solves the conjecture proposed in [45, Remark 1.1]. The difficulty to study (1.7) lies in the drift b living in a weighted Besov space, which prevents us from using a fixed point argument in the same space. It is possible to use the technique in [30] to solve the problem, by which the solution stays in a Besov space with exponential weights. This seems not easy to deduce a uniform $L^\infty(\rho_\delta)$ estimate for the solution to (1.8). In Sect. 3 we develop a new technique to establish a sublinear growth bound for the solutions to equation (1.7). The key idea is to use a new characterization of the weighted Hölder space (see Lemma 3.8) to localize the problem with coefficients in unweighted Besov spaces, for which we obtain the Schauder estimate depends polynomially on the norm of the coefficients compared to the exponential dependence by the usual Gronwall type argument. To this end, we add a new damping term λu_1 to (1.7) and use the classical maximum principle. We also mention that Eq. (1.7) on the torus has been studied in [10], where the difficulty with weights does not appear. In a subsequent work [35], we also apply the localization technique developed in this paper to singular kinetic equations.

In Sects. 4 and 5, we study (1.8). Compared to (1.5) the distribution-valued f becomes function-valued. To treat the distribution-valued transport term $b \cdot \nabla u$, we use Zvonkin's transformation to kill the singular part and transform (1.8) into the following general HJB equation (see Sect. 5)

$$\partial_t v = \mathrm{tr}(a \cdot \nabla^2 v) + B \cdot \nabla v + \widetilde{H}(v, \nabla v),\ \ v(0) = \varphi, \tag{1.9}$$

where the matrix $a \in L^\infty_T \mathbf{C}^{1-\alpha}$ is symmetric and uniformly elliptic, $B \in \mathbb{L}^\infty_T(\rho_{\delta_1})$ for some $\delta_1 \in (0, 1]$.

More precisely, assume that $\mathbf{u}$ solves

$$(\partial_t - \Delta + \lambda)\mathbf{u} = b \cdot (\nabla \mathbf{u} + \mathbb{I}). \tag{1.10}$$

If $\Phi(t, x) = x + \mathbf{u}(t, x)$ is a diffeomorphism in the x variable, then $u_2(t, \Phi^{-1}(t, x))$ will solve (1.9). All the coefficients of (1.9) are function-valued with the cost that (1.9) is given in a non-divergence form PDE. This procedure is usually called Zvonkin's transformation, which was originally used for treating SDEs with irregular drifts (see

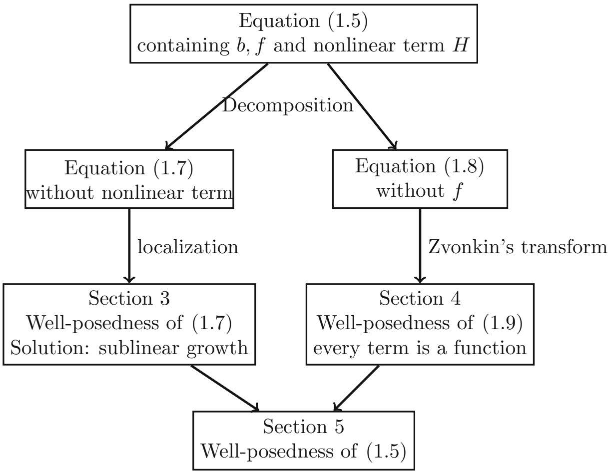

Fig. 1 Steps of solving (1.5)

[56]). However, due to the presence of the weights, this argument needs to be refined. To this end, we use [18, Lemma 2.6] to decompose b into a singular term $b_>$ in the unweighted Besov space and a function-valued term $b_\leq$ with polynomial growth. Then we use Zvonkin's transformation to kill the singular part $b_>$ by subtracting a new term (see Remark 5.4 for more details on this point). The idea comes from Zvonkin's transformation for SDEs, but our Zvonkin's transformation is different from the normal one. To the best of our knowledge, it is the first time to use Zvonkin's transformation to deal with the nonlinear PDE (1.8) with singular drift b.

Section 4 is devoted to the global well-posedness of Eq. (1.9) (see Theorem 4.2). We first establish a maximum principle in Sect. 4.1 with the help of Feymann-Kac's formula. For the subcritical case,[1] the global estimate follows from the $L^\infty(\rho_\delta)$-estimate and the L^p theory of PDEs. For the critical case, the proof is more involved. In this case by taking the spatial derivative on both sides of (1.9), we obtain a divergence PDE, which only holds for $d = 1$. Then the $L^\infty(\rho_\delta)$-bound and energy estimate yield the $\mathbb{H}^{2,p}_T(\rho_\eta)$-estimate of the solution to Eq. (1.9). By using this and Zvonkin's transformation we finally establish a priori global estimates for solutions to (1.8) as well as the well-posedness of (1.5) in Sect. 5.

In the above Fig. 1, we outline the main idea and steps of solving Eq. (1.5).

In Sect. 6 we apply our main result to the KPZ type Eqs. (1.3) and (1.4). Finally, in Appendix 1, we prove the uniqueness of solutions to (1.5) based on the exponential weight approach developed in [30]. Appendix 1 is then devoted to an exponential moment estimate for SDEs used in Sect. 4.

[1] We refer to Sect. 4 for the meaning of subcritical and critical, which is different from the meaning in [27].

1.3 Conventions and notations

Throughout this paper, we use C or c with or without subscripts to denote an unrelated constant, whose value may change in different places. We also use := as a way of definition. By $A \lesssim_C B$ and $A \asymp_C B$ or simply $A \lesssim B$ and $A \asymp B$, we mean that for some constant $C \geq 1$,

$$A \leq CB, \quad C^{-1}B \leq A \leq CB.$$

For convenience, we collect some commonly used notations and definitions below.

$\mathscr{C}^\alpha(\rho)$: weighted Hölder space (Def. 2.3)	$\mathscr{C}^\alpha := \mathscr{C}^\alpha(1)$
$\mathbf{B}^\alpha_{p,q}(\rho)$: weighted Besov space (Def. 2.5)	$\mathbf{B}^\alpha_{p,q} := \mathbf{B}^\alpha_{p,q}(1)$
$\mathbf{C}^\alpha(\rho)$: weighted Hölder-Zygmund space (Def. 2.5)	$\mathbf{C}^\alpha := \mathbf{C}^\alpha(1)$
$\mathbb{S}^\alpha_T(\rho)$: Paracontrolled solution space (2.3)	$\mathbb{S}^\alpha_T := \mathbb{S}^\alpha_T(1)$
$\mathbb{B}^\alpha_T(\rho)$: Space of renormalized pair (Def. 2.14)	$\mathbb{B}^\alpha_T := \mathbb{B}^\alpha_T(1)$
$f \prec g$, $f \succ g$, $f \circ g$: Paraproduct (Sec. 2.3)	$f \succcurlyeq g := f \succ g + f \circ g$
$f \prec\!\!\prec g$: Modified paraproduct (Sec. 2.3)	$\mathscr{L}_\lambda := \partial_t - \Delta + \lambda$
$\mathrm{com}(f, g, h) := (f \prec g) \circ h - f(g \circ h)$ (Sec. 2.3)	$\mathscr{I}_\lambda := (\partial_t - \Delta + \lambda)^{-1}$
$\mathscr{V}_> f$, $\mathscr{V}_\leq f$: Localization operator (Sec. 2.3)	$\mathscr{L} := \mathscr{L}_0$, $\mathscr{I} := \mathscr{I}_0$
$P_t f(x) := (4\pi t)^{-d/2} \int_{\mathbb{R}^d} f(y) \mathrm{e}^{-\lvert x-y\rvert^2/(4t)} \mathrm{d}y$	$B_r := \{x : \lvert x\rvert \leq r\}$
$\mathscr{I}^t_s f(x) := \int_s^t P_{t-r} f(r, x) \mathrm{d}r$	$\langle x\rangle := (1 + \lvert x\rvert^2)^{1/2}$
Commutator: $[\mathscr{A}_1, \mathscr{A}_2] f := \mathscr{A}_1(\mathscr{A}_2 f) - \mathscr{A}_2(\mathscr{A}_1 f)$	$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

2 Preliminaries

2.1 Weighted Besov spaces

In this section we introduce the weighted Besov spaces which will be used in the sequel. Recall the following definition of admissible weight introduced in [51].

Definition 2.1 A C^∞-smooth function $\rho : \mathbb{R}^d \to (0, \infty)$ is called an admissible weight if for each $j \in \mathbb{N}$, there is a constant $C_j > 0$ such that

$$|\nabla^j \rho(x)| \leq C_j \rho(x), \quad \forall x \in \mathbb{R}^d,$$

and for some $C, \beta > 0$,

$$\rho(x) \leq C\rho(y)(1 + |x - y|)^\beta, \quad \forall x, y \in \mathbb{R}^d.$$

The set of all the admissible weights is denoted by $\mathscr{W}$.

Example 2.2 Let $\rho_\delta(x) = \langle x\rangle^{-\delta} = (1+|x|^2)^{-\delta/2}$, where $\delta \in \mathbb{R}$. It is easy to see that $\rho_\delta \in \mathscr{W}$. Such a weight is called polynomial weight.

We introduce the following weighted Hölder space for later use.

Definition 2.3 (Weighted Hölder spaces) Let $\rho \in \mathscr{W}$ and $k \in \mathbb{N}_0$. For $\alpha \in [0, 1)$, we define the weighted Hölder space $\mathscr{C}^{k+\alpha}(\rho)$ by the norm

$$\|f\|_{\mathscr{C}^{k+\alpha}(\rho)} := \sum_{j=0}^{k} \|\nabla^j(\rho f)\|_{L^\infty} + \sup_{x\neq y} \frac{|\nabla^k(\rho f)(x) - \nabla^k(\rho f)(y)|}{|x-y|^\alpha} < \infty.$$

Remark 2.4 Note that the k-order derivative of a function in $\mathscr{C}^k(\rho)$ is not necessarily continuous. By the properties of admissible weights and elementary calculations, it is easy to see that for some $C = C(d, \rho) \geq 1$,

$$\begin{aligned}\|f\|_{\mathscr{C}^{k+\alpha}(\rho)} &\asymp_C \sum_{j=0}^{k} \|\rho\nabla^j f\|_{L^\infty} + \sup_{|x-y|\leq 1} \frac{|(\rho\nabla^k f)(x) - (\rho\nabla^k f)(y)|}{|x-y|^\alpha}\\ &\asymp_C \sum_{j=0}^{k} \|\rho\nabla^j f\|_{L^\infty} + \sup_{|x-y|\leq 1} \frac{\rho(x)|\nabla^k f(x) - \nabla^k f(y)|}{|x-y|^\alpha}.\end{aligned} \tag{2.1}$$

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of Schwartz functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, which is the dual space of $\mathcal{S}(\mathbb{R}^d)$. The Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined through

$$\widehat{f}(z) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\mathrm{e}^{-iz\cdot x}\mathrm{d}x.$$

For $j \geq -1$, let Δ_j be the usual block operator used in the Littlewood-Paley decomposition so that for any $f \in \mathcal{S}'(\mathbb{R}^d)$ (see [1]),

$$\Delta_j f \in \mathcal{S}, \ \ \mathrm{supp}(\widehat{\Delta_j f}) \subset B_{2^{j+2}/3} \setminus B_{2^{j-1}}, \ \ j \in \mathbb{N}_0,$$

and

$$\mathrm{supp}(\widehat{\Delta_{-1} f}) \subset B_1, \ \ f = \sum_{j\geq -1} \Delta_j f.$$

We also introduce the following weighted Besov spaces (cf. [51]):

Definition 2.5 Let $\rho \in \mathscr{W}$ and $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$. The weighted Besov space $\mathbf{B}^\alpha_{p,q}(\rho)$ is defined by

$$\mathbf{B}^\alpha_{p,q}(\rho) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathbf{B}^\alpha_{p,q}(\rho)} := \left(\sum_j 2^{\alpha jq} \|\Delta_j f\|^q_{L^p(\rho)} \right)^{1/q} < \infty \right\},$$

where

$$\|f\|_{L^p(\rho)} := \|\rho f\|_p := \left(\int_{\mathbb{R}^d} |\rho(x)f(x)|^p \mathrm{d}x\right)^{1/p}.$$

The weighted Hölder-Zygmund space is defined by

$$\mathbf{C}^\alpha(\rho) := \mathbf{B}^\alpha_{\infty,\infty}(\rho).$$

Remark 2.6 Let $\rho \in \mathscr{W}$. For any $0 < \beta \notin \mathbb{N}$ and $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, it is well known that (see [51, Theorem 6.5, Theorem 6.9], [1, page99])

$$\|f\|_{\mathbf{C}^\beta(\rho)} \asymp \|f\|_{\mathscr{C}^\beta(\rho)}, \ \|f\|_{\mathbf{B}^\alpha_{p,q}(\rho)} \asymp \|f\rho\|_{\mathbf{B}^\alpha_{p,q}}. \tag{2.2}$$

For $T > 0$, $\alpha \in \mathbb{R}$ and an admissible weight $\rho \in \mathscr{W}$, let $L^\infty_T \mathbf{C}^\alpha(\rho)$ be the space of space-time distributions with finite norm

$$\|f\|_{L^\infty_T \mathbf{C}^\alpha(\rho)} := \sup_{0 \le t \le T} \|f(t)\|_{\mathbf{C}^\alpha(\rho)} < \infty.$$

For $\alpha \in (0, 1)$, we denote by $C^\alpha_T L^\infty(\rho)$ the space of α-Hölder continuous mappings $f : [0, T] \to L^\infty(\rho)$ with finite norm

$$\|f\|_{C^\alpha_T L^\infty(\rho)} := \sup_{0 \le t \le T} \|f(t)\|_{L^\infty(\rho)} + \sup_{0 \le s \ne t \le T} \frac{\|f(t) - f(s)\|_{L^\infty(\rho)}}{|t - s|^\alpha}.$$

The following space will be used frequently: for $\alpha \in (0, 2)$,

$$\mathbb{S}^\alpha_T(\rho) := \Big\{ f : \|f\|_{\mathbb{S}^\alpha_T(\rho)} := \|f\|_{L^\infty_T \mathbf{C}^\alpha(\rho)} + \|f\|_{C^{\alpha/2}_T L^\infty(\rho)} < \infty \Big\}. \tag{2.3}$$

We have the following simple fact (see [45, Lemma 2.11]): for $\alpha \in (0, 1)$,

$$\|\nabla f\|_{\mathbb{S}^\alpha_T(\rho)} \lesssim \|f\|_{\mathbb{S}^{\alpha+1}_T(\rho)}. \tag{2.4}$$

For $p \in [1, \infty]$, $k \in \mathbb{N}_0$ and $T > 0$, we also need the following Sobolev space:

$$\mathbb{H}^{k,p}_T := \Big\{ f : \|f\|_{\mathbb{H}^{k,p}_T} := \|f\|_{\mathbb{L}^p_T} + \|\nabla^k f\|_{\mathbb{L}^p_T} < \infty \Big\},$$

where, with the usual modification when $p = \infty$,

$$\|f\|_{\mathbb{L}^p_T} := \left(\int_0^T \int_{\mathbb{R}^d} |f(t,x)|^p \mathrm{d}x \mathrm{d}t\right)^{\frac{1}{p}}.$$

For an admissible weight ρ, we also introduce the weighted Sobolev space

$$\mathbb{H}^{k,p}_T(\rho) := \Big\{ f : \|f\|_{\mathbb{H}^{k,p}_T(\rho)} := \|f\rho\|_{\mathbb{H}^{k,p}_T} < \infty \Big\},$$

and local space $\mathbb{H}^{k,p}_{\rm loc}$:

$$\mathbb{H}^{k,p}_{\rm loc} := \Big\{ f : f\chi_R \in \mathbb{H}^{k,p}_T, \ \ \forall T, R > 0 \Big\},$$

where $\chi_R(x) = \chi(x/R)$ and $\chi \in C^\infty_c(\mathbb{R}^d)$ with $\chi = 1$ on B_1.

The following interpolation inequality will be used frequently, which is an easy consequence of Hölder's inequality and the corresponding definition. (see [19, Lemma A.3] for a discrete version).

Lemma 2.7 *Let* $\rho \in \mathscr{W}$ *and* $\theta \in [0, 1]$. *Let* $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ *and* $\delta, \delta_1, \delta_2 \in \mathbb{R}$ *satisfy*

$$\delta = \theta\delta_1 + (1-\theta)\delta_2, \ \ \alpha = \theta\alpha_1 + (1-\theta)\alpha_2,$$

and $p, q, p_1, q_1, p_2, q_2 \in [1, \infty]$ *satisfy*

$$\tfrac{1}{p} = \tfrac{\theta}{p_1} + \tfrac{1-\theta}{p_2}, \ \ \tfrac{1}{q} = \tfrac{\theta}{q_1} + \tfrac{1-\theta}{q_2}.$$

Then we have

$$\|f\|_{\mathbf{B}^{\alpha}_{p,q}(\rho^\delta)} \le \|f\|^{\theta}_{\mathbf{B}^{\alpha_1}_{p_1,q_1}(\rho^{\delta_1})} \|f\|^{1-\theta}_{\mathbf{B}^{\alpha_2}_{p_2,q_2}(\rho^{\delta_2})}. \tag{2.5}$$

Moreover, for any $0 < \alpha < \beta < 2$ *with* $\theta = \alpha/\beta$, *we also have*

$$\|f\|_{\mathbb{S}^{\alpha}_T(\rho^\delta)} \lesssim \|f\|^{\theta}_{\mathbb{S}^{\beta}_T(\rho^{\delta_1})} \|f\|^{1-\theta}_{\mathbb{L}^{\infty}_T(\rho^{\delta_2})}. \tag{2.6}$$

2.2 Estimates of Gaussian heat semigroups

We proceed with the Schauder estimate for the heat semigroup. For $t > 0$, let P_t be the Gaussian heat semigroup defined by

$$P_t f(x) := (4\pi t)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)} f(y) \mathrm{d}y.$$

Let ρ be an admissible weight. It is well know that there is a constant $C = C(\rho, d) > 0$ such that (see [43, Lemma 2.10])

$$\|\Delta_j P_t f\|_{L^\infty(\rho)} \lesssim_C \mathrm{e}^{-2^{2j}t} \|\Delta_j f\|_{L^\infty(\rho)}, \ \ j \ge 0, t \ge 0. \tag{2.7}$$

The following lemma provides some quantified estimates for the Gaussian heat semigroup in weighted Hölder spaces.

Lemma 2.8 *Let ρ be an admissible weight and $T > 0$.*

(i) For any $\theta > 0$ and $\alpha \in \mathbb{R}$, there is a constant $C = C(\rho, d, \alpha, \theta, T) > 0$ such that for all $t \in (0, T]$,

$$\|P_t f\|_{\mathbf{C}^{\theta+\alpha}(\rho)} \lesssim_C t^{-\theta/2} \|f\|_{\mathbf{C}^\alpha(\rho)}. \tag{2.8}$$

(ii) For any $m \in \mathbb{N}_0$ and $\theta < m$, there is a constant $C = C(\rho, d, m, \theta, T) > 0$ such that for all $t \in (0, T]$,

$$\|\nabla^m P_t f\|_{L^\infty(\rho)} \lesssim_C t^{(\theta-m)/2} \|f\|_{\mathbf{C}^\theta(\rho)}. \tag{2.9}$$

(iii) For any $0 < \theta < 2$, there is a constant $C = C(\rho, d, \theta, T) > 0$ such that for all $t \in [0, T]$,

$$\|P_t f - f\|_{L^\infty(\rho)} \lesssim_C t^{\theta/2} \|f\|_{\mathbf{C}^\theta(\rho)}. \tag{2.10}$$

Proof (i) By the definition and (2.7), we have

$$\begin{aligned}
\|P_t f\|_{\mathbf{C}^{\theta+\alpha}(\rho)} &= \sup_{j \geq -1} 2^{(\theta+\alpha)j} \|\Delta_j P_t f\|_{L^\infty(\rho)} \\
&\lesssim \sup_{j \geq 0} 2^{(\theta+\alpha)j} \mathrm{e}^{-2^{2j}t} \|\Delta_j f\|_{L^\infty(\rho)} + \|\Delta_{-1} P_t f\|_{L^\infty(\rho)} \\
&\lesssim \sup_{j \geq 0} 2^{\theta j} \mathrm{e}^{-2^{2j}t} \|f\|_{\mathbf{C}^\alpha(\rho)} + \|f\|_{\mathbf{C}^\alpha(\rho)} \lesssim t^{-\theta/2} \|f\|_{\mathbf{C}^\alpha(\rho)}.
\end{aligned}$$

(ii) For $m \in \mathbb{N}_0$ and $\theta < m$, by (2.7) we have

$$\begin{aligned}
\|\nabla^m P_t f\|_{L^\infty(\rho)} &\leq \sum_{j \geq -1} \|\nabla^m \Delta_j P_t f\|_{L^\infty(\rho)} \\
&\lesssim \sum_{j \geq 0} 2^{mj} \mathrm{e}^{-2^{2j}t} \|\Delta_j f\|_{L^\infty(\rho)} + \|\Delta_{-1} f\|_{L^\infty(\rho)} \\
&\lesssim \sum_{j \geq 0} (2^{mj} \mathrm{e}^{-2^{2j}t} 2^{-\theta j}) \|f\|_{\mathbf{C}^\theta(\rho)} + \|f\|_{\mathbf{C}^\theta(\rho)} \lesssim_T t^{(\theta-m)/2} \|f\|_{\mathbf{C}^\theta(\rho)}.
\end{aligned}$$

(iii) By (2.9), we have

$$\|P_t f - f\|_{L^\infty(\rho)} = \left\| \int_0^t \Delta P_s f \mathrm{d}s \right\|_{L^\infty(\rho)} \lesssim \int_0^t s^{-1+\theta/2} \|f\|_{\mathbf{C}^\theta(\rho)} \mathrm{d}s \lesssim t^{\theta/2} \|f\|_{\mathbf{C}^\theta(\rho)}.$$

The proof is complete. □

For given $\lambda \geq 0$ and $f \in L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$, we consider the following heat equation:

$$\mathscr{L}_\lambda u := (\partial_t - \Delta + \lambda) u = f, \quad u(0) = 0.$$

The unique solution to this equation is given by

$$u(t,x)=\int_0^t \mathrm{e}^{-\lambda(t-s)}P_{t-s}f(s,x)\mathrm{d}s =: \mathscr{I}_\lambda f(t,x).$$

In other words, $\mathscr{I}_\lambda$ is the inverse of $\mathscr{L}_\lambda$.

The following Schauder estimate is well known for $q=\infty$ and $\theta=2$ (see [18]). Here we spell out how the implicit constants depend on λ.

Lemma 2.9 (Schauder estimates in weighted spaces) *Let $\rho\in\mathscr{W}$ and*

$$\alpha\in(0,1],\quad \theta\in(\alpha,2].$$

For any $q\in[\frac{2}{2-\theta},\infty]$, $T>0$, there is a constant $C=C(\rho,d,\alpha,\theta,q,T)>0$ such that for all $\lambda\geq 0$ and $f\in L^q_T\mathbf{C}^{-\alpha}(\rho)$,

$$\|\mathscr{I}_\lambda f\|_{\mathbb{S}^{\theta-\alpha}_T(\rho)}\lesssim_C (\lambda\vee 1)^{\frac{\theta}{2}+\frac{1}{q}-1}\|f\|_{L^q_T\mathbf{C}^{-\alpha}(\rho)}. \tag{2.11}$$

Proof Let $q\in[\frac{2}{2-\theta},\infty]$ and $\frac{1}{p}+\frac{1}{q}=1$. For $t\in(0,T]$, by (2.7) and Hölder's inequality, we have for $j\geq 0$

$$\begin{aligned}
2^{j(\theta-\alpha)}\|\Delta_j\mathscr{I}_\lambda f(t)\|_{L^\infty(\rho)} &\lesssim 2^{j(\theta-\alpha)}\int_0^t \mathrm{e}^{-(\lambda+2^{2j})(t-s)}\|\Delta_j f(s)\|_{L^\infty(\rho)}\mathrm{d}s\\
&\lesssim 2^{j\theta}\left(\int_0^t \mathrm{e}^{-p(\lambda+2^{2j})(t-s)}\mathrm{d}s\right)^{\frac{1}{p}}\left(\int_0^t \|f(s)\|^q_{\mathbf{C}^{-\alpha}(\rho)}\mathrm{d}s\right)^{\frac{1}{q}}\\
&\lesssim 2^{j\theta}\left(\int_0^t \mathrm{e}^{-p(\lambda+2^{2j})s}\mathrm{d}s\right)^{\frac{1}{p}}\|f\|_{L^q_T\mathbf{C}^{-\alpha}(\rho)}\\
&\lesssim 2^{j\theta}(2^{2j}+\lambda)^{-\frac{1}{p}}\|f\|_{L^q_T\mathbf{C}^{-\alpha}(\rho)}\lesssim(\lambda\vee 1)^{\frac{\theta}{2}-\frac{1}{p}}\|f\|_{L^q_T\mathbf{C}^{-\alpha}(\rho)},
\end{aligned}$$

and

$$\begin{aligned}
\|\Delta_{-1}\mathscr{I}_\lambda f(t)\|_{L^\infty(\rho)} &\lesssim \int_0^t \mathrm{e}^{-\lambda(t-s)}\|f(s)\|_{\mathbf{C}^{-\alpha}(\rho)}\mathrm{d}s\\
&\lesssim\left(\int_0^t \mathrm{e}^{-\lambda p(t-s)}\mathrm{d}s\right)^{\frac{1}{p}}\|f\|_{L^q_T\mathbf{C}^{-\alpha}(\rho)}\\
&\lesssim(\lambda\vee 1)^{-\frac{1}{p}}\|f\|_{L^q_T\mathbf{C}^{-\alpha}(\rho)},
\end{aligned}$$

which implies by the definition of Besov spaces that

$$\|\mathscr{I}_\lambda f\|_{L^\infty_T\mathbf{C}^{\theta-\alpha}(\rho)}\lesssim_C (\lambda\vee 1)^{\frac{\theta}{2}+\frac{1}{q}-1}\|f\|_{L^q_T\mathbf{C}^{-\alpha}(\rho)}. \tag{2.12}$$

On the other hand, let $u = \mathscr{I}_\lambda f$. For $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned} u(t_2) - u(t_1) &= \int_0^{t_1} (\mathrm{e}^{-\lambda(t_2-s)} - \mathrm{e}^{-\lambda(t_1-s)}) P_{t_2-s} f(s)\mathrm{d}s \\ &\quad + (P_{t_2-t_1} - I)\mathscr{I}_\lambda f(t_1) + \int_{t_1}^{t_2} \mathrm{e}^{-\lambda(t_2-s)} P_{t_2-s} f(s)\mathrm{d}s \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For I_1, by (2.8) and Hölder's inequality, we have

$$\begin{aligned} \|I_1\|_{L^\infty(\rho)} &\le |\mathrm{e}^{-\lambda(t_2-t_1)} - 1| \int_0^{t_1} \mathrm{e}^{-\lambda(t_1-s)} \|P_{t_2-s} f(s)\|_{L^\infty(\rho)} ds \\ &\lesssim \Big((\lambda(t_2-t_1)) \wedge 1\Big) \int_0^{t_1} \mathrm{e}^{-\lambda(t_1-s)} (t_2-s)^{-\frac{\alpha}{2}} \|f(s)\|_{\mathbf{C}^{-\alpha}(\rho)} ds \\ &\le (\lambda(t_2-t_1))^{\frac{\theta}{2}} (t_2-t_1)^{-\frac{\alpha}{2}} \left(\int_0^{t_1} \mathrm{e}^{-\lambda(t_1-s)p} ds\right)^{1/p} \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho)} \\ &\lesssim (t_2-t_1)^{\frac{\theta-\alpha}{2}} \lambda^{\frac{\theta}{2}-\frac{1}{p}} \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho)}. \end{aligned}$$

For I_2, by (2.10) and (2.12) we have

$$\begin{aligned} \|I_2\|_{L^\infty(\rho)} &\lesssim (t_2-t_1)^{\frac{\theta-\alpha}{2}} \|\mathscr{I}_\lambda f\|_{L^\infty_T \mathbf{C}^{\theta-\alpha}(\rho)} \\ &\lesssim (t_2-t_1)^{\frac{\theta-\alpha}{2}} (\lambda \vee 1)^{\frac{\theta}{2}-\frac{1}{p}} \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho)}. \end{aligned}$$

For I_3, by (2.9) and the change of variable, we have

$$\begin{aligned} \|I_3\|_{L^\infty(\rho)} &\lesssim \lambda^{\frac{\alpha}{2}-\frac{1}{p}} \left(\int_0^{\lambda(t_2-t_1)} \mathrm{e}^{-sp} s^{-\frac{\alpha p}{2}} \mathrm{d}s\right)^{\frac{1}{p}} \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho)} \\ &\lesssim (t_2-t_1)^{\frac{\theta-\alpha}{2}} \lambda^{-1+\frac{\theta}{2}+\frac{1}{q}} \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho)}, \end{aligned}$$

where we used $\mathrm{e}^{-sp} s^{-\frac{\alpha p}{2}} \le s^{\frac{(\theta-\alpha)p}{2}-1}$ for all $s > 0$. Therefore,

$$\|\mathscr{I}_\lambda f\|_{C_T^{(\theta-\alpha)/2} L^\infty(\rho)} \lesssim_C (\lambda \vee 1)^{\frac{\theta}{2}+\frac{1}{q}-1} \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho)}, \tag{2.13}$$

which together with (2.12) yields (2.11). □

2.3 Paracontrolled calculus

In this subsection we recall some basic ingredients in the paracontrolled calculus developed by Bony [5] and [20]. The first important fact is that the product fg of two distributions $f \in \mathbf{C}^\alpha$ and $g \in \mathbf{C}^\beta$ is well defined if and only if $\alpha + \beta > 0$. In terms of Littlewood-Paley's block operator Δ_j, the product fg of two distributions f and g can be formally decomposed as

$$fg = f \prec g + f \circ g + f \succ g,$$

where

$$f \prec g = g \succ f := \sum_{j\geq -1}\sum_{i<j-1} \Delta_i f \Delta_j g, \quad f \circ g := \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g.$$

In the following we collect some important estimates from [18] about the paraproducts in weighted Besov spaces, that will be used below.

