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Stationary stochastic Higher Spin Six Vertex Model and q-Whittaker measure

Takashi Imamura[1] · **Matteo Mucciconi**[2] · **Tomohiro Sasamoto**[2]

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Abstract

In this paper we consider the Higher Spin Six Vertex Model on the lattice $\mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$. We first identify a family of translation invariant measures and subsequently we study the one point distribution of the height function for the model with certain random boundary conditions. Exact formulas we obtain prove to be useful in order to establish the asymptotic of the height distribution in the long space-time limit for the stationary Higher Spin Six Vertex Model. In particular, along the characteristic line we recover Baik–Rains fluctuations with size of characteristic exponent $1/3$. We also consider some of the main degenerations of the Higher Spin Six Vertex Model and we adapt our analysis to the relevant cases of the q-Hahn particle process and of the Exponential Jump Model.

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Contents

✉ Matteo Mucciconi
matteomucciconi@gmail.com

Takashi Imamura
imamura@math.s.chiba-u.ac.jp

Tomohiro Sasamoto
sasamoto@phys.titech.ac.jp

[1] Department of Mathematics and Informatics, Chiba University, Chiba, Japan

[2] Department of Physics, Tokyo Institute of Technology, Tokyo, Japan

1 Introduction

1.1 Background

During the last two decades, the study of one dimensional integrable systems, related to random growth of interfaces or to particle transport, has produced a number of fundamental results. This wave of interest was surely fostered by breakthroughs like that of Johansson [43], which around year 2000 successfully provided an exact description of the current of particles in the Totally Asymmetric Simple Exclusion Process (TASEP). Methods used by Johansson, which were drawing inspiration from combinatorics and random matrix theory, soon proved to be parallel to a more algebraic framework in terms of free fermions [51], leading to the definition of the Schur processes in [52]. These last are probability measures weighting sequences of partitions of integers expressed in terms of Schur functions, a class of special symmetric functions more commonly used in representation theory. The richness of techniques and possibilities given by the intersection of so many apparently distinct fields gathered immediately the attention of the community of mathematicians and physicists and gave rise to a new field of its own that today bears the name of Integrable Probability [17]. Over the years, purely determinantal processes like the TASEP or the Schur processes have seen a number of generalizations and methods introduced in [43,52] have been extended and applied to these more general models, which are not necessarily free fermionic.

Conceptually relevant deformations of the TASEP are exclusion processes like the q-TASEP [13], the q-Hahn TASEP [56] or also the long standing Asymmetric Simple Exclusion Process (ASEP) [62]. The first of these models was introduced first by Borodin and Corwin as a marginal projection of the Macdonald processes [13], which as of today, happen also to be the among the richest generalization of the Schur processes. On the other hand, models like the q-Hahn TASEP or the ASEP had not been immediately identified as particular cases of general integrable models, like the Macdonald processes, and the study of their properties was carried out by different authors employing different techniques [14,15,29,67].

A unifying picture was offered by Corwin and Petrov in [30] using the language of vertex models. Here, authors, taking advantage of recent developments on algebraic theories concerning the Yang–Baxter equation [49], introduced the Higher Spin Six Vertex Model, which they used to construct a random dynamics of particles on the lattice where the update rules of position of particles at each time were given in terms of what are usually called stochastic $\mathcal{R}$-matrices. These are operators, which we denote with the symbol L, solving the celebrated Yang–Baxter equation [42]

$$\mathsf{L}^{(1,2)}\mathsf{L}^{(1,3)}\mathsf{L}^{(2,3)} = \mathsf{L}^{(1,3)}\mathsf{L}^{(2,3)}\mathsf{L}^{(1,2)}, \tag{1.1}$$

which also satisfy the property of having positive entries and sum-to-one condition for rows. In (1.1), the stochastic $\mathcal{R}$-matrices depend on a number of parameters, by specializing which one can degenerate the Higher Spin Six Vertex Model to previously mentioned models including the q-TASEP, the q-Hahn TASEP or the ASEP. In [18] a description of the Higher Spin Six Vertex Model complementary to that of [30] was offered using a language closer to that of the Schur processes. Here, the model was studied through a family of symmetric rational functions, introduced in [12], whose properties descended from the commutation relation (1.1) and that are in fact multi-parameter generalizations of the Schur functions. An alternative framework to compute observables of the Higher Spin Six Vertex Model uses its stochastic self-duality [15,30,60]. Connections between duality and Yang–Baxter integrability were further developed in [46].

On top of offering a unified theory embracing the majority of methods used in previously studied models, spanning from asymmetric exclusion processes to random partitions, the Higher Spin Six Vertex Model also admits as particular cases new integrable systems, including inhomogeneous traffic models as the Exponential Jump Model [19] or the Hall–Littlewood Push-TASEP [35].

The idea of utilizing the Yang–Baxter integrability to produce and solve stochastic particle systems can be traced back to the early work [37] by Gwa and Spohn. There authors interpreted a particular degeneration of the Six Vertex Model, of which the Higher Spin Six Vertex Model is a generalization, as a cellular automata in the same fashion as in [21,30] and they were able to compute the roughening exponent of a random interface associated with the current of particles. More recently, following the example of [30], the formalism of vertex models has shown other promising applications producing dynamical version of the Higher Spin Six Vertex Model [3,24] or multi-species integrable particle processes in the very recent work [20].

1.2 KPZ universality, integrability and initial conditions

What drives the field of Integrable Probability is the broader context of the KPZ universality class [27]. This is often vaguely defined as a class of random processes describing the stochastic evolution of interfaces that, in the long time limit, possess a characteristic 3:2:1 scaling. That is to say that the profile of the random surfaces in question grows linearly in the time of the system t, while the range of spatial correlations and the size of fluctuations around its expected shape scale asymptotically as $t^{2/3}$ and $t^{1/3}$.

The principal example of a model exhibiting such properties is the KPZ equation itself, that is the simplest stochastic partial differential equation describing the random profile of an interface where both relaxation and lateral growth are allowed. It was first introduced in [44], where authors after deducing its characteristic scaling, conjectured that its “nontrivial relaxation patterns” must be shared by a large class of growth processes. During the last 30 years extensive work has been done in order to understand the properties and the boundaries of this universality class, including also experimental confirmations [64,65] of the predictions of [44].

The first result of this sort is the one obtained in [43], which predicts that, under narrow wedge initial conditions (step initial conditions for the TASEP), the limiting fluctuations of the height function obey the GUE Tracy–Widom distribution. Confirmations of the fact that this limiting law is in fact universal for the class are given in a number of other papers (see [14,29,34,53] and references therein) and in particular this result is established for the solution of the KPZ equation in [4,59].

For flat initial conditions the limiting fluctuations of the height profile are believed to be ruled by the GOE Tracy–Widom distribution and this again is based on results on the polynuclear growth model [7,57] and on the TASEP [58]. The validation of this conjecture for other models and in particular for the KPZ equation has proven to be rather troublesome, although some steps forward were made recently in [54] for the particular case of the ASEP with alternating initial data.

From the point of view of non-equilibrium statistical physics or stochastic interacting particle systems, arguably the most relevant class of initial conditions one can consider is represented by the stationary ones and this is indeed the case we pursue in this paper. For continuous models these can be regarded as Brownian motions and the height function possesses in the long time limit two different regimes. Around the characteristic line of the Burgers equation associated with the dynamics, that one can understand as the direction of the growth, the one point distribution of the height is believed to be governed by the Baik–Rains distribution F_0 (see Definition 6.2). Alternatively, when we move away from the characteristic line the size of fluctuations coming from the stationary initial data overwhelms any possible nontrivial behavior produced by the random dynamics and the height performs a gaussian process. The special law F_0 was introduced first in [8], where authors identified it as limiting distribution of the height of a polynuclear growth model with critical boundary conditions.

An analogous result was obtained for the TASEP in [33]. More recently, using certain limiting properties of the q-TASEP, Baik–Rains fluctuations were established for the solution of the KPZ equation first in [22] and then in [40].

Our main result is the confirmation of the KPZ scaling theory for the stationary Higher Spin Six Vertex Model (that we define in the Sect. 1.3) and hence for the hierarchy of models it generalizes. Prior to our work, in [2], the stationary Six Vertex Model was studied by Aggarwal, who was also able to solve the long standing problem of characterizing the asymptotic fluctuations of the current in the stationary ASEP. It is also important to mention that the convergence of the Higher Spin Six Vertex Model to the KPZ equation (under weak asymmetric limit) was considered in [28, 47].

Very much related to this paper is the work by two of the authors on the stationary continuous time q-TASEP [39], which can in fact be considered as a part one of a two parts effort.

1.3 The model

In this subsection we give a definition of the Higher Spin Six Vertex Model [30]. This represents a generalization of the Six Vertex Model, a classical integrable system in statistical physics first introduced by Pauling in 1935 [55]. Although the Six Vertex Model served originally to study geometric configurations of water molecules in an ice layer, it is today often presented in literature along with other exactly solvable models describing different physical phenomena, such as one dimensional quantum spin chains (see [11]). The reason behind this association is that such models all share the same integrability structure, that pivots around the notion of Yang–Baxter equation. In particular such algebraic structures allow for general constructions, leading for example to higher spin generalizations of the Heisenberg XXZ model [45]. We refer to [36] for an extended review of these results.

We will consider the stochastic Higher Spin Six Vertex Model, presented here as an ensemble of directed paths in a quadrant of the two dimensional lattice $\mathbb{Z} \times \mathbb{Z}$. Define the lattice $\Lambda_{1,0}$ and its boundary $\partial\Lambda_{1,0}$ as the sets

$$\Lambda_{1,0} = \big(\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}\big) \setminus (1,0) \qquad \text{and} \qquad \partial\Lambda_{1,0} = \big(\mathbb{Z}_{\geq 2} \times \{0\}\big) \cup \big(\{1\} \times \mathbb{Z}_{\geq 1}\big).$$

We see that $\Lambda_{1,0}$ is the union of the quadrant $\mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$ with the set of its nearest neighbor vertices $\partial\Lambda_{1,0}$ and we refer to the interior $\mathring{\Lambda}_{1,0} = \Lambda_{1,0} \setminus \partial\Lambda_{1,0}$ as the bulk of the lattice. We use the symbol $\mathfrak{P}(\Lambda_{1,0})$ to denote the set of up right directed paths in $\Lambda_{1,0}$. That is the generic element $\mathfrak{p}$ of $\mathfrak{P}(\Lambda_{1,0})$ is a collection of up right directed paths emanating from the boundary $\partial\Lambda_{1,0}$, as those represented in Fig. 1.

A natural way to encode the information contained in the single configuration $\mathfrak{p}$ is to record how many times each edge of the lattice is shared by its paths. For this we introduce the collection of occupancy numbers

$$\mathsf{m}_x^t = \text{number of paths exiting the vertex } (x,t) \text{ in the upward direction}, \tag{1.2}$$
$$\mathsf{j}_x^t = \text{number of paths exiting the vertex } (x,t) \text{ in the rightward direction}, \tag{1.3}$$

Fig. 1 A possible configuration of up right directed paths in the lattice $\Lambda_{1,0}$

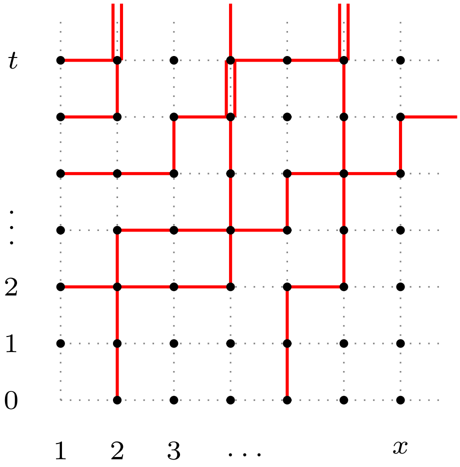
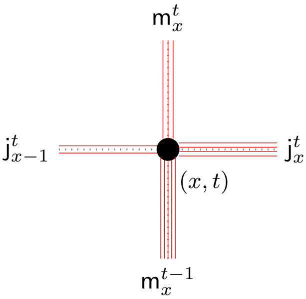


Fig. 2 An arrangement of paths across the vertex (x, t). Random variables $\mathsf{m}_x^{t-1}, \mathsf{j}_{x-1}^t$ represent paths entering respectively from below and from the left. Random variables $\mathsf{m}_x^t, \mathsf{j}_x^t$ describe the number of paths exiting the vertex respectively in the upward direction and to the right



of which a graphical representation is given in Fig. 2. We think at the specific $\mathfrak{p}$ as a realization of sequences (1.2), (1.3), that we express with the notation $\{m_x^t, j_x^t\}_{(x,t)\in\Lambda_{1,0}}$. Since paths only generate at $\partial\Lambda_{1,0}$, quantities $\mathsf{m}_2^0, \mathsf{m}_3^0, \dots, \mathsf{j}_1^1, \mathsf{j}_1^2, \dots$ describe the boundary conditions of configurations $\mathfrak{p}$, while in the bulk, around the generic vertex (x, t), we necessarily have the conservation law

$$\mathsf{m}_x^{t-1} + \mathsf{j}_{x-1}^t = \mathsf{m}_x^t + \mathsf{j}_x^t. \tag{1.4}$$

We associate now, to each vertex (x, t) in $\mathring{\Lambda}_{1,0}$ a *stochastic weight* $L_{(x,t)}$. This is a non-negative valued function

$$L_{(x,t)}\left(m_x^{t-1}, j_{x-1}^t \,|\, m_x^t, j_x^t\right) \tag{1.5}$$

of the vertex configuration, that is zero when the occupancy numbers do not fulfill (1.4) and that satisfies, for any fixed m_x^{t-1}, j_{x-1}^t, the sum-to-one condition

$$\sum_{m_x^t, j_x^t \geq 0} L_{(x,t)}\left(m_x^{t-1}, j_{x-1}^t \,|\, m_x^t, j_x^t\right) = 1. \tag{1.6}$$

An *up right directed path stochastic vertex model* on $\Lambda_{1,0}$ is a probability measure $\mathcal{P}$ on the set $\mathfrak{P}(\Lambda_{1,0})$ where, for all vertices (x,t) in the bulk $\mathring{\Lambda}_{1,0}$, the joint law of $\mathsf{m}_x^t, \mathsf{j}_x^t$, conditioned to $\mathsf{m}_x^{t-1}, \mathsf{j}_{x-1}^t$ and independently of $\mathsf{m}_{x-k}^{t+k}, \mathsf{j}_{x-k}^{t+k}$ for $k = \pm 1, \pm 2, \dots$ is written as

$$\mathcal{P}\Big(\mathsf{m}_x^t = i', \mathsf{j}_x^t = j' \,|\, \mathsf{m}_x^{t-1} = i, \mathsf{j}_{x-1}^t = j\Big) = L_{(x,t)}(i,j|\,i',j'), \tag{1.7}$$

for all 4-tuples of non-negative integers i, j, i', j'. When this is the case, we interpret (1.5) as the probabilities ruling how paths propagate in the lattice in the up-right direction while crossing single vertices. To complete the definition of the measure $\mathcal{P}$ we need to assign a probability law to boundary random variables $\mathsf{m}_2^0, \mathsf{m}_3^0, \dots, \mathsf{j}_1^1, \mathsf{j}_1^2, \dots$. For the sake of this paper we will always consider them as mutually independent random variables that are a.s. finite and we can denote their joint law with the symbol $\mathcal{P}_B$. It is rather clear that, once we specify $\mathcal{P}_B$, the *vertex model rule* (1.7) uniquely defines the measure $\mathcal{P}$ on any bounded subset of $\Lambda_{1,0}$. Through this observation, by making use of standard techniques of measure theory we could at this point prove that, given stochastic vertex weights $\{L_{(x,t)}\}_{(x,t)\in\mathring{\Lambda}_{1,0}}$ and boundary conditions $\mathcal{P}_B$, the measure $\mathcal{P}$ is uniquely defined on the full set $\mathfrak{P}(\Lambda_{1,0})$. This is a consequence of the fact that paths are up-right directed and the distribution of $\mathsf{m}_x^t, \mathsf{j}_x^t$ only depends on $\mathsf{m}_x^{t-1}, \mathsf{j}_{x-1}^t$ and therefore one can view the generic configuration $\mathfrak{p}$ as result of a markovian propagation as done in [30].

In this paper we focus on the particular class of vertex weights $\mathsf{L}_{\xi_x u_t, s_x}$, defined in Table 1. Collectively, the family $\{\mathsf{L}_{\xi_x u_t, s_x}\}_{(x,t)\in\mathring{\Lambda}_{1,0}}$ depends on a number q and on the sets of values

$$\Xi = (\xi_2, \xi_3, \dots), \qquad \mathbf{S} = (s_2, s_3, \dots), \qquad \mathbf{U} = (u_1, u_2, \dots),$$

which are called respectively *inhomogeneity*, *spin* and *spectral* parameters. Unless otherwise specified we will always assume that

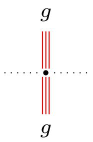
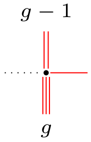
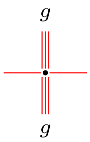
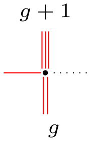
$$0 \le q < 1, \qquad 0 \le s_x < 1, \qquad \xi_x > 0, \qquad u_t < 0, \qquad \text{for all } x, t. \tag{1.8}$$

Under condition (1.8) we can easily verify that $\mathsf{L}_{\xi_x u_t, s_x}$ are positive quantities fulfilling the sum-to-one condition (1.6) and hence we regard them as bona fide stochastic vertex weights. We refer to the directed path stochastic vertex model with choice $L_{(x,t)} = \mathsf{L}_{\xi_x u_t, s_x}$ as the *Stochastic Higher Spin Six Vertex Model.*

From Table 1 we see that the only vertex configurations having positive probability are those where different paths do not share any of the horizontal edges of the lattice. This limitation can be removed by means of a procedure called *fusion*, that consists in collapsing together a number of different rows of vertices and that we review in Sect. 2.3. By fusing together a column of J vertices one can construct the vertex weight $\mathsf{L}^{(J)}_{\xi_x u_t, s_x}$, which in this case takes a rather complicated form, stated below in (2.17). We refer to the directed path stochastic vertex model with choice of weights $L_{(x,t)} = \mathsf{L}^{(J)}_{\xi_x u_t, s_x}$ as the *fused stochastic Higher Spin Six Vertex Model.*

Boundary conditions for the model we study in this paper are given in terms of a special family of probability distribution, that we call *q-negative binomial.* A random

Table 1 In the top row we see all acceptable configurations of paths entering and exiting a vertex; below we reported the corresponding stochastic weights $\mathsf{L}_{\xi_x u_t, s_x}(m, j | m', j')$. Spin and inhomogeneity parameters s and ξ will depend on the x coordinate of a vertex, whereas spectral parameters u will depend on the t coordinate

				
$\mathsf{L}_{\xi_x u_t, s_x}$	$\frac{1-q^g\xi_x s_x u_t}{1-s_x\xi_x u_t}$	$\frac{-s_x\xi_x u_t+q^g\xi_x s_x u_t}{1-s_x\xi_x u_t}$	$\frac{-s_x\xi_x u_t+s_x^2 q^g}{1-s_x\xi_x u_t}$	$\frac{1-s_x^2 q^g}{1-s_x\xi_x u_t}$

variable X is said to be q-negative binomial if its probability mass function is expressed as

$$\mathbb{P}(X = n) = p^n \frac{(b;q)_n}{(q;q)_n} \frac{(p;q)_\infty}{(pb;q)_\infty}, \qquad \text{for all } n \in \mathbb{Z}_{\geq 0}, \tag{1.9}$$

for some parameters p, b and we use the notation $X \sim q\text{NB}(b, p)$. In case p, b belong to the interval $[0, 1)$, then X is supported on $\mathbb{Z}_{\geq 0}$, whereas if $p < 0$ and $b = q^{-L}$ for some positive integer L, then X only takes values on the set $\{0, \dots, L\}$. We define the *double sided q-negative binomial Higher Spin Six Vertex Model* as the (fused) Higher Spin Six Vertex Model where boundary random variables $\mathsf{m}_2^0, \mathsf{m}_3^0, \dots, \mathsf{j}_1^1, \mathsf{j}_1^2, \dots$ are independently distributed with laws

$$\mathsf{m}_x^0 \sim q\text{NB}(s_x^2, v/(\xi_x s_x)), \qquad \mathsf{j}_1^t \sim q\text{NB}(q^{-J}, q^J u_t d), \tag{1.10}$$

for parameters d, v satisfying $d > 0$ and

$$0 \leq v < \inf_x \{\xi_x s_x\}. \tag{1.11}$$

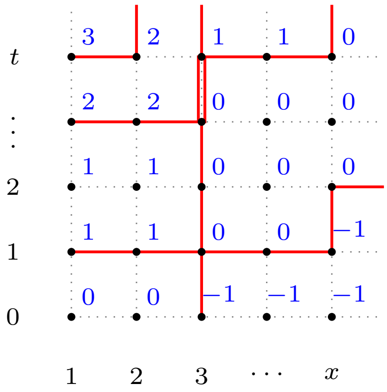
A special case of conditions (1.10) will be given setting $v = d$ and we will refer to the model with this choice as the *stationary Higher Spin Six Vertex Model.*

Exact properties of the model will be described by means of the *height function* $\mathcal{H}$, an observable that we define as

$$\mathcal{H}(x, t) = -\mathsf{m}_2^0 - \cdots - \mathsf{m}_x^0 + \mathsf{j}_x^1 + \cdots + \mathsf{j}_x^t. \tag{1.12}$$

Pictorially we might look at paths in the generic configuration $\mathfrak{p}$ as the contours of an irregular staircase with steps one unit length tall, that we climb down as we move in the down-right direction. By centering the value of the height of the staircase $\mathcal{H}(1, 0) = 0$, we see that (1.12) describes the vertical displacement that we encounter going from $(1, 0)$ to (x, t) (see Fig. 3). The main results of this this paper is the exact description of the one point distribution of $\mathcal{H}$ under a restricted class of q-negative binomial boundary conditions, that includes the stationary case.

