Probability Theory and Related Fields (2023) 185:391–447
https://doi.org/10.1007/s00440-022-01126-z

Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model driven by space-time white noise on the torus in the full L^1-regime

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Received: 20 July 2021 / Revised: 7 March 2022 / Accepted: 9 March 2022 /
Published online: 13 May 2022

Abstract

The present paper is a continuation of our previous work (Hoshino et al., J Evol Equ 21:339–375, 2021) on the stochastic quantization of the $\exp(\Phi)_2$-quantum field model on the two-dimensional torus. Making use of key properties of Gaussian multiplicative chaos and refining the method for singular SPDEs introduced in the previous work, we construct a unique time-global solution to the corresponding parabolic stochastic quantization equation in the full "L^1-regime" $|\alpha| < \sqrt{8\pi}$ of the charge parameter α. We also identify the solution with an infinite-dimensional diffusion process constructed by the Dirichlet form approach.

Keywords Stochastic quantization · Høegh-Krohn model · Gaussian multiplicative chaos · Singular SPDE · Dirichlet form

Mathematics Subject Classification 81S20 · 60H17 · 35R60 · 60J46 · 81T08 · 81T40

Dedicated to Professor Shigeki Aida on the occasion of his 60th birthday.

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1 Introduction

1.1 Background

In the present paper, we study stochastic quantization associated with space-time quantum fields with interactions of exponential type, called the $\exp(\Phi)_2$-*quantum field model* in the Euclidean quantum field theory, in finite volume. The $\exp(\Phi)_2$-quantum field (or the $\exp(\Phi)_2$-measure) $\mu^{(\alpha)}$ is a probability measure on $\mathcal{D}'(\Lambda)$, the space of distributions on the two-dimensional torus $\Lambda = \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$, which is given by

$$\mu^{(\alpha)}(\mathrm{d}\phi) = \frac{1}{Z^{(\alpha)}} \exp\Big(- \int_\Lambda \exp^\diamond(\alpha\phi)(x)\,\mathrm{d}x \Big)\mu_0(\mathrm{d}\phi),$$

where the massive Gaussian free field μ_0 is the Gaussian measure on $\mathcal{D}'(\Lambda)$ with zero mean and the covariance operator $(1-\triangle)^{-1}$, $\triangle$ being the Laplacian in $L^2(\Lambda)$ with the periodic boundary conditions, $\alpha (\in \mathbb{R})$ is called the *charge parameter*, the *Wick exponential* $\exp^\diamond(\alpha\phi)(x)$ is formally introduced by the expression

$$\exp^\diamond(\alpha\phi)(x) = \exp\Big(\alpha\phi(x) - \frac{\alpha^2}{2}\mathbb{E}^{\mu_0}[\phi(x)^2]\Big), \qquad x \in \Lambda,$$

and

$$Z^{(\alpha)} = \int_{\mathcal{D}'(\Lambda)} \exp\Big(- \int_\Lambda \exp^\diamond(\alpha\phi)(x)\,\mathrm{d}x \Big)\mu_0(\mathrm{d}\phi) > 0$$

is the normalizing constant. We remark that the diverging term $\mathbb{E}^{\mu_0}[\phi(x)^2]$ plays a role of the Wick renormalization. Since this quantum field model was first introduced by Høegh-Krohn [32] in the "L^2-regime"

$$|\alpha| < \sqrt{4\pi},$$

it is also called the *Høegh-Krohn model*. For a physical background and related early works of this model, see e.g., [2, 3, 47] and references therein. Kahane [35] constructed a random measure

$$\nu_\phi^{(\alpha)}(\mathrm{d}x) := \exp^\diamond(\alpha\phi)(x)\,\mathrm{d}x, \qquad x \in \Lambda,$$

called the *Gaussian multiplicative chaos*, in the "L^1-regime"

$$|\alpha| < \sqrt{8\pi}.$$

It implies the existence of the $\exp(\Phi)_2$-measure $\mu^{(\alpha)}$ under $|\alpha| < \sqrt{8\pi}$, which gives a generalization of the early works mentioned above. After that, the relevance of both the Gaussian multiplicative chaos and the $\exp(\Phi)_2$-quantum field model has been received much attention by many people in connection with topics like the Liouville conformal field theory and the stochastic Ricci flow. See e.g., [12–14, 21, 22, 34, 35, 45] and references therein. We should also mention that Kusuoka [38] independently studied the $\exp(\Phi)_2$-quantum field model under $|\alpha| < \sqrt{8\pi}$.

By heuristic calculations, we observe that the $\exp(\Phi)_2$-measure $\mu^{(\alpha)}$ is an invariant measure of the following two-dimensional parabolic stochastic partial differential equation (SPDE in short) involving exponential nonlinearity:

$$\partial_t \Phi_t(x) = \frac{1}{2}(\triangle - 1)\Phi_t(x) - \frac{\alpha}{2}\exp^{\diamond}\big(\alpha\Phi_t\big)(x) + \dot{W}_t(x), \qquad t > 0,\ x \in \Lambda, \tag{1.1}$$

where $(\dot{W}_t)_{t\geq 0}$ is an $\mathbb{R}$-valued Gaussian space-time white noise, that is, the time derivative of a standard $L^2(\Lambda)$-cylindrical Brownian motion $\{W_t = (W_t(x))_{x\in\Lambda}\}_{t\geq 0}$. We call (1.1) the $\exp(\Phi)_2$-*stochastic quantization equation* associated with $\mu^{(\alpha)}$. Due to the singularity of the nonlinear drift term, the interpretation and construction of a solution to this *singular*-SPDE have been a challenging problem over the past years. For a concise overview on stochastic quantization equations, we refer to [1, 5–7] and references therein. Albeverio and Röckner [9] first solved (1.1) (in the case where Λ is replaced by $\mathbb{R}^2$) weakly under $|\alpha| < \sqrt{4\pi}$ by using the Dirichlet form theory. Inspired by recent quick developments of singular SPDEs based on Hairer's groundbreaking work on *regularity structures* [29] and the related work, called *paracontrolled calculus*, due to Gubinelli et al. [28], Garban [26] constructed a unique strong solution to (1.1) (for the case where $(\triangle - 1)$ is replaced by $\triangle$, i.e., massless case) in a more restrictive condition than $|\alpha| < \sqrt{4\pi}$. In our previous paper [33], we constructed the time-global and pathwise-unique solution to the SPDE (1.1) under $|\alpha| < \sqrt{4\pi}$ by splitting the original equation (1.1) into the Ornstein–Uhlenbeck process

$$\partial_t X_t(x) = \frac{1}{2}(\triangle - 1)X_t(x) + \dot{W}_t(x),$$

and the shifted equation

$$\partial_t Y_t(x) = \frac{1}{2}(\triangle - 1)Y_t(x) - \frac{\alpha}{2}\exp(\alpha Y_t(x))\exp^{\diamond}(\alpha X_t)(x). \tag{1.2}$$

This split is based on the idea of Da Prato and Debussche [17], which is now called the *Da Prato–Debussche trick*. By the uniqueness of the solution, we also obtained the identification with the limit of the solutions to the stochastic quantization equations generated by the approximating measures to the $\exp(\Phi)_2$-measure $\mu^{(\alpha)}$, and with the process obtained by the Dirichlet form approach. Our construction of the solution to the shifted equation (1.2) is different from the standard fixed-point argument applied in [17, 26]. To be more precise, we proved convergence of solutions to approximating equations of (1.1) by using compact embedding, and then identified the limit as the solution. We should mention that, after Hoshino et al. [33], Oh et al. [43] independently constructed the time-global unique solution to (1.1) in the same regime in [33]. Later in [42], together with Tzvetkov, they studied the massless case on two-dimensional compact Riemannian manifolds in the L^2-regime. Besides, elliptic SPDEs, which also realize the $\exp(\Phi)_2$-quantum field model have been studied in e.g., [1].

The main purpose of the present paper is to construct the time-global and pathwise-unique solution to the parabolic SPDE (1.1) in the full "L^1-regime" $|\alpha| < \sqrt{8\pi}$. Although the present paper builds on our previous work [33], we significantly improve the arguments of [33] in several ways. To apply the Da Prato–Debussche trick, we need to construct the Wick exponential of the Ornstein–Uhlenbeck process $\{\exp^\diamond(\alpha X_t)\}_{t\geq 0}$ as a driving noise of the shifted equation (1.2). Since the Gaussian free field μ_0 is the stationary measure of the Ornstein–Uhlenbeck process $\{X_t\}_{t\geq 0}$, this problem is reduced to the construction of the Wick exponential $\exp^\diamond(\alpha\phi)$. In [33, Theorem 2.2], we constructed it under $|\alpha| < \sqrt{4\pi}$ by combining the Wick calculus of the Gaussian free field μ_0 with the standard Fourier expansion on a negative order L^2-Sobolev space $H^s(\Lambda)$ $(s < 0)$. However, this kind of argument does not work beyond the L^2-regime. Refining existing results on the convergence of the Gaussian multiplicative chaos $\nu_\phi^{(\alpha)}(\mathrm{d}x)$ in [12, 22, 45], we construct the Wick exponential $\exp^\diamond(\alpha\phi)$ on a suitable Besov space under $|\alpha| < \sqrt{8\pi}$ (see Theorem 2.1). This is one of the important contributions of the present paper. On the other hand, in this case, since the Wick exponential $\exp^\diamond(\alpha\phi)$ does not have L^2-integrability with respect to μ_0 unlike the case of $|\alpha| < \sqrt{4\pi}$, we need to modify our arguments mentioned above into L^p-setting for the construction of the time-global and pathwise-unique solution to (1.1). Besides, due to the lack of the L^2-integrability, we cannot follow the argument as in [5, 8, 9, 33] to show the closability of the associated Dirichlet form. To overcome this difficulty, in Corollary 2.4, we prove that the Wick exponential $\exp^\diamond(\alpha\phi)$ has the L^2-integrability with respect not to μ_0, but to $\mu^{(\alpha)}$. This key property plays a significant role not only for the closability of the Dirichlet form, but also for the identication of the diffusion process obtained by the Dirichlet form approach with the solution to the SPDE (1.1).

We should mention here that our model is closely connected with the *sine-Gordon model* (or $\cos(\Phi)_2$-quantum field model), which has been studied for a long period by many authors. See e.g., [4, 23–25] for the early works. Since the sine-Gordon model is formally obtained by replacing the nonlinearity $e^{\alpha\phi}$ by $e^{\sqrt{-1}\alpha\phi}$, it has some similarities with the $\exp(\Phi)_2$-model. Indeed, it can be constructed rigorously in the same way as the $\exp(\Phi)_2$-model in the case $|\alpha| < \sqrt{4\pi}$. On the other hand, for large values of $|\alpha|$ up to $\sqrt{8\pi}$, further renormalization by counter-terms is required (see [11, 20] for details). To make a rigorous meaning to stochastic quantization equations associated with both the Φ^4_3-model and the sine-Gordon model, we require further renormalization procedures beyond the Wick renormalization, and recent developments of regularity structure and paracontrolled calculus enable us to study such singular SPDEs rigorously. In [15, 31], Hairer, Shen and Chandra proved local well-posedness of (the massless version of) the sine-Gordon stochastic quantization equation by applying regularity structure. Hence, at first sight, one might guess that regularity structure or paracontrolled calculus is applicable to the $\exp(\Phi)_2$-stochastic quantization equation (1.1) beyond the L^2-regime. To apply such general theories, we usually assume that the inputs of the solution map to the shifted equation of a given singular SPDE take values in a Besov space $B^s_{\infty,\infty}(\Lambda)$ with some $s < 0$. (We mention here that the reconstruction theorem in $B^s_{p,q}(\mathbb{R}^d)$ was also studied by Hairer and Labbé [30], but they considered only the models with $B^s_{\infty,\infty}$-type bounds.) In contrast, the Wick exponential of the Ornstein–

Uhlenbeck process $\{\exp^\diamond(\alpha X_t)\}_{t\geq 0}$, which plays a role of an input in our case, belongs to $B^s_{p,p}(\Lambda)$ for some $p \in [1, 2)$, but does not to $B^s_{\infty,\infty}(\Lambda)$ (see Theorem 3.2). Moreover, since the nonlinear term of the SPDE (1.1) has exponential growth, it is out of results by these general theories. Alternatively, by making use of the nonnegativity of $\exp^\diamond(\alpha X_t)$, we may define a product between two rough objects $\exp(\alpha Y_t)$ and $\exp^\diamond(\alpha X_t)$ on the right-hand side of the shifted equation (1.2) (see Theorem 4.3). This is the most crucial point in our argument. We remark that the nonnegativity of the Wick exponential is a remarkable and useful property, and is also applied in proofs of previous results (see e.g [3, 26, 33, 42, 43]).

The organization of the rest of the present paper is as follows: In Sect. 1.2, we present the framework and state the main theorems (Theorems 1.1, 1.5 and 1.7). In Sect. 1.3, we fix some notations and summarize several basic properties on Besov spaces. In Sect. 2, we introduce an approximation of the Wick exponentials of the Gaussian free fields and show its almost-sure convergence in an appropriate Besov space (see Theorem 2.1). For later use, we modify the argument of Berestycki [12] to obtain a stronger estimate than existing results. Moreover, we also prove that the exp(Φ)$_2$-measure $\mu^{(\alpha)}$ is well-defined and the Wick exponential $\exp^\diamond(\alpha\phi)$ has the L^2-integrability with respect to $\mu^{(\alpha)}$ (see Corollaries 2.3 and 2.4). In Sect. 3, we prove the almost-sure convergence of the Wick exponential of the infinite-dimensional Ornstein–Uhlenbeck process (see Theorem 3.2). In Sect. 4, we prove Theorem 1.1 using the result of Sect. 3. In Sects. 5 and 6, we prove Theorems 1.5 and 1.7, respectively. Since some parts of Sects. 4–6 go in very similar ways to the arguments of the previous paper [33], we sometimes omit the details in the present paper. Finally, in Appendix, we give several estimates on the approximation of the Green function of $(1 - \triangle)$, which are used in Sect. 2.

1.2 Statement of the main theorems

We begin with introducing some notations and objects. Let $\Lambda = \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ be the two-dimensional torus equipped with the Lebesgue measure $\mathrm{d}x$. Let $L^p(\Lambda)$ ($p \in [1, \infty]$) be the usual real-valued Lebesgue space. In particular, $L^2(\Lambda)$ is a Hilbert space equipped with the usual inner product

$$\langle f, g\rangle = \int_\Lambda f(x)g(x)\,\mathrm{d}x, \qquad f, g \in L^2(\Lambda).$$

Let $C^\infty(\Lambda)$ be the space of real-valued smooth functions on Λ equipped with the topology given by the convergence $f_n \to f$ in $C^\infty(\Lambda)$:

$$\sup_{(x_1,x_2)\in\Lambda}\left|\frac{\partial^{i+j} f_n}{\partial x_1^i \partial x_2^j}(x_1, x_2) - \frac{\partial^{i+j} f}{\partial x_1^i \partial x_2^j}(x_1, x_2)\right| \to 0 \quad \text{as } n \to \infty$$

for all $i, j \in \mathbb{N} \cup \{0\}$. We denote by $\mathcal{D}'(\Lambda)$ the topological dual space of $C^\infty(\Lambda)$. We have $L^p(\Lambda) \subset \mathcal{D}'(\Lambda)$ for all $p \in [1, \infty]$ by identification of $f \in L^p(\Lambda)$ with the map $C^\infty(\Lambda) \ni \varphi \mapsto \int_\Lambda f(x)\varphi(x)\,\mathrm{d}x \in \mathbb{R}$. Since $C^\infty(\Lambda) \subset L^2(\Lambda) \subset \mathcal{D}'(\Lambda)$, the L^2-inner product $\langle\cdot,\cdot\rangle$ is naturally extended to the pairing of $C^\infty(\Lambda)$ and its dual space $\mathcal{D}'(\Lambda)$.

For $k = (k_1, k_2) \in \mathbb{Z}^2$ and $x = (x_1, x_2) \in \Lambda$, we write $|k| = (k_1^2 + k_2^2)^{1/2}$ and $k \cdot x = k_1 x_1 + k_2 x_2$. Although we work in the framework of real-valued functions, it is sometimes easier to do computations by using a system of complex-valued functions $\{\mathbf{e}_k\}_{k\in\mathbb{Z}^2}$ defined by

$$\mathbf{e}_k(x) = \frac{1}{2\pi} e^{\sqrt{-1}k\cdot x}, \qquad k \in \mathbb{Z}^2,\ x \in \Lambda.$$

For $f \in \mathcal{D}'(\Lambda)$, we define the k-th Fourier coefficient $\hat{f}(k)$ ($k \in \mathbb{Z}^2$) by

$$\hat{f}(k) := \langle f, \mathbf{e}_{-k}\rangle = \int_\Lambda f(x)\overline{\mathbf{e}_k(x)}\,\mathrm{d}x.$$

Note that, since f is real-valued, $\hat{f}(-k) = \overline{\hat{f}(k)}$ for $k \in \mathbb{Z}^2$.

For $s \in \mathbb{R}$, we define the real L^2-Sobolev space of order s with periodic boundary condition by

$$H^s(\Lambda) = \left\{ u \in \mathcal{D}'(\Lambda)\,;\, \|u\|_{H^s}^2 := \sum_{k\in\mathbb{Z}^2} (1 + |k|^2)^s |\hat{u}(k)|^2 < \infty \right\}.$$

This space is a Hilbert space equipped with the inner product

$$(u, v)_{H^s} := \sum_{k\in\mathbb{Z}^2} (1 + |k|^2)^s \hat{u}(k)\overline{\hat{v}(k)}, \qquad u, v \in H^s(\Lambda).$$

Note that $H^0(\Lambda)$ coincides with $L^2(\Lambda)$ and we regard $H^{-s}(\Lambda)$ as the dual space of $H^s(\Lambda)$ through the standard chain $H^s(\Lambda) \subset L^2(\Lambda) \subset H^{-s}(\Lambda)$ for $s \geq 0$.

We now define the massive Gaussian free field measure μ_0 by the centered Gaussian measure on $\mathcal{D}'(\Lambda)$ with covariance $(1 - \triangle)^{-1}$, that is, determined by the formula

$$\int_{\mathcal{D}'(\Lambda)} \langle\phi, \mathbf{e}_k\rangle\overline{\langle\phi, \mathbf{e}_\ell\rangle}\, \mu_0(\mathrm{d}\phi) = (1 + |k|^2)^{-1}\mathbf{1}_{k=\ell}, \qquad k, \ell \in \mathbb{Z}^2, \tag{1.3}$$

where $\triangle$ is the Laplacian acting on $L^2(\Lambda)$ with periodic boundary condition. This formula implies

$$\int_{\mathcal{D}'(\Lambda)} \|\phi\|_{H^{-\varepsilon}}^2 \mu_0(\mathrm{d}\phi) < \infty$$

for any $\varepsilon > 0$, and thus the Gaussian free field measure μ_0 has a full support on $H^{-\varepsilon}(\Lambda)$. For a charge parameter $\alpha \in (-\sqrt{8\pi}, \sqrt{8\pi})$, we then define the $\exp(\Phi)_2$-*quantum field* (or the $\exp(\Phi)_2$-*measure*) $\mu^{(\alpha)}$ on $\mathcal{D}'(\Lambda)$ by

$$\mu^{(\alpha)}(\mathrm{d}\phi) := \frac{1}{Z^{(\alpha)}} \exp\left(- \int_\Lambda \exp^\diamond(\alpha\phi)(x)\,\mathrm{d}x \right) \mu_0(\mathrm{d}\phi),$$

where $Z^{(\alpha)} > 0$ is the normalizing constant and $\exp^\diamond(\alpha\phi)$ is the *Wick exponential* which will be rigorously constructed in Sect. 2. We prove in Theorem 2.1 that the function $\phi \mapsto \int_\Lambda \exp^\diamond(\alpha\phi)(x)\,\mathrm{d}x$ is a positive measurable function for all $|\alpha| < \sqrt{8\pi}$. Hence, we may also regard $\mu^{(\alpha)}$ as a probability measure on $\mathcal{D}'(\Lambda)$ (see Corollary 2.3).

In the present paper, we consider the stochastic quantization equation (1.1) associated with exp(Φ)$_2$-measure $\mu^{(\alpha)}$, that is a parabolic SPDE

$$\partial_t \Phi_t(x) = \frac{1}{2}(\triangle - 1)\Phi_t(x) - \frac{\alpha}{2} \exp^\diamond(\alpha\Phi_t)(x) + \dot{W}_t(x), \qquad t > 0,\ x \in \Lambda,$$

where $W = \{W_t(x); t \geq 0, x \in \Lambda\}$ is an $L^2(\Lambda)$-cylindrical Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $(\dot{W}_t)_{t\geq 0}$ is its time derivative in weak sense. This driving noise has a convenient Fourier series representation

$$W_t(x) = \sum_{k\in\mathbb{Z}^2} w_t^{(k)} e_k(x), \qquad t \geq 0,\ x \in \Lambda,$$

where $\{e_k\}_{k\in\mathbb{Z}^2}$ is a real-valued complete orthonormal system (CONS) of $L^2(\Lambda)$ defined by $e_{(0,0)}(x) = (2\pi)^{-1}$ and

$$e_k(x) = \frac{1}{\sqrt{2}\pi} \begin{cases} \cos(k\cdot x), & k \in \mathbb{Z}^2_+, \\ \sin(k\cdot x), & k \in \mathbb{Z}^2_-, \end{cases} \tag{1.4}$$

with $\mathbb{Z}^2_+ = \{(k_1, k_2) \in \mathbb{Z}^2\,; k_1 > 0\} \cup \{(0, k_2)\,; k_2 > 0\}$ and $\mathbb{Z}^2_- = -\mathbb{Z}^2_+$, and $\{w^{(k)}\}_{k\in\mathbb{Z}^2}$ is a family of independent one-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions starting at origin. See [18, Chapter 4] for the precise definition of cylindrical Brownian motions. For later use, we note here that

$$e_k(x) = \begin{cases} \dfrac{\sqrt{2}}{2}\big(\mathbf{e}_k(x) + \mathbf{e}_{-k}(x)\big), & k \in \mathbb{Z}^2_+, \\ \dfrac{\sqrt{2}}{2\sqrt{-1}}\big(\mathbf{e}_k(x) - \mathbf{e}_{-k}(x)\big), & k \in \mathbb{Z}^2_-. \end{cases}$$

The exponential term of the SPDE (1.1) is difficult to treat as it is, because the solution Φ_t is expected to take values in $\mathcal{D}'(\Lambda) \setminus C(\Lambda)$. For this reason, we need to give a rigorous meaning of this SPDE by the renormalization. We assume some properties for the multiplier function.

Hypothesis 1 $\psi : \mathbb{R}^2 \to [0, 1]$ *is a function satisfying the following properties:*

(i) $\psi(0) = 1$ *and* $\psi(x) = \psi(-x)$ *for any* $x \in \mathbb{R}^2$.
(ii) $\sup_{x\in\mathbb{R}^2} |x|^{2+\kappa}|\psi(x)| < \infty$ *for some* $\kappa > 0$.

(iii) $\sup_{x\in\mathbb{R}^2} |x|^{-\zeta}|\psi(x)-1| < \infty$ *for some* $\zeta > 0$.

Note that ψ does not need to be continuous except the origin. For a function ψ satisfying Hypothesis 1, we define the Fourier cut-off operator P_N on $\mathcal{D}'(\Lambda)$ by

$$P_N f(x) = \sum_{k\in\mathbb{Z}^2} \psi(2^{-N}k)\hat{f}(k)\mathbf{e}_k(x), \qquad N \in \mathbb{N},\ x \in \Lambda. \tag{1.5}$$

From Hypothesis 1, we have the following.

- P_N maps $H^{-1-\varepsilon}(\Lambda)$ to $H^{1+\varepsilon}(\Lambda)$ for small $\varepsilon > 0$. Since $H^{1+\varepsilon}(\Lambda) \subset C(\Lambda)$, the regularized cylindrical Brownian motion $(P_N W_t)_{t\geq 0}$ is a continuous function almost surely.
- $\lim_{N\to\infty} \|P_N f - f\|_{H^s} = 0$ for any $s \in \mathbb{R}$ and $f \in H^s(\Lambda)$.

By introducing approximating equations driven by the regularized white noise $(P_N \dot{W}_t)_{t\geq 0}$, we obtain the following theorem in the full L^1-regime of the charge parameter α. See Sect. 1.3 below for the definition of the Besov space $B^{-\varepsilon}_{p,p}(\Lambda)$.

Theorem 1.1 *Assume that ψ satisfies Hypothesis 1. Let $|\alpha| < \sqrt{8\pi}$, $p \in (1, \frac{8\pi}{\alpha^2} \wedge 2)$, and $\varepsilon > 0$. For any $N \in \mathbb{N}$, consider the initial value problem*

$$\begin{cases} \partial_t \Phi_t^N = \frac{1}{2}(\Delta - 1)\Phi_t^N - \frac{\alpha}{2}\exp\left(\alpha\Phi_t^N - \frac{\alpha^2}{2}C_N\right) + P_N\dot{W}_t, & t > 0, \\ \Phi_0^N = P_N\phi, \end{cases} \tag{1.6}$$

where $\phi \in \mathcal{D}'(\Lambda)$ and

$$C_N := \frac{1}{4\pi^2}\sum_{k\in\mathbb{Z}^2} \frac{\psi(2^{-N}k)^2}{1+|k|^2}.$$

Then for μ_0-almost every $\phi \in \mathcal{D}'(\Lambda)$, the unique time-global classical solution Φ^N converges as $N \to \infty$ to a $B^{-\varepsilon}_{p,p}(\Lambda)$-valued stochastic process Φ in the space $C([0,T]; B^{-\varepsilon}_{p,p}(\Lambda))$ for any $T > 0$, $\mathbb{P}$-almost surely. Moreover, the limit Φ is independent of the choice of ψ.