Lemma 2.10 *Let ρ_1, ρ_2 be two admissible weights. We have for any $\beta \in \mathbb{R}$,*

$$\|f \prec g\|_{\mathbf{C}^\beta(\rho_1\rho_2)} \lesssim \|f\|_{L^\infty(\rho_1)}\|g\|_{\mathbf{C}^\beta(\rho_2)}, \tag{2.14}$$

and for any $\alpha < 0$ and $\beta \in \mathbb{R}$,

$$\|f \prec g\|_{\mathbf{C}^{\alpha+\beta}(\rho_1\rho_2)} \lesssim \|f\|_{\mathbf{C}^\alpha(\rho_1)}\|g\|_{\mathbf{C}^\beta(\rho_2)}. \tag{2.15}$$

Moreover, for any $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$,

$$\|f \circ g\|_{\mathbf{C}^{\alpha+\beta}(\rho_1\rho_2)} \lesssim \|f\|_{\mathbf{C}^\alpha(\rho_1)}\|g\|_{\mathbf{C}^\beta(\rho_2)}. \tag{2.16}$$

In particular, if $\alpha + \beta > 0$, then

$$\|fg\|_{\mathbf{C}^{\alpha\wedge\beta}(\rho_1\rho_2)} \lesssim \|f\|_{\mathbf{C}^\alpha(\rho_1)}\|g\|_{\mathbf{C}^\beta(\rho_2)}. \tag{2.17}$$

Proof See [18, Lemma 2.14]. □

Lemma 2.11 *Let ρ_1, ρ_2, ρ_3 be three admissible weights. For any $\alpha \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$, there exists a bounded trilinear operator* com *on $\mathbf{C}^\alpha(\rho_1) \times \mathbf{C}^\beta(\rho_2) \times \mathbf{C}^\gamma(\rho_3)$ such that*

$$\|\mathrm{com}(f, g, h)\|_{\mathbf{C}^{\alpha+\beta+\gamma}(\rho_1\rho_2\rho_3)} \lesssim \|f\|_{\mathbf{C}^\alpha(\rho_1)}\|g\|_{\mathbf{C}^\beta(\rho_2)}\|h\|_{\mathbf{C}^\gamma(\rho_3)}, \tag{2.18}$$

where for smooth functions f, g, h,

$$\mathrm{com}(f, g, h) := (f \prec g) \circ h - f(g \circ h).$$

Proof See [18, Lemma 2.16]. □

Moreover, we will make use of the time-mollified paraproducts as introduced in [20, Section 5]. Let $Q : \mathbb{R} \to \mathbb{R}_+$ be a smooth function with support in $[-1, 1]$ and

$\int_{\mathbb{R}} Q(s)\mathrm{d}s = 1$. For $T > 0$ and $j \geq -1$, we define an operator $Q_j : L^\infty_T \mathbf{C}^\alpha(\rho) \to L^\infty_T \mathbf{C}^\alpha(\rho)$ by

$$Q_j f(t) := \int_{\mathbb{R}} 2^{2j} Q(2^{2j}(t-s)) f((s \wedge T) \vee 0)\mathrm{d}s,$$

and the modified paraproduct of $f, g \in L^\infty_T \mathbf{C}^\alpha(\rho)$ by

$$f \prec\!\!\prec g := \sum_{j \geq -1} (S_{j-1} Q_j f)\Delta_j g \ \text{ with } S_j f = \sum_{i \leq j-1} \Delta_i f.$$

Note that for $\alpha < 0$, $\beta \in \mathbb{R}$ and $\rho_1, \rho_2 \in \mathscr{W}$,

$$\|f \prec\!\!\prec g\|_{L^\infty_T \mathbf{C}^{\alpha+\beta}(\rho_1\rho_2)} \lesssim \|f\|_{L^\infty_T \mathbf{C}^\alpha(\rho_1)} \|g\|_{L^\infty_T \mathbf{C}^\beta(\rho_2)}. \tag{2.19}$$

Lemma 2.12 *Let ρ_1, ρ_2 be two admissible weights and $T > 0$. For any $\alpha \in (0, 1)$ and $\beta \in \mathbb{R}$, there is a constant $C = C(\rho_1, \rho_2, d, \alpha, \beta) > 0$ such that for all $\lambda \geq 0$*

$$\left\|[\mathscr{L}_\lambda, f \prec\!\!\prec] g\right\|_{L^\infty_T \mathbf{C}^{\alpha+\beta-2}(\rho_1\rho_2)} \lesssim_C \|f\|_{\mathbb{S}^\alpha_T(\rho_1)} \|g\|_{L^\infty_T \mathbf{C}^\beta(\rho_2)}, \tag{2.20}$$

and

$$\|f \prec g - f \prec\!\!\prec g\|_{L^\infty_T \mathbf{C}^{\alpha+\beta}(\rho_1\rho_2)} \lesssim_C \|f\|_{C^{\alpha/2}_T L^\infty(\rho_1)} \|g\|_{L^\infty_T \mathbf{C}^\beta(\rho_2)}. \tag{2.21}$$

Moreover, for any $\varepsilon > 0$, we also have for some $C = C(\varepsilon, \rho_1, \rho_2, d, \alpha, \beta, T)$,

$$\|[\nabla \mathscr{I}_\lambda, f \prec] g\|_{L^\infty_T \mathbf{C}^{\alpha+\beta+1-\varepsilon}(\rho_1\rho_2)} \lesssim_C \|f\|_{\mathbb{S}^\alpha_T(\rho_1)} \|g\|_{L^\infty_T \mathbf{C}^\beta(\rho_2)}. \tag{2.22}$$

Proof The estimates (2.20) and (2.21) can be found in [18, Lemma 2.17]. We only prove (2.22). By definition, we have

$$\begin{aligned}
&[\nabla \mathscr{I}_\lambda, f \prec] g(t) \\
&= \int_0^t \mathrm{e}^{-\lambda(t-s)} P_{t-s} \nabla(f(s) \prec g(s))\mathrm{d}s - f(t) \prec \int_0^t \mathrm{e}^{-\lambda(t-s)} \nabla P_{t-s} g(s)\mathrm{d}s \\
&= \int_0^t \mathrm{e}^{-\lambda(t-s)} P_{t-s}(\nabla f(s) \prec g(s))\mathrm{d}s + \int_0^t \mathrm{e}^{-\lambda(t-s)} [P_{t-s}, f(s) \prec] \nabla g(s)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{-\lambda(t-s)} (f(s) - f(t)) \prec P_{t-s} \nabla g(s)\mathrm{d}s =: I_1(t) + I_2(t) + I_3(t).
\end{aligned}$$

For I_1, by (2.12) with $\theta = 2$ and $q = \infty$ and (2.15), we have

$$\|I_1(t)\|_{L^\infty_T \mathbf{C}^{\alpha+\beta+1}(\rho_1\rho_2)} \lesssim \|\nabla f \prec g\|_{L^\infty_T \mathbf{C}^{\alpha+\beta-1}(\rho_1\rho_2)} \lesssim \|f\|_{L^\infty_T \mathbf{C}^\alpha(\rho_1)} \|g\|_{L^\infty_T \mathbf{C}^\beta(\rho_2)}.$$

For I_2, by a modification of [9, Lemma A.1] we have by $\mathrm{e}^{-\lambda(t-s)} \leq 1$

$$\begin{aligned}\|I_2(t)\|_{\mathbf{C}^{\alpha+\beta+1-\varepsilon}(\rho_1\rho_2)} &\lesssim \int_0^t (t-s)^{-1+\frac{\varepsilon}{2}} \|f(s)\|_{\mathbf{C}^\alpha(\rho_1)} \|g(s)\|_{\mathbf{C}^\beta(\rho_2)} \mathrm{d}s \\ &\lesssim \|f\|_{L_T^\infty \mathbf{C}^\alpha(\rho_1)} \|g\|_{L_T^\infty \mathbf{C}^\beta(\rho_2)}.\end{aligned}$$

For I_3, by (2.14) and (2.8) we have by $\mathrm{e}^{-\lambda(t-s)} \leq 1$

$$\begin{aligned}\|I_3(t)\|_{\mathbf{C}^{\alpha+\beta+1-\varepsilon}(\rho_1\rho_2)} &\lesssim \int_0^t \|f(s)-f(t)\|_{L^\infty(\rho_1)} \|\nabla P_{t-s} g(s)\|_{\mathbf{C}^{\alpha+\beta+1-\varepsilon}(\rho_2)} \mathrm{d}s \\ &\lesssim \|f\|_{C_T^{\alpha/2} L^\infty(\rho_1)} \|g\|_{L_T^\infty \mathbf{C}^\beta(\rho_2)} \int_0^t (t-s)^{-1+\frac{\varepsilon}{2}} \mathrm{d}s.\end{aligned}$$

The proof is complete. □

Finally we recall the localization operators from [18]. Let $\sum_{k\geq -1} w_k = 1$ be a smooth dyadic partition of unity on $\mathbb{R}^d$, where w_{-1} is supported in a ball containing zero and there exists an annulus $\mathscr{A}$ such that for each $k \geq 0$, w_k is supported on the annulus $2^k\mathscr{A}$. Let $(v_m)_{m\geq -1}$ be a smooth dyadic partition of unity on $[0,\infty)$ such that v_{-1} is supported in a ball containing zero and for each $m \geq 0$, v_m is supported on the annulus of size 2^m. For a given sequence $(L_{k,m})_{k,m\geq -1}$, we define localization operators $\mathscr{V}_>$, $\mathscr{V}_\leq$ as in [18]

$$\begin{aligned}\mathscr{V}_> f(t,x) &= \sum_{k,m} w_k(x) v_m(t) \sum_{j>L_{k,m}} \Delta_j f(t,\cdot)(x), \\ \mathscr{V}_\leq f(t,x) &= \sum_{k,m} w_k(x) v_m(t) \sum_{j\leq L_{k,m}} \Delta_j f(t,\cdot)(x).\end{aligned} \tag{2.23}$$

Lemma 2.13 *Let ρ be an admissible weight. For given $L > 0, T > 0$, there exists a (universal) choice of parameters $(L_{k,m})_{k,m\geq -1}$ such that for all $\alpha, \beta, \kappa \in \mathbb{R}$ and $\gamma, \delta > 0$*

$$\begin{aligned}\|\mathscr{V}_> f\|_{L_T^\infty \mathbf{C}^{-\alpha-\delta}(\rho^{\beta-\delta})} &\lesssim 2^{-\delta L} \|f\|_{L_T^\infty \mathbf{C}^{-\alpha}(\rho^\beta)}, \\ \|\mathscr{V}_\leq f\|_{L_T^\infty \mathbf{C}^{\gamma-\alpha}(\rho^{\beta+\gamma})} &\lesssim 2^{\gamma L} \|f\|_{L_T^\infty \mathbf{C}^{-\alpha}(\rho^\beta)},\end{aligned}$$

where the proportional constant depends on $\alpha, \beta, \delta, \gamma$ but is independent of f.

Proof The proof is exactly the same as in [18, Lemma 2.6] although $\alpha > 0$ is required therein. □

2.4 Renormalized pairs

In this section we introduce the renormalized pairs, which correspond to the stochastic objects in the theory of singular SPDEs. Fix $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and an admissible weight

$\rho \in \mathscr{W}$. For $T > 0$, let $b = (b_1, \cdots, b_d)$ and f be $(d+1)$-distributions in $L^\infty_T \mathbf{C}^{-\alpha}(\rho)$. First of all, we introduce two quantities for later use

$$\ell^b_T(\rho) := \sup_{\lambda \geq 0} \|b \circ \nabla \mathscr{I}_\lambda b\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho^2)} + \|b\|^2_{L^\infty_T \mathbf{C}^{-\alpha}(\rho)} + 1, \tag{2.24}$$

and for $q \in [1, \infty]$,

$$\mathbb{A}^{b,f}_{T,q}(\rho) := \sup_{\lambda \geq 0} \|b \circ \nabla \mathscr{I}_\lambda f\|_{L^q_T \mathbf{C}^{1-2\alpha}(\rho^2)} + (\|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho)} + 1)\|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho)}. \tag{2.25}$$

By (2.16), except for $\alpha < \frac{1}{2}$, in general, $b(t) \circ \nabla \mathscr{I}_\lambda f(t)$ is not well-defined since by Schauder's estimate, we only have (see Lemma 2.9)

$$\nabla \mathscr{I}_\lambda f \in L^\infty_T \mathbf{C}^{1-\alpha}(\rho).$$

However, in the probabilistic sense, it is possible to give a meaning for $b \circ \nabla \mathscr{I}_\lambda f$ when b, f belong to the chaos of Gaussian noise (see Sect. 6 below). This motivates us to introduce the following notion.

Definition 2.14 We call the above $(b, f) \in L^\infty_T \mathbf{C}^{-\alpha}(\rho)$ a renormalized pair if there exist $b_n, f_n \in L^\infty_T \mathscr{C}^\infty(\rho)$ with $\sup_{n \in \mathbb{N}} \big(\ell^{b_n}_T(\rho) + \mathbb{A}^{b_n, f_n}_{T,\infty}(\rho)\big) < \infty$ and such that (b_n, f_n) converges to (b, f) in $L^\infty_T \mathbf{C}^{-\alpha}(\rho)$. Moreover, for each $\lambda \geq 0$, there are functions $g_\lambda, h_\lambda \in L^\infty_T \mathbf{C}^{1-2\alpha}(\rho^2)$ such that

$$\lim_{n \to \infty} \|b_n \circ \nabla \mathscr{I}_\lambda f_n - g_\lambda\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho^2)} = 0 \tag{2.26}$$

and

$$\lim_{n \to \infty} \|b_n \circ \nabla \mathscr{I}_\lambda b_n - h_\lambda\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho^2)} = 0. \tag{2.27}$$

For notational convenience, we shall write

$$g_\lambda =: b \circ \nabla \mathscr{I}_\lambda f, \ h_\lambda =: b \circ \nabla \mathscr{I}_\lambda b.$$

The set of all the above renormalized pair is denoted by $\mathbb{B}^\alpha_T(\rho)$.

Remark 2.15 (i) Let $b \in \mathbb{L}^\infty_T(\rho)$ and $f \in L^\infty_T \mathbf{C}^{-\alpha}(\rho)$. Define $b_n(t, x) := b(t, \cdot) * \varrho_n(x)$ and $f_n(t, x) := f(t, \cdot) * \varrho_n(x)$ with ϱ_n being the usual mollifier. By Definition 2.14 and (2.16), it is easy to see that $(b, f) \in \mathbb{B}^\alpha_T(\rho)$. Moreover, if $(b, f) \in \mathbb{B}^\alpha_T(\rho)$ and $b' \in \mathbb{L}^\infty_T(\rho)$, then $(b + b', f) \in \mathbb{B}^\alpha_T(\rho)$.

(ii) To make the convergence in (2.26) and (2.27) hold, we need to subtract some terms containing renormalization constants in the approximation $b_n \circ \nabla \mathscr{I}_\lambda f_n$ and $b_n \circ \nabla \mathscr{I}_\lambda b_n$. In Definition 2.14, we suppose that the renormalization constants are zero for simplicity. Indeed in concrete examples we can choose symmetric mollifiers for approximations, which make the renormalization constants vanish (see Sect. 6). In general we only use the uniform bounds $\sup_{n\in\mathbb{N}}\big(\ell_T^{b_n}(\rho)+\mathbb{A}_{T,\infty}^{b_n,f_n}(\rho)\big)<\infty$ and the convergence in (2.26), (2.27). In particular, the renormalization constants do not affect our analysis and calculations.

An integration by parts allows to eliminate the parameter λ in (2.26) and (2.27) as shown in the following lemma, where the right hand side can be estimated by some probabilistic calculations (see Sect. 6).

Lemma 2.16 *Let $\mathscr{I}_s^t(f)=\int_s^t P_{t-r}f(r)\mathrm{d}r$. For any $t>0$, we have*

$$\sup_{\lambda\geq 0}\|b(t)\circ\nabla\mathscr{I}_\lambda f(t)\|_{\mathbf{C}^{1-2\alpha}(\rho)}\leq 2\sup_{s\in[0,t]}\|b(t)\circ\nabla\mathscr{I}_s^t(f)\|_{\mathbf{C}^{1-2\alpha}(\rho)}. \tag{2.28}$$

Proof Note that by integration by parts formula,

$$\begin{aligned}\int_0^t \mathrm{e}^{-\lambda(t-s)}P_{t-s}f(s)\mathrm{d}s&=\int_0^t P_{t-s}f(s)\mathrm{d}s-\lambda\int_0^t\mathrm{e}^{-\lambda(t-s)}\int_0^s P_{t-r}f(r)\mathrm{d}r\mathrm{d}s\\&=\mathrm{e}^{-\lambda t}\int_0^t P_{t-s}f(s)\mathrm{d}s+\lambda\int_0^t\mathrm{e}^{-\lambda(t-s)}\int_s^t P_{t-r}f(r)\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Thus,

$$b(t)\circ\nabla\mathscr{I}_\lambda f(t)=\mathrm{e}^{-\lambda t}b(t)\circ\nabla\mathscr{I}_0^t f+\lambda\int_0^t\mathrm{e}^{-\lambda(t-s)}b(t)\circ\nabla\mathscr{I}_s^t(f)\mathrm{d}s.$$

From this we get the desired estimate. □

The following lemma is used to deal with the localization of renormalized terms.

Lemma 2.17 *Let $T>0$, $\rho,\bar\rho\in\mathscr{W}$, $\varepsilon\in(0,1)$ and $\alpha\in(\frac{1}{2},\frac{2}{3})$. Suppose that*

$$\phi\in\mathbf{C}^{\alpha+\varepsilon}(\bar\rho\rho^{-2}),\ \ \psi\in\mathbb{S}_T^{\alpha+\varepsilon},\ \ (b,f)\in\mathbb{B}_T^\alpha(\rho).$$

Then there is a constant $C>0$ depending only on $T,\varepsilon,\alpha,d,\rho,\bar\rho$ such that for all $\lambda\geq 0$ and $t\in[0,T]$,

$$\|((b\phi)\circ\nabla\mathscr{I}_\lambda(f\psi))(t)\|_{\mathbf{C}^{1-2\alpha}(\bar\rho)}\lesssim_C\|\phi\|_{\mathbf{C}^{\alpha+\varepsilon}(\bar\rho\rho^{-2})}\|\psi\|_{\mathbb{S}_t^{\alpha+\varepsilon}}\mathbb{A}_{t,\infty}^{b,f}(\rho). \tag{2.29}$$

Proof We only prove the estimate (2.29). For simplicity, we drop the time variable. By using paraproduct, we have

$$\begin{aligned}(b\phi)\circ\nabla\mathscr{I}_\lambda(f\psi)&=(b\phi)\circ\nabla\mathscr{I}_\lambda(\psi\succcurlyeq f)+(b\phi)\circ\nabla\mathscr{I}_\lambda(\psi\prec f)\\&=(b\phi)\circ\nabla\mathscr{I}_\lambda(\psi\succcurlyeq f)+(b\phi)\circ[\nabla\mathscr{I}_\lambda,\psi\prec]f\\&\quad+\mathrm{com}(\psi,\nabla\mathscr{I}_\lambda f,b\phi)+\psi((b\phi)\circ\nabla\mathscr{I}_\lambda f)\\&=(b\phi)\circ\nabla\mathscr{I}_\lambda(\psi\succcurlyeq f)+(b\phi)\circ[\nabla\mathscr{I}_\lambda,\psi\prec]f\end{aligned}$$

$$+ \mathrm{com}(\psi, \nabla \mathscr{I}_\lambda f, b\phi) + \psi((\phi \succcurlyeq b) \circ \nabla \mathscr{I}_\lambda f)$$
$$+ \psi \mathrm{com}(\phi, b, \nabla \mathscr{I}_\lambda f) + \psi\phi(b \circ \nabla \mathscr{I}_\lambda f).$$

Let $\varepsilon > 0$ small enough. We estimate each term as follows:

- By (2.16), (2.11) and (2.15), we have

$$\begin{aligned}\|(b\phi) \circ \nabla \mathscr{I}_\lambda(\psi \succcurlyeq f)\|_{\mathbf{C}^0(\bar\rho)} &\lesssim \|b\phi\|_{\mathbf{C}^{-\alpha}(\bar\rho\rho^{-1})} \|\nabla \mathscr{I}_\lambda(\psi \succcurlyeq f)\|_{L^\infty_t \mathbf{C}^{\alpha+\varepsilon}(\rho)} \\ &\lesssim \|b\phi\|_{\mathbf{C}^{-\alpha}(\bar\rho\rho^{-1})} \|\psi \succ f + \psi \circ f\|_{L^\infty_t \mathbf{C}^{\alpha-1+\varepsilon}(\rho)} \\ &\lesssim \|\phi\|_{\mathbf{C}^{\alpha+\varepsilon}(\bar\rho\rho^{-2})} \|b\|_{\mathbf{C}^{-\alpha}(\rho)} \|f\|_{L^\infty_t \mathbf{C}^{-\alpha}(\rho)} \|\psi\|_{L^\infty_t \mathbf{C}^{\alpha+\varepsilon}}.\end{aligned}$$

- By (2.16), (2.17) and (2.22), we have

$$\begin{aligned}\|(b\phi) \circ [\nabla \mathscr{I}_\lambda, \psi \prec] f\|_{\mathbf{C}^0(\bar\rho)} &\lesssim \|b\phi\|_{\mathbf{C}^{-\alpha}(\bar\rho\rho^{-1})} \|[\nabla \mathscr{I}_\lambda, \psi \prec] f\|_{L^\infty_t \mathbf{C}^{\alpha+\varepsilon}(\rho)} \\ &\lesssim \|\phi\|_{\mathbf{C}^{\alpha+\varepsilon}(\bar\rho\rho^{-2})} \|b\|_{\mathbf{C}^{-\alpha}(\rho)} \|\psi\|_{\mathbb{S}^{2\alpha-1+2\varepsilon}_t} \|f\|_{L^\infty_t \mathbf{C}^{-\alpha}(\rho)}.\end{aligned}$$

- By (2.18), (2.11) and (2.17), we have

$$\begin{aligned}\|\mathrm{com}(\psi, \nabla \mathscr{I}_\lambda f, b\phi)\|_{\mathbf{C}^0(\bar\rho)} &\lesssim \|\psi\|_{\mathbf{C}^{2\alpha-1+\varepsilon}} \|\nabla \mathscr{I}_\lambda f\|_{L^\infty_t \mathbf{C}^{1-\alpha}(\rho)} \|b\phi\|_{\mathbf{C}^{-\alpha}(\bar\rho\rho^{-1})} \\ &\lesssim \|\psi\|_{\mathbf{C}^{2\alpha-1+\varepsilon}} \|f\|_{L^\infty_t \mathbf{C}^{-\alpha}(\rho)} \|b\|_{\mathbf{C}^{-\alpha}(\rho)} \|\phi\|_{\mathbf{C}^{\alpha+\varepsilon}(\bar\rho\rho^{-2})}.\end{aligned}$$

- By (2.17), (2.16), (2.11) and (2.15), we have

$$\begin{aligned}\|\psi((\phi \succcurlyeq b) \circ \nabla \mathscr{I}_\lambda f)\|_{\mathbf{C}^0(\bar\rho)} &\lesssim \|\psi\|_{L^\infty} \|\phi \succcurlyeq b\|_{\mathbf{C}^{\alpha-1+\varepsilon}(\bar\rho\rho^{-1})} \|\nabla \mathscr{I}_\lambda f\|_{\mathbf{C}^{1-\alpha}(\rho)} \\ &\lesssim \|\psi\|_{L^\infty} \|\phi\|_{\mathbf{C}^{\alpha+\varepsilon}(\bar\rho\rho^{-2})} \|b\|_{\mathbf{C}^{-\alpha}(\rho)} \|f\|_{L^\infty_t \mathbf{C}^{-\alpha}(\rho)}.\end{aligned}$$

- By (2.17) and (2.18), we have

$$\|\psi \mathrm{com}(\phi, b, \nabla \mathscr{I}_\lambda f)\|_{\mathbf{C}^0(\bar\rho)} \lesssim \|\psi\|_{L^\infty} \|\phi\|_{\mathbf{C}^{2\alpha-1+\varepsilon}(\bar\rho\rho^{-2})} \|b\|_{\mathbf{C}^{-\alpha}(\rho)} \|f\|_{L^\infty_t \mathbf{C}^{-\alpha}(\rho)}.$$

- By (2.17), we have

$$\|\psi\phi(b \circ \nabla \mathscr{I}_\lambda f)\|_{\mathbf{C}^{1-2\alpha}(\bar\rho)} \lesssim \|\psi\phi\|_{\mathbf{C}^{2\alpha-1+\varepsilon}(\bar\rho\rho^{-2})} \|b \circ \nabla \mathscr{I}_\lambda f\|_{\mathbf{C}^{1-2\alpha}(\rho^2)}.$$

Combining the above calculations, we obtain the desired estimate. □

3 A study of linear parabolic equations in weighted Hölder spaces

In this section we consider the following linear parabolic equation:

$$\mathscr{L}_\lambda u = (\partial_t - \Delta + \lambda) u = b \cdot \nabla u + f, \quad u(0) = u_0 \in \cup_{\epsilon>0} \mathbf{C}^{1+\alpha+\epsilon}, \tag{3.1}$$

where $\lambda \geq 0, b = (b_1, \cdots, b_d)$ is a vector-valued distribution and f is a scalar-valued distribution. Suppose that for some $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and admissible weight $\rho \in \mathscr{W}$,

$$(b, f) \in \mathbb{B}_T^\alpha(\rho), \quad T > 0. \tag{3.2}$$

The aim of this section is to show the well-posedness of PDE (3.1) under (3.2). We first give the definition of the paracontrolled solutions to (3.1). We then establish the Schauder estimate with the coefficients in unweighted Besov spaces by choosing λ large enough. Then by a classical maximum principle, we obtain the Schauder estimate for (3.1) depending polynomially on the coefficients. In Sect. 3.3 we establish global well-posedness of equation (3.1) under (3.2) and obtain a uniform estimate of solution to (3.1) in Besov spaces with sublinear weights, where the proofs are based on a new characterization of weighted Hölder spaces and a localization argument.

3.1 Paracontrolled solutions

To introduce the paracontrolled solution of PDE (3.1), by Bony's decomposition, we make the following paracontrolled ansatz as in [20]:

$$u = \nabla u \lll \mathscr{I}_\lambda b + u^\sharp + \mathscr{I}_\lambda f + \mathrm{e}^{-\lambda t} P_t u_0, \tag{3.3}$$

where $u^\sharp$ solves the following PDE in the weak sense

$$\mathscr{L}_\lambda u^\sharp = \nabla u \prec b - \nabla u \lll b + \nabla u \succ b + b \circ \nabla u - [\mathscr{L}_\lambda, \nabla u \lll] \mathscr{I}_\lambda b \tag{3.4}$$

with $u^\sharp(0) = 0$. Note that $b \circ \nabla u$ is not well-defined in the classical sense. By the paracontrolled ansatz (3.3), we have

$$\begin{aligned} b \circ \nabla u &= b \circ \nabla(\nabla u \lll \mathscr{I}_\lambda b) + b \circ \nabla u^\sharp + b \circ \nabla \mathscr{I}_\lambda f + b \circ \nabla P_t u_0 \mathrm{e}^{-\lambda t} \\ &= b \circ \nabla(\nabla u \prec \mathscr{I}_\lambda b) + \mathrm{com}_1 + b \circ \nabla u^\sharp + b \circ \nabla \mathscr{I}_\lambda f + b \circ \nabla P_t u_0 \mathrm{e}^{-\lambda t} \\ &= b \circ (\nabla^2 u \prec \mathscr{I}_\lambda b) + (b \circ \nabla \mathscr{I}_\lambda b) \cdot \nabla u + \mathrm{com} \\ &\quad + \mathrm{com}_1 + b \circ \nabla u^\sharp + b \circ \nabla \mathscr{I}_\lambda f + b \circ \nabla P_t u_0 \mathrm{e}^{-\lambda t}, \end{aligned} \tag{3.5}$$

where

$$\mathrm{com}_1 := b \circ \nabla[\nabla u \lll \mathscr{I}_\lambda b - \nabla u \prec \mathscr{I}_\lambda b]$$

and

$$\mathrm{com} := \mathrm{com}(\nabla u, \nabla \mathscr{I}_\lambda b, b).$$

Definition 3.1 Let $\rho, \bar{\rho} \in \mathscr{W}$ be two bounded admissible weights and $\varepsilon \geq 0$. For given $(b, f) \in \mathbb{B}^\alpha_T(\rho)$ and $u_0 \in \cup_{\epsilon>0}\mathbf{C}^{1+\alpha+\epsilon}$ for $\epsilon > 0$, with notation (2.3), a pair of functions

$$(u, u^\sharp) \in \mathbb{S}^{2-\alpha}_T(\bar{\rho}) \times \mathbb{S}^{3-2\alpha}_T(\rho^{2+\varepsilon}\bar{\rho}) \tag{3.6}$$

is called a paracontrolled solution of PDE (3.1) if $(u, u^\sharp)$ satisfies (3.3) and (3.4) in the weak sense, where $b \circ \nabla u$ is well-defined by (3.5) and Lemma 3.3 below.

Remark 3.2 For $b, f \in L^\infty_T \mathscr{C}^2(\rho)$ with $\rho(x) = \langle x \rangle^{-1}$, it is well known that PDE (3.1) has a unique classical solution. From Definition 3.1, it is not hard to see that classical solutions are paracontrolled solutions.

The following lemma makes the above definition more transparent.

Lemma 3.3 *Let $T, \varepsilon \geq 0$ and $(u, u^\sharp)$ be a paracontrolled solution of (3.1) in the sense of Definition 3.1. For any $\gamma, \beta \in (\alpha, 2-2\alpha]$, there is a constant $C > 0$ depending only on $T, \varepsilon, \alpha, \gamma, \beta, d, \rho, \bar{\rho}$ such that for all $\lambda \geq 0$ and $t \in [0, T]$,*

$$\begin{aligned}\|(b \circ \nabla u)(t)\|_{\mathbf{C}^{1-2\alpha}(\rho^{2+\varepsilon}\bar{\rho})} \lesssim_C \ell^b_t(\rho)\|u\|_{\mathbb{S}^{\alpha+\gamma}_t(\bar{\rho})} + \sqrt{\ell^b_t(\rho)}\|u^\sharp(t)\|_{\mathbf{C}^{\beta+1}(\rho^{1+\varepsilon}\bar{\rho})} \\ + \|(b \circ \nabla \mathscr{I}_\lambda f)(t)\|_{\mathbf{C}^{1-2\alpha}(\rho^{2+\varepsilon}\bar{\rho})} + \|u_0\|_{\mathbf{C}^{1+\alpha+\epsilon}}.\end{aligned} \tag{3.7}$$

Proof Below we drop the time variable t and fix

$$\gamma, \beta \in (\alpha, 2-2\alpha].$$

Recall $1-2\alpha < 0$. We now estimate each term in (3.5) as follows:

- By (2.16) and (2.9), we have

$$\|b \circ \nabla P_t u_0\|_{\mathbf{C}^{1-2\alpha}(\rho^{2+\varepsilon}\bar{\rho})} \lesssim \|u_0\|_{\mathbf{C}^{1+\alpha+\epsilon}}\|b\|_{\mathbf{C}^{-\alpha}(\rho)}.$$

- Since $\gamma > \alpha$, by (2.15), (2.16) and (2.11), we have

$$\begin{aligned}\|b \circ (\nabla^2 u \prec \mathscr{I}_\lambda b)\|_{\mathbf{C}^{1-2\alpha}(\rho^2\bar{\rho})} &\lesssim \|b\|_{\mathbf{C}^{-\alpha}(\rho)}\|\nabla^2 u \prec \mathscr{I}_\lambda b\|_{\mathbf{C}^\gamma(\rho\bar{\rho})} \\ &\lesssim \|b\|_{\mathbf{C}^{-\alpha}(\rho)}\|\nabla^2 u\|_{\mathbf{C}^{\gamma+\alpha-2}(\bar{\rho})}\|\mathscr{I}_\lambda b\|_{\mathbf{C}^{2-\alpha}(\rho)} \\ &\lesssim \|b\|^2_{L^\infty_t\mathbf{C}^{-\alpha}(\rho)}\|u\|_{\mathbf{C}^{\gamma+\alpha}(\bar{\rho})} \lesssim \ell^b_t(\rho)\|u\|_{\mathbf{C}^{\alpha+\gamma}(\bar{\rho})}.\end{aligned}$$

- By (2.17), we have

$$\begin{aligned}\|\nabla u(b \circ \nabla \mathscr{I}_\lambda b)\|_{\mathbf{C}^{1-2\alpha}(\rho^2\bar{\rho})} &\lesssim \|\nabla u\|_{\mathbf{C}^{\gamma+\alpha-1}(\bar{\rho})}\|b \circ \nabla \mathscr{I}_\lambda b\|_{\mathbf{C}^{1-2\alpha}(\rho^2)} \\ &\lesssim \ell^b_t(\rho)\|u\|_{\mathbf{C}^{\alpha+\gamma}(\bar{\rho})}.\end{aligned}$$

- Since $\gamma > \alpha$, by (2.18) and (2.11), we have

$$\begin{aligned}\|\mathrm{com}\|_{\mathbf{C}^{1-2\alpha}(\rho^2\bar\rho)} &\lesssim \|b\|_{\mathbf{C}^{-\alpha}(\rho)}\|\nabla u\|_{\mathbf{C}^{\gamma+\alpha-1}(\bar\rho)}\|\nabla \mathscr{I}_\lambda b\|_{\mathbf{C}^{1-\alpha}(\rho)}\\ &\lesssim \|b\|^2_{L^\infty_t\mathbf{C}^{-\alpha}(\rho)}\|u\|_{\mathbf{C}^{\gamma+\alpha}(\bar\rho)} \lesssim \ell^b_t(\rho)\|u\|_{\mathbf{C}^{\alpha+\gamma}(\bar\rho)}.\end{aligned}$$

- By Lemma 2.10, (2.4) (2.21) and (2.11), we have

$$\begin{aligned}\|\mathrm{com}_1\|_{\mathbf{C}^{1-2\alpha}(\rho^2\bar\rho)} &\lesssim \|b\|_{\mathbf{C}^{-\alpha}(\rho)}\|\nabla u \lll \mathscr{I}_\lambda b - \nabla u \prec \mathscr{I}_\lambda b\|_{\mathbf{C}^{\gamma+1}(\rho\bar\rho)}\\ &\lesssim \|b\|_{\mathbf{C}^{-\alpha}(\rho)}\|\nabla u\|_{C_t^{(\gamma+\alpha-1)/2}L^\infty(\bar\rho)}\|\mathscr{I}_\lambda b\|_{L^\infty_t\mathbf{C}^{2-\alpha}(\rho)}\\ &\lesssim \|b\|^2_{L^\infty_t\mathbf{C}^{-\alpha}(\rho)}\|u\|_{\mathbb{S}^{\alpha+\gamma}_t(\bar\rho)} \lesssim \ell^b_t(\rho)\|u\|_{\mathbb{S}^{\alpha+\gamma}_t(\bar\rho)}.\end{aligned}$$

- Since $\beta > \alpha$, by (2.16), we have

$$\|b\circ\nabla u^\sharp\|_{L^\infty(\rho^{2+\varepsilon}\bar\rho)} \lesssim \|b\|_{\mathbf{C}^{-\alpha}(\rho)}\|\nabla u^\sharp\|_{\mathbf{C}^\beta(\rho^{1+\varepsilon}\bar\rho)} \le \sqrt{\ell^b_t(\rho)}\|u^\sharp\|_{\mathbf{C}^{\beta+1}(\rho^{1+\varepsilon}\bar\rho)}.$$

Combining the above calculations and by (3.5), we obtain the estimate. □

3.2 Schauder's estimate for paracontrolled solutions without weights

As the first step towards the Schauder estimate for (3.1), we assume that the coefficients are in an unweighted Besov space. More precisely, we assume $(b, f) \in \mathbb{B}^\alpha_T := \mathbb{B}^\alpha_T(1)$, and for simplicity, we shall write

$$\ell^b_T = \ell^b_T(1), \quad \mathbb{A}^{b,f}_{T,q} = \mathbb{A}^{b,f}_{T,q}(1).$$

The proof will be divided into two steps. We first prove a Schauder estimate depending polynomially on the coefficients for λ large enough. Then by a classical maximum principle we extend the result to all $\lambda \ge 0$. The following Schauder estimate is a consequence of Lemmas 2.9 and 3.3.

Lemma 3.4 *Assume $u_0 = 0$. For any $\theta \in (1+\frac{3\alpha}{2}, 2)$, $q \in (\frac{2}{2-\theta}, \infty]$ and $T > 0$, there exist constants $c_0, c_1 > 0$ only depending on θ, α, d, q, T such that for all $\lambda \ge c_0(\ell^b_T)^{1/(1-\frac{\theta}{2}-\frac{1}{q})}$ and any paracontrolled solution $u_\lambda = u$ to PDE (3.1),*

$$\|u_\lambda\|_{\mathbb{S}^{\theta-\alpha}_T} \le c_1\mathbb{A}^{b,f}_{T,q}. \tag{3.8}$$

Moreover, there is a constant $c_2 > 0$ such that for all $\lambda \ge 0$,

$$\|u_\lambda\|_{\mathbb{S}^{2-\alpha}_T} + \|u^\sharp_\lambda\|_{\mathbb{S}^{3-2\alpha}_T} \le c_2(\ell^b_T)^{\frac{4}{2-3\alpha}}\Big(\|u_\lambda\|_{\mathbb{L}^\infty_T} + \mathbb{A}^{b,f}_{T,\infty}\Big). \tag{3.9}$$

Proof Below we fix

$$\theta \in (1+\tfrac{3\alpha}{2}, 2], \quad q \in [\tfrac{2}{2-\theta}, \infty], \quad \gamma, \beta \in (\alpha, \theta - 2\alpha].$$

By (2.11), (2.15) and (2.14), we clearly have

$$
\begin{aligned}
(\lambda \vee 1)^{1-\frac{\theta}{2}-\frac{1}{q}} \|u\|_{\mathbb{S}_T^{\theta-\alpha}} &\lesssim \|b \prec \nabla u + b \succ \nabla u + b \circ \nabla u + f\|_{L_T^q \mathbf{C}^{-\alpha}} \\
&\lesssim \|b\|_{L_T^\infty \mathbf{C}^{-\alpha}} \|\nabla u\|_{L_T^q L^\infty} + \|b \circ \nabla u\|_{L_T^q \mathbf{C}^{-\alpha}} + \mathbb{A}_{T,q}^{b,f},
\end{aligned}
\tag{3.10}
$$

and by Lemma 2.12,

$$
\begin{aligned}
(\lambda \vee 1)^{1-\frac{\theta}{2}-\frac{1}{q}} \|u^\sharp\|_{\mathbb{S}_T^{\theta+\gamma-1}} &\lesssim \|\nabla u \prec b - \nabla u \prec\!\!\prec b\|_{L_T^\infty \mathbf{C}^{\gamma-1}} + \|\nabla u \succ b\|_{L_T^\infty \mathbf{C}^{\gamma-1}} \\
&\quad + \|[\mathscr{L}_\lambda, \nabla u \prec\!\!\prec] \mathscr{I}_\lambda b\|_{L_T^\infty \mathbf{C}^{\gamma-1}} + \|b \circ \nabla u\|_{L_T^q \mathbf{C}^{\gamma-1}} \\
&\lesssim \|u\|_{\mathbb{S}_T^{\gamma+\alpha}} \|b\|_{L_T^\infty \mathbf{C}^{-\alpha}} + \|b \circ \nabla u\|_{L_T^q \mathbf{C}^{1-2\alpha}},
\end{aligned}
$$

where we used (2.4), (2.21), (2.22) and (2.15) in the second inequality. Moreover, by (3.7), we also have

$$
\|b \circ \nabla u\|_{L_T^q \mathbf{C}^{1-2\alpha}} \lesssim \ell_T^b \|u\|_{\mathbb{S}_T^{\gamma+\alpha}} + \sqrt{\ell_T^b} \|u^\sharp\|_{L_T^q \mathbf{C}^{\beta+1}} + \mathbb{A}_{T,q}^{b,f}.
$$

Thus, we obtain that for all $\lambda \geq 0$,

$$
\begin{aligned}
&(\lambda \vee 1)^{1-\frac{\theta}{2}-\frac{1}{q}} \left(\|u\|_{\mathbb{S}_T^{\theta-\alpha}} + \|u^\sharp\|_{\mathbb{S}_T^{\theta+\gamma-1}} \right) \\
&\lesssim \ell_T^b \|u\|_{\mathbb{S}_T^{\gamma+\alpha}} + \sqrt{\ell_T^b} \|u^\sharp\|_{L_T^\infty \mathbf{C}^{\beta+1}} + \mathbb{A}_{T,q}^{b,f}.
\end{aligned}
\tag{3.11}
$$

In particular, letting $\gamma = \theta - 2\alpha$ and $\beta = 2\theta - 2\alpha - 2$, we get for some $c = c(\theta, \alpha, d, q, T)$,

$$
(\lambda \vee 1)^{1-\frac{\theta}{2}-\frac{1}{q}} \left(\|u\|_{\mathbb{S}_T^{\theta-\alpha}} + \|u^\sharp\|_{\mathbb{S}_T^{2\theta-2\alpha-1}} \right) \lesssim_c \ell_T^b \left(\|u\|_{\mathbb{S}_T^{\theta-\alpha}} + \|u^\sharp\|_{\mathbb{S}_T^{2\theta-2\alpha-1}} \right) + \mathbb{A}_{T,q}^{b,f}.
$$

Choosing λ such that $\lambda^{1-\frac{\theta}{2}-\frac{1}{q}} \geq c\ell_T^b$, we obtain (3.8).