Fig. 3 A possible set of up right paths in the Higher Spin Six Vertex Model. The numbers in blue reported next to each vertex are the values of the height function $\mathcal{H}$ defined in (1.12) (color figure online)



1.4 Methods

In this paper we study the double sided q-negative binomial Higher Spin Six Vertex Model by expressing the distribution of the height function $\mathcal{H}$ in terms of certain q-Whittaker measures. These measures arise from the formalism of the Macdonald processes and are known to describe the joint law of a class of dynamics for the q-TASEP [13,50]. Under a more restricted set of boundary conditions analogies between the Higher Spin Six Vertex Model and the q-TASEPs were studied first in [53], where authors provided a coupling between the height function and the position of tagged particles along *time-like paths*, that are up right paths in $\Lambda_{0,1}$ with the vertical direction read as the time for the q-TASEP dynamics. Here we extend part of Orr and Petrov's argument [53] to the case of q-negative binomial boundary conditions.

Random boundary conditions we will consider for our model will be produced through a fusion procedure. In the case of the Stochastic Six Vertex Model, such techniques were adopted first in [2] to generate independent Bernoulli random entries from the horizontal axis. In this regard we find that the q-negative binomial boundary conditions represent the natural generalization of the independent Bernoulli ones for a higher spin version of the model.

Information describing the probability distribution of $\mathcal{H}$ are encoded in the q-Laplace transform (see "Appendix A")

$$\mathbb{E}_{\mathrm{HS}(v,\mathscr{d})}\left(\frac{1}{(\zeta q^{\mathcal{H}(x,t)};q)_\infty}\right), \tag{1.13}$$

where the subscript $\mathrm{HS}(v,\mathscr{d})$ refers to boundary conditions (1.10). Ever since the introduction of the q-Whittaker processes in [13], the study of q-Laplace transforms such as (1.13) has proven to be successful in order to derive rigorous asymptotic analysis in a number of examples, especially for cases corresponding to step boundary conditions ($v = 0$) [10,34]. In the language of the q-TASEP, determinantal structures for the q-Laplace transform of the probability density of a tagged particle y_x were obtained in [39] employing elliptic analogs of the Cauchy determinants after considering the expansion

$$\mathbb{E}\left(\frac{1}{(\zeta q^{y_x+x};q)_\infty}\right)=\sum_{l\in\mathbb{Z}}\mathbb{P}(y_x+x=l)\frac{1}{(\zeta q^l;q)_\infty}.$$

This is the strategy we follow here, bringing computations and asymptotic analysis of [39] to the more general setting of the Higher Spin Six Vertex Model.

A different approach and also a more established one in the context of Integrable Probability, would have been that of studying the q-Laplace transform (1.13) by means of a q-moments expansion as done in [2,22]. Although in the case of random boundary conditions such type of expansion would be ill posed as the height $\mathcal{H}$ assumes also negative values and its q-moments diverge, one can still make sense of it after employing certain analytic continuation in parameters governing the initial measure.

An interesting finding is that, in terms of exact determinantal formulas, these two strategies produce similar yet different results. In particular the Cauchy determinants approach offers a Fredholm determinant representation for (1.13), where the kernel has finite rank and it admits a biorthogonal expansion reminiscent of those found in random matrix theory for the study of gaussian ensembles. Due to the finiteness of the rank of the kernel, formulas we obtain are rather easy to manipulate in concrete examples, when for instance one wants to compute the q-Laplace transform (1.13) at a vertex (x,t) reasonably close to the origin. On the other hand, through a q-moments approach one would obtain a representation of (1.13) in terms of a Fredholm determinant of an infinite rank operator. Such procedure, which often requires a certain amount of guesswork for the choice of the kernel, has nevertheless shown its advantages too as expressions one gets are amenable to rigorous asymptotics with relatively weak assumptions on parameters defining the system. The question on how to move from one representation to the other is indeed an interesting one and it remains open, although we plan to address this issue in a forthcoming paper.

In order to establish Baik–Rains asymptotic fluctuations for the height function we employ techniques closer to works on the TASEP with deterministic initial conditions [16], than to more recent ones [10,34] on more general models. The Baik–Rains limit, compared to GOE or GUE Tracy–Widom limits often requires an extra amount of care, conceptually because the procedure involves the exchange of a limit and of a derivative sign. Throughout Sect. 6.2 we take care of such technical difficulties by a detailed analysis of the remainder terms in the asymptotic limit that presents some novel aspects.

1.5 Results

Our first result is a characterization of the stationary Higher Spin Six Vertex Model, which we recall was defined above as the model with double sided q-negative binomial boundary conditions (1.10) with parameters $v=\mathscr{d}$. We find that the probability measures with these particular choices of boundary conditions are the only one to satisfy a certain translation invariance that we call *Burke's property*.

Definition 1.1 We say that a probability measure $\mathcal{P}$ on $\mathfrak{P}(\Lambda_{1,0})$ satisfies the *Burke's property* if there exist families $\{P^{(x)}\}_{x\geq 2}$, $\{\widetilde{P}^{(t)}\}_{t\geq 1}$ of probability distributions such

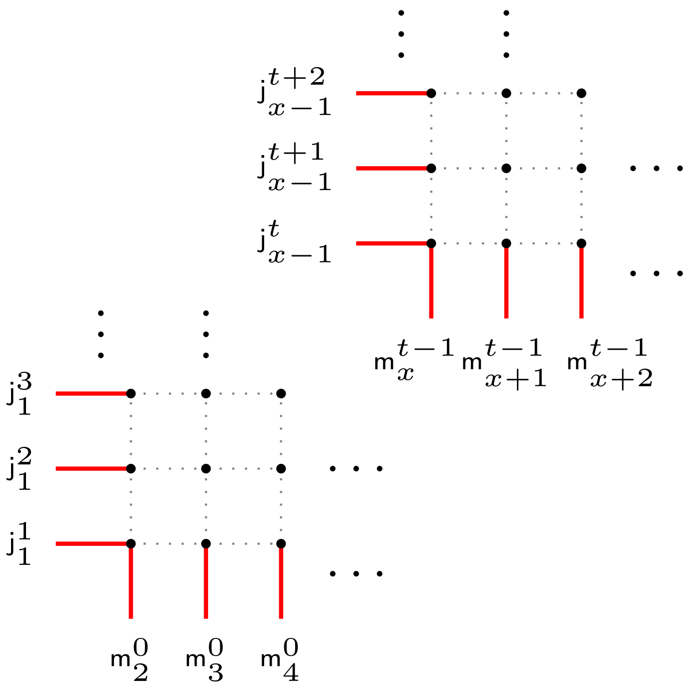


Fig. 4 An illustration of the Burke's property

that, for all $(x', t') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$ we have, independently

$$\begin{aligned} \mathsf{m}^{t'}_{x'+k} &\sim P^{(x'+k)}, \qquad \text{for all } k \geq 1, \\ \mathsf{j}^{t'+k}_{x'} &\sim \widetilde{P}^{(t'+k)}, \qquad \text{for all } k \geq 1. \end{aligned}$$

In words, the Burke's property states that, for any choice of a vertex (x', t'), the Higher Spin Six Vertex Model on the shifted lattice $\Lambda_{x',t'} = (\mathbb{Z}_{\geq x'} \times \mathbb{Z}_{\geq t'}) \setminus (x', t')$, obtained as a marginal process of the model on $\Lambda_{1,0}$, possesses boundary conditions that are always described by the same family of probability laws $P^{(x)}$, $\widetilde{P}^{(t)}$ after appropriately shifting indices x, t (see Fig. 4).

Proposition 1.2 *The Higher Spin Six Vertex Model on the lattice $\Lambda_{1,0}$ satisfies the Burke's property if and only if boundary conditions are taken as*

$$m_x^0 \sim q\mathrm{NB}(s_x^2, \mathscr{d}/(\xi_x s_x)), \qquad j_1^t \sim q\mathrm{NB}(q^{-J}, q^J \mathscr{d} u_t), \tag{1.14}$$

independently of each other, for all $x \geq 2$, $t \geq 1$, where $\mathscr{d}$ is a parameter that meets the condition

$$0 \leq \mathscr{d} < \inf_x \{\xi_x s_x\}. \tag{1.15}$$

Result of Proposition 1.2 can be compared to the well known characterization of translation invariant measures for a general class zero range processes on $\mathbb{Z}$ obtained in [5], that states that these come in the form of factorized measures. In our case we define a notion of translation invariance for factorized measures on vertex models inhomogeneous both in the spatial and time coordinates and subsequently we describe the entire family of measures satisfying such properties.

The next result we present offers a Fredholm determinant representation of the q-Laplace transform (1.13) in a model with double sided q-negative binomial boundary conditions with parameters $v < \mathscr{d}$. In the following we refer to a q-Poisson random variable with parameter p, in short $q\mathrm{Poi}(p)$, as a q-negative binomial (1.9) with parameters $0 < p < 1$ and $b = 0$. For the sake of the following analytical statements we will assume that parameters $\Xi, \mathbf{S}$ are placed in such a way that

$$q \sup_i \{\xi_i s_i\} < \mathscr{d} < \inf_i \{\xi_i s_i\} \leq \sup_i \{\xi_i s_i\} < \inf_i \{\xi_i / s_i\}. \tag{1.16}$$

Theorem 1.3 *Consider a double sided q-negative binomial Higher Spin Six Vertex Model on $\Lambda_{1,0}$ with parameters $v, \mathscr{d}, \Xi, \mathbf{S}$ satisfying (1.16) and $v < \mathscr{d}$. Also set m to be an independent q-Poisson random variable of parameter $v/\mathscr{d}$. Then we have*

$$\mathbb{E}_{\mathrm{HS}(v,\mathscr{d})\otimes \mathsf{m}} \left(\frac{1}{(\zeta q^{\mathcal{H}(x,t)-\mathsf{m}}; q)_\infty} \right) = \det(\mathbf{1} - fK)_{l^2(\mathbb{Z})}. \tag{1.17}$$

The kernel fK on the right hand side is given in (5.2), (5.3) and it is finite dimensional.

In order to employ the statement of Theorem 1.3 for the study of the stationary model one needs to remove from expression (1.17) the contribution of the q-Poisson random variable m, that becomes a.s. infinite in the limit $v \to \mathscr{d}$. In Sect. 5.2 it is shown how such decoupling procedure provides us with determinantal formulas which describe the height function $\mathcal{H}$ in the case of q-negative binomial boundary conditions with parameters $v, \mathscr{d}$ satisfying $qv < \mathscr{d} < v/q$. This is in fact the most general range of boundary conditions we will state exact results for.

Theorem 1.4 *Consider the double sided q-negative binomial Higher Spin Six Vertex Model with parameters $v, \mathscr{d}, \Xi, \mathbf{S}$ satisfying (1.16) and*

$$qv < \mathscr{d} < v/q. \tag{1.18}$$

Then we have

$$\begin{aligned} \mathbb{E}_{\mathrm{HS}(v,\mathscr{d})} \left(\frac{1}{(\zeta q^{\mathcal{H}(x,t)}; q)_\infty} \right) = \frac{1}{(qv/\mathscr{d}; q)_\infty} \sum_{k \geq 0} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \\ \times \left(\frac{v}{\mathscr{d}} \right)^k V_{x;v,\mathscr{d}}(\zeta q^{-k}), \end{aligned} \tag{1.19}$$

where the function $V_{x;v,d}$ *is defined as*

$$V_{x;v,d}(\zeta) = \frac{1}{1-v/d}\det(\mathbf{1} - fK)_{l^2(\mathbb{Z})}.$$

Remarkably, expression (1.19) is amenable to rigorous asymptotic analysis and in particular we pursue the case when the model is stationary. By taking the limit $v \to d$, the expression of function $V_{x;v,d}$ takes a rather complicated form, stated below in (5.26) and we devote Sect. 6.2 to establish its behavior in the large x limit. As already mentioned in Sect. 1.2, when the measure is stationary, the characteristic 3:2:1 scaling of the model is only observed along a specific direction, which is usually referred to as the *characteristic line*. The scaling of the height function along this line is conjectured to be universal and it is described by the KPZ scaling theory [63], that we explain briefly in Sect. 6.1.

For the stationary Higher Spin Six Vertex Model we now want to give the exact expression of scaling parameters defining the characteristic line and the expected behavior of the height function $\mathcal{H}$. We make use of q-polygamma type functions υ_k defined in "Appendix A". For non-negative integers k consider the functions

$$\begin{aligned} a_k(d) &= \upsilon_k(q^J u d) - \upsilon_k(u d), \\ h_k(d) &= \frac{1}{x}\sum_{y=2}^{x}\big(\upsilon_k(d/(\xi_y s_y)) - \upsilon_k(d s_y/\xi_y)\big) \end{aligned} \tag{1.20}$$

and, depending on the parameter d, define the quantities

$$\kappa_0 = \frac{h_1(d)}{a_1(d)}, \qquad \eta_0 = \kappa_0 a_0(d) - h_0(d), \qquad \gamma = -\left(\frac{1}{2}(\kappa_0 a_2(d) - h_2(d))\right)^{1/3}. \tag{1.21}$$

We assume that the functions h_k always converge in the large x limit and we refer to the curve $(x, \kappa_0 x)$ as the *characteristic line* of the stationary Higher Spin Six Vertex Model. For random growth models usually the characteristic line is expressed as a function of the time t, rather than of the coordinate x, but in our case, since the system exhibits spatial inhomogeneities we find more natural to adopt the notation $(x, \kappa_0 x)$. The parameter η_0 multiplied by x is readily understood as the expectation $\mathbb{E}(\mathcal{H}(x, \kappa_0 x))$, whereas γ will be used to describe the size of the characteristic fluctuations of $\mathcal{H}$ around η_0. By slightly perturbing quantities κ_0, η_0 we can analyze the asymptotic behavior of $\mathcal{H}$ in a region of size $x^{2/3}$ around the characteristic line. For this we extend the definitions given in (1.21) setting

$$\kappa_\varpi = \kappa_0 + \frac{h_2 a_1 - h_1 a_2}{a_1^2}\frac{\varpi}{\gamma x^{1/3}} \tag{1.22}$$

$$\eta_\varpi = \eta_0 + \frac{a_0(h_2 a_1 - h_1 a_2)}{a_1^2}\frac{\varpi}{\gamma x^{1/3}} + \frac{h_2 a_1 - h_1 a_2}{a_1}\frac{\varpi^2}{\gamma^2 x^{2/3}}, \tag{1.23}$$

where ϖ is a real number parameterizing the displacement from the characteristic line and functions a_k, h_k are evaluated at $\mathscr{d}$. We come now to state our main result.

Theorem 1.5 *Consider the stationary Higher Spin Six Vertex Model with parameters* $q, \mathscr{d}, \Xi, \mathbf{S}$ *fulfilling conditions stated in Definition 6.4. Then, for any real numbers* ϖ, r *we have*

$$\lim_{x\to\infty} \mathbb{P}_{\mathrm{HS}(\mathscr{d},\mathscr{d})}\left(\frac{\mathcal{H}(x,\kappa_\varpi x)-\eta_\varpi x}{\gamma x^{1/3}} > -r\right) = F_\varpi(r), \tag{1.24}$$

where $F_\varpi(r)$ *is the Baik–Rains distribution presented in Definition 6.2.*

Assumptions on parameters made in the statement of Theorem 1.5 are technical and they substantially require q to be sufficiently close to zero. These arise while establishing the steep descent property of integration contours of the integral kernel K (see "Appendix C"). Such conditions can be considerably weakened employing certain determinant preserving transformations of the kernel that involve deforming integration contours to regions containing poles of the integrand function. As such procedures are rather technical, we postpone their description to a future work and for the sake of this paper we stick to the small q assumption.

Techniques used in the proof of Theorem 1.5 can be employed also to establish Tracy–Widom asymptotic fluctuations of the height function $\mathcal{H}$ when the model has step Bernoulli boundary conditions ($v = 0$). This result was already proved in [53] using a certain matching between q-Whittaker measures and Schur measures.

Additional results we obtain are stated in Sect. 7 and they are adaptations of Fredholm determinant formulas (1.17), (1.19) and of the universal limit (1.24) to two of the main degenerations of the Higher Spin Six Vertex Models, the q-Hahn TASEP and the Exponential Jump Model.