In this paper we call this Φ the *strong solution* of the SPDE (1.1), because in view of Theorem 1.1 we have the mapping from the initial value ϕ and the driving noise $\dot{W}_t$ to the process Φ.

Remark 1.2 The key ingredient of the proof is Theorem 2.1 below, which is the almost-sure convergence of Gaussian multiplicative chaos (GMC in short). The law of GMC was first constructed by Kahane [35], and Robert and Vargas [46] extended it for general convolution approximations of the covariance kernel. Although these results give only convergence in law, some stronger convergence results were also obtained: almost-sure convergence for the circle average and standard Fourier projection [22] and the convergence in probability for general convolution approximations [12]. See

[13, 45] for the reviews of these theories. Our proof of Theorem 2.1 is a modification of [12]. We remark that Hypothesis 1 is prepared for the main theorems on singular SPDEs (for e.g. Theorem 1.1), and the circle average approximation contained in [12] does not satisfy Hypothesis 1. However, our construction of Wick exponentials of the Gaussian free field in Sect. 2 includes the case of the approximations by averaging treated in [12], in particular the circle average approximation, because the estimates (2.5) and (2.6) below hold also for the approximations by averaging (see Sect. A.3). See Sect. 2 for our construction of Wick exponentials (GMC).

Remark 1.3 Since the $\exp(\Phi)_2$-measure $\mu^{(\alpha)}$ is absolutely continuous with respect to μ_0 (see Corollary 2.3), "μ_0-almost every ϕ" can be replaced by "$\mu^{(\alpha)}$-almost every ϕ".

Remark 1.4 We can refine the state space of the strong solution obtained in Theorem 1.1. Precisely, the strong solution is in $C([0,T]; H^{-\varepsilon}(\Lambda))$ almost surely (see Corollary 1.6 for detail).

To introduce another approach to the SPDE (1.1), we define the regularized $\exp(\Phi)_2$-measure

$$\mu_N^{(\alpha)}(\mathrm{d}\phi) := \frac{1}{Z_N^{(\alpha)}} \exp\left\{-\int_\Lambda \exp\left(\alpha P_N\phi(x) - \frac{\alpha^2}{2}C_N\right)\mathrm{d}x\right\}\mu_0(\mathrm{d}\phi), \qquad N \in \mathbb{N}, \tag{1.7}$$

where $Z_N^{(\alpha)} > 0$ is the normalizing constant, and consider the SPDE associated with this measure. The sequence $\{\mu_N^{(\alpha)}\}_{N\in\mathbb{N}}$ of probability measures weakly converges to $\mu^{(\alpha)}$ (see Corollary 2.3).

Hypothesis 2 *The operators P_N defined by (1.5) satisfy the following properties.*

(i) *P_N is nonnegative, that is, $P_N f \geq 0$ if $f \geq 0$.*
(ii) *For any $p \in (1,2)$, $s \in \mathbb{R}$, there exists a constant $C > 0$ such that*

$$\sup_{N\in\mathbb{N}} \|P_N f\|_{B^s_{p,p}} \leq C\|f\|_{B^s_{p,p}}, \qquad \lim_{N\to\infty} \|P_N f - f\|_{B^s_{p,p}} = 0$$

for any $f \in B^s_{p,p}(\Lambda)$.

If ψ is a Schwartz function and the inverse Fourier transform of ψ is a nonnegative function, then Hypothesis 2 holds. See e.g., [10, Proposition 2.78].

Theorem 1.5 *Assume that ψ satisfies Hypotheses 1 and 2. Let $|\alpha| < \sqrt{8\pi}$ and $\varepsilon > 0$. For any $N \in \mathbb{N}$, consider the solution $\widetilde{\Phi}^N = \widetilde{\Phi}^N(\phi)$ of the SPDE*

$$\begin{cases} \partial_t \widetilde{\Phi}_t^N = \dfrac{1}{2}(\triangle - 1)\widetilde{\Phi}_t^N - \dfrac{\alpha}{2}P_N \exp\left(\alpha P_N \widetilde{\Phi}_t^N - \dfrac{\alpha^2}{2}C_N\right) + \dot{W}_t, & t > 0, \\ \widetilde{\Phi}_0^N = \phi \in \mathcal{D}'(\Lambda). \end{cases} \tag{1.8}$$

Let ξ_N be a random variable with the law $\mu_N^{(\alpha)}$ independent of W. Then $\widetilde{\Phi}^{N,\mathrm{stat}} = \widetilde{\Phi}^N(\xi_N)$ is a stationary process and converges in law as $N \to \infty$ to the strong solution Φ^{stat} of (1.1) with an initial law $\mu^{(\alpha)}$, on the space $C([0,T]; H^{-\varepsilon}(\Lambda))$ for any $T > 0$. Moreover, the law of the random variable Φ_t^{stat} is $\mu^{(\alpha)}$ for any $t \geq 0$.

Corollary 1.6 *The strong solution Φ of the SPDE (1.1) belongs to the space $C([0,T]; H^{-\varepsilon}(\Lambda))$, $\mathbb{P}$-almost surely, for μ_0-almost every (or $\mu^{(\alpha)}$-almost every) initial value $\phi \in \mathcal{D}'(\Lambda)$.*

Finally, we discuss a connection between the SPDE (1.1) and the Dirichlet form theory. Let $s \in (0,1)$ be an exponent fixed later (see Corollary 2.4) and set $H = L^2(\Lambda)$ and $E = H^{-s}(\Lambda)$. Recall that $\{e_k\}_{k\in\mathbb{Z}^2}$ is a real-valued CONS of H defined by (1.4). We then denote by $\mathfrak{F}C_b^\infty$ the space of all smooth cylinder functions $F : E \to \mathbb{R}$ having the form

$$F(\phi) = f(\langle \phi, l_1\rangle, \dots, \langle \phi, l_n\rangle), \qquad \phi \in E,$$

with $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$ and $l_1, \dots, l_n \in \mathrm{Span}\{e_k; k \in \mathbb{Z}^2\}$. Since $\mathrm{supp}(\mu^{(\alpha)}) = E$, two different functions in $\mathfrak{F}C_b^\infty$ are also different in $L^p(\mu^{(\alpha)})$-sense. Moreover, $\mathfrak{F}C_b^\infty$ is dense in $L^p(\mu^{(\alpha)})$ for all $p \geq 1$. For $F \in \mathfrak{F}C_b^\infty$, we define the H-derivative $D_H F : E \to H$ by

$$D_H F(\phi) := \sum_{j=1}^n \partial_j f\big(\langle \phi, l_1\rangle, \dots, \langle \phi, l_n\rangle\big) l_j, \qquad \phi \in E.$$

We then consider a pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ which is given by

$$\mathcal{E}(F,G) = \frac{1}{2}\int_E \big(D_H F(\phi), D_H G(\phi)\big)_H \mu^{(\alpha)}(\mathrm{d}\phi), \qquad F, G \in \mathfrak{F}C_b^\infty, \tag{1.9}$$

where $(\cdot,\cdot)_H$ is the inner product of H. Applying the integration by parts formula for $\mu^{(\alpha)}$, we obtain that $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ is closable on $L^2(\mu^{(\alpha)})$ (see Proposition 6.1 below for detail), so we can define $\mathcal{D}(\mathcal{E})$ as the completion of $\mathfrak{F}C_b^\infty$ with respect to $\mathcal{E}_1^{1/2}$-norm. Thus, by directly applying the general methods in the theory of Dirichlet forms (cf. [16, 40]), we can prove quasi-regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the existence of a diffusion process $\mathbb{M} = (\Theta, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, (\Psi_t)_{t\geq 0}, (\mathbb{Q}_\phi)_{\phi\in E})$ properly associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

The following theorem says that the diffusion process $\Psi = (\Psi_t)_{t\geq 0}$ coincides with the strong solution Φ obtained in Theorem 1.1.

Theorem 1.7 *Let $|\alpha| < \sqrt{8\pi}$. Then for $\mu^{(\alpha)}$-almost every ϕ, the diffusion process Ψ coincides $\mathbb{Q}_\phi$-almost surely with the strong solution Φ of the SPDE (1.1) with the initial value ϕ, driven by some $L^2(\Lambda)$-cylindrical $(\mathcal{G}_t)_{t\geq 0}$-Brownian motion $\mathcal{W} = (\mathcal{W}_t)_{t\geq 0}$.*

1.3 Notations and preliminaries

Throughout this paper, we use the notation $A \lesssim B$ for two functions $A = A(\lambda)$ and $B = B(\lambda)$ of a variable λ, if there exists a constant $c > 0$ independent of λ such that $A(\lambda) \le cB(\lambda)$ for any λ. We write $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$. We write $A \lesssim_\mu B$ if we want to emphasize that the constant c depends on another variable μ.

For a measure space $(\mathfrak{M}, m)$ and a Banach space B, denote by $L^p(\mathfrak{M}, m; B)$ the usual L^p-space, where $\mathfrak{M}$ or m may be omitted if they are obvious in the context. If $B = \mathbb{R}$, then we write it by $L^p(\mathfrak{M}, m)$ simply. If $\mathfrak{M}$ is a compact topological space, denote by $C(\mathfrak{M}; B)$ the space of continuous functions with the supremum norm.

We collect several basic facts on function spaces used through this paper. Below we usually denote $L^p(\Lambda)$, $H^s(\Lambda)$ and $B^s_{p,q}(\Lambda)$ by L^p, H^s and $B^s_{p,q}$, respectively, for the sake of brevity. Denote by $\mathcal{S}(\mathbb{R}^2)$ for the space of real-valued Schwartz functions on $\mathbb{R}^2$ and denote its dual by $\mathcal{S}'(\mathbb{R}^2)$, which is the space of tempered distributions. The Fourier transform $\mathcal{F}$ is defined by

$$(\mathcal{F}f)(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)e^{-\sqrt{-1}x\cdot\xi}\,\mathrm{d}x, \qquad f \in \mathcal{S}(\mathbb{R}^2),\ \xi \in \mathbb{R}^2,$$

and so the inverse Fourier transform is given by $\mathcal{F}^{-1}f(z) = \mathcal{F}f(-z)$ ($z \in \mathbb{R}^2$). Also for the distribution $f \in \mathcal{S}'(\mathbb{R}^2)$, the usual generalization of the Fourier transform is considered.

Let (χ, ρ) be a dyadic partition of unity, that is, they satisfy the following:

- $\chi, \rho : \mathbb{R}^2 \to [0, 1]$ are smooth radial functions,
- $\mathrm{supp}(\chi) \subset B(0, 4/3)$, $\mathrm{supp}(\rho) \subset B(0, 8/3) \setminus B(0, 3/4)$,
- $\chi(\xi) + \sum_{j=0}^{\infty} \rho(2^{-j}\xi) = 1$ for any $\xi \in \mathbb{R}^2$,

where $B(x, r)$ stands for the open ball in $\mathbb{R}^2$ centered at x and with radius r. We then set $\rho_{-1} := \chi$ and $\rho_j := \rho(2^{-j}\cdot)$ for $j \ge 0$. We define the *Littlewood–Paley blocks* (or the Littlewood–Paley operator) $\{\Delta_j\}_{j=-1}^{\infty}$ by

$$(\Delta_j f)(x) := \sum_{k\in\mathbb{Z}^2} \rho_j(k)\hat{f}(k)\mathbf{e}_k(x), \qquad f \in \mathcal{D}'(\Lambda),\ x \in \Lambda.$$

We then define the inhomogeneous Besov norm $\|\cdot\|_{B^s_{p,q}}$ and the Besov space $B^s_{p,q}(\Lambda)$ ($s \in \mathbb{R}$, $p, q \in [1, \infty]$) by

$$\|f\|_{B^s_{p,q}} := \begin{cases} \Big(\sum\limits_{j=-1}^{\infty} 2^{jsq}\|\Delta_j f\|_{L^p}^q\Big)^{1/q}, & q \in [1, \infty), \\ \sup\limits_{j\ge -1} \big(2^{js}\|\Delta_j f\|_{L^p}\big), & q = \infty, \end{cases}$$

and

$$B^s_{p,q} = B^s_{p,q}(\Lambda) := \{f \in \mathcal{D}'(\Lambda)\,;\, \|f\|_{B^s_{p,q}} < \infty\},$$

respectively. The Besov space $B^s_{p,q}$ is a Banach space. Moreover, $B^s_{p,q}$ is separable if $q < \infty$ (see [10, Lemma 2.73]).

We recall mainly from Bahouri et al. [10] some basic properties of Besov spaces. We remark that the setting in [10] is not on a torus but on the Euclidean spaces. However, it is known that most results in [10] also follow in the case of function spaces on a torus, and are proved by a parallel argument or by extending functions on a torus to those on the Euclidean spaces periodically (see e.g. [28, Appendix A]). In view of this fact we refer associate results in [10] below, though there is a difference between a torus and the Euclidean spaces. The following embeddings are immediate consequences of the definition.

- If $s_1 \le s_2$, then $B^{s_2}_{p,q} \subset B^{s_1}_{p,q}$.
- If $p_1 \le p_2$, then $B^s_{p_2,q} \subset B^s_{p_1,q}$.
- If $q_1 \le q_2$, then $B^s_{p,q_2} \supset B^s_{p,q_1}$. However, $B^s_{p,q_2} \subset B^{s-\varepsilon}_{p,q_1}$ for any $\varepsilon > 0$.

It is important to note that $B^s_{2,2}$ coincides with the Sobolev space H^s for any $s \in \mathbb{R}$, and $B^s_{\infty,\infty}$ coincides with the Hölder space $C^s(\Lambda)$ for any $s \in \mathbb{R} \setminus \mathbb{N}$ with the equivalent norms [10, Page 99]. The second and third properties above implies that $H^s \subset B^{s-\varepsilon}_{p,p}$ for any $p \in [1, 2]$ and $\varepsilon > 0$.

The following is an immediate consequence of the interpolations of L^p-spaces and of ℓ^p-spaces.

Proposition 1.8 *Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1,\infty]$. Let $\theta \in [0, 1]$ and set $s = (1-\theta)s_1 + \theta s_2$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, and $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$. Then one has*

$$\|f\|_{B^s_{p,q}} \le \|f\|^{1-\theta}_{B^{s_1}_{p_1,q_1}} \|f\|^{\theta}_{B^{s_2}_{p_2,q_2}}.$$

Proposition 1.9 [10, Proposition 2.71] *For any $s \in \mathbb{R}$, $p_1, p_2, q_1, q_2 \in [1,\infty]$ such that $p_1 \le p_2$ and $q_1 \le q_2$, one has the embedding*

$$B^s_{p_1,q_1} \hookrightarrow B^{s-2(1/p_1-1/p_2)}_{p_2,q_2}.$$

In particular, the space $B^s_{p,p}$ is embedded into $C(\Lambda)$ if $s > \frac{2}{p}$.

The following equivalence of norms plays an important role in Corollary 2.4.

Proposition 1.10 [49, Theorem 9 and Remark 26] *For any $s > 0$ and $p, q \in [1,\infty]$,*

$$\|\xi\|_{B^{-s}_{p,q}} \asymp \|e^{\triangle}\xi\|_{L^p} + \left\| t^{s/2}\|e^{t\triangle}\xi\|_{L^p} \right\|_{L^q((0,1],t^{-1}\mathrm{d}t)},$$

where $e^{t\triangle}$ denotes the heat semigroup of the Laplacian $\triangle$ on Λ.

A distribution $\xi \in \mathcal{D}'(\Lambda)$ is said to be *nonnegative* if $\xi(\varphi) = \langle \xi, \varphi \rangle \geq 0$ for any nonnegative $\varphi \in C^\infty(\Lambda)$. Let $B^{s,+}_{p,q}$ be the set of all nonnegative elements in $B^s_{p,q}$. Thanks to the following theorem, a nonnegative distribution is regarded as a nonnegative Borel measure. This fact plays a crucial role in Sect. 4.

Theorem 1.11 [39, Theorem 6.22] *For any nonnegative $\xi \in \mathcal{D}'(\Lambda)$, there exists a unique nonnegative Borel measure μ_ξ such that*

$$\xi(\varphi) = \int_\Lambda \varphi(x)\mu_\xi(\mathrm{d}x), \qquad \varphi \in C^\infty(\Lambda).$$

Consequently, the domain of ξ can be extended to whole $C(\Lambda)$.

2 Wick exponentials of GFFs

In this section, we construct the Wick exponentials of Gaussian free fields (GFFs in short) on Λ, that is, the so-called Gaussian multiplicative chaos (see [12–14, 21, 22, 34, 35, 45]). For some specific approximations for Gaussian multiplicative chaos (e.g., usual Fourier cut-off and circle average), the almost-sure convergence was obtained in [13, 22]. In the present paper, we consider the approximation with general Fourier multiplier operators as in (1.5). Since we need a stronger convergence for our purpose, we give a self-contained proof of the construction in this section.

As mentioned in Remark 2.6 below, our arguments work on more general approximations than previous results.

2.1 GFFs and Wick exponentials

Recall that μ_0 is the centered Gaussian measure on $\mathcal{D}'(\Lambda)$ with covariance $(1-\triangle)^{-1}$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a $\mathcal{D}'(\Lambda)$-valued random variable $\mathbb{X}$ with the law μ_0 is called a (massive) Gaussian free field. Recalling (1.3), we have the covariance formula of the random field $\mathbb{X}$:

$$\mathbb{E}\left[\mathbb{X}(x)\mathbb{X}(y)\right] = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}^2}\frac{1}{1+|k|^2}\mathbf{e}_k(x-y) = G_\Lambda(x,y), \qquad x, y \in \Lambda, \tag{2.1}$$

where G_Λ stands for the Green function of $1-\triangle$ on Λ. Since G_Λ depends on only the difference $x-y$, the law of $\mathbb{X}$ is shift invariant, that is, $\mathbb{X} \overset{d}{=} \mathbb{X}(\cdot + h)$ for any fixed $h \in \Lambda$.

The aim of this section is to define the formal exponential

$$\exp(\alpha\mathbb{X})$$

for any GFF $\mathbb{X}$ and any α with $|\alpha| < \sqrt{8\pi}$. Since $\mathbb{X}$ is $\mathcal{D}'(\Lambda)$-valued, we need a renormalization procedure to give a rigorous meaning to it. Recall that ψ satisfies Hypothesis 1, and the Fourier cut-off operator P_N on $\mathcal{D}'(\Lambda)$ is defined by (1.5):

$$P_N f(x) = \sum_{k\in\mathbb{Z}^2} \psi_N(k)\hat{f}(k)\mathbf{e}_k(x),$$

where $\psi_N := \psi(2^{-N}\cdot)$. Since P_N maps $H^{-1-\varepsilon}$ to $C(\Lambda)$ for small $\varepsilon > 0$ as mentioned before (after Hypothesis 1), the approximation $\mathbb{X}_N := P_N\mathbb{X}$ is a continuous function, so the exponential $\exp(\alpha\mathbb{X}_N)$ is well-defined. However, to take a limit as $N \to \infty$, we need an approximation with renormalization

$$\exp_N^\diamond(\alpha\mathbb{X})(x) := \exp\left(\alpha\mathbb{X}_N(x) - \frac{\alpha^2}{2}C_N\right), \qquad N \in \mathbb{N}, \tag{2.2}$$

where

$$C_N := \mathbb{E}[\mathbb{X}_N(x)^2] = \frac{1}{4\pi^2}\sum_{k\in\mathbb{Z}^2}\frac{\psi_N(k)^2}{1+|k|^2}.$$

The following is the main theorem of this section.

Theorem 2.1 *Assume that ψ satisfies Hypothesis 1. Let $|\alpha| < \sqrt{8\pi}$ and choose parameters p, β such that*

$$p \in \left(1, \frac{8\pi}{\alpha^2} \wedge 2\right), \qquad \beta \in \left(\frac{\alpha^2}{4\pi}(p-1), \frac{2}{p}(p-1)\right). \tag{2.3}$$

Then the sequence $\{\exp_N^\diamond(\alpha\mathbb{X})\}_{N\in\mathbb{N}}$ converges in the space $B_{p,p}^{-\beta}$, $\mathbb{P}$-almost surely and in $L^p(\mathbb{P})$. Moreover, by regarding $\exp_N^\diamond(\alpha\mathbb{X})$ as the random nonnegative Borel measure $\exp_N^\diamond(\alpha\mathbb{X})(x)\,\mathrm{d}x$ on Λ for $N \in \mathbb{N}$, one has the weak convergence of $\{\exp_N^\diamond(\alpha\mathbb{X})\}_{N\in\mathbb{N}}$ almost surely. The limits obtained by different ψ's coincide with each other almost surely.

Remark 2.2 The conclusion of Theorem 2.1 holds under the estimates (2.5) and (2.6) in Proposition 2.5 below, even without Hypothesis 1. See Remark 2.6 below for details. In most references, approximations with continuous parameter are used for the convergence in probability and in $L^p(\mathbb{P})$. It is associated to adopt $\psi_\varepsilon := \psi(\varepsilon\cdot)$ instead of ψ_N for the approximation. For almost-sure convergence we need discretization of the approximation parameter and sufficiently high speed of the approximation with respect to the parameter in order to control the $\mathbb{P}$-null sets. This is the reason why we choose approximations with discrete parameter as appeared in the definition of ψ_N in Theorem 2.1. Here, we remark that for the convergence in $L^p(\mathbb{P})$ (in particular the convergence in probability), we do not need to discretize the approximation parameter. Furthermore we remark that we choose the exponential speed 2^{-N} for the definition ψ_N because of the simplicity of the proof, and N^{-r} with sufficiently large $r > 0$ instead of 2^{-N} is also sufficient for the almost-sure convergence. See the proof of Theorem 2.1 in the last part of Sect. 2.4.

We denote the ($\mathbb{P}$-almost-sure) unique limit by

$$\exp^\diamond(\alpha\mathbb{X}).$$

When the probability space $(\Omega, \mathbb{P})$ is $(\mathcal{D}'(\Lambda), \mu_0)$, the canonical map $\phi \mapsto \phi$ is obviously a GFF. We denote by $\exp^\diamond(\alpha\phi)$ the associated Wick exponential. Since the approximation (2.2) is nonnegative, we can define the $\exp(\Phi)_2$-measure $\mu^{(\alpha)}$ as follows.

Corollary 2.3 *On the Borel probability space $(\mathcal{D}'(\Lambda), \mu_0)$, the probability measure*

$$\mu^{(\alpha)}(\mathrm{d}\phi) = \frac{1}{Z^{(\alpha)}} \exp\left(-\int_\Lambda \exp^\diamond(\alpha\phi)(x)\,\mathrm{d}x\right)\mu_0(\mathrm{d}\phi)$$

is defined as the limit of the approximating measures $\{\mu_N^{(\alpha)}\}_{N\in\mathbb{N}}$ given by (1.7) in weak topology. Moreover, the following holds.

(i) *The Radon-Nikodym derivatives $\left\{\frac{\mathrm{d}\mu_N^{(\alpha)}}{\mathrm{d}\mu_0}\right\}_{N\in\mathbb{N}}$ are uniformly bounded.*

(ii) *$\frac{\mathrm{d}\mu^{(\alpha)}}{\mathrm{d}\mu_0}$ is bounded and strictly positive μ_0-almost everywhere. Hence $\mu^{(\alpha)}$ and μ_0 are absolutely continuous with respect to each other.*

Proof Denote $M_{\phi,N}^{(\alpha)} = \exp_N^\diamond(\alpha\phi)$ and $M_\phi^{(\alpha)} = \exp^\diamond(\alpha\phi)$ in short, and regard them as the corresponding random nonnegative Borel measures on Λ, according to Theorem 1.11.

Although the proof of (i) is completely the same as Hoshino et al. [33, Corollary 2.3], we note the fact on the uniform positivity of the normalizing constants

$$Z_N^{(\alpha)} := \int_{\mathcal{D}'(\Lambda)} \exp\big(-M_{\phi,N}^{(\alpha)}(\Lambda)\big)\mu_0(\mathrm{d}\phi), \qquad N \in \mathbb{N},$$

which is used in the next corollary. By Jensen's inequality,

$$Z_N^{(\alpha)} \geq \exp\Big(-\int_{\mathcal{D}'(\Lambda)} M_{\phi,N}^{(\alpha)}(\Lambda)\mu_0(\mathrm{d}\phi)\Big) = \exp\Big(-\int_\Lambda \mathrm{d}x\Big) > 0.$$

Here we used the fact that $\int_{\mathcal{D}'(\Lambda)} M_{\phi,N}^{(\alpha)}(x)\mu_0(\mathrm{d}\phi) = 1$ for any $x \in \Lambda$, which follows from the definition.