On the other hand, letting $\theta = 2$ and $q = \infty$ in (3.11), we obtain that for any $\gamma, \beta \in (\alpha, 2 - 2\alpha]$,

$$
\|u\|_{\mathbb{S}_T^{2-\alpha}} + \|u^\sharp\|_{\mathbb{S}_T^{1+\gamma}} \lesssim \ell_T^b \|u\|_{\mathbb{S}_T^{\gamma+\alpha}} + \sqrt{\ell_T^b} \|u^\sharp\|_{L_T^\infty \mathbf{C}^{\beta+1}} + \mathbb{A}_{T,\infty}^{b,f}.
\tag{3.12}
$$

If $\alpha < \beta < \gamma < 2 - 2\alpha$, then by (2.6) and Young's inequality, we have for any $\varepsilon \in (0, 1)$,

$$
\begin{aligned}
\|u\|_{\mathbb{S}_T^{2-\alpha}} + \|u^\sharp\|_{\mathbb{S}_T^{1+\gamma}} &\leq \varepsilon \left(\|u\|_{\mathbb{S}_T^{2-\alpha}} + \|u^\sharp\|_{\mathbb{S}_T^{1+\gamma}} \right) + C_\varepsilon (\ell_T^b)^{\frac{2-\alpha}{2-\gamma-2\alpha}} \|u\|_{\mathbb{L}_T^\infty} \\
&\quad + C_\varepsilon (\ell_T^b)^{\frac{1+\gamma}{2(\gamma-\beta)}} \|u^\sharp\|_{\mathbb{L}_T^\infty} + C\mathbb{A}_{T,\infty}^{b,f}.
\end{aligned}
\tag{3.13}
$$

Note that by (3.3),

$$\|u^\sharp\|_{\mathbb{L}^\infty_T} = \|u - \nabla u \prec\!\!\prec \mathscr{I}_\lambda b - \mathscr{I}_\lambda f\|_{\mathbb{L}^\infty_T}$$
$$\lesssim \|u\|_{\mathbb{L}^\infty_T}(1 + \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}}) + \|f\|_{L^\infty_T \mathbf{C}^{-\alpha}} \lesssim \|u\|_{\mathbb{L}^\infty_T}\sqrt{\ell^b_T} + \mathbb{A}^{b,f}_{T,\infty}.$$

Substituting it into (3.13) and taking $\varepsilon = 1/2$, we obtain

$$\|u\|_{\mathbb{S}^{2-\alpha}_T} + \|u^\sharp\|_{\mathbb{S}^{1+\gamma}_T} \lesssim (\ell^b_T)^{\frac{2-\alpha}{2-\gamma-2\alpha} \vee (\frac{1+\gamma}{2(\gamma-\beta)} + \frac{1}{2})} \Big(\|u\|_{\mathbb{L}^\infty_T} + \mathbb{A}^{b,f}_{T,\infty}\Big),$$

which, by choosing $\gamma = 2/3$ and β close to α, yields that

$$\|u\|_{\mathbb{S}^{2-\alpha}_T} + \|u^\sharp\|_{\mathbb{S}^{5/3}_T} \lesssim (\ell^b_T)^{\frac{8-3\alpha}{2(2-3\alpha)}} \Big(\|u\|_{\mathbb{L}^\infty_T} + \mathbb{A}^{b,f}_{T,\infty}\Big).$$

Moreover, by (3.12) with $\gamma = 2 - 2\alpha$ and $\beta = 2/3$, we get

$$\|u^\sharp\|_{\mathbb{S}^{3-2\alpha}_T} \lesssim \ell^b_T \|u\|_{\mathbb{S}^{2-\alpha}_T} + \sqrt{\ell^b_T}\|u^\sharp\|_{\mathbb{S}^{5/3}_T} + \mathbb{A}^{b,f}_{T,\infty} \lesssim (\ell^b_T)^{\frac{4}{2-3\alpha}} \Big(\|u\|_{\mathbb{L}^\infty_T} + \mathbb{A}^{b,f}_{T,\infty}\Big).$$

The proof is complete. □

Now we can show the main result of this section, where the key point is to obtain an estimate depending polynomially on the quantity ℓ^b_T. Note that a simple Gronwall argument will lead to the exponential dependence on ℓ^b_T.

Theorem 3.5 *Let $T > 0$ and $u_0 = 0$. For any $(b, f) \in \mathbb{B}^\alpha_T$, there is a unique paracontrolled solution $u_\lambda = u$ to PDE (3.1) in the sense of Definition 3.1. Moreover, there exist $q = q(\alpha) > 1$ and a constant $c_3 = c_3(\alpha, d, T) > 0$ such that for all $\lambda \geq 0$,*

$$\|u_\lambda\|_{\mathbb{L}^\infty_T} \leq c_3 (\ell^b_T)^{\frac{5}{2-3\alpha}} \mathbb{A}^{b,f}_{T,q},$$
$$\|u_\lambda\|_{\mathbb{S}^{2-\alpha}_T} + \|u^\sharp_\lambda\|_{\mathbb{S}^{3-2\alpha}_T} \leq c_3 (\ell^b_T)^{\frac{9}{2-3\alpha}} \mathbb{A}^{b,f}_{T,\infty}.$$

Proof We first assume that

$$b, f \in L^\infty_T \mathscr{C}^2, \quad \forall T > 0.$$

Fix $\lambda \geq 0$. For any $\lambda' > 0$, it is well known that there is a unique classical solution w to the following PDE:

$$\partial_t w = \Delta w - (\lambda' + \lambda) w + b \cdot \nabla w + f, \quad w(0) = 0. \tag{3.14}$$

In particular, for any $\theta \in (1 + \frac{3}{2}\alpha, 2)$ and $q \in (\frac{2}{2-\theta}, \infty]$, by (3.8), we have for $\lambda' \geq c_0 (\ell^b_T)^{1/(1-\frac{\theta}{2}-\frac{1}{q})}$,

$$\|w\|_{\mathbb{L}^\infty_T} \leq \|w\|_{L^\infty_T \mathbf{C}^{\theta-\alpha}} \leq c_1 \cdot \mathbb{A}^{b,f}_{T,q}.$$

Now let u be the unique classical solution to PDE (3.1) with $u_0 = 0$. Let $\bar{u} = u - w$. Then $\bar{u}$ solves the following PDE:

$$\partial_t \bar{u} = \Delta \bar{u} - \lambda \bar{u} + b \cdot \nabla \bar{u} + \lambda' w, \quad \bar{u}(0) = 0.$$

By the classical maximum principle, we have

$$\|\bar{u}\|_{\mathbb{L}^\infty_T} \leq \lambda' T \|w\|_{\mathbb{L}^\infty_T}.$$

Hence, by taking θ close to $1 + \frac{3\alpha}{2}$ and q large enough, we obtain

$$\|u\|_{\mathbb{L}^\infty_T} \leq (\lambda' T + 1)\|w\|_{\mathbb{L}^\infty_T} \leq (c_0 T (\ell^b_T)^{1/(1-\frac{\theta}{2}-\frac{1}{q})} + 1)c_1 \cdot \mathbb{A}^{b,f}_{T,q} \lesssim (\ell^b_T)^{\frac{5}{2-3\alpha}} \cdot \mathbb{A}^{b,f}_{T,q},$$

which together with (3.9) yields

$$\|u\|_{\mathbb{S}^{2-\alpha}_T} + \|u^\sharp\|_{\mathbb{S}^{3-2\alpha}_T} \lesssim (\ell^b_T)^{\frac{9}{2-3\alpha}} \mathbb{A}^{b,f}_{T,\infty}. \tag{3.15}$$

(Existence) Let b_n and f_n be the smoothing approximations of b and f in $\mathbb{B}^\alpha_T$. We consider the following approximation equation:

$$\partial_t u_n = \Delta u_n - \lambda u_n + b_n \cdot \nabla u_n + f_n, \quad u_n(0) = 0.$$

By the assumption and (3.15), we have the following uniform estimate:

$$\sup_{n\in\mathbb{N}} \left(\|u_n\|_{\mathbb{S}^{2-\alpha}_T} + \|u^\sharp_n\|_{\mathbb{S}^{3-2\alpha}_T} \right) \lesssim 1.$$

Using this uniform estimate and by a standard compact and weak convergence method, we can show the existence of a paracontrolled solution (see [18]).

(Uniqueness) Let u_1 and u_2 be two paracontrolled solution of PDE (3.1). Let $\bar{u} := u_1 - u_2$. Clearly, $\bar{u}$ is a paracontrolled solution of

$$\partial_t \bar{u} = \Delta \bar{u} - \lambda \bar{u} + b \cdot \nabla \bar{u}, \quad u(0) = 0.$$

Let $\theta \in (1 + \alpha, 2)$ and $q = \frac{2}{2-\theta}$. By (2.11), we have

$$\|\bar{u}\|^q_{\mathbb{S}^{\theta-\alpha}_T} \leq C \int_0^T \|(b \cdot \nabla \bar{u})(t)\|^q_{\mathbf{C}^{-\alpha}} \mathrm{d}t. \tag{3.16}$$

On the other hand, by (2.14), (2.15) and Lemma 3.3 we have

$$\begin{aligned} \|(b \cdot \nabla \bar{u})(t)\|_{\mathbf{C}^{-\alpha}} &\leq \|(b \prec \nabla \bar{u})(t)\|_{\mathbf{C}^{-\alpha}} + \|(b \succ \nabla \bar{u})(t)\|_{\mathbf{C}^{-\alpha}} + \|(b \circ \nabla \bar{u})(t)\|_{\mathbf{C}^{-\alpha}} \\ &\lesssim \|b(t)\|_{\mathbf{C}^{-\alpha}} \|\nabla \bar{u}(t)\|_{L^\infty} + \|(b \circ \nabla \bar{u})(t)\|_{\mathbf{C}^{1-2\alpha}} \end{aligned}$$

$$\lesssim \|\nabla\bar{u}(t)\|_{L^\infty} + \|\bar{u}\|_{\mathbb{S}^{2-\alpha}_t} + \|\bar{u}^\sharp\|_{L^\infty_t\mathbf{C}^{3-2\alpha}} \overset{(3.9)}{\lesssim} \|\nabla\bar{u}\|_{\mathbb{L}^\infty_t} + \|\bar{u}\|_{\mathbb{L}^\infty_t}.$$

Substituting this into (3.16) and by $\theta - \alpha > 1$, we obtain

$$\|\bar{u}\|^q_{L^\infty_T\mathbf{C}^{\theta-\alpha}} \leq C\int_0^T \|\bar{u}\|^q_{L^\infty_t\mathbf{C}^{\theta-\alpha}}\mathrm{d}t,$$

which in turn implies that $\bar{u} = 0$. The uniqueness is proven. □

Remark 3.6 The polynomial dependence on ℓ^b_T in Theorem 3.5 together with a new characterization for weighted Hölder spaces in Lemma 3.8 below shall be used to establish the Schauder estimate in sublinear weighted Hölder spaces (see [45, Remark 1.1]).

3.3 Schauder's estimate for paracontrolled solutions with weights

In this section we show the well-posedness of PDE (3.1) in weighted Hölder spaces. Recall that for $\delta \in \mathbb{R}$,

$$\rho_\delta(x) := (1 + |x|^2)^{-\delta/2} =: \langle x\rangle^{-\delta}.$$

Now we give the main result of this section.

Theorem 3.7 *Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and $\vartheta := \frac{9}{2-3\alpha}$. Choose $\kappa > 0$ so that*

$$\delta := (2\vartheta + 2)\kappa \leq 1, \quad \delta_0 := (\frac{55}{27}\vartheta + 4)\kappa.$$

For any $T > 0$ and $\lambda \geq 0$, $(b, f) \in \mathbb{B}^\alpha_T(\rho_\kappa)$ and $u_0 \in \cup_{\epsilon>0}\mathbf{C}^{1+\alpha+\epsilon}$, there exists a unique paracontrolled solution $(u, u^\sharp)$ to PDE (3.1) in the sense of Definition 3.1 with

$$\|u\|_{\mathbb{S}^{2-\alpha}_T(\rho_\delta)} + \|u^\sharp\|_{\mathbb{S}^{3-2\alpha}_T(\rho_{\delta_0})} \lesssim_C \mathbb{A}^{b,f}_{T,\infty}(\rho_\kappa), \tag{3.17}$$

where $C = C(T, d, \alpha, \kappa, \ell^b_T(\rho_\kappa)) > 0$.

To prove the result we first prove a characterization of weighted Hölder spaces. To this end, we introduce the following notations. Let $\chi \in C^\infty_c(\mathbb{R}^d)$ with

$$\chi(x) = 1, \quad |x| \leq 1/8, \quad \chi(x) = 0, \quad |x| > 1/4,$$

and for $r > 0$ and $z \in \mathbb{R}^d$,

$$\chi^z_r(x) := \chi((x - z)/r), \quad \phi^z_r(x) := \chi^z_{r(1+|z|)}(x).$$

To show the existence of a paracontrolled solution, we need the following simple characterization of weighted Hölder spaces.

Lemma 3.8 *Let $\alpha \geq 0$ and $r \in (0, 1]$. For any $\delta, \kappa \in \mathbb{R}$, there is a constant $C = C(r, \alpha, d, \delta, \kappa) > 0$ such that*

$$\|f\|_{\mathscr{C}^\alpha(\rho_\delta\rho_\kappa)} \asymp_C \sup_z \left(\rho_\delta(z)\|\phi_r^z f\|_{\mathscr{C}^\alpha(\rho_\kappa)}\right). \tag{3.18}$$

Moreover, for any $m \in \mathbb{N}_0$, we also have

$$\sup_z \|\nabla\phi_r^z\|_{\mathscr{C}^m(\rho_1^{-1})} < \infty. \tag{3.19}$$

Proof Without loss of generality, we may assume $\kappa = 0$ by noting that

$$\sup_z \left(\rho_\delta(z)\|\phi_r^z f\|_{\mathscr{C}^\alpha(\rho_\kappa)}\right) \asymp \sup_z \left(\rho_\delta(z)\|\phi_r^z \rho_\kappa f\|_{\mathscr{C}^\alpha}\right) \asymp \|\rho_\delta\rho_\kappa f\|_{\mathscr{C}^\alpha}.$$

First of all, for fixed $r \in (0, 1]$, $\delta \in \mathbb{R}$ and any $m \in \mathbb{N}$, we have for some $C = C(m, r, \delta, d) > 0$,

$$\|\rho_\delta\phi_r^z\|_{\mathscr{C}^m} \lesssim_C \rho_\delta(z), \quad \forall z \in \mathbb{R}^d. \tag{3.20}$$

Indeed, let $B_r(z)$ be the ball with radius r centered at z. Noting that for any $\delta \in \mathbb{R}$ and $x \in B_{(1+|z|)/2}(z)$,

$$\rho_\delta(x) \leq 2^{|\delta|}(1 + |x|)^{-\delta} \leq 4^{|\delta|}(1 + |z|)^{-\delta} = 4^{|\delta|}\rho_\delta(z), \tag{3.21}$$

we have by definition and the chain rule,

$$\begin{aligned}\|\rho_\delta\phi_r^z\|_{\mathscr{C}^m} &= \sum_{k=0}^m \|\nabla^k(\rho_\delta\phi_r^z)\|_{L^\infty} \lesssim \sum_{k=0}^m\sum_{j=0}^k \|\nabla^{k-j}\rho_\delta\nabla^j\phi_r^z\|_{L^\infty} \\ &\lesssim \sum_{k=0}^m\sum_{j=0}^k \|\rho_\delta\nabla^j\phi_r^z\|_{L^\infty} \leq 4^{|\delta|}\sum_{k=0}^m\sum_{j=0}^k \rho_\delta(z)\|\nabla^j\phi_r^z\|_{L^\infty} \lesssim \rho_\delta(z).\end{aligned} \tag{3.22}$$

Moreover, since the definition of $\mathscr{C}^\alpha$ is local,

$$\|f\|_{\mathscr{C}^\alpha} \lesssim \sup_z \|f\|_{\mathscr{C}^\alpha(B_{r/16}(z))} \lesssim \sup_z \|\chi_{r/2}^z f\|_{\mathscr{C}^\alpha}.$$

Thus by (3.20) and $\chi_{r/2}^z\phi_r^z = \chi_{r/2}^z$, we have

$$\begin{aligned}\|f\|_{\mathscr{C}^\alpha(\rho_\delta)} &= \|\rho_\delta f\|_{\mathscr{C}^\alpha} \lesssim \sup_z \|\chi_{r/2}^z\rho_\delta f\|_{\mathscr{C}^\alpha} = \sup_z \|\chi_{r/2}^z\phi_r^z\rho_\delta\phi_r^z f\|_{\mathscr{C}^\alpha} \\ &\lesssim \sup_z \left(\|\chi_{r/2}^z\phi_r^z\rho_\delta\|_{\mathscr{C}^{[\alpha]+1}}\|\phi_r^z f\|_{\mathscr{C}^\alpha}\right) \lesssim \sup_z \left(\rho_\delta(z)\|\phi_r^z f\|_{\mathscr{C}^\alpha}\right),\end{aligned}$$

and

$$\sup_z \big(\rho_\delta(z)\|\phi_r^z f\|_{\mathscr{C}^\alpha}\big) \lesssim \sup_z \Big(\rho_\delta(z)\|\phi_r^z \rho_\delta^{-1}\|_{\mathscr{C}^{[\alpha]+1}}\|\rho_\delta f\|_{\mathscr{C}^\alpha}\Big) \lesssim \|\rho_\delta f\|_{\mathscr{C}^\alpha}.$$

So, (3.18) is proven. On the other hand, note that for any $m \in \mathbb{N}_0$,

$$\|\nabla \phi_r^z\|_{\mathscr{C}^m} \lesssim (1+|z|)^{-1}.$$

As in (3.22), by (3.21) we have (3.19). □

Remark 3.9 Estimate (3.19) provides an extra weight ρ_1 and helps us to obtain the a priori estimate for the solutions in Besov spaces with polynomial weights.

Now we give the proof of Theorem 3.7.

Proof of Theorem 3.7 (Existence). Without loss of generality we may assume $\lambda = 0$ and $u_0 = 0$. In fact, for general initial data $u_0 \in \cup_{\epsilon>0}\mathbf{C}^{1+\alpha+\epsilon}$, by considering $\bar{u} = u - u_0$, we can reduce the nonzero initial value to zero initial value with f replaced by $\bar{f} = f + \Delta u_0 + b\cdot\nabla u_0 \in \mathbf{C}^{-\alpha}(\rho_\kappa)$. In this case, by Lemma 2.10,

$$\|b\circ\nabla\mathscr{I}(\Delta u_0)\|_{L^\infty_T\mathbf{C}^\epsilon(\rho_\kappa)} \lesssim 1,$$

and by Lemma 2.17 with $\psi = \nabla u_0$, $f = b$, $\phi = 1$, $\bar\rho = \rho_{2\kappa}$, $\rho = \rho_\kappa$,

$$\|b\circ\nabla\mathscr{I}(b\cdot\nabla u_0)\|_{L^\infty_T\mathbf{C}^{1-2\alpha}(\rho_{2\kappa})} \lesssim 1.$$

Hence, we still have

$$(b, \bar f) \in \mathbb{B}^\alpha_T(\rho_\kappa).$$

Now, let $T > 0$ and $b_n, f_n \in L^\infty_T\mathscr{C}^\infty(\rho_\kappa)$ be as in the definition of $\mathbb{B}^\alpha_T(\rho_\kappa)$. For every n, define

$$\bar b_n(t,x) := b_n(t,x)\chi_n(x), \quad \bar f_n(t,x) := f_n(t,x)\chi_n(x),$$

where $\chi_n(x) = \chi(x/n)$ and $\chi \in C_c^\infty(\mathbb{R}^d)$ equals to 1 on B_1. It is well known that there is a unique classical solution $u_n \in L^\infty_T\mathscr{C}^2$ solving (3.1) with $(b, f) = (\bar b_n, \bar f_n)$. Our main aim is to show that there is a constant $C > 0$ independent of n such that

$$\|u_n\|_{\mathbb{S}^{2-\alpha}_T(\rho_\delta)} + \|u_n^\sharp\|_{\mathbb{S}^{3-2\alpha}_T(\rho_{\delta_0})} \lesssim_C \mathbb{A}^{\bar b_n,\bar f_n}_{T,\infty}(\rho_\kappa) \tag{3.23}$$

On the other hand, by (2.29) with $\bar\rho = \rho^2 = \rho_{2\kappa}$ and $\phi = \psi = \chi_n$, we also have for some C independent of n,

$$\mathbb{A}^{\bar b_n,\bar f_n}_{T,\infty}(\rho_\kappa) \lesssim_C \mathbb{A}^{b_n,f_n}_{T,\infty}(\rho_\kappa), \quad \ell_T^{\bar b_n}(\rho_\kappa) \lesssim_C \ell_T^{b_n}(\rho_\kappa).$$

Hence,

$$\sup_n \Big(\|u_n\|_{\mathbb{S}_T^{2-\alpha}(\rho_\delta)} + \|u_n^\sharp\|_{\mathbb{S}_T^{3-2\alpha}(\rho_{\delta_0})}\Big) < \infty.$$

Thus, by a standard compact argument, we can show the existence of a paracontrolled solution (see [18]).

Now we prove (3.23) by a localization technique. For simplicity, we drop the bar and subscript n and assume $b, f \in L_T^\infty \mathscr{C}^2$. We fix $0 < r < 1/2$. Note that $\phi_{2r}^z = 1$ on the support of ϕ_r^z. For each $z \in \mathbb{R}^d$, it is easy to see that $u_z := u\phi_r^z$ satisfies the following PDE:

$$\partial_t u_z = \Delta u_z + b_z \cdot \nabla u_z + F_z, \quad u_z(0) = 0,$$

where $b_z := b\phi_{2r}^z$ and

$$F_z := f\phi_r^z - 2\nabla u \cdot \nabla\phi_r^z - u\Delta\phi_r^z - b \cdot \nabla\phi_r^z u.$$

Let q be the same as in Theorem 3.5. By Theorem 3.5, there is constant $C > 0$ such that for all $z \in \mathbb{R}^d$,

$$\|u_z\|_{\mathbb{S}_T^{2-\alpha}} \le C(\ell_T^{b_z})^\vartheta \mathbb{A}_{T,\infty}^{b_z,F_z}, \quad \|u_z\|_{\mathbb{L}_T^\infty} \le C(\ell_T^{b_z})^\vartheta \mathbb{A}_{T,q}^{b_z,F_z}. \tag{3.24}$$

Let $\varepsilon > 0$ be small enough. By the definition of F_z, using $\phi_{2r}^z \nabla\phi_r^z u = \nabla\phi_r^z u$ and (2.17), we have

$$\begin{aligned}
\|F_z\|_{\mathbf{C}^{-\alpha}} &\le \|f\phi_r^z\|_{\mathbf{C}^{-\alpha}} + 2\|\nabla u \cdot \nabla\phi_r^z\|_{L^\infty} + \|u\Delta\phi_r^z\|_{L^\infty} + \|b\phi_{2r}^z \cdot \nabla\phi_r^z u\|_{\mathbf{C}^{-\alpha}} \\
&\lesssim \|f\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi_r^z\|_{\mathbf{C}^{\alpha+\varepsilon}(\rho_\kappa^{-1})} + \|\nabla u\|_{L^\infty(\rho_1)}\|\nabla\phi_r^z\|_{L^\infty(\rho_1^{-1})} \\
&\quad + \|u\|_{L^\infty(\rho_1)}\|\Delta\phi_r^z\|_{L^\infty(\rho_1^{-1})} + \|b\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi_{2r}^z\nabla\phi_r^z u\|_{\mathbf{C}^{\alpha+\varepsilon}(\rho_\kappa^{-1})} \\
&\lesssim \|f\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi_r^z\|_{\mathscr{C}^1(\rho_\kappa^{-1})} + \|u\|_{\mathscr{C}^1(\rho_1)}\|\nabla\phi_r^z\|_{\mathscr{C}^1(\rho_1^{-1})} \\
&\quad + \|b\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|u\|_{\mathscr{C}^1(\rho_1)}\|\nabla\phi_r^z\|_{\mathscr{C}^1(\rho_1^{-1})}\|\phi_{2r}^z\|_{\mathscr{C}^1(\rho_\kappa^{-1})}, \\
&\overset{(3.19)}{\lesssim} \|f\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi_r^z\|_{\mathscr{C}^1(\rho_\kappa^{-1})} + \|u\|_{\mathscr{C}^1(\rho_1)}\Big(1 + \|b\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi_{2r}^z\|_{\mathscr{C}^1(\rho_\kappa^{-1})}\Big).
\end{aligned} \tag{3.25}$$

In particular,

$$\begin{aligned}
\|F_z\|_{L_T^q\mathbf{C}^{-\alpha}} &\lesssim \|f\|_{L_T^q\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi_r^z\|_{\mathscr{C}^1(\rho_\kappa^{-1})} \\
&\quad + \Big(1 + \|b\|_{L_T^\infty\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi_{2r}^z\|_{\mathscr{C}^1(\rho_\kappa^{-1})}\Big)\left(\int_0^T \|u(t)\|_{\mathscr{C}^1(\rho_1)}^q \mathrm{d}t\right)^{1/q}.
\end{aligned} \tag{3.26}$$

Similarly, we also have

$$\begin{aligned}\|(b_z\circ\nabla\mathscr{I}_\lambda F_z)\|_{\mathbf{C}^{1-2\alpha}}&\le\|b_z\circ\nabla\mathscr{I}_\lambda(f\phi^z_r)\|_{\mathbf{C}^{1-2\alpha}}+\|b_z\circ\nabla\mathscr{I}_\lambda(b\cdot\nabla\phi^z_r u)\|_{\mathbf{C}^{1-2\alpha}}\\&\quad+\|b_z\circ\nabla\mathscr{I}_\lambda(u\Delta\phi^z_r+2\nabla u\cdot\nabla\phi^z_r)\|_{L^\infty}=:I^z_1+I^z_2+I^z_3.\end{aligned}$$

For I^z_1, by (2.29) with $\bar\rho\equiv1$, $\rho=\rho_\kappa$ and $\psi=\phi^z_r$, we have

$$I^z_1\lesssim\|\phi^z_{2r}\|_{\mathbf{C}^{\alpha+\varepsilon}(\rho_\kappa^{-2})}\|\phi^z_r\|_{\mathbf{C}^{\alpha+\varepsilon}}\mathbb{A}^{b,f}_{t,\infty}(\rho_\kappa)\lesssim\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-2})}\mathbb{A}^{b,f}_{t,\infty}(\rho_\kappa).$$

For I^z_2, by (2.29) with $\bar\rho\equiv1$, $\rho=\rho_\kappa$ and $\psi=\nabla\phi^z_r u$, we have

$$\begin{aligned}I^z_2&\lesssim\|\phi^z_{2r}\|_{\mathbf{C}^{\alpha+\varepsilon}(\rho_\kappa^{-2})}\|\nabla\phi^z_r u\|_{\mathbb{S}^{\alpha+\varepsilon}_t}\mathbb{A}^{b,b}_{t,\infty}(\rho_\kappa)\\&\lesssim\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-2})}\|\nabla\phi^z_r\|_{\mathscr{C}^1(\rho_1^{-1})}\|u\|_{\mathbb{S}^1_t(\rho_1)}\ell^b_t(\rho_\kappa)\\&\overset{(3.19)}{\lesssim}\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-2})}\|u\|_{\mathbb{S}^1_t(\rho_1)}\ell^b_t(\rho_\kappa).\end{aligned}$$

For I^z_3, as in (3.25), since

$$\|b_z\|_{\mathbf{C}^{-\alpha}}\lesssim\|b\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-1})},\tag{3.27}$$

by (2.17), we have

$$\begin{aligned}I^z_3&\lesssim\|b_z\|_{\mathbf{C}^{-\alpha}}\|\nabla\mathscr{I}_\lambda(u\Delta\phi^z_r+2\nabla u\cdot\nabla\phi^z_r)\|_{\mathbf{C}^{\alpha+\varepsilon}}\\&\lesssim\|b\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-1})}\|u\Delta\phi^z_r+2\nabla u\cdot\nabla\phi^z_r\|_{L^\infty_tL^\infty}\\&\lesssim\|b\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-1})}\|u\|_{L^\infty_t\mathscr{C}^1(\rho_1)}\|\nabla\phi^z_r\|_{\mathscr{C}^1(\rho_1^{-1})}\\&\overset{(3.19)}{\lesssim}\|b\|_{\mathbf{C}^{-\alpha}(\rho_\kappa)}\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-1})}\|u\|_{L^\infty_t\mathscr{C}^1(\rho_1)}.\end{aligned}$$

Combining the above calculations, by the definition of $\mathbb{A}^{b_z,F_z}_{T,q}$, (3.19), (3.26) and (3.27), we get

$$\begin{aligned}&\mathbb{A}^{b_z,F_z}_{T,q}=\sup_\lambda\|b_z\circ\nabla\mathscr{I}_\lambda F_z\|_{L^q_T\mathbf{C}^{1-2\alpha}}+(\|b_z\|_{L^\infty_T\mathbf{C}^{-\alpha}}+1)\|F_z\|_{L^q_T\mathbf{C}^{-\alpha}}\\&\lesssim\Big(\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-2})}+\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-1})}(\|\phi^z_r\|_{\mathscr{C}^1(\rho_\kappa^{-1})}+\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-1})})+\|\phi^z_r\|_{\mathscr{C}^1(\rho_\kappa^{-1})}+1\Big)\\&\quad\times\Big(\mathbb{A}^{b,f}_{T,\infty}(\rho_\kappa)+\ell^b_T(\rho_\kappa)\Big(\int_0^T\|u\|^q_{\mathbb{S}^{2\alpha}_t(\rho_1)}\mathrm{d}t\Big)^{1/q}\Big).\end{aligned}$$

By Lemma 3.8, we have

$$\sup_z\rho_\kappa(z)\|\phi^z_{2r}\|_{\mathscr{C}^1(\rho_\kappa^{-1})}\lesssim1.$$

On the other hand, by Lemma 3.8 and (2.29) with $\bar\rho = 1$, $\bar\rho = \rho_\kappa$, we have

$$\sup_z \rho_\kappa^2(z)\ell_T^{b_z} \lesssim \sup_z \rho_\kappa^2(z)(\|\phi_{2r}^z\|_{\mathscr{C}^1(\rho_\kappa^{-2})} + \|\phi_{2r}^z\|^2_{\mathscr{C}^1(\rho_\kappa^{-1})})\ell_T^b(\rho_\kappa) \lesssim \ell_T^b(\rho_\kappa),$$

which together with the above estimate implies that for $\delta = (2\vartheta + 2)\kappa \le 1$,

$$\begin{aligned}\sup_z \rho_\delta(z)(\ell_T^{b_z})^\vartheta \mathbb{A}_{T,q}^{b_z,F_z} &\le \left(\sup_z \rho_\kappa^2(z)\ell_T^{b_z}\right)^\vartheta \sup_z \rho_\kappa^2(z)\mathbb{A}_{T,q}^{b_z,F_z}\\ &\le \left(\ell_T^b(\rho_\kappa)\right)^{\vartheta+1}\left(\mathbb{A}_{T,\infty}^{b,f}(\rho_\kappa) + \left(\int_0^T \|u\|^q_{\mathbb{S}_t^{2\alpha}(\rho_1)}\mathrm{d}t\right)^{1/q}\right).\end{aligned}$$

Note that by (2.6) and Young's inequality,

$$\|u\|_{\mathbb{S}_t^{2\alpha}(\rho_1)} \le \varepsilon\|u\|_{\mathbb{S}_t^{2-\alpha}(\rho_1)} + C_\varepsilon\|u\|_{\mathbb{L}_t^\infty(\rho_1)}.$$

Hence, multiplying both sides of (3.24) by $\rho_\delta(z)$ we arrive at

$$\|u\|_{\mathbb{S}_T^{2-\alpha}(\rho_\delta)} \le \varepsilon\|u\|_{\mathbb{S}_T^{2-\alpha}(\rho_\delta)} + C_\varepsilon\|u\|_{\mathbb{L}_T^\infty(\rho_1)} + C_\varepsilon\mathbb{A}_{T,\infty}^{b,f}(\rho_\kappa),$$

and

$$\|u\|_{\mathbb{L}_T^\infty(\rho_\delta)} \lesssim \mathbb{A}_{T,\infty}^{b,f}(\rho_\kappa) + \left(\int_0^T \|u\|^q_{\mathbb{S}_t^{2-\alpha}(\rho_1)}\mathrm{d}t\right)^{1/q}.$$

The above two estimates imply that

$$\|u\|_{\mathbb{L}_T^\infty(\rho_1)} \le \|u\|_{\mathbb{L}_T^\infty(\rho_\delta)} \lesssim \mathbb{A}_{T,\infty}^{b,f}(\rho_\kappa) + \left(\int_0^T \|u\|^q_{\mathbb{L}_t^\infty(\rho_1)}\mathrm{d}t\right)^{1/q}.$$

Finally, we use Gronwall's inequality to deduce the first estimate in (3.23).
By (3.3), (2.19) and (2.12) we have for weight $\rho, \bar\rho \in \mathscr{W}$

$$\begin{aligned}\|u^\sharp\|_{L_T^\infty\mathbf{C}^{2-\alpha}(\rho\bar\rho)} &\lesssim \|u\|_{L_T^\infty\mathbf{C}^{2-\alpha}(\rho\bar\rho)} + \|\nabla u \prec\!\!\prec \mathscr{I}_\lambda b\|_{L_T^\infty\mathbf{C}^{2-\alpha}(\rho\bar\rho)} + \|\mathscr{I}_\lambda f\|_{L_T^\infty\mathbf{C}^{2-\alpha}(\rho\bar\rho)}\\ &\lesssim \|u\|_{L_T^\infty\mathbf{C}^{2-\alpha}(\bar\rho)} + \|\nabla u\|_{\mathbb{L}_T^\infty(\bar\rho)}\|b\|_{L_T^\infty\mathbf{C}^{-\alpha}(\rho)} + \|f\|_{L_T^\infty\mathbf{C}^{-\alpha}(\rho)}\\ &\lesssim \sqrt{\ell_T^b(\rho)}\|u\|_{L_T^\infty\mathbf{C}^{2-\alpha}(\bar\rho)} + \|f\|_{L_T^\infty\mathbf{C}^{-\alpha}(\rho)}.\end{aligned} \tag{3.28}$$

Next we estimate each term on the right hand side of (3.4) by using Lemma 2.10.