1.6 Outline of the paper

In Sect. 2 we describe some further properties of the Higher Spin Six Vertex Model, that were left out in Sect. 1.3. Especially we recall nested contour integral formulas for q-moments of the model with step boundary conditions. In Sect. 3 we recall the definition and main properties of the q-Whittaker process. In Sect. 4 we prove the Burke's property of the stationary Higher Spin Six Vertex Model and we establish its integrability. In Sect. 5 we employ elliptic determinant computations from [39] in order to compute the q-Laplace transform of the probability mass function of the height function in the case of double sided q-negative binomial boundary conditions. In Sect. 6 we specialize determinantal expression obtained in Sect. 5 to the stationary model and we compute the asymptotics of the one point distribution of the height function along the critical line. Finally, in Sect. 7 we consider the main degenerations of the Higher Spin Six Vertex Model and and we establish determinantal formulas and Baik–Rains fluctuations for these models.

2 Stochastic Higher Spin Six Vertex Model

In this section we give a review on the Higher Spin Six Vertex Model. We take the chance to fix some notations and recall major results which will be used throughout the rest of the paper.

2.1 Directed paths picture

A description of the Higher Spin Six Vertex Model as an up right directed path ensemble in the lattice $\Lambda_{1,0}$ was given in Sect. 1.3. The choice of the set $\Lambda_{1,0}$ was made only to keep our notation consistent with that introduced in previous works [18,30] and we could extend the notion of the model to the generic lattice $\Lambda_{x',t'}$ with boundary $\partial\Lambda_{x',t'}$, defined as

$$\Lambda_{x',t'} = \left(\mathbb{Z}_{\geq x'} \times \mathbb{Z}_{\geq t'}\right) \setminus (x', t') \qquad \text{and} \qquad \partial\Lambda_{x',t'} \\ = \left(\mathbb{Z}_{\geq x'+1} \times \{t'\}\right) \cup \left(\{x'\} \times \mathbb{Z}_{\geq t'+1}\right),$$

for a generic (x', t') in $\mathbb{Z} \times \mathbb{Z}$. When this is the case boundary conditions are given specifying the laws of $\mathsf{m}_{x'+1}^{t'}, \mathsf{m}_{x'+2}^{t'}, \dots, \mathsf{j}_{x'}^{t'+1}, \mathsf{j}_{x'}^{t'+2}, \dots$ and the measure depends on parameters q, $\{u_t\}_{t>t'}$, $\{\xi_x, s_x\}_{x>x'}$. In Sect. 1.3 such parameters were assumed to satisfy condition (1.8), so to guarantee the stochasticity of vertex weights $\mathsf{L}_{\xi_x u_t, s_x}$ of Table 1 and the same assumption is made now. Although in this paper we will not investigate range of parameters different than (1.8), we want to point out that there exist also different conditions that would make all $\mathsf{L}_{\xi_x u_t, s_x}$ non-negative quantities. The full list of stochasticity conditions is given imposing for all x, t one of the following:

1. $0 < q < 1$, $-1 < s_x < 1$ and $s_x \xi_x u_t < 0$,
2. $0 < q < 1$, $q^{-G} = s_x^2 \leq s_x \xi_x u_t$ with $G \in \mathbb{Z}_{>0}$ and $g = 0, 1, \dots, G$,
3. $-1 < q < 0$, $q^{-1} \leq \xi_x s_x u_t \leq 0$ and $s_x^2 \leq \min\left(1, q^{-1}\xi_x s_x u_t\right)$,
4. $q > 1$, $0 \leq s_x \xi_x u_t \leq s_x^2 = q^{-G}$, with $G \in \mathbb{Z}_{>0}$ and $g = 0, 1, \dots, G$,

where the integer g appearing in 2, 4 is the number of vertical path entering the vertex as in Table 1. Choice 1 corresponds to (1.8), where in (1.8), with no loss of generality, we fixed the signs of u_t, ξ_x, s_x, as the weights L only depend on the product $\xi_x s_x u_t$ and s_x^2. Choice 2, with $G = 1$, produces the Six Vertex Model and the only path configurations with positive measure are those where both horizontal and vertical edges are crossed at most by one path. A list of stochasticity conditions analogous to that presented above appeared in [30], where authors used a slightly different notation.

In Sect. 1.3 we described the double sided q-negative binomial boundary conditions, introduced in (1.10) for the model in the lattice $\Lambda_{1,0}$. Clearly the same definition can be adapted also for a Higher Spin Six Vertex Model in $\Lambda_{x',t'}$. A very special case of boundary conditions is obtained when we set $J = 1$, $v = 0$ and $d = \infty$, obtaining

$$\mathsf{j}_{x'}^{t} = 1 \quad \text{a.s.}, \qquad \mathsf{m}_{x}^{t'} = 0 \quad \text{a.s.}, \tag{2.1}$$

for all $t > t'$, $x > x'$. This is to say that from the horizontal axis no path originates and at each vertex (x', t) exactly one path enters the system. We refer to (2.1) as *step boundary conditions*, as they are an analogous version of the step initial conditions for simple exclusion processes on the infinite lattice, where vertical segments of paths m_x^t are interpreted as gaps between consecutive particles when the time of the system is t (more in Sect. 2.4).

Other relevant choices of boundary conditions are given setting $v = 0$, but leaving d as a finite quantity. In this case too, paths can only enter the system from the vertical axis and they do so randomly with q-negative binomial distribution of parameters $(q^{-J}, q^J d u_t)$. We refer to these as *step q-negative binomial boundary conditions* and when $J = 1$ we also use the name *step Bernoulli boundary conditions*, considered in [2,53].

So far we considered a model where at each vertex the weight L was depending both on the x and t coordinate. A slight simplification is given by choosing parameters u_t, s_x to be constant numbers u, s for all x, t and to set the inhomogeneity parameters $\xi_x = 1$ for all x. This specialization takes the name of *Homogeneous Higher Spin Six Vertex Model* and it was considered in the original paper [30].

2.2 One line dynamical picture

In this Section we focus on the Higher Spin Six Vertex Model with $J = 1$. A possible alternative to presenting it as a static ensemble of directed paths is to interpret the arrangements of paths along each row of vertices as a dynamical process. As in Sect. 1.3, denote with m_x^t the number of paths exiting vertex (x, t) from above in a particular realization of the model. The information contained in the sequence $\{m_x^t\}_x$ can be encoded in the symbol

$$\lambda(t) = \prod_x x^{m_x^t}. \tag{2.2}$$

When $\sum_x m_x^t$ is a finite number, or equivalently, when only finitely many paths populate the region with ordinate $t' \leq t$, then $\lambda(t)$ can be thought as a signature written in multiplicative notation.

If $\lambda(t+1)$ is the symbol generated by the path configuration on vertical edges joining vertices $(x, t+1)$ and $(x, t+2)$ for $x \geq 1$, we say that $\lambda(t)$ transitions to $\lambda(t+1)$ and we want, at least formally, to describe the probability of such transition to take place. For this assume first that the probability of the event $\{\mathsf{j}_x^t = 1, \text{ eventually for } x \gg 0\}$ is zero, which is to say that paths traveling on horizontal lines will almost surely turn upward. To ensure this condition we can take parameters ξ, s, u such that

$$\sup_{x,t} \mathsf{L}_{\xi_x u_t, s_x}(0, 1|\, 0, 1) < 1. \tag{2.3}$$

In case we consider a model in the lattice $\Lambda_{0,0}$, conservation law (1.4) implies that, when we specify the number j_0^{t+1} of paths emanating the boundary vertex $(0, t+1)$, there exists at most one choice of $\{j_x^{t+1}\}$ such that $\lambda(t)$ transitions to $\lambda(t+1)$ and the

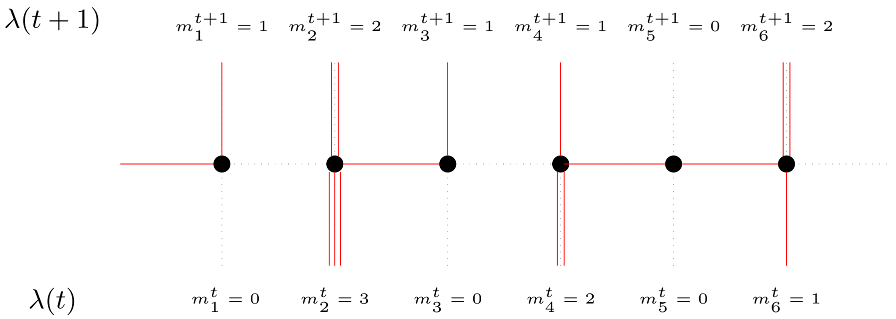


Fig. 5 As the paths cross the horizontal line of vertices the signature $\lambda(t) = 1^0 2^3 3^0 4^2 5^0 6^1$ transitions to $\lambda(t+1) = 1^1 2^2 3^1 4^1 5^0 6^2$

probability of such transition is formally given by

$$\mathfrak{X}_{u_{t+1}}(j_0^{t+1};\lambda(t)\to\lambda(t+1)) = \mathbb{P}\Big(\mathsf{j}_0^{t+1} = j_0^{t+1}\Big)\prod_{x\geq 1}\mathsf{L}_{\xi_x u_{t+1}, s_x}$$

$$\Big(m_x^t, j_{x-1}^{t+1}|\, m_x^{t+1}, j_x^{t+1}\Big). \tag{2.4}$$

Operator $\mathfrak{X}$ takes the name of *transfer operator* and one can possibly define it rigorously through an inverse limit procedure of its action on finite path configurations as done in [30], Definition 2.6. For example, when $\lambda(t)$ and $\lambda(t+1)$ describe configurations of a finite number of paths, expression (2.4) is well posed, as the infinite product on weights L on the right hand side contains almost surely only finitely many factors different from $\mathsf{L}_{\xi u,s}(0,0|\,0,0) = 1$ (Fig. 5).

We can possibly remove the dependence of $\lambda(t+1)$ from the boundary value j_0^{t+1}. To do so we simply need to set $\mathsf{m}_0^1 = \infty$ a.s., which is to say that each vertex $(1,t)$ is vertically crossed by infinitely many paths. From Table 1 we see that, assuming (1.8), when $g=\infty$, we have

$$\mathsf{L}_{\xi_1 u_t, s_1}(\infty,0|\,\infty,0) = \mathsf{L}_{\xi_1 u_t, s_1}(\infty,1|\,\infty,0) = \frac{1}{1-\xi_1 s_1 u_t} \tag{2.5}$$

$$\mathsf{L}_{\xi_1 u_t, s_1}(\infty,0|\,\infty,1) = \mathsf{L}_{\xi_1 u_t, s_1}, (\infty,1|\,\infty,1) = \frac{-\xi_1 s_1 u_t}{1-\xi_1 s_1 u_t}. \tag{2.6}$$

This means that the choice $\mathsf{m}_0^1 = \infty$ a.s. implies that random variables $\{\mathsf{j}_1^t\}_{t\geq 1}$ become mutually independent Bernoulli distributed as

$$\mathsf{j}_1^t \sim \mathrm{Ber}\left(\frac{-\xi_1 s_1 u_t}{1-\xi_1 s_1 u_t}\right) \qquad \text{for all } t\geq 1. \tag{2.7}$$

By looking at configurations of paths in the resticted lattice $\Lambda_{1,0}$, (2.7) can be regarded as a boundary condition, so that setting $\mathsf{m}_x^0 = 0$ a.s. for each $x\geq 2$ we produce the step Bernoulli boundary conditions, considered above (see Fig. 7a).

2.3 Fused transfer operator $\mathfrak{X}^{(J)}$

In Sect. 2.2 we considered the unfused Higher Spin Six Vertex Model and each horizontal edge of the lattice could be crossed by no more than one path. We see now how it is possible to exploit combinatorial properties of weights $\mathsf{L}_{\xi_x u_t, s_x}$ in order to take away this restriction. The strategy consists of collapsing together multiple horizontal lines of vertices.

Suppose we aim to allow up to J paths to travel an edge horizontally. For a given probability distribution P on $\{0,1\}^J$, consider the quantity

$$\sum_{\substack{\mathbf{h},\mathbf{h}' \in \{0,1\}^J \\ |\mathbf{h}|=j, |\mathbf{h}'|=j',}} P(\mathbf{h}) \prod_{k=1}^{J} \mathsf{L}_{\xi_x u_k, s_x}(i_{k-1}, h_k | \, i_k, h'_k), \tag{2.8}$$

where

$$i_k = i_{k-1} + h_k - h'_k, \qquad i_0 = i, \qquad i_J = i' \qquad \text{and} \qquad i + j = i' + j'. \tag{2.9}$$

Naturally, (2.8) is the probability that in a column of J vertices i paths enter from below, i' exit from above and, independently on their arrangement j of them enter from the left and j' of them exit from the right. We ask under what conditions on P, expression (2.8) can be written in the form

$$\widetilde{P}(j) \mathsf{L}^{(J)}_{\xi_x u, s_x}(i, j | \, i', j'), \tag{2.10}$$

for some probability distribution $\widetilde{P}$ on $\{0, \dots, J\}$ and some weight $\mathsf{L}^{(J)}_{\xi_x u, s_x}$. A possible answer is essentially contained in the following

Definition 2.1 A probability distribution P on $\{0,1\}^J$ is said to be *q-exchangeable* if it is of the form

$$P(\mathrm{h}) = \widetilde{P}(|\mathrm{h}|) \frac{q^{\sum_{k=1}^{J} h_k (k-1)}}{Z_J(|\mathrm{h}|)}, \tag{2.11}$$

where $\widetilde{P}$ is a probability distribution on $\{0, \dots, J\}$, $|\mathrm{h}| = h_1 + \cdots + h_J$ and

$$Z_J(j) = q^{\frac{j(j-1)}{2}} \frac{(q;q)_J}{(q;q)_j (q;q)_{J-j}}.$$

In the previous definition we made use of the common notation of q-Pochhammer symbol $(x,q)_n$, whose definition is recalled in "Appendix A". For the next result we set the parameters $(u_1, \dots, u_J) = (u, qu, \dots, q^{J-1}u)$.

Proposition 2.2 ([18], Proposition 5.4) *Fix non-negative integers i, i' and let P be a q-exchangeable probability distribution on $\{0,1\}^J$. Then also*

$$P'(\mathrm{h}') = \sum_{\mathrm{h}\in\{0,1\}^J} P(\mathrm{h}) \prod_{k=1}^{J} L_{\xi_x q^{k-1}u, s_x}(i_{k-1}, h_k |\, i_k, h'_k) \tag{2.12}$$

is q-exchangeable. Here numbers i_k are defined as in (2.9).

A way to rephrase result of Proposition 2.2 is to say that, in expression (2.12), assuming $|\mathbf{h}'| = j'$ and assuming that the probability distribution P has the form (2.11), then we can write

$$P'(\mathbf{h}') = \widetilde{P}(i'+j'-i)\mathsf{L}^{(J)}_{\xi_x u, s_x}(i, i'+j'-i |\, i', j') \frac{q^{\sum_{k=1}^{J} h'_k(k-1)}}{Z_J(j')} \tag{2.13}$$

where the exact form of the probability weight $\mathsf{L}^{(J)}_{\xi_x u, s_x}$ can be also computed and it is given below in (2.17). This can be easily exploited to collapse together J different rows of vertices. Assume that in the leftmost column paths enter with q-exchangeable distribution P (see Fig. 6), then

$$\begin{aligned} &\sum_{\substack{\mathbf{h}\in\{0,1\}^J \\ |\mathbf{h}|=j_0}} \sum_{\nu_1,\nu_2,\dots,\nu_{J-1}} P(\mathbf{h}) \mathfrak{X}_u(h_1; \lambda \to \nu_1) \cdots \mathfrak{X}_{q^{J-1}u}(h_J; \nu_{J-1} \to \lambda') \\ &\quad = \widetilde{P}(j_0) \prod_{x\geq 1} \mathsf{L}^{(J)}_{\xi_x u, s_x}(m_x, j_{x-1} |\, m'_x, j_x), \end{aligned} \tag{2.14}$$

where $\lambda = 1^{m_1}2^{m_2}\cdots$ and $\lambda' = 1^{m'_1}2^{m'_2}\cdots$ are symbols indicating configurations entering and exiting respectively the bottom and the top row. In analogy with expression (2.4), we formally define the fused transfer operator

$$\begin{aligned} &\mathfrak{X}^{(J)}_{u_{t+1}}(j_0^{t+1}; \lambda(t) \to \lambda(t+1)) \\ &\quad = \mathbb{P}(\mathsf{j}_0^{t+1} = j_0^{t+1}) \prod_{x\geq 1} \mathsf{L}^{(J)}_{\xi_x u_{t+1}, s_x}(m_x^t, j_{x-1}^{t+1} |\, m_x^{t+1}, j_x^{t+1}), \end{aligned} \tag{2.15}$$

where again, numbers j_x^{t+1} satisfy the conservation law

$$m_x^t + j_{x-1}^{t+1} = m_x^{t+1} + j_x^{t+1}. \tag{2.16}$$

Here, with a little abuse of notation, we assume that j_0^{t+1} is a random variable taking values in the set $\{0, 1, \dots, J\}$. It is clear that, thanks to (2.16), once we specify $\{j_0^t\}_{t\geq 1}$ and $\{m_x^t\}_{x,t\geq 1}$, we automatically obtain quantities $\{j_x^t\}_{x,t\geq 1}$, which we interpret as occupancy numbers of collapsed horizontal edges. In this way, definition of random variables $\mathsf{m}_x^t, \mathsf{j}_x^t$ given in (1.2), (1.3) has been extended to include the case where multiple paths can share horizontal edges.

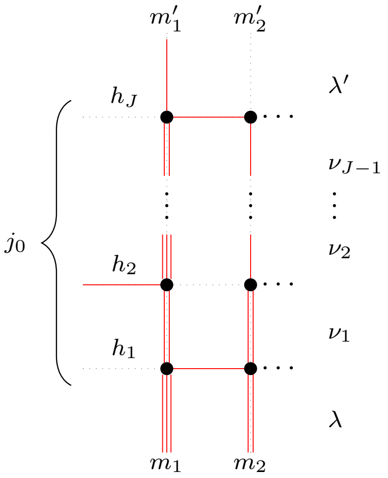


Fig. 6 A schematic representation of the fusion of J rows reported in (2.14). Here from the leftmost J vertices the total number of entering paths is $j_0 = h_1 + h_2 + \cdots + h_J$. Symbols λ, λ' indicate the initial and final configurations, whereas intermediate ones are described by symbols $\nu_1, \dots, \nu_{J-1}$

The closed expression of weights $\mathsf{L}^{(J)}$ ([18], Formula 5.6 or [30], Theorem 3.15) is considerably more involved than that of the $J = 1$ case presented in Table 1 and it is

$$\mathsf{L}^{(J)}_{u,s}(i_1, j_1 | \, i_2, j_2) = \mathbb{1}_{i_1+j_1=i_2+j_2} \frac{(-1)^{i_1} q^{\frac{1}{2} i_1 (i_1+2j_1-1)} u^{i_1} s^{j_1+j_2-i_2} (us^{-1}; q)_{j_2-i_1}}{(q;q)_{i_2} (su;q)_{i_2+j_2} (q^{J+1-j_1}; q)_{j_1-j_2}}$$
$$\times \, {}_4\bar{\phi}_3 \left(\begin{matrix} q^{-i_2}, \; q^{-i_1}, \; suq^J, \quad qs/u \\ s^2, \quad q^{1+j_2-i_1}, \; q^{J+1-i_2-j_2} \end{matrix} \middle| q, \; q \right). \qquad (2.17)$$

Here the function ${}_4\bar{\phi}_3$ is a particular instance of the regularized q-hypergeometric series defined in "Appendix A, (A.10)".