Next we show (ii). Let p and β be as in Theorem 2.1. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}\mu_0\left(M_\phi^{(\alpha)}(\Lambda) \geq n\right) &\leq \frac{1}{n^p}\int_{\mathcal{D}'(\Lambda)} \big(M_\phi^{(\alpha)}(\Lambda)\big)^p \mu_0(\mathrm{d}\phi) \\ &\lesssim \frac{1}{n^p}\int_{\mathcal{D}'(\Lambda)} \big\|M_\phi^{(\alpha)}\big\|_{B_{p,p}^{-\beta}}^p \mu_0(\mathrm{d}\phi),\end{aligned}$$

since $\mathbf{1}_\Lambda \in C^\infty(\Lambda) \subset B_{p',p'}^\beta$ $(1/p + 1/p' = 1)$ and $B_{p',p'}^\beta$ is a dual space of $B_{p,p}^{-\beta}$ (see e.g., [10, Proposition 2.76]). Letting $n \to \infty$, we have $\mu_0(M_\phi^{(\alpha)}(\Lambda) = \infty) = 0$. Since $Z^{(\alpha)} := \int_{\mathcal{D}'(\Lambda)} \exp\big(-M_\phi^{(\alpha)}(\Lambda)\big)\mu_0(\mathrm{d}\phi)$ is positive by the above estimate of $Z_N^{(\alpha)}$ and the dominated convergence theorem, this implies $\frac{\mathrm{d}\mu^{(\alpha)}}{\mathrm{d}\mu_0}$ is bounded and strictly positive μ_0-almost everywhere. □

Even though Theorem 2.1 and Corollary 2.3 imply that the random variable $\phi \mapsto \exp^{\diamond}(\alpha\phi)$ belongs to $L^p(\mu^{(\alpha)}; B_{p,p}^{-\beta})$, the state space can be chosen smaller. The following fact plays a crucial role in Sects. 5 and 6.

Corollary 2.4 *If $|\alpha| < \sqrt{8\pi}$, then there exists an exponent $s \in (0,1)$ such that*

$$\sup_{N\in\mathbb{N}} \int_{\mathcal{D}'(\Lambda)} \| \exp_N^{\diamond}(\alpha\phi)\|_{H^{-s}}^2 \mu_N^{(\alpha)}(\mathrm{d}\phi) < \infty. \tag{2.4}$$

Moreover, the random variable $\phi \mapsto \exp^{\diamond}(\alpha\phi)$ belongs to $L^2(\mu^{(\alpha)}; H^{-s})$.

Proof Recall that $H^{-s} = B_{2,2}^{-s}$ for $s \in \mathbb{R}$. By the interpolation between Besov spaces (Proposition 1.8),

$$\|M_{\phi,N}^{(\alpha)}\|_{H^{-s}} \le \|M_{\phi,N}^{(\alpha)}\|_{B_{p,p}^{-\beta}}^{p/2} \|M_{\phi,N}^{(\alpha)}\|_{B_{\infty,\infty}^{-2}}^{1-p/2}$$

for p, β in (2.3), and $-s := -\beta\frac{p}{2} - 2(1-\frac{p}{2}) > -1$. Since $M_{\phi,N}^{(\alpha)}$ is nonnegative, we have

$$\|e^{t\triangle} M_{\phi,N}^{(\alpha)}\|_{L^\infty} \lesssim t^{-1} M_{\phi,N}^{(\alpha)}(\Lambda), \quad t \in (0,1]$$

by the bound of the heat kernel in spacial component. By Proposition 1.10 we have

$$\|M_{\phi,N}^{(\alpha)}\|_{B_{\infty,\infty}^{-2}} \lesssim M_{\phi,N}^{(\alpha)}(\Lambda).$$

Since the function $x^{2-p}e^{-x}$ is bounded on $x \in (0,\infty)$,

$$\begin{aligned}\int_{\mathcal{D}'(\Lambda)} \|M_{\phi,N}^{(\alpha)}\|_{H^{-s}}^2 e^{-M_{\phi,N}^{(\alpha)}(\Lambda)} \mu_0(\mathrm{d}\phi) &\lesssim \int_{\mathcal{D}'(\Lambda)} \|M_{\phi,N}^{(\alpha)}\|_{B_{p,p}^{-\beta}}^p \big(M_{\phi,N}^{(\alpha)}(\Lambda)\big)^{2-p} e^{-M_{\phi,N}^{(\alpha)}(\Lambda)} \mu_0(\mathrm{d}\phi) \\ &\lesssim \int_{\mathcal{D}'(\Lambda)} \|M_{\phi,N}^{(\alpha)}\|_{B_{p,p}^{-\beta}}^p \mu_0(\mathrm{d}\phi).\end{aligned}$$

Since $\{M_{\phi,N}^{(\alpha)}\}_{N\in\mathbb{N}}$ are bounded in the space $L^p(\mu_0; B_{p,p}^{-\beta})$ as in Theorem 2.1, and $\{Z_N^{(\alpha)}\}_{N\in\mathbb{N}}$ are uniformly positive as stated in the proof of Corollary 2.3, we have the uniform bound (2.4). Since $\langle M_{\phi,N}^{(\alpha)}, \mathbf{e}_k\rangle \to \langle M_{\phi}^{(\alpha)}, \mathbf{e}_k\rangle$ for any $k \in \mathbb{Z}^2$ almost everywhere, by using Fatou's lemma we have

$$\int_{\mathcal{D}'(\Lambda)} \|M_{\phi}^{(\alpha)}\|_{H^{-s}}^2 \mu^{(\alpha)}(\mathrm{d}\phi) \le \liminf_{N\to\infty} \int_{\mathcal{D}'(\Lambda)} \|M_{\phi,N}^{(\alpha)}\|_{H^{-s}}^2 \mu_N^{(\alpha)}(\mathrm{d}\phi) < \infty.$$

Thus we complete the proof. □

Below, we give a self-contained proof of Theorem 2.1. For the proof we prepare a lot of technical results, and in the end of Sect. 2, Theorem 2.1 is proved.

2.2 Approximation of the Green function

By definition, the random field $\mathbb{X}_N = P_N\mathbb{X}$ has the covariance function

$$G_{M,N}(x, y) := \mathbb{E}[\mathbb{X}_M(x)\mathbb{X}_N(y)] = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}^2}\frac{\psi_M(k)\psi_N(k)}{1+|k|^2}\mathbf{e}_k(x-y)$$

for any $M, N \in \mathbb{N}$. Recall that $\psi_N = \psi(2^{-N}\cdot)$. By definition, $C_N = \mathbb{E}[\mathbb{X}_N^2(x)] = G_{N,N}(x, x)$. The function $G_{M,N}$ approximates the Green function G_Λ defined by (2.1). In the following proposition, we summarize the properties of the function $G_{M,N}$ used in the proof of Theorem 2.1. We regard $G_{M,N}$ as a periodic function on $\mathbb{R}^2\times\mathbb{R}^2$, rather than a function on $\Lambda\times\Lambda$.

Proposition 2.5 *Assume that ψ satisfies Hypothesis 1. Then for any $x, y \in \mathbb{R}^2$ with $|x-y| < 1$ and any $M, N \in \mathbb{N}$,*

$$G_{M,N}(x, y) = -\frac{1}{2\pi}\log\left(|x-y|\vee 2^{-M}\vee 2^{-N}\right) + R_{M,N}(x, y), \tag{2.5}$$

where the remainder term $R_{M,N}(x, y)$ is uniformly bounded over $x, y, M,$ and N. Moreover, there exist constants $C > 0$ and $\theta > 0$ such that, for any $M, N \in \mathbb{N}$,

$$\iint_{\Lambda\times\Lambda}\left|G_{M,N+1}(x, y) - G_{M,N}(x, y)\right|\mathrm{d}x\,\mathrm{d}y \le C2^{-\theta N}. \tag{2.6}$$

Since the proof of Proposition 2.5 is long and technical, we provide it in Appendix A. We remark that (2.6) can be improved by L^p-estimate for all $p \in [1,\infty)$ (see Proposition A.5).

Remark 2.6 Theorem 2.1 holds true for any multiplier ψ such that the function $G_{M,N}$ defined from ψ satisfies the estimates (2.5) and (2.6). Indeed, in the proof of Theorem 2.1 after Proposition 2.5, we use only (2.5) and (2.6), but do not use Hypothesis 1. The class of approximations satisfying (2.5) and (2.6) is quite large, and includes the approximations by averaging, treated in [12], in particular the circle average approximation (see Sect. A.3). Moreover, our proofs would go similarly even if we replace the torus Λ with the Lebesgue measure dx and the Gaussian field $\mathbb{X}$ generated by free field measure, by a two-dimensional compact Riemannian manifold $\mathcal{M}$ with its volume measure σ and a Gaussian random field $\mathbb{X}^{\mathcal{M}}$ on $\mathcal{M}$ with covariance function $G_{\mathcal{M}}$ satisfying (2.5) and (2.6) with replacement of $|x-y|$ by the metric $d(x, y)$ in $\mathcal{M}$, respectively. However, in the case of $\mathcal{M}$ and $\mathbb{X}_N^{\mathcal{M}}$, $C_N(x) := \mathbb{E}[\mathbb{X}_N^{\mathcal{M}}(x)^2]$ appeared in (2.2) for renormalization, which is a constant in the case of the torus with the Lebesgue measure dx, will depend on $x \in \mathcal{M}$ generally. We are also able to extend it to compact Riemannian manifold with other dimensions. In the case the range of the charge constant α should be changed according to the dimension. Even though we have such extensions, for simplicity, we discuss our problem only on the torus Λ with the Lebesgue measure dx in the present paper.

2.3 Uniform integrability

Using the first property (2.5) of Proposition 2.5, we first prove the uniform bound of $\{\exp_N^\diamond(\alpha\mathbb{X})\}_{N\in\mathbb{N}}$ in $L^p(\mathbb{P}; B_{p,p}^{-\beta})$. Below, we usually denote

$$\mathbb{M}_N^{(\alpha)} = \exp_N^\diamond(\alpha\mathbb{X})$$

in short. At the beginning, we present *Kahane's convexity inequality* (cf. [35]), which plays a significant role in the proof.

Lemma 2.7 (See [14, Proposition 5.6]) *Let D be an open and bounded subset of $\mathbb{R}^2$. Let φ_1, φ_2 be continuous Gaussian random fields on D with mean zero and with covariance functions $C_1, C_2 : D \times D \to \mathbb{R}$, respectively. If $C_1(x, y) \leq C_2(x, y)$ for any $x, y \in D$, then one has*

$$\mathbb{E}\left[\left\{\int_D \exp\left(\varphi_1(x) - \frac{1}{2}C_1(x,x)\right)\mathrm{d}x\right\}^p\right] \leq \mathbb{E}\left[\left\{\int_D \exp\left(\varphi_2(x) - \frac{1}{2}C_2(x,x)\right)\mathrm{d}x\right\}^p\right]$$

for any $p \in [1, \infty)$.

The following estimate is useful to determine the regularity of $\mathbb{M}_N^{(\alpha)}$. The estimate is called a multifractal property and is proved also in previous results (see e.g. [13, Theorem 3.23], [26, Proposition 3.9] and [45, Theorem 2.14]). As mentioned in Remark 2.6, our arguments work in the case of more general approximations than those treated in the previous results.

Proposition 2.8 *For any $\alpha \in \mathbb{R}$ and $p \in [1, \infty)$ there exists a constant $C > 0$ such that, for any $N \in \mathbb{N}$ and $\lambda \in (0, 1]$,*

$$\mathbb{E}\left[\left(\int_{B(0,\lambda/2)} \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^p\right] \leq C\lambda^{2p-\alpha^2 p(p-1)/4\pi}\,\mathbb{E}\left[\left(\int_\Lambda \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^p\right].$$

Proof Consider the random field $x \mapsto \mathbb{X}_N(\lambda x)$. The inequality

$$\log\left(|\lambda x| \vee 2^{-N}\right) \geq \log\left(|x| \vee 2^{-N}\right) + \log\lambda$$

is easily checked by considering the three cases separately; $\lambda|x| < |x| \leq 2^{-N}$, $\lambda|x| \leq 2^{-N} < |x|$, and $2^{-N} < \lambda|x| < |x|$. By the estimate (2.5), for $x, y \in \mathbb{R}^2$ with $|x| \vee |y| < 1/2$,

$$\begin{aligned}
\mathbb{E}[\mathbb{X}_N(\lambda x)\mathbb{X}_N(\lambda y)] &= -\frac{1}{2\pi}\log\left(|\lambda(x-y)| \vee 2^{-N}\right) + O(1) \\
&\leq -\frac{1}{2\pi}\log\left(|x-y| \vee 2^{-N}\right) - \frac{1}{2\pi}\log\lambda + O(1) \\
&\leq \mathbb{E}[\mathbb{X}_N(x)\mathbb{X}_N(y)] - \frac{1}{2\pi}\log\lambda + c
\end{aligned}$$

for some constant $c > 0$ independent of λ, x, y, and N. Hence by introducing a centered Gaussian random variable Y_λ with variance $-(1/2\pi)\log\lambda + c$, independent of $\mathbb{X}$, we have

$$\mathbb{E}[\mathbb{X}_N(\lambda x)\mathbb{X}_N(\lambda y)] \le \mathbb{E}[(\mathbb{X}_N(x) + Y_\lambda)(\mathbb{X}_N(y) + Y_\lambda)].$$

Then Lemma 2.7 yields

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{|x|<1/2} \mathbb{M}_N^{(\alpha)}(\lambda x)\mathrm{d}x\right)^p\right] \\
&\quad \le \mathbb{E}\left[\exp\left(\alpha p Y_\lambda - \frac{\alpha^2 p}{2}\mathbb{E}[Y_\lambda^2]\right)\right]\mathbb{E}\left[\left(\int_{|x|<1/2} \mathbb{M}_N^{(\alpha)}(x)\mathrm{d}x\right)^p\right] \\
&\quad = C\exp\left(-\frac{\alpha^2 p(p-1)}{4\pi}\log\lambda\right)\mathbb{E}\left[\left(\int_{|x|<1/2} \mathbb{M}_N^{(\alpha)}(x)\mathrm{d}x\right)^p\right]
\end{aligned}$$

for some constant $C > 0$. By changing the variable $y = \lambda x$ we obtain the assertion. □

The following lemmas are useful to show the uniform integrability of $\int_\Lambda \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x$.

Lemma 2.9 *For $\alpha \in \mathbb{R}$ and $p \in [1, 2]$ there exists a constant $C > 0$ such that, for any $N \in \mathbb{N}$ and $\delta \in (0, 1/4]$,*

$$\begin{aligned}
&\mathbb{E}\left[\left(\iint_{|x|\vee|y|<1/2,\ |x-y|<\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\,\mathrm{d}x\,\mathrm{d}y\right)^{p/2}\right] \\
&\quad \le C\delta^{(2-\alpha^2 p/4\pi)(p-1)}\mathbb{E}\left[\left(\int_\Lambda \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^p\right].
\end{aligned}$$

Proof For any $\delta \in (0, 1/4]$ we can choose $\{x_i;\, i = 1, 2, \ldots, n_\delta\}$ such that

$$B(0, 1/2) \subset \bigcup_{i=1}^{n_\delta} B(x_i, \delta), \qquad n_\delta \le c\delta^{-2},$$

where c is an absolute constant. Since

$$\begin{aligned}
&\iint_{|x|\vee|y|<1/2,\ |x-y|<\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \le \int_{|x|<1/2} \mathbb{M}_N^{(\alpha)}(x)\left(\int_{B(x,\delta)} \mathbb{M}_N^{(\alpha)}(y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\quad \le \sum_{i=1}^{n_\delta}\int_{B(x_i,\delta)} \mathbb{M}_N^{(\alpha)}(x)\left(\int_{B(x,\delta)} \mathbb{M}_N^{(\alpha)}(y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\quad \le \sum_{i=1}^{n_\delta}\left(\int_{B(x_i,\delta)} \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)\left(\int_{B(x_i,2\delta)} \mathbb{M}_N^{(\alpha)}(y)\,\mathrm{d}y\right) \\
&\quad \le \sum_{i=1}^{n_\delta}\left(\int_{B(x_i,2\delta)} \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^2,
\end{aligned}$$

we have by the elementary inequality $(a+b)^{p/2} \le a^{p/2} + b^{p/2}$ for $a, b \ge 0$ and the shift invariance of the law of $\mathbb{M}_N^{(\alpha)}$,

$$\mathbb{E}\left[\left(\iint_{|x|\vee|y|<1/2,\,|x-y|<\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\,\mathrm{d}x\,\mathrm{d}y\right)^{p/2}\right]$$
$$\le c\delta^{-2}\mathbb{E}\left[\left(\int_{B(0,2\delta)} \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^{p}\right].$$

Hence Proposition 2.8 yields the conclusion. □

Lemma 2.10 *For any $\alpha \in \mathbb{R}$ there exists a constant $C > 0$ such that, for any $N \in \mathbb{N}$ and $\delta \in (0, 1/4]$,*

$$\mathbb{E}\left[\iint_{|x|\vee|y|<1/2,\,|x-y|\ge\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\,\mathrm{d}x\,\mathrm{d}y\right] \le C(1+\delta^{2-\alpha^2/2\pi}).$$

Proof By the estimate (2.5),

$$\begin{aligned}
&\mathbb{E}\left[\iint_{|x|\vee|y|<1/2,\,|x-y|\ge\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\,\mathrm{d}x\,\mathrm{d}y\right]\\
&= e^{-\alpha^2 C_N}\iint_{|x|\vee|y|<1/2,\,|x-y|\ge\delta} \mathbb{E}\big[\exp\left(\alpha(\mathbb{X}_N(x)+\mathbb{X}_N(y))\right)\big]\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{|x|\vee|y|<1/2,\,|x-y|\ge\delta} e^{\alpha^2 G_{N,N}(x,y)}\,\mathrm{d}x\,\mathrm{d}y\\
&\lesssim \iint_{|x|\vee|y|<1/2,\,|x-y|\ge\delta} |x-y|^{-\alpha^2/2\pi}\,\mathrm{d}x\,\mathrm{d}y \lesssim 1+\delta^{2-\alpha^2/2\pi}.
\end{aligned}$$

□

By using above estimates, we prove L^p-boundedness, in particular the uniform integrability, of $\int_\Lambda \mathbb{M}_N^{(\alpha)}(x)\mathrm{d}x$. It has also proved in previous results (see e.g. [13, Theorem 3.26] and [46, Proposition 3.5]). As mentioned in Remark 2.6, our arguments work in the case of more general approximations than those treated in the previous results.

Proposition 2.11 *For any $|\alpha| < \sqrt{8\pi}$ and $p \in (1, 8\pi/\alpha^2) \cap (1, 2]$,*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left(\int_\Lambda \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^p\right] < \infty.$$

Proof Choosing finite points $\{x_i\}$ such that $\Lambda = [-\pi,\pi)^2 \subset \bigcup_i B(x_i, 1/2)$ and using the shift invariance of the law of $\mathbb{M}_N^{(\alpha)}$,

$$\mathbb{E}\left[\left(\int_\Lambda \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^p\right] \le C^p\mathbb{E}\left[\left(\int_{B(0,1/2)} \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^p\right]$$

for an absolute constant $C > 0$. Let $\delta \in (0, 1/4]$ and we decompose

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{B(0,1/2)} \mathbb{M}_N^{(\alpha)}(x)\mathrm{d}x\right)^p\right] \\
&\quad\le \mathbb{E}\left[\left(\iint_{|x|\vee|y|<1/2,\ |x-y|<\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\mathrm{d}x\mathrm{d}y\right)^{p/2}\right] \\
&\qquad + \mathbb{E}\left[\left(\iint_{|x|\vee|y|<1/2,\ |x-y|\ge\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\mathrm{d}x\mathrm{d}y\right)^{p/2}\right] \\
&\quad\le \mathbb{E}\left[\left(\iint_{|x|\vee|y|<1/2,\ |x-y|<\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\mathrm{d}x\mathrm{d}y\right)^{p/2}\right] \\
&\qquad + \mathbb{E}\left[\iint_{|x|\vee|y|<1/2,\ |x-y|\ge\delta} \mathbb{M}_N^{(\alpha)}(x)\mathbb{M}_N^{(\alpha)}(y)\mathrm{d}x\mathrm{d}y\right]^{p/2}.
\end{aligned}$$

In the second inequality, we use $p \le 2$ and the nonnegativity of $\mathbb{M}_N^{(\alpha)}$. Applying Lemmas 2.9 and 2.10, we have

$$\mathbb{E}\left[\left(\int_\Lambda \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^p\right] \le C'\delta^{(2-\alpha^2 p/4\pi)(p-1)}\mathbb{E}\left[\left(\int_\Lambda \mathbb{M}_N^{(\alpha)}(x)\,\mathrm{d}x\right)^p\right] + C'\delta^{p(1-\alpha^2/4\pi)},$$

where the constant C' is independent of N and δ. Since $\alpha^2 p < 8\pi$, by choosing sufficiently small δ, we complete the proof. □

Corollary 2.12 *For any parameters p and β as in (2.3), one has*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\|\mathbb{M}_N^{(\alpha)}\|^p_{B^{-\beta}_{p,p}}\right] < \infty.$$

Proof By definition of the Besov norm,

$$\begin{aligned}
\mathbb{E}\left[\|\mathbb{M}_N^{(\alpha)}\|^p_{B^{-\beta}_{p,p}}\right] &= \sum_{j=-1}^{\infty} 2^{-j\beta p}\mathbb{E}\left[\|\Delta_j\mathbb{M}_N^{(\alpha)}\|^p_{L^p}\right] \\
&= \sum_{j=-1}^{\infty} 2^{-j\beta p}\int_\Lambda \mathbb{E}\left[|\Delta_j\mathbb{M}_N^{(\alpha)}(x)|^p\right]\mathrm{d}x.
\end{aligned}$$

By the shift invariance of the law of $\mathbb{M}_N^{(\alpha)}$, it is sufficient to consider $\mathbb{E}\big[|\Delta_j\mathbb{M}_N^{(\alpha)}(0)|^p\big]$. The bounds for $j = -1, 0$ are obvious in view of Proposition 2.11. For $j \ge 1$, by using Mikowski's inequality, rapid decay of the Schwartz function $\mathcal{F}^{-1}\rho$, and the shift invariance of the law of $\mathbb{M}_N^{(\alpha)}$,

$$\begin{aligned}
\mathbb{E}\left[|\Delta_j\mathbb{M}_N^{(\alpha)}(0)|^p\right]^{1/p} &= \left\|\int_{\mathbb{R}^2} (\mathcal{F}^{-1}\rho)(x)\mathbb{M}_N^{(\alpha)}(2^{-j}x)\,\mathrm{d}x\right\|_{L^p(\mathbb{P})} \\
&\lesssim \sum_{k\in\mathbb{Z}^2} (1+|k|)^{-3}\left\|\int_{B(k,1)} \mathbb{M}_N^{(\alpha)}(2^{-j}x)\,\mathrm{d}x\right\|_{L^p(\mathbb{P})}
\end{aligned}$$

$$\lesssim \left\| \int_{B(0,1)} \mathbb{M}_N^{(\alpha)}(2^{-j}x)\,\mathrm{d}x \right\|_{L^p(\mathbb{P})}.$$

Hence by Proposition 2.8,

$$\mathbb{E}\left[\left|\Delta_j \mathbb{M}_N^{(\alpha)}(0)\right|^p\right]^{1/p} \lesssim (2^{-j})^{-\alpha^2(p-1)/4\pi}.$$

Therefore, we obtain $\mathbb{E}[\|\mathbb{M}_N^{(\alpha)}\|_{B_{p,p}^{-\beta}}^p] \lesssim 1$ for $\beta > \alpha^2(p-1)/4\pi$. □

2.4 Almost-sure convergence

In this subsection, we show the almost-sure weak convergence of $\mathbb{M}_N^{(\alpha)}$ as $N \to \infty$ in the space of positive Borel measures on Λ, and we finally complete the proof of Theorem 2.1. We apply the following proposition several times, which follows from direct computation.

Proposition 2.13 *Let X be an n-dimensional centered Gaussian random vector with a covariance matrix V. Then, for $a \in \mathbb{R}^n$ and a Borel function f on $\mathbb{R}^n$,*

$$\mathbb{E}\left[e^{a\cdot X} f(X)\right] = e^{a\cdot(Va)/2}\mathbb{E}\left[f(X+Va)\right].$$

The following theorem plays a crucial role to prove Theorem 2.1.

Theorem 2.14 *Let $|\alpha| < \sqrt{8\pi}$. Then, there exist positive constants c and C such that*

$$\mathbb{E}[|\langle f, \mathbb{M}_{N+1}^{(\alpha)}\rangle - \langle f, \mathbb{M}_N^{(\alpha)}\rangle|] \leq C\|f\|_{C(\Lambda)} 2^{-cN} \tag{2.7}$$

for any $N \in \mathbb{N}$ and $f \in C(\Lambda)$.

Proof Our proof is based on the same spirit as [12, Sects. 3 and 4]. It is well-known that the limiting measure $\mathbb{M}^{(\alpha)}$ must be supported on the points x such that

$$\lim_{N\to\infty} \frac{\mathbb{X}_N(x)}{C_N} = \alpha,$$

called α-*thick points*. An essential point of [12] is to decompose $\mathbb{M}_N^{(\alpha)}$ into two parts:

$$\mathbb{M}_N^{<}(x) := \mathbb{M}_N^{(\alpha)}(x) \prod_{n_0 \leq n \leq N} \mathbf{1}_{\{\mathbb{X}_n(x) \leq \alpha(1+\delta)C_n\}}, \qquad \mathbb{M}_N^{>}(x) := \mathbb{M}_N^{(\alpha)}(x) - \mathbb{M}_N^{<}(x),$$

for some fixed $n_0 \in \mathbb{N}$ and $\delta > 0$. Then, L^1 contribution of $\mathbb{M}_N^{>}$ can be eliminated, while $\mathbb{M}_N^{<}$ has a good control in L^2 depending on the choice of n_0 and δ. However, we need the following modifications to obtain the stronger estimate (2.7).