- By (2.21), (2.4) we have

$$\|\nabla u \prec b - \nabla u \prec\!\!\prec b\|_{L_T^\infty\mathbf{C}^{1-2\alpha}(\rho\bar\rho)} \lesssim \|u\|_{\mathbb{S}_T^{2-\alpha}(\bar\rho)}\|b_{L_T^\infty\mathbf{C}^{-\alpha}(\rho)}.$$

- By (2.15), we have

$$\|\nabla u \succ b\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho\bar\rho)} \lesssim \|u\|_{L^\infty_T \mathbf{C}^{2-\alpha}(\bar\rho)} \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho)}.$$

- By (2.20) and (2.12) we have

$$\|[\mathscr{L}, \nabla u \prec\!\!\prec] \mathscr{I} b\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho\bar\rho)} \lesssim \|u\|_{\mathbb{S}^{2-\alpha}_T(\bar\rho)} \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho)}.$$

- By Lemma 3.3 with $\gamma = 2 - 2\alpha$, $\beta \in (\alpha, 2 - 2\alpha)$, we have

$$\|b \circ \nabla u\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho^{2+\varepsilon}\bar\rho)} \lesssim \|u\|_{\mathbb{S}^{2-\alpha}_T(\bar\rho)} + \|u^\sharp\|_{L^\infty_T \mathbf{C}^{\beta+1}(\rho^{1+\varepsilon}\bar\rho)} + \mathbb{A}^{b,f}_{T,\infty}(\rho).$$

Combining the above calculations and by (3.4) and (2.11) with $\theta = 2$ and $q = \infty$, we obtain

$$\|u^\sharp\|_{\mathbb{S}^{3-2\alpha}_T(\rho^{2+\varepsilon}\bar\rho)} \lesssim \|u\|_{\mathbb{S}^{2-\alpha}_T(\bar\rho)} + \|u^\sharp\|_{L^\infty_T \mathbf{C}^{\beta+1}(\rho^{1+\varepsilon}\bar\rho)} + \mathbb{A}^{b,f}_{T,\infty}(\rho). \tag{3.29}$$

On the other hand, for $\varepsilon > \frac{2\alpha-1}{2-3\alpha}$, one can choose β close to α so that

$$\theta := \tfrac{\varepsilon}{1+\varepsilon} = \tfrac{\alpha+\beta-1}{1-\alpha}.$$

Thus by interpolation inequality (2.5), Young's inequality and (3.28), for any $\delta > 0$,

$$\begin{aligned}\|u^\sharp\|_{L^\infty_T \mathbf{C}^{\beta+1}(\rho^{1+\varepsilon}\bar\rho)} &\lesssim \|u^\sharp\|^{\theta}_{L^\infty_T \mathbf{C}^{3-2\alpha}(\rho^{2+\varepsilon}\bar\rho)} \|u^\sharp\|^{1-\theta}_{L^\infty_T \mathbf{C}^{2-\alpha}(\rho\bar\rho)} \\ &\leq \delta \|u^\sharp\|_{L^\infty_T \mathbf{C}^{3-2\alpha}(\rho^{2+\varepsilon}\bar\rho)} + C_\delta \Big(\|u\|_{\mathbb{S}^{2-\alpha}_T(\bar\rho)} + \mathbb{A}^{b,f}_{T,\infty}(\rho) \Big).\end{aligned}$$

Substituting this into (3.29), we obtain the second estimate in (3.23) by taking $\rho = \rho_\kappa$, $\bar\rho = \rho_\delta$.

(Uniqueness). It follows by Theorem A.2 in the appendix. □

4 Hamilton–Jacobi–Bellman equations

The next two sections are devoted to a priori estimates on solutions to Eq. (1.8). The proof is divided into two steps. First we construct a C^1-diffeomorphism and perform a Zvonkin transformation through this diffeomorphism. After this transform the singular part in (1.8) disappears and we obtain an HJB equation in non-divergence form. We then obtain a priori estimates for this HJB equation, which leads to global uniform bounds for solutions to (1.8). To this end, in this section we consider the following general HJB equation:

$$\partial_t v = \mathrm{tr}(a \cdot \nabla^2 v) + B \cdot \nabla v + H(v, \nabla v), \;\; v(0) = v_0, \tag{4.1}$$

where $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a symmetric matrix-valued measurable function, and $B : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a vector-valued measurable function, and

$$H(t, x, v, Q) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$

is a real-valued measurable function, and continuous in v, Q for each t, x.

For instance, for any $\zeta \in [1, 2]$, the equation

$$\mathscr{L}v = |\nabla v|^\zeta + B \cdot \nabla v + f \tag{4.2}$$

is a typical HJB equation. Note that for $\lambda > 0$, if we define

$$v_\lambda(t, x) := v(\lambda^2 t, \lambda x),\ \ B_\lambda(t, x) := \lambda B(\lambda^2 t, \lambda x),\ \ f_\lambda(t, x) := \lambda^2 f(\lambda^2 t, \lambda x),$$

then

$$\mathscr{L}v_\lambda = \lambda^{2-\zeta}|\nabla v_\lambda|^\zeta + B_\lambda \cdot \nabla v_\lambda + f_\lambda.$$

In particular, if $\zeta = 2$, then the nonlinear term has the same order as the Laplacian term in scaling level. In this case, we say that HJB Eq. (4.2) is *critical*. While for $\zeta < 2$, the nonlinear term can be controlled well by the Laplacian term. In this case, we say that HJB equation (4.2) is *subcritical*[2].

Throughout this section we use the following polynomial weight function

$$\rho_\delta(x) := \langle x\rangle^{-\delta} = (1 + |x|^2)^{-\delta/2} \Rightarrow \rho_\delta^\gamma = \rho_{\gamma\delta},\ \ \delta, \gamma \in \mathbb{R},$$

and make the following elliptic assumption on a:

($\mathbf{H}_1^\alpha$) $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a symmetric $d \times d$-matrix-valued measurable function and satisfies that for some $c_0 \in (0, 1)$,

$$c_0|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(t, x)\xi_i\xi_j \le c_0^{-1}|\xi|^2,\ \ \forall \xi \in \mathbb{R}^d, \tag{4.3}$$

and for some $\alpha \in (0, 1)$ and $c_1 \ge 1$,

$$|a(t, x) - a(t, y)| \le c_1|x - y|^\alpha.$$

About the nonlinear term H, we separately consider two cases: subcritical case for all $d \in \mathbb{N}$ and critical case only for $d = 1$, and assume

($\mathbf{H}_{\rm sub}^{\delta,\zeta}$) Suppose that for some $\delta, \zeta \in [0, 2)$ and $c_2 > 0$,

$$|H(t, x, v, Q)| \lesssim_{c_2} \langle x\rangle^\delta + |Q|^\zeta. \tag{4.4}$$

[2] Here the critical and subcritical conditions are different from the meaning in [27].

($\mathbf{H}^{\delta,\beta}_{\text{crit}}$) Suppose that $d = 1$ and H can be decomposed as $H_s + H_c$ with H_s satisfying ($\mathbf{H}^{\delta,\zeta}_{\text{sub}}$) and H_c satisfying for some $\delta \in [0, 2)$ and $c_2 > 0$,

$$|H_c(t, x, v, Q)| \lesssim_{c_2} \langle x\rangle^\delta + |Q|^2, \quad |\partial_v H_c(t, x, v, Q)| \lesssim_{c_2} \langle x\rangle^\delta + |v|^2 + |Q|, \tag{4.5}$$

and for some $\beta \in (0, 1]$ and all $|x - y| \leq 1$,

$$|H_c(t, x, v, Q) - H_c(t, y, v, Q)| \lesssim_{c_2} |x - y|^\beta(\langle x\rangle^\delta + \langle y\rangle^\delta + |v|^2 + |Q|^2). \tag{4.6}$$

We introduce the following definition of strong solution to HJB Eq. (4.1).

Definition 4.1 We call a function $v \in \cap_{p\geq 2}\mathbb{H}^{2,p}_{loc}$ strong solution to (4.1) if for all $\psi \in C^\infty_c(\mathbb{R}^d)$ and $t \geq 0$,

$$\langle v(t), \psi\rangle = \langle v_0, \psi\rangle + \int_0^t \Big\langle\Big(\mathrm{tr}(a \cdot \nabla^2 v) + B \cdot \nabla v + H(v, \nabla v)\Big)(s), \psi\Big\rangle \mathrm{d}s,$$

where $\langle v_0, \psi\rangle := \int v_0\psi$. In particular, for all $t \geq 0$ and Lebesgue almost all $x \in \mathbb{R}^d$,

$$v(t, x) = v_0(x) + \int_0^t \Big(\mathrm{tr}(a \cdot \nabla^2 v) + B \cdot \nabla v + H(v, \nabla v)\Big)(s, x)\mathrm{d}s.$$

The aim of this section is to establish the following well-posedness for HJB Eq. (4.1). For simplicity of notation, we introduce the following parameter set for the dependence of constants:

$$\Theta := (T, d, \alpha, \beta, \zeta, \delta, c_0, c_1, c_2).$$

Theorem 4.2 *Let $T > 0$, $\delta \in (0, 2)$ and $\alpha, \beta, \delta_1 \in (0, 1]$. Suppose that ($\mathbf{H}^\alpha_1$), $B \in \mathbb{L}^\infty_T(\rho_{\delta_1})$ and ($\mathbf{H}^{\delta,\zeta}_{\text{sub}}$) or ($\mathbf{H}^{\delta,\beta}_{\text{crit}}$) hold. Let*

$$\begin{cases} \eta > \frac{\zeta\delta}{2-\zeta} \vee [2\delta_1 + \delta], & \text{under } (\mathbf{H}^{\delta,\zeta}_{\text{sub}}); \\ \eta > 2\left(\frac{(1+2\beta)\delta}{\beta} \vee (\delta_1 + \delta) \vee \frac{\delta\vee(2\delta-1)}{2-\zeta}\right), & \text{under } (\mathbf{H}^{\delta,\beta}_{\text{crit}}). \end{cases} \tag{4.7}$$

(Existence) *For any initial value $v_0 \in \mathscr{C}^2(\rho_\delta)$, there exist p_0 large enough and strong solution v to HJB Eq. (4.1), which satisfies the following estimate: for any $p \geq p_0$, there is a constant $C = C(\Theta, p, \eta, \delta_1, \|B\|_{\mathbb{L}^\infty_T(\rho_{\delta_1})}, \|v_0\|_{\mathscr{C}^2(\rho_\delta)}) > 0$ such that*

$$\|v\|_{\mathbb{L}^\infty_T(\rho_\delta)} + \|\partial_t v\|_{\mathbb{L}^p_T(\rho_\eta)} + \|v\|_{\mathbb{H}^{2,p}_T(\rho_\eta)} \leq C. \tag{4.8}$$

In particular, for any $0 \leq \varepsilon' < \varepsilon \leq 2$,

$$\|v\|_{C_T^{\varepsilon'/2}\mathbf{C}^{2-\varepsilon}(\rho_\eta)} \leq C.$$

(Uniqueness) *If, in addition, for some* $C > 0$,

$$|\partial_v H(t, x, v, Q)|^{1/2} + |\partial_Q H(t, x, v, Q)| \lesssim_C \langle x\rangle + |v|^{1/\delta} + |Q|^{1/\eta}, \tag{4.9}$$

then there is a unique strong solution with regularity (4.8).

Remark 4.3 (i) When $a \in L_T^\infty \mathscr{C}^1$, the above regularity result could be obtained by De-Giorgi's iteration method since it can be written in the divergence form (cf. [39]). However, there seems no literature studying this problem when a is only Hölder continuous. Moreover, the unbounded B and H cause some difficulties for obtaining the global estimates, which is crucial for a-priori estimate for (1.8) and KPZ type equations. We believe that the above theorem is of its own interest.

(ii) The condition in (4.7) on η comes from the energy estimate and the integrability of the weights in $\mathbb{R}^d$ (see Theorems 4.6 and 4.7 below).

In the following we first establish a maximum principle in Sect. 4.1. The subcritical case is treated in Sect. 4.2 by using $L^\infty(\rho_\delta)$-estimate and L^p-theory for PDEs. For the critical case, we take spatial derivative on both sides of (4.1) and obtain a PDE in divergence form. Then using the $L^\infty(\rho_\delta)$-bound and energy estimate we obtain the $\mathbb{H}_T^{2,p}(\rho_\eta)$-estimate in Sect. 4.3.

4.1 Maximum principle in weighted spaces

We first show the following maximum principle in weighted spaces by an exponential transform and a probabilistic method.

Theorem 4.4 (Maximum principle) *Let* $T > 0$ *and* $\delta \in (0, 2)$. *Suppose (4.3) and for some* $c_2, c_3 > 0$,

$$|H(t, x, v, Q)| \leq c_2\langle x\rangle^\delta + c_3|Q|^2, \quad B \in \mathbb{L}_T^\infty(\rho_1).$$

For any $v_0 \in L^\infty(\rho_\delta)$, *there is a function* $C(r) = C_\Theta(r) > 0$ *with* $C(0) = 0$ *such that for any strong solution* $v \in \cap_{p\geq 2}\mathbb{H}_{loc}^{2,p} \cap \mathbb{L}_T^\infty(\rho_\delta)$ *to (4.1) with initial value* v_0,

$$\|v\|_{\mathbb{L}_T^\infty(\rho_\delta)} \leq C(c_2 + \|v_0\|_{L^\infty(\rho_\delta)}). \tag{4.10}$$

Proof We use a probabilistic method. For $\lambda > 0$, define

$$w(t, x) := \mathrm{e}^{\lambda v(t,x)}.$$

By the chain rule, it is easy to see that w satisfies

$$\partial_t w = \mathrm{tr}(a \cdot \nabla^2 w) + B \cdot \nabla w + \lambda w\Big(H(v, \nabla v) - \lambda \mathrm{tr}(a \cdot \nabla v \otimes \nabla v)\Big).$$

For simplicity of notations, we write

$$F_\delta(x) := c_2\langle x\rangle^\delta,\ \ U_\lambda := \lambda w\Big(H(v,\nabla v) - \lambda \mathrm{tr}(a\cdot\nabla v\otimes\nabla v) - F_\delta\Big).$$

Next we reverse the time variable. For a space-time function f, we set

$$f^T(t,x) := f(T-t,x).$$

It is easy to see that $w^T(t,x) = w(T-t,x)$ solves the following backward equation:

$$\partial_t w^T + \mathrm{tr}(a^T\cdot\nabla^2 w^T) + B^T\cdot\nabla w^T + U^T_\lambda + \lambda w^T F_\delta = 0, \tag{4.11}$$

with subjected to the final condition

$$w^T(T,x) = w(0,x) = \mathrm{e}^{\lambda v_0(x)}. \tag{4.12}$$

Under (4.3) and $B\in\mathbb{L}^\infty_T(\rho_1)$, for each $(t,x)\in[0,T]\times\mathbb{R}^d$, it is well known that the following SDE has a (probabilistically) weak solution starting from x at time t (see [37, page 87, Theorem 1])

$$X^{t,x}_s = x + \int_t^s \sqrt{2a^T}(r,X^{t,x}_r)\mathrm{d}W_r + \int_t^s B^T(r,X^{t,x}_r)\mathrm{d}r,\ \ \forall s\in[t,T],$$

where W is a d-dimensional Brownian motion on some stochastic basis $(\Omega',\mathcal{F}',\mathbb{P})$. For $R>0$, define a stopping time

$$\tau_R := \inf\{s\geq t: |X^{t,x}_s| > R\}.$$

It is well known that the following Krylov estimate holds ([37, page 52, Theorem 2]): for any $p\geq d+1$,

$$\mathbb{E}\left(\int_t^{T\wedge\tau_R} f(s,X^{t,x}_s)\mathrm{d}s\right) \leq C_R\left(\int_t^T\int_{B_R}|f(s,x)|^p\mathrm{d}x\mathrm{d}s\right)^{1/p}.$$

Since $v\in\cap_{p\geq 2}\mathbb{H}^{2,p}_{loc}\cap\mathbb{L}^\infty_T(\rho_\delta)$, it is easy to see that

$$w^T\in\cap_{p\geq 2}\mathbb{H}^{2,p}_{\mathrm{loc}},\ \ \partial_t w^T\in\cap_{p\geq 2}\mathbb{L}^p_{\mathrm{loc}}.$$

Thus, for each fixed (t,x), by generalized Itô's formula (see [37, page 122, Theorem 1]), we have

$$\begin{aligned}\mathrm{d}_s w^T(s,X^{t,x}_s) &= (\partial_s w^T + \mathrm{tr}(a^T\cdot\nabla^2 w^T) + B^T\cdot\nabla w^T)(s,X^{t,x}_s)\mathrm{d}s\\ &\quad + (\sqrt{2a^T}\cdot\nabla w^T)(s,X^{t,x}_s)\mathrm{d}W_s,\end{aligned}$$

and by (4.11) and (4.12),

$$\begin{aligned}
&\mathrm{e}^{\int_t^{t'} \lambda F_\delta(X_s^{t,x})\mathrm{d}s} w^T(t', X_{t'}^{t,x}) \\
&= w^T(t,x) + \int_t^{t'} \mathrm{e}^{\int_t^s \lambda F_\delta(X_r^{t,x})\mathrm{d}r} \mathrm{d}_s w^T(s, X_s^{t,x}) \\
&\quad + \int_t^{t'} \mathrm{e}^{\int_t^s \lambda F_\delta(X_r^{t,x})\mathrm{d}r} (\lambda F_\delta w^T)(s, X_s^{t,x})\mathrm{d}s \\
&= w^T(t,x) - \int_t^{t'} \mathrm{e}^{\int_t^s \lambda F_\delta(X_r^{t,x})\mathrm{d}r} U_\lambda^T(s, X_s^{t,x})\mathrm{d}s + M_{t'},
\end{aligned}$$

where

$$M_{t'} := \int_t^{t'} \mathrm{e}^{\int_t^s \lambda F_\delta(X_r^{t,x})\mathrm{d}r} (\sqrt{2a^T} \cdot \nabla w^T)(s, X_s^{t,x})\mathrm{d}W_s.$$

By (4.3) and $|H(v,Q)| \le F_\delta + c_3|Q|^2$, one can choose $\lambda = c_3/c_0$ so that

$$U_\lambda^T \le \lambda w\Big(c_3|\nabla v|^2 - \lambda c_0|\nabla v|^2\Big) = 0.$$

Hence, for $\lambda = (c_3/c_0) \vee 1$,

$$\mathrm{e}^{\lambda v(T-t,x)} = w^T(t,x) \le \mathrm{e}^{\int_t^{t'} \lambda F_\delta(X_s^{t,x})\mathrm{d}s} w^T(t', X_{t'}^{t,x}) - M_{t'}.$$

Since $t' \mapsto M_{t' \wedge \tau_R}$ is a martingale, we have

$$\mathrm{e}^{\lambda v(T-t,x)} \le \mathbb{E}\left(\mathrm{e}^{\int_t^{T\wedge\tau_R} \lambda F_\delta(X_s^{t,x})\mathrm{d}s} w^T(T\wedge\tau_R, X_{T\wedge\tau_R}^{t,x})\right).$$

On the other hand, by Lemma B.1 in appendix, for any $\gamma \ge 0$ and $\alpha \in [0,2)$,

$$\mathbb{E}\left(\mathrm{e}^{\gamma \sup_{s\in[t,T]} \langle X_s^{t,x}\rangle^\alpha}\right) \le C(\gamma)\mathrm{e}^{C_2\gamma\langle x\rangle^\alpha}.$$

Since $w^T(t,x) \le \mathrm{e}^{\lambda\|v\|_{\mathbb{L}_T^\infty(\rho_\delta)}\langle x\rangle^\delta}$, letting $R \to \infty$ and by the dominated convergence theorem, we get

$$\begin{aligned}
\mathrm{e}^{\lambda v(T-t,x)} &\le \mathbb{E}\left(\mathrm{e}^{\int_t^T \lambda F_\delta(X_s^{t,x})\mathrm{d}s} w^T(T, X_T^{t,x})\right) = \mathbb{E}\left(\mathrm{e}^{\int_t^T \lambda F_\delta(X_s^{t,x})\mathrm{d}s + \lambda v_0(X_T^{t,x})}\right) \\
&\le \mathbb{E}\left(\mathrm{e}^{\ell_0 \sup_{s\in[t,T]}\langle X_s^{t,x}\rangle^\delta}\right) \le C(\ell_0)\mathrm{e}^{\ell_0\langle x\rangle^\delta},
\end{aligned}$$

where $\ell_0 := \lambda(c_2 T + \|v_0\|_{L^\infty(\rho_\delta)})$. Hence,

$$v(T-t,x) \le C(\ell_0)\langle x\rangle^\delta.$$

By applying the above estimate to $-v$, we obtain the desired estimate. □

4.2 Subcritical case

In this section we consider the subcritical case ($\mathbf{H}^{\delta,\zeta}_{\text{sub}}$) and prove some a priori regularity estimates. To this end, we first show the following interpolation inequalities in weighted spaces, which will play important roles in dealing with the weights.

Lemma 4.5 *(i) For any $p \geq 2$ and $r, p \in [1, \infty]$ satisfying $\frac{2}{p} = \frac{1}{r} + \frac{1}{q}$, and $\delta, \delta_1, \delta_2 \in \mathbb{R}$ with $\delta_1 + \delta_2 = 2\delta$, there is a constant $C = C(p, r, q, \delta, \delta_1, \delta_2) > 0$ such that*

$$\|\nabla v \rho_\delta\|_{L^p} \lesssim_C \|\nabla^2 v \rho_{\delta_1}\|_{L^q}^{1/2} \|v \rho_{\delta_2}\|_{L^r}^{1/2} + \|v \rho_{\delta+1}\|_{L^p}. \tag{4.13}$$

(ii) For any $p, q \in [2, \infty)$, $r \in [2, \infty]$ satisfying $\frac{q+2}{p} = 1 + \frac{2}{r}$, and $\delta, \delta_1, \delta_2 \in \mathbb{R}$ with $\delta = \frac{q\delta_1}{q+2} + \frac{2\delta_2}{q+2}$, there is a constant $C = C(p, q, r, \delta, \delta_1, \delta_2) > 0$ such that

$$\|\nabla v \rho_\delta\|_{L^p} \lesssim_C \left(\int |\nabla^2 v|^2 |\nabla v|^{q-2} \rho_{\delta_1}^q\right)^{\frac{1}{q+2}} \|v \rho_{\delta_2}\|_{L^r}^{\frac{2}{q+2}} + \|v \rho_{\delta+1}\|_{L^p}. \tag{4.14}$$

Proof By definition and the integration by parts, we have

$$\begin{aligned}\|\nabla v \rho_\delta\|_{L^p}^p &= \int |\nabla v|^p \rho_{\delta p} = \int \langle \nabla v, \nabla v |\nabla v|^{p-2} \rho_{\delta p}\rangle \\ &\lesssim \int |v| \Big(|\nabla^2 v| |\nabla v|^{p-2} \rho_{\delta p} + |\nabla v|^{p-1} |\nabla \rho_{\delta p}|\Big).\end{aligned} \tag{4.15}$$

(i) By Hölder's inequality we have

$$\int |v| |\nabla^2 v| |\nabla v|^{p-2} \rho_{\delta p} \leq \|v \rho_{\delta_2}\|_{L^r} \|\nabla^2 v \rho_{\delta_1}\|_{L^q} \|\nabla v \rho_\delta\|_{L^p}^{p-2},$$

and by $|\nabla \rho_\delta| \lesssim \rho_{\delta+1}$,

$$\int |v| |\nabla v|^{p-1} |\nabla \rho_{\delta p}| \leq \|\nabla v \rho_\delta\|_{L^p}^{p-1} \|v \rho_{\delta+1}\|_{L^p}. \tag{4.16}$$

Therefore,

$$\|\nabla v \rho_\delta\|_{L^p}^p \lesssim \|v \rho_{\delta_2}\|_{L^r} \|\nabla^2 v \rho_{\delta_1}\|_{L^q} \|\nabla v \rho_\delta\|_{L^p}^{p-2} + \|\nabla v \rho_\delta\|_{L^p}^{p-1} \|v \rho_{\delta+1}\|_{L^p}.$$

Thus by Young's inequality, we obtain (4.13).
(ii) On the other hand, by Hölder's inequality we have

$$\int |v| |\nabla^2 v| |\nabla v|^{p-2} \rho_{\delta p} \leq \left(\int |\nabla^2 v|^2 |\nabla v|^{q-2} \rho_{\delta_1 q}\right)^{1/2} \|\nabla v \rho_\delta\|_{L^p}^{p-\frac{q}{2}-1} \|v \rho_{\delta_2}\|_{L^r},$$

which together with (4.15) and (4.16) yields (4.14). □

We now prove the following a priori L^p-regularity estimate by the $\mathbb{L}^\infty_T(\rho_\delta)$ estimate obtained in Theorem 4.4.

Theorem 4.6 *Let $T > 0$, $\delta \in (0, 2)$ and $\alpha, \delta_1 \in (0, 1]$. Suppose ($\mathbf{H}_1^\alpha$), $B \in \mathbb{L}^\infty_T(\rho_{\delta_1})$ and ($\mathbf{H}_{\rm sub}^{\delta,\zeta}$). Then for any $\eta > (2\delta_1 + \delta) \vee \frac{\zeta\delta}{2-\zeta}$ and $v_0 \in \mathscr{C}^2(\rho_\delta)$, there is a p_0 large enough so that for all $p > p_0$ and any strong solution v of HJB (4.1),*

$$\|\partial_t(v\rho_\eta)\|_{\mathbb{L}^p_T} + \|v\rho_\eta\|_{\mathbb{H}^{2,p}_T} \le C,$$

where $C = C(\Theta, \eta, p, \delta_1, \|B\|_{\mathbb{L}^\infty_T(\rho_{\delta_1})}, \|v_0\|_{\mathscr{C}^2(\rho_\delta)})$.

Proof Multiplying both sides of (4.1) by ρ_η, we get

$$\partial_t(v\rho_\eta) = \mathrm{tr}(a\cdot\nabla^2(v\rho_\eta)) - \Gamma_\rho + (B\cdot\nabla v)\rho_\eta + H(v,\nabla v)\rho_\eta, \tag{4.17}$$

where

$$\Gamma_\rho = \mathrm{tr}(a\cdot(2\nabla v\otimes\nabla\rho_\eta + v\nabla^2\rho_\eta)).$$

Fix

$$p > \frac{(2-\zeta)d}{(2-\zeta)\eta - \zeta\delta} \vee \frac{d}{\eta - 2\delta_1 - \delta} =: p_0.$$

By the L^p-theory of PDEs (see [38]), there is a constant $C = C(\Theta, p)$ such that

$$\|\partial_t(v\rho_\eta)\|_{\mathbb{L}^p_T} + \|v\rho_\eta\|_{\mathbb{H}^{2,p}_T} \lesssim_C \|H(v,\nabla v)\rho_\eta + (B\cdot\nabla v)\rho_\eta - \Gamma_\rho\|_{\mathbb{L}^p_T} + \|v_0\rho_\eta\|_{H^{2,p}}.$$

Since $p(\eta-\delta) > d$, we have

$$\|v_0\rho_\eta\|_{H^{2,p}} \lesssim \|v_0\rho_\delta\|_{\mathscr{C}^2}\left(\int_{\mathbb{R}^d}\rho^p_{\eta-\delta}(x)\mathrm{d}x\right)^{1/p} \lesssim \|v_0\|_{\mathscr{C}^2(\rho_\delta)},$$

and by (4.4),

$$\|H(v,\nabla v)\rho_\eta\|_{\mathbb{L}^p_T} \lesssim \|\rho_{\eta-\delta}\|_{L^p} + \||\nabla v|^\zeta\rho_\eta\|_{\mathbb{L}^p_T} \lesssim 1 + \|\nabla v\rho_{\eta/\zeta}\|^\zeta_{\mathbb{L}^{\zeta p}_T}.$$

By interpolation inequality (4.13) and using $|\nabla\rho_\delta| \lesssim \rho_{\delta+1}$, we have

$$\|\nabla v\rho_{\eta/\zeta}\|^\zeta_{\mathbb{L}^{\zeta p}_T} \le \|\nabla^2 v\rho_\eta\|^{\zeta/2}_{\mathbb{L}^p_T}\|v\rho_{\eta(2/\zeta-1)}\|^{\zeta/2}_{\mathbb{L}^q_T} + \|v\rho_{\eta/\zeta+1}\|^\zeta_{\mathbb{L}^{\zeta p}_T},$$

where $q = p\zeta/(2-\zeta)$. Since $p(\eta - \zeta\delta/(2-\zeta)) > d$, by (4.10), we have

$$\|v\rho_{2\eta/\zeta-\eta}\|^q_{\mathbb{L}^q_T} = \int_0^T\int_{\mathbb{R}^d}|v(t,x)|^q\rho_{\eta p}(x)\mathrm{d}x\mathrm{d}t$$

$$\lesssim \int_{\mathbb{R}^d} \rho_\delta(x)^{-p\zeta/(2-\zeta)}\rho_{\eta p}(x)\mathrm{d}x$$
$$\lesssim \int_{\mathbb{R}^d} (1+|x|)^{\frac{p\zeta\delta}{2-\zeta}-\eta p}\mathrm{d}x \lesssim 1,$$

and also,

$$\|v\rho_{\eta/\zeta+1}\|^{\zeta}_{\mathbb{L}^{\zeta p}_T} \lesssim \|\rho_{\eta/\zeta+1-\delta}\|^{\zeta}_{\mathbb{L}^{\zeta p}_T} \lesssim 1.$$

Thus, for any $\varepsilon \in (0,1)$, by Young's inequality,

$$\|H(v,\nabla v)\rho_\eta\|_{\mathbb{L}^p_T} \lesssim \varepsilon\|\nabla^2 v\rho_\eta\|_{\mathbb{L}^p_T} + 1.$$

Since $B \in \mathbb{L}^\infty_T(\rho_{\delta_1})$ and $\eta > 2\delta_1 + \delta$ and $p(\eta - 2\delta_1 - \delta) > d$, we also have by (4.13) and (4.10)

$$\|(B\cdot\nabla v)\rho_\eta\|_{\mathbb{L}^p_T} \lesssim \|\rho_{\eta-\delta_1}|\nabla v|\|_{\mathbb{L}^p_T} \lesssim \|\nabla^2 v\rho_\eta\|^{1/2}_{\mathbb{L}^p_T}\|v\rho_{\eta-2\delta_1}\|^{1/2}_{\mathbb{L}^p_T} + \|v\rho_{\eta+1}\|_{\mathbb{L}^p_T}$$
$$\lesssim \varepsilon\|\nabla^2 v\rho_\eta\|_{\mathbb{L}^p_T} + 1.$$

Moreover, noting that

$$|\Gamma_\rho| \lesssim |\nabla v||\nabla\rho_\eta| + |v||\nabla^2\rho_\eta| \lesssim \rho_\eta|\nabla v| + \rho_\eta|v|,$$

we have by (4.13) and (4.10)

$$\|\Gamma_\rho\|_{\mathbb{L}^p_T} \lesssim \|\nabla v\rho_\eta\|_{\mathbb{L}^p_T} + \|v\rho_\eta\|_{\mathbb{L}^p_T} \lesssim \|\nabla^2 v\rho_\eta\|^{1/2}_{\mathbb{L}^p_T} + 1.$$

Combining the above calculations, by Young's inequality, we get

$$\|\partial_t(v\rho_\eta)\|_{\mathbb{L}^p_T} + \|v\rho_\eta\|_{\mathbb{H}^{2,p}_T} \lesssim 1.$$

The result now follows. □

4.3 Critical one dimensional case

In this section we consider the critical one dimensional case and prove the following a priori estimate.

Theorem 4.7 *Let $T > 0$ and $\alpha, \delta_1 \in (0,1]$, $\delta \in (0,2)$. Suppose ($\mathbf{H}^\alpha_1$), $B \in \mathbb{L}^\infty_T(\rho_{\delta_1})$ and ($\mathbf{H}^{\delta,\beta}_{\mathrm{crit}}$) . For any $\eta > 2\big(\frac{(1+2\beta)\delta}{\beta} \vee (\delta_1+\delta) \vee \frac{\delta\vee(2\delta-1)}{2-\zeta}\big)$ and $v_0 \in \mathscr{C}^2(\rho_\delta)$, there is a p_0 large enough so that for all $p > p_0$ and any strong solution v of HJB (4.1),*

$$\|\partial_t(v\rho_\eta)\|_{\mathbb{L}^p_T} + \|v\rho_\eta\|_{\mathbb{H}^{2,p}_T} \leq C,$$

where $C = C(\Theta, \eta, p, \delta_1, \|B\|_{\mathbb{L}^\infty_T(\rho_{\delta_1})}, \|v_0\|_{\mathscr{C}^2(\rho_\delta)})$.

The key point of the proof of this theorem is that we can use the Hölder regularity of H in x and integration by parts to treat the quadratic growth of H in Q for the equation obtained in divergence form (see (4.20) below) by taking partial derivatives on both sides of the Eq. (4.1).