In expression (2.17), we notice the rational dependence of $\mathsf{L}^{(J)}_{u,s}$ on q^J, so that one can provide an analytic continuation in this parameter. Substituting q^J with a generic complex number not belonging to the set $q^{\mathbb{Z}}$, we see that the fused weights are well defined for each choice of j_1, j_2, so that paths of the Higher Spin Six Vertex Model no more undergo any limitation as far as number of horizontal edges they can simultaneously cross.

As explained in the last paragraph of Sect. 2.2, we can decouple boundary random variables $\{\mathsf{j}_0^t\}_t$ and $\{\lambda(t)\}_t$ by setting $\mathsf{m}_1^0 = \infty$ a.s. This is still true after the fusion of rows procedure and what we obtain is a fused version of the step Bernoulli boundary conditions for the Higher Spin Six Vertex Model on the restricted lattice $\Lambda_{1,0}$. In this case, to express the probability distribution of random variables j_1^t we need the following

Proposition 2.3 *Consider $Y_1, \dots Y_J$ independent Bernoulli random variables respectively of mean $p/(1+p), qp/(1+qp), \dots, q^{J-1}p/(1+q^{J-1}p)$, with $p \in \mathbb{R}_{>0}$. Then, defining $X_J = Y_1 + \cdots + Y_J$ we have $X_J \sim q\mathrm{NB}(q^{-J}, -q^J p)$.*

Proof For any k in the set $\{0, 1, \dots, , J\}$, we have

$$\mathbb{P}(X_J = k) = \frac{p^k}{(-p;q)_J} \sum_{\substack{\mathsf{h}\in\{0,1\}^J: \\ |\mathsf{h}|=k}} q^{\sum_{i=1}^J (i-1)h_i}. \tag{2.18}$$

The sum involving powers of q in the right hand side of (2.18) is easily expressed as

$$\sum_{\substack{\mathsf{h}\in\{0,1\}^J: \\ |\mathsf{h}|=k}} q^{\sum_{i=1}^J (i-1)h_i} = (-1)^k q^{Jk} \frac{(q^{-J};q)_k}{(q;q)_k}, \tag{2.19}$$

as a result of the two different notable expansions for the q-Pochhammer symbol

$$(z;q)_J = \sum_{k=0}^J z^k (-1)^k \sum_{\substack{\mathsf{h}\in\{0,1\}^J: \\ |\mathsf{h}|=k}} q^{\sum_{i=1}^J (i-1)h_i} \qquad \text{and}$$

$$(z;q)_J = \sum_{k=0}^J \left(zq^J\right)^k \frac{(q^{-J};q)_k}{(q;q)_k}.$$

Combining (2.18) and (2.19) we conclude the proof. □

As we already made clear, the fusion of rows procedure consists in taking the Higher Spin Six Vertex Model as defined in Sects. 2.1, 2.2, specializing spectral parameters in geometric progressions of ratio q and tracing out over configurations of paths sharing the same number of occupied horizontal edges at a each column of vertices. When we do so, after the choice $\mathsf{m}_1^0 = \infty$ a.s., recalling (2.7) and utilizing result of Proposition 2.3, we obtain

$$\mathbb{P}(\mathsf{j}_1^t = k) = \left(q^J s_1 \xi_1 u_t\right)^k \frac{\left(q^{-J};q\right)_k}{(q;q)_k} \frac{\left(q^J \xi_1 s_1 u_t;q\right)_\infty}{(\xi_1 s_1 u_t;q)_\infty}. \tag{2.20}$$

Therefore we can conclude that in the Higher Spin Six Vertex Model, after a fusion of J rows, when in the leftmost column of vertices flow infinitely many paths, $\mathsf{j}_1^1, \mathsf{j}_1^2, \mathsf{j}_1^3, \dots,$ become mutually independent q-negative binomial random variables respectively with parameters $(q^{-J}, q^J s_1 \xi_1 u_t)$ for $t = 1, 2, \dots$. The special case when $\mathsf{m}_0^x = 0$ a.s. for $x \geq 2$ generates the step q-negative binomial boundary conditions introduced in Sect. 2.1 (see Fig. 7b).

We close this subsection remarking that the history of expression (2.17) is actually longer than how it might seem from reading this brief overview of results. More complicated expressions for a quantity analogous to the transition matrix $\mathsf{L}^{(J)}$ had been known in the context of quantum integrable systems for almost three decades since [45]. Relatively compact expressions such as that presented in (2.17) became available only in more recent times after the work [49]. A detailed probabilistic derivation of the stochastic weight $\mathsf{L}^{(J)}$ can be found in [30].

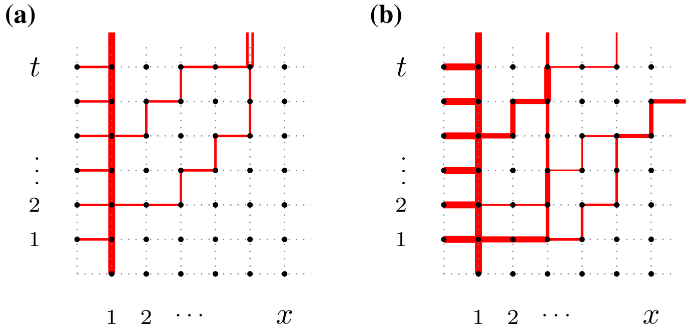


Fig. 7 **a** A possible configuration of paths in the Higher Spin Six Vertex Model on $\Lambda_{0,0}$ and $J = 1$. Here boundary conditions are taken as $\mathsf{m}_1^0 = \infty$ and $\mathsf{j}_0^t = 1$ $\mathsf{m}_x^0 = 0$ a.s. for all $x, t > 0$. Such choice produces the step Bernoulli boundary conditions in $\Lambda_{1,0}$. **b** A possible configuration of paths in the Higher Spin Six Vertex Model on $\Lambda_{0,0}$ and $J > 1$. The thickness of red traits indicates multiple occupations at edges. Here boundary conditions are takes as $\mathsf{m}_1^0 = \infty$ and $\mathsf{j}_0^t = J$ $\mathsf{m}_x^0 = 0$ a.s. for all $x, t > 0$. Such choice produces the step q-negative binomial boundary conditions in $\Lambda_{1,0}$ (color figure online)

2.4 Observables in the higher spin six vertex model with step boundary conditions

After discussing the fusion procedure in Sect. 2.3 we come back to the unfused model, with $J = 1$, defined in the lattice $\Lambda_{0,0}$. Recall that with step boundary conditions (2.1), no path enters from the x axis and each vertex on the t axis has a path entering to its left. When this is the case $\lambda(t)$ is at each level associated with a signature in the set

$$\mathrm{Sign}_t^{>0} = \{\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_t > 0) |\ \nu_i \in \mathbb{Z}_{>0}\}.$$

A relevant quantity for which we possess exact formulas is given in the following

Definition 2.4 Consider the Higher Spin Six Vertex Model on $\Lambda_{0,0}$ with step boundary conditions. We refer to the quantity

$$\mathfrak{h}(x,t) = \sum_{y \geq x} \mathsf{m}_y^t = \#\{j|\ \lambda_j(t) \geq x\} \tag{2.21}$$

as the *height* function $\mathfrak{h}$ at the vertex (x, t) (for an example see Fig. 8).

The two different height functions $\mathfrak{h}$ and $\mathcal{H}$, defined respectively in (2.21) and (1.12) are clearly related quantities. In particular they are connected by the trivial relation

$$\mathcal{H}(x,t) = \mathfrak{h}(x+1,t).$$

We like to keep their notation distinct as $\mathfrak{h}$ only refers to a model with step boundary conditions, while the definition of $\mathcal{H}$ also makes sense when paths emanate randomly from the horizontal axis.

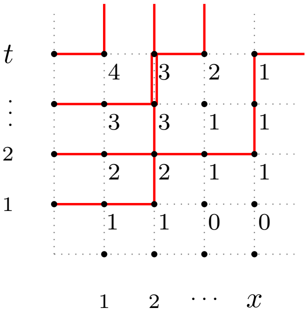


Fig. 8 A possible set of up right paths in the Higher Spin Six Vertex Model with step boundary conditions. The numbers reported next to each vertex are the values of the height function $\mathfrak{h}$ defined in (2.21)

In [30] the height function is thought of as the position of a specific particle evolving in a certain totally asymmetric exclusion process, or as the current of a totally asymmetric zero range process. Authors derive a closed expression for the multi point q-moments in case of step initial conditions (for the exclusion process), exploiting Markov self duality of the Higher Spin Six Vertex Model. The same result is achieved in [18] following a rather algebraic approach.

What follows is a nested contour integral expression for the single point q-moments of the height function of the (unfused) Higher Spin Six Vertex Model with step boundary conditions.

Proposition 2.5 *Consider the unfused Higher Spin Six Vertex Model and assume conditions*[1] (1.8)*, with* $q \neq 0$. *Let*

$$\max_{i=1,\dots,t}\{u_i^{-1}\} < q \min_{i=1,\dots,t}\{u_i^{-1}\} \tag{2.22}$$

and let products $\xi_i s_i$ *be strictly positive for all* i. *Then for all* $l \in \mathbb{Z}_{\geq 0}$ *and* $x \in \mathbb{Z}_{\geq 0}$, *we have*

$$\begin{aligned}\mathbb{E}\left(q^{l\mathfrak{h}(x+1,t)}\right) = \frac{q^{\frac{l(l-1)}{2}}}{(2\pi i)^l} \oint_{\overline{\gamma_1}[\overline{\mathbf{U}}|1]} \cdots \oint_{\overline{\gamma_l}[\overline{\mathbf{U}}|l]} \prod_{1\leq A<B\leq l} \frac{z_A - z_B}{z_A - q z_B} \\ \times \prod_{i=1}^{l} \left(\prod_{j=1}^{x} \frac{\xi_j - s_j z_i}{\xi_j - s_j^{-1} z_i} \prod_{j=1}^{t} \frac{1 - q u_j z_i}{1 - u_j z_i} \frac{dz_i}{z_i} \right)\end{aligned} \tag{2.23}$$

where the variable z_i *is integrated along the path* $\overline{\gamma_i}[\overline{\mathbf{U}}|i] = \gamma_i[\overline{\mathbf{U}}] \cup r^i C_0$ *where* $r > q^{-1}$, C_0 *is a small contour around 0 and* $\overline{\gamma_i}[\overline{\mathbf{U}}]$ *encircles the set* $\{u_1^{-1}, \dots, u_t^{-1}\}$, $q^{-1}\overline{\gamma_{i-1}}[\overline{\mathbf{U}}]$ *and no other singularities. Moreover* $q\overline{\gamma_i}[\overline{\mathbf{U}}]$ *doesn't intersect any* $r^j C_0$ *and* $r^l C_0$ *doesn't contain any point of the set* $\{\xi_j s_j\}_j$ *(Fig. 9).*

Proof This Proposition is a consequence of Corollary 9.9 of [18] (with $J = 1$), where authors state nested contour integral formulas for the multi-point q-moments of the

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[1] In [18] authors consider parameters s_i, u_j which have opposite sign compared to our choice (1.8). This is just a convention and hence not a problem, as the stochastic weights L depend on $s_i u_j$ and s_i^2.

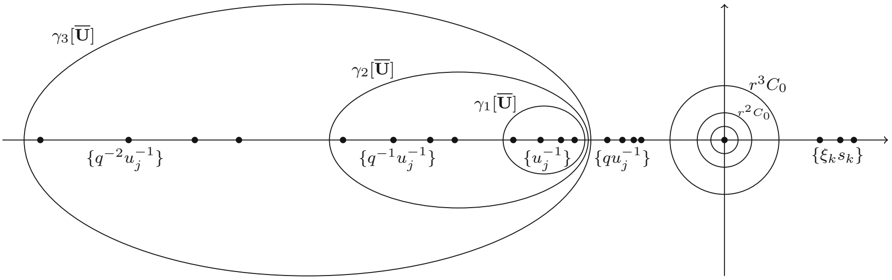

Fig. 9 An example of nested integration contours for (2.23) in the case $l = 3$

higher spin six vertex model $\mathbb{E}\left(\prod_{i=1}^{l} q^{\mathfrak{h}(x_i,t)}\right)$. These reduce to the single-point q-moments in the left hand side of (2.23) after setting $x_i = x + 1$ for $i = 1, \ldots, l$ (we precise that in [18] the height $\mathfrak{h}(x, t)$ is denoted by the slightly different symbol $\mathfrak{h}_\nu(x)$ and $t = n$). Moreover, in the notation of [18] contours $\gamma_j[\overline{\mathbf{U}}]$ are denoted by $\gamma_j^-[\overline{\mathbf{u}}]$, the circle around the origin C_0 is c_0 and the union $\overline{\gamma}_j[\overline{\mathbf{U}}|j] = \gamma_j[\overline{\mathbf{U}}] \cup r^j C_0$ is $\gamma_j^-[\overline{\mathbf{u}}|j] = \gamma_j^-[\overline{\mathbf{u}}] \cup r^j c_0$.

To prove this fact, first let us assume that

$$\min_{j=1,\ldots,x}\{s_j/\xi_j\} > q \max_{j=1,\ldots,x}\{s_j/\xi_j\}, \tag{2.24}$$

$$\min_{j=1,\ldots,x}\{1/(s_j\xi_j)\} > \max_{j=1,\ldots,x}\{s_j/\xi_j\}, \tag{2.25}$$

$$0 < s_j < 1, \qquad 0 < \xi_j < \infty, \qquad \text{for all } j = 1, \ldots, x. \tag{2.26}$$

We can match conditions on parameters given so far with the hypothesis of Corollary 9.9 of [18] (modulo the change sign of parameters u_i, s_j). Namely (2.26) and (1.8) with $q \neq 0$ correspond to (5.1), (5.2) of [18],[2] (2.24), (2.25) correspond to (7.4) of [18] and (2.22) corresponds to (8.18) of [18] (with $J = 1$). This implies the statement of Proposition 2.5 under assumptions (2.24), (2.25), (2.26).

We now come to remove assumptions (2.24), (2.25) and to include the case when $s_j \to 0, \xi_j \to \infty$ simultaneously keeping $\xi_j s_j$ a positive finite quantity. We do this through an analytic continuation argument. First we notice that the left hand side of (2.23) is a finite sum of a finite product of weights $\mathsf{L}_{\xi_i u_j, s_i}$ and hence it is a rational function of all parameters ξ_i, s_i, u_j. This is because the height function $\mathfrak{h}(x + 1, t)$ only depends on path configurations on the finite lattice $\{0, \ldots, x\} \times \{0, \ldots, t\}$ and for step boundary conditions there exists only a finite number of such configurations. The right hand side of (2.23) is also a rational function of ξ_i, s_i, u_j since the integrand function depends rationally on these parameters. Therefore equality (2.23) holds whenever weights $\mathsf{L}_{\xi_i u_j, s_j}$ are all positive quantities and at the same time one can construct

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[2] In (5.1), (5.2) of [18], authors think of s_j and ξ_j as infinite sequences. They require that these parameters are uniformly (in j) bounded away from the boundaries of their domain of definition. When it comes to stating formulas like (2.22) the clarification "uniformly" is redundant since only finitely many of the s_j, ξ_j are used.

integration contours $\overline{\gamma}_j[\overline{\mathbf{U}}|j]$. At this point it is easy to see that hypothesis made in the statement of the Proposition fulfill these conditions. This completes the proof. □

The next Corollary is a reformulation of Proposition 2.5 with a different choice of integration contours in (2.23). It is stated for the unfused model, whereas its generalization to the fused model is discussed in Remark 2.7.

Corollary 2.6 *For the unfused Higher Spin Six Vertex Model assume conditions* (1.8) *and let*

$$q \sup_i \{\xi_i s_i\} < \inf_i \{\xi_i s_i\}.$$

Then we have

$$\begin{aligned}\mathbb{E}\left(q^{l\mathfrak{h}(x+1,n)}\right) = \frac{(-1)^l q^{\frac{l(l-1)}{2}}}{(2\pi \mathrm{i})^l} \oint_{\overline{C}[\Xi\mathbf{S}|1]} \cdots \oint_{\overline{C}[\Xi\mathbf{S}|l]} \prod_{1 \le A < B \le l} \frac{z_A - z_B}{z_A - q z_B} \\ \times \prod_{i=1}^{l} \Big(\prod_{j=1}^{x} \frac{\xi_j s_j - s_j^2 z_i}{\xi_j s_j - z_i} \prod_{j=1}^{n} \frac{1 - q u_j z_i}{1 - u_j z_i} \frac{dz_i}{z_i} \Big)\end{aligned} \tag{2.27}$$

where the integration contour of z_i is $\overline{C}[\Xi\mathbf{S}|i]$ and is the disjoint union of two curves $C[\Xi\mathbf{S}|i]$ and $r^{i-1}\partial D$. Here $C[\Xi\mathbf{S}|i]$ is a counterclockwise contour encircling the set $\{\xi_i s_i\}_i$ and $qC[\Xi\mathbf{S}|i+1]$, but not 0 or any number u_i^{-1}. The disk D is centered at 0 and it is sufficiently large to contain every $C[\Xi\mathbf{S}|i]$ and the set $\{u_i^{-1}\}_i$. The coefficient r is bigger than q^{-1} and $r^{i-1}\partial D$ is clockwise oriented. A visualization of such contours is given in Fig. 10.