- Let $n_0 = \delta^3 N$ be a *variable* depending on δ and N.

- Replace the indicator function $\mathbf{1}$ with some Lipschitz function.

Now we start the proof of this theorem. Denote by $\tilde{B}(x,r)$ the open ball in Λ centered at x and with radius r under the canonical metric of Λ. It is sufficient to show (2.7) for $f \in C(\Lambda)$ with $\operatorname{supp} f \subset \tilde{B}(0,1/2)$. Indeed, we obtain the assertion for general $f \in C(\Lambda)$, once we apply the finite decomposition $f = \sum_k f_k$ with f_k supported in some ball $\tilde{B}(x_k, 1/2)$ and the shift invariance of the law of $\mathbb{M}_N^{(\alpha)}$. Hence we assume $|x| \vee |y| < 1/2$ throughout this proof.

As introduced in Sect. 2.2, we set $G_{M,N}(x,y) = \mathbb{E}[\mathbb{X}_M(x)\mathbb{X}_N(y)]$ and set $C_{M,N} = G_{M,N}(x,x)$ for $M, N \in \mathbb{N}$. By the estimate (2.5), for any $x, y \in \mathbb{R}^2$ with $|x| \vee |y| < 1/2$ and any $M, N \in \mathbb{N}$ with $M \leq N$, we have

$$G_{M,N}(x,y) = -\frac{1}{2\pi}\log\left(|x-y| \vee 2^{-M}\right) + O(1), \qquad C_{M,N} = \frac{M}{2\pi}\log 2 + O(1).$$

These yield the following: for any sufficiently small $\delta > 0$, there exists an integer N'_δ depending on δ such that, for any $N'_\delta \leq M \leq N$ and $|x| \vee |y| < 1/2$

$$\frac{1}{\tilde{C}_M} \leq \delta^3, \qquad \left|\frac{C_{M,N} - \tilde{C}_M}{\tilde{C}_M}\right| \leq \delta^3, \qquad \left|\frac{G_{M,N}(x,y) - \tilde{G}_M(x,y)}{\tilde{C}_M}\right| \leq \delta^3, \tag{2.8}$$

where

$$\tilde{C}_M := \frac{M}{2\pi}\log 2, \qquad \tilde{G}_M(x,y) := -\frac{1}{2\pi}\log\left(|x-y| \vee 2^{-M}\right).$$

The parameter δ is to be chosen later, as a sufficiently small number compared with $1 - \alpha^2/8\pi$ and the exponent θ in the estimate (2.6).

Furthermore, let χ_δ be a function on $\mathbb{R}$ such that

$$\chi_\delta(\tau) = \begin{cases} 1, & \tau \leq \delta, \\ -\tau/\delta + 2, & \delta \leq \tau \leq 2\delta, \\ 0, & \tau \geq 2\delta. \end{cases}$$

Then we define for each $N, i \in \mathbb{N}$ such that $N \leq i$ (actually we will let $i = N$ or $N+1$),

$$\begin{aligned} \mathbb{M}^<_{N,i}(x) &:= \mathbb{M}_i^{(\alpha)}(x) \prod_{\delta^3 N \leq n \leq i} \chi_\delta\left(\frac{\mathbb{X}_n(x) - \alpha C_{n,i}}{\alpha \tilde{C}_n}\right), \\ \mathbb{M}^>_{N,i}(x) &:= \mathbb{M}_i^{(\alpha)}(x) - \mathbb{M}^<_{N,i}(x). \end{aligned}$$

Let N_δ be an integer such that $N_\delta \geq N'_\delta/\delta^3$. From (2.8) we have that, if $N \geq N_\delta$, then for any integers m, n with $\delta^3 N \leq m \leq n$ and $|x| \vee |y| < 1/2$,

$$\frac{1}{\tilde{C}_m} \leq \delta^3, \qquad \left|\frac{C_{m,n} - \tilde{C}_m}{\tilde{C}_m}\right| \leq \delta^3, \qquad \left|\frac{G_{m,n}(x,y) - \tilde{G}_m(x,y)}{\tilde{C}_m}\right| \leq \delta^3. \tag{2.9}$$

We assume $N \geq N_\delta$ throughout this proof, and decompose

$$\mathbb{M}_{N+1}^{(\alpha)} - \mathbb{M}_N^{(\alpha)} = (\mathbb{M}_{N,N+1}^{<} - \mathbb{M}_{N,N}^{<}) + (\mathbb{M}_{N,N+1}^{>} - \mathbb{M}_{N,N}^{>}).$$

(1) ***The terms*** $\mathbb{M}_{N,N+1}^{>}$ ***and*** $\mathbb{M}_{N,N}^{>}$**.** For any fixed $i \in \{N, N+1\}$ and $x \in \Lambda$, we apply Proposition 2.13 to the $(i - [\delta^3 N] + 1)$-dimensional random vector $X = (\mathbb{X}_n(x))_{\delta^3 N \leq n \leq i}$ and a fixed vector $a = (0, \ldots, 0, \alpha)$. Then, since $Va = (\alpha C_{n,i})_{\delta^3 N \leq n \leq i}$, we have

$$\begin{aligned}
\mathbb{E}[\mathbb{M}_{N,i}^{>}(x)] &= \mathbb{E}\left[e^{a\cdot X - a\cdot Va/2}\left\{1 - \prod_{\delta^3 N \leq n \leq i} \chi_\delta\left(\frac{\mathbb{X}_n(x) - \alpha C_{n,i}}{\alpha \tilde{C}_n}\right)\right\}\right] \\
&= \mathbb{E}\left[1 - \prod_{\delta^3 N \leq n \leq i} \chi_\delta\left(\frac{\mathbb{X}_n(x)}{\alpha \tilde{C}_n}\right)\right] \\
&\leq \sum_{\delta^3 N \leq n \leq i} \mathbb{E}\left[1 - \chi_\delta\left(\frac{\mathbb{X}_n(x)}{\alpha \tilde{C}_n}\right)\right] \\
&\leq \sum_{\delta^3 N \leq n \leq i} \mathbb{P}\left(\mathbb{X}_n(x) \geq \delta\alpha \tilde{C}_n\right),
\end{aligned}$$

where we used the elementary inequality

$$1 - \prod_{n=1}^{K} a_n \leq \sum_{n=1}^{K} (1 - a_n), \qquad a_1, \ldots, a_K \in [0, 1].$$

Since $\mathbb{X}_n(x)$ has a variance $C_{n,n}$ and (2.9) implies $C_{n,n} = (1 + o(\delta))\tilde{C}_n$, we have by the tail estimate of the normal distribution,

$$\begin{aligned}
\mathbb{E}[\mathbb{M}_{N,i}^{>}(x)] &\leq \sum_{\delta^3 N \leq n \leq i} \mathbb{P}\left(\frac{\mathbb{X}_n(x)}{\sqrt{C_{n,n}}} \geq \alpha(\delta + o(\delta))\sqrt{\tilde{C}_n}\right) \\
&\leq C_\delta \sum_{\delta^3 N \leq n \leq i} e^{-\alpha^2(\delta + o(\delta))^2 \tilde{C}_n/2} \leq C'_\delta 2^{-\alpha^2(\delta + o(\delta))^2 \delta^3 N/4\pi}
\end{aligned}$$

for some positive constants C_δ and C'_δ depending on δ. Therefore, we obtain the exponential decay (2.7) for $\mathbb{E}\left[|\langle f, \mathbb{M}_{N,N+1}^{>}\rangle - \langle f, \mathbb{M}_{N,N}^{>}\rangle|\right]$.

(2) ***The difference*** $\mathbb{M}_{N,N+1}^{<} - \mathbb{M}_{N,N}^{<}$**.** We actually show the stronger estimate

$$\mathbb{E}\left[\left|\langle f, \mathbb{M}_{N,N+1}^{<}\rangle - \langle f, \mathbb{M}_{N,N}^{<}\rangle\right|^2\right] \leq C_\delta \|f\|_{C(\Lambda)}^2 2^{-c_\delta N}$$

than (2.7) with replacement of $\mathbb{M}_{N+1}^{(\alpha)}$ and $\mathbb{M}_N^{(\alpha)}$ by $\mathbb{M}_{N,N+1}^{<}$ and $\mathbb{M}_{N,N}^{<}$, respectively. We write the expectation as the form $\iint_{\Lambda^2} f(x)f(y)\mathfrak{M}_N(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where

$$\mathfrak{M}_N(x,y)=\mathbb{E}\left[\left(\mathbb{M}_{N,N+1}^{<}(x)-\mathbb{M}_{N,N}^{<}(x)\right)\left(\mathbb{M}_{N,N+1}^{<}(y)-\mathbb{M}_{N,N}^{<}(y)\right)\right],$$

and consider the integral

$$\iint_{|x|\vee|y|<1/2}|\mathfrak{M}_N(x,y)|\,\mathrm{d}x\,\mathrm{d}y. \tag{2.10}$$

Moreover, we decompose the integrand by

$$\mathfrak{M}_N(x,y)=I_{N+1,N+1}(x,y)-I_{N+1,N}(x,y)-I_{N,N+1}(x,y)+I_{N,N}(x,y),$$

where $I_{i,j}(x,y):=\mathbb{E}[\mathbb{M}_{N,i}^{<}(x)\mathbb{M}_{N,j}^{<}(y)]$ $(i,j=N,N+1)$. For any fixed $x,y\in\Lambda$, we apply Proposition 2.13 to the multidimensional Gaussian random variable

$$\mathfrak{X}=\Big((\mathbb{X}_n(x))_{\delta^3N\le n\le i},(\mathbb{X}_m(y))_{\delta^3N\le m\le j}\Big)$$

and a fixed vector $a\in\mathbb{R}^{(i-[\delta^3N]+1)+(j-[\delta^3N]+1)}$ such that $a\cdot\mathfrak{X}=\alpha(\mathbb{X}_i(x)+\mathbb{X}_j(y))$. Since the covariance matrix V of $\mathfrak{X}$ is given by

$$\begin{aligned}
&Va=\alpha\Big((C_{n,i}+G_{n,j}(x,y))_{\delta^3N\le n\le i},(G_{m,i}(x,y)+C_{m,j})_{\delta^3N\le m\le j}\Big),\\
&a\cdot Va=\alpha^2(C_{i,i}+C_{j,j}+2G_{i,j}(x,y)),
\end{aligned}$$

Proposition 2.13 yields

$$\begin{aligned}
&I_{i,j}(x,y)\\
&=e^{\alpha^2G_{i,j}(x,y)}\mathbb{E}\Bigg[e^{a\cdot\mathfrak{X}-a\cdot Va/2}\prod_{\delta^3N\le n\le i}\chi_\delta\left(\frac{\mathbb{X}_n(x)-\alpha C_{n,i}}{\alpha\tilde{C}_n}\right)\prod_{\delta^3N\le m\le j}\chi_\delta\left(\frac{\mathbb{X}_m(x)-\alpha C_{m,j}}{\alpha\tilde{C}_m}\right)\Bigg]\\
&=e^{\alpha^2G_{i,j}(x,y)}\mathbb{E}\Bigg[\prod_{\delta^3N\le n\le i}\chi_\delta\left(\frac{\mathbb{X}_n(x)+\alpha G_{n,j}(x,y)}{\alpha\tilde{C}_n}\right)\prod_{\delta^3N\le m\le j}\chi_\delta\left(\frac{X_m(y)+\alpha G_{m,i}(x,y)}{\alpha\tilde{C}_m}\right)\Bigg].
\end{aligned}$$

We decompose the integral (2.10) into the two regions

$$|x-y|<2^{-\delta^3N},\qquad 2^{-\delta^3N}\le|x-y|<1.$$

(2-1) ***The integral over*** $|x-y|<2^{-\delta^3N}$**.** We estimate each $I_{i,j}$ $(i,j=N,N+1)$ separately. Assume $i\le j$ without loss of generality. We further decompose the integral into two regions

$$|x-y|<2^{-i},\qquad 2^{-i}\le|x-y|<2^{-\delta^3N}.$$

(2-1-1) ***The integral over*** $|x-y| < 2^{-i}$**.** Since $\chi_\delta \le 1$,

$$I_{i,j}(x,y) \le e^{\alpha^2 G_{i,j}(x,y)} \mathbb{E}\left[\chi_\delta\left(\frac{\mathbb{X}_i(x)+\alpha G_{i,j}(x,y)}{\alpha \tilde{C}_i}\right)\right].$$

Since $|x-y| < 2^{-i}$, (2.9) implies that

$$G_{i,j}(x,y) = \tilde{G}_i(x,y) + o(\delta)\tilde{C}_i = (1+o(\delta))\tilde{C}_i.$$

Hence we have

$$\begin{aligned}\mathbb{E}\left[\chi_\delta\left(\frac{\mathbb{X}_i(x)+\alpha G_{i,j}(x,y)}{\alpha \tilde{C}_i}\right)\right] &\le \mathbb{P}\left(\mathbb{X}_i(x) \le (-1+2\delta+o(\delta))\alpha\tilde{C}_i\right)\\ &\lesssim e^{-(1+O(\delta))\alpha^2\tilde{C}_i/2} \lesssim 2^{-(1+O(\delta))\alpha^2 N/4\pi}.\end{aligned}$$

Since $e^{\alpha^2 G_{i,j}(x,y)} \lesssim |x-y|^{-\alpha^2/2\pi} \lesssim 2^{\alpha^2 N/2\pi}$ by the estimate (2.5), we obtain

$$\begin{aligned}\iint_{|x|\vee|y|<1/2,\ |x-y|<2^{-i}} I_{i,j}(x,y)\,\mathrm{d}x\,\mathrm{d}y &\lesssim \iint_{|x|\vee|y|<1/2,\ |x-y|<2^{-i}} 2^{(1+O(\delta))\alpha^2 N/4\pi}\,\mathrm{d}x\,\mathrm{d}y\\ &\lesssim 2^{N\left(\alpha^2/4\pi-2+O(\delta)\right)}.\end{aligned}$$

This decays exponentially if $\alpha^2 < 8\pi$ and δ is chosen sufficiently small.
(2-1-2) ***The integral over*** $2^{-i} \le |x-y| < 2^{-\delta^3 N}$**.** The argument is similar to (**2-1-1**). For any x, y in this region, there exists an integer $n_{x,y} \in [\delta^3 N, i]$ satisfying $2^{-n_{x,y}} \le |x-y| < 2^{-n_{x,y}+1}$. For such $n_{x,y}$, we have

$$I_{i,j}(x,y) \le e^{\alpha^2 G_{i,j}(x,y)} \mathbb{E}\left[\chi_\delta\left(\frac{\mathbb{X}_{n_{x,y}}(x)+\alpha G_{n_{x,y},j}(x,y)}{\alpha \tilde{C}_{n_{x,y}}}\right)\right].$$

Since (2.9) implies

$$G_{n_{x,y},j}(x,y) = \tilde{G}_{n_{x,y}}(x,y) + o(\delta)\tilde{C}_{n_{x,y}} = (1+o(\delta))\tilde{C}_{n_{x,y}},$$

similarly to the argument in **(2-1-1)** we have

$$\begin{aligned}\mathbb{E}\left[\chi_\delta\left(\frac{\mathbb{X}_{n_{x,y}}(x)+\alpha G_{n_{x,y},j}(x,y)}{\alpha \tilde{C}_{n_{x,y}}}\right)\right] &\le \mathbb{P}\left(\mathbb{X}_{n_{x,y}}(x) \le (-1+2\delta+o(\delta))\alpha\tilde{C}_{n_{x,y}}\right)\\ &\lesssim e^{-(1+O(\delta))\alpha^2\tilde{C}_{n_{x,y}}/2}\\ &\lesssim 2^{-(1+O(\delta))\alpha^2 n_{x,y}/4\pi} \asymp |x-y|^{(1+O(\delta))\alpha^2/4\pi}.\end{aligned}$$

On the other hand, by the estimate (2.5), $e^{\alpha^2 G_{i,j}(x,y)} \lesssim |x-y|^{-\alpha^2/2\pi}$. Hence we have

$$\iint_{|x|\vee|y|<1/2,\ 2^{-i}\le|x-y|<2^{-\delta^3 N}} I_{i,j}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$
\begin{aligned}
&\lesssim \iint_{|x|\vee|y|<1/2,\, 2^{-i}\le|x-y|<2^{-\delta^3 N}} |x-y|^{-\alpha^2/4\pi+O(\delta)}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \int_{2^{-i}\le|x|<2^{-\delta^3 N}} |x|^{-\alpha^2/4\pi+O(\delta)}\,\mathrm{d}x \\
&\lesssim \int_{2^{-i}}^{2^{-\delta^3 N}} r^{-\alpha^2/4\pi+1+O(\delta)}\,\mathrm{d}r \\
&\lesssim 2^{\delta^3 N(\alpha^2/4\pi-2+O(\delta))}
\end{aligned}
$$

if $\alpha^2 < 8\pi$. This decays exponentially if δ is chosen sufficiently small.

(2-2) ***The integral over*** $|x-y| \ge 2^{-\delta^3 N}$**.** We have to consider combinations of I terms. We consider only $I_{N+1,N} - I_{N,N}$, since the other difference $I_{N+1,N+1} - I_{N,N+1}$ is estimated by a similar way. For simplicity, we write

$$
\chi_n^j(x) = \chi_\delta\left(\frac{\mathbb{X}_n(x) + \alpha G_{n,j}(x,y)}{\alpha \tilde{C}_n}\right), \quad \chi_m^i(y) = \chi_\delta\left(\frac{\mathbb{X}_m(y) + \alpha G_{m,i}(x,y)}{\alpha \tilde{C}_n}\right).
$$

Now we decompose

$$
\begin{aligned}
&I_{N+1,N}(x,y) - I_{N,N}(x,y) \\
&\quad = e^{\alpha^2 G_{N+1,N}(x,y)}\mathbb{E}\Big[\prod_{\delta^3 N\le n\le N+1} \chi_n^N(x) \prod_{\delta^3 N\le m\le N} \chi_m^{N+1}(y)\Big] \\
&\qquad - e^{\alpha^2 G_{N,N}(x,y)}\mathbb{E}\Big[\prod_{\delta^3 N\le n\le N} \chi_n^N(x) \prod_{\delta^3 N\le m\le N} \chi_m^{N}(y)\Big] \\
&\quad = \left(e^{\alpha^2 G_{N+1,N}(x,y)} - e^{\alpha^2 G_{N,N}(x,y)}\right)\mathbb{E}\Big[\prod_{\delta^3 N\le n\le N+1} \chi_n^N(x) \prod_{\delta^3 N\le m\le N} \chi_m^{N+1}(y)\Big] \\
&\qquad + e^{\alpha^2 G_{N,N}(x,y)}\mathbb{E}\Big[\left(\chi_{N+1}^N(x) - 1\right)\prod_{\delta^3 N\le n\le N} \chi_n^N(x) \prod_{\delta^3 N\le m\le N} \chi_m^{N+1}(y)\Big] \\
&\qquad + e^{\alpha^2 G_{N,N}(x,y)}\mathbb{E}\Big[\sum_{\delta^3 N\le m_0\le N}\Big\{\Big(\prod_{\delta^3 N\le n\le N} \chi_n^N(x)\Big)\left(\chi_{m_0}^{N+1}(y) - \chi_{m_0}^N(y)\right) \\
&\qquad\qquad \times \prod_{\delta^3 N\le m< m_0} \chi_m^N(y) \prod_{m_0<m\le N} \chi_m^{N+1}(y)\Big\}\Big] \\
&\quad =: J_1(x,y) + J_2(x,y) + J_3(x,y).
\end{aligned}
$$

In the region $|x-y| \ge 2^{-\delta^3 N}$, we have no choice but to do

$$
\prod_{\delta^3 N\le n\le N} \chi_n^N(z) \le 1.
$$

However, we can use the estimate (2.6). Indeed,

$$\begin{aligned}
&\left|e^{\alpha^2 G_{N+1,N}(x,y)} - e^{\alpha^2 G_{N,N}(x,y)}\right| \\
&\quad \lesssim |G_{N+1,N}(x,y) - G_{N,N}(x,y)| \left(e^{\alpha^2 G_{N+1,N}(x,y)} \vee e^{\alpha^2 G_{N,N}(x,y)}\right) \\
&\quad \lesssim |G_{N+1,N}(x,y) - G_{N,N}(x,y)| \cdot |x-y|^{-\alpha^2/2\pi}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\chi_{m_0}^{N+1}(y) - \chi_{m_0}^{N}(y)\right| \\
&= \left|\chi_\delta\left(\frac{\mathbb{X}_{m_0}(y) + \alpha G_{m_0,N+1}(x,y)}{\alpha \tilde{C}_{m_0}}\right) - \chi_\delta\left(\frac{\mathbb{X}_{m_0}(y) + \alpha G_{m_0,N}(x,y)}{\alpha \tilde{C}_{m_0}}\right)\right| \\
&\lesssim_\delta \left|G_{m_0,N+1}(x,y) - G_{m_0,N}(x,y)\right|.
\end{aligned}$$

Hence by the estimate (2.6) we have

$$\begin{aligned}
&\iint_{|x|\vee|y|<1/2,\ |x-y|\geq 2^{-\delta^3 N}} (|J_1(x,y)| + |J_3(x,y)|)\, \mathrm{d}x\, \mathrm{d}y \\
&\lesssim \sum_{\delta^3 N \leq m_0 \leq N} \iint_{|x|\vee|y|<1/2,\ |x-y|\geq 2^{-\delta^3 N}} |G_{m_0,N+1}(x,y) - G_{m_0,N}(x,y)|\, |x-y|^{-\alpha^2/2\pi}\, \mathrm{d}x\, \mathrm{d}y \\
&\lesssim 2^{\delta^3 N \alpha^2/2\pi} \sum_{\delta^3 N \leq m_0 \leq N} \iint_{|x|\vee|y|<1/2} |G_{m_0,N+1}(x,y) - G_{m_0,N}(x,y)|\, \mathrm{d}x\, \mathrm{d}y \\
&\lesssim N 2^{N(\delta^3\alpha^2/2\pi - \theta)}.
\end{aligned}$$

Since $\theta > 0$, this decays exponentially if δ is chosen sufficiently small.