Lemma 4.8 *Under the assumptions of Theorem 4.7, for any* $\eta > \frac{(1+2\beta)\delta}{\beta} \vee (\delta_1 + \delta) \vee \frac{\delta \vee (2\delta - 1)}{2-\zeta}$*, there is a* p_0 *large enough so that for all* $p > p_0$ *and any strong solution* v *of HJB (4.1),*

$$\|\partial_x v \rho_\eta\|_{L^\infty_T L^p} + \int_0^T \int |\partial_x^2 v|^2 |\partial_x v|^{p-2} \rho_\eta^p \leq C. \tag{4.18}$$

Proof Let $p \geq 2$ be fixed, whose value will be determined below. Define

$$w(t,x) := \partial_x v(t,x), \quad \mathbb{A}_p^w := \int |\partial_x w|^2 |w|^{p-2} \rho_\eta^p.$$

For given $q \in [\frac{p}{2}+1, p+2]$ and $\gamma \in \mathbb{R}$, by (4.14) and (4.10) and $|\nabla \rho_\delta| \lesssim \rho_{\delta+1}$ we have

$$\begin{aligned}\left(\int |w|^q \rho_{p\eta+\gamma}\right)^{1/q} &\lesssim \left(\int |\partial_x w|^2 |w|^{p-2} \rho_{p\eta}\right)^{\frac{1}{p+2}} \|v \rho_{\delta_2}\|_{L^r}^{\frac{2}{p+2}} + \|v \rho_{\frac{p\eta+\gamma}{q}+1}\|_{L^q} \\ &\lesssim \left(\mathbb{A}_p^w\right)^{\frac{1}{p+2}} \|\rho_{\delta_2 - \delta}\|_{L^r}^{\frac{2}{p+2}} + \|\rho_{\frac{p\eta+\gamma}{q}+1-\delta}\|_{L^q},\end{aligned}$$

where

$$\delta_2 := \tfrac{(p+2-q)p\eta}{2q} + \tfrac{(p+2)\gamma}{2q}, \quad r := \tfrac{2q}{p+2-q} \in [2, \infty].$$

Recalling $\rho_\delta(x) = \langle x \rangle^{-\delta}$ and $d = 1$, we have for $q = p+2$ and $\gamma = 2\delta$, or $q \in [\frac{p}{2}+1, p+2)$ and $\gamma > \frac{2q\delta}{p+2} + (1 - p\eta)(1 - \frac{q}{p+2}) =: \gamma_0$,

$$\|\rho_{\delta_2 - \delta}\|_{L^r} + \|\rho_{\frac{p\eta+\gamma}{q}+1-\delta}\|_{L^q} < \infty.$$

Thus we always have

$$\int |w|^q \rho_{p\eta+\gamma} \lesssim \begin{cases} \mathbb{A}_p^w + 1, & q = p+2, \gamma = 2\delta, \\ (\mathbb{A}_p^w)^{\frac{q}{p+2}} + 1, & q \in [\frac{p}{2}+1, p+2), \gamma > \gamma_0. \end{cases} \tag{4.19}$$

Now by (4.1), one sees that

$$\partial_t w = \partial_x (a \cdot \partial_x w) + \partial_x (Bw) + \partial_x H(v, w). \tag{4.20}$$

Since $\eta > \big(\frac{1+2\beta}{\beta}\big)\delta \vee (\delta_1+\delta) \vee \frac{\delta\vee(2\delta-1)}{2-\zeta}$, we can choose p large enough such that

$$\begin{aligned}\eta > &\Big(\Big[2\tfrac{p+1}{p}+\tfrac{p+2}{\beta p}\Big]\delta+\tfrac{1}{p}\Big)\vee\Big((1+\tfrac{2}{p})\delta_1+\tfrac{1}{p}+\delta\Big)\\ &\vee\Big(\tfrac{p+2\zeta-2}{(2-\zeta)p}\delta+\tfrac{1}{p}\Big)\vee\Big(\tfrac{p+\zeta}{(2-\zeta)p}(2\delta-1)\Big).\end{aligned}\tag{4.21}$$

Multiplying both sides of (4.20) by $w|w|^{p-2}\rho_{p\eta}$ and integrating on $\mathbb{R}$, we obtain

$$\begin{aligned}\frac{1}{p}\partial_t\int|w\rho_\eta|^p &= -\int a\partial_x w\partial_x(w|w|^{p-2}\rho_{p\eta})-\int Bw\partial_x(w|w|^{p-2}\rho_{p\eta})\\ &\quad-\int H_s(v,w)\partial_x(w|w|^{p-2}\rho_{p\eta})-\int H_c(v,w)\partial_x(w|w|^{p-2}\rho_{p\eta})\\ &=: I_1+I_2+I_3+I_4.\end{aligned}$$

For I_1, since $a\ge c_0$ and $\eta>\frac{1}{p}+\delta$, by (4.19) with $q=p$ and $\gamma=0$, we have

$$\begin{aligned}I_1 &\le -c_0\int|\partial_x w|^2|w|^{p-2}\rho_{p\eta}+C\int|\partial_x w||w|^{p-1}\rho_{p\eta}\\ &\le -\frac{c_0}{2}\mathbb{A}_p^w+C\int|w|^p\rho_{p\eta}\le -\frac{c_0}{4}\mathbb{A}_p^w+C.\end{aligned}$$

For I_2, since $|B|\le\|B\|_{\mathbb{L}_T^\infty(\rho_{\delta_1})}\rho_{\delta_1}^{-1}$ and $\eta>(1+\frac{2}{p})\delta_1+\frac{1}{p}+\delta$, by (4.19) with $q=p$ and $\gamma=-2\delta_1$, we have

$$\begin{aligned}I_2 &\lesssim \int|\partial_x w||w|^{p-1}\rho_{p\eta-\delta_1}+\int|w|^p\rho_{p\eta+1-\delta_1}\\ &\lesssim \Big(\mathbb{A}_p^w\Big)^{1/2}\Big(\int|w|^p\rho_{p\eta-2\delta_1}\Big)^{1/2}+\int|w|^p\rho_{p\eta}\\ &\lesssim \Big(\mathbb{A}_p^w\Big)^{(p+1)/(p+2)}+1.\end{aligned}$$

For I_3 since $\eta>\Big(\frac{p+2\zeta-2}{(2-\zeta)p}\delta+\frac{1}{p}\Big)\vee\Big(\frac{p+\zeta}{(2-\zeta)p}(2\delta-1)\Big)$, by (4.19) with $q=p-2+2\zeta$, $\gamma=0$ and $q=p+\zeta$, $\gamma=1$

$$\begin{aligned}I_3 &\lesssim \int\rho_{-\delta}|\partial_x(w|w|^{p-2}\rho_{p\eta})|+\int|w|^\zeta|\partial_x(w|w|^{p-2}\rho_{p\eta})|\\ &\lesssim \Big(\mathbb{A}_p^w\Big)^{1/2}\Big(\int|w|^{p-2}\rho_{p\eta-2\delta}\Big)^{1/2}+\int|w|^{p-1}\rho_{p\eta+1-\delta}\\ &\quad+\Big(\mathbb{A}_p^w\Big)^{1/2}\Big(\int|w|^{p-2+2\zeta}\rho_{p\eta}\Big)^{1/2}+\int|w|^{p+\zeta}\rho_{p\eta+1}\\ &\lesssim \Big(\mathbb{A}_p^w\Big)^{(p+\zeta)/(p+2)}+1.\end{aligned}$$

Now we treat the most difficult term I_4. The key idea is to use regularity of H w.r.t the spatial variable and integration by parts. To balance the weight, we use a convolution approximation. Let $\phi_\varepsilon(y) = \varepsilon^{-1}\phi(y/\varepsilon)$, where $\phi \in C_c^\infty((-1, 1))$ is a smooth density function. Define for given $t \in [0, T]$ and $v, Q \in \mathbb{R}$,

$$H_\varepsilon(t, x, v, Q) = \int H_c(t, y, v, Q)\phi_{\varepsilon\rho_{\delta/\beta}(x)}(x - y)\mathrm{d}y. \tag{4.22}$$

We make the following decomposition for I_4:

$$\begin{aligned} I_4 &= \int (H_\varepsilon(v, w) - H_c(v, w))\partial_x(w|w|^{p-2}\rho_{p\eta}) \\ &\quad - (p-1)\int H_\varepsilon(v, w)\partial_x w|w|^{p-2}\rho_{p\eta} \\ &\quad - \int H_\varepsilon(v, w)w|w|^{p-2}\partial_x\rho_{p\eta} \\ &:= I_{41} - I_{42} - I_{43}. \end{aligned}$$

For I_{41}, noting that by (4.6), (4.22) and (4.10),

$$\begin{aligned} |H_\varepsilon(x, v, w) - H_c(x, v, w)| &\le \int |H(y, v, w) - H_c(x, v, w)|\phi_{\varepsilon\rho_{\delta/\beta}(x)}(x - y)\mathrm{d}y \\ &\lesssim \varepsilon^\beta\rho_\delta(x)\int (\langle x\rangle^\delta + \langle y\rangle^\delta + |v|^2 + |w|^2)\phi_{\varepsilon\rho_{\delta/\beta}(x)}(x - y)\mathrm{d}y \\ &\lesssim \varepsilon^\beta\rho_\delta(x)\Big(\langle x\rangle^\delta + \langle x\rangle^{2\delta} + |w|^2\Big) \lesssim \rho_\delta^{-1}(x) + \varepsilon^\beta\rho_\delta(x)|w|^2, \end{aligned}$$

we have

$$I_{41} \lesssim \int \rho_\delta^{-1}|\partial_x(w|w|^{p-2}\rho_{p\eta})| + \varepsilon^\beta\int \rho_\delta w^2|\partial_x(w|w|^{p-2}\rho_{p\eta})| =: I_{311} + I_{312}.$$

For I_{411}, noting that by the chain rule and $|\nabla\rho_{p\eta}| \lesssim \rho_{p\eta+1}$,

$$|\partial_x(w|w|^{p-2}\rho_{p\eta})| \lesssim |w|^{p-2}|\partial_x w|\rho_{p\eta} + |w|^{p-1}\rho_{p\eta+1}, \tag{4.23}$$

since $\eta > \frac{1}{p} + \delta$, we have by (4.19) and Hölder's inequality,

$$\begin{aligned} I_{411} &\lesssim \int |w|^{p-2}|\partial_x w|\rho_{p\eta-\delta} + \int |w|^{p-1}\rho_{p\eta+1-\delta} \\ &\lesssim \left(\mathbb{A}_p^w\right)^{1/2}\left(\int |w|^{p-2}\rho_{p\eta-2\delta}\right)^{1/2} + \int |w|^{p-1}\rho_{p\eta+1-\delta} \\ &\lesssim \left(\mathbb{A}_p^w\right)^{p/(p+2)} + 1. \end{aligned}$$

For I_{412}, due to $\eta > \frac{1}{p}+\delta$, by (4.23), (4.19), Hölder's inequality and Young's inequality, we have

$$
\begin{aligned}
I_{412} &\lesssim \varepsilon^\beta \int |w|^p|\partial_x w|\rho_{p\eta+\delta} + \int \varepsilon^\beta |w|^{p+1}\rho_{p\eta+1+\delta}\\
&\lesssim \varepsilon^\beta \int (|w|^{p-2}|\partial_x w|^2\rho_{p\eta} + |w|^{p+2}\rho_{p\eta+2\delta}) + \int |w|^{p+1}\rho_{p\eta+1+\delta}\\
&\lesssim \varepsilon^\beta \mathbb{A}_p^w + \left(\mathbb{A}_p^w\right)^{(p+1)/(p+2)} + 1.
\end{aligned}
$$

For I_{42}, noting that by the chain rule,

$$
\begin{aligned}
H_\varepsilon(v,w)\partial_x w|w|^{p-2} &= \partial_x\Big(\int_0^w H_\varepsilon(v,r)|r|^{p-2}\mathrm{d}r\Big)\\
&\quad - \int_0^w (\partial_x H_\varepsilon(v,r) + \partial_v H_\varepsilon(v,r)w)|r|^{p-2}\mathrm{d}r,
\end{aligned}
$$

by the integration by parts, we have

$$
\begin{aligned}
I_{42} &\lesssim \int \left(\int_0^w |H_\varepsilon(v,r)||r|^{p-2}\mathrm{d}r\right)|\partial_x\rho_{p\eta}|\\
&\quad + \int \left(\int_0^w |\partial_x H_\varepsilon(v,r)||r|^{p-2}\mathrm{d}r\right)\rho_{p\eta}\\
&\quad + \int \left(\int_0^w |\partial_v H_\varepsilon(v,r)w||r|^{p-2}\mathrm{d}r\right)\rho_{p\eta}\\
&=: I_{421} + I_{422} + I_{423}.
\end{aligned}
$$

For I_{421}, by (4.5) and (4.19) we have

$$
\begin{aligned}
I_{421} &\lesssim \int \left(\int_0^w (\rho_\delta^{-1} + |r|^2)|r|^{p-2}\mathrm{d}r\right)\rho_{p\eta+1}\\
&\lesssim \int (\rho_\delta^{-1}|w|^{p-1} + |w|^{p+1})\rho_{p\eta+1}\\
&\lesssim (\mathbb{A}_p^w)^{\frac{p+1}{p+2}} + 1.
\end{aligned}
$$

For I_{422}, noting that

$$
|\partial_x H_\varepsilon(x,v,w)| \lesssim \varepsilon^{-1}\rho_{\delta/\beta}^{-1}(x)(\langle x\rangle^\delta + w^2),
$$

and

$$
\eta > \Big[2\tfrac{p+1}{p} + \tfrac{p+2}{\beta p}\Big]\delta + \tfrac{1}{p},
$$

by (4.19) with $q = p + 1$, $\gamma = -\delta/\beta$ and $q = p - 1$, $\gamma = -\delta - \delta/\beta$, we have

$$I_{422} \lesssim \varepsilon^{-1} \int (\rho_{\delta+\delta/\beta}^{-1} |w|^{p-1} + \rho_{\delta/\beta}^{-1} |w|^{p+1}) \rho_{p\eta} \lesssim (\mathbb{A}_p^w)^{\frac{p+1}{p+2}} + 1.$$

For I_{423}, by (4.5), (4.10) and (4.19) with $q = p$, $\gamma = -2\delta$, we have

$$I_{423} \lesssim \int (|w|^{p+1} + \rho_\delta^{-2} |w|^p) \rho_{p\eta} \lesssim 1 + (\mathbb{A}_p^w)^{\frac{p+1}{p+2}}.$$

Finally, for I_{43}, by (4.5) and (4.19), we similarly have

$$I_{43} \lesssim \int (|w|^{p-1} \rho_\delta^{-1} + |w|^{p+1}) \rho_{p\eta+1} \lesssim (\mathbb{A}_p^w)^{\frac{p+1}{p+2}} + 1.$$

Combining the above calculations, choosing ε small enough and by Young's inequality, we obtain

$$\frac{1}{2} \partial_t \|w \rho_\eta\|_{L^p}^p \lesssim -\frac{c_0}{8} \mathbb{A}_p^w + 1.$$

Integrating both sides from 0 to T, we obtain the desired estimate. □

Now we can give the proof of Theorem 4.7.

Proof of Theorem 4.7 We follow the proof of Theorem 4.6. Fix $p > 1/(\eta - \delta)$. By the L^p-theory of PDEs (cf. [38]), we have

$$\|\partial_t (v \rho_\eta)\|_{\mathbb{L}_T^p} + \|v \rho_\eta\|_{\mathbb{H}_T^{2,p}} \lesssim_C \|H(v, \nabla v) \rho_\eta + (B \cdot \nabla v) \rho_\eta - \Gamma_\rho\|_{\mathbb{L}_T^p} + \|v_0 \rho_\eta\|_{H^{2,p}},$$

with Γ_ρ defined in the proof of Theorem 4.6. Since $p > 1/(\eta - \delta)$, by $|H(v, Q)| \lesssim \langle x \rangle^\delta + |Q|^2$, we have

$$\|H(v, \nabla v) \rho_\eta\|_{\mathbb{L}_T^p} \lesssim \|\rho_{\eta-\delta}\|_{L^p} + \||\nabla v|^2 \rho_\eta\|_{\mathbb{L}_T^p} \lesssim 1 + \|\nabla v \rho_{\eta/2}\|_{\mathbb{L}_T^{2p}}^2.$$

We have by Hölder's inequality and Sobolev's embedding,

$$\begin{aligned} \|\nabla v \rho_{\eta/2}\|_{\mathbb{L}_T^{2p}} \leq & \|\nabla v \rho_\eta\|_{\mathbb{L}_T^\infty}^\theta \|\nabla v \rho_{\eta_0}\|_{L_T^\infty L^r}^{1-\theta} \\ & \lesssim \|\nabla (\nabla v \rho_\eta)\|_{\mathbb{L}_T^p}^\theta \|\nabla v \rho_{\eta_0}\|_{L_T^\infty L^r}^{1-\theta} + \|\nabla v \rho_\eta\|_{\mathbb{L}_T^p}^\theta \|\nabla v \rho_{\eta_0}\|_{L_T^\infty L^r}^{1-\theta}, \end{aligned}$$

where $\theta \in (0, 1/2)$ and

$$r = 2p(1 - \theta), \quad \eta_0 = \tfrac{1-2\theta}{2(1-\theta)} \eta.$$

Let p_0 be as in Lemma 4.8. Since $\eta > 2\Big(\frac{1+2\beta}{\beta}\delta \vee (\delta_1+\delta) \vee \frac{\delta\vee(2\delta-1)}{2-\zeta}\Big)$, one can choose θ close to zero and p large enough so that

$$\eta_0 = \tfrac{1-2\theta}{2(1-\theta)}\eta > \tfrac{1+2\beta}{\beta}\delta \vee (\delta_1+\delta), \quad r, p \geq p_0.$$

Thus by (4.18), we obtain

$$\|\nabla v \rho_{\eta_0}\|_{L^\infty_T L^r} + \|\nabla v \rho_\eta\|_{\mathbb{L}^p_T} \leq C,$$

and therefore,

$$\|H(v, \nabla v)\rho_\eta\|_{\mathbb{L}^p_T} \leq \varepsilon \|\nabla^2(v\rho_\eta)\|_{\mathbb{L}^p_T} + C.$$

Moreover, as in the proof of Theorem 4.6, one has

$$\|(B\cdot\nabla v)\rho_\eta - \Gamma_\rho\|_{\mathbb{L}^p_T} \leq C.$$

Thus we obtain the desired estimate as in the proof of Theorem 4.6. □

4.4 Proof of Theorem 4.2

The existence proof follows by the previous a priori estimates and standard compact method. For the uniqueness part we use a probabilistic method.

(Existence). Let $T > 0$. For fixed $m \in \mathbb{N}$, let $\chi^m_n(x) := \chi^m(x/n), n \in \mathbb{N}$ be the cutoff function in $\mathbb{R}^m$, and $\varrho^m_n(x) := n^m \varrho^m(nx), n \in \mathbb{N}$ be the mollifiers in $\mathbb{R}^m$, where $\chi^m \in C^\infty_c(\mathbb{R}^m)$ with $\chi^m = 1$ for $|x| \leq 1$ and $\chi^m = 0$ for $|x| > 2$, and $\varrho^m \in C^\infty_c(\mathbb{R}^m)$ is a density function. Define

$$B_n(t,x) := B(t,x)\mathbf{1}_{|x|\leq n}, \quad \varphi_n(x) := v_0(x)\chi^d_n(x).$$

For nonlinear term H, we construct the approximation H_n as follows:

$$H_n(t,x,v,Q) := ((H(t,x,\cdot,\cdot)\chi^{d+1}_n) * \varrho^{d+1}_n)(v,Q)\chi^d_n(x). \tag{4.24}$$

We consider the following approximation equation:

$$\partial_t v_n = \mathrm{tr}(a\cdot\nabla^2 v_n) + B_n\cdot\nabla v_n + H_n(v_n, \nabla v_n), \quad v_n(0) = \varphi_n. \tag{4.25}$$

Note that by the assumptions of Theorem 4.2,

$$B_n \in \cap_{p\in[1,\infty]}\mathbb{L}^p_T, \quad \varphi_n \in \cap_{p\in[1,\infty]} H^{2,p},$$

and

$$\|H_n\|_{\mathbb{L}^\infty_T} + \|\partial_v H_n\|_{\mathbb{L}^\infty_T} + \|\partial_Q H_n\|_{\mathbb{L}^\infty_T} < \infty.$$

It is well known that the approximation Eq. (4.25) admits a unique strong solution $v_n \in \cap_{p\geq 2}\mathbb{H}^{2,p}_T$ (cf. [38]). Moreover, by definition, we have the following uniform estimates:

$$\|B_n\rho_{\delta_1}\|_{\mathbb{L}^\infty_T} \leq \|B\rho_{\delta_1}\|_{\mathbb{L}^\infty_T},$$

and for some C independent of n, in the subcritical case,

$$|H_n(v,Q)| \lesssim_C \langle x\rangle^\delta + |Q|^\zeta,$$

and in the critical case $d=1$,

$$|H_n(t,x,v,Q)| \lesssim_C \langle x\rangle^\delta + |Q|^2, \quad |\partial_v H_n(t,x,v,Q)| \lesssim_C \langle x\rangle^\delta + |v|^2 + |Q|,$$
$$|H_n(t,x,v,Q) - H_n(t,y,v,Q)| \lesssim_C |x-y|^\beta(\langle x\rangle^\delta + \langle y\rangle^\delta + |v|^2 + |Q|^2),$$

Thus by Theorems 4.4, 4.6 and 4.7, we have the following uniform estimates: for η being as in (4.7) and p large enough,

$$\|v_n\rho_\delta\|_{\mathbb{L}^\infty_T} + \|\partial_t(v_n\rho_\eta)\|_{\mathbb{L}^p_T} + \|v_n\rho_\eta\|_{\mathbb{H}^{2,p}_T} \leq C,$$

where C is independent of n. By Sobolev's embedding (cf. [12, Lemma 2.3]), for any $\beta' \in (0, 2-\frac{2}{p})$ and $\gamma = 1 - \frac{\beta'}{2} - \frac{1}{p}$,

$$\|v_n\rho_\eta\|_{C^\gamma_T\mathbf{C}^{\beta'-d/p}} \lesssim \|v_n\rho_\eta\|_{C^\gamma_T H^{\beta',p}}$$
$$\lesssim \|\partial_t(v_n\rho_\eta)\|_{\mathbb{L}^p_T} + \|v_n\rho_\eta\|_{\mathbb{H}^{2,p}_T} + \|v_0\rho_\eta\|_{H^{\beta',p}} \leq C.$$

Thus by Ascolli-Arzela's lemma, there are subsequence n_k and $v \in \mathbb{L}^\infty_T(\rho_\delta)\cap\mathbb{H}^{2,p}_T(\rho_\eta)$ such that for all t, x,

$$\nabla^j v_{n_k}(t,x) \to \nabla^j v(t,x), \quad j = 0, 1, \tag{4.26}$$

and for any $R > 0$,

$$\nabla^2 v_n \to \nabla^2 v \ \text{ weakly in } \ L^2([0,T]\times B_R). \tag{4.27}$$

By taking limits for (4.25), one finds that v is a strong solution to (4.1) in the sense of Definition 4.1. Indeed, for any $\psi \in C^\infty_c(\mathbb{R}^d)$, by (4.27) we have

$$\lim_{n\to\infty}\int_0^t \langle \mathrm{tr}(a\cdot\nabla^2 v_n), \psi\rangle \mathrm{d}s = \int_0^t \langle \mathrm{tr}(a\cdot\nabla^2 v), \psi\rangle \mathrm{d}s$$

and by (4.26) and the dominated convergence theorem,

$$\lim_{n\to\infty}\int_0^t \langle B_n\cdot\nabla v_n, \psi\rangle \mathrm{d}s = \int_0^t \langle B\cdot\nabla v, \psi\rangle \mathrm{d}s$$

Moreover, since for each $(t,x)\in[0,T]\times\mathbb{R}^d$ and $R>0$,

$$\lim_{n\to\infty}\sup_{|(v,Q)|\leq R}|H_n(t,x,v,Q)-H(t,x,v,Q)|=0,$$

by (4.26) and the dominated convergence theorem, we also have

$$\lim_{n\to\infty}\int_0^t\langle H_n(s,\cdot,v_n,\nabla v_n),\psi\rangle\mathrm{d}s=\int_0^t\langle H(s,\cdot,v,\nabla v),\psi\rangle\mathrm{d}s.$$

Thus we obtain the existence of a strong solution.

(Uniqueness). We prove the uniqueness on the time interval [0, 1] by a probabilisitic method. Let v_1, v_2 be two strong solutions of HJB Eq. (4.1) with the same initial value v_0. By (4.8), we have

$$v_1,v_2\in\mathbb{L}_1^\infty(\rho_\delta)\cap L_1^\infty\mathscr{C}^1(\rho_\eta). \tag{4.28}$$

Let $V:=v_1-v_2$. Then V is a strong solution of the following linear PDE:

$$\partial_t V=\mathrm{tr}(a\cdot\nabla^2V)+B\cdot\nabla V+G\cdot\nabla V+K\cdot V,\ \ V(0)=0,$$

where

$$G:=\int_0^1\partial_Q H(v_1,\nabla v_1+\theta\nabla(v_2-v_1))\mathrm{d}\theta,$$

and

$$K:=\int_0^1\partial_v H(v_1+\theta(v_2-v_1),\nabla v_2)\mathrm{d}\theta.$$

By (4.28) and (4.9), there is a constant $C_0>0$ such that for all $(t,x)\in[0,1]\times\mathbb{R}^d$,

$$|G(t,x)|\lesssim_{C_0}\langle x\rangle,\ \ |K(t,x)|\lesssim_{C_0}\langle x\rangle^2. \tag{4.29}$$

Let $T\in(0,1]$ be fixed and determined below. For a space-time function F, let

$$F^T(t,x):=F(T-t,x).$$

Thus under $(\mathbf{H}_1^\alpha)$ and $B\in\mathbb{L}_1^\infty(\rho_{\delta_1})$, for each $(t,x)\in[0,T]\times\mathbb{R}^d$, the following SDE admits a unique weak solution starting from x at time t (see [37]):

$$X_s^{t,x}=x+\int_t^s\sqrt{2a^T}(r,X_r^{t,x})\mathrm{d}W_r+\int_t^s(B^T+G^T)(r,X_r^{t,x})\mathrm{d}r,\ \ \forall s\in[t,T].$$

As in the proof of Theorem 4.4, by Itô's formula, we have

$$\mathrm{e}^{\int_t^{t'} K^T(s, X_s^{t,x})\mathrm{d}s} V^T(t', X_{t'}^{t,x}) = V^T(t,x) + M_{t'},\ t' \in [t, T],$$

where $M_{t'}$ is a continuous local martingale. Note that by (4.29) and [54, Lemma 2.2], for $T = T(C_0, d, c_0, \|B\|_{\mathbb{L}_1^\infty(\rho_{\delta_1})})$ small enough,

$$\mathbb{E}\mathrm{e}^{2\int_t^T K^T(s, X_s^{t,x})\mathrm{d}s} \le \mathbb{E}\mathrm{e}^{2C_0 \sup_{s\in[t,T]} |X_s^{t,x}|^2} < \infty.$$

By using stopping time technique as in the proof of Theorem 4.4 and taking expectations, we find that for T being small enough, $0 \le t \le T$

$$V^T(t,x) = \mathbb{E}\mathrm{e}^{\int_t^T K^T(s, X_s^{t,x})\mathrm{d}s} V(0, X_T^{t,x}) \equiv 0.$$

Thus we obtain the uniqueness on small time interval $[0, T]$. We can proceed to consider $[T, 2T]$ and so on. The proof is complete.

5 HJB equations with distribution-valued coefficients

In this section we focus on Eq. (1.5). Our strategy is summarized as follows: we first decompose Eq. (1.5) into two equations: the linear one with singular f and the nonlinear one without f. For the linear equation, we can obtain the desired estimate by Theorem 3.7. For the nonlinear equation, we introduce Zvonkin's transformation to kill the singular part so that we can use the results in Sect. 4 to deduce a priori estimates for solutions to the nonlinear equation. Finally we employ the standard compactness argument to construct a solution to (1.5).

Now we fix $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and $\kappa \in (0,1)$ being small enough so that

$$\bar\alpha := \alpha + \widetilde\kappa \in (\tfrac{1}{2}, \tfrac{2}{3}),\quad \widetilde\kappa := \kappa^{1/4},\quad \delta := 2(\tfrac{9}{2-3\alpha} + 1)\kappa < 1. \tag{5.1}$$

We consider the following singular HJB equation:

$$\mathscr{L}u = (\partial_t - \Delta)u = b\cdot\nabla u + H(u, \nabla u) + f,\quad u(0) = \varphi, \tag{5.2}$$

where $(b, f) \in \cap_{T>0}\mathbb{B}_T^\alpha(\rho_\kappa)$ and

$$H(t,x,u,Q) : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$

satisfies $(\mathbf{H}_{\rm sub}^{\delta,\zeta})$ or $(\mathbf{H}_{\rm crit}^{\delta,\beta})$ with $\zeta \in [0,2)$, $\beta \in (0,1]$ and for some $C > 0$,

$$|\partial_u H(t,x,u,Q)| + |\partial_Q H(t,x,u,Q)| \lesssim_C \langle x\rangle^\delta + |u| + |Q|. \tag{5.3}$$

To understand HJB Eq. (5.2), we use the paracontrolled calculus:

$$u = \nabla u \lll \mathscr{I} b + \mathscr{I} f + u^\sharp + P_t\varphi, \tag{5.4}$$

where $u^\sharp$ solves the following equation

$$\begin{cases} \mathscr{L} u^\sharp = \nabla u \prec b - \nabla u \lll b + \nabla u \succ b + b \circ \nabla u \\ \qquad\quad + H(u, \nabla u) - [\mathscr{L}, \nabla u \lll]\mathscr{I} b, \\ u^\sharp(0) = 0, \end{cases} \tag{5.5}$$

with $b \circ \nabla u$ being defined by (3.5) for $\lambda = 0$.

Our aim of this section is to prove the following result.

Theorem 5.1 *Let $T > 0$, $\beta \in (0, 1 - \bar\alpha]$, $\zeta \in [0, 2)$ and $\alpha, \bar\alpha, \kappa, \delta$ be as in (5.1). Suppose that $(b, f) \in \mathbb{B}^\alpha_T(\rho_\kappa)$ and ($\mathbf{H}^{\delta,\zeta}_{\text{sub}}$) or ($\mathbf{H}^{\delta,\beta}_{\text{crit}}$) as well as (5.3) hold. Let*

$$\begin{cases} \eta > \frac{2\zeta\delta}{2-\zeta} \vee [2\widetilde\kappa + 2\delta], & \text{under } (\mathbf{H}^{\delta,\zeta}_{\text{sub}}); \\ \eta > 2\left[\frac{2(1+2\beta)\delta}{\beta} \vee (\widetilde\kappa + 2\delta) \vee \frac{(2\delta)\vee(4\delta-1)}{2-\zeta}\right], & \text{under } (\mathbf{H}^{\delta,\beta}_{\text{crit}}). \end{cases} \tag{5.6}$$

Fix $\varepsilon \in (0, 1)$. For any initial value $\varphi \in \mathbf{C}^{1+\alpha+\varepsilon}(\rho_{\varepsilon\delta})$, there is a paracontrolled solution $(u, u^\sharp)$ solving (5.4) and (5.5) with regularity

$$u \in \mathbb{S}^{2-\bar\alpha}_T(\rho_\eta) \cap \mathbb{L}^\infty_T(\rho_{2\delta}), \quad u^\sharp \in \mathbb{S}^{3-2\bar\alpha}_T(\rho_{2\eta}) \cap \mathbb{L}^\infty_T(\rho_{2\delta+\kappa}).$$

Moreover, if $\eta < \frac{1-\alpha}{2}$, then the paracontrolled solution $(u, u^\sharp)$ is unique.

Remark 5.2 (i) Typical examples satisfying ($\mathbf{H}^{\delta,\beta}_{\text{crit}}$) as well as (5.3) are given by

$$H(x, u) = g_1(x)|\nabla u|^2 + (g_2(x) + F(u))\nabla u + g_3(u) + g_4(x),$$

where $g_1 \in \mathbf{C}^\beta$, $g_2 \in L^\infty(\rho_{\delta_0})$, $\delta_0 < \delta$, $g_4 \in L^\infty(\rho_\delta)$, and $g_3, F \in \mathscr{C}^1$.

(ii) By (5.6), one sees that η can be arbitrarily small as long as κ is small.

To show the existence of a paracontrolled solution, we use the approximation method. More precisely, since $(b, f) \in \mathbb{B}^\alpha_T(\rho_\kappa)$, by the very definition, there is a sequence of $(b_n, f_n) \in L^\infty_T\mathscr{C}^\infty(\rho_\kappa)$ with

$$\sup_n \left(\ell^{b_n}_T(\rho_\kappa) + \mathbb{A}^{b_n, f_n}_{T,\infty}(\rho_\kappa)\right) \le c_0,$$

and such that for $\lambda \ge 0$,

$$\begin{cases} \lim_{n\to\infty} \left(\|b_n - b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)} + \|f_n - f\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)}\right) = 0, \\ \lim_{n\to\infty} \|b_n \circ \nabla \mathscr{I}_\lambda b_n - b \circ \nabla \mathscr{I}_\lambda b\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_\kappa)} = 0, \\ \lim_{n\to\infty} \|b_n \circ \nabla \mathscr{I}_\lambda f_n - b \circ \nabla \mathscr{I}_\lambda f\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_\kappa)} = 0. \end{cases} \tag{5.7}$$

Moreover, let φ_n be the the convolution of φ with smooth mollifier so that

$$\sup_n \|\varphi_n\|_{\mathbf{C}^{1+\alpha+\varepsilon}(\rho_{\varepsilon\delta})} \lesssim \|\varphi\|_{\mathbf{C}^{1+\alpha+\varepsilon}(\rho_{\varepsilon\delta})}.$$

We consider the following approximation equation:

$$\mathscr{L}u_n = b_n \cdot \nabla u_n + H(u_n, \nabla u_n) + f_n, \quad u_n(0) = \varphi_n. \tag{5.8}$$

By Theorem 4.2, it is well known that approximation equation (5.8) admits a unique strong solution u_n with

$$\|u_n\|_{\mathbb{L}^\infty_T(\rho_\delta)} + \|\partial_t u_n\|_{\mathbb{L}^p_T(\rho_\eta)} + \|u_n\|_{\mathbb{H}^{2,p}_T(\rho_\eta)} \leq C_n.$$

Our aim is of course to establish the following uniform estimate:

$$\sup_n \Big(\|u_n\|_{\mathbb{S}^{2-\bar{\alpha}}_T(\rho_\eta)} + \|u_n\|_{\mathbb{L}^\infty_T(\rho_{2\delta})} + \|u_n^\sharp\|_{\mathbb{S}^{3-2\bar{\alpha}}_T(\rho_{2\eta})} + \|u_n^\sharp\|_{\mathbb{L}^\infty_T(\rho_{2\delta+\kappa})} \Big) \leq C, \tag{5.9}$$

where $u_n^\sharp$ is defined by (5.4) with (b, f) being replaced by (b_n, f_n).