Proof We first assume also hypothesis (2.22) from Proposition 2.5. In this case for the q-moments of the height functions we possess the integral representation (2.23). We use inductively the residue theorem to turn the integrations around contours $\overline{\gamma_i}[\overline{\mathbf{U}}|i]$ in (2.23) into integrations around $\overline{C}[\Xi\mathbf{S}|i]$. Let's start with the most external contour $\overline{\gamma_l}[\overline{\mathbf{U}}|l]$. We see that the integrand in the rhs of (2.23) has in the variable z_l poles at

$$q^{-1}z_1, \dots, q^{-1}z_{l-1}, \xi_1 s_1, \dots, \xi_x s_x, u_1^{-1}, \dots, u_t^{-1}, 0, \infty,$$

so that since $\overline{\gamma_l}[\overline{\mathbf{U}}|l]$ only leaves outside $\xi_1 s_1, \dots, \xi_x s_x$ and ∞ and we have

$$\oint_{\overline{\gamma_l}[\overline{\mathbf{U}}|l]} = -\oint_{\overline{C}[\Xi\mathbf{S}|l]}.$$

We move now to the integration in z_{l-1}. Here poles are at

$$q z_l, q^{-1}z_1, \dots, q^{-1}z_{l-2}, \xi_1 s_1, \dots, \xi_x s_x, u_1, \dots, u_t, 0, \infty$$

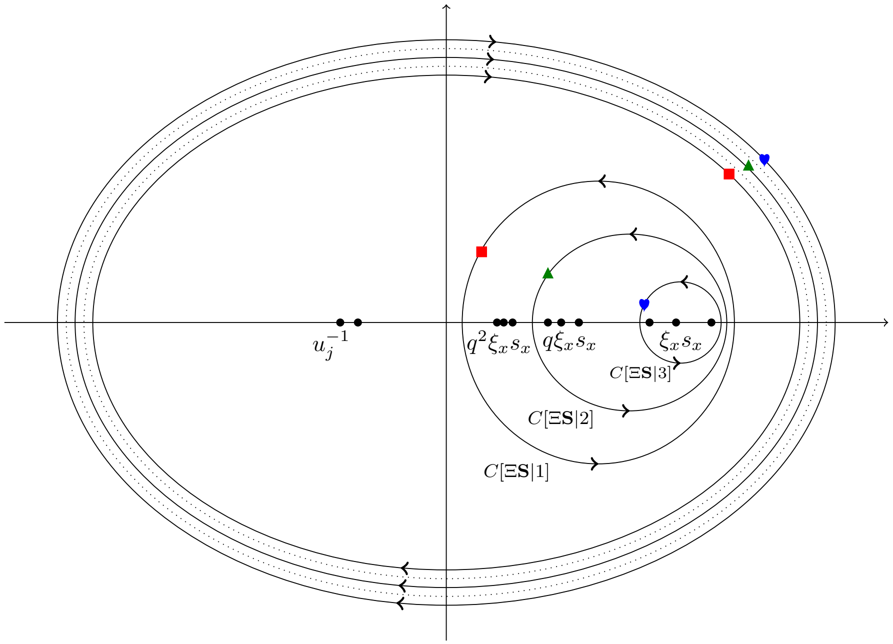


Fig. 10 An example of nested integration contours in (2.27) for the case $l = 3$. We see the integration contours for z_1, z_2, z_3 labeled respectively with 🟥, ▲, ♥. Dotted lines between the external clockwise oriented contours ∂D, $r\partial D$, $r^2\partial D$ are the shifted paths $q^{-1}\partial D$, $q^{-2}\partial D$ (color figure online)

and $\overline{\gamma_{l-1}}[\overline{\mathbf{U}}|l-1]$ leaves out $\infty, \xi_1 s_1, \dots, \xi_x s_x$ and the shifted contours $qC[\Xi\mathbf{S}|l]$, $qr^{l-1}\partial D$ where qz_l lies. Since $C[\Xi\mathbf{S}|l-1]$ contains $\xi_1 s_1, \dots, \xi_x s_x$, $qC[\Xi\mathbf{S}|l]$ and no other pole and $r^{l-2}\partial D$ encircles $qr^{l-1}\partial D$ and ∞ we also get

$$\oint_{\overline{\gamma_{l-1}}[\overline{\mathbf{U}}|l-1]} = -\oint_{\overline{C}[\Xi\mathbf{S}|l-1]}.$$

By repeating the same procedure for all the remaining integration variables we can prove (2.27), under the hypothesis (2.22). To remove this last condition we use an analytic continuation argument. In fact the q-moments in the left hand side of (2.27) are rational functions of the parameters u_j, because so are the stochastic weights $\mathsf{L}_{\xi_i u_j, s_i}$. Also, the integral expression in the right hand side of (2.27) depends analytically on the u_j's as long as they stay negative, because they do not cross the integration contours. Therefore (2.27) holds for any $u_j < 0$ and this completes the proof. □

Remark 2.7 Expression for q-moments (2.27) does not require any hypothesis on the spectral parameters u_i other than $u_i < 0$, as stated in (1.8). In particular, in (2.27) we can also take the u_i's in geometric progressions of ratio q therefore obtaining the same statement for general spin number J. In such case the q-moments are expressed as in formula (2.27) with the only change

$$\frac{1 - q u_j z_i}{1 - u_j z_i} \to \frac{1 - q^J u_j z_i}{1 - u_j z_i}. \tag{2.28}$$

This is in contrast with expression (2.23), where the further assumption (2.22) was made and therefore the general J case (that was still stated in Corollary 9.9 of [18]) would not follow from analytic continuation.

Following a consolidated approach developed in [13,15], in the particular case of step initial conditions, by using the result of Proposition 2.5 or Corollary 2.6, one can obtain a determinantal expression for the quantity

$$\mathbb{E}\left(\frac{1}{(\zeta q^{\mathfrak{h}(x+1,t)}; q)_\infty}\right),$$

which is known to be the q-Laplace transform of the probability mass function of $\mathfrak{h}(x+1,t)$. This type of result has proven to be fruitful (see [34]) when it comes to the study of asymptotics of $\mathfrak{h}(x+1,t)$ as x and t go to infinity.

We remark that results like those of Proposition 2.5 or Corollary 2.6 essentially hold only for step boundary conditions, which at the current state are the only boundary conditions exhibiting a nice enough underlying algebraic structure to derive exact formulas for observables. It is nonetheless possible, after a suitable choice of parameters Ξ, $\mathbf{S}$, $\mathbf{U}$, to produce other initial conditions, as those considered in [53] or [2,25]. In particular we will use a similar approach to that developed in [2] to produce and study the stationary model.

3 *q*-Whittaker processes

In this section we give a brief review on q-Whittaker processes and present main results which will be used in the remainder of the paper.

3.1 Macdonald processes and *q*-Whittaker processes

The q-Whittaker processes and the q-Whittaker measure have been first introduced in [13] as particular cases of the more general Macdonald processes. This is a family of measures on the Gelfand–Tsetlin[3] cone $\mathbb{GT}_n^{\geq 0}$ (Fig. 11), the set of sequences of partitions of integers

$$\emptyset \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(n)},$$

where every $\lambda^{(i)}$ is an element of

$$\mathrm{Part}_i^{\geq 0} = \{\mu = (\mu_1 \geq \cdots \geq \mu_i \geq 0) | \ \mu_j \in \mathbb{Z}_{\geq 0}\}.$$

---

[3] The notation $\mathbb{GT}^{\geq 0}$ refers to Gelfand–Tsetlin cones of partitions. One can define the same object with generic signatures instead of partitions. We refer to the two sided Gelfand–Tsetlin cone with the notation $\mathbb{GT}$ as in (3.4).

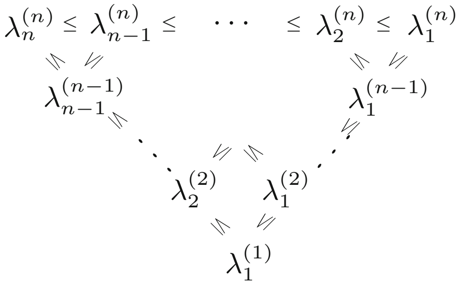


Fig. 11 A triangular array in the Gelfand–Tsetlin cone $\mathbb{GT}_n^{\geq 0}$

The interlacing relation between two partitions λ, μ, denoted with $\mu \prec \lambda$, means that

$$\lambda_{k+1} \leq \mu_k \leq \lambda_k \quad \text{for each } k.$$

The Macdonald process is defined as the measure

$$\mathbb{M}\big(\emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)}\big) = \frac{1}{\Pi(\mathbf{a}, \rho)} \prod_{i=1}^{n} P_{\lambda^{(i)}/\lambda^{(i-1)}}(a_i; q, t) Q_{\lambda^{(n)}}(\rho; q, t),$$

where P and Q are Macdonald functions ([48], Chapter IV) and Π is a normalization constant and its value can be expressed through the known Cauchy sums for symmetric functions. Here, q, t are parameters in [0,1), $\mathbf{a} = (a_1, \dots a_n)$ denotes a set of numerical values at which Macdonald polynomials are evaluated, whereas ρ can be a generic complex algebra homomorphism on the algebra of symmetric functions. In this sense the quantity $\Pi(\mathbf{a}, \rho)$ can be thought as the generating function of functions $Q_\lambda(\rho; q, t)$.

The Macdonald measure is a particular case of the Macdonald process, obtained by projecting the measure $\mathbb{M}$ on the last partition $\lambda^{(n)}$. The branching rule

$$\sum_{\varkappa} P_{\varkappa/\mu}(a_1, \dots, a_{k-1}; q, t) P_{\lambda/\varkappa}(a_k; q, t) = P_{\lambda/\mu}(a_1, \dots, a_k; q, t),$$

of Macdonald functions allows us to write

$$\begin{aligned}\mathbb{M}(\lambda) &= \sum_{\lambda^{(1)}, \dots, \lambda^{(n-1)}} \mathbb{M}\Big(\emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n-1)} \prec \lambda\Big) \\ &= \frac{1}{\Pi(\mathbf{a}, \rho)} P_\lambda(\mathbf{a}; q, t) Q_\lambda(\rho; q, t).\end{aligned}$$

The definition itself of the Macdonald functions depends on two parameters q, t and letting these parameters vary one can obtain numerous other families of symmetric functions such as the Schur polynomials or the Hall–Littlewood functions. The q-Whittaker functions arise when we set $t = 0$. An exact expression for the one variable skew q-Whittaker polynomials $P_{\lambda/\mu}$, which we denote dropping the explicit dependence on q, is

$$P_{\lambda/\mu}(a) = \prod_{i=1}^{N} a^{\lambda_i} \prod_{i=1}^{N-1} a^{-\mu_i} \binom{\lambda_i - \lambda_{i+1}}{\lambda_i - \mu_i}_q ,$$

and the generic q-Whittaker polynomial is

$$P_\lambda(\mathbf{a}) = \sum_{\substack{\lambda_i^{(k)}:1\le i\le k\le N-1 \\ \lambda_{i+1}^{(k+1)}\le\lambda_i^{(k)}\le\lambda_i^{(k+1)}}} \prod_{j=1}^{N} P_{\lambda^{(j)}/\lambda^{(j-1)}}(a_j),$$

where $\lambda^{(n)}$ is meant to be λ. In case the Macdonald measure is considered with $t = 0$ we use the notation

$$\mathbb{W}(\lambda) = \frac{1}{\Pi(\mathbf{a}, \rho)} P_\lambda(\mathbf{a}) Q_\lambda(\rho). \tag{3.1}$$

Macdonald functions possess an important orthogonality property with respect to the so called *torus scalar product*. For $t = 0$, it is defined as the n-fold integral

$$\langle f, g\rangle_n = \int_{\mathbb{T}^n} \prod_{j=1}^{n} \frac{dz_j}{z_j} f\big(\mathbf{z}^{-1}\big) g(\mathbf{z}) m_n^q(\mathbf{z}),$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the complex circle and

$$m_n^q(\mathbf{z}) = \frac{1}{(2\pi \mathrm{i})^n n!} \prod_{1\le i\ne j\le n} (z_i/z_j; q)_\infty \tag{3.2}$$

is the q-Sklyanin weight. For the q-Whittaker polynomials this relation reads as

$$\langle P_\lambda, P_\mu\rangle_n = c_\lambda \delta_{\lambda,\mu}$$

and c_λ is some non zero constant. This property can be exploited to give an indirect definition of the dual q-Whittaker function Q as

$$\begin{aligned} Q_\lambda(\rho) &= \frac{1}{c_\lambda} \Big\langle P_\lambda, \sum_\mu P_\mu Q_\mu(\rho) \Big\rangle_n \\ &= \frac{1}{c_\lambda} \langle P_\lambda, \Pi(\bullet, \rho)\rangle_n . \end{aligned} \tag{3.3}$$

An additional remarkable structural feature of Macdonald polynomials is the shifting property,

$$P_{\lambda+r^n}(\mathbf{x}; q, t) = (x_1 \cdots x_n)^r P_\lambda(\mathbf{x}; q, t),$$

where r is a non-negative integer and $\lambda + r^n = (\lambda_1 + r \geq \cdots \geq \lambda_n + r)$. Combining this shifting invariance with (3.3), we are allowed to extend the notion of q-Whittaker functions P_λ and Q_λ to the set of signatures

$$\mathrm{Sign}_i = \{\lambda = (\lambda_1 \geq \cdots \geq \lambda_i) |\, \lambda_j \in \mathbb{Z}\}.$$

This observation is used in [39] to extend the definition of the q-Whittaker processes on the two sided Gelfand–Tsetlin cone

$$\mathbb{GT}_n = \left\{ \{\lambda_i^{(k)}\}_{i=1,\dots,k}^{k=1,\dots n} \in \mathbb{Z}^{\binom{n+1}{2}} |\, \lambda_{i+1}^{(k)} \leq \lambda_i^{(k-1)} \leq \lambda_i^{(k)} \right\}. \tag{3.4}$$

3.2 *q*-moments of the corner coordinate

Macdonald polynomials can be constructed as eigenfunctions of the Macdonald operator D_n ([48], Sect. VI.3-4), whose action on the generic symmetric function F is

$$D_n F(\mathbf{x}) = \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{x_i - t x_j}{x_i - x_j} F(x_1, \dots, x_{j-1}, q x_j, x_{j+1}, \dots x_n)$$

and the eigenvalue relative to the function P_λ is

$$\sum_{j=1}^{n} q^{\lambda_j} t^{n-j}.$$

As such eigenvalues are all distinct, the function P_λ is uniquely identified imposing the condition

$$P_\lambda(\mathbf{x}; q, t) = m_\lambda(\mathbf{x}) + \sum_{\mu < \lambda} C_{\lambda,\mu} m_\mu(\mathbf{x}),$$

where m_λ is the monomial symmetric function, the summation in the right hand side is taken over all partitions $\mu < \lambda$ in the lexicographic order and $C_{\lambda,\mu}$ are constants. In the simple case where the Macdonald operator acts on a product function $F(\mathbf{x}) = f(x_1) \cdots f(x_n)$, it can be written in the integral form

$$\frac{D_n F(\mathbf{x})}{F(\mathbf{x})} = \frac{(t-1)^{-1}}{2\pi \mathrm{i}} \oint \prod_{j=1}^{n} \frac{x_j - t z}{x_j - z} \frac{f(qz)}{f(z)} \frac{dz}{z},$$

being the integration contour a path encircling $x_1, \dots, x_n$ and no other singularity. This last formula has been used in [13] to determine the q-moments of the observable $\lambda_n^{(n)}$ in the q-Whittaker process. We report this result in the next

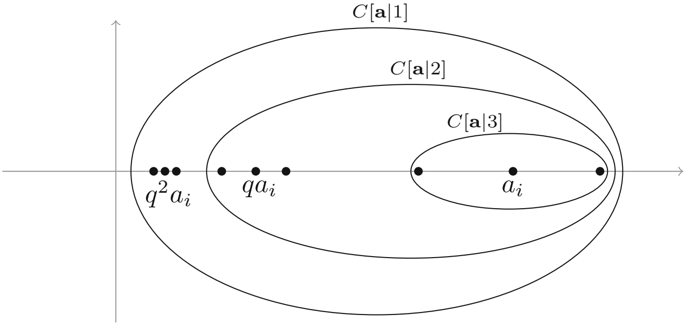


Fig. 12 A possible choice of integration contrours for (3.5) in the case $k = 3$

Proposition 3.1 ([13], Proposition 3.1.5) *Let ρ be the specialization of symmetric functions for which the normalization constant Π of the q-Whittaker measure* (3.1) *assumes the form*

$$\Pi(\mathbf{a}; \rho) = \prod_{i=1}^{n} e^{\tau a_i} \prod_{i \geq 1} \frac{(1 + \beta_i a_j)}{(\alpha_i a_j; q)_\infty},$$

for some sets of non-negative real numbers τ, $\{\alpha_i\}_{i\geq 1}$, $\{\beta_i\}_{i\geq 1}$ satisfying

$$\sum_{i\geq 1}(\alpha_i + \beta_i) < \infty, \qquad \sup_{i,j} |\alpha_i a_j| < 1.$$

Then, for any non-negative integer k we have

$$\begin{aligned}\mathbb{E}_{\mathbb{W}}\left(q^{k\lambda_n^{(n)}}\right) = \frac{(-1)^k q^{\binom{k}{2}}}{(2\pi \mathrm{i})^k} \oint_{C[\mathbf{a}|k]} \cdots \oint_{C[\mathbf{a}|1]} \prod_{1\leq i<j\leq k} \frac{z_i - z_j}{z_i - qz_j} \\ \times \prod_{j=1}^{k} \left(\prod_{m=1}^{n} \frac{a_m}{a_m - z_j} e^{\tau(q-1)z_j} \prod_{i\geq 1}(1 - \alpha_i z_j) \frac{1 + q\beta_i z_j}{1 + \beta_i z_j} \frac{dz_j}{z_j} \right),\end{aligned} \tag{3.5}$$

where $C[\mathbf{a}|j]$ is the integration contour for the complex variable z_j and contains $a_1, \ldots, a_n$, each shifted contour $qC[\mathbf{a}|l]$ for $l > j$ and no other pole of the integrand (Fig. 12).

Naturally, under some suitable assumption on the growth of the q-moments, they completely determine the distribution of $\lambda_n^{(n)}$ as they are generated by the q-Laplace transform (see (A.15))

$$\mathbb{E}_{\mathbb{W}}\left(\frac{1}{(\zeta q^{\lambda_n^{(n)}}; q)_\infty}\right).$$

This is the case for expressions (3.5), stated only for step initial conditions, where all $q^{k\lambda_n^{(n)}}$ are positive quantities bounded above by 1.

So far we presented the exact expression of the q-moments of two different observables of two different stochastic processes. Despite the difference between the Higher Spin Six Vertex Model and the q-Whittaker processes, expression of the q-moments respectively of the height function $\mathfrak{h}$ and of the corner coordinate $\lambda_n^{(n)}$, given in (2.27) and (3.5) indeed show similarities. This fact was noticed before in [53], where authors described the correspondence between these two models under step Bernoulli boundary conditions for the Higher Spin Six Vertex Model. The matching reported in the following Proposition traces that of Theorem 4.11 of [53]. We consider the unfused model, granted that a generalization to the fused one can be given simply specializing spectral parameters in geometric progression of ratio q.

Proposition 3.2 *Set parameters of the unfused Higher Spin Six Vertex Model* $\Xi, \mathbf{S}, \mathbf{U}$ *to be as in Corollary 2.6. Moreover let the following bounds*

$$\sup_i \{\xi_i s_i\} < 1, \qquad \sup_i \{s_i/\xi_i\} < 1, \qquad \sup_{i,j} |\xi_i s_i u_j| < 1, \tag{3.6}$$

hold. Then, we have

$$\mathbb{E}_{\text{HS}}\left(q^{l\mathfrak{h}(x+1,n)}\right) = \mathbb{E}_{\mathbb{W}_{\Xi,\mathbf{S},\mathbf{U}}}\left(\left(q^{\lambda_x} + q^n \prod_{j=1}^{x} s_j^2\right)_q^l\right), \tag{3.7}$$

where $(a+b)_q^l = \sum_{k=0}^{l} \binom{l}{k}_q a^k b^{l-k}$ *and the q-Whittaker measure in the right hand side is given by*

$$\mathbb{W}_{\Xi,\mathbf{S},\mathbf{U}}(\lambda) = \prod_{i=1}^{x} \frac{\prod_{j=1}^{x}(s_i s_j \xi_i/\xi_j; q)_\infty}{\prod_{j=1}^{n}(1 - u_j \xi_i s_i)} P_\lambda(\xi_1 s_1, \dots, \xi_x s_x)$$
$$\times Q_\lambda(\underbrace{s_1/\xi_1, \dots, s_x/\xi_x}_{\alpha-\textit{specializations}}, \underbrace{-u_1, \dots, -u_n}_{\beta-\textit{specializations}}). \tag{3.8}$$

In (3.8) we made use of the common terminology of α-specializations and β-specializations, justified by the fact that the q-Whittaker measure $\mathbb{W}_{\Xi,\mathbf{S},\mathbf{U}}$ is obtained setting in Proposition 3.1 $\alpha_i = s_i/\xi_i$, $\beta_j = -u_j$, $\tau = 0$ and $a_i = \xi_i s_i$.