Finally we consider J_2. The estimate (2.5) implies that for $2^{-\delta^3 N} \leq |x-y| \leq 1$,

$$\begin{aligned}
\left|\frac{G_{N+1,N}(x,y)}{\tilde{C}_{N+1}}\right| &\leq \frac{1}{\tilde{C}_{N+1}}\left(-\frac{1}{2\pi}\log|x-y| + O(1)\right) \\
&\leq \frac{1}{\tilde{C}_{N+1}}\left(\frac{\delta^3 N}{2\pi}\log 2 + O(1)\right) \\
&= o(\delta).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathbb{E}\left[\left|\chi_{N+1}^{N}(x) - 1\right|\right] &= \mathbb{E}\left[\left|\chi_\delta\left(\frac{\mathbb{X}_{N+1}(x) + \alpha G_{N+1,N}(x,y)}{\alpha \tilde{C}_{N+1}}\right) - 1\right|\right] \\
&\leq \mathbb{P}\left(\mathbb{X}_{N+1}(x) + \alpha G_{N+1,N}(x,y) \geq \delta\alpha\tilde{C}_{N+1}\right) \\
&\leq \mathbb{P}\left(\mathbb{X}_{N+1}(x) \geq (\delta + o(\delta))\alpha\tilde{C}_{N+1}\right) \\
&\lesssim_\delta e^{-(\delta+o(\delta))^2\alpha^2\tilde{C}_{N+1}/2} \\
&\lesssim 2^{-(\delta+o(\delta))^2 N\alpha^2/4\pi}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\iint_{|x|\vee|y|<1/2,\,|x-y|\geq 2^{-\delta^3 N}} |J_2(x,y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\lesssim \iint_{|x|\vee|y|<1/2,\,|x-y|\geq 2^{-\delta^3 N}} |x-y|^{-\alpha^2/2\pi} 2^{-(\delta+o(\delta))^2 N\alpha^2/4\pi}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\lesssim 2^{-c_\delta N}
\end{aligned}$$

with $c_\delta = (\delta + o(\delta))^2\alpha^2/4\pi - \delta^3\alpha^2/2\pi$. Since c_δ is positive for sufficiently small δ, this completes the proof. □

Corollary 2.15 *For any $f \in C(\Lambda)$, the sequence $\{\langle f, \mathbb{M}_N^{(\alpha)}\rangle\}_{N\in\mathbb{N}}$ converges almost surely and in $L^1(\mathbb{P})$. This limit is independent to the choice of ψ.*

Proof Almost-sure convergence follows from Theorem 2.14. Denote by $\langle f, \mathbb{M}_\infty^{(\alpha)}\rangle$ the limit. The uniqueness follows completely in the same way as the argument in [12, Sect. 5], but we provide a sketch of the proof for readers' convenience. Let $\bar{\psi} = \mathbf{1}_{B(0,1)}$, the indicator function of the ball $B(0,1)$, and define $\bar{P}_N$ and $\bar{\mathbb{M}}_N^{(\alpha)}$ in a similarly way to P_N and $\mathbb{M}_N^{(\alpha)}$, respectively, by $\bar{\psi}$ instead of ψ. Since $\bar{\psi}$ satisfies Hypothesis 1, there exists an almost-sure and L^1-limit $\langle f, \bar{\mathbb{M}}_\infty^{(\alpha)}\rangle$. Denote by $\mathcal{F}_n$ the filtration generated by $\{\hat{\mathbb{X}}(k)\}_{|k|<2^n}$. Since $(1-\bar{P}_n)\mathbb{X}_N$ is independent of $\mathcal{F}_n$, we have

$$\mathbb{E}[\langle f, \mathbb{M}_N^{(\alpha)}\rangle|\mathcal{F}_n] = \langle f, \bar{\mathbb{M}}_{N,n}^{(\alpha)}\rangle,$$

where

$$\bar{\mathbb{M}}_{N,n}^{(\alpha)} := \exp\left(\alpha\bar{P}_n\mathbb{X}_N - \frac{\alpha^2}{2}\bar{C}_{N,n}\right), \quad \bar{C}_{N,n} = \mathbb{E}[(\bar{P}_n\mathbb{X}_N(x))^2].$$

Since $\bar{P}_n\mathbb{X}_N$ converges as $N\to\infty$ to $\bar{P}_n\mathbb{X}$ uniformly in $x\in\Lambda$ almost surely for each n, we have

$$\langle f, \bar{\mathbb{M}}_n^{(\alpha)}\rangle = \lim_{N\to\infty}\langle f, \bar{\mathbb{M}}_{N,n}^{(\alpha)}\rangle = \lim_{N\to\infty}\mathbb{E}[\langle f, \mathbb{M}_N^{(\alpha)}\rangle|\mathcal{F}_n] = \mathbb{E}[\langle f, \mathbb{M}_\infty^{(\alpha)}\rangle|\mathcal{F}_n].$$

Letting $n\to\infty$, we have $\langle f, \bar{\mathbb{M}}_\infty^{(\alpha)}\rangle = \langle f, \mathbb{M}_\infty^{(\alpha)}\rangle$ almost surely. □

Corollary 2.16 *Regard $\mathbb{M}_N^{(\alpha)}$ as a measure as in Theorem 2.1. Then, the sequence $\{\mathbb{M}_N^{(\alpha)}\}_{N\in\mathbb{N}}$ converges in the weak topology, almost surely.*

Proof Let D be a countable dense set in $C(\Lambda)$ which includes the constant function 1. Then, by Corollary 2.15 we have

$$\mathbb{P}\left(\lim_{N\to\infty}\langle f, \mathbb{M}_N^{(\alpha)}\rangle \text{ exists for all } f\in D\right) = 1.$$

From now, in order to clarify the dependence of the randomness $\omega \in \Omega$, we denote $\mathbb{M}_N^{(\alpha)}$ with a sample $\omega \in \Omega$ by $\mathbb{M}_N^{(\alpha)}(\omega)$. Let $\mathcal{N} \in \mathcal{F}$ be the event that $\lim_{N\to\infty}\langle f, \mathbb{M}_N^{(\alpha)}(\omega)\rangle$ does not exists for some $f \in D$. For each $\omega \in \Omega \setminus \mathcal{N}$, define an operator $\mathring{\mathbb{M}}_\infty^{(\alpha)}(\omega)$ on $C(\Lambda)$ with domain D by

$$\mathring{\mathbb{M}}_\infty^{(\alpha)}(\omega)(f) := \lim_{N\to\infty}\langle f, \mathbb{M}_N^{(\alpha)}(\omega)\rangle, \quad f \in D.$$

Then, for $\omega \in \Omega\setminus\mathcal{N}$, it is easy to see that $\mathring{\mathbb{M}}_\infty^{(\alpha)}(\omega)$ can be extended to a linear operator on the space linearly spanned by D. Moreover, since D includes the constant function 1,

$$\sup_{N\in\mathbb{N}}\int_\Lambda \mathbb{M}_N^{(\alpha)}(\omega)\,\mathrm{d}x < \infty, \tag{2.11}$$

and hence, for $f \in D$

$$\left|\mathring{\mathbb{M}}_\infty^{(\alpha)}(\omega)(f)\right| = \lim_{N\to\infty}\left|\langle f, \mathbb{M}_N^{(\alpha)}(\omega)\rangle\right| \le \|f\|_{C(\Lambda)}\sup_{N\in\mathbb{N}}\int_\Lambda \mathbb{M}_N^{(\alpha)}(\omega)\,\mathrm{d}x \lesssim \|f\|_{C(\Lambda)}.$$

In view of these facts, for $\omega \in \Omega\setminus\mathcal{N}$, $\mathring{\mathbb{M}}_\infty^{(\alpha)}(\omega)$ is extended to a bounded linear operator $\mathbb{M}_\infty^{(\alpha)}(\omega)$ on $C(\Lambda)$. By the deneness of D in $C(\Lambda)$ and (2.11), we have for $\omega \in \Omega\setminus\mathcal{N}$, $\mathbb{M}_\infty^{(\alpha)}(\omega)(f) = \lim_{N\to\infty}\langle f, \mathbb{M}_N^{(\alpha)}(\omega)\rangle$ for $f \in C(\Lambda)$. Nonnegativity of $\mathbb{M}_\infty^{(\alpha)}$ follows from that of $\{\mathbb{M}_N^{(\alpha)}\}_{N\in\mathbb{N}}$. $\square$

Proof of Theorem 2.1 Since convergence of the corresponding measures follows from Corollary 2.16, we prove convergence in the Besov space and independence of the limit in ψ.

First we show the convergence of $\mathbb{M}_N^{(\alpha)}$ in $B_{p,p}^{-\beta}$. By Theorem 2.14, for small $\delta > 0$ and any $N \ge N_\delta$,

$$\begin{aligned}\mathbb{E}\left[\|\Delta_j\mathbb{M}_{N+1}^{(\alpha)} - \Delta_j\mathbb{M}_N^{(\alpha)}\|_{L^1}\right] &= \int_\Lambda \mathbb{E}[|\langle\check{\rho}_j(x-\cdot), \mathbb{M}_{N+1}^{(\alpha)} - \mathbb{M}_N^{(\alpha)}\rangle|]\,\mathrm{d}x \\ &\lesssim C_\delta 2^{2j}2^{-c_\delta N},\end{aligned}$$

where $\check{\rho}_j = \sum_{k\in\mathbb{Z}^2}(\mathcal{F}^{-1}\rho_j)(\cdot + 2\pi k)$. This means

$$\mathbb{E}\left[\|\mathbb{M}_{N+1}^{(\alpha)} - \mathbb{M}_N^{(\alpha)}\|_{B_{1,1}^{-\gamma}}\right] \lesssim C_\delta 2^{-c_\delta N}$$

for any $\gamma > 2$. On the other hand, by Corollary 2.12, for any parameters p', β' satisfying (2.3),

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left[\|\mathbb{M}_{N+1}^{(\alpha)} - \mathbb{M}_N^{(\alpha)}\|_{B_{p',p'}^{-\beta'}}^{p'}\right] < \infty.$$

Fix parameters p and β satisfying (2.3). For any $\varepsilon \in (0,1)$, let p_ε and β_ε be parameters defined by $1/p = \varepsilon + (1-\varepsilon)/p_\varepsilon$ and $\beta = \varepsilon\gamma + (1-\varepsilon)\beta_\varepsilon$. Since $p_\varepsilon \to p$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0$, p_ε and β_ε satisfy (2.3) for small $\varepsilon > 0$. For such ε, by the interpolation (Proposition 1.8), we have

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbb{M}_{N+1}^{(\alpha)} - \mathbb{M}_N^{(\alpha)}\|_{B_{p,p}^{-\beta}}^{p}\right] \\
&\quad \le \mathbb{E}\left[\|\mathbb{M}_{N+1}^{(\alpha)} - \mathbb{M}_N^{(\alpha)}\|_{B_{1,1}^{-\gamma}}\right]^{\varepsilon p} \mathbb{E}\left[\|\mathbb{M}_{N+1}^{(\alpha)} - \mathbb{M}_N^{(\alpha)}\|_{B_{p_\varepsilon,p_\varepsilon}^{-\beta_\varepsilon}}^{p_\varepsilon}\right]^{(1-\varepsilon)p/p_\varepsilon} \le C_\delta' 2^{-c_\delta' N}
\end{aligned}$$

for some constants $C_\delta', c_\delta' > 0$ depending on p, β, and ε. This implies the $L^p(\mathbb{P})$-convergence of $\{\mathbb{M}_N^{(\alpha)}\}$ in $B_{p,p}^{-\beta}(\Lambda)$. Moreover, since

$$\sum_{N=N_\delta}^{\infty} \mathbb{E}\left[\|\mathbb{M}_{N+1}^{(\alpha)} - \mathbb{M}_N^{(\alpha)}\|_{B_{p,p}^{-\beta}}\right] < \infty,$$

by the Borel-Cantelli lemma we obtain the almost-sure convergence of $\{\mathbb{M}_N^{(\alpha)}\}_{N\in\mathbb{N}}$.

Finally we show the uniqueness of the limit. Consider two multipliers ψ and $\bar{\psi}$ satisfying Hypothesis 1 and define the limits $\mathbb{M}_\infty^{(\alpha)}$ and $\bar{\mathbb{M}}_\infty^{(\alpha)}$, respectively. By Corollary 2.15, $\langle \mathbb{M}_\infty^{(\alpha)}, \mathbf{e}_k\rangle = \langle \bar{\mathbb{M}}_\infty^{(\alpha)}, \mathbf{e}_k\rangle$ for any $k \in \mathbb{Z}^2$ almost surely, so $\Delta_j \mathbb{M}_\infty^{(\alpha)} = \Delta_j \bar{\mathbb{M}}_\infty^{(\alpha)}$ for any $j \ge -1$ almost surely. Hence $\mathbb{M}_\infty^{(\alpha)} = \bar{\mathbb{M}}_\infty^{(\alpha)}$ in $B_{p,p}^{-\beta}$ almost surely. □

3 Wick exponentials of Ornstein–Uhlenbeck processes

For $\phi \in \mathcal{D}'(\Lambda)$ and an $L^2(\Lambda)$-cylindrical Brownian motion W, let $X = X(\phi)$ be the unique solution of the initial value problem

$$\begin{cases} \partial_t X_t = \dfrac{1}{2}(\triangle - 1)X_t + \dot{W}_t, & t > 0, \\ X_0 = \phi. \end{cases} \tag{3.1}$$

In this section, we consider the Wick exponential of the infinite-dimensional Ornstein–Uhlenbeck (OU in short) process X. First we recall the basic estimate of X in [33].

Proposition 3.1 *For $\varepsilon > 0$, $\delta \in (0,1)$, $m \in \mathbb{N}$, and $T > 0$, there exists a constant $C > 0$ such that one has the a priori estimate*

$$\mathbb{E}\left[\|X(\phi)\|_{C([0,T];H^{-\varepsilon})\cap C^{\delta/2}([0,T];H^{-\varepsilon-\delta})}^m\right] \le C(1 + \|\phi\|_{H^{-\varepsilon}}^m). \tag{3.2}$$

Moreover, for any $\varepsilon > 0$ and $\phi_1, \phi_2 \in H^{-\varepsilon}$,

$$\|X(\phi_1) - X(\phi_2)\|_{C([0,T];H^{-\varepsilon})} \le \|\phi_1 - \phi_2\|_{H^{-\varepsilon}}. \tag{3.3}$$

Proof See [33, Proposition 2.1] for the proof of (3.2). The estimate (3.3) is obtained by writing down the mild form of (3.1). □

It is known that the GFF measure μ_0 is the invariant measure of the process X (see e.g., [19, Theorem 6.2.1]). Therefore, the random variable

$$\Omega \times \mathcal{D}'(\Lambda) \ni (\omega, \phi) \mapsto X_t(\phi)(\omega) \in \mathcal{D}'(\Lambda)$$

is also a GFF under the probability measure $\mathbb{P} \otimes \mu_0$ for any $t > 0$. Thus the existence of the Wick exponential of X is an immediate consequence of Theorem 2.1.

Theorem 3.2 *Assume that ψ satisfies Hypothesis 1. Let $|\alpha| < \sqrt{8\pi}$ and choose parameters p and β as in (2.3). Then the functions*

$$\mathcal{X}_t^N(\phi)(x) := \exp\left(\alpha\big(P_N X_t(\phi)\big)(x) - \frac{\alpha^2}{2} C_N\right), \qquad N \in \mathbb{N}$$

are uniformly bounded in the space $L^p(\mathbb{P} \otimes \mu_0; L^p([0,T]; B_{p,p}^{-\beta}))$ for any $T > 0$. Moreover, the function $\mathcal{X}^N$ converges as $N \to \infty$ in the space $L^p([0,T]; B_{p,p}^{-\beta})$, $\mathbb{P} \otimes \mu_0$-almost surely and in $L^p(\mathbb{P} \otimes \mu_0)$. The limits obtained by different ψ's coincide with each other, $\mathbb{P} \otimes \mu_0$-almost surely.

Proof Using the invariance of μ_0 with respect to X_t and using Theorems 2.1 and 2.14, we have the exponential decay

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}\otimes\mu_0}\left[\left\|\mathcal{X}^{N+1} - \mathcal{X}^N\right\|_{L^p([0,T];B_{p,p}^{-\beta})}^p\right] \\
&\quad = \int_{\mathcal{D}'(\Lambda)} \int_0^T \mathbb{E}\left[\left\|\mathcal{X}_t^{N+1}(\phi) - \mathcal{X}_t^N(\phi)\right\|_{B_{p,p}^{-\beta}}^p\right] \mathrm{d}t\, \mu_0(\mathrm{d}\phi) \\
&\quad = T\mathbb{E}\left[\left\|\exp_{N+1}^{\diamond}(\alpha\mathbb{X}) - \exp_N^{\diamond}(\alpha\mathbb{X})\right\|_{B_{p,p}^{-\beta}}^p\right] \\
&\quad \le TC2^{-cN}
\end{aligned}$$

for some positive constants c and C, where $\mathbb{X}$ is a GFF under the probability $\mathbb{P}$. Then the assertion is obtained by a similar way to the proof of Theorem 2.1. □

Denote by $\mathcal{X}^\infty := \lim_{N\to\infty} \mathcal{X}^N$ the $\mathbb{P} \otimes \mu_0$-almost-sure limit. The following result is an immediate consequence of the $\mathbb{P} \otimes \mu_0$-almost-sure convergence in Theorem 3.2.

Corollary 3.3 *For μ_0-almost every $\phi \in \mathcal{D}'(\Lambda)$, the random function $\mathcal{X}^N(\phi)$ converges to $\mathcal{X}^\infty(\phi)$ in the space $L^p([0,T]; B_{p,p}^{-\beta})$ almost surely.*

In Sect. 5, the following "stability" result of $\mathcal{X}^\infty(\phi)$ with respect to ϕ makes an important role.

Lemma 3.4 *Let $\varepsilon > 0$, and let $\{\xi_N\}_{N\in\mathbb{N}\cup\{\infty\}}$ be $H^{-\varepsilon}$-valued random variables independent of W. Assume that the law ν_N of ξ_N is absolutely continuous with respect to μ_0 for any $N \in \mathbb{N}\cup\{\infty\}$, and Radon–Nikodym derivatives $\left\{\frac{\mathrm{d}\nu_N}{\mathrm{d}\mu_0}\right\}_{N\in\mathbb{N}\cup\{\infty\}}$ are uniformly bounded. If $\lim_{N\to\infty}\xi_N = \xi_\infty$ in $H^{-\varepsilon}$ almost surely, then for any $T > 0$,*

$$\lim_{N\to\infty}\mathcal{X}^\infty(\xi_N) = \mathcal{X}^\infty(\xi_\infty)$$

in $L^p([0,T]; B^{-\beta}_{p,p})$ in probability.

Proof The proof is very similar to [33, Lemma 2.5] and done by a slight modification. For any fixed $M \in \mathbb{N}$, by the estimate (3.3),

$$\begin{aligned}\|P_M X(\xi_N) - P_M X(\xi_\infty)\|_{C([0,T];C(\Lambda))} &\lesssim_M \|X(\xi_N) - X(\xi_\infty)\|_{C([0,T];H^{-\varepsilon})}\\ &\lesssim \|\xi_N - \xi_\infty\|_{H^{-\varepsilon}} \xrightarrow{N\to\infty} 0,\end{aligned}$$

almost surely. In the first inequality, we use the fact that P_M sends $H^{-\varepsilon}$ to $C(\Lambda)$, as mentioned after Hypothesis 1. Hence for any fixed $M \in \mathbb{N}$,

$$\lim_{N\to\infty}\|\mathcal{X}^M(\xi_N) - \mathcal{X}^M(\xi_\infty)\|_{C([0,T];C(\Lambda))} = 0$$

almost surely, from the definition of the Wick exponential $\mathcal{X}^M$. On the other hand, since Radon–Nikodym derivatives $\frac{\mathrm{d}\nu_N}{\mathrm{d}\mu_0}$ are uniformly bounded, by using Theorem 3.2,

$$\begin{aligned}&\sup_{N\in\mathbb{N}\cup\{\infty\}}\mathbb{E}\Big[\|\mathcal{X}^M(\xi_N) - \mathcal{X}^\infty(\xi_N)\|^p_{L^p([0,T];B^{-\beta}_{p,p})}\Big]\\ &\lesssim \sup_{N\in\mathbb{N}\cup\{\infty\}}\mathbb{E}\left[\int_{\mathcal{D}'(\Lambda)}\|\mathcal{X}^M(\phi) - \mathcal{X}^\infty(\phi)\|^p_{L^p([0,T];B^{-\beta}_{p,p})}\nu_N(\mathrm{d}\phi)\right] \xrightarrow{M\to\infty} 0.\end{aligned}$$

Hence, by using the inequality $(a+b)\wedge 1 \le a + (b\wedge 1)$ for $a, b \ge 0$, we have

$$\begin{aligned}&\mathbb{E}\left[\|\mathcal{X}^\infty(\xi_N) - \mathcal{X}^\infty(\xi_\infty)\|_{L^p([0,T];B^{-\beta}_{p,p})}\wedge 1\right]\\ &\le 2\sup_{N\in\mathbb{N}\cup\{\infty\}}\mathbb{E}\Big[\|\mathcal{X}^M(\xi_N) - \mathcal{X}^\infty(\xi_N)\|_{L^p([0,T];B^{-\beta}_{p,p})}\Big]\\ &\quad + \mathbb{E}\left[\|\mathcal{X}^M(\xi_N) - \mathcal{X}^M(\xi_\infty)\|_{L^p([0,T];B^{-\beta}_{p,p})}\wedge 1\right].\end{aligned}$$

Letting $N \to \infty$ first and then $M \to \infty$, we have

$$\lim_{N\to\infty}\mathbb{E}\left[\|\mathcal{X}^\infty(\xi_N) - \mathcal{X}^\infty(\xi_\infty)\|_{L^p([0,T];B^{-\beta}_{p,p})}\wedge 1\right] = 0.$$

Thus we have the assertion. □

4 Global well-posedness of the strong solution

In this section, we consider the approximating equation (1.6):

$$\begin{cases} \partial_t \Phi_t^N = \frac{1}{2}(\triangle - 1)\Phi_t^N - \frac{\alpha}{2}\exp\left(\alpha\Phi_t^N - \frac{\alpha^2}{2}C_N\right) + P_N\dot{W}_t, & t > 0, \\ \Phi_0^N = P_N\phi, \end{cases}$$

and prove Theorem 1.1. The proof goes in a similar way to Hoshino et al. [33, Sect. 3] with a slight modification. Similarly to the previous paper, we use the Da Prato–Debussche trick, that is, we decompose the solution of (1.6) by $\Phi^N = X^N + Y^N$, where X^N and Y^N solve

$$\begin{cases} \partial_t X_t^N = \frac{1}{2}(\triangle - 1)X_t^N + P_N\dot{W}_t, & t > 0, \\ X_0^N = P_N\phi, \end{cases} \tag{4.1}$$

$$\begin{cases} \partial_t Y_t^N = \frac{1}{2}(\triangle - 1)Y_t^N - \frac{\alpha}{2}\exp(\alpha Y_t^N)\exp\left(\alpha X_t^N - \frac{\alpha^2}{2}C_N\right), & t > 0, \\ Y_0^N = 0. \end{cases} \tag{4.2}$$

Note that $X^N = P_N X(\phi)$, where $X(\phi)$ is the solution of (3.1) with the initial value ϕ. Hence the renormalized exponential of X^N in (4.2) is equal to

$$\exp\left(\alpha X_t^N - \frac{\alpha^2}{2}C_N\right) = \mathcal{X}_t^N(\phi).$$

Since $\mathcal{X}^N$ converges to $\mathcal{X}^\infty$ in $L^p([0,T]; B_{p,p}^{-\beta})$ as stated in Corollary 3.3, we consider the solution map of the *deterministic* equation

$$\partial_t \Upsilon_t = \frac{1}{2}(\triangle - 1)\Upsilon_t - \frac{\alpha}{2}e^{\alpha\Upsilon_t}\mathcal{X}_t, \qquad t \in [0, T]$$

for any generic nonnegative $\mathcal{X} \in L^p([0,T]; B_{p,p}^{-\beta})$.

4.1 Products of continuous functions and nonnegative distributions

Since any nonnegative distribution is regarded as a nonnegative Borel measure by Theorem 1.11, the product of a function $f \in C(\Lambda)$ and a nonnegative distribution $\xi \in \mathcal{D}'(\Lambda)$ is well-defined as a Borel measure.

Definition 4.1 For any $f \in C(\Lambda)$ and any nonnegative $\xi \in \mathcal{D}'(\Lambda)$, we define the signed Borel measure

$$\mathcal{M}(f, \xi)(\mathrm{d}x) := f(x)\mu_\xi(\mathrm{d}x),$$

where $\mu_\xi(\mathrm{d}x)$ is the Borel measure associated with ξ, as in Theorem 1.11.

We recall some properties of the product map $\mathcal{M}$ from [33, Sect. 3.1]. Recall that $B^{s,+}_{p,q}(\Lambda)$ denotes the set of nonnegative elements in $B^s_{p,q}(\Lambda)$.

Theorem 4.2 [33, Theorems 3.4 and 3.5] *Let $s > 0$ and $p, q \in [1, \infty]$. The map*

$$\mathcal{M} : C(\Lambda) \times B^{-s,+}_{p,q} \to B^{-s}_{p,q}$$

is continuous, and bounded in the sense that

$$\|\mathcal{M}(f, \xi)\|_{B^{-s}_{p,q}} \lesssim \|f\|_{C(\Lambda)} \|\xi\|_{B^{-s}_{p,q}}$$

for any $f \in C(\Lambda)$ and $\xi \in B^{-s,+}_{p,q}$.

Theorem 4.3 [33, Theorem 3.6] *Let $s > 0$, $p, q \in [1, \infty]$, and $r \in (1, \infty]$. For any space-time functions $(Y, \mathcal{X}) \in L^1([0, T]; C(\Lambda)) \times L^r([0, T]; B^{-s,+}_{p,q})$ and any function $f \in C^1_b(\mathbb{R})$, consider the time-dependent distribution*

$$\mathcal{M}(f(Y), \mathcal{X})(t) := \mathcal{M}(f(Y_t), \mathcal{X}_t).$$

Then the correspondence $(Y, \mathcal{X}) \mapsto \mathcal{M}(f(Y), \mathcal{X})$ is well-defined as a map

$$L^1([0, T]; C(\Lambda)) \times L^r([0, T]; B^{-s,+}_{p,q}) \to L^{r'}([0, T]; B^{-s}_{p,q}).$$

for any $r' \in [1, r]$. Moreover, if $r' < r$, this map is continuous.

4.2 Global well-posedness of Y

In this part, we can consider more general parameters

$$p \in (1, \infty), \qquad \beta \in \left(0, \frac{2}{p}(p-1)\right), \tag{4.3}$$

than those in (2.3). We fix such parameters p, β and the time interval $[0, T]$. We consider the initial value problem

$$\begin{cases} \partial_t \Upsilon_t = \frac{1}{2}(\triangle - 1)\Upsilon_t - \frac{\alpha}{2}\mathcal{M}(e^{\alpha \Upsilon_t}, \mathcal{X}_t), & t \in (0, T], \\ \Upsilon_0 = \upsilon, \end{cases} \tag{4.4}$$

for any given $\mathcal{X} \in L^p([0, T]; B^{-\beta,+}_{p,p})$ and $\upsilon \in B^{2-\beta}_{p,p}$. To solve the equation (4.4), we introduce the space

$$\mathscr{Y}_T = \left\{\Upsilon \in L^p([0, T]; C(\Lambda)) \cap C([0, T]; L^p) \,;\, e^{\alpha\Upsilon} \in L^\infty([0, T]; C(\Lambda))\right\}$$

as a solution space. The purpose of this section is showing the following theorem:

Theorem 4.4 *Assume that p and β satisfy (4.3). Let $\mathcal{X} \in L^p([0,T]; B_{p,p}^{-\beta,+})$ and $\upsilon \in B_{p,p}^{2-\beta}$. Then there exists the unique mild solution $\Upsilon \in \mathscr{Y}_T$ of (4.4), that is, Υ satisfies the equation*

$$\Upsilon_t = e^{t(\Delta-1)/2}\upsilon - \frac{\alpha}{2}\int_0^t e^{(t-s)(\Delta-1)/2}\mathcal{M}(e^{\alpha\Upsilon_s}, \mathcal{X}_s)\,\mathrm{d}s \tag{4.5}$$

for any $t \in (0,T]$. Moreover, this solution belongs to the space

$$L^p([0,T]; B_{p,p}^{2/p+\delta}) \cap C([0,T]; B_{p,p}^{\delta})$$

for any $\delta \in (0, \frac{2}{p}(p-1)-\beta)$, and the mapping

$$\mathcal{S} : B_{p,p}^{2-\beta} \times L^p([0,T]; B_{p,p}^{-\beta,+}) \ni (\upsilon, \mathcal{X}) \mapsto \Upsilon \in L^p([0,T]; B_{p,p}^{2/p+\delta}) \cap C([0,T]; B_{p,p}^{\delta})$$

is continuous.