To show the uniform estimate (5.9), our approach is to transform (5.8) into HJB equation studied in Sect. 4. In the following, for simplicity, we drop the subscript n and use the convention that all the constants appearing below only depend on the parameter set

$$\Theta := (T, d, \alpha, \beta, \eta, \zeta, \kappa, c_0, \varepsilon, \|\varphi\|_{\mathbf{C}^{1+\alpha+\varepsilon}(\rho_{\varepsilon\delta})}).$$

First of all, by Lemma 2.13, one can make the following decomposition for the initial value $\varphi \in \mathbf{C}^{1+\alpha+\varepsilon}(\rho_{\varepsilon\delta})$: for $\varepsilon_0 \in (0, \frac{\varepsilon\alpha}{1-\varepsilon})$,

$$\varphi = \varphi_1 + \varphi_2, \quad \varphi_1 \in \mathbf{C}^{1+\alpha+\varepsilon_0}, \ \varphi_2 \in \mathscr{C}^2(\rho_\delta).$$

Next we make the following decomposition for u:

$$u = u_1 + u_2,$$

where u_1 solves the following linear equation with non-homogeneous term f

$$\mathscr{L}u_1 = b \cdot \nabla u_1 + f, \quad u_1(0) = \varphi_1, \tag{5.10}$$

while u_2 solves the following HJB equation

$$\mathscr{L}u_2 = b \cdot \nabla u_2 + H(u_1 + u_2, \nabla u_1 + \nabla u_2), \quad u_2(0) = \varphi_2. \tag{5.11}$$

Clearly, the linear Eq. (5.10) can be uniquely solved by Theorem 3.7 with the solution $u_1 \in \mathbb{S}^{2-\alpha}_T(\rho_\delta)$. Thus it remains to solve (5.11). However, since b is a distribution, we cannot directly apply Theorem 4.2. We use (2.23) and Zvonkin's transformation to kill the singular part of b.

5.1 Zvonkin's transformation for HJB equations

In this section we introduce a transformation of phase space to kill the distributional part in the drift of the HJB equation (5.11) so that we can apply the result in Sect. 4. Such a transformation was firstly used by Zvonkin in [56] to study the SDEs with singular drifts. In the literature, it is also called Zvonkin's transformation. Below we always assume

$$b \in L^\infty_T(\mathscr{C}^\infty(\rho_\kappa)),\ \ell^b_T(\rho_\kappa) \le c_0. \tag{5.12}$$

The key step for Zvonkin's transform is to construct a C^1-diffeomorphism such that the solutions to Eq. (5.11) composed with this diffeomorphism satisfy a new equation without the singular part of the drift b. However, a diffeomorphism does not allow polynomial growth for C^1-norm as $|x| \to \infty$. To this end, we decompose b into two parts by Lemma 2.13. By Lemma 2.13 we make the following decomposition:

$$b = b_> + b_\le := \mathscr{V}_> b + \mathscr{V}_\le b,$$

We are goint to construct a C^1-differmophism to kill the $b_>$ part. Furthermore, we define

$$\bar{b} := b_> \circ \nabla \mathscr{I}_\lambda b_>,\quad \bar{b}_> := \mathscr{V}_> \bar{b},\quad \bar{b}_\le := \mathscr{V}_\le \bar{b}. \tag{5.13}$$

Lemma 5.3 *For any $m \in \mathbb{N}$ and $\varepsilon > 0$, it holds that*

$$b_> \in L^\infty_T \mathscr{C}^m,\quad \bar{b}_\le \in L^\infty_T \mathscr{C}^m(\rho_{2\kappa+\varepsilon}). \tag{5.14}$$

For some $C = C(d, \alpha, \kappa) > 0$, it holds that

$$\|b_>\|_{L^\infty_T \mathbf{C}^{-\alpha-\tilde{\kappa}}} + \|b_\le\|_{\mathbb{L}^\infty_T(\rho_{\tilde{\kappa}})} \lesssim_C \sqrt{\ell^b_T(\rho_\kappa)}, \tag{5.15}$$

where $\tilde{\kappa} = \kappa^{1/4}$, and

$$\|\bar{b}\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_{\tilde{\kappa}})} + \|\bar{b}_>\|_{L^\infty_T \mathbf{C}^{1-2\alpha-\tilde{\kappa}}} + \|\bar{b}_\le\|_{\mathbb{L}^\infty_T(\rho_{\tilde{\kappa}})} \lesssim_C \ell^b_T(\rho_\kappa). \tag{5.16}$$

Proof (i) Since $b \in L^\infty_T \mathscr{C}^\infty(\rho_\kappa)$, by Lemma 2.13 one sees that (5.14) holds.
(ii) We use Lemma 2.13 with weight $\rho_{\kappa^{1/2}}$ to conclude

$$\|b_>\|_{L^\infty_T \mathbf{C}^{-\alpha-\tilde{\kappa}}} \lesssim \|b_>\|_{L^\infty_T \mathbf{C}^{-\alpha-\kappa^{1/2}}} \lesssim \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)} \le \sqrt{\ell^b_T(\rho_\kappa)}.$$

Since $\alpha < 1$, we can choose $\varepsilon > 0$ being small enough so that

$$\bar{\kappa} := \kappa + \kappa^{1/2}(\alpha + \varepsilon) \le \kappa^{1/2} - \kappa < \tfrac{2}{3}\widetilde{\kappa} - \kappa.$$

Noting that

$$\rho_{\bar{\kappa}}(x) = \langle x \rangle^{-\kappa^{1/2}(\kappa^{1/2}+\alpha+\varepsilon)} = \rho_{\kappa^{1/2}}^{\kappa^{1/2}+\alpha+\varepsilon}(x),$$

by Lemma 2.13 again, we have

$$\begin{aligned}\|b_{\le}\|_{\mathbb{L}^\infty_T(\rho_{\widetilde{\kappa}})} \le \|b_{\le}\|_{\mathbb{L}^\infty_T(\rho_{\bar{\kappa}})} = \|b_{\le}\|_{\mathbb{L}^\infty_T(\rho_{\kappa^{1/2}}^{\kappa^{1/2}+\alpha+\varepsilon})} \\ \lesssim \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_{\kappa^{1/2}}^{\kappa^{1/2}})} = \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)}.\end{aligned}$$

(iii) Note that by definition (5.13),

$$\bar{b} = b \circ \nabla \mathscr{I}_\lambda b - b \circ \nabla \mathscr{I}_\lambda (b_{\le}) - b_{\le} \circ \nabla \mathscr{I}_\lambda b_{>}$$

and

$$\|b \circ \nabla \mathscr{I}_\lambda b\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_{2\kappa})} \le \ell^b_T(\rho_\kappa).$$

By (2.16), (2.12) and (5.15), we have for $\varepsilon \in (0, 1-\alpha)$,

$$\|b \circ \nabla \mathscr{I}_\lambda (b_{\le})\|_{L^\infty_T \mathbf{C}^0(\rho_{\kappa+\bar{\kappa}})} \lesssim \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)} \|b_{\le}\|_{L^\infty_T \mathbf{C}^{\alpha+\varepsilon-1}(\rho_{\bar{\kappa}})} \lesssim \ell^b_T(\rho_\kappa),$$

and

$$\|b_{\le} \circ \nabla \mathscr{I}_\lambda (b_{>})\|_{L^\infty_T \mathbf{C}^{1-\alpha-\widetilde{\kappa}}(\rho_{\widetilde{\kappa}})} \lesssim \|b_{\le}\|_{\mathbb{L}^\infty_T(\rho_{\widetilde{\kappa}})} \|b_{>}\|_{L^\infty_T \mathbf{C}^{-\alpha-\widetilde{\kappa}}} \lesssim \ell^b_T(\rho_\kappa).$$

Combining the above estimate we get

$$\|\bar{b}\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_{\widetilde{\kappa}})} \lesssim \|\bar{b}\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_{\kappa+\bar{\kappa}})} \lesssim \ell^b_T(\rho_\kappa).$$

(iii) As for the other two estimates in (5.16), we use Lemma 2.13 with the weight $\rho_{\widetilde{\kappa}}$ to have

$$\|\bar{b}_{>}\|_{L^\infty_T \mathbf{C}^{1-2\alpha-\widetilde{\kappa}}} \le \|\bar{b}_{>}\|_{L^\infty_T \mathbf{C}^{1-2\alpha-\frac{\bar{\kappa}+\kappa}{\widetilde{\kappa}}}} \lesssim \|\bar{b}\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_{\kappa+\bar{\kappa}})} \lesssim \ell^b_T(\rho_\kappa),$$

and for $\varepsilon > 0$ small enough

$$\|\bar{b}_{\le}\|_{\mathbb{L}^\infty_T(\rho_{\widetilde{\kappa}})} \le \|\bar{b}_{\le}\|_{\mathbb{L}^\infty_T(\rho_{\bar{\kappa}+\kappa+\widetilde{\kappa}(2\alpha-1+\varepsilon)})} \lesssim \|\bar{b}\|_{L^\infty_T \mathbf{C}^{1-2\alpha}(\rho_{\kappa+\bar{\kappa}})} \lesssim \ell^b_T(\rho_\kappa).$$

The proof is complete. □

To construct a C^1-diffeomorphism for killing the singular $b_>$, we consider the following vector-valued parabolic equation:

$$\mathscr{L}_\lambda \mathbf{u} = (b_> - \bar{b}_\leq) \cdot (\nabla \mathbf{u} + \mathbb{I}), \quad \mathbf{u}(0) = \mathbf{0} \in \mathbb{R}^d. \tag{5.17}$$

Remark 5.4 The reason for considering $b_> - \bar{b}_\leq$ rather than $b_>$ is the following: in order to use (5.17) to construct a C^1-diffeomorphism, the solution $\mathbf{u}$ to (5.17) must be in unweighted spaces, which requires $\ell_T^{b_>}(1) < \infty$. However, by (5.16), $\bar{b} = b_> \circ \nabla \mathscr{I}_\lambda b_>$ stays in a weighted space. Hence, we shall use $\bar{b}_\leq$ to cancel the weight contained in the decomposition of renormalizing $b_> \circ \nabla \mathbf{u}$. It should also be also noticed that since $b_> - \bar{b}_\leq$ still stays in some weighted space, one cannot directly use Lemma 3.4 to construct a C^1-diffeomorphism. Fortunately, one still has the following result.

Lemma 5.5 *Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and $\kappa \in (0, (\frac{2}{3} - \alpha)^4)$. Under (5.12), for $\bar{\alpha} = \alpha + \widetilde{\kappa}$, there exist $\lambda = \lambda(\Theta)$ large enough and a constant $C = C(\Theta) > 0$ such that*

$$\|\mathbf{u}\|_{\mathscr{C}^1} \leq 1/2, \quad \|\mathbf{u}\|_{\mathbb{S}_T^{2-\bar{\alpha}}} \leq C. \tag{5.18}$$

Proof We use the paracontrolled ansatz as in (3.3) and write

$$\mathbf{u} = \nabla \mathbf{u} \lll \mathscr{I}_\lambda b_> + \mathscr{I}_\lambda b_> + \mathbf{u}^\sharp,$$

where

$$\mathbf{u}^\sharp := \mathscr{I}_\lambda \big(\nabla \mathbf{u} \prec b_> - \nabla \mathbf{u} \lll b_> + \nabla \mathbf{u} \succ b_> + \Gamma_{\mathbf{u}}^b - [\mathscr{L}_\lambda, \nabla \mathbf{u} \lll] \mathscr{I}_\lambda b_> \big)$$

with

$$\Gamma_{\mathbf{u}}^b := b_> \circ \nabla \mathbf{u} - \bar{b}_\leq \cdot (\nabla \mathbf{u} + \mathbb{I}).$$

Recalling that $\bar{b} = b_> \circ \nabla \mathscr{I}_\lambda b_>$, as in (3.5), we have

$$\begin{aligned} \Gamma_{\mathbf{u}}^b &= b_> \circ (\nabla^2 \mathbf{u} \prec \mathscr{I}_\lambda b_>) + b_> \circ (\nabla \mathbf{u} \prec \nabla \mathscr{I}_\lambda b_>) + b_> \circ \nabla \mathscr{I}_\lambda b_> \\ &\quad + \mathrm{com}_1 + b_> \circ \nabla \mathbf{u}^\sharp - \bar{b}_\leq \cdot (\nabla \mathbf{u} + \mathbb{I}) \\ &= b_> \circ (\nabla^2 \mathbf{u} \prec \mathscr{I}_\lambda b_>) + \mathrm{com}(\nabla \mathbf{u}, \nabla \mathscr{I}_\lambda b_>, b_>) \\ &\quad + \mathrm{com}_1 + b_> \circ \nabla \mathbf{u}^\sharp + \bar{b}_> \cdot (\nabla \mathbf{u} + \mathbb{I}), \end{aligned}$$

where

$$\mathrm{com}_1 := b_> \circ \nabla [\nabla \mathbf{u} \lll \mathscr{I}_\lambda b_> - \nabla \mathbf{u} \prec \mathscr{I}_\lambda b_>].$$

Let

$$\gamma, \beta \in (\bar{\alpha}, 2 - 2\bar{\alpha}].$$

Except for the last term $\bar{b}_> \cdot (\nabla \mathbf{u} + \mathbb{I})$, we estimate each term of $\Gamma^b_{\mathbf{u}}$ as in Lemma 3.3 and obtain

$$\begin{aligned}\|\Gamma^b_{\mathbf{u}}\|_{L^\infty_T \mathbf{C}^{1-2\bar{\alpha}}} &\lesssim \|b_>\|^2_{L^\infty_T \mathbf{C}^{-\bar{\alpha}}} \|\mathbf{u}\|_{\mathbb{S}^{\bar{\alpha}+\gamma}_T} + \|b_>\|_{L^\infty_T \mathbf{C}^{-\bar{\alpha}}} \|\nabla \mathbf{u}^\sharp\|_{L^\infty_T \mathbf{C}^\beta} \\ &\quad + \|\bar{b}_> \cdot (\nabla \mathbf{u} + \mathbb{I})\|_{L^\infty_T \mathbf{C}^{1-2\bar{\alpha}}} \\ &\lesssim \ell^b_T(\rho_\kappa)\Big(\|\mathbf{u}\|_{\mathbb{S}^{\bar{\alpha}+\gamma}_T} + 1\Big) + \sqrt{\ell^b_T(\rho_\kappa)} \|\mathbf{u}^\sharp\|_{L^\infty_T \mathbf{C}^{\beta+1}},\end{aligned}$$

where we have used (5.15), (5.16) and (2.17). As in Lemma 3.4, for any $\theta \in (1+\frac{3\bar{\alpha}}{2}, 2)$, there is a constant $C > 0$ independent of λ such that for all $\lambda \geq 1$,

$$\lambda^{1-\frac{\theta}{2}}(\|\mathbf{u}\|_{\mathbb{S}^{\theta-\bar{\alpha}}_T} + \|\mathbf{u}^\sharp\|_{\mathbb{S}^{2\theta-2\bar{\alpha}-1}_T}) \leq c\ell^b_T(\rho_\kappa)(\|\mathbf{u}\|_{\mathbb{S}^{\theta-\bar{\alpha}}_T} + \|\mathbf{u}^\sharp\|_{\mathbb{S}^{2\theta-2\bar{\alpha}-1}_T} + 1).$$

Taking λ large enough, we get the first estimate in (5.18). The second estimate follows from the same argument as in Lemma 3.4. □

Now, let us define

$$\Phi(t, x) := x + \mathbf{u}(t, x).$$

By Lemma 5.5, it is easy to see that for each $t \in [0, T]$ and $x, y \in \mathbb{R}^d$,

$$\tfrac{1}{2}|x - y| \leq |\Phi(t, x) - \Phi(t, y)| \leq \tfrac{3}{2}|x - y| \tag{5.19}$$

and

$$\partial_t \Phi = \Delta \Phi - \lambda \mathbf{u} + (b_> - \bar{b}_\leq) \cdot \nabla \Phi. \tag{5.20}$$

In particular,

$$x \mapsto \Phi(t, x) \text{ is a } C^1\text{-diffeomorphism.}$$

Let $\Phi^{-1}(t, x)$ be the inverse of $x \mapsto \Phi(t, x)$ and define

$$v(t, x) := u_2(t, \Phi^{-1}(t, x)) \Rightarrow v(t, \Phi(t, x)) = u_2(t, x),$$

where u_2 solves HJB Eq. (5.11).

In the rest of this subsection, if there is no confusion, we also use $\circ$ to denote the composition of two functions. By the chain rule, we have

$$\partial_t v \circ \Phi + \partial_t \Phi \cdot (\nabla v \circ \Phi) = \partial_t u_2, \quad \nabla u_2 = \nabla \Phi \cdot (\nabla v \circ \Phi)$$

and

$$\Delta u_2 = \Delta \Phi \cdot (\nabla v \circ \Phi) + \mathrm{tr}(\widetilde{a} \cdot \nabla^2 v \circ \Phi),$$

where $\widetilde{a}_{ij} := \sum_{k=1}^{d}(\partial_k \Phi^i \partial_k \Phi^j)$, which implies by (5.11) and (5.20) that

$$(\partial_t v)\circ\Phi = \mathrm{tr}(\widetilde{a}\cdot\nabla^2 v\circ\Phi) + H(u_1+u_2, \nabla u_1+\nabla u_2) + ((b_{\le}+\bar{b}_{\le})\cdot\nabla\Phi+\lambda\mathbf{u})\cdot(\nabla v\circ\Phi).$$

Thus we obtain the following key lemma for solving HJB equation (5.11).

Lemma 5.6 *The v defined above solves the following HJB equation:*

$$\partial_t v = \mathrm{tr}\left(a\cdot\nabla^2 v\right) + B\cdot\nabla v + \widetilde{H}(v,\nabla v), \quad v(0)=\varphi_2, \tag{5.21}$$

where $a_{ij} := \sum_{k=1}^{d}(\partial_k\Phi^i\partial_k\Phi^j)\circ\Phi^{-1}$ and

$$B := ((b_{\le}+\bar{b}_{\le})\cdot\nabla\Phi+\lambda\mathbf{u})\circ\Phi^{-1},$$

and for $(t,x,v,Q)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$,

$$\widetilde{H}(t,x,v,Q) := H\big(t,\cdot,u_1(t,\cdot)+v,\nabla u_1(t,\cdot)+\nabla\Phi(t,\cdot)\cdot Q\big)\circ\Phi^{-1}(t,x).$$

Moreover, a satisfies $(\mathbf{H}_a^{1-\bar{\alpha}})$, $B\in\mathbb{L}_T^\infty(\rho_{\widetilde{\kappa}})$, and under $(\mathbf{H}_{\mathrm{sub}}^{\delta,\zeta})$ or $(\mathbf{H}_{\mathrm{crit}}^{\delta,\beta})$ for $\beta\le 1-\bar{\alpha}$, $\widetilde{H}$ still satisfies $(\mathbf{H}_{\mathrm{sub}}^{2\delta,\zeta})$ or $(\mathbf{H}_{\mathrm{crit}}^{2\delta,\beta})$.

Proof (i) By (5.19) and (5.18), we have $\frac{1}{2}\mathbb{I}\le\widetilde{a}\le 2\mathbb{I}$ and

$$|a(t,x)-a(t,y)| \lesssim |\nabla\mathbf{u}(t,\Phi^{-1}(t,x))-\nabla\mathbf{u}(t,\Phi^{-1}(t,y))| \lesssim |\Phi^{-1}(t,x)-\Phi^{-1}(t,y)|^{1-\bar{\alpha}} \lesssim |x-y|^{1-\bar{\alpha}}.$$

(ii) Note that for some $C\ge 1$,

$$C^{-1}\langle x\rangle \le \langle\Phi(t,x)\rangle \le C\langle x\rangle,\ \ \forall t\in[0,T]. \tag{5.22}$$

The assertion $B\in\mathbb{L}_T^\infty(\rho_{\widetilde{\kappa}})$ follows by (5.18) and Lemma 5.3.
(iii) We only check that under $(\mathbf{H}_{\mathrm{crit}}^{\delta,\beta})$, $\widetilde{H}$ satisfies $(\mathbf{H}_{\mathrm{crit}}^{2\delta,\beta})$. For simplicity, we drop the time variable and we only consider H_c part. By (4.5), we have

$$|H_c\big(x,u_1(x)+v,\nabla u_1(x)+\nabla\Phi(x)\cdot Q\big)| \le c_2\langle x\rangle^\delta + c_3'(|Q|^2+|\nabla u_1(x)|^2) \le c_2'\langle x\rangle^{2\delta}+c_3'|Q|^2,$$

where we used $u_1\in\mathbb{S}_T^{2-\alpha}(\rho_\delta)$. By (4.6) and (5.3), we have for $|x-y|\le 1$, $\beta\le 1-\bar{\alpha}$

$$|H_c\big(x,u_1(x)+v,\nabla u_1(x)+\nabla\Phi(x)\cdot Q\big)-H_c\big(y,u_1(y)+v,\nabla u_1(y)+\nabla\Phi(y)\cdot Q\big)| \lesssim |x-y|^\beta\Big(\langle x\rangle^\delta+\langle y\rangle^\delta+|u_1(x)+v|^2+|\nabla u_1(x)+\nabla\Phi(x)\cdot Q|^2\Big)$$

$$
\begin{aligned}
&+ |u_1(x) - u_1(y)|\Big(\langle y\rangle^\delta + |v| + |u_1(x)| + |u_1(y)| + |\nabla u_1(x)| + |Q|\Big)\\
&\quad + (|\nabla u_1(x) - \nabla u_1(y)| + |\nabla\Phi(x) - \nabla\Phi(y)||Q|)\\
&\quad \times (\langle y\rangle^\delta + |u_1(y)| + |v| + |\nabla u_1(x)| + |\nabla u_1(y)| + |Q|)\\
&\lesssim |x-y|^\beta(\langle x\rangle^{2\delta} + \langle y\rangle^{2\delta} + |v|^2 + |Q|^2).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&|\partial_v H_c\big(x, u_1(x) + v, \nabla u_1(x) + \nabla\Phi(x)\cdot Q\big)|\\
&\quad \lesssim \langle x\rangle^\delta + |u_1(x)| + |v| + |\nabla u_1(x)| + |Q| \lesssim \langle x\rangle^\delta + |v| + |Q|.
\end{aligned}
$$

Therefore, $\widetilde{H}$ satisfies ($\mathbf{H}^{2\delta,\beta}_{\rm crit}$) by definition and (5.19), (5.22). □

5.2 Proof of Theorem 5.1

By Lemma 5.6 and Theorem 4.2 we can derive the following a priori estimate for the solution to (5.2).

Lemma 5.7 *Under (5.12), there is a constant $C = C(\Theta) > 0$ such that*

$$
\|u\|_{\mathbb{L}^\infty_T(\rho_{2\delta})} + \|u\|_{\mathbb{S}^{2-\bar\alpha}_T(\rho_\eta)} \le C. \tag{5.23}
$$

Proof Recall $u = u_1 + u_2$, where u_1 solves Eq. (5.10) and u_2 solves Eq. (5.11). By Theorem 3.7, one has

$$
\|u_1\|_{\mathbb{S}^{2-\bar\alpha}_T(\rho_\delta)} \lesssim 1.
$$

Hence, to prove (5.23), due to $\eta \ge 2\delta$, it suffices to prove that

$$
\|u_2\|_{\mathbb{L}^\infty_T(\rho_{2\delta})} + \|u_2\|_{\mathbb{S}^{2-\bar\alpha}_T(\rho_\eta)} \lesssim 1. \tag{5.24}
$$

Note that by Lemma 5.6, $v = u_2(\Phi)$ solves (5.21). In particular, by Lemma 5.6 and Theorem 4.2, for p large enough and η satisfying (5.6),

$$
\|v\|_{\mathbb{L}^\infty_T(\rho_{2\delta})} + \|\partial_t v\|_{\mathbb{L}^p_T(\rho_\eta)} + \|v\|_{\mathbb{H}^{2,p}_T(\rho_\eta)} \lesssim 1, \tag{5.25}
$$

which implies by [12, Lemma 2.3],

$$
\|v\|_{C^{(2-\bar\alpha)/2}_T L^\infty(\rho_\eta)} \lesssim 1. \tag{5.26}
$$

By (5.22), we have

$$
\|u_2\|_{\mathbb{L}^\infty_T(\rho_{2\delta})} = \|v(\Phi)\rho_{2\delta}\|_{\mathbb{L}^\infty_T} \asymp \|v(\Phi)\rho_{2\delta}(\Phi)\|_{\mathbb{L}^\infty_T} = \|v\|_{\mathbb{L}^\infty_T(\rho_{2\delta})},
$$

and by (2.17), (5.25) and (5.18),

$$\begin{aligned}\|\nabla u_2\|_{L^\infty_T\mathbf{C}^{1-\bar\alpha}(\rho_\eta)} &= \|\nabla v\circ\Phi\cdot\nabla\Phi\|_{L^\infty_T\mathbf{C}^{1-\bar\alpha}(\rho_\eta)}\\ &\lesssim \|\nabla v(\Phi)\|_{L^\infty_T\mathbf{C}^{1-\bar\alpha}(\rho_\eta)}\|\nabla\Phi\|_{L^\infty_T\mathbf{C}^{1-\bar\alpha}}\\ &\lesssim \|\nabla v\|_{L^\infty_T\mathbf{C}^{1-\bar\alpha}(\rho_\eta)}(\|\mathbf{u}\|_{L^\infty_T\mathbf{C}^{2-\bar\alpha}}+1)\lesssim 1,\end{aligned}$$

where in the second inequality, we have used that for $|x-y|\leq 1$,

$$\begin{aligned}\rho_\eta(x)|\nabla v(\Phi(x))-\nabla v(\Phi(y))| &\overset{(5.22)}{\lesssim} \rho_\eta(\Phi(x))|\nabla v(\Phi(x))-\nabla v(\Phi(y))|\\ &\overset{(2.1),(5.19)}{\lesssim} |\Phi(x)-\Phi(y)|^{1-\bar\alpha}\|\nabla v\|_{L^\infty_T\mathbf{C}^{1-\bar\alpha}(\rho_\eta)}.\end{aligned}$$

Moreover, by (5.22), we also have

$$\begin{aligned}\|u_2(t)-u_2(s)\|_{L^\infty(\rho_\eta)} &\lesssim \|v(t,\Phi(t))-v(t,\Phi(s))\|_{L^\infty(\rho_\eta)}+\|v(t)-v(s)\|_{L^\infty(\rho_\eta)}\\ &\leq \|\Phi(t)-\Phi(s)\|_{L^\infty}\int_0^1\|\nabla v(t,\Gamma^{t,s}_r)\|_{L^\infty(\rho_\eta)}\mathrm{d}r\\ &\quad+\|v(t)-v(s)\|_{L^\infty(\rho_\eta)},\end{aligned}$$

where $\Gamma^{t,s}_r(x):=r\Phi(t,x)+(1-r)\Phi(s,x)$. Since for any $r\in[0,1]$ and $t,s\in[0,T]$,

$$\Gamma^{t,s}_r(x)=x+r\mathbf{u}(t,x)+(1-r)\mathbf{u}(s,x),$$

by (5.18), we have

$$\rho_\eta(\Gamma^{t,s}_r(x))\asymp\rho_\eta(x).$$

Hence, by (5.18) and (5.26),

$$\frac{\|u_2(t)-u_2(s)\|_{L^\infty(\rho_\eta)}}{|t-s|^{(2-\bar\alpha)/2}}\lesssim 1.$$

Combining the above estimates, we obtain (5.24). The proof is complete. □

Next we apply (5.23), (5.4) and (5.5) to derive the following a priori estimate for $u^\sharp$ as done in Lemma 3.3.

Lemma 5.8 *Under (5.12), there is a constant $C=C(\Theta)>0$ such that*

$$\|u^\sharp\|_{\mathbb{L}^\infty_T(\rho_{2\delta+\kappa})}+\|u^\sharp\|_{\mathbb{S}^{3-2\bar\alpha}_T(\rho_{2\eta})}\leq C. \tag{5.27}$$

Proof First of all, by (5.4) and (5.23), we have

$$\|u^\sharp\|_{\mathbb{L}^\infty_T(\rho_{2\delta+\kappa})}+\|u^\sharp\|_{L^\infty_T\mathbf{C}^{2-\bar\alpha}(\rho_{\eta+\kappa})}\lesssim 1. \tag{5.28}$$

Next we estimate each term on the right hand side of (5.5) by using Lemma 2.10.

- By (2.21), (2.4), and $\bar\alpha = \alpha + \widetilde\kappa$, we have

$$\|\nabla u \prec b - \nabla u \prec\!\!\prec b\|_{L^\infty_T \mathbf{C}^{1-2\bar\alpha}(\rho_{\eta+\kappa})} \lesssim \|u\|_{\mathbb{S}^{2-\bar\alpha}_T(\rho_\eta)} \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)} \lesssim 1.$$

- By (2.15) we have

$$\|\nabla u \succ b\|_{L^\infty_T \mathbf{C}^{1-2\bar\alpha}(\rho_{\eta+\kappa})} \lesssim \|u\|_{L^\infty_T \mathbf{C}^{2-\bar\alpha}(\rho_\eta)} \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)} \lesssim 1.$$

- By (2.20) and (2.12) we have

$$\|[\mathscr{L}, \nabla u \prec\!\!\prec]\mathscr{I} b\|_{L^\infty_T \mathbf{C}^{1-2\bar\alpha}(\rho_{\eta+\kappa})} \lesssim \|u\|_{\mathbb{S}^{2-\bar\alpha}_T(\rho_\eta)} \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)} \lesssim 1.$$

- By the growth of H and (5.23), we have

$$\|H(u, \nabla u)\|_{\mathbb{L}^\infty_T(\rho_{2\eta})} \lesssim 1 + \|\nabla u\|^2_{\mathbb{L}^\infty_T(\rho_\eta)} \lesssim 1.$$

- By Lemma 3.3 with $\gamma = 2 - 2\bar\alpha$, $\beta \in (\bar\alpha, 2 - 2\bar\alpha)$, we have

$$\|b \circ \nabla u\|_{L^\infty_T \mathbf{C}^{1-2\bar\alpha}(\rho_{2\eta})} \lesssim \|u\|_{\mathbb{S}^{2-\bar\alpha}_T(\rho_{2\eta-2\kappa})} + \|u^\sharp\|_{L^\infty_T \mathbf{C}^{\beta+1}(\rho_{2\eta-\kappa})} + 1,$$

  and by interpolation inequality (2.5) with $\theta = \frac{\eta-2\kappa}{\eta-\kappa}$, (5.28) and Young's inequality,

$$\begin{aligned}\|u^\sharp\|_{L^\infty_T \mathbf{C}^{\beta+1}(\rho_{2\eta-\kappa})} &\lesssim \|u^\sharp\|^\theta_{L^\infty_T \mathbf{C}^{3-2\bar\alpha}(\rho_{2\eta})} \|u^\sharp\|^{1-\theta}_{L^\infty_T \mathbf{C}^{2-\bar\alpha}(\rho_{\eta+\kappa})} \\ &\lesssim \varepsilon \|u^\sharp\|_{L^\infty_T \mathbf{C}^{3-2\bar\alpha}(\rho_{2\eta})} + 1,\end{aligned}$$

  where we choose β such that $\beta \leq (1 - \bar\alpha)(\theta + 1)$ since κ is small enough.

Combining the above calculations and by (2.11) with $\theta = 2$ and $q = \infty$, we obtain

$$\|u^\sharp\|_{\mathbb{S}^{3-2\bar\alpha}_T(\rho_{2\eta})} \lesssim \varepsilon \|u^\sharp\|_{L^\infty_T \mathbf{C}^{3-2\bar\alpha}(\rho_{2\eta})} + 1,$$

which in turn implies the desired estimate. □

Now we are in a position to give the proof of Theorem 5.1.

Proof of Theorem 5.1 (Existence) By (5.23) and (5.27), we obtain the uniform estimate (5.9). Now by Ascoli-Arzelà's lemma, there are a subsequence still denoted by n and

$$(u, u^\sharp) \in \mathbb{S}^{2-\bar\alpha}_T(\rho_\eta) \times \mathbb{S}^{3-2\bar\alpha}_T(\rho_{2\eta})$$

such that for each $\gamma > 0$,

$$(u_n, u^\sharp_n) \to (u, u^\sharp) \text{ in } \mathbb{S}^{2-\bar\alpha-\gamma}_T(\rho_{\eta+\gamma}) \times \mathbb{S}^{3-2\bar\alpha-\gamma}_T(\rho_{2\eta+\gamma}).$$

By (5.7) and taking weak limits for approximation equation (5.4) and (5.5) with (b, f) being replaced by (b_n, f_n), one sees that $(u, u^\sharp)$ solves (5.4) and (5.5) (see [18] for more details).

(Uniqueness) Let $u, \bar{u}$ be two paracontrolled solutions to (5.2) in the sense of Theorem 5.1 starting from the same initial value. Let $U := u - \bar{u}$. It is easy to see that U is a paracontrolled solution to the following linear equation

$$\partial_t U = \Delta U + (b + R) \cdot \nabla U + K \cdot U, \quad U(0) = 0, \tag{5.29}$$

where

$$R := \int_0^1 \nabla_Q H(u, \nabla u + s\nabla(\bar{u} - u))\mathrm{d}s,$$
$$K := \int_0^1 \partial_u H(u + s(\bar{u} - u), \nabla \bar{u})\mathrm{d}s.$$

Note that by (5.3) and $u, \bar{u} \in \mathbb{S}_T^{2-\bar{\alpha}}(\rho_\eta)$,

$$|R| + |K| \lesssim \rho_\delta^{-1} + |u| + |\bar{u}| + |\nabla \bar{u}| + |\nabla u| \lesssim \rho_\eta^{-1}.$$

Then uniqueness follows from Theorem A.2. □

6 Applications

In this section we apply the main results in Sect. 5 to the KPZ type Eqs. (1.3) and (1.4).

6.1 KPZ type equations

Consider the following KPZ type equation:

$$\mathscr{L}h = (\partial_x h)^{\diamond 2} + g(h) + \xi, \quad h(0) = h_0 \tag{6.1}$$

where $g \in \mathscr{C}^1$ and ξ is a space-time white noise on $\mathbb{R}^+ \times \mathbb{R}$ on some stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Here the nonlinear term $(\partial_x h)^{\diamond 2} =$ "$(\partial_x h)^2 - \infty$" with $\infty = \lim_{n\to\infty} c_n^{\curlyvee}$ for $c_n^{\curlyvee}$ and the approximation ξ_n being defined below.

For $g \equiv 0$ Eq. (6.1) is the classical KPZ equation. The motivation for adding the nonlinear term g comes from geometric stochastic heat equations with values in a Riemannian manifold M studied in [4] via regularity structure theory and in [11, 49] by Dirichlet form, which, in local coordinates, can be written as

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\partial_x u^\beta \partial_x u^\gamma + h^\alpha(u) + \sigma_i^\alpha(u)\xi_i, \tag{6.2}$$

where Γ denotes the Christoffel symbols for the Levi-Civita connection and ξ_i are i.i.d. space-time white noises and σ_i are a collection of vector fields on the manifold. We use Einstein's convention of summation over repeated indices. The first three terms in (6.2) correspond to Eells-Sampson's harmoic map flow [14] and h^α corresponds to the $\frac{1}{8}\nabla R$ with R the scalar curvature of M. For more background on (6.2) we refer to [4] and [11, 49]. As (6.2) is driven by multiplicative noise, there are more than forty terms required for renormalization and regularity structure theory is required to derive local well-posedness of (6.2). It is also interesting to study (6.2) on the whole line (see [11] for different long time behavior compared to the finite volume case). As directly obtaining global well-posedness to Eq. (6.2) by PDE argument is out of reach by the techniques so far, we study (6.1) and apply our main result.