Proof Comparing expressions (3.5) and (2.27), after properly substituting parameters $\tau, \alpha_i, \beta_i, a_i$ we see that they only differ by the choice of integration contours. In fact in (2.27) the contours are $\overline{C}[\Xi\mathbf{S}|i] = C[\Xi\mathbf{S}|i] \cup r^{i-1}\partial D$ and we see that performing the integration over the large circle $r^{i-1}\partial D$ corresponds to the evaluation of the residue at $z_i = \infty$. In order to express (2.27) only in terms of integrals over $C[\Xi\mathbf{S}|i]$, we choose $1 \le i_1 < \dots < i_{l-k} \le l$ for a $k \in \{0, 1, \dots, l\}$ and for all variables z with indices in $\{i_1, \dots, i_{l-k}\}$ we evaluate the correspoding residue at ∞. A computation shows that for any choice of $i_1, \dots, i_{l-k}$ the residue is

$$\begin{aligned} & q^{\binom{l}{2}+(1-i_1)+\cdots+(1-i_{l-k})} \left(q^n \prod_{j=1}^{x} s_j^2 \right)^{l-k} \frac{(-1)^k}{(2\pi \mathrm{i})^k} \oint_{C[\Xi \mathbf{S}|1]} \\ & \cdots \oint_{C[\Xi \mathbf{S}|k]} \prod_{1 \le A < B \le k} \frac{z_A - z_B}{z_A - q z_B} \\ & \times \prod_{i=1}^{k} \left(\prod_{j=1}^{x} \frac{\xi_j s_j}{\xi_j s_j - z_i} (1 - s_j \xi_j^{-1} z_i) \prod_{j=1}^{n} \frac{1 - q u_j z_i}{1 - u_j z_i} \frac{d z_i}{z_i} \right) \\ & = q^{\binom{l}{2}+(1-i_1)+\cdots(1-i_{l-k})-\binom{k}{2}} \left(q^n \prod_{j=1}^{x} s_j^2 \right)^{l-k} \mathbb{E}_{\mathbb{W}_{\Xi,\mathbf{S},\mathbf{U}}} \big(q^{k \lambda_x} \big). \end{aligned} \tag{3.9}$$

In the last equality we used the fact that, assuming the bounds (3.6), the q-moments of λ_x in the q-Whittaker measure $\mathbb{W}_{\Xi,\mathbf{S},\mathbf{U}}$ defined as in (3.8) are written as nested contour integrals. We can now take the summation over all choices of $i_1, \dots, i_{l-k}$ and we express the term depending on these indeces as

$$\begin{aligned} q^{\binom{l}{2}+l-k-\binom{k}{2}} \sum_{1 \le i_1 < \cdots i_{l-k} \le l} q^{-i_1 - \cdots - i_{l-k}} &= q^{\binom{l+1}{2}-\binom{k+1}{2}-\binom{l-k+1}{2}} \binom{l}{l-k}_{q^{-1}} \\ &= \binom{l}{l-k}_q, \end{aligned}$$

where in the first equality we used (A.5). The proof of identity (3.7) is concluded once we take the summation over $k = 0, \dots, l$. □

Remark 3.3 The same matching discussed in Proposition 3.2 holds if we consider the fused Higher Spin Six Vertex Model with spin number J. In this case the β-specialization of the q-Whittaker measure become $-u_j \rightarrow (-u_j, -q u_j, \dots, -q^{J-1} u_j)$ and subsequently in the right hand side of (3.7) one should substitute $q^n \rightarrow q^{Jn}$.

Remark 3.4 The result of Proposition 3.2 (and [53, Theorem 4.11]) establishes an equivalence in distribution between a marginal of the q-Whittaker process and the Higher Spin Six Vertex Model. For the case when all $s_x = 0$ the Higher Spin Six Vertex Model reduces to the Bernoulli q-TASEP, that was obtained as a marginal of the q-Whittaker process in [50]. For parameters $s_x \neq 0$, although this matching is clear at the level of formulas we are not able to give an intuitive argument to explain why this should hold.

3.3 Explicit distribution of $\lambda_n^{(n)}$

By making use of (3.3) and additional combinatorical properties of q-Whittaker functions, in [39] it is shown to be possible to express in a compact form the probability distribution of $\lambda_n^{(n)}$ in a two sided q-Whittaker process, which has been briefly defined at the end of Sect. 3.1. We have the following

Proposition 3.5 ([39], Proposition 3.14) *Let $\overline{\rho}$ be a specialization of two sided q-Whittaker functions, such that*

$$\Pi(\mathbf{a};\overline{\rho}) = \prod_{i=1}^{n} e^{\tau a_i} \prod_{i,j=1}^{n} \frac{1}{(\gamma_i/a_j;q)_\infty},$$

for parameters satisfying $|\gamma_i| < |a_j|$. Then, for any integer l, we have

$$\mathbb{P}_{q\mathrm{W}}(\lambda_n^{(n)} = l) = (q;q)_\infty^{n-1} \int_{\mathbb{T}^n} \left(\frac{A}{Z}\right)^l m_n^q(\mathbf{z}) \frac{\Pi(\mathbf{z};\overline{\rho})}{\Pi(\mathbf{a};\overline{\rho})} \times \frac{(A/Z;q)_\infty}{\prod_{i,j=1}^{n}(a_i/z_j;q)_\infty} \prod_{j=1}^{n} \frac{dz_j}{z_j}, \tag{3.10}$$

where $A = a_1 \cdots a_n$, $Z = z_1 \cdots z_n$ and the integration is performed over the n dimensional torus $\mathbb{T}^n$.

With very little changes in the proof of this last proposition one can allow the normalization constant Π to be of a more general form.

Proposition 3.6 *Expression* (3.10) *also holds for $\overline{\rho}$ being a specialization of two sided q-Whittaker functions such that*

$$\Pi(\mathbf{a};\overline{\rho}) = \prod_{i=1}^{n} e^{a_i \tau} \prod_{j\geq 1} \frac{(1+\beta_j a_i)}{(\alpha_j a_i;q)_\infty} \prod_{j=1}^{n} \frac{1}{(\gamma_j/a_i;q)_\infty},$$

for non-negative real parameters τ, $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$ satisfying

$$\sum_{j\geq 1} \left(\alpha_j + \beta_j\right) < \infty, \quad |\alpha_i a_j| < 1, \quad |\gamma_i| < |a_j|.$$

4 Boundary conditions

The aim of this section is twofold. First, in Sect. 4.1 we prove Proposition 1.2, which characterizes the family of probability measures satisfying a certain translational symmetry, which we denoted as *Burke's property* (see Definition 1.1). By means of this property we define the *full plane Stationary Higher Spin Six Vertex Model* and this is done in Proposition 4.5. Subsequently, in Sect. 4.2, we give a description of a family of boundary conditions which one can construct from the step one (2.1) and that will be suitable to study the model in the stationary case.

4.1 Burke's property in the higher spin six vertex model

We like to start this Subsection by giving the proof of Proposition 1.2. First we show our results for the simpler case where the model has unfused rows, corresponding to the choice $J = 1$. Subsequently we extend our proof to the general J case. In particular, when $J = 1$, each horizontal edge is crossed by either zero or one path and therefore random variables j_x^t are Bernoulli distributed. We also recall the sequential update mechanism produced by the transfer operator $\mathfrak{X}_{u_t}$, which has been described in Sect. 2 (and in [18], Section 6.4.2). Let $\lambda(t-1) = 2^{m_2^{t-1}} 3^{m_3^{t-1}} \cdots$ be a configuration of paths entering the row of vertices with ordinate t and assume that, conditionally to the value of $\lambda(t-1)$ and j_1^t, random variables $\mathsf{m}_2^t, \dots, \mathsf{m}_{x-1}^t, \mathsf{j}_2^t, \dots, \mathsf{j}_{x-1}^t$ assumed respectively the values $m_2^t, \dots, m_{x-1}^t, j_2^t, \dots, j_{x-1}^t$. Then we have

$$\mathbb{P}\left(\mathsf{m}_x^t = m_x^t | \lambda(t-1), \{m_y^t, j_y^t\}_{y<x}\right) = \mathsf{L}_{u_t \xi_x, s_x}\left(m_x^{t-1}, j_{x-1}^t | \, m_x^t, j_x^t\right), \tag{4.1}$$

where the definition of L is given in Table 1 and at the boundaries (that is for $x = 2$ or $t = 1$), the law of $\{\mathsf{m}_x^0\}_{x \geq 2}, \{\mathsf{j}_1^t\}_{t \geq 1}$ is assumed to be known. This update is called sequential since it can be regarded as a sequence of moves propagating from the leftmost vertex to the right.

The update produced by the fused transfer operator $\mathfrak{X}_{u_t}^{(J)}$ naturally follows the same rule, with weights L being replaced by weights $\mathsf{L}^{(J)}$ given in (2.17).

Lemma 4.1 *Assume that the Higher Spin Six Vertex Model, with $J = 1$, satisfies the Burke's property. Then, setting*

$$p_t = \mathbb{P}\big(j_x^t = 1\big) \quad \text{and} \quad \pi_{M,x} = \mathbb{P}\big(m_x^t = M\big) \tag{4.2}$$

we have

$$\begin{aligned} \pi_{M,x} = {} & \pi_{M-1,x}\, p_t \frac{1 - s_x^2 q^{M-1}}{1 - s_x \xi_x u_t} \\ & + \pi_{M,x} \left[(1-p_t) \frac{1 - s_x \xi_x u_t q^M}{1 - s_x \xi_x u_t} + p_t \frac{-s_x \xi_x u_t + s_x^2 q^M}{1 - s_x \xi_x u_t} \right] \\ & + \pi_{M+1,x} (1-p_t) \frac{-s_x \xi_x u_t + s_x \xi_x u_t q^{M+1}}{1 - s_x \xi_x u_t} \end{aligned} \tag{4.3}$$

for each $M \geq 0$, with $\pi_{-1,x} = 0$.

Proof From the definition (4.1) of the update rule given by $\mathfrak{X}_{u_t}$ we have

$$\begin{aligned} \mathbb{P}(\mathsf{m}_x^t = M) = {} & \mathbb{P}(\mathsf{m}_x^{t-1} = M-1, \mathsf{j}_{x-1}^t = 1) \mathsf{L}_{u_t \xi_x, s_x}(M-1, 1 | \, M, 0) \\ & + \mathbb{P}(\mathsf{m}_x^{t-1} = M, \mathsf{j}_{x-1}^t = 0) \mathsf{L}_{u_t \xi_x, s_x}(M, 0 | \, M, 0) \\ & + \mathbb{P}(\mathsf{m}_x^{t-1} = M, \mathsf{j}_{x-1}^t = 1) \mathsf{L}_{u_t \xi_x, s_x}(M, 1 | \, M, 1) \\ & + \mathbb{P}(\mathsf{m}_x^{t-1} = M+1, \mathsf{j}_{x-1}^t = 0) \mathsf{L}_{u_t \xi_x, s_x}(M+1, 0 | \, M, 1), \end{aligned} \tag{4.4}$$

which becomes (4.3) when we substitute the definition of weights L given in Table 1 and use the Burke's property to express the joint law of m_x^{t-1} and j_{x-1}^t through quantities $\pi_{M,x}$, p_t. □

An interesting feature of the recurrence relations (4.3) is that it admits an exact solution in terms of the Al Salam–Chihara polynomials ([41]).

Lemma 4.2 *Assume that the Higher Spin Six Vertex Model, with* $J = 1$*, satisfies the Burke's property. Then, for each* $(x, t) \in \Lambda_{1,0}$ *random variables* $\mathsf{m}_x^t, \mathsf{j}_x^t$ *have laws*

$$\mathsf{m}_x^t \sim q\mathrm{NB}(s_x^2, \mathscr{d}/(\xi_x s_x)), \qquad \mathsf{j}_x^t \sim \mathrm{Ber}(-\mathscr{d}u_t/(1-\mathscr{d}u_t)), \tag{4.5}$$

where $\mathscr{d}$ *is a parameter independent of* x *or* t *that satisfies* (1.15).

Proof We use results of Lemma 4.1. We claim that

$$\pi_{M,x} = \left(\frac{p_t}{-s_x \xi_x u_t (1-p_t)}\right)^M \frac{(s_x^2; q)_M}{(q;q)_M} \frac{\left(\frac{-s_x p_t}{(1-p_t)\xi_x u_t}; q\right)_\infty}{\left(\frac{p_t}{-s_x \xi_x (1-p_t) u_t}; q\right)_\infty} \tag{4.6}$$

is solution of the recurrence (4.3). Such expression for $\pi_{M,x}$ is relatively simple, so that plugging it into (4.3) one could easily verify that indeed our claim holds. The assumption that the probability measure satisfies the Burke's property implies that values of $\pi_{M,x}$ cannot depend on the t coordinate and therefore t dependent quantities u_t, p_t must satisfy the relation

$$\frac{p_t}{-u_t(1-p_t)} = \mathscr{d}, \tag{4.7}$$

for some parameter $\mathscr{d}$ independent on x or t, which necessarily has to meet condition (1.15) as well. By inverting (4.7) and recalling the definition of p_t given in (4.2) we complete the proof of (4.5).

This checking style argument might not be the most elegant, so we now quickly show how this solution was obtained. Setting $s_x = s$, $\xi_x u_t = u$, $p_t = p$ and defining the auxiliary sequence f_M as

$$\pi_M = \beta^{-M} \frac{(s^2; q)_M}{(q;q)_M} f_M,$$

the recurrence (4.3) becomes

$$\left(1-q^M\right)\frac{p\beta^2}{(1-p)su} f_{M-1} + \left[\beta \frac{su - p - sup}{(1-p)su} \right.$$
$$\left. -\beta \frac{su - sup - s^2 p}{(1-p)su} q^M \right] f_M - \left(1 - s^2 q^M\right) f_{M+1} = 0. \tag{4.8}$$

This last expression has to be compared with the general recurrence relation

$$-t_1^2(1-q^n)g_{n-1}+t_1\big[z+1/z-(t_1+t_2)q^n\big]g_n-(1-t_1t_2q^n)g_{n+1}=0, \tag{4.9}$$

with initial conditions $p_{-1}=0$, $p_0=1$, which is known to be satisfied by the Al Salam–Chihara polynomials ([41], (15.1.6))

$$g_n(z;t_1,t_2|q)={}_3\phi_2\left(\begin{matrix}q^{-n}, & t_1z, & t_1/z\\ & t_1t_2, & 0\end{matrix}\;\middle|\, q,\ q\right).$$

Equating term by term (4.8) and (4.9) we get, in terms of variables z, t_1, t_2, β, the second order system

$$\begin{cases}(z+1/z)t_1=\beta\frac{su-p-sup}{(1-p)su},\\ t_1^2=-\beta^2\frac{p}{(1-p)su},\\ t_1^2+t_1t_2=\beta\frac{su-sup-s^2p}{(1-p)su},\\ t_1t_2=s^2,\end{cases} \tag{4.10}$$

whose solution is

$$t_1=-s\sqrt{\frac{-sp}{(1-p)u}},\qquad t_2=-s\sqrt{\frac{(1-p)u}{-sp}},\qquad z=t_2,\qquad \beta=s^2.$$

With these choices of values the Al Salam–Chihara polynomial assumes a simple form, so that

$$f_M=g_M(t_2;t_1,t_2|q)={}_2\phi_1\left(\begin{matrix}q^{-M}, & -sp/(u-up)\\ & 0\end{matrix}\;\middle|\, q,\ q\right)=\left(\frac{-sp}{u(1-p)}\right)^M,$$

where in the last equality we used the q-analog Chu–Vandermonde identity (see "Appendix A"). Using the definition of f_M we finally obtain (4.6). □

Result of Lemma 4.2 suffices to prove the "only if" part of the statement of Proposition 1.2 in the particular case of a model with $J=1$. The next two Lemmas address the "if" part of Proposition 1.2.

Lemma 4.3 *Consider the Higher Spin Six Vertex Model on $\Lambda_{1,0}$, with $J=1$ and boundary conditions*

$$\mathsf{m}_x^0\sim q\mathrm{NB}(s_x^2,\text{ɗ}/(\xi_xs_x)),\qquad \mathsf{j}_1^t\sim \mathrm{Ber}(-\text{ɗ}u_t/(1-\text{ɗ}u_t)), \tag{4.11}$$

where $\mathsf{m}_2^0,\mathsf{m}_3^0,\mathsf{m}_4^0,\dots,\mathsf{j}_1^1,\mathsf{j}_1^2,\mathsf{j}_1^3,\dots$ are independent random variables. Then for all $x\geq 2$ the sequence $\mathsf{m}_x^0,\mathsf{m}_{x+1}^0,\mathsf{m}_{x+2}^0,\dots,\mathsf{j}_{x-1}^1,\mathsf{j}_{x-1}^2,\mathsf{j}_{x-1}^3,\dots$ is a family of independent random variables and for each $t\geq 1$ we have $\mathsf{j}_{x-1}^t\sim \mathrm{Ber}(-\text{ɗ}u_t/(1-\text{ɗ}u_t))$.

Proof We start observing that, due to the choice of boundary conditions and due to the fact that paths propagate in the lattice in the up right direction the family of random variables $\mathsf{m}_x^0, \mathsf{m}_{x+1}^0, \mathsf{m}_{x+2}^0, \dots$ is always independent of the family $\mathsf{j}_{x-1}^1, \mathsf{j}_{x-1}^2, \mathsf{j}_{x-1}^3, \dots$ and hence we only need to show that $\mathsf{j}_{x-1}^1, \mathsf{j}_{x-1}^2, \mathsf{j}_{x-1}^3, \dots$ are mutually independent and that their distributions follow the law described in the statement of Lemma 4.3. We prove this claim for the $x = 3$ case, as the general x case would simply follow by induction procedure. This means that, for all $t \geq 1$ and for all choices of $(j_1, \dots, j_t) \in \{0, 1\}^t$, we need to show that

$$\mathbb{P}\left(\mathsf{j}_2^1 = j_1, \dots, \mathsf{j}_2^t = j_t\right) = \prod_{k=1}^{t} \frac{(-\mathscr{d} u_k)^{j_k}}{1 - \mathscr{d} u_k}. \tag{4.12}$$

To do so we follow a rather algebraic approach. Introduce the 2×2 matrices

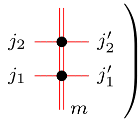
$$\mathcal{A}_k = \frac{1}{1 - s_2\xi_2 u_k} \begin{pmatrix} 1 & -s_2\xi_2 u_k \\ 1 & -s_2\xi_2 u_k \end{pmatrix}, \qquad \mathcal{B}_k = \frac{1}{1 - s_2\xi_2 u_k} \begin{pmatrix} -s_2\xi_2 u_k & s_2\xi_2 u_k \\ -s_2^2 & s_2^2 \end{pmatrix},$$

through which we can express the stochastic weight L, using the classical bra-ket notation,[4] as

$$\mathsf{L}_{\xi_2 u_k, s_2}(m, j \,|\, m + j - j', j') = \langle e_j | \left(\mathcal{A}_k + q^m \mathcal{B}_k\right) | e_{j'} \rangle,$$

for each $j, j' = 0, 1$. It is not hard to convince oneself that it is possible to describe the weight of any admissible configuration of paths around a column of two vertices as

$\mathsf{L}_{\{\xi_2 u_1, \xi_2 u_k\}, s_2}\Bigg($



$$\Bigg) = \sum_{l_1, l_2 \geq 0} \mathsf{L}_{\xi_2 u_1, s_2}(m, j_1 |\; l_1, j_1') \mathsf{L}_{\xi_2 u_2, s_2}(l_1, j_2 |\; l_2, j_2')$$

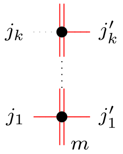
$$= \langle e_{j_1} | \otimes \langle e_{j_2} | \left[(\mathcal{A}_1 + q^m \mathcal{B}_1) \otimes \mathcal{A}_2 + q^m \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) (\mathcal{A}_1 + q^m \mathcal{B}_1) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/q \end{smallmatrix}\right) \otimes \mathcal{B}_2 \right] | e_{j_1'} \rangle \otimes | e_{j_2'} \rangle.$$

More in general, defining the sequence

$$\begin{cases} T_k^{(m)} = T_{k-1}^{(m)} \otimes \mathcal{A}_k + q^m \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{\otimes(k-1)} T_{k-1}^{(m)} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/q \end{smallmatrix}\right)^{\otimes(k-1)} \otimes \mathcal{B}_k \\ T_1^{(m)} = \mathcal{A}_1 + q^m \mathcal{B}_1, \end{cases} \tag{4.13}$$

one can show that, for all admissible configurations of paths around a column of k vertices, we have

$\mathsf{L}_{\{\xi_x u_1, \dots, \xi_x u_k\}, s_x}\Bigg($



$$\Bigg) = (\langle e_{j_1} | \otimes \cdots \otimes \langle e_{j_k} |) \cdot T_k^{(m)} \cdot (| e_{j_1'} \rangle \otimes \cdots \otimes | e_{j_k'} \rangle).$$

---

[4] The numbering of rows and column starts from zero rather than from one.