Recall the following Schauder estimates for the heat semigroup.

Proposition 4.5 [37, Lemma 2.2] and [41, Proposition 6] *Let $s \in \mathbb{R}$ and $p, q \in$* $[1,\infty]$.

(i) *For every $\delta \geq 0$, $\|e^{t(\Delta-1)/2}u\|_{B_{p,q}^{s+2\delta}} \lesssim t^{-\delta}\|u\|_{B_{p,q}^{s}}$ uniformly over $t > 0$.*

(ii) *For every $\delta \in [0,1]$, $\|(e^{t(\Delta-1)/2}-1)u\|_{B_{p,q}^{s-2\delta}} \lesssim t^{\delta}\|u\|_{B_{p,q}^{s}}$ uniformly over $t > 0$.*

Remark 4.6 We remark that, if $\Delta - 1$ is replaced by Δ, then

$$\|e^{t\Delta/2}u\|_{B_{p,q}^{s+2\delta}} \lesssim (1+t^{-\delta})\|u\|_{B_{p,q}^{s}}$$

is the right t-uniform estimate ([37, Lemma 2.2]). The constant 1 comes from the bound of $e^{t\Delta/2}\Delta_{-1}u$. In the above proposition, we can omit this constant by using the factor e^{-t}.

Proposition 4.7 [33, Proposition A.3] *Let $\theta \in \mathbb{R}$, $p, q \in [1,\infty]$, and $r \in (1,\infty]$. Let U be an element of $L^r([0,T]; B_{p,q}^{\theta})$, and let u be the mild solution of the equation*

$$\partial_t u = \frac{1}{2}(\Delta - 1)u + U, \qquad t > 0,$$

with initial value $u_0 \in B_{p,q}^{\theta+2}$. Then for any $\varepsilon > 0$ and $\delta \in (0, 2/r')$, one has

$$\begin{aligned}&\|u\|_{L^r([0,T]; B_{p,q}^{\theta+2-\varepsilon}) \cap C([0,T]; B_{p,q}^{\theta+2/r'-\varepsilon}) \cap C^{\delta/2}([0,T]; B_{p,q}^{\theta+2/r'-\varepsilon-\delta})}\\ &\quad\lesssim \|u_0\|_{B_{p,q}^{\theta+2}} + \|U\|_{L^r([0,T]; B_{p,q}^{\theta})},\end{aligned}$$

where $r' \in [1,\infty)$ is such that $1/r + 1/r' = 1$.

We first show the uniqueness of the solution, by following Hoshino et al. [33, Lemma 3.8]. Since the function $x \mapsto |x|^p$ is not twice differentiable if $p < 2$, we need to modify the previous argument.

Lemma 4.8 *For any $\mathcal{X} \in L^p([0,T]; B^{-\beta,+}_{p,p})$ and $\upsilon \in B^{2-\beta}_{p,p}$, there is at most one mild solution $\Upsilon \in \mathscr{Y}_T$ of (4.4).*

Proof Let $\Upsilon, \Upsilon' \in \mathscr{Y}_T$ be two solutions of (4.4) with the same $\mathcal{X}$ and υ. Then $Z = \Upsilon - \Upsilon'$ solves the equation

$$\left\{\partial_t - \frac{1}{2}(\triangle - 1)\right\} Z_t = -\frac{\alpha}{2}\mathcal{M}(e^{\alpha\Upsilon_t} - e^{\alpha\Upsilon'_t}, \mathcal{X}_t) =: D_t,$$

where $D \in L^p([0,T]; B^{-\beta}_{p,p})$, because of definition of $\mathscr{Y}_T$ and Theorem 4.2. Let $\varepsilon > 0$ and define $Z^\varepsilon = e^{\varepsilon\triangle}Z$. Then Z^ε solves the equation

$$\left\{\partial_t - \frac{1}{2}(\triangle - 1)\right\} Z^\varepsilon = e^{\varepsilon\triangle}D.$$

By the regularizing effect of the heat semigroup (Proposition 4.5), $e^{\varepsilon\triangle}D$ belongs to $L^p([0,T]; C^\infty(\Lambda))$. Then by the Schauder estimate (Proposition 4.7), we have that Z^ε belongs to $C([0,T]; C^\infty(\Lambda))$. Hence for any $f \in C^2(\mathbb{R})$, we have

$$\begin{aligned}
&\int_\Lambda f(Z^\varepsilon_t(x))\,\mathrm{d}x \\
&= f(0)|\Lambda| + \int_0^t \int_\Lambda f'(Z^\varepsilon_s(x))\partial_s Z^\varepsilon_s(x)\,\mathrm{d}x\mathrm{d}s \\
&= f(0)|\Lambda| + \frac{1}{2}\int_0^t \int_\Lambda f'(Z^\varepsilon_s(x))(\triangle - 1)Z^\varepsilon_s(x)\,\mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^t \int_\Lambda f'(Z^\varepsilon_s(x))e^{\varepsilon\triangle}D_s(x)\,\mathrm{d}x\mathrm{d}s \\
&= f(0)|\Lambda| - \frac{1}{2}\int_0^t \int_\Lambda f''(Z^\varepsilon_s(x))|\nabla Z^\varepsilon_s(x)|^2\,\mathrm{d}x\mathrm{d}s \\
&\quad - \frac{1}{2}\int_0^t \int_\Lambda f'(Z^\varepsilon_s(x))Z^\varepsilon_s(x)\,\mathrm{d}x\mathrm{d}s + \int_0^t \int_\Lambda f'(Z^\varepsilon_s(x))e^{\varepsilon\triangle}D_s(x)\,\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

where the first equality is justified as a Riemann–Stieltjes integral, because

$$\partial_s Z^\varepsilon = \frac{1}{2}(\triangle - 1)Z^\varepsilon + e^{\varepsilon\triangle}D \in L^p([0,T]; C^\infty(\Lambda)).$$

For $\lambda > 0$, let

$$f_\lambda(x) = (\lambda^2 + x^2)^{p/2}, \qquad x \in \mathbb{R},$$

and for $R > 0$, let $\varphi_R \in C^\infty(\mathbb{R})$ be a nonnegative even smooth function such that

$$\varphi_R(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| > R+1. \end{cases}$$

Then we define $f_{\lambda,R} \in C^2(\mathbb{R})$ by the function determined by

$$\begin{cases} f''_{\lambda,R}(x) = f''_\lambda(x)\varphi_R(x), \qquad x \in \mathbb{R}, \\ f'_{\lambda,R}(0) = 0, \\ f_{\lambda,R}(0) = \lambda^p. \end{cases}$$

Since we easily have the properties

- $f''_{\lambda,R} \geq 0$,
- $f'_{\lambda,R}$ is bounded and $xf'_{\lambda,R}(x) \geq 0$,
- $f_{\lambda,R}(x) \uparrow f_\lambda(x)$ as $R \to \infty$,

we have the inequality

$$\int_\Lambda f_{\lambda,R}(Z_t^\varepsilon(x))\,\mathrm{d}x \leq \lambda^p|\Lambda| + \int_0^t \int_\Lambda f'_{\lambda,R}(Z_s^\varepsilon(x))e^{\varepsilon\triangle}D_s(x)\,\mathrm{d}x\,\mathrm{d}s.$$

Once we let $\varepsilon \to 0$, $e^{\varepsilon\triangle}D \to D$ in $L^p([0,T]; B_{p,p}^{-\beta-\kappa})$ for any $\kappa > 0$ by Proposition 4.5, and hence $Z^\varepsilon \to Z$ in $L^p([0,T]; B_{p,p}^{2-\beta-2\kappa})$ by Proposition 4.7. Since $B_{p,p}^{2-\beta-2\kappa} \subset C(\Lambda)$ for small $\kappa > 0$, by using Theorem 4.3 we have

$$\int_\Lambda f_{\lambda,R}(Z_t(x))\,\mathrm{d}x \leq \lambda^p|\Lambda| + \int_0^t \int_\Lambda f'_{\lambda,R}(Z_s(x))D_s(x)\,\mathrm{d}x\,\mathrm{d}s \tag{4.6}$$

for almost every t. Here, we used the boundedness of $f'_{\lambda,R}$ and that $e^{\varepsilon\triangle}D$ is a difference of two nonnegative functions, for the convergence of the second term of the right-hand side. We can deduce the term as

$$\begin{aligned} &\int_\Lambda f'_{\lambda,R}(Z_s(x))D_s(x)\mathrm{d}x \\ &= -\frac{\alpha}{2}\int_\Lambda (e^{\alpha\Upsilon_s(x)} - e^{\alpha\Upsilon'_s(x)})f'_{\lambda,R}(Z_s(x))\mathcal{X}_s(x)\mathrm{d}x \\ &= -\frac{\alpha^2}{2}\int_\Lambda e^{A(\alpha\Upsilon_s(x),\alpha\Upsilon'_s(x))}Z_s(x)f'_{\lambda,R}(Z_s(x))\mathcal{X}_s(x)\mathrm{d}x \leq 0, \end{aligned}$$

where $A(x,y)$ is a continuous function on $\mathbb{R}^2$ defined by

$$A(x,y) = \begin{cases} \log\frac{e^x-e^y}{x-y}, & x \neq y, \\ x & x = y. \end{cases}$$

Hence letting $R \to \infty$ in (4.6), we have

$$\int_\Lambda f_\lambda(Z_t(x))\,\mathrm{d}x \leq \lambda^p|\Lambda|$$

for almost every t. Letting $\lambda \to 0$, we have $\|Z_t\|_{L^p(\Lambda)} = 0$, which implies $\Upsilon = \Upsilon'$ for almost every (t, x), thus $\Upsilon = \Upsilon'$ in $\mathscr{Y}_T$. □

Next we show the existence of the solution, by following Hoshino et al. [33, Lemma 3.10]. Since the only difference is that we use Besov spaces instead of Sobolev spaces, we omit some details in this part. The following embedding theorem is frequently used below.

Lemma 4.9 [48, Corollary 5] *Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$ be Banach spaces such that the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ is compact. Then for any $p \in [1, \infty]$ and $s > 0$, the embeddings*

$$L^p([0,T];\mathcal{A}) \cap C^s([0,T];\mathcal{C}) \hookrightarrow L^p([0,T];\mathcal{B}),$$
$$C([0,T];\mathcal{A}) \cap C^s([0,T];\mathcal{C}) \hookrightarrow C([0,T];\mathcal{B})$$

are compact.

Lemma 4.10 *For any $\mathcal{X} \in L^p([0,T]; B^{-\beta,+}_{p,p})$ and $\upsilon \in B^{2-\beta}_{p,p}$, there is at least one mild solution $\Upsilon \in \mathscr{Y}_T$. Moreover, for any $\delta \in (0, \frac{2}{p}(p-1)-\beta)$, there exists a constant $C > 0$ independent of $\mathcal{X}$ and υ such that one has the a priori estimate*

$$\begin{aligned}
&\|\Upsilon\|_{L^p([0,T];B^{2/p+\delta}_{p,p})\cap C([0,T];B^{\delta}_{p,p})\cap C^{\delta/2}([0,T];L^p)} \\
&\quad \le C\left\{\|\upsilon\|_{B^{2-\beta}_{p,p}} + e^{|\alpha|\|\upsilon\|_{C(\Lambda)}}\|\mathcal{X}\|_{L^p([0,T];B^{-\beta}_{p,p})}\right\}. \qquad (4.7)
\end{aligned}$$

Proof As discussed in [33, Lemma 3.10], for any $\mathcal{X} \in L^p([0,T]; B^{-\beta,+}_{p,p})$, there exists a family $\{\mathcal{X}^N\}_{N\in\mathbb{N}}$ of nonnegative continuous functions on $[0,T]\times\Lambda$ such that $\mathcal{X}^N \to \mathcal{X}$ in $L^p([0,T]; B^{-\beta,+}_{p,p})$ as $N \to \infty$. For such $\mathcal{X}^N$, we consider the classical global solutions of the approximating equations

$$\begin{cases} \partial_t \Upsilon^N_t = \dfrac{1}{2}(\triangle - 1)\Upsilon^N_t - \dfrac{\alpha}{2} e^{\alpha\Upsilon^N_t}\mathcal{X}^N_t, \\ \Upsilon^N_0 = \upsilon. \end{cases}$$

Note that $\alpha\Upsilon^N_t \le |\alpha|\|\upsilon\|_{C(\Lambda)}$ follows from the comparison principle. By applying the Schauder estimate (Proposition 4.7) and Theorem 4.2, for any $\delta' \in (\delta, \frac{2}{p}(p-1)-\beta)$ we have

$$\begin{aligned}
&\|\Upsilon^N\|_{L^p([0,T];B^{2/p+\delta'}_{p,p})\cap C^{\delta'/2}([0,T];L^p)} \\
&\lesssim \|\upsilon\|_{B^{2-\beta}_{p,p}} + \|\mathcal{M}(e^{\alpha\Upsilon^N}, \mathcal{X}^N)\|_{L^p([0,T];B^{-\beta}_{p,p})} \\
&\lesssim \|\upsilon\|_{B^{2-\beta}_{p,p}} + \|e^{\alpha\Upsilon^N}\|_{L^\infty([0,T];C(\Lambda))}\|\mathcal{X}^N\|_{L^p([0,T];B^{-\beta}_{p,p})} \\
&\lesssim \|\upsilon\|_{B^{2-\beta}_{p,p}} + e^{|\alpha|\|\upsilon\|_{C(\Lambda)}}\|\mathcal{X}^N\|_{L^p([0,T];B^{-\beta}_{p,p})}.
\end{aligned}$$

By Lemma 4.9, the embeddings

$$\begin{aligned} &L^p([0,T];B_{p,p}^{2/p+\delta'})\cap C^{\delta'/2}([0,T];L^p)\hookrightarrow L^p([0,T];B_{p,p}^{2/p+\delta}),\\ &C([0,T];B_{p,p}^{\delta'})\cap C^{\delta'/2}([0,T];L^p)\hookrightarrow C([0,T];B_{p,p}^{\delta}) \end{aligned}$$

are compact. Here, recall that the embedding $B_{p,p}^{s}\hookrightarrow B_{p,p}^{s'}$ is compact for any $s'<s$ (see [10, Corollary 2.96]). Hence there exists a subsequence $\{N_k\}$ such that

$$\Upsilon^{N_k}\to\Upsilon \qquad \text{in} \quad L^p([0,T];B_{p,p}^{2/p+\delta})\cap C([0,T];B_{p,p}^{\delta}).$$

This yields the bound (4.7) for Υ, thus in particular $\Upsilon\in\mathscr{Y}_T$.

We have that Υ solves the mild equation (4.5) by a similar argument to Hoshino et al. [33, Lemma 3.10]. Since $\alpha\Upsilon^N$ is uniformly bounded from above, we can apply Theorem 4.3 to the function $f\in C_b^1(\mathbb{R})$ such that $f(x)=e^x$ on some half line $x\in(-\infty,a]$ and obtain

$$\mathcal{M}(e^{\alpha\Upsilon^{N_k}},\mathcal{X}^{N_k})\to\mathcal{M}(e^{\alpha\Upsilon},\mathcal{X}) \qquad \text{in} \quad L^q([0,T];B_{p,p}^{-\beta})$$

for any $q<p$. Then letting $N_k\to\infty$ on both sides of (4.5) and applying the Schauder estimate (Proposition 4.7), we have that $(\Upsilon,\mathcal{X})$ solves the same equation in the space $C([0,T];B_{p,p}^{\delta})$. □

By Lemmas 4.8 and 4.10, the solution map $\mathcal{S}:(\upsilon,\mathcal{X})\mapsto\Upsilon$ is well-defined. The continuity of the map

$$\mathcal{S}:B_{p,p}^{2-\beta}\times L^p([0,T];B_{p,p}^{-\beta,+})\ni(\upsilon,\mathcal{X})\mapsto\Upsilon\in L^p([0,T];B_{p,p}^{2/p+\delta})\cap C([0,T];B_{p,p}^{\delta})$$

follows from a similar compactness argument as above, and from uniqueness of the solution. Indeed, by the a priori estimate (4.7), any convergent sequence of $B_{p,p}^{2-\beta}\times L^p([0,T];B_{p,p}^{-\beta,+})$ is sent to a bounded sequence of $L^p([0,T];B_{p,p}^{2/p+\delta''})\cap C([0,T];B_{p,p}^{\delta''})$ by the map $\mathcal{S}$, for any $\delta''\in(\delta,\frac{2}{p}(p-1)-\beta)$. This sequence is precompact in $L^p([0,T];B_{p,p}^{2/p+\delta})\cap C([0,T];B_{p,p}^{\delta})$. By the same argument as before, we see that any accumulation point solves (4.5), which is unique. Hence this precompact sequence converges. This completes the proof of Theorem 4.4.

4.3 Proof of Theorem 1.1

From Theorem 4.4, the first main result of this paper (Theorem 1.1) immediately follows.

Proof of Theorem 1.1 By the Da Prato–Debussche decomposition (4.1)–(4.2), the solution $\Phi^N(\phi)$ of (1.6) satisfies

$$\Phi^N(\phi) = P_N X(\phi) + \mathcal{S}(0, \mathcal{X}^N(\phi)).$$

For μ_0-almost every ϕ, $X(\phi) \in C([0, T]; H^{-\varepsilon})$ for any $\varepsilon > 0$, in view of Proposition 3.1. Hence the first term $P_N X(\phi)$ of the right-hand side converges almost surely to $X(\phi)$ in $C([0, T]; H^{-\varepsilon})$ for any $\varepsilon > 0$, under Hypothesis 1. The second term $\mathcal{S}(0, \mathcal{X}^N(\phi))$ converges almost surely to $\mathcal{S}(0, \mathcal{X}^\infty(\phi))$ in $C([0, T]; B^\delta_{p,p})$ (see Theorems 3.2 and 4.4). Hence $\Phi^N(\phi)$ converges to

$$\Phi(\phi) = X(\phi) + \mathcal{S}(0, \mathcal{X}^\infty(\phi))$$

in the space $C([0, T]; B^{-\varepsilon}_{p,p})$ for any $\varepsilon > 0$ almost surely, for μ_0-almost every ϕ. □

5 Stationary solution

In this section, we prove Theorem 1.5 and Corollary 1.6 by assuming that ψ satisfies Hypotheses 1 and 2. Recall that $\widetilde{\Phi}^N = \widetilde{\Phi}^N(\phi)$ is a unique solution of the SPDE (1.8):

$$\begin{cases} \partial_t \widetilde{\Phi}^N_t = \dfrac{1}{2}(\triangle - 1)\widetilde{\Phi}^N_t - \dfrac{\alpha}{2} P_N \exp\left(\alpha P_N \widetilde{\Phi}^N_t - \dfrac{\alpha^2}{2} C_N\right) + \dot{W}_t, & t > 0, \\ \widetilde{\Phi}^N_0 = \phi \in \mathcal{D}'(\Lambda), \end{cases}$$

and $\Phi = \Phi(\phi)$ is the strong solution obtained by Theorem 1.1. Since the nonlinear term of (1.8) is given by the log-derivative of the approximating measure $\mu^{(\alpha)}_N$ defined by (1.7), it is easy to show that $\mu^{(\alpha)}_N$ is an invariant measure of the process $\widetilde{\Phi}^N$ (see [33, Sect. 4] for details). Therefore, if ξ_N is a random variable with the law $\mu^{(\alpha)}_N$ and independent of W, then

$$\widetilde{\Phi}^{N,\mathrm{stat}} := \widetilde{\Phi}^N(\xi_N)$$

is a stationary process. Let ξ be a $\mathcal{D}'(\Lambda)$-valued random variable with the law $\mu^{(\alpha)}$ and independent of W, and define

$$\Phi^{\mathrm{stat}} := \Phi(\xi).$$

The proof of Theorem 1.5 consists of showing the following two facts:

(i) $\{\widetilde{\Phi}^{N,\mathrm{stat}}\}_{N\in\mathbb{N}}$ is tight in the space $C([0, T]; H^{-\varepsilon})$ for any $\varepsilon > 0$.
(ii) $\widetilde{\Phi}^{N,\mathrm{stat}}$ converges in law to Φ^{stat} in the space $C([0, T]; B^{-\varepsilon}_{p,p})$ for any $\varepsilon > 0$.

Once they are proved, Theorem 1.5 is obtained as follows: (i) implies that there exists a subsequence $\{\widetilde{\Phi}^{N_k,\mathrm{stat}}\}_{k\in\mathbb{N}}$ converging in law to a stochastic process Ψ in the space $C([0, T]; H^{-\varepsilon})$. On the other hand, $\{\widetilde{\Phi}^{N_k,\mathrm{stat}}\}_{k\in\mathbb{N}}$ converges to Φ^{stat} in $C([0, T]; B^{-\varepsilon}_{p,p})$

by (ii). Since $C([0,T]; H^{-\varepsilon})$ is continuously embedded into $C([0,T]; B_{p,p}^{-\varepsilon'})$ for any $\varepsilon' > \varepsilon$, the laws of Ψ and Φ^{stat} in $C([0,T]; B_{p,p}^{-\varepsilon'})$ coincide. Since $H^{-\varepsilon}$ and $B_{p,p}^{-\varepsilon'}$ are separable, by Lusin–Souslin's theorem (cf. [36, Theorem 15.1]), $C([0,T]; H^{-\varepsilon})$ is a measurable subset of $C([0,T]; B_{p,p}^{-\varepsilon'})$. Therefore,

$$\mathbb{P}\left(\Phi^{\mathrm{stat}} \in C([0,T]; H^{-\varepsilon})\right) = \mathbb{P}\left(\Psi \in C([0,T]; H^{-\varepsilon})\right) = 1,$$

and hence $\Psi \stackrel{d}{=} \Phi^{\mathrm{stat}}$ in $C([0,T]; H^{-\varepsilon})$. This implies that the accumulation point of the laws of $\{\widetilde{\Phi}^{N,\mathrm{stat}}\}_{N\in\mathbb{N}}$ in $C([0,T]; H^{-\varepsilon})$ is unique, therefore $\widetilde{\Phi}^{N,\mathrm{stat}}$ converges in law to Φ^{stat} in the space $C([0,T]; H^{-\varepsilon})$. For any bounded continuous function f on $H^{-\varepsilon}$, by Corollary 2.3,

$$\begin{aligned}\mathbb{E}\big[f(\Phi_t^{\mathrm{stat}})\big] &= \lim_{k\to\infty} \mathbb{E}\big[f(\widetilde{\Phi}_t^{N_k,\mathrm{stat}})\big]\\ &= \lim_{k\to\infty} \int_{H^{-\varepsilon}} f(\phi)\mu_{N_k}^{(\alpha)}(\mathrm{d}\phi) = \int_{H^{-\varepsilon}} f(\phi)\mu^{(\alpha)}(\mathrm{d}\phi)\end{aligned}$$

for any $t \geq 0$. This means that Φ_t^{stat} has a law $\mu^{(\alpha)}$ for any $t > 0$.

Corollary 1.6 is obtained as follows. Since

$$\int_{\mathcal{D}'(\Lambda)} \mathbb{P}\left(\Phi(\phi) \in C([0,T]; H^{-\varepsilon})\right)\mu^{(\alpha)}(\mathrm{d}\phi) = \mathbb{P}\left(\Phi^{\mathrm{stat}} \in C([0,T]; H^{-\varepsilon})\right) = 1,$$

we have

$$\mathbb{P}\left(\Phi(\phi) \in C([0,T]; H^{-\varepsilon})\right) = 1$$

for $\mu^{(\alpha)}$-almost every $\phi \in \mathcal{D}'(\Lambda)$. Since $\mu^{(\alpha)}$ and μ_0 are absolutely continuous with respect to each other (Corollary 2.3), "$\mu^{(\alpha)}$-almost every ϕ" can be replaced by "μ_0-almost every ϕ".

We now turn to proofs of (i) and (ii). The proofs go in very similar ways to Hoshino et al. [33, Sect. 4].