We define the $2n$ periodization of ξ by

$$\tilde{\xi}_n(\psi) = \xi(\psi_n) \text{ where } \psi_n(t,x) = \mathbf{1}_{[-n,n)}(x) \sum_{y \in 2n\mathbb{Z}} \psi(t, x+y).$$

Let $\varphi \in C_c^\infty(\mathbb{R})$ be even and such that $\varphi(0) = 1$ and define the spatial regularization of $\tilde{\xi}_n$

$$\xi_n = \varphi(n^{-1}\partial_x)\tilde{\xi}_n = \mathcal{F}^{-1}(\varphi(n^{-1}\cdot)\mathcal{F}\tilde{\xi}_n).$$

The regularity of the space-time white noise ξ is more rough than the coefficient f given in (1.5). To apply Theorem 5.1 we need to introduce the following random fields and use Da Prato-Debussche trick (cf. [13]) to decompose (6.1) into (1.5) and the following equations, which is the usual way for the KPZ equation (cf. [23, 26, 45]):

$$\begin{aligned} \mathscr{L}Y_n &= \xi_n & \mathscr{L}Y &= \xi \\ \mathscr{L}Y_n^{\vee} &= (\partial_x Y_n)^2 - c_n^{\vee} & \mathscr{L}Y_n^{\curlyvee} &= 2\partial_x Y_n \partial_x Y_n^{\vee} \\ \mathscr{L}Y_n^{\curlyvee\!\vee} &= 2\partial_x Y_n^{\curlyvee} \circ \partial_x Y_n + c_n^{\triangledown} & \mathscr{L}Y_n^{\triangledown} &= (\partial_x Y_n^{\vee})^2 - c_n^{\triangledown} \\ \mathscr{L}Y_n^{\prime} &= \partial_x Y_n, \end{aligned} \tag{6.3}$$

all with zero initial conditions except $Y(0)(x) = Cx + B(x)$ and $Y_n(0)$ defined similarly as ξ_n with ξ replaced by $Cx + B(x)$, where B is a two sided Brownian motion, which is independent of space-time white noise ξ, and $C \in \mathbb{R}$. The choice of the initial condition is due to our interest in the KPZ equation starting from its invariant measure (cf. [48, Section 1.4] and [16]). Here $c_n^{\vee}$ and $c_n^{\triangledown}$ are renormalization constants. For simplicity of notation we also set

$$X_n = \partial_x Y_n, \quad X = \partial_x Y, \quad X^{(\cdot)} = \partial_x Y^{(\cdot)},$$

where $(\cdot)$ stands for the above trees. In the following we draw a table for the regularity of each $Y^{(\cdot)}$. For $\gamma > 0$ the homogeneities $\alpha_\tau \in \mathbb{R}$ are given by

τ	$=$	Y	$Y^{\<2>}$	$Y^{\<3>}$	$Y^{\<4>}$	$Y^{\<22>}$
α_τ	$=$	$\frac{1}{2}-\gamma$	$1-\gamma$	$\frac{3}{2}-\gamma$	$2-\gamma$	$2-\gamma$
τ	$=$	X	$Y^{\<1>}$	$\partial_x Y^{\<1>}\circ\partial_x Y$	$\mathscr{L}Y^{\<4>}$	$\mathscr{L}Y^{\<22>}$
α_τ	$=$	$-\frac{1}{2}-\gamma$	$\frac{3}{2}-\gamma$	$-\gamma$	$-\gamma$	$-\gamma$

Lemma 6.1 *With the above notations, there exist random distributions*

$$\mathscr{Y} := \left\{Y^{\<2>}, Y^{\<3>}, Y^{\<4>}, Y^{\<22>}, X, Y^{\<1>}, \partial_x Y^{\<1>} \circ \partial_x Y, \mathscr{L}Y^{\<4>}, \mathscr{L}Y^{\<22>}\right\}$$

and divergence constants $c_n^{\<2>}$, $c_n^{\<22>}$ *such that for every* $\tau \in \mathscr{Y}$,

$$\tau \in \cap_{\kappa>0}\mathbb{S}_T^{\alpha_\tau}(\rho_\kappa),$$

for α_τ *given in the above table. Moreover, for* τ_n *defined in (6.3)* $\tau_n \to \tau$ *in* $L^p(\Omega, \mathbb{S}_T^{\alpha_\tau}(\rho_\kappa))$ *for every* $p \in [1,\infty)$ *and every* $\kappa > 0$*. Furthermore,* $Y_n \to Y$ *in* $L^p(\Omega, \mathbb{S}_T^{\frac{1}{2}-\gamma}(\rho_{1+\kappa})$ *for every* $p \in [1,\infty)$*. Finally, there exist random distribution* $\nabla\mathscr{I}_s^t(X)\circ X$ *such that*

$$\sup_{0\le s\le t\le T}\|\nabla\mathscr{I}_s^t(X_n)\circ X_n(t) - \nabla\mathscr{I}_s^t(X)\circ X(t)\|_{\mathbf{C}^{-\gamma}(\rho_\kappa)} \to 0 \text{ in } L^p(\Omega).$$

Proof Most terms except $\mathscr{L}Y^{\<4>}$, $\mathscr{L}Y^{\<22>}$ in (6.3) have been considered in [45, Theorem 3.6]. These two terms can also been obtained by similar calculation as in [23, Theorem 9.3] (see also [55, Section 3.3.1, Section A.2]). The last convergence result for $\nabla\mathscr{I}_s^t(X)\circ X(t)$ can be obtained similarly as in [45, Lemma C.1]. For reader's convenience we spell out more details for completeness and we follow the notation of [23, Section 9].

Let W be the space-time white noise in Fourier space. We write $\nabla\mathscr{I}_s^t(X_n)\circ X_n(t)$ as

$$\nabla\mathscr{I}_s^t(X_n)\circ X_n(t) = \int e^{ik_{[12]}x}\psi_0(k_1,k_2)H_{t-s_1}^n(k_1) \times \int_s^t \mathrm{d}\sigma\, H_{t-\sigma}(k_2)H_{\sigma-s_2}^n(k_2)W(\mathrm{d}\eta_1)W(\mathrm{d}\eta_2),$$

with $H_t(k) = ik\mathrm{e}^{-k^2t}\mathbf{1}_{t\ge0}$, $H_t^n(k) = H_t(k)\varphi(n^{-1}k)$, $\eta_i = (s_i,k_i)$, $s_{-i} = s_i, k_{-i} = k_i$, $\mathrm{d}\eta_i = \mathrm{d}s_i\mathrm{d}k_i$, $k_{[12]} = k_1+k_2$, $\psi_0(k_1,k_2) = \sum_{|i-j|\le1}\theta_i(k_1)\theta_j(k_2)$ for θ_i being the dyadic partition of unity. By Wiener chaos decomposition the term in the zeroth order chaos is given by

$$\int H_{t-s_1}^n(k_1)\int_s^t \mathrm{d}\sigma\, H_{t-\sigma}(-k_1)H_{\sigma-s_1}^n(-k_1)\mathrm{d}\eta_1,$$

which is zero by using the fact that the integrand is antisymmetric under the change of variables $k_1 \to -k_1$. For the second order chaos we calculate for $0 \le s \le r \le t$

$$\begin{aligned}
&\mathbb{E}|\Delta_q(\nabla \mathscr{I}_s^t(X_n) \circ X_n - \nabla \mathscr{I}_r^t(X_n) \circ X_n)|^2 \\
&\lesssim \int |\theta_q(k_{[12]})|^2 \psi_0(k_1,k_2)^2 |H_{t-s_1}(k_1)|^2 \left| \int_s^r \mathrm{d}\sigma\, H_{t-\sigma}(k_2) H^n_{\sigma-s_2}(k_2)\right|^2 \mathrm{d}\eta_1 \mathrm{d}\eta_2 \\
&\lesssim |r-s|^{\varepsilon} \int_{E^2} |\theta_q(k_{[12]})|^2 \psi_0(k_1,k_2)^2 (|k_2|+1)^{-2+2\varepsilon} \mathrm{d}k_1 \mathrm{d}k_2 \\
&\lesssim |r-s|^{\varepsilon} 2^{2q\varepsilon},
\end{aligned}$$

where the implicit constant is independent of n. The rest of the proof follows by standard arguments as in [23]. □

We make the following decomposition

$$h = Y + Y^{\vee} + Y^{\vee\vee} + \widetilde{h},$$

where $\widetilde{h}$ satisfies the following equation

$$\begin{cases} \mathscr{L}\widetilde{h} = 2\partial_x \widetilde{h}(X + X^{\vee} + X^{\vee\vee}) + (\partial_x \widetilde{h})^2 + \mathscr{L}Y^{\vee\vee} + \mathscr{L}Y^{\vee\vee\vee} \\ \qquad + (X^{\vee\vee})^2 + 2X^{\vee\vee}X^{\vee} + 2(XX^{\vee\vee} - X \circ X^{\vee\vee}) \\ \qquad + g(Y + Y^{\vee} + Y^{\vee\vee} + \widetilde{h}), \\ \widetilde{h}(0) = h_0 - Y(0). \end{cases} \tag{6.4}$$

Here we use (6.3).

Using Lemma 6.1, we obtain the following lemma.

Lemma 6.2 *There exists a measurable set Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that for every $\kappa > 0$, $\gamma \in (0, \frac{1}{4})$ and $\omega \in \Omega_0$*

$$\begin{aligned}
&b := 2(X + X^{\vee} + X^{\vee\vee}) \in L_T^\infty \mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa), \\
&f := \mathscr{L}Y^{\vee\vee} + \mathscr{L}Y^{\vee\vee\vee} + (X^{\vee\vee})^2 + 2X^{\vee\vee}X^{\vee} + 2(XX^{\vee\vee} - X \circ X^{\vee\vee}) \in L_T^\infty \mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa).
\end{aligned}$$

Proof By Lemma 2.10 and (2.11) we have that

$$\begin{aligned}
\|(X^{\vee\vee})^2\|_{\mathbf{C}^{\frac{1}{2}-\gamma}(\rho_\kappa)} &\lesssim \|X^{\vee\vee}\|^2_{\mathbf{C}^{\frac{1}{2}-\gamma}(\rho_{\kappa/2})}, \\
\|X^{\vee\vee}X^{\vee}\|_{\mathbf{C}^{-\gamma}(\rho_\kappa)} &\lesssim \|X^{\vee\vee}\|_{\mathbf{C}^{\frac{1}{2}-\gamma}(\rho_{\kappa/2})} \|X^{\vee}\|_{\mathbf{C}^{-\gamma}(\rho_{\kappa/2})},
\end{aligned}$$

and

$$XX^{\vee\vee} - X \circ X^{\vee\vee} = X \succ X^{\vee\vee} - X \prec X^{\vee\vee},$$

to have

$$\|XX^{\curlyvee} - X \circ X^{\curlyvee}\|_{\mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa)} \lesssim \|X\|_{\mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_{\kappa/2})}\|X^{\curlyvee}\|_{\mathbf{C}^{\frac{1}{2}-\gamma}(\rho_{\kappa/2})}.$$

Other terms follows directly from Lemma 6.1. □

As a result $\widetilde{h}$ satisfies (1.5) with b, f given above. We say that h is a paracontrolled solution to (6.1) if $\widetilde{h}$ is a paracontrolled solution to (6.4) in the sense of (5.4) and (5.5).

Since γ can be arbitrary small, we apply Theorem 5.1 to obtain the following result.

Theorem 6.3 *Suppose $g \in \mathscr{C}^1$. For every initial condition $\widetilde{h}(0) \in \mathbf{C}^{\frac{3}{2}+\varepsilon+\gamma}(\rho_{\varepsilon\delta})$ where $0 < \varepsilon < 1, \gamma \in (0, \frac{1}{4}), 0 < \delta := 40\kappa < 1$, there exists a unique paracontrolled solution*

$$(\widetilde{h}, \widetilde{h}^\sharp) \in (\mathbb{S}_T^{\frac{3}{2}-\kappa^{1/4}-\gamma}(\rho_\eta) \cap \mathbb{L}_T^\infty(\rho_{2\delta}), \mathbb{S}_T^{2-2\kappa^{1/4}-2\gamma}(\rho_{2\eta}) \cap \mathbb{L}_T^\infty(\rho_{2\delta+\kappa}))$$

to (6.4), where

$$2(\kappa^{1/4} + 80\kappa) < \eta < \tfrac{1}{4}.$$

Proof In the following we check other conditions of Theorem 5.1. The condition for $H = H_c + H_s$ is satisfied easily by Lemma 6.1 where $H_c = Q^2$, $H_s = g(Y + Y^{\vee} + Y^{\curlyvee} + Q)$. In the following we prove $(b, f) \in \mathbb{B}_T^\alpha(\rho_\kappa)$. The approximation $\{(b_n, f_n)\}_n$ for (b, f) is given as in Lemma 6.2 with the corresponding tree τ replaced by τ_n in Lemma 6.1. In the following we prove that for every $\kappa > 0$

$$\sup_n(\ell_T^{b_n}(\rho_\kappa) + \mathbb{A}_{T,\infty}^{b_n,f_n}(\rho_\kappa)) < \infty, \tag{6.5}$$

with $\ell_T^{b_n}(\rho_\kappa)$ and $\mathbb{A}_{T,\infty}^{b_n,f_n}(\rho_\kappa))$ defined in (2.25) and (2.24), respectively. In the following we omit the subscript n for simplicity and all the following bounds are uniform in n and λ. Note that

$$\frac{1}{4}\nabla\mathscr{I}_\lambda(b) \circ b = \nabla\mathscr{I}_\lambda(X + X^{\vee} + X^{\curlyvee}) \circ (X + X^{\vee} + X^{\curlyvee}).$$

By the last result in Lemma 6.1 and Lemma 2.16 we deduce the first term

$$\|\nabla\mathscr{I}_\lambda X \circ X\|_{L_T^\infty\mathbf{C}^{-\gamma}(\rho_\kappa)} \lesssim 1.$$

Other terms on the right hand side can be calculated by Lemma 2.10 and (2.11) to have

$$\begin{aligned}&\|\nabla\mathscr{I}_\lambda(X^{\vee} + X^{\curlyvee}) \circ b\|_{L_T^\infty\mathbf{C}^{-\gamma}(\rho_{2\kappa})}\\ &\quad\lesssim \Big(\|Y^{\vee}\|_{L_T^\infty\mathbf{C}^{1-\gamma}(\rho_\kappa)} + \|Y^{\curlyvee}\|_{L_T^\infty\mathbf{C}^{\frac{3}{2}-\gamma}(\rho_\kappa)}\Big)\|b\|_{L_T^\infty\mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa)} \lesssim 1,\end{aligned}$$

and

$$\begin{aligned}&\|\nabla \mathscr{I}_\lambda X \circ (X^{\vee} + X^{\vee\vee})\|_{L_T^\infty \mathbf{C}^{-\gamma}(\rho_{2\kappa})}\\&\lesssim \|Y\|_{L_T^\infty \mathbf{C}^{\frac{1}{2}-\gamma}(\rho_\kappa)}(\|Y^{\vee}\|_{L_T^\infty \mathbf{C}^{1-\gamma}(\rho_\kappa)} + \|Y^{\vee\vee}\|_{L_T^\infty \mathbf{C}^{\frac{3}{2}-\gamma}(\rho_\kappa)}) \lesssim 1.\end{aligned}$$

On the other hand, note that

$$\nabla \mathscr{I}_\lambda f \circ b = \nabla \mathscr{I}_\lambda f_1 \circ b + \nabla \mathscr{I}_\lambda (X^{\vee\vee} \prec X) \circ 2(X + X^{\vee} + X^{\vee\vee}),$$

with $f_1 = f - X^{\vee\vee} \prec X \in L_T^\infty \mathbf{C}^{-2\gamma}(\rho_\kappa)$. By Lemma 2.10 and (2.11) we know

$$\|\nabla \mathscr{I}_\lambda f_1 \circ b\|_{L_T^\infty \mathbf{C}^{-\gamma}(\rho_{2\kappa})} \lesssim \|f_1\|_{L_T^\infty \mathbf{C}^{-2\gamma}(\rho_\kappa)} \|b\|_{L_T^\infty \mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa)} \lesssim 1,$$

and

$$\begin{aligned}&\|\nabla \mathscr{I}_\lambda (X^{\vee\vee} \prec X) \circ (X^{\vee} + X^{\vee\vee})\|_{L_T^\infty \mathbf{C}^{-\gamma}(\rho_{2\kappa})}\\&\lesssim \|X^{\vee\vee}\|_{L_T^\infty \mathbf{C}^{\frac{1}{2}-\gamma}(\rho_{\kappa/2})} \|X\|_{L_T^\infty \mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_{\kappa/2})} (\|X^{\vee}\|_{L_T^\infty \mathbf{C}^{-\gamma}(\rho_\kappa)} + \|X^{\vee\vee}\|_{L_T^\infty \mathbf{C}^{\frac{1}{2}-\gamma}(\rho_\kappa)}) \lesssim 1.\end{aligned}$$

It remains to consider the term $\nabla \mathscr{I}_\lambda (X^{\vee\vee} \prec X) \circ X$ and we use the commutator introduced in Lemma 2.11 and Lemma 2.12 to have

$$\begin{aligned}\nabla \mathscr{I}_\lambda (X^{\vee\vee} \prec X) \circ X &= ([\nabla \mathscr{I}_\lambda, X^{\vee\vee} \prec] X) \circ X\\&\quad + \mathrm{com}(X^{\vee\vee}, \nabla \mathscr{I}_\lambda X, X) + X^{\vee\vee} (\nabla \mathscr{I}_\lambda X \circ X).\end{aligned}$$

By Lemmas 2.12, 2.11 and Lemma 6.1 we have

$$\|\nabla \mathscr{I}_\lambda (X^{\vee\vee} \prec X) \circ X\|_{L_T^\infty \mathbf{C}^{-\gamma}(\rho_\kappa)} \lesssim 1,$$

where we used time regularity of $X^{\vee\vee}$, which follows from (2.4). Combining all the above estimates, we deduce (6.5) follows. Furthermore, we know that the convergence in Definition 2.14 also holds by Lemma 6.1 and Lemma 2.16, which gives that $(b, f) \in \mathbb{B}_T^\alpha(\rho_\kappa)$. Then the result follows from Theorem 5.1. □

Remark 6.4 1. The exponent η of the weight could be arbitrarily small since κ is arbitrarily small. This result improves the weight for the solution to the KPZ equation obtained in [45].

2. In the finite volume case, the initial value φ could be more rough, and the fixed point argument allows $\varphi \in \cup_{\varepsilon>0} \mathbf{C}^\varepsilon$ (see [26]). In the infinite volume case, the singularity near $t = 0$ seems to break the energy estimate in Lemma 4.8. We shall study this problem in the future.

3. We may also consider the following more general singular SPDEs

$$\partial_t h = \partial_x^2 h + |\partial_x h|^2 + g(h) + K(h)\partial_x h + \xi, \tag{6.6}$$

for $g, K \in \mathscr{C}^1$. We have the decomposition

$$h = Y + Y^{\vee} + Y^{\vee^{\vee}} + \widetilde{h},$$

with $\widetilde{h}$ satisfying

$$\begin{aligned} \partial_t \widetilde{h} = b \cdot \nabla \widetilde{h} + f + (\partial_x \tilde{h})^2 + g(Y + Y^{\vee} + Y^{\vee^{\vee}} + \widetilde{h}) \\ + K(Y + Y^{\vee} + Y^{\vee^{\vee}} + \widetilde{h})(\partial_x \tilde{h} + X + X^{\vee} + X^{\vee^{\vee}}), \end{aligned} \tag{6.7}$$

for b, f given in Lemma 6.2. Since $K(Y + Y^{\vee} + Y^{\vee^{\vee}} + \widetilde{h})X$ requires further renormalization and in this paper we mainly concentrate on the singular renormalized terms from $|\partial_x h|^2$, we consider the following simplified equation

$$\begin{aligned} \partial_t \widetilde{h} = b \cdot \nabla \widetilde{h} + f + (\partial_x \tilde{h})^2 + g(Y + Y^{\vee} + Y^{\vee^{\vee}} + \widetilde{h}) \\ + K(Y + Y^{\vee} + Y^{\vee^{\vee}} + \widetilde{h})(\partial_x \tilde{h} + X^{\vee^{\vee}}), \end{aligned} \tag{6.8}$$

where $K, g \in \mathscr{C}^1$ and the most singular terms coming from $(\partial_x h)^2$ in (6.6) have been included. We can apply Theorem 5.1 to obtain the same global well-posedness for Eq. (6.8).

4. A challenging question is whether PDE arguments can be used to deduce global well-posedness of vector-valued generalized KPZ equations since it is not clear whether the maximum principle can be extended to cover such a situation. We leave this for our future work.

6.2 Modified KPZ equations

In this subsection we consider the following modified KPZ equation:

$$\mathscr{L} h = g(x)(\partial_x h)^{\diamond 2} + K(x)\partial_x h + \xi, \quad h(0) = h_0 \tag{6.9}$$

where $g, K \in \mathscr{C}^1$ and ξ is a space-time white noise on $\mathbb{R}^+ \times \mathbb{R}$ on some stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P})$. This model can be derived similarly like KPZ equation as surface growth model where the growth rate also depending on position x (c.f. [36]). We emphasize that for this model we cannot use Cole–Hopf's transformation to directly obtain the well-posedness since for $w = e^{gh}$ there exists some new nonlinear terms in the equation of w which cannot be cancelled.

Here the nonlinear term requires renormalization and we define the spatial regularization of ξ as in Sect. 6.1. To apply Theorem 5.1 we also introduce the following random fields as in Sect. 6.1 and use Da Prato-Debussche trick to decompose (6.9) into (1.5) and the following equations:

$$
\begin{aligned}
&\mathscr{L}Y_n = \xi_n && \mathscr{L}Y = \xi \\
&\mathscr{L}\bar Y_n^{\vee} = g[(\partial_x Y_n)^2 - c_n^{\vee}] && \mathscr{L}\bar Y_n^{\curlyvee} = g[2\partial_x Y_n \partial_x \bar Y_n^{\vee}] \\
&\mathscr{L}\bar Y_n^{\Upsilon} = g[2\partial_x \bar Y_n^{\curlyvee} \circ \partial_x Y_n + g^2 c_n^{\nabla}] && \mathscr{L}\bar Y_n^{\nabla} = g[(\partial_x \bar Y_n^{\vee})^2 - g^2 c_n^{\nabla}] \\
&\mathscr{L}Y_n^{\prime} = \partial_x Y_n,
\end{aligned}
\tag{6.10}
$$

all with zero initial conditions except $Y(0)(x) = B(x)+Cx, C \in \mathbb{R}$, and $Y_n(0)$ defined similarly as ξ_n with ξ replaced by $B(x)+Cx$, where B is a two sided Brownian motion, which is independent of space-time white noise ξ. Here $c_n^{\vee}$ and c_n^{∇} are renormalization constants as in (6.3). We also set

$$
X_n = \partial_x Y_n, \quad X = \partial_x Y, \quad \bar X^{(\cdot)} = \partial_x \bar Y^{(\cdot)},
$$

where $(\cdot)$ stands for the above tree. The regularity and the homogeneities of each $\bar Y^{(\cdot)}$ are the same as the corresponding $Y^{(\cdot)}$ if the trees in the superscript are the same.

Lemma 6.5 *With the above notations, there exist random distributions*

$$
\mathscr{Y} := \left\{ \bar Y^{\vee}, \bar Y^{\curlyvee}, \bar Y^{\Upsilon}, \bar Y^{\nabla}, X, Y^{\prime}, \partial_x Y^{\prime} \circ \partial_x Y, \mathscr{L}\bar Y^{\Upsilon}, \mathscr{L}\bar Y^{\nabla} \right\}
$$

and divergence constants $c_n^{\vee}$, c_n^{∇} such that for every $\tau \in \mathscr{Y}$,

$$
\tau \in \cap_{\kappa>0} \mathbb{S}_T^{\alpha_\tau}(\rho_\kappa),
$$

for α_τ given in the above table. Moreover, for τ_n defined in (6.10) $\tau_n \to \tau$ in $L^p(\Omega, \mathbb{S}_T^{\alpha_\tau}(\rho_\kappa))$ for every $p \in [1,\infty)$ and every $\kappa > 0$.

Proof If terms in the bracket of (6.10) converge in the corresponding space as $n \to \infty$, we can obtain results easily by Schauder estimate. However, $[(\partial_x Y_n)^2 - c_n^{\vee}]$ does not converge in spatial distribution space and we have to do probabilistic calculation again. We follow the method and notation in [15]. Let $K_{j,x}(y) = 2^j K(2^j(x-y))$ be the kernel associated with the j-th Littlewood-Paley block Δ_j on $\mathbb{R}$. For a function f we write $\Delta_j f(x) = \int K_{j,x}(y) f(y)\mathrm{d}y$. We also use P to denote the heat kernel on $\mathbb{R}\times\mathbb{R}$, i.e. $P(t,x) = (4\pi t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{|x|^2}{4t}} 1_{t\geq 0}$. For fixed $\bar\zeta = (t, \bar x)$ and for $j \geq -1$ define the measure

$$
\mu_j(\mathrm{d}\zeta) := \left[\int K_{j,\bar x}(x) P(t-s, x-y)\mathrm{d}x\right] 1_{s\geq 0}\mathrm{d}\zeta,
$$

with $\zeta = (s, y)$. For $\zeta_i = (s_i, x_i)$, set $|\zeta_1 - \zeta_2| := |s_1 - s_2|^{\frac{1}{2}} + |x_1 - x_2|$. Then by similar calculation as in [27, Section 10] and using [27, Lemma 10.14] we know

$$\mathbb{E}|\Delta_j \bar{Y}_n^{\vee}|^2 \lesssim \int |\zeta_1 - \zeta_1'|^{-2} |\mu_j(\mathrm{d}\zeta_1)||\mu_j(\mathrm{d}\zeta_1')|. \tag{6.11}$$

By [15, (87)] we find

$$\left| \int K_{j,\bar{x}}(x) P(t - s_1, x - x_1) \mathrm{d}x \right| \lesssim \frac{2^{-(1-\varepsilon/2)j}}{(|\bar{x} - x_1| + |t - s_1|^{1/2} + 2^{-j})^{2-\varepsilon/2}},$$

which combined with (6.11) and [27, Lemma 10.14] implies that $\mathbb{E}|\Delta_j \bar{Y}_n^{\vee}|^2$ could be controlled by $2^{-(2-\varepsilon)j}$ for $\varepsilon > 0$ small enough. Then the desired estimate for $\bar{Y}_n^{\vee}$ follows by standard techniques (c.f. [23, Lemma 9.8]).

We also give more details for the most complicated term $\partial_x \bar{Y}_n^{\curlyvee} \circ \partial_x Y_n + g^2 c_n^{\triangledown}$ (see also [55, Section 3.3.1] for the calculation of a similar term). For fixed $\bar{\zeta} = (t, \bar{x}) \in \mathbb{R} \times \mathbb{R}^3$ and $q \in \mathbb{Z}$, $q \geq -1$, define the measure

$$\mu_q(\mathrm{d}\zeta_1, \mathrm{d}\zeta_2) := \left[\int K_{q,\bar{x}}(x) \sum_{|i-j| \leq 1} K_{i,x}(y) K_{j,x}(x_2) \partial_x P(t - s_1, y - x_1) \mathrm{d}x \mathrm{d}y \right]$$
$$\delta(t - s_2) 1_{s_1 \geq 0} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2,$$

with $\zeta_i = (s_i, x_i) \in \mathbb{R} \times \mathbb{R}$ for $i = 1, 2$. $\tilde{\mu}_q(\mathrm{d}\zeta_1, \mathrm{d}\zeta_2)$ is defined similarly with $\partial_x P$ replaced $\partial_x P * \partial_x P$.

We decompose $\partial_x \bar{Y}_n^{\curlyvee} \circ \partial_x Y_n + g^2 c_n^{\triangledown} = I_4 + I_2 + I_0$ with I_i in the space of i-th Wiener chaos. Then by similar calulation as in [27, Section 10] and using [27, Lemma 10.14] we know

$$\mathbb{E}|\Delta_q I_4|^2 \lesssim \int |\zeta_1 - \zeta_1'|^{-1-\varepsilon} |\zeta_2 - \zeta_2'|^{-1} |\mu_q(\mathrm{d}\zeta_1, \mathrm{d}\zeta_2)||\mu_q(\mathrm{d}\zeta_1', \mathrm{d}\zeta_2')|,$$

which by [15, Lemma A.19][3] can be contolled by $2^{q\varepsilon}$ for $\varepsilon > 0$ small. For I_2 we have the decomposition

$$I_2 = 2\bar{Y}_n^{\flat} + \bar{Y}_n^{\curlyvee} + 2\bar{Y}_n^{\diamond} := \sum_{j=1}^{3} I_{2j},$$

where we refer to [27, Section 10], [23, Section 9] for the meaning of the graph. We use [27, Lemmas 10.14, 10.16] to have

$$\mathbb{E}|\Delta_q I_{21}|^2 \lesssim \int |\zeta_1 - \zeta_1'|^{-1-\varepsilon} |\zeta_2 - \zeta_2'|^{-1} |\mu_q(\mathrm{d}\zeta_1, \mathrm{d}\zeta_2)||\mu_q(\mathrm{d}\zeta_1', \mathrm{d}\zeta_2')|,$$

[3] In [15] the result is proved for $d = 3$, which could be extended to $d = 1$ by exactly the same argument.

and use [27, Lemma 10.16] to have

$$\mathbb{E}|\Delta_q I_{22}|^2 \lesssim \int |\zeta_1-\zeta_1'|^{-3\varepsilon}|\zeta_1-\zeta_2|^{-1+\varepsilon}|\zeta_1'-\zeta_2'|^{-1+\varepsilon}|\mu_q(\mathrm{d}\zeta_1,\mathrm{d}\zeta_2)||\mu_q(\mathrm{d}\zeta_1',\mathrm{d}\zeta_2')|,$$

and use [27, Lemma 10.14, (10.37 a)] to have

$$\begin{aligned}\mathbb{E}|\Delta_q I_{23}|^2 \lesssim &\int |\zeta_1-\zeta_1'|^{-2}|\zeta_1-\zeta_2|^{-\varepsilon}|\zeta_1'-\zeta_2'|^{-\varepsilon}|\mu_q(\mathrm{d}\zeta_1,\mathrm{d}\zeta_2)||\mu_q(\mathrm{d}\zeta_1',\mathrm{d}\zeta_2')|\\ &+\int |\zeta_1-\zeta_1'|^{-2}|\zeta_1-\zeta_2|^{-1}|\zeta_1'-\zeta_2'|^{-1}|\tilde{\mu}_q(\mathrm{d}\zeta_1,\mathrm{d}\zeta_2)||\tilde{\mu}_q(\mathrm{d}\zeta_1',\mathrm{d}\zeta_2'|.\end{aligned}$$

Then by [15, Lemma A.19] $\mathbb{E}|\Delta_q I_{2i}|^2, i=1,2,3$, can be contolled by $2^{q\varepsilon}$ for $\varepsilon>0$ small.

Different from the classical case in Sect. 6.1, I_0 contains g. We use $g\in\mathscr{C}^2$ to have $|g(x)-g(y)|\lesssim|x-y|$. Then we can shift g to the vertex x_2 and use [27, Lemmas 10.14, 10.16] to obtain

$$\begin{aligned}|I_0-g^2c_n^{\vee\!\vee}|\lesssim 1+\sum_{|i-j|\le 1}\int\Big|\int K_{i,x}(y)\partial_x P(t-s_1,y-x_1)\mathrm{d}y\Big|\\ \delta(t-s_2)|K_{j,x}(x_2)||g^2(x_2)-g^2(x)||\zeta_1-\zeta_2|^{-1}\mathrm{d}\zeta_1\mathrm{d}\zeta_2.\end{aligned}$$

By [15, Lemma A.16], we find for $\delta\in(0,1)$

$$|\int K_{i,x}(y)\partial_x P(t-s_1,y-x_1)\mathrm{d}y|\lesssim 2^{-i\delta}(|t-s_1|^{1/2}+|x-x_1|)^{-2-\delta},$$

which combined with $|g^2(x_2)-g^2(x)|\lesssim|x-x_1|+|x_2-x_1|$ and [15, Lemma A.16], [27, Lemmas 10.14] implies that $|I_0-g^2c_n^{\vee\!\vee}|\lesssim 1$.

Then the required regularity of $\mathscr{L}\bar{Y}^{\vee\vee}$ follows by standard argument (c.f. [23]). □

We make the following decomposition

$$h=Y+\bar{Y}^{\vee}+\bar{Y}^{\vee\vee}+\widetilde{h},$$

where $\widetilde{h}$ satisfies the following equation

$$\begin{cases}\mathscr{L}\widetilde{h}=2g\partial_x\widetilde{h}(X+\bar{X}^{\vee}+\bar{X}^{\vee\vee})+g(\partial_x\widetilde{h})^2+\mathscr{L}\bar{Y}^{\vee\vee}+\mathscr{L}\bar{Y}^{\vee\!\vee}\\ \qquad+g(\bar{X}^{\vee\vee})^2+2g\bar{X}^{\vee\vee}\bar{X}^{\vee}+2g(X\bar{X}^{\vee\vee}-X\circ\bar{X}^{\vee\vee})\\ \qquad+K(x)(X+\bar{X}^{\vee}+\bar{X}^{\vee\vee}+\partial_x\widetilde{h}),\\ \widetilde{h}(0)=h_0-Y(0).\end{cases}\tag{6.12}$$

Using Lemma 6.5, we obtain the following lemma.

Lemma 6.6 *There exists a measurable set* Ω_0 *with* $\mathbb{P}(\Omega_0) = 1$ *such that for every* $\kappa > 0$, $\gamma > 0$ *and* $\omega \in \Omega_0$

$$\begin{aligned} &b := 2g(X + \bar{X}^{\vee} + \bar{X}^{\vee\!\vee}) + K \in L_T^\infty \mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa), \\ &f := \mathscr{L}\bar{Y}^{\vee\!\vee} + \mathscr{L}\bar{Y}^{\vee\!\vee\!\vee} + g(\bar{X}^{\vee\!\vee})^2 + 2g\bar{X}^{\vee\!\vee}\bar{X}^{\vee} + 2g(X\bar{X}^{\vee\!\vee} - X \circ \bar{X}^{\vee\!\vee}) \\ &+ K(X + \bar{X}^{\vee} + \bar{X}^{\vee\!\vee}) \in L_T^\infty \mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa). \end{aligned}$$

Proof The proof follows from the proof of Lemma 6.2, Lemma 6.5 and $g, K \in \mathscr{C}^1$. □

As a result $\widetilde{h}$ satisfies (1.5) with b, f given above. We say that h is a paracontrolled solution to (6.9) if $\widetilde{h}$ is a paracontrolled solution to (6.12) in the sense of (5.4) and (5.5).