Define also the vector

$$\langle \mathbf{v}_k | = \frac{1}{1-\mathscr{d} u_k} \langle e_0 | + \frac{-\mathscr{d} u_k}{1-\mathscr{d} u_k} \langle e_1 |,$$

so that

$$\mathbb{P}\left(\mathsf{j}_1^1 = j_1, \dots, \mathsf{j}_1^t = j_t \right) = \langle \mathbf{v}_1 | \otimes \cdots \otimes \langle \mathbf{v}_t | \cdot | e_{j_1} \rangle \otimes \cdots \otimes | e_{j_t} \rangle.$$

Adopting this matrix notation we can translate equality (4.12) into the eigenrelation

$$\langle \mathbf{v}_1 | \otimes \cdots \otimes \langle \mathbf{v}_t | \cdot \sum_{m \geq 0} \pi_m^{(2)} T_t^{(m)} = \langle \mathbf{v}_1 | \otimes \cdots \otimes \langle \mathbf{v}_t |, \tag{4.14}$$

where we used the shorthand

$$\pi_m^{(k)} = \mathbb{P}\big(\mathsf{m}_k^0 = m\big) = \left(\frac{\mathscr{d}}{\xi_x s_x} \right)^m \frac{(s_k^2; q)_m}{(q;q)_m} \frac{(\mathscr{d}/(\xi_k s_k); q)_\infty}{(\mathscr{d} s_k/\xi_k; q)_\infty}. \tag{4.15}$$

We prove (4.14) by induction. When $t = 1$, we have

$$\langle \mathbf{v}_1 | \cdot \sum_{m \geq 0} \pi_m^{(2)} \left(\mathcal{A}_1 + q^m \mathcal{B}_1 \right) = \langle \mathbf{v}_1 | \cdot \left(\mathcal{A}_1 + \frac{1 - \mathscr{d}/(s_2 \xi_2)}{1 - \mathscr{d} s_2/\xi_2} \mathcal{B}_1 \right) = \langle \mathbf{v}_1 |, \tag{4.16}$$

where the summation with respect to m was performed using the expression (4.15) for $\pi_\bullet^{(2)}$ and the q-binomial theorem (A.7) and the second equality follows by direct inspection of the matrix product. We now assume that (4.14) is true for $t-1$ and from this we would like to show that the t case follows. Using recursion (4.13), we write

$$\begin{aligned} &\langle \mathbf{v}_1 | \otimes \cdots \otimes \langle \mathbf{v}_t | \cdot \sum_{m \geq 0} \pi_m^{(2)} T_t^{(m)} \\ &\quad = \langle \mathbf{v}_1 | \otimes \cdots \otimes \langle \mathbf{v}_t | \cdot \Bigg(\sum_{m \geq 0} \pi_m^{(2)} T_{t-1}^{(m)} \otimes \mathcal{A}_t \\ &\qquad + \sum_{m \geq 0} \pi_m^{(2)} q^m \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix} \right)^{\otimes (t-1)} T_{t-1}^{(m)} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/q \end{smallmatrix} \right)^{\otimes (t-1)} \otimes \mathcal{B}_t \Bigg) \end{aligned} \tag{4.17}$$

and we see that the more complicated term to analyze is the second addend in the right hand side, as the first one becomes

$$\langle \mathbf{v}_1 | \otimes \cdots \otimes \langle \mathbf{v}_{t-1} | \otimes \left(\langle \mathbf{v}_t | \cdot \mathcal{A}_t \right), \tag{4.18}$$

using the inductive hypothesis and the Kronecker rule for multiplication of tensor products of matrices and vectors. From the computation of q moments of the probability measure $\pi_{\bullet}^{(2)}$ it is easy to see that

$$\sum_{m\geq 0} q^{nm}\pi_m^{(2)} = \frac{1-\mathscr{d}/(s_2\xi_2)}{1-\mathscr{d}s_2/\xi_2}\sum_{m\geq 0} q^{(n-1)m}\pi_m^{(2)}\Big|_{\mathscr{d}\to q\mathscr{d}},$$

which implies the identity

$$\sum_{m\geq 0}\pi_m^{(2)}q^m T_{t-1}^{(m)} = \frac{1-\mathscr{d}/(s_2\xi_2)}{1-\mathscr{d}s_2/\xi_2}\sum_{m\geq 0}\pi_m^{(2)}\Big|_{\mathscr{d}\to q\mathscr{d}} T_{t-1}^{(m)}, \tag{4.19}$$

where the subscript $\big|_{\mathscr{d}\to q\mathscr{d}}$ in the previous two equations denotes that in $\pi_m^{(2)}$ we substitute every $\mathscr{d}$ with $q\mathscr{d}$. Using (4.19) and the inductive hypothesis again, we can evaluate the second addend in the right hand side of (4.17) as

$$\begin{aligned}
&\langle \mathbf{v}_1| \otimes \cdots \otimes \langle \mathbf{v}_{t-1}| \\
&\quad\cdot \left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)^{\otimes(t-1)} \left(\frac{1-\mathscr{d}/(s_2\xi_2)}{1-\mathscr{d}s_2/\xi_2}\sum_{m\geq 0}\pi_m^{(2)}\Big|_{\mathscr{d}\to q\mathscr{d}} T_{t-1}^{(m)}\right)\left(\begin{smallmatrix}1&0\\0&1/q\end{smallmatrix}\right)^{\otimes(t-1)} \otimes (\mathbf{v}_t\cdot\mathcal{B}_t) \\
&\quad= \langle \mathbf{v}_1| \otimes \cdots \otimes \langle \mathbf{v}_{t-1}| \otimes \left(\frac{1-\mathscr{d}/(s_2\xi_2)}{1-\mathscr{d}s_2/\xi_2}\langle \mathbf{v}_t|\cdot\mathcal{B}_t\right).
\end{aligned} \tag{4.20}$$

So far we were able to transform the right hand side of (4.17) in the sum of (4.18) and of the right hand side of (4.20). Employing identity (4.16) we recover (4.14) for the general t case, which completes the proof. □

Statement of Lemma 4.3 implies a certain propagation of the boundary conditions (4.11) in the horizontal direction. The next Lemma addresses their propagation in the vertical direction.

Lemma 4.4 *Consider the Higher Spin Six Vertex Model on* $\Lambda_{1,0}$ *with* $J = 1$ *and boundary conditions as in Lemma 4.3. Then, for all* $t \geq 1$ *the sequence* $\mathsf{m}_2^{t-1}, \mathsf{m}_3^{t-1}, \mathsf{m}_4^{t-1}, \dots, \mathsf{j}_1^t, \mathsf{j}_1^{t+1}, \mathsf{j}_1^{t+2}, \dots$ *is a family of independent random variables and for each* $x \geq 2$ *we have* $\mathsf{m}_x^{t-1} \sim q\mathrm{NB}(s_x^2, \mathscr{d}/(\xi_x s_x))$.

Proof By similar argument as in the proof of Lemma 4.3 the two families of random variables $\mathsf{m}_2^{t-1}, \mathsf{m}_3^{t-1}, \mathsf{m}_4^{t-1}, \dots$ and $\mathsf{j}_1^t, \mathsf{j}_1^{t+1}, \mathsf{j}_1^{t+2}, \dots$ are always independent and in order to prove Lemma 4.4 it is sufficient to show that, for all $x \geq 2$ and for all $(m_2, \dots, m_x) \in \mathbb{Z}_{\geq 0}^{x-1}$ we have

$$\mathbb{P}(\mathsf{m}_2^1 = m_2, \dots, \mathsf{m}_x^1 = m_x) = \prod_{k=2}^{x}\pi_{m_k}^{(k)}, \tag{4.21}$$

where the quantities $\pi^{(k)}_m$'s were introduced in (4.15). Also in this case we follow a rather algebraic approach. First introduce the operator

$$\mathcal{U}^{(k)} = \sum_{m,m' \geq 0} |e_m\rangle \begin{pmatrix} \mathsf{L}_{\xi_k u, s_k}(m,0|\,m',0) & \mathsf{L}_{\xi_k u, s_k}(m,0|\,m',1) \\ \mathsf{L}_{\xi_k u, s_k}(m,1|\,m',0) & \mathsf{L}_{\xi_k u, s_k}(m,1|\,m',1) \end{pmatrix} \langle e_{m'}|$$

and vectors

$$\langle \mathbf{w}_k| = \sum_{m \geq 0} \pi^{(k)}_m \langle e_k|, \qquad \mathbf{v}^T = \frac{1}{1-\mathscr{d}u} \begin{pmatrix} 1 \\ -\mathscr{d}u \end{pmatrix}.$$

Notice that operator $\mathcal{U}^{(k)}$ is the one vertex analog of the transfer operator $\mathfrak{X}_u$ and that the distribution of $\mathsf{m}^1_2, \ldots, \mathsf{m}^1_x$ is given by

$$\begin{aligned} &\mathbb{P}(\mathsf{m}^1_2 = m_2, \ldots, \mathsf{m}^1_x = m_x) \\ &\quad = \mathbf{v} \cdot \left(\bigotimes_{k=2}^{x} \langle \mathbf{w}_k| \right) \cdot \left(\bigotimes_{k=2}^{x} \mathcal{U}^{(k)} \right) \cdot \left(\bigotimes_{k=2}^{x} |e_{m_k}\rangle \right) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{4.22}$$

By direct inspection we easily see that

$$\langle \mathbf{w}_k| \cdot \mathcal{U}^{(k)} = \sum_{m' \geq 0} \left(\hat{\mathcal{A}}_k + q^{m'} \hat{\mathcal{B}}_k \right) \pi^{(k)}_{m_k} \langle e_{m'}|, \tag{4.23}$$

where matrices $\hat{\mathcal{A}}_k, \hat{\mathcal{B}}_k$ are given by

$$\hat{\mathcal{A}}_k = \frac{1}{1-s_k\xi_k u} \begin{pmatrix} 1 & -\mathscr{d}u \\ \xi_k s_k/\mathscr{d} & -\xi_k s_k u \end{pmatrix}, \qquad \hat{\mathcal{B}}_k = \frac{1}{1-s_k\xi_k u} \begin{pmatrix} -\xi_k s_k u & \mathscr{d}u s_k^2 \\ -\xi_k s_k/\mathscr{d} & s_k^2 \end{pmatrix}.$$

By means of (4.23) and of the Kronecker rule for multiplication of tensor products we see that the right hand side of (4.22) reduces to

$$\mathbf{v} \cdot \prod_{k=2}^{x} \left(\hat{\mathcal{A}}_k + q^{m_k} \hat{\mathcal{B}}_k \right) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \times \prod_{k=2}^{x} \pi^{(k)}_{m_k}, \tag{4.24}$$

which would prove (4.21) in case the scalar product $\mathbf{v} \cdot \prod_{k=2}^{x} (\hat{\mathcal{A}}_k + q^{m'} \hat{\mathcal{B}}_k) \cdot \binom{1}{1}$ is equal to one. This is a consequence of certain product identities involving matrices $\hat{\mathcal{A}}_k, \hat{\mathcal{B}}_k$. In particular, for all k, k' we have

$$\hat{\mathcal{A}}_k \hat{\mathcal{A}}_{k'} = \hat{\mathcal{A}}_k, \quad \hat{\mathcal{A}}_k \hat{\mathcal{B}}_{k'} = 0, \quad \hat{\mathcal{B}}_k \hat{\mathcal{A}}_{k'} = c_{k,k'} \hat{\mathcal{C}}, \quad \hat{\mathcal{B}}_k \hat{\mathcal{B}}_{k'} = d_{k,k'} \hat{\mathcal{B}}_{k'},$$

where $c_{k,k'}, d_{k,k'}$ are constants depending on parameters s, ξ, u and

$$\hat{\mathcal{C}} = \begin{pmatrix} 1 & -\mathscr{d}u \\ 1/(\mathscr{d}u) & -1 \end{pmatrix}.$$

Since $\mathbf{v}\cdot\hat{\mathcal{C}}\cdot\binom{1}{1}=\mathbf{v}\cdot\hat{\mathcal{B}}_k\cdot\binom{1}{1}=0$ for all k, we can now compute (4.24) as

$$(4.24)=\mathbf{v}\cdot\hat{\mathcal{A}}_2\cdot\begin{pmatrix}1\\1\end{pmatrix}\times\prod_{k=2}^{x}\pi^{(k)}_{m_k}=\prod_{k=2}^{x}\pi^{(k)}_{m_k},$$

which proves (4.21). ◻

We can now summarize results obtained so far in this Subsection and extend them to the slightly more general setting of the model with fused rows.

Proof of Proposition 1.2 The case $J=1$ of Proposition 1.2 is obtained combining Lemmas 4.2, 4.3, 4.4. To prove the general J case we need to show that $\mathsf{j}^1_{x-1},\mathsf{j}^2_{x-1},\mathsf{j}^3_{x-1},\dots$ are independent q-negative binomial random variables respectively of parameters $(q^{-J},q^J u_t\mathscr{d})$ for $t=1,2,3,\dots$, even when $J\neq 1$. Recall that the fusion of rows is obtained by collapsing together J rows of vertices with spectral parameters taken in geometric progression of ratio q (see Proposition 2.2). Since, as a result of Lemma 4.3, for each $x\geq 2$, random variables $\mathsf{j}^1_{x-1},\mathsf{j}^2_{x-1},\mathsf{j}^3_{x-1},\dots$ are independently distributed, the proof reduces to show that, in the unfused model

$$\mathbb{P}\left(\mathsf{j}^1_{x-1}+\cdots,\mathsf{j}^J_{x-1}=k\right)=\left(q^J\mathscr{d}u\right)^k\frac{(q^{-J},q)_k}{(q;q)_k}\frac{(q^J\mathscr{d}u;q)_\infty}{(\mathscr{d}u;q)_\infty},\qquad\text{for all } k\geq 0,$$

when $u_1=u, u_2=qu,\dots,u_J=q^{J-1}u$, which is the statement of Proposition 2.3. This concludes the proof. ◻

Employing result of Proposition 1.2 we are able to extend the Higher Spin Six Vertex Model to the full lattice $\mathbb{Z}\times\mathbb{Z}$.

Proposition 4.5 *Take paramaters*

$$\mathbf{U}=(\dots,u_{-1},u_0,u_1,\cdots),\quad\mathbf{S}=(\dots,s_{-1},s_0,s_1,\dots),\quad\Xi=(\dots,\xi_{-1},\xi_0,\xi_1,\dots)$$

fulfilling conditions (1.8)*. Take also a parameter* $\mathscr{d}$ *to fulfill condition* (1.15)*. Then there exists a probability measure on the set of directed up right paths on* $\mathbb{Z}\times\mathbb{Z}$*, such that, for each choice of* (x,t)*,*

$$\mathbb{P}\left(\mathsf{m}^t_x=m',\mathsf{j}^t_x=j'|\,\mathsf{m}^{t-1}_x=m,\mathsf{j}^t_{x-1}=j\right)=\mathsf{L}^{(J)}_{u_t\xi_x,s_x}\left(m,j|\,m',j'\right)$$

and $\mathsf{j}^t_{x-1},\mathsf{j}^{t+1}_{x-1},\mathsf{j}^{t+2}_{x-1},\dots,\mathsf{m}^{t-1}_x,\mathsf{m}^{t-1}_{x+1},\mathsf{m}^{t-1}_{x+2},\dots$ *are independent random variables distributed as*

$$\mathsf{m}^{t-1}_{x+k}\sim q\mathrm{NB}(s^2_{x+k},\mathscr{d}/(\xi_{x+k}s_{x+k})),\qquad\mathsf{j}^{t+k}_{x-1}\sim q\mathrm{NB}(q^{-J},q^J\mathscr{d}u_{t+k}),\quad(4.25)$$

for each k*. In other words it is possible to define the Higher Spin Six Vertex Model on the lattice* $\mathbb{Z}\times\mathbb{Z}$ *in such a way that it satisfies the Burke's property and such* *that, for each choice of vertex* (x, t)*, paths entering the restricted lattice* $\Lambda_{x-1,t-1}$ *are distributed as in* (4.25)*. We refer to this model as the* full plane Stationary Higher Spin Six Vertex Model.

Proof The procedure we follow to prove this extension result is fairly standard and it has been utilized, for the translation invariant Six Vertex Model case in [2], “Appendix A”.

We call $\mathbb{P}_N$ the probability measure of the Higher Spin Six Vertex Model defined on the lattice $\Lambda_{-N,-N}$ with boundary conditions given by

$$\mathsf{m}_{-N+k}^{-N} \sim q\mathrm{NB}(s_{-N+k}^2, \text{đ}/(\xi_{-N+k}s_{-N+k})), \qquad \mathsf{j}_{-N}^{-N+k} \sim q\mathrm{NB}(q^{-J}, q^J \text{đ} u_{-N+k}).$$

Let E_N be an event involving only configurations of paths in $\Lambda_{-N,-N}$. It is clear, from Proposition 1.2, that, for each $N^* > N$,

$$\mathbb{P}_{N^*}(E_N) = \mathbb{P}_N(E_N)$$

and therefore $\mathbb{P}_{N^*}$ extends $\mathbb{P}_N$. By using the Caratheodory’s extension theorem we can take the limit $N^* \to \infty$ and deduce the existence of a measure $\mathbb{P}_\infty$ to finally define the translation invariant Higher Spin Six Vertex Model on $\mathbb{Z}^2$. ◻

We close this Subsection explaining the reason behind the use of the terminology “Burke’s property” of Definition 1.1. Here we rephrase a generalization by Ferrari and Fontes of a theorem by Burke [26,32], stated for queuing systems in a language more familiar to us.

Theorem 4.6 (Burke) *Let* $\{y_x\}_{x\in\mathbb{Z}}$ *be a totally asymmetric simple exclusion process where, at time* $t = 0$, $y_1 = -1$ *a.s. and consecutive particles are spaced independently with geometric distribution of parameter* đ*. Then, the distribution of gaps is stationary in time and the marginal distribution of* y_1 *is that of a Poisson process with rate* $1 - \text{đ}$.