Proof of (i) By the definition (3.1) of the OU process X, we can decompose $\widetilde{\Phi}^{N,\mathrm{stat}} = X(\xi_N) + \mathbf{Y}^N$, where $\mathbf{Y}^N$ solves

$$\begin{cases}\partial_t \mathbf{Y}_t^N = \dfrac{1}{2}(\triangle - 1)\mathbf{Y}_t^N - \dfrac{\alpha}{2} P_N \exp_N^{\diamond}(\alpha \widetilde{\Phi}_t^{N,\mathrm{stat}}),\\ \mathbf{Y}_0^N = 0.\end{cases}$$

For $X(\xi_N)$, it is easy to check that

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\|X_0(\xi_N)\|_{H^{-\varepsilon}}\right] + \sup_{N\in\mathbb{N}} \mathbb{E}\left[\sup_{s,t\in[0,T]} \frac{\|X_t(\xi_N) - X_s(\xi_N)\|_{H^{-\varepsilon}}}{|t-s|^{\delta}}\right] < \infty$$

for sufficiently small $\delta, \varepsilon > 0$, by the a priori estimate of the OU process (Proposition 3.1) and the uniform bound of Radon–Nikodym derivatives $\frac{d\mu_N^{(\alpha)}}{d\mu_0}$ (Corollary 2.3). For $\mathbf{Y}^N$, by the Schauder estimate (Proposition 4.7), the invariance of $\mu_N^{(\alpha)}$ under $\widetilde{\Phi}^N$, and Corollary 2.4, for any small $\delta > 0$ we have

$$
\begin{aligned}
\mathbb{E}\big[\|\mathbf{Y}^N\|^2_{C^\delta([0,T];L^2)}\big] &\lesssim \mathbb{E}\Big[\Big\|P_N\left\{\exp_N^\diamond(\alpha\widetilde{\Phi}^{N,\mathrm{stat}})\right\}\Big\|^2_{L^2([0,T];H^{-s})}\Big] \\
&\leq \mathbb{E}\Big[\Big\|\exp_N^\diamond(\alpha\widetilde{\Phi}^{N,\mathrm{stat}})\Big\|^2_{L^2([0,T];H^{-s})}\Big] \\
&= \int_{\mathcal{D}'(\Lambda)}\mu_N^{(\alpha)}(d\phi)\int_0^T \mathbb{E}\Big[\Big\|\exp_N^\diamond(\alpha\widetilde{\Phi}_t^N(\phi))\Big\|^2_{H^{-s}}\Big]dt \\
&= T\int_{\mathcal{D}'(\Lambda)}\|\exp_N^\diamond(\alpha\phi)\|^2_{H^{-s}}\mu_N^{(\alpha)}(d\phi) \lesssim 1.
\end{aligned}
$$

Then by a similar argument to [33, Theorem 4.2], we have the tightness of $\{\widetilde{\Phi}^{N,\mathrm{stat}}\}_{N\in\mathbb{N}}$ in $C([0,T];H^{-\varepsilon})$. □

Proof of (ii) By a similar argument to the proof of [33, Theorem 1.3], we can assume that ξ_N converges to ξ in $H^{-\varepsilon}$ almost surely. Then we can complete the proof of (ii) by showing that

$$
\widetilde{\Phi}^{N,\mathrm{stat}} \to \Phi^{\mathrm{stat}}
$$

in $C([0,T];B^{-\varepsilon}_{p,p})$, in probability. To do this, we decompose $\widetilde{\Phi}^{N,\mathrm{stat}} = X(\xi_N) + \mathbf{Y}^N$, as in the proof of (i), and decompose $\Phi^{\mathrm{stat}} = X(\xi) + \mathbf{Y}$, where $\mathbf{Y} = \mathcal{S}(0, \mathcal{X}^\infty(\xi))$. Since

$$
\begin{aligned}
X(\xi_N) &\to X(\xi), &&\text{in } C([0,T];H^{-\varepsilon}) \quad \text{almsot surely}, \\
\mathcal{X}^N(\xi_N) &\to \mathcal{X}^\infty(\xi), &&\text{in } L^p([0,T];B^{-\beta}_{p,p}) \quad \text{in probability},
\end{aligned}
$$

by (3.3) of Proposition 3.1 and Lemma 3.4, we consider the solution $\Upsilon^N = \mathcal{S}_N(0, \mathcal{X}^N)$ of the deterministic initial value problem

$$
\begin{cases}
\partial_t\Upsilon_t^N = \dfrac{1}{2}(\triangle - 1)\Upsilon_t^N - \dfrac{\alpha}{2}P_N\left(e^{\alpha P_N\Upsilon_t^N}\mathcal{X}_t^N\right), \\
\Upsilon_0^N = 0
\end{cases}
$$

for any nonnegative functions $\{\mathcal{X}^N\}_{N\in\mathbb{N}} \subset C([0,T]\times\Lambda)$. Then, the proof completes, once we show that; if

$$
\mathcal{X}^N \to \mathcal{X} \quad \text{in} \quad L^p([0,T];B^{-\beta}_{p,p}),
$$

then

$$
\mathcal{S}_N(0,\mathcal{X}^N) \to \mathcal{S}(0,\mathcal{X}) \quad \text{in} \quad C([0,T];B^\delta_{p,p}).
$$

This is obtained by a similar way to Lemma 4.10. Indeed, the a priori estimate (4.7) holds for Υ^N uniformly over N, since $\{P_N\}$ are nonnegative and uniformly bounded as operators on $B_{p,p}^{-\beta}$, in view of Hypothesis 2. If $\{\Upsilon^{N_k}\}_{k\in\mathbb{N}}$ is a convergent subsequence, then the limit Υ solves (4.5) as a consequence of the continuity of P_N as $N \to \infty$, which is assumed by Hypothesis 2. □

6 Relation with Dirichlet form theory

In this section, we prove Theorem 1.7. Although the proof goes in a very similar way to one in [33], we provide a sketch of the proof for readers' convenience.

We fix the parameter $s \in (0,1)$ appearing in Corollary 2.4 and set $D = \mathrm{Span}\{e_k; k \in \mathbb{Z}^2\}$, $H = L^2$ and $E = H^{-s}$. In what follows, $\langle\cdot,\cdot\rangle$ stands for the pairing of E and its dual space $E^* = H^s$. By Corollary 2.4, the map $\phi \mapsto \exp^\diamond(\alpha\phi)$ can be regarded as a $\mathcal{B}(E)/\mathcal{B}(E)$-measurable map. Let $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ be the pre-Dirichlet form defined by (1.9), that is,

$$\mathcal{E}(F,G) = \frac{1}{2}\int_E \big(D_H F(\phi), D_H G(\phi)\big)_H \mu^{(\alpha)}(\mathrm{d}\phi), \qquad F, G \in \mathfrak{F}C_b^\infty.$$

Then we obtain the following:

Proposition 6.1 *It holds that*

$$\mathcal{E}(F,G) = -\int_E \mathcal{L}F(\phi)G(\phi)\mu^{(\alpha)}(\mathrm{d}\phi), \qquad F, G \in \mathfrak{F}C_b^\infty, \tag{6.1}$$

where $\mathcal{L}F \in L^2(\mu^{(\alpha)})$ is given by

$$\begin{aligned}\mathcal{L}F(\phi) &= \frac{1}{2}\sum_{i,j=1}^n \partial_i\partial_j f(\langle\phi,l_1\rangle,\dots,\langle\phi,l_n\rangle)\langle l_i,l_j\rangle \\ &\quad - \frac{1}{2}\sum_{j=1}^n \partial_j f(\langle\phi,l_1\rangle,\dots,\langle\phi,l_n\rangle)\cdot\Big\{\big\langle(1-\triangle)\phi,l_j\big\rangle + \alpha\big\langle\exp^\diamond(\alpha\phi),l_j\big\rangle\Big\}\end{aligned}$$

for $F(\phi) = f(\langle\phi,l_1\rangle,\dots,\langle\phi,l_n\rangle)$ with $f \in C_b^\infty(\mathbb{R}^n)$, $l_1,\dots l_n \in D$.

Proof Let $\psi = \mathbf{1}_{[-1,1]^2}$, which satisfies Hypothesis 1. Applying the Gaussian integration by parts formula with respect to μ_0 (see [27, page 207]), we have

$$\begin{aligned}&\int_E \big(D_H F(\phi),h\big)_H \exp\Big(-\int_\Lambda \exp_N^\diamond(\alpha\phi)(x)\,\mathrm{d}x\Big)\mu_0(\mathrm{d}\phi) \\ &= \int_E F(\phi)\Big(\langle\phi,(1-\triangle)h\rangle - \alpha\langle\exp_N^\diamond(\alpha\phi),P_N h\rangle\Big)\exp\Big(-\int_\Lambda \exp_N^\diamond(\alpha\phi)(x)\,\mathrm{d}x\Big)\mu_0(\mathrm{d}\phi)\end{aligned}$$

for all $F \in \mathfrak{F}C_b^\infty$, $h \in D$ and $N \in \mathbb{N}$.

Now we recall Theorem 2.1, Corollary 2.4 and $\lim_{N\to\infty} Z_N^{(\alpha)} = Z^{(\alpha)} > 0$. Taking the limit $N \to \infty$ on both sides of the above equality, we obtain

$$\int_E \big(D_H F(\phi), h\big)_H \mu^{(\alpha)}(\mathrm{d}\phi) = \int_E F(\phi)\Big(\langle \phi, (1-\triangle)h\rangle - \alpha\langle \exp^{\diamond}(\alpha\phi), h\rangle\Big)\mu^{(\alpha)}(\mathrm{d}\phi)$$

and this leads us to the desired integration by parts formula (6.1). Besides, applying Corollary 2.4 again, we obtain $\mathcal{L}F \in L^2(\mu^{(\alpha)})$. This completes the proof. □

Proposition 6.1 implies that $(\mathcal{E}, \mathfrak{F}C_b^{\infty})$ is closable on $L^2(\mu^{(\alpha)})$. We denote the closure by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. As mentioned in Sect. 1.2, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(\mu^{(\alpha)})$, and thus we obtain an E-valued diffusion process $\mathbb{M} = (\Theta, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, (\Psi_t)_{t\geq 0}, (\mathbb{Q}_\phi)_{\phi\in E})$ properly associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. By recalling Corollary 2.4 and applying [9, Lemma 4.2], we have

$$\mathbb{E}^{\mathbb{Q}_\phi}\Big[\int_0^T \|\exp^{\diamond}(\alpha\Psi_t)\|_E\,\mathrm{d}t\Big] < \infty, \qquad T > 0,\ \mu^{(\alpha)}\text{-a.e. } \phi.$$

In particular,

$$\mathbb{Q}_\phi\Big(\int_0^T \|\exp^{\diamond}(\alpha\Psi_t)\|_E\,\mathrm{d}t < \infty\Big) = 1, \qquad T > 0,\ \mu^{(\alpha)}\text{-a.e. } \phi.$$

Thus we are able to apply [9, Lemma 6.1 and Theorem 6.2] and [44, Theorem 13] as in [33]. As a result, there exists an H-cylindrical $(\mathcal{G}_t)$-Brownian motion $\mathcal{W} = (\mathcal{W}_t)_{t\geq 0}$ defined on $(\Theta, \mathcal{G}, \mathbb{Q}_\phi)$ such that

$$\begin{aligned}\Psi_t &= e^{t(\Delta-1)/2}\phi - \frac{\alpha}{2}\int_0^t e^{(t-s)(\Delta-1)/2}\exp^{\diamond}(\alpha\Psi_s)\,\mathrm{d}s \\ &\quad + \int_0^t e^{(t-s)(\Delta-1)/2}\,\mathrm{d}\mathcal{W}_s, \qquad t\geq 0,\ \mathbb{Q}_\phi\text{-a.s.},\ \mu^{(\alpha)}\text{-a.e. } \phi.\end{aligned}$$

Now we are going to prove Theorem 1.7. Precisely, we prove that the process Ψ coincides with the strong solution Φ driven by the cylindrical Brownian motion $\mathcal{W}$. We decompose $\Psi = \mathfrak{X}(\phi) + \mathfrak{Y}$, where

$$\begin{aligned}\mathfrak{X}(\phi)_t &:= e^{t(\Delta-1)/2}\phi + \int_0^t e^{(t-s)(\Delta-1)/2}\,\mathrm{d}\mathcal{W}_s, \\ \mathfrak{Y}_t &:= -\frac{\alpha}{2}\int_0^t e^{(t-s)(\Delta-1)/2}\exp^{\diamond}(\alpha\Psi_s)\,\mathrm{d}s, \qquad t\geq 0.\end{aligned} \tag{6.2}$$

From the Da Prato–Debussche trick as used in Sect. 4, it is sufficient to show that

$$\mathbb{Q}_\phi\Big(\mathfrak{Y} = \mathcal{S}(0, \exp^{\diamond}(\alpha\mathfrak{X}(\phi)))\Big) = 1, \qquad \mu^{(\alpha)}\text{- a.e. } \phi.$$

We prepare the following lemma.

Lemma 6.2 *Assume that the mollifier ψ satisfies Hypothesis 1. Let E_0 be the set of all $\phi \in E$ such that the convergence*

$$\exp^\diamond(\alpha\phi) = \lim_{N\to\infty} \exp_N^\diamond(\alpha\phi)$$

holds in $B_{p,p}^{-\beta}$. Then, for any $f \in H^{1+\varepsilon}$ and $\phi \in E_0$ such that $f + \phi \in E_0$, one has

$$\exp^\diamond(\alpha(f+\phi)) = \exp(\alpha f)\exp^\diamond(\alpha\phi).$$

Proof Let $f \in H^{1+\varepsilon}$ and $\phi \in E_0$. $P_N f$ converges to f in $H^{1+\varepsilon}$ by Hypothesis 1. Since $H^{1+\varepsilon} \subset C(\Lambda)$, $\exp(\alpha P_N f)$ converges to $\exp(\alpha f)$ in $C(\Lambda)$. Therefore, by Theorem 4.2,

$$\exp_N^\diamond(\alpha(f+\phi)) = \exp(\alpha P_N f)\exp_N^\diamond(\alpha\phi) \xrightarrow{N\to\infty} \exp(\alpha f)\exp^\diamond(\alpha\phi)$$

in $B_{p,p}^{-\beta}$. If $f + \phi \in E_0$, $\exp_N^\diamond(\alpha(f+\phi))$ converges to $\exp^\diamond(\alpha(f+\phi))$. From these convergences the assertion follows. ◻

Proof of Theorem 1.7 It is sufficient to check that $\mathfrak{Y}$ belongs to the space $\mathscr{Y}_T$ and solves the mild equation (4.5). By the invariance of $\mu^{(\alpha)}$ under Ψ and Corollary 2.4,

$$\int_E \mathbb{E}^{\mathbb{Q}_\phi}\Big[\|\exp^\diamond(\alpha\Psi)\|_{L^2([0,T];H^{-s})}^2\Big]\mu^{(\alpha)}(\mathrm{d}\phi) = \int_0^T \mathrm{d}t\int_E \|\exp^\diamond(\alpha\phi)\|_{H^{-s}}^2\mu^{(\alpha)}(\mathrm{d}\phi) < \infty.$$

In particular,

$$\mathbb{Q}_\phi\Big(\exp^\diamond(\alpha\Psi) \in L^2([0,T];H^{-s})\Big) = 1, \qquad \mu^{(\alpha)}\text{-a.e. }\phi.$$

By the Schauder estimate (Proposition 4.7),

$$\mathbb{Q}_\phi\Big(\mathfrak{Y} \in L^2([0,T];H^{1+\kappa}) \cap C([0,T];H^\kappa)\Big) = 1, \qquad \mu^{(\alpha)}\text{-a.e. }\phi \tag{6.3}$$

for small $\kappa > 0$. Since $\alpha\mathfrak{Y}$ is nonpositive, we have

$$\mathbb{Q}_\phi\big(\mathfrak{Y} \in \mathscr{Y}_T\big) = 1, \qquad \mu^{(\alpha)}\text{-a.e. }\phi.$$

Finally we show that $\mathfrak{Y}$ solves the mild equation (4.5) with $(\upsilon, \mathcal{X}) = (0, \mathfrak{X})$. By the definition (6.2) of $\mathfrak{Y}$, it is sufficient to show that

$$\mathbb{Q}_\phi\Big(\exp^\diamond(\alpha\Psi_t) = e^{\alpha\mathfrak{Y}_t}\cdot\exp^\diamond(\alpha\mathfrak{X}_t),\ \text{a.e. }t\Big) = 1, \qquad \mu_0\text{-a.e. }\phi. \tag{6.4}$$

Recall the definition of the subset E_0 in Lemma 6.2. Then $\mu_0(E_0) = 1$, so $\mu^{(\alpha)}(E_0) = 1$ by the absolute continuity (see Corollary 2.3). By using the invariance of $\mu^{(\alpha)}$ under Ψ,

$$\int_E \mathbb{E}^{\mathbb{Q}_\phi}\left[\int_0^T \mathbf{1}_{E_0^c}(\Psi_t)\,\mathrm{d}t\right]\mu^{(\alpha)}(\mathrm{d}\phi) = \int_0^T \mathrm{d}t\int_E \mathbf{1}_{E_0^c}(\phi)\mu^{(\alpha)}(\mathrm{d}\phi)$$
$$= T\mu^{(\alpha)}(E_0^c) = 0.$$

Similarly, by the invariance of μ_0 under $\mathfrak{X}$,

$$\int_E \mathbb{E}^{\mathbb{Q}_\phi}\left[\int_0^T \mathbf{1}_{E_0^c}(\mathfrak{X}_t)\,\mathrm{d}t\right]\mu_0(\mathrm{d}\phi) = \int_0^T \mathrm{d}t\int_E \mathbf{1}_{E_0^c}(\phi)\mu_0(\mathrm{d}\phi)$$
$$= T\mu_0(E_0^c) = 0.$$

From these equalities and (6.3), we have

$$\mathbb{Q}_\phi\Big(\Psi_t \in E_0,\ \mathfrak{X}_t \in E_0,\ \mathfrak{Y}_t \in H^{1+\kappa},\ \text{a.e. } t\Big) = 1$$

for μ_0-almost every ϕ. Therefore, Lemma 6.2 implies (6.4). □

Acknowledgements The authors are grateful to Professor Makoto Nakashima for helpful discussions on Gaussian multiplicative chaos, and to Professor Ryo Takada for helpful comments on the references on Besov spaces. They also thank anonymous two referees for helpful suggestions that improved the quality of the present paper. This work was partially supported by JSPS KAKENHI Grant Numbers 17K05300, 17K14204, 19K14556, 20K03639 and 21H00988.

Data Availability Statement All data generated or analysed during this study are included in this published article.

A Green functions and their approximation

In this appendix, we provide some properties of Green functions and their approximation on the whole space and the torus. In the end, we prove a proposition, which yields Proposition 2.5.

A.1 Green function on the whole plane

Recall that ψ is a function satisfying Hypothesis 1, $\psi_N = \psi(2^{-N}\cdot)$, and

$$G_{M,N}(x,y) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}^2}\frac{\psi_M(k)\psi_N(k)}{1+|k|^2}\mathbf{e}_k(x-y), \qquad M, N \in \mathbb{N}.$$

We regard $G_{M,N}$ as a periodic function on $\mathbb{R}^2 \times \mathbb{R}^2$, rather than a function on $\Lambda \times \Lambda$. Then by the Poisson summation formula, we can write it as an infinite sum of decreasing functions

$$G_{M,N}(x, y) = \sum_{k \in \mathbb{Z}^2} K_{M,N}(x - y + 2\pi k), \qquad K_{M,N} := \frac{1}{2\pi} \mathcal{F}^{-1} \left(\frac{\psi_M \psi_N}{1 + |\cdot|^2} \right).$$

Hence we need to observe the behavior of $K_{M,N}$ for our purpose. Setting $\rho_{M,N} = \frac{1}{2\pi} \mathcal{F}^{-1} (\psi_M \psi_N)$, we can write $K_{M,N}$ as a convolution

$$K_{M,N}(x) = (1 - \triangle_{\mathbb{R}^2})^{-1} \rho_{M,N}(x) = \int_{\mathbb{R}^2} K(x - y) \rho_{M,N}(y)\, \mathrm{d}y,$$

where $\triangle_{\mathbb{R}^2}$ is the Laplacian on $\mathbb{R}^2$, and K is the Green function of $1 - \triangle_{\mathbb{R}^2}$.

Proposition A.1 *The function $K : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is positive and has the estimates*

$$K(x) \begin{cases} = -\dfrac{1}{2\pi} \log |x| + O(1), & |x| < 1, \\ \lesssim e^{-|x|/2}, & |x| \geq 1. \end{cases} \tag{A.1}$$

Proof By the relation between the heat semigroup and the resolvent kernel, we have

$$K(x) = \frac{1}{4\pi} \int_0^\infty \exp\left(-t - \frac{|x|^2}{4t} \right) \frac{\mathrm{d}t}{t}$$

for $x \neq 0$. Since the integral over $(0, |x|/2)$ and $(|x|/2, \infty)$ are equal in view of the change of variables by $s = |x|^2/4t$, we have

$$K(x) = \frac{1}{2\pi} \int_{|x|/2}^\infty \exp\left(-t - \frac{|x|^2}{4t} \right) \frac{\mathrm{d}t}{t} = \frac{1}{2\pi} \int_1^\infty \exp\left(-\frac{|x|}{2} \left(t + \frac{1}{t} \right) \right) \frac{\mathrm{d}t}{t}.$$

Hence we observe the behavior of the function

$$g(r) = \int_1^\infty \exp\left(-\frac{r}{2} \left(s + \frac{1}{s} \right) \right) \frac{\mathrm{d}s}{s}, \qquad r \in (0, \infty).$$

Since the integrand is bounded by $e^{-rs/2}$ on $s \geq 1$, we have $g(r) \lesssim e^{-r/2}$ for $r \geq 1$, so the latter part of (A.1) follows. To consider the estimate on $r < 1$, we decompose

$$g(r) = \int_1^{1/r} \frac{\mathrm{d}s}{s} + \int_1^{1/r} \left\{ \exp\left(-\frac{r}{2} \left(s + \frac{1}{s} \right) \right) - 1 \right\} \frac{\mathrm{d}s}{s} + \int_{1/r}^\infty \exp\left(-\frac{r}{2} \left(s + \frac{1}{s} \right) \right) \frac{\mathrm{d}s}{s}.$$

The first term is equal to $-\log r$. The other terms are $O(1)$, since

$$\int_1^{1/r} \left| \exp\left(-\frac{r}{2} \left(s + \frac{1}{s} \right) \right) - 1 \right| \frac{\mathrm{d}s}{s} \leq \int_1^{1/r} \frac{r}{2} \left(s + \frac{1}{s} \right) \frac{\mathrm{d}s}{s} \leq \frac{1}{2},$$

and

$$\int_{1/r}^{\infty} \exp\left(-\frac{r}{2}\left(s+\frac{1}{s}\right)\right)\frac{\mathrm{d}s}{s} \le \int_{1/r}^{\infty} r\exp\left(-\frac{r}{2}s\right)\mathrm{d}s \le 2e^{-1/2}.$$

Thus we have the former part of (A.1). □

Next we consider the convolution of K and a function with sufficient decay.

Lemma A.2 [46, Lemma 4.1] *For any function ρ on $\mathbb{R}^2$ such that*

$$|\rho(x)| \le C(1+|x|)^{-2-\gamma}$$

for some $C > 0$ and $\gamma > 0$, one has

$$\sup_{|x|>1}\left|\int_{\mathbb{R}^2} |\rho(y)| \log\frac{|x|}{|x-y|}\,\mathrm{d}y\right| < \infty.$$

Lemma A.3 *Let ρ be a function on $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2}\rho(x)\,\mathrm{d}x = 1$ and*

$$|\rho(x)| \le C(1+|x|)^{-4-2\gamma} \tag{A.2}$$

for some $C > 0$ and $\gamma > 0$. Set $\rho_N = 2^{2N}\rho(2^N\cdot)$ for $N \in \mathbb{N}$. Then for any $|x| < 1$ and $N \in \mathbb{N}$,

$$K * \rho_N(x) = -\frac{1}{2\pi}\log\left(|x| \vee 2^{-N}\right) + O(1). \tag{A.3}$$

Moreover, for any $x \in \mathbb{R}^2$ and $N \in \mathbb{N}$,

$$|K * \rho_N(x)| \lesssim |x|^{-2-\gamma}. \tag{A.4}$$

Proof First we prove (A.3). By Proposition A.1, we can decompose

$$K(x) = -\frac{1}{2\pi}\log(|x| \wedge 1) + R(x),$$

where R is a bounded function with rapid decay as $|x| \to \infty$. Since $R*\rho_N$ is bounded, it is sufficient to show that

$$(\rho_N * \log(|\cdot| \wedge 1))(x) = \log\left(|x| \vee 2^{-N}\right) + O(1).$$

We decompose

$$(\rho_N * \log(|\cdot| \wedge 1))(x) = \int_{\mathbb{R}^2} \rho_N(y)\log|x-y|\,\mathrm{d}y - \int_{|x-y|>1} \rho_N(y)\log|x-y|\,\mathrm{d}y. \tag{A.5}$$

Since $|x| < 1$, the second term of the right-hand side is bounded. Indeed, since $1 < |x-y| \le 1+|y|$,

$$\begin{aligned}\int_{|x-y|>1} \rho_N(y) \log|x-y|\,\mathrm{d}y &\le \int_{\mathbb{R}^2} |\rho_N(y)| \log(1+|y|)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} |\rho(z)| \log(1+2^{-N}|z|)\,\mathrm{d}z \\ &\le \int_{\mathbb{R}^2} |\rho(z)| \log(1+|z|)\,\mathrm{d}z < \infty.\end{aligned}$$

Consider the first term of the right-hand side of (A.5). When $|x| > 2^{-N}$, by Lemma A.2,

$$\begin{aligned}\int_{\mathbb{R}^2} \rho_N(y) \log|x-y|\,\mathrm{d}y &= \log 2^{-N} + \int_{\mathbb{R}^2} \rho(y) \log|2^N x - y|\,\mathrm{d}y \\ &= \log 2^{-N} + \log|2^N x| + O(1) = \log|x| + O(1).\end{aligned}$$

When $|x| \le 2^{-N}$, by the calculation

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} \rho(y) \log|2^N x - y|\,\mathrm{d}y\right| \\ &\le \left|\int_{|2^N x - y|<1} \rho(y) \log|2^N x - y|\,\mathrm{d}y\right| + \left|\int_{|2^N x - y|\ge 1} \rho(y) \log|2^N x - y|\,\mathrm{d}y\right| \\ &\le \left(\int_{\mathbb{R}^2} \rho^2(y)\,\mathrm{d}y\right)^{1/2} \left(\int_{|y|<1} (\log|y|)^2\,\mathrm{d}y\right)^{1/2} + \int_{\mathbb{R}^2} \rho(y)\log(1+|y|)\,\mathrm{d}y < \infty,\end{aligned}$$

we have

$$\int_{\mathbb{R}^2} \rho_N(y) \log|x-y|\,\mathrm{d}y = \log 2^{-N} + O(1).$$

Thus, we have (A.3).