Since γ can be arbitrarily small, we apply Theorem 5.1 to obtain the following result.

Theorem 6.7 *Let* $g, K \in \mathscr{C}^1$. *For every initial condition* $\widetilde{h}(0) \in \mathbf{C}^{\frac{3}{2}+\varepsilon+\gamma}(\rho_{\varepsilon\delta})$ *where* $0 < \varepsilon < 1, \gamma \in (0, \frac{1}{4}), 0 < \delta := 40\kappa < 1$, *there exists a unique paracontrolled solution*

$$(\widetilde{h}, \widetilde{h}^\sharp) \in (\mathbb{S}_T^{\frac{3}{2}-\kappa^{1/4}-\gamma}(\rho_\eta) \cap \mathbb{L}_T^\infty(\rho_{2\delta}), \mathbb{S}_T^{2-2\kappa^{1/4}-2\gamma}(\rho_{2\eta}) \cap \mathbb{L}_T^\infty(\rho_{2\delta+\kappa}))$$

to (6.12), where

$$2(\kappa^{1/4} + 80\kappa) < \eta < \tfrac{1}{4}.$$

Proof In the following we check other conditions of Theorem 5.1. The condition for H is satisfied easily. In the following we prove $(b, f) \in \mathbb{B}_T^\alpha(\rho_\kappa)$. The approximation $\{(b_n, f_n)\}_n$ for (b, f) is given as in Lemma 6.6 with the corresponding tree τ replaced by τ_n in Lemma 6.5. In the following we prove that for every $\kappa > 0$

$$\sup_n(\ell_T^{b_n}(\rho_\kappa) + \mathbb{A}_{T,\infty}^{b_n,f_n}(\rho_\kappa)) < \infty, \tag{6.13}$$

with $\ell_T^{b_n}(\rho_\kappa)$ and $\mathbb{A}_{T,\infty}^{b_n,f_n}(\rho_\kappa))$ defined in (2.25) and (2.24), respectively. In the following we omit the subscript n for simplicity and all the following bounds are uniform in n and λ. We first consider

$$\nabla\mathscr{I}_\lambda(b) \circ b = \nabla\mathscr{I}_\lambda(2g(X + X^{\vee} + X^{\vee\!\vee}) + K) \circ [2g(X + X^{\vee} + X^{\vee\!\vee}) + K]. \tag{6.14}$$

For the first term $\nabla\mathscr{I}_\lambda(gX) \circ [gX]$ we use Lemma 2.17 and Lemma 6.1 to have

$$\|\nabla\mathscr{I}_\lambda(gX) \circ (gX)\|_{L_T^\infty \mathbf{C}^{-\gamma}(\rho_{2\kappa})} \lesssim 1.$$

Other terms on the right hand side of (6.14) can be calculated by Lemma 2.10 and (2.11):

$$\begin{aligned}
&\|\nabla\mathscr{I}_\lambda(2g(\bar{X}^{\vee}+\bar{X}^{\check{\vee}})+K)\circ b\|_{L_T^\infty\mathbf{C}^{-\gamma}(\rho_{2\kappa})}\\
&\quad\lesssim(\|\bar{Y}^{\vee}\|_{L_T^\infty\mathbf{C}^{1-\gamma}(\rho_\kappa)}+\|\bar{Y}^{\check{\vee}}\|_{L_T^\infty\mathbf{C}^{\frac{3}{2}-\gamma}(\rho_\kappa)}+1)\|b\|_{L_T^\infty\mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa)}\lesssim 1,
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla\mathscr{I}_\lambda(gX)\circ(2g(\bar{X}^{\vee}+\bar{X}^{\check{\vee}})+K)\|_{L_T^\infty\mathbf{C}^{-\gamma}(\rho_{2\kappa})}\\
&\quad\lesssim\|Y\|_{L_T^\infty\mathbf{C}^{\frac{1}{2}-\gamma}(\rho_\kappa)}(\|\bar{Y}^{\vee}\|_{L_T^\infty\mathbf{C}^{1-\gamma}(\rho_\kappa)}+\|\bar{Y}^{\check{\vee}}\|_{L_T^\infty\mathbf{C}^{\frac{3}{2}-\gamma}(\rho_\kappa)}+1)\lesssim 1.
\end{aligned}$$

On the other hand, we know

$$\nabla\mathscr{I}_\lambda f\circ b=\nabla\mathscr{I}_\lambda f_1\circ b+\nabla\mathscr{I}_\lambda(2g(\bar{X}^{\check{\vee}}\prec X)+KX)\circ(2g(X+\bar{X}^{\vee}+\bar{X}^{\check{\vee}})+K),$$

with $f_1=f-g(\bar{X}^{\check{\vee}}\prec X)-KX\in L_T^\infty\mathbf{C}^{-2\gamma}(\rho_\kappa)$. By Lemma 2.10 and (2.11) we know

$$\|\nabla\mathscr{I}_\lambda f_1\circ b\|_{L_T^\infty\mathbf{C}^{-\gamma}(\rho_{2\kappa})}\lesssim\|f_1\|_{L_T^\infty\mathbf{C}^{-2\gamma}(\rho_\kappa)}\|b\|_{L_T^\infty\mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa)}\lesssim 1,$$

and

$$\begin{aligned}
&\|\nabla\mathscr{I}_\lambda(2g(\bar{X}^{\check{\vee}}\prec X)+KX)\circ(2g\bar{X}^{\vee}+2g\bar{X}^{\check{\vee}}+K)\|_{L_T^\infty\mathbf{C}^{-\gamma}(\rho_{2\kappa})}\\
&\quad\lesssim(1+\|\bar{X}^{\check{\vee}}\|_{L_T^\infty\mathbf{C}^{\frac{1}{2}-\gamma}(\rho_\kappa)})\|X\|_{L_T^\infty\mathbf{C}^{-\frac{1}{2}-\gamma}(\rho_\kappa)}\\
&\qquad\times(\|\bar{X}^{\vee}\|_{L_T^\infty\mathbf{C}^{-\gamma}(\rho_\kappa)}+\|\bar{X}^{\check{\vee}}\|_{L_T^\infty\mathbf{C}^{\frac{1}{2}-\gamma}(\rho_\kappa)}+1)\lesssim 1.
\end{aligned}$$

We use Lemma 2.17 and Lemma 6.1 to have

$$\|\nabla\mathscr{I}_\lambda(KX)\circ(gX)\|_{L_T^\infty\mathbf{C}^{-\gamma}(\rho_{2\kappa})}\lesssim 1.$$

It remains to consider the term $\nabla\mathscr{I}_\lambda(g(\bar{X}^{\check{\vee}}\prec X))\circ(gX)$ and we use the commutator introduced in Lemma 2.11 and Lemma 2.12 to have

$$\begin{aligned}
&\nabla\mathscr{I}_\lambda(g(\bar{X}^{\check{\vee}}\prec X))\circ(gX)\\
&\quad=\nabla\mathscr{I}_\lambda(g\succcurlyeq(\bar{X}^{\check{\vee}}\prec X))\circ(gX)+\nabla\mathscr{I}_\lambda(g\prec(\bar{X}^{\check{\vee}}\prec X))\circ(gX)\\
&\quad=\nabla\mathscr{I}_\lambda(g\succcurlyeq(\bar{X}^{\check{\vee}}\prec X))\circ(gX)+([\nabla\mathscr{I}_\lambda,g\prec](\bar{X}^{\check{\vee}}\prec X))\circ(gX)\\
&\qquad+\Big[g\prec([\nabla\mathscr{I}_\lambda,\bar{X}^{\check{\vee}}\prec]X)\Big]\circ(gX)+\mathrm{com}(g,\bar{X}^{\check{\vee}}\prec\nabla\mathscr{I}_\lambda(X),gX)
\end{aligned}$$

$$+ g\Big([\bar{X}^{\curlyvee} \prec \nabla \mathscr{I}_\lambda(X)] \circ (gX)\Big).$$

We have further decomposition for the last term

$$\begin{aligned}
&[\bar{X}^{\curlyvee} \prec \nabla \mathscr{I}_\lambda(X)] \circ (gX) \\
&\quad = \mathrm{com}(\bar{X}^{\curlyvee}, \nabla \mathscr{I}_\lambda(X), gX) + \bar{X}^{\curlyvee}(\nabla \mathscr{I}_\lambda(X) \circ (gX)) \\
&\quad = \mathrm{com}(\bar{X}^{\curlyvee}, \nabla \mathscr{I}_\lambda(X), gX) + \bar{X}^{\curlyvee}(\nabla \mathscr{I}_\lambda(X) \circ (g \succcurlyeq X)) \\
&\qquad + g\bar{X}^{\curlyvee}(\nabla \mathscr{I}_\lambda(X) \circ X) + \bar{X}^{\curlyvee}\mathrm{com}(g, X, \nabla \mathscr{I}_\lambda(X)).
\end{aligned}$$

By Lemmas 2.12, 2.11 and Lemma 6.1

$$\|\nabla \mathscr{I}_\lambda(g\bar{X}^{\curlyvee} \prec X) \circ (gX)\|_{L_T^\infty \mathbf{C}^{-\gamma}(\rho_\kappa)} \lesssim 1,$$

where we used time regularity of $\bar{X}^{\curlyvee}$, which follows from (2.4). Combining all the above estimates, we deduce that (6.13) follows. Furthermore, we know that the convergence in Definition 2.14 also holds by using Lemma 6.1 and Lemma 2.16, which gives that $(b, f) \in \mathbb{B}_T^\alpha(\rho_\kappa)$. Then the result follows from Theorem 5.1. □

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Appendix A: Uniqueness of paracontrolled solutions

In this subsection we use Hairer and Labbé's argument [30] to show the uniqueness of paracontrolled solutions. For this aim, we use the following time-dependent exponential weight: for $\ell \in (0, 1)$,

$$\mathbf{e}_t^\ell(x) := \exp(-(1+t)\langle x\rangle^\ell), \ t \geq 0, \ x \in \mathbb{R}^d.$$

We can similarly define the Hölder space with weight $\mathbf{e}^\ell$ (see [45]). For instance,

$$\|f\|_{L_T^\infty \mathbf{C}^\alpha(\mathbf{e}^\ell)} := \sup_{t\in[0,T]} \|f(t,\cdot)\|_{\mathbf{C}^\alpha(\mathbf{e}_t^\ell)},$$

and for $\alpha \in (0,1)$,

$$\|f\|_{C^\alpha_T L^\infty(\mathbf{e}^\ell)} := \sup_{0\le t\le T} \|f(t)\mathbf{e}^\ell_t\|_{L^\infty} + \sup_{0\le s\ne t\le T} \frac{\|f(t)-f(s)\|_{L^\infty(\mathbf{e}^\ell_{t\vee s})}}{|t-s|^\alpha}.$$

In particular, for $\alpha \in (0,2)$, we also set

$$\mathbb{S}^\alpha_T(\mathbf{e}^\ell) := \|f\|_{L^\infty_T \mathbf{C}^\alpha(\mathbf{e}^\ell)} + \|f\|_{C^{\alpha/2}_T L^\infty(\mathbf{e}^\ell)}.$$

By [43, Lemma 2.10], for any $T>0$, there is a $C = C(T,\ell,d) > 0$ such that for all $s,t \in [0,T]$ and $j \ge -1$,

$$\|\Delta_j P_t f\|_{L^\infty(\mathbf{e}^\ell_s)} \lesssim \mathrm{e}^{-2^{2j}t} \|\Delta_j f\|_{L^\infty(\mathbf{e}^\ell_s)}. \tag{A.1}$$

Moreover, Lemmas 2.8, 2.10, 2.11 and 2.12 still hold for exponential weight $\mathbf{e}^\ell_t$ (see [45]). The following result corresponds to Lemma 2.9.

Lemma A.1 *Let $\alpha,\ell \in (0,1)$, $\kappa \in (0,(1-\frac{\alpha}{2})\ell)$. For any $q \in (\frac{1}{1-\alpha/2-\kappa/\ell},\infty]$ and $T>0$, there is a constant $C = C(T,d,\alpha,\ell,\theta,\kappa,q) > 0$ such that*

$$\|\mathscr{I} f\|_{\mathbb{S}_T^{2-\frac{2}{q}-\frac{2\kappa}{\ell}-\alpha}(\mathbf{e}^\ell)} \lesssim_C \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho_\kappa \mathbf{e}^\ell)}.$$

Proof First of all we have the following simple observation:

$$\mathbf{e}^\ell_t(x) \lesssim \langle x\rangle^{-\kappa} \mathbf{e}^\ell_s(x)/|t-s|^{\kappa/\ell}, \quad 0\le s<t<\infty. \tag{A.2}$$

Let $\frac{1}{p}+\frac{1}{q} = 1$ and $t \in (0,T]$. By (A.1) and Hölder's inequality, we have for $j \ge -1$,

$$\begin{aligned}
\|\Delta_j \mathscr{I} f(t)\|_{L^\infty(\mathbf{e}^\ell_t)} &\lesssim \int_0^t \mathrm{e}^{-2^{2j}(t-s)} \|\Delta_j f(s)\|_{L^\infty(\mathbf{e}^\ell_t)} \mathrm{d}s \\
&\lesssim \int_0^t \frac{\mathrm{e}^{-2^{2j}(t-s)}}{|t-s|^{\kappa/\ell}} \|\Delta_j f(s)\|_{L^\infty(\rho_\kappa \mathbf{e}^\ell_s)} \mathrm{d}s \\
&\lesssim 2^{\alpha j} \left(\int_0^t \frac{\mathrm{e}^{-p2^{2j}(t-s)}}{|t-s|^{p\kappa/\ell}} \mathrm{d}s\right)^{1/p} \|f\|_{L^q_t \mathbf{C}^{-\alpha}(\rho_\kappa \mathbf{e}^\ell)} \\
&\lesssim 2^{-(\frac{2}{p}-\frac{2\kappa}{\ell}-\alpha)j} \|f\|_{L^q_t \mathbf{C}^{-\alpha}(\rho_\kappa \mathbf{e}^\ell)},
\end{aligned}$$

which in turn gives that

$$\|\mathscr{I} f\|_{L^\infty_T \mathbf{C}^{2-\frac{2}{q}-\frac{2\kappa}{\ell}-\alpha}(\mathbf{e}^\ell)} \lesssim \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho_\kappa \mathbf{e}^\ell)}. \tag{A.3}$$

On the other hand, for $0 \le t_1 < t_2 \le T$, we have

$$\|\mathscr{I} f(t_2) - \mathscr{I} f(t_1)\|_{L^\infty(\mathbf{e}^\ell_{t_2})} \le \|(P_{t_2-t_1} - I)\mathscr{I} f(t_1)\|_{L^\infty(\mathbf{e}^\ell_{t_2})} + \left\| \int_{t_1}^{t_2} P_{t_2-s} f(s) \mathrm{d}s \right\|_{L^\infty(\mathbf{e}^\ell_{t_2})} =: I_1 + I_2.$$

For I_1, by (2.10) and (A.3) we have

$$\begin{aligned} I_1 &\lesssim (t_2 - t_1)^{1-\frac{\alpha}{2}-\frac{1}{q}-\frac{\kappa}{\ell}} \|\mathscr{I} f(t_1)\|_{\mathbf{C}^{2-\alpha-\frac{2}{q}-\frac{2\kappa}{\ell}}(\mathbf{e}^\ell_{t_2})} \\ &\lesssim (t_2 - t_1)^{1-\frac{\alpha}{2}-\frac{1}{q}-\frac{\kappa}{\ell}} \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho_\kappa \mathbf{e}^\ell)}. \end{aligned}$$

For I_2, by (2.8), (A.2) and Hölder's inequality, we have

$$\begin{aligned} I_2 &\lesssim \int_{t_1}^{t_2} (t_2 - s)^{-\frac{\alpha}{2}} \|f(s)\|_{\mathbf{C}^{-\alpha}(\mathbf{e}^\ell_{t_2})} \mathrm{d}s \\ &\lesssim \int_{t_1}^{t_2} (t_2 - s)^{-\frac{\alpha}{2}-\frac{\kappa}{\ell}} \|f(s)\|_{\mathbf{C}^{-\alpha}(\rho_\kappa \mathbf{e}^\ell_s)} \mathrm{d}s \\ &\lesssim (t_2 - t_1)^{1-\frac{\alpha}{2}-\frac{1}{q}-\frac{\kappa}{\ell}} \|f\|_{L^q_T \mathbf{C}^{-\alpha}(\rho_\kappa \mathbf{e}^\ell)}. \end{aligned}$$

Combining the above estimates, we obtain the desired estimate. □

Now we consider the following linear equation:

$$\mathscr{L} u = (b + \bar{b}) \cdot \nabla u + h u, \quad u(0) \equiv 0, \tag{A.4}$$

where $b \in \cap_{T>0} \mathbb{B}^\alpha_T(\rho_\kappa)$ and $\bar{b}, h \in \cap_{T>0} L^\infty_T(\rho_\eta)$. Let

$$(u, u^\sharp) \in \cap_{T>0} \mathbb{S}^{2-\alpha}_T(\rho_\eta) \times \mathbb{S}^{3-2\alpha}_T(\rho_{2\eta})$$

be the paracontrolled solution of PDE (A.4). That is,

$$u = \nabla u \prec\!\!\prec \mathscr{I} b + u^\sharp, \tag{A.5}$$

with $u^\sharp$ solving the following PDE in weak sense

$$\begin{aligned} \mathscr{L} u^\sharp = &\nabla u \prec b - \nabla u \prec\!\!\prec b + \nabla u \succ b + b \circ \nabla u \\ &+ \bar{b} \cdot \nabla u + h u - [\mathscr{L}, \nabla u \prec\!\!\prec] \mathscr{I} b, \end{aligned} \tag{A.6}$$

where

$$b \circ \nabla u = b \circ (\nabla^2 u \prec \mathscr{I} b) + (b \circ \nabla \mathscr{I} b) \cdot \nabla u + \mathrm{com}$$

$$+ \mathrm{com}_1 + b \circ \nabla u^\sharp, \tag{A.7}$$

and

$$\mathrm{com}_1 := b \circ \nabla[\nabla u \lll \mathscr{I} b - \nabla u \prec \mathscr{I} b]$$

and

$$\mathrm{com} := \mathrm{com}(\nabla u, \nabla \mathscr{I} b, b).$$

Theorem A.2 *Let $\ell \in (0,1)$ and $\kappa \in (0, \frac{(2-3\alpha)\ell}{6})$, $\eta \in (0, \frac{(1-\alpha)\ell}{2})$. Suppose that*

$$b \in \cap_{T>0} \mathbb{B}^\alpha_T(\rho_\kappa), \quad \bar b, h \in \cap_{T>0} \mathbb{L}^\infty_T(\rho_\eta),$$
$$\beta \in (\alpha, (2-2\alpha-\tfrac{6\kappa}{\ell}) \wedge (1-\tfrac{2\eta}{\ell})), \quad \gamma \in (\alpha, 2-2\alpha-\tfrac{4\kappa}{\ell}).$$

The unique paracontrolled solution to PDE (A.4) in the sense of Definition 3.1 with

$$(u, u^\sharp) \in \mathbb{S}^{\gamma+\alpha}_T(\mathbf{e}^\ell) \times L^\infty_T \mathbf{C}^{\beta+1}(\mathbf{e}^\ell)$$

is zero.

Proof Let $T > 0$. Choose q large enough such that

$$\alpha < \gamma \le 2 - 2\alpha - \tfrac{2}{q} - \tfrac{4\kappa}{\ell}, \quad \alpha < \beta \le (2 - 2\alpha - \tfrac{2}{q} - \tfrac{6\kappa}{\ell}) \wedge (1 - \tfrac{2\eta}{\ell}).$$

First of all, by Lemmas A.1 and 2.10, we have

$$\begin{aligned}
&\|u\|_{\mathbb{S}_T^{2-\alpha-\frac{2}{q}-\frac{4\kappa}{\ell}}(\mathbf{e}^\ell)} \\
&\lesssim \|b \prec \nabla u + b \succ \nabla u + b \circ \nabla u\|_{L^q_T \mathbf{C}^{-\alpha}(\rho_{2\kappa}\mathbf{e}^\ell)} + \|\bar b \cdot \nabla u + hu\|_{L^q_T L^\infty(\rho_\eta \mathbf{e}^\ell)} \\
&\lesssim \|b\|_{L^\infty_T \mathbf{C}^{-\alpha}(\rho_\kappa)} \|\nabla u\|_{L^q_T L^\infty(\mathbf{e}^\ell)} + \|b \circ \nabla u\|_{L^q_T \mathbf{C}^{-\alpha}(\rho_{2\kappa}\mathbf{e}^\ell)} \\
&+ \|\bar b\|_{\mathbb{L}^\infty_T(\rho_\eta)} \|\nabla u\|_{L^q_T L^\infty(\mathbf{e}^\ell)} + \|h\|_{\mathbb{L}^\infty_T(\rho_\eta)} \|u\|_{L^q_T L^\infty(\mathbf{e}^\ell)},
\end{aligned}$$

and by the corresponding version of Lemma 2.12 for exponential weight $\mathbf{e}^\ell$ (see [45, Lemma 2.10]),

$$\begin{aligned}
\|u^\sharp\|_{L^\infty_T \mathbf{C}^{\beta+1}(\mathbf{e}^\ell)} &\lesssim \|\nabla u \prec b - \nabla u \lll b + \nabla u \succ b - [\mathscr{L}, \nabla u \lll] \mathscr{I} b\|_{L^\infty_T \mathbf{C}^{1-2\alpha-\frac{2}{q}-\frac{4\kappa}{\ell}}(\rho_\kappa \mathbf{e}^\ell)} \\
&\quad + \|b \circ \nabla u\|_{L^q_T \mathbf{C}^{1-2\alpha}(\rho_{2\kappa}\mathbf{e}^\ell)} + \|\bar b \cdot \nabla u + hu\|_{\mathbb{L}^\infty_T(\rho_\eta \mathbf{e}^\ell)} \\
&\lesssim \|b \circ \nabla u\|_{L^q_T \mathbf{C}^{1-2\alpha}(\rho_{2\kappa}\mathbf{e}^\ell)} + \|u\|_{\mathbb{S}_T^{2-\alpha-\frac{2}{q}-\frac{4\kappa}{\ell}}(\mathbf{e}^\ell)} \\
&\quad + \|\bar b\|_{\mathbb{L}^\infty_T(\rho_\eta)} \|\nabla u\|_{\mathbb{L}^\infty_T(\mathbf{e}^\ell)} + \|h\|_{\mathbb{L}^\infty_T(\rho_\eta)} \|u\|_{\mathbb{L}^\infty_T(\mathbf{e}^\ell)} \\
&\lesssim \|u\|_{\mathbb{S}_T^{2-\alpha-\frac{2}{q}-\frac{4\kappa}{\ell}}(\mathbf{e}^\ell)} + \|b \circ \nabla u\|_{L^q_T \mathbf{C}^{1-2\alpha}(\rho_{2\kappa}\mathbf{e}^\ell)}.
\end{aligned}$$

Moreover, by Lemma 3.3 with $(\rho, \bar{\rho}) = (\rho_\kappa, \mathbf{e}_t^\ell)$,

$$\|(b \circ \nabla u)(t)\|_{\mathbf{C}^{1-2\alpha}(\rho_{2\kappa}\mathbf{e}_t^\ell)} \lesssim \|u\|_{\mathbb{S}_t^{\gamma+\alpha}(\mathbf{e}^\ell)} + \|u^\sharp(t)\|_{\mathbf{C}^{\beta+1}(\rho_\kappa \mathbf{e}_t^\ell)}.$$

Combining the above three estimates, we obtain

$$\begin{aligned}
&\|u\|_{\mathbb{S}_T^{\gamma+\alpha}(\mathbf{e}^\ell)} + \|u^\sharp\|_{L_T^\infty \mathbf{C}^{\beta+1}(\mathbf{e}^\ell)} \\
&\lesssim \|\nabla u\|_{L_T^q L^\infty(\mathbf{e}^\ell)} + \|u\|_{L_T^q L^\infty(\mathbf{e}^\ell)} + \|b \circ \nabla u\|_{L_T^q \mathbf{C}^{1-2\alpha}(\rho_{2\kappa}\mathbf{e}^\ell)} \\
&\lesssim \left(\int_0^T \left(\|u\|^q_{\mathbb{S}_t^{\gamma+\alpha}(\mathbf{e}^\ell)} + \|u^\sharp(t)\|^q_{\mathbf{C}^{\beta+1}(\rho_\kappa \mathbf{e}_t^\ell)}\right)\mathrm{d}t\right)^{1/q},
\end{aligned}$$

which implies $u \equiv 0$ by Gronwall's inequality. □

Appendix B: Exponential moment estimates for SDEs

In this section we consider the following SDE:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \ \ X_0 = x.$$

We have the following exponential moment estimates for X_t.

Lemma B.1 *Suppose that σ is bounded and b is linear growth. Then for any $\alpha \in [0, 2)$ and $T, \gamma > 0$, there is a constant $C > 0$ such that for all $x \in \mathbb{R}^d$,*

$$\mathbb{E}\mathrm{e}^{\gamma \sup_{t\in[0,T]}\langle X_t\rangle^\alpha} \le C\mathrm{e}^{\langle x\rangle^\alpha}.$$

Proof Let $\beta \in (\alpha, 2)$. Recall $\langle x\rangle^\beta = (1 + |x|^2)^{\beta/2}$. By Itô's formula, we have

$$M_t := \mathrm{e}^{-\lambda t}\langle X_t\rangle^\beta = \langle x\rangle^\beta + \int_0^t \eta_s \mathrm{d}s + \int_0^t \xi_s \mathrm{d}W_s,$$

where

$$\begin{aligned}
\eta_s :=\ & \mathrm{e}^{-\lambda s}\beta\Big[X_s \cdot b(s, X_s) + \mathrm{tr}(\sigma\sigma^*)(s, X_s)/2\Big]\langle X_s\rangle^{\beta-2} \\
&+ \beta(\tfrac{\beta}{2} - 1)\mathrm{e}^{-\lambda s}|\sigma^*(s, X_s)X_s|^2\langle X_s\rangle^{\beta-4} - \lambda \mathrm{e}^{-\lambda s}\langle X_s\rangle^\beta,
\end{aligned}$$

and

$$\xi_s := \beta \mathrm{e}^{-\lambda s}\sigma^*(s, X_s)X_s\langle X_s\rangle^{\beta-2}.$$

By the linear growth of b and the boundedness of σ, there is a λ large enough so that

$$\eta_s \le 0$$

and

$$|\xi_s|^2 \leq C\mathrm{e}^{-\lambda s}\langle X_s\rangle^{2(\beta-1)} \leq CM_s^{2-\frac{2}{\beta}}.$$

Now by [33, Theorem 1.1], we obtain the desired estimate. □

References

1. Bahouri, H., Chemin, J.-Y., Danchin, R.: Fourier analysis and nonlinear partial differential equations. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343. Springer, Heidelberg (2011)
2. Bertini, L., Cancrini, N., Jona-Lasinio, G.: The stochastic Burgers equation. Commun. Math. Phys. **165**(2), 211–232 (1994)
3. Bertini, L., Giacomi, G.: Stochastic Burgers and KPZ equations from particle systems. Commun. Math. Phys. **183**(3), 571–607 (1997)
4. Bruned, Y., Gabriel, F., Hairer, M., Zambotti, L.: Geometric stochastic heat equations. J. Am. Math. Soc. **35**(1), 1–80 (2021)
5. Bony, J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. École Norm. Sup. (4) **14**(2), 209–246 (1981)
6. Bergh, J., Löfström, J.: Interpolation Spaces: An Introduction. Springer, Berlin (1976)
7. Cole, J.D.: On a Quasi-Linear parabolic equation occurring in aerodynamics. Quart. Appl. Math. **9**, 225–236 (1951)
8. Corwin, I.: The Kardar–Parisi–Zhang equation and universality class. Random Matrices Theory Appl. **1**(1), 113000176 (2012)
9. Catellier, R., Chouk, K.: Paracontrolled distributions and the 3-dimensional stochastic quantization equation. Ann. Probab. **46**(5), 2621–2679 (2018)
10. Cannizzaro, G., Chouk, K.: Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential. Ann. Probab. **46**(3), 1710–1763 (2018)
11. Chen, X., Wu, B., Zhu, R., Zhu, X.: Stochastic heat equations for infinite strings with values in a manifold. Trans. Am. Math. Soc. **374**(1), 407–452 (2021)
12. Chen, Z.Q., Zhang, X.: Uniqueness of stable-like processes. arXiv:1604.02681 (2016)
13. Da Prato, G., Debussche, A.: Strong solutions to the stochastic quantization equations. Ann. Probab. **31**(4), 1900–1916 (2003)
14. Eells, J., Sampson, H.: Harmonic mappings of Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)
15. Furlan, M., Gubinelli, M.: Weak universality for a class of 3d stochastic reaction–diffusion models. Probab. Theory Relat. Fields **173**, 1099–1164 (2019)
16. Funaki, T., Quastel, J.: KPZ equation, its renormalization and invariant measures. Stoch. PDE. Anal. Comp. **3**, 159–220 (2015)
17. Fleming, W.H., Soner, H.M.: Controlled Markov Processes and Viscosity Solutions. Springer (2006)
18. Gubinelli, M., Hofmanová, M.: Global solutions to elliptic and parabolic Φ^4 models in Euclidean space. Commun. Math. Phys. **368**(3), 1201–1266 (2019)
19. Gubinelli, M., Hofmanová, M.: A PDE construction of the Euclidean Φ^4 quantum field theory. Commun. Math. Phys. **384**(1), 1–75 (2021)
20. Gubinelli, M., Imkeller, P., Perkowski, N.: Paracontrolled distributions and singular PDEs. Forum Math. Pi **3**(6), 1 (2015)
21. Gubinelli, M., Jara, M.: Regularization by noise and stochastic Burgers equations. Stoch. Part. Differ. Equ.: Anal. Comput. **1**(2), 325–350 (2013)
22. Goncalves, P., Jara, M.: Nonlinear fluctuations of weakly asymmetric interacting particle systems. Arch. Ration. Mech. Anal. **212**(2), 597–644 (2014)
23. Gubinelli, M., Perkowski, N.: KPZ reloaded. Commun. Math. Phys. **349**(1), 165–269 (2017)
24. Gubinelli, M., Perkowski, N.: Energy solutions of KPZ are unique. J. Am. Math. Soc. **31**(2), 427–471 (2018)

25. Gubinelli, M., Perkowski, N.: The infinitesimal generator of the stochastic Burgers equation. **178**, 1067–1124 (2020)
26. Hairer, M.: Solving the KPZ equation. Ann. Math. (2) **178**(2), 559–664 (2013)
27. Hairer, M.: A theory of regularity structures. Invent. Math. **198**(2), 269–504 (2014)
28. Hopf, E.: The Partial Differential Equation $u_t + uu_x = \mu u_{xx}$. Commun. Pure Appl. Math. **3**, 201–230 (1950)
29. Hairer, M., Labbé, C.: A simple construction of the continuum parabolic Anderson model on $\mathbb{R}^2$. Electron. Commun. Probab. **20**(43), 11 (2015)
30. Hairer, M., Labbé, C.: Multiplicative stochastic heat equations on the whole space. J. Eur. Math. Soc. (JEMS) **20**(4), 1005–1054 (2018)
31. Hairer, M., Pardoux, E.: A Wong–Zakai theorem for stochastic PDEs. J. Math. Soc. Jpn. **67**(4), 1551–1604 (2015)
32. Hairer, M., Quastel, J.: A class of growth models rescaling to KPZ. Forum Math. Pi (2018)
33. Huang, W.: Exponential integrability of Ito's processes. J. Math. Anal. Appl. **358**, 427–433 (2009)
34. Hairer, M., Xu, W.: Large-scale limit of interface fluctuation models. Ann. Probab. **47**(6), 3478–3550 (2019)
35. Hao, Z., Zhang, X., Zhu, R., Zhu, X.: Singular kinetic equations and applications, arXiv:2108.05042 (2021)
36. Kardar, M., Parisi, G., Zhang, Y.: Dynamic scaling of growing interfaces. Phys. Rev. Lett. **56**, 889–892 (1986)
37. Krylov, N.V.: Controlled Diffusion Processes. Springer (1980)
38. Krylov, N. V.: Lectures on Elliptic and Parabolic Equations in Sobolev Spaces. AMS, Graduate Studies in Mathematics, Vol. 96, (2008)
39. Ladyzenskaja, O.A., Solonnikov, V.A., UralCeva, N.N.: Linear and quasilinear equations of parabolic type, Monographs. Amer. Math. Soc. (1968)
40. Martin, J., Perkowski, N.: Paracontrolled distributions on Bravais lattices and weak universality of the 2d parabolic Anderson model. Ann. Inst. Henri Poincaré Probab. Stat. **55**(4), 2058–2110 (2019)
41. Matetski, K., Quastel, J., Remenik, D.: The KPZ fixed point. arXiv:1701.00018v1 (2017)
42. Menoukeu-Pamen, O., Tangpi, L.: Maximum principle for stochastic control of SDEs with measurable drifts. arXiv:2101.06206 (2021)
43. Mourrat, J.-C., Weber, H.: Global well-posedness of the dynamic Φ^4 model in the plane. Ann. Probab. **45**(4), 2398–2476 (2017)
44. Mourrat, J.-C., Weber, H.: The dynamic Φ^4_3 model comes down from infinity. Commun. Math. Phys. **356**(3), 673–753 (2017)
45. Perkowski, N., Rosati, T.C.: The KPZ equation on the real line. Electron. J. Probab. **24**(117), 1–56 (2019)
46. Perkowski, N., Rosati, T. C.: A rough super-Brownian motion. arXiv:1905.05825 (2019)
47. Quastel, J.: Introduction to KPZ. In Current developments in mathematics, 2011, pp. 125– 194. Int. Press (2012)
48. Quastel, J., Spohn, H.: The One-dimensional KPZ equation and its universality class. J. Stat. Phys. **160**, 965–984 (2015)
49. Röckner, M., Wu, B., Zhu, R., Zhu, X.: Stochastic heat equations with values in a manifold via Dirichlet forms. SIAM J. Math. Anal. **52**(3), 2237–2274 (2020)
50. Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. North-Holland Mathematical Library 18. North-Holland Publishing Co (1978)
51. Triebel, H.: Theory of Function Spaces III. Basel
52. Walsh, J.B.: An introduction to stochastic partial differential equations. In École d'été de probabilités de Saint-Flour, XIV-1984, volume 1180 of Lecture Notes in Math., pp. 265–439. Springer (1986)
53. Yong, J., Zhou, X.: Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer (1999)
54. Zhang, X.: Stochastic flows and Bismut formulas for stochastic Hamiltonian systems. Stoch. Proc. Appl. **120**, 1929–1949 (2010)
55. Zhu, R., Zhu, X.: Three-dimensional Navier–Stokes equations driven by space-time white noise. J. Differ. Equ. **259**(9), 4443–4508 (2015)
56. Zvonkin, A.K.: A transformation of the phase space of a diffusion process that removes the drift. Math. Sb. **93**(135), 129–149 (1974)