Next we prove (A.4). By Proposition A.1, $K \in L^p(\mathbb{R}^2)$ for any $p \in [1,\infty)$ and

$$\sup_{x\in\mathbb{R}^2} |x|^{2+\gamma} K(x) < \infty.$$

Hence we have

$$\begin{aligned}&|x|^{2+\gamma}|K * \rho_N(x)| \\ &\lesssim \int_{\mathbb{R}^2} |y|^{2+\gamma} K(y)|\rho_N(x-y)|\,\mathrm{d}y + \int_{\mathbb{R}^2} |x-y|^{2+\gamma} K(y)|\rho_N(x-y)|\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} |y|^{2+\gamma} K(y)|\rho_N(x-y)|\,\mathrm{d}y + \int_{\mathbb{R}^2} K(x-y)|y|^{2+\gamma}|\rho_N(y)|\,\mathrm{d}y\end{aligned}$$

$$\lesssim \int_{\mathbb{R}^2} |\rho_N(y)|\,\mathrm{d}y + \left(\int_{\mathbb{R}^2} \left(|y|^{2+\gamma}|\rho_N(y)|\right)^q \,\mathrm{d}y\right)^{1/q}$$

for any $q \in (1, \infty)$. By the condition (A.2),

$$\int_{\mathbb{R}^2} \left(|y|^{2+\gamma}|\rho_N(y)|\right)^q \,\mathrm{d}y = 2^{-N(2+\gamma q)} \int_{\mathbb{R}^2} \left(|y|^{2+\gamma}|\rho(y)|\right)^q \,\mathrm{d}y$$
$$\lesssim \int_{\mathbb{R}^2} (1+|y|)^{-(2+\gamma)q}\,\mathrm{d}y < \infty.$$

Thus we have (A.4). □

A.2 Green function on the torus

We return to the proof of Proposition 2.5.

Lemma A.4 *Let ψ be a function satisfying Hypothesis 1. Then there exists a smooth function $\bar{\psi}$ with the following properties:*

- *$\bar{\psi}$ satisfies Hypothesis 1.*
- *$0 \le \psi \le \bar{\psi}$ on $\mathbb{R}^2$.*
- *For any $k \in \mathbb{N}^2$ there exists a constant C_k such that*

$$|\partial^k \bar{\psi}(x)| \le C_k (1+|x|)^{-2-\kappa-|k|_1} \tag{A.6}$$

for any $x \in \mathbb{R}^2$, where κ is a constant as in Hypothesis 1(ii) *and $|k|_1 := |k_1| + |k_2|$ for each $k = (k_1, k_2) \in \mathbb{N}^2$.*

Proof By Hypothesis 1(ii),

$$|\psi(x)| \le C(1+|x|)^{-2-\kappa}$$

for some constants $C, \kappa > 0$. Then, there exists a radial smooth function $\bar{\psi}$ such that $0 \le \psi \le \bar{\psi}$ on $\mathbb{R}^2$, $\bar{\psi}(x) = 1$ on $x \in B(0, r)$ for some $r > 0$, and

$$\bar{\psi}(x) = C(1+|x|)^{-2-\kappa}$$

on $x \in B(0, R)^c$ for some $R > r$. Obviously, $\bar{\psi}$ satisfies all the required properties. □

Now we prove the following proposition, which yields Proposition 2.5. The estimate (A.8) in the following proposition is better than (2.6), because (A.8) is L^p-estimate for all $p \in [1, \infty)$.

Proposition A.5 *Assume that ψ satisfies Hypothesis 1. Then for any $x, y \in \mathbb{R}^2$ with $|x - y| < 1$ and any $M, N \in \mathbb{N}$,*

$$G_{M,N}(x, y) = -\frac{1}{2\pi} \log\left(|x-y| \vee 2^{-M} \vee 2^{-N}\right) + R_{M,N}(x, y), \tag{A.7}$$

where the remainder term $R_{M,N}(x, y)$ is uniformly bounded over $x, y, M,$ and N. Moreover, for any $p \in [1, \infty)$, there exist constants $C > 0$ and $\theta > 0$ such that, for any $M, N \in \mathbb{N}$,

$$\iint_{\Lambda\times\Lambda} |G_{M,N+1}(x, y) - G_{M,N}(x, y)|^p \, \mathrm{d}x\mathrm{d}y \le C2^{-\theta N}. \tag{A.8}$$

Proof First, we prove (A.7). Let $M \le N$ without loss of generality. First we assume that ψ satisfies (A.6) in addition to Hypothesis 1. By (A.6), the function $\rho_0 = \frac{1}{2\pi}\mathcal{F}^{-1}\psi$ satisfies that for all $n \in \mathbb{N}$

$$\begin{aligned}(1 + |x|^2)^n |\rho_0(x)| &= \frac{1}{2\pi}\left|\mathcal{F}^{-1}\left\{(1-\triangle)^n\psi\right\}(x)\right| \\ &\le \frac{1}{2\pi}\int_{\mathbb{R}^2}\left|(1-\triangle)^n\psi(\xi)\right| \mathrm{d}\xi < \infty.\end{aligned}$$

Recall that $\rho_{M,N} = \frac{1}{2\pi}\mathcal{F}^{-1}(\psi_M\psi_N)$. Let $\rho_N = \frac{1}{2\pi}\mathcal{F}^{-1}\psi_N$. Since $\rho_N = 2^{2N}\rho_0(2^N\cdot)$, we have

$$\rho_{M,N} = \rho_M * \rho_N = 2^{2M}(\rho_0 * \rho_{N-M})(2^M\cdot).$$

Let $\widetilde{\rho}_{M,N} = \rho_0 * \rho_{N-M}$. Then from the above estimate of ρ_0, we have the uniform estimates

$$|\widetilde{\rho}_{M,N}(x)| \lesssim (1 + |x|)^{-6}.$$

Indeed, for $|x| < 1$, since $\widetilde{\rho}_{M,N}$ is uniformly bounded, this estimate is obvious. For $|x| \ge 1$,

$$\begin{aligned}&|x|^6|\widetilde{\rho}_{M,N}(x)| \\ &\quad\lesssim \int_{\mathbb{R}^2}|y|^6|\rho_0(y)||\rho_{N-M}(x-y)|\,\mathrm{d}y + \int_{\mathbb{R}^2}|x-y|^6|\rho_0(y)||\rho_{N-M}(x-y)|\,\mathrm{d}y \\ &\quad\lesssim \int_{\mathbb{R}^2}|\rho_{N-M}(y)|\,\mathrm{d}y + \int_{\mathbb{R}^2}|y|^6|\rho_{N-M}(y)|\,\mathrm{d}y \\ &\quad\lesssim \int_{\mathbb{R}^2}\Big(|\rho_0(y)| + |y|^6|\rho_0(y)|\Big)\,\mathrm{d}y < \infty.\end{aligned}$$

Since $\int_{\mathbb{R}^2}\widetilde{\rho}_{M,N}(x)\,\mathrm{d}x = 1$, $\widetilde{\rho}_{M,N}$ satisfies the conditions of Lemma A.3 with $\gamma = 1$. Therefore, the estimates (A.3) and (A.4) yield

$$G_{M,N}(x, y) = \sum_{k\in\mathbb{Z}^2}(K * \rho_{M,N})(x - y + 2\pi k) = -\frac{1}{2\pi}\log\Big(|x-y| \vee 2^{-M}\Big) + O(1)$$

for $|x - y| < 1$.

Next let ψ be an arbitrary function satisfying Hypothesis 1. Let $\bar{\psi}$ be a smooth function in Lemma A.4, and define the function $\bar{G}_{M,N}$ similarly to $G_{M,N}$ with replacement ψ by $\bar{\psi}$. As shown above, $\bar{G}_{M,N}$ satisfies the estimate

$$\bar{G}_{M,N}(x,y) = -\frac{1}{2\pi}\log\left(|x-y| \vee 2^{-M}\right) + O(1).$$

Since $0 \le \psi \le \bar{\psi}$,

$$\begin{aligned}|G_{M,N}(x,y) - \bar{G}_{M,N}(x,y)| \le \frac{1}{4\pi^2}&\sum_{k\in\mathbb{Z}^2;\,|k|\le 2^M}\frac{\bar{\psi}_M(k)\bar{\psi}_N(k)-\psi_M(k)\psi_N(k)}{1+|k|^2}\\ &+\frac{1}{4\pi^2}\sum_{k\in\mathbb{Z}^2;\,|k|>2^M}\frac{\bar{\psi}_M(k)\bar{\psi}_N(k)-\psi_M(k)\psi_N(k)}{1+|k|^2}.\end{aligned}\tag{A.9}$$

Hypothesis 1(iii) and the property of $\bar{\psi}$ imply that for sufficiently small $\zeta > 0$,

$$|\psi(x)-1| + |\bar{\psi}(x)-1| \le C|x|^{\zeta}, \qquad x \in \mathbb{R}^2,$$

with a positive constant C. Hence, by the boundedness of ψ and $\bar{\psi}$,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^2;\,|k|\le 2^M}\frac{\bar{\psi}_M(k)\bar{\psi}_N(k)-\psi_M(k)\psi_N(k)}{1+|k|^2}\\ &\lesssim \sum_{k\in\mathbb{Z}^2;\,|k|\le 2^M}\frac{|\bar{\psi}(2^{-M}k)\bar{\psi}(2^{-N}k)-1| + |\psi(2^{-M}k)\psi(2^{-N}k)-1|}{1+|k|^2}\\ &\lesssim \sum_{k\in\mathbb{Z}^2;\,|k|\le 2^M}\frac{2^{-M\zeta}|k|^{\zeta}}{1+|k|^2}\\ &\lesssim 1.\end{aligned}$$

Besides, since $0 \le \psi \le \bar{\psi}$ and $\bar{\psi}(x) \lesssim (1+|x|)^{-2-\kappa}$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2;\,|k|>2^M}\frac{\bar{\psi}_M(k)\bar{\psi}_N(k)-\psi_M(k)\psi_N(k)}{1+|k|^2} &\le \sum_{k\in\mathbb{Z}^2;\,|k|>2^M}\frac{\bar{\psi}_M(k)}{1+|k|^2}\\ &\lesssim \sum_{k\in\mathbb{Z}^2;\,|k|>2^M}\frac{1}{(1+|k|^2)(1+|2^{-M}k|)^{2+\kappa}}\\ &\lesssim \sum_{k\in\mathbb{Z}^2;\,|k|>2^M}\frac{2^{(2+\kappa)M}}{(1+|k|)^{4+\kappa}}\\ &\lesssim 2^{(2+\kappa)M}\int_{|x|>2^M}\frac{1}{|x|^{4+\kappa}}\,\mathrm{d}x\\ &\lesssim 1.\end{aligned}$$

These inequalities and (A.9) yield the estimate (A.7) for $G_{M,N}$.

Finally, we prove (A.8). Let $p \in [1,\infty)$. In view of the shift invariance of $G_{M,N}(\cdot,\cdot)$ and the compactness of Λ, it is sufficient to show

$$\int_\Lambda \left|G_{M,N+1}(x,0) - G_{M,N}(x,0)\right|^p \,\mathrm{d}x \lesssim 2^{-\theta N} \tag{A.10}$$

for some $\theta > 0$. Hypothesis 1(iii) implies that for sufficiently small $\zeta > 0$,

$$|\psi(x) - 1| \le C|x|^\zeta, \qquad x \in \mathbb{R}^2,$$

with another positive constant C. Then, by Plancherel's formula we have

$$\begin{aligned}
\left\|G_{M,N+1}(\cdot,0) - G_{M,N}(\cdot,0)\right\|_{H^{1-\zeta}}^2 &\lesssim \sum_{k\in\mathbb{Z}^2} \frac{|\psi_{N+1}(k) - \psi_N(k)|^2}{(1+|k|^2)^{2(1-\zeta)}} \\
&\lesssim \sum_{k\in\mathbb{Z}^2} \frac{|\psi(2^{-N-1}k) - 1|^2 + |\psi(2^{-N}k) - 1|^2}{(1+|k|^2)^{2(1-\zeta)}} \\
&\lesssim 2^{-2N\zeta} \sum_{k\in\mathbb{Z}^2} \frac{|k|^{2\zeta}}{(1+|k|^2)^{2(1-\zeta)}} \\
&\lesssim 2^{-2N\zeta} \sum_{k\in\mathbb{Z}^2} \frac{1}{(1+|k|^2)^{2-3\zeta}} \\
&\lesssim 2^{-2N\zeta}
\end{aligned}$$

for sufficiently small ζ. Since the Sobolev embedding theorem implies $H^{1-\zeta} \subset L^p$ for $\zeta \le 2/p$, by talking ζ sufficiently small we have the estimate (A.10). □

A.3 Approximations by averaging

We introduce a class of approximations of the Gaussian free field, which contains the circle average (see e.g. [12, 13, 22]), and show that the associated kernels also satisfy (2.5) and (2.6) in Proposition 2.5. This implies that our construction of Wick exponentials of the Gaussian free field in Sect. 2 includes the circle averaging approximation.

Let $\mathbb{X}$ be the Gaussian free field on $\Lambda = \mathbb{T}^2$ as defined in Sect. 2.1, and extend $\mathbb{X}$ on $\mathbb{R}^2$ periodically. Let ν be a probability measure on $\mathbb{R}^2$ supported in the unit ball $B(0,1)$ such that

$$\sup_{|x|\le 2} \int_{\mathbb{R}^2} |\log(x-y)|\nu(\mathrm{d}y) < \infty. \tag{A.11}$$

For $N \in \mathbb{N}$ denote by ν_N the measure given by $\nu_N(A) = \nu(2^N A)$ for a Borel set A. Define the approximation $\mathbb{X}_N$ of $\mathbb{X}$ by

$$\mathbb{X}_N(x) := \mathbb{X} * \nu_N(x) := \int_{\mathbb{R}^2} \mathbb{X}(x-y)\nu_N(\mathrm{d}y).$$

Then the random field $\mathbb{X}_N$ has the covariance function

$$G_{M,N}(x,y) := \mathbb{E}[\mathbb{X}_M(x)\mathbb{X}_N(y)] = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}^2} \frac{(\mathcal{F}\nu_M)(k)(\mathcal{F}\nu_N)(k)}{1+|k|^2} \mathbf{e}_k(x-y)$$

for $M, N \in \mathbb{N}$, where

$$(\mathcal{F}\mu)(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-\sqrt{-1}\xi\cdot x} \mu(\mathrm{d}x), \quad \xi \in \mathbb{R}^2$$

for a probability measure μ.

Proposition A.6 *The sequence $G_{M,N}(x,y)$ defined as above satisfies (2.5) and (2.6) in Proposition 2.5.*

Proof The estimate (2.5) is obtained in [12, Lemma 3.5]. We show the estimate (2.6). It is easy to see

$$\begin{aligned} (\mathcal{F}\nu_N)(k) &= (\mathcal{F}\nu)(2^{-N}k), \quad k \in \mathbb{Z}^2, \\ |(\mathcal{F}\nu)(\xi)| &\le \frac{1}{2\pi} \int_{\mathbb{R}^2} \nu(\mathrm{d}y), \quad \xi \in \mathbb{R}^2, \\ |(\mathcal{F}\nu)(\xi_1) - (\mathcal{F}\nu)(\xi_2)| &\le \frac{1}{2\pi} \int |e^{-\sqrt{-1}\xi_1\cdot y} - e^{-\sqrt{-1}\xi_2\cdot y}|\nu(\mathrm{d}y) \\ &\lesssim |\xi_1 - \xi_2| \int_{|y|\le 1} |y|\nu(\mathrm{d}y), \quad \xi_1, \xi_2 \in \mathbb{R}^2. \end{aligned}$$

From these inequalities it follows that $\mathcal{F}\nu$ is bounded and ζ-Hölder continuous for any $\zeta \in (0,1]$. Hence, the estimate (2.6) is obtained in the same way as the proof of (A.8) in Proposition A.5. ◻

References

1. Albeverio, S., De Vecchi, F.C., Gubinelli, M.: The elliptic stochastic quantization of some two dimensional Euclidean QFTs. Ann. Inst. H. Poincaré Probab. Stat. **57**, 2372–2414 (2021)
2. Albeverio, S., Høegh-Krohn, R.: Uniqueness of the physical vacuum and the Wightman functions in the infinite volume limit for some non-polynomial interactions. Commun. Math. Phys. **30**, 171–200 (1973)
3. Albeverio, S., Høegh-Krohn, R.: The Wightman axioms and the mass gap for strong interactions of exponential type in two-dimensional space-time. J. Funct. Anal. **16**, 39–82 (1974)
4. Albeverio, S., Høegh-Krohn, R.: Uniqueness of the global Markov property for Euclidean fields. The case of trigonometric interactions. Commun. Math. Phys. **68**, 95–128 (1979)

5. Albeverio, S., Kawabi, H., Mihalache, S.-R., Röckner, M.: Strong uniqueness for Dirichlet operators related to stochastic quantization under exponential/trigonometric interactions on the two-dimensional torus. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) (to appear). https://doi.org/10.2422/2036-2145.202105_106. Original version: arXiv:2004.12383
6. Albeverio, S., Kusuoka, S.: The invariant measure and the flow associated to the Φ^4_3-quantum field model. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **20**, 1359–1427 (2020)
7. Albeverio, S., Ma, Z.-M., Röckner, M.: Quasi regular Dirichlet forms and the stochastic quantization problem. In: Festschrift Masatoshi Fukushima, Interdiscip. Math. Sci. World Scientific Publishing, Hackensack, vol. 17, pp. 27–58 (2015)
8. Albeverio, S., Röckner, M.: Classical Dirichlet forms on topological vector spaces—closability and a Cameron-Martin formula. J. Funct. Anal. **88**, 395–436 (1990)
9. Albeverio, S., Röckner, M.: Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. Probab. Theory Relat. Fields **89**, 347–386 (1991)
10. Bahouri, H., Chemin, J.-Y., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343. Springer, Heidelberg (2011)
11. Benfatto, G., Gallavotti, G., Nicoló, F.: On the massive sine-Gordon equation in the first few regions of collapse. Commun. Math. Phys. **83**, 387–410 (1982)
12. Berestycki, N.: An elementary approach to Gaussian multiplicative chaos. Electron. Commun. Probab. **22**(27), 1–12 (2017)
13. Berestycki, N., Powell, E.: Gaussian free field, Liouville quantum gravity and Gaussian multiplicative chaos, Lecture Note (2021). Available at the author's webpage (https://homepage.univie.ac.at/nathanael.berestycki/Articles/master.pdf)
14. Biskup, M.: Extrema of the two-dimensional discrete Gaussian free field. In: Random Graphs, Phase Transitions, and the Gaussian Free Field. Springer Proc. Math. Stat. Springer, vol. 304, pp. 163–407 (2020)
15. Chandra, A., Hairer, M., Shen, H.: The dynamical sine-Gordon model in the full subcritical regime. Preprint (2018). arXiv:1808.02594
16. Chen, Z.-Q., Fukushima, M.: Symmetric Markov Processes, Time change, and Boundary Theory, London Mathematical Society Monographs Series, vol. 35. Princeton University Press, Princeton (2012)
17. Da Prato, G., Debussche, A.: Strong solutions to the stochastic quantization equations. Ann. Probab. **31**, 1900–1916 (2003)
18. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (1992)
19. Da Prato, G., Zabczyk, J.: Ergodicity for infinite-dimensional systems, London Mathematical Society Lecture Note Series, vol. 229. Cambridge University Press, Cambridge (1996)
20. Dimock, J., Hurd, T.R.: Construction of the two-dimensional sine-Gordon model for $\beta < 8\pi$. Commun. Math. Phys. **156**, 547–580 (1993)
21. Dubédat, J., Shen, H.: Stochastic Ricci flow on compact surfaces. Int. Math. Res. Not. (to appear). https://doi.org/10.1093/imrn/rnab015. Original version: arXiv:1904.10909
22. Duplantier, B., Sheffield, S.: Liouville quantum gravity and KPZ. Invent. Math. **185**, 333–393 (2011)
23. Fröhlich, J.: Classical and quantum statistical mechanics in one and two dimensions: two-component Yukawa- and Coulomb systems. Commun. Math. Phys. **47**, 233–268 (1976)
24. Fröhlich, J., Park, Y.M.: Remarks on exponential interactions and the quantum sine-Gordon equation in two space–time dimensions. Helv. Phys. Acta **50**, 315–329 (1977)
25. Fröhlich, J., Seiler, E.: The massive Thirring–Schwinger model (QED2): convergence of perturbation theory and particle structure. Helv. Phys. Acta **49**, 889–924 (1976)
26. Garban, C.: Dynamical Liouville. J. Funct. Anal. **278**(108351), 1–54 (2020)
27. Glimm, J., Jaffe, A.: Quantum Physics: A Functional Integral Point of View. Springer, Berlin (1986)
28. Gubinelli, M., Imkeller, P., Perkowski, N.: Paracontrolled distributions and singular PDEs. Forum Math. Pi **3**(e6), 1–75 (2015)
29. Hairer, M.: A theory of regularity structures. Invent. Math. **198**, 269–504 (2014)
30. Hairer, M., Labbé, C.: The reconstruction theorem in Besov spaces. J. Funct. Anal. **273**, 2578–2618 (2017)
31. Hairer, M., Shen, H.: The dynamical Sine-Gordon model. Commun. Math. Phys. **341**, 933–989 (2016)

32. Høegh-Krohn, R.: A general class of quantum fields without cut-offs in two space-time dimensions. Commun. Math. Phys. **21**, 244–255 (1971)
33. Hoshino, M., Kawabi, H., Kusuoka, S.: Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model driven by space-time white noise on the torus. J. Evol. Equ. **21**, 339–375 (2021)
34. Junnila, J., Saksman, E.: Uniqueness of critical Gaussian chaos. Electron. J. Probab. **22**(11), 1–31 (2017)
35. Kahane, J.-P.: Sur le chaos multiplicatif. Ann. Sci. Math. Québec **9**, 105–150 (1985)
36. Kechris, A.-S.: Classical Descriptive Set Theory, Graduate Texts in Mathematics, vol. 156. Springer, New York (1995)
37. Kozono, H., Ogawa, T., Taniuchi, Y.: Navier–Stokes equations in the Besov space near L^∞ and BMO. Kyushu J. Math. **57**, 303–324 (2003)
38. Kusuoka, S.: Høegh-Krohn's model of quantum fields and the absolute continuity of measures. In: Ideas and Methods in Quantum and Statistical Physics (Oslo. 1988). Cambridge University Press, Cambridge, pp. 405–424 (1992)
39. Lieb, E.H., Loss, M.: Analysis, Graduate Studies in Mathematics, 2nd edn. American Mathematical Society, Providence, vol. 14 (2001)
40. Ma, Z.-M., Röckner, M.: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Springer, Berlin, Heidelberg (1992)
41. Mourrat, J.-C., Weber, H.: Global well-posedness of the dynamic Φ^4 model in the plane. Ann. Probab. **45**, 2398–2476 (2017)
42. Oh, T., Robert, T., Tzvetkov, N., Wang, Y.: Stochastic quantization of Liouville conformal field theory (2020). arXiv:2004.04194
43. Oh, T., Robert, T., Wang, Y.: On the parabolic and hyperbolic Liouville equations. Commun. Math. Phys. **387**, 1281–1351 (2021)
44. Ondreját, M.: Uniqueness for stochastic evolution equations in Banach spaces. Dissertaiones Math. (Rozprawy Mat.) **426**, 63pp (2004)
45. Rhodes, R., Vargas, V.: Gaussian multiplicative chaos and applications: a review. Probab. Surv. **11**, 315–392 (2014)
46. Robert, R., Vargas, V.: Gaussian multiplicative chaos revisited. Ann. Probab. **38**, 605–631 (2010)
47. Simon, B.: The $P(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton Series in Physics. Princeton University Press, Princeton (1974)
48. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. (4) **146**, 65–96 (1987)
49. Triebel, H.: Characterizations of Besov–Hardy–Sobolev spaces: a unified approach. J. Approx. Theory **52**, 162–203 (1988)