In the Higher Spin Six Vertex Model on the full plane $\mathbb{Z} \times \mathbb{Z}$ defined by Proposition 4.5, regarding the generic occupancy number m_x^t as the gap between the $(x-1)$-th and the x-th particle of a process $\{y_x\}_{x\in\mathbb{Z}}$ at time t, we obtain a discrete time generalization of the totally asymmetric simple exclusion process [30]. A consequence of Proposition 4.5 is that, in this generalized model, when at time $t = 0$ we set $y_1 = -1$ a.s. and consecutive particles are independently spaced with q-negative binomial distribution of parameters $(s_x^2, \text{đ}/(\xi_x, s_x))$, then the marginal process y_1 is equivalent, in distribution, to a sequence of independent jumps with q-negative binomial distribution of parameters $(q^{-J}, q^J \text{đ} u_t)$. This analogy should justify our choice of words.

A concept analogous to the Burke’s property stated in Definition 1.1 appeared already in literature in the context of random polymers [9,61]. In particular, in [61] the author considers the log-Gamma directed polymers model with random external sources. This sort of model is known to be described by the so called α-Whittaker processes [13], of which the q-Whittaker processes presented in Sect. 3 represent a “quantized” generalization. Although we do not describe here relations between the model studied in [61] and the Stochastic Higher Spin Six Vertex Model, we will say that the role played by random external sources in polymer models is analogous to that played by boundary conditions in the Higher Spin Six Vertex Model or to that played by random initial conditions for totally asymmetric simple exclusion processes.

4.2 Exactly solvable boundary conditions

In the previous Subsection we characterized the family of Higher Spin Six Vertex Models satisfying the Burke's property. Here we explain how the study of the model with double sided q-negative binomial boundary conditions is accessible by properly specializing parameters $\Xi, \mathbf{S}, \mathbf{U}$ starting from a Higher Spin Six Vertex Model with step boundary conditions.

For later purpose we now introduce the quantities

$$\boldsymbol{\ell}^{(i)}_{\wp,v}\Big(M;\overline{M}\Big)=\left(\frac{v}{\xi_i s_i}\right)^M \frac{\Big(s_i^2,\wp q^{\overline{M}};q\Big)_M}{\Big(v\wp q^{\overline{M}}s_i/\xi_i,q;q\Big)_M}\frac{\Big(v\wp q^{\overline{M}}s_i/\xi_i,v/(\xi_i s_i);q\Big)_\infty}{\Big(v\wp q^{\overline{M}}/(\xi_i s_i),vs_i/\xi_i;q\Big)_\infty}, \tag{4.26}$$

which are families of probability mass functions (in M), provided $\overline{M}$ is a non-negative integer and parameters $\wp$, v satisfy one of the two conditions

$$v \text{ as in (1.11)} \quad \text{and} \quad \wp<1, \tag{4.27}$$

or

$$v<0 \quad \text{and} \quad \wp=q^{-K}, \quad \text{for} \quad K\in\mathbb{Z}_{\geq\overline{M}}. \tag{4.28}$$

In case v, $\wp$ are taken according to (4.28), $\boldsymbol{\ell}^{(i)}_{\wp,v}(\bullet;\overline{M})$ is supported on the set $\{0,K-\overline{M}\}$, whereas when they are taken as in (4.27), expression (4.26) takes positive values for each $M\in\mathbb{Z}_{\geq 0}$. In both cases the sum-to-one condition in M is guaranteed by the q-Gauss summations (A.8).

The next definition is rather technical and aims to describe the most general set of boundary conditions we will cover in the remaining part of the paper.

Definition 4.7 Consider a random variable $\mathsf{m}\sim qNB(\wp,v/d)$, with $d>\max(0,v)$. We denote with the symbol $\mathbb{P}_{\wp,v,d}$ the probability measure of a coupling of m with a Higher Spin Six Vertex Model on $\Lambda_{1,0}$, in which $\mathsf{j}_1^t\sim qNB(q^{-J},q^J u_t d)$ for $t=1,2,3\ldots$ are independent random variables and $\mathsf{m}_2^0,\mathsf{m}_3^0,\mathsf{m}_4^0,\ldots$, conditionally to m, have law

$$\mathbb{P}_{\wp,v,d}\Big(\mathsf{m}_2^0=m_2,\ldots,\mathsf{m}_x^0=m_x|\mathsf{m}=m_1\Big)=\prod_{i=2}^{x}\boldsymbol{\ell}^{(i)}_{\wp,v}\left(m_i;\sum_{j=1}^{i-1}m_j\right). \tag{4.29}$$

For this particular model we introduce the shifted height function

$$\overline{\mathcal{H}}(x,t) = \mathcal{H}(x,t) - \mathsf{m}. \tag{4.30}$$

When parameters $\wp, v$ are taken as in (4.27) and $\wp = 0$, m and the Higher Spin Six Vertex Model become independent processes and in this case we use the notation $\mathbb{P}_{0,v,\mathscr{d}} = \mathbb{P}_{\mathrm{HS}(v,\mathscr{d})\otimes\mathsf{m}}$.

Indeed the probability measure $\mathbb{P}_{\wp,v,\mathscr{d}}$ introduced in Definition 4.7 represents a generalization of the double sided q-negative binomial Higher Spin Six Vertex Model. The reason why the choice $\wp = 0$ decouples $\mathsf{m}, \mathsf{m}_2^0, \mathsf{m}_3^0, \dots$ comes from the exact expression (4.26) of weights $\boldsymbol{\ell}^{(i)}_{\wp,v}$. In fact, for any k, the law of m_k^0 depends on the outcome of $\mathsf{m}_2^0, \mathsf{m}_3^0, \dots, \mathsf{m}_{k-1}^0$ and m only when the factor $\wp q^{\overline{M}}$ is different than zero. By setting $\wp = 0$ we see that $\mathsf{m}_2^0, \mathsf{m}_3^0, \dots$ become independent q-negative binomials of parameters respectively $(s_k^2, v/(\xi_k s_k))$ for $k = 2, 3, \dots$, whereas m becomes a q-Poisson random variable of parameter $v/\mathscr{d}$ independent of the rest of the process. The reason why we consider a coupling between the Higher Spin Six Vertex Model and the random variable m becomes clear with the construction we present next. We claim in fact that the measure $\mathbb{P}_{\wp,v,\mathscr{d}}$ is obtained as a marginal process of a certain specialization of the Higher Spin Six Vertex Model with step boundary conditions. We give the following

Proposition 4.8 *Let $K \in \mathbb{Z}_{\geq 1}$ and consider the Higher Spin Six Vertex Model on the lattice $\Lambda_{0,-K}$ with step boundary conditions (that is $j_0^t = 1$ a.s. for $t \geq -K+1$ and $m_x^{-K} = 0$ a.s. for $x \geq 1$). Spectral parameters are taken as*

$$\hat{\mathbf{U}} = (q/v, q^2/v, \dots, q^K/v) \cup \mathbf{U}, \tag{4.31}$$

where the set $\mathbf{U} = (u_1, \dots, q^{J-1}u_1, u_2, \dots, q^{J-1}u_2, \dots)$ refers to vertices with positive abscisse. Here $v < 0$ and at $x = 1$, ξ_1, s_1 are given setting

$$s_1 = 1/N, \quad \xi_1 = \mathscr{d} N \quad \textit{and taking the limit } N \to \infty. \tag{4.32}$$

Then, the marginal process on the lattice $\Lambda_{1,0}$ is described by the law $\mathbb{P}_{\wp,v,\mathscr{d}}$, presented in Definition 4.7 with parameters $v, \wp$ as is (4.28).

The proof of Proposition 4.8 boils down to finding the following simplified expression for the fused vertex weight (2.17).

Lemma 4.9 *We have*

$$\begin{aligned} &L^{(K)}_{\xi q v^{-1}, s}(0, j_1 | \, i_2, j_1 - i_2) \\ &= \left(\frac{v}{\xi s}\right)^{i_2} \frac{(s^2, q^{-j_1}; q)_{i_2}}{(v q^{-j_1} s/\xi, q; q)_{i_2}} \frac{(v q^{-j_1} s/\xi, v/(\xi s), q)_\infty}{(v q^{-j_1}/(\xi s), v s/\xi; q)_\infty}, \end{aligned} \tag{4.33}$$

where, remarkably, the right hand side is independent of K.

Proof From the exact expression of the weight $\mathsf{L}^{(K)}$ in (2.17), setting the number i_1 of paths entering the vertex from below to zero, its rather complicated formula simplifies to

$$\mathsf{L}^{(K)}_{\xi q v^{-1}, s}(0, j_1 | \, i_2, j_1 - i_2) = \frac{\left(s^2; q\right)_{i_2}}{(q; q)_{i_2}} \frac{\left(q^{1+j_1-i_2}; q\right)_{i_2} s^{2(j_1-i_2)} (q\xi/(vs); q)_{j_1-i_2}}{(q\xi s/v; q)_{j_1}}.$$

By multiplying and dividing this last expression by

$$\prod_{l=0}^{i_2-1} (s^2 - \xi s q^{j_1-i_2+l+1}/v)$$

and taking out of the product all factors depending only on j_1 we get a term proportional to

$$\left(\frac{v}{\xi s}\right)^{i_2} \frac{(s^2; q)_{i_2} (q^{-j_1}; q)_{i_2}}{(q; q)_{i_2} (v q^{-j_1} s/\xi; q)_{i_2}}. \tag{4.34}$$

Result (4.33) is obtained normalizing (4.34) so that its sum over all i_2 is one and this is done by means of the q-Gauss summation (A.8). □

Proof of Proposition 4.8 First we observe that choice (4.32) generates q-negative binomial random entries in the vertical boundary of the lattice $\Lambda_{1,0}$. By substituting the values of ξ_1, s_1 in the definition of transition probabilities L, we have

$$\lim_{N \to \infty} \mathsf{L}_{u_t \mathscr{d} N, 1/N}(m, 1 | \, m, 1) = \frac{-u_t \mathscr{d}}{1 - u_t \mathscr{d}}, \tag{4.35}$$

which also implies, using Proposition 2.3,

$$\lim_{N \to \infty} \mathsf{L}^{(J)}_{u_t \mathscr{d} N, 1/N}(m, J | \, m + J - l, l) = \left(q^J \mathscr{d} u_t\right)^l \frac{\left(q^{-J}, q\right)_l}{(q; q)_l} \frac{\left(q^J \mathscr{d} u_t; q\right)_\infty}{(\mathscr{d} u_t; q)_\infty}. \tag{4.36}$$

This procedure of obtaining independent random entries at column $\{(2, t)\}_{t \geq 1}$ is alternative to that presented in Sect. 2, where in (2.5), (2.6) a result analogous to (4.35) was achieved setting $\mathsf{m}_1^0 = \infty$ a.s. Here the value of m_1^0 depends on the process on the strip $\mathbb{Z}_{\geq 1} \times \{-K + 1, \dots, 0\}$ and it is in general not infinite. From (4.31) we see that the first K spectral parameters (those related to non-positive ordinates t) are in geometric progression of ratio q and therefore we can use the notion of fused transfer operator $\mathfrak{X}^{(K)}_{q/v}$, formally given by (2.15), to calculate the probability

$$\begin{aligned}\mathbb{P}\left(\mathsf{m}^0_1 = m_1, \dots, \mathsf{m}^0_x = m_x\right) &= \sum_{m_{x+1}, m_{x+2}, \dots} \mathfrak{X}^{(K)}_{q/v}(K, \emptyset \to 1^{m_1} 2^{m_2} \cdots) \\ &= \mathsf{L}^{(K)}_{q\xi_1 v^{-1}, s_1}(0, K | \, m_1, j_1) \prod_{i=2}^{x} \mathsf{L}^{(K)}_{q\xi_i v^{-1}, s_i} \\ &\quad \times (0, j_{i-1} | \, m_i, j_i),\end{aligned} \tag{4.37}$$

where $j_i = K - m_1 - \dots - m_i$ for $i = 1, \dots, x$. In the last expression we took account of the boundary conditions and we let no path enter the axis $\mathbb{Z}_{\geq 1} \times \{-K+1\}$ from below and exactly K paths entered the region $\mathbb{Z}_{\geq 1} \times \{-K+1, \dots, 0\}$ from the leftmost column of vertices. All factors in the right hand side of (4.37) are of the form

$$\mathsf{L}^{(K)}_{\xi q v^{-1}, s}(0, K - \overline{M}, M, K - \overline{M} - M),$$

for some integers M, $\overline{M}$, so that using result of Lemma 4.33 and expression of weights $\boldsymbol{\ell}^{(i)}_{\wp, v}$ we obtain

$$\mathbb{P}(\mathsf{m}^0_1 = m_1, \mathsf{m}^0_2 = m_2, \dots, \mathsf{m}^0_x = m_x) = \prod_{i=1}^{x} \boldsymbol{\ell}^{(i)}_{\wp, v} \left(m_i; \sum_{j=1}^{i-1} m_j \right),$$

which completes the proof, after identifying m with the random variable m^0_1. □

Result of Proposition 4.8 opens the door to study the measure $\mathbb{P}_{\wp, v, \mathscr{d}}$, at least when $\wp = q^{-K}$, using integral formulas for q-moments presented in Sect. 2.4. We recall that results like (2.5) are available only for the particular choice of step boundary conditions and following construction presented in Proposition 4.8 they are extended to boundary conditions given by Definition 4.7. We remark that at this stage we are not yet ready to study the Higher Spin Six Vertex Model in the case of double sided q-negative binomial boundary conditions, but only when the distribution of $\mathsf{m}^0_2, \mathsf{m}^0_3, \dots$ is of the form (4.29) in which probability weights $\boldsymbol{\ell}^{(i)}_{\wp, v}$ are considered with $\wp$, v as in (4.28). We devote the remaining part of this Subsection to extend integrability results of the measure $\mathbb{P}_{\wp, v, \mathscr{d}}$ also to the region of parameters $\wp$, v in (4.27). The strategy we follow is an analytic continuation of the probability distribution of the shifted height function $\overline{\mathcal{H}}$.

Following the construction provided in Proposition 4.8, we recover the equality

$$\mathfrak{h}(x+1, t) - K \overset{\mathcal{D}}{=} \overline{\mathcal{H}}(x, t), \tag{4.38}$$

for all meaningful x, t, where the left hand side refers to a Higher Spin Six Vertex Model on $\Lambda_{0,-K}$ with step boundary conditions and employing relation (4.38) we write the one point probability distribution of $\overline{\mathcal{H}}$ using that of $\mathfrak{h}$. Following techniques analogous to those used in [2,23], we now provide a description of the probability mass function of $\overline{\mathcal{H}}(x, t)$ when the probability measure is considered both with choices of parameters (4.27) or (4.28).

Proposition 4.10 *Consider the probability measure* $\mathbb{P}_{\wp,v,\mathscr{d}}$ *introduced in Definition* 4.7 *and assume that parameters* $\Xi, \mathbf{S}, \mathbf{U}$ *satisfy* (1.8) *and the additional bounds*

$$\sup_i\{\xi_i s_i s_j/\xi_j\} < 1, \quad \sup_i\{s_i/\xi_i\} < 1, \quad |\wp| < |1/v| \times \inf_i\{\xi_i s_i\}.$$

Then, we have

$$\mathbb{P}_{\wp,v,\mathscr{d}}\left(\overline{\mathcal{H}}(x,t) = l\right) = (q;q)^{x-1} \int_{\mathbb{T}^x} \prod_{j=1}^{x} \frac{dz_j}{z_j} m_x^q(\mathbf{z}) \frac{\widetilde{\Pi}\left(\mathbf{z},\, \Xi^{-1}\mathbf{S}, \mathbf{U}\right)}{\widetilde{\Pi}\left(\Xi\mathbf{S},\, \Xi^{-1}\mathbf{S}, \mathbf{U}\right)}$$
$$\times \frac{\left(\frac{\Xi S}{Z}; q\right)_\infty}{\prod_{i,j=1}^{x}\left(\frac{\xi_i s_i}{z_j}; q\right)_\infty} \left(\frac{\Xi S}{Z}\right)^l \prod_{j=1}^{x} \frac{\left(\wp\frac{v}{z_j}; q\right)_\infty \left(\frac{v}{\xi_j s_j}; q\right)_\infty}{\left(\wp\frac{v}{\xi_j s_j}; q\right)_\infty \left(\frac{v}{z_j}; q\right)_\infty}, \tag{4.39}$$

where $\Xi S/Z = \mathscr{d}(\prod_{i=2}^{x}\xi_i s_i)/(\prod_{i=1}^{n} z_i)$, m_x^q *is the* q*-Sklyanin measure* (3.2) *and the the factor* $\widetilde{\Pi}$ *is given by*

$$\widetilde{\Pi}\left(\mathrm{z},\, \Xi^{-1}\mathbf{S}, \mathbf{U}\right) = \prod_{j=1}^{x}\left(\prod_{i=2}^{x}(z_j s_i/\xi_i; q)_\infty^{-1} \prod_{i=1}^{t}(z_j u_i; q)_J\right). \tag{4.40}$$

The proof of Proposition 4.10 makes use of the matching of q-moments between the height in the Higher Spin Six Vertex Model and the corner coordinate in a q-Whittaker process stated in Proposition 3.2 and is based on a result of [53]. We have the following.

Lemma 4.11 *Consider the probability measure* $\mathbb{P}_{\wp,v,\mathscr{d}}$ *as in Proposition* 4.10 *with parameters* $\wp$, v *as in* (4.28)*. Then, we have*

$$\mathbb{E}_{q^{-K},v,\mathscr{d}}\left(q^{l(\overline{\mathcal{H}}(x,t)+K)}\right) = \mathbb{E}_{\mathbb{W}_{\Xi,\mathbf{S},\hat{\mathbf{U}}}}\left(q^{l\lambda_x}\right), \tag{4.41}$$

where the right hand side refers to the q*-Whittaker measure* (3.8) *specialized as in* (4.31)*,* (4.32)*.*

Proof We know, from Proposition 4.8 that the probability measure $\mathbb{P}_{q^{-K},v,\mathscr{d}}$ is obtained as a marginal process from a Higher Spin Six Vertex Model on $\Lambda_{0,-K}$ with step boundary conditions and parameters specialized as (4.31), (4.32). From this equivalence of models relation (4.38) follows and we see that (4.41) is obtained as a corollary of Proposition 3.2, since choice (4.32) annihilates the term $q^n \prod_{j=1}^{x} s_j^2$ in (3.7). □

Lemma 4.12 *Consider the probability measure* $\mathbb{P}_{\wp,v,\mathscr{d}}$ *as in Proposition* 4.10 *with parameters* $\wp$, v *as in* (4.28)*. Then, we have*

$$\mathbb{P}_{q^{-K},v,\mathscr{d}}\left(\overline{\mathcal{H}}(x,t)+K=l\right) = (q;q)_\infty^{x-1}\int_{\mathbb{T}^x}\prod_{j=1}^{x}\frac{dz_j}{z_j}m_x^q(\mathbf{z})\frac{\Pi(\mathbf{z};\,\Xi^{-1}\mathbf{S},\mathbf{U})}{\Pi(\Xi\mathbf{S};\,\Xi^{-1}\mathbf{S},\mathbf{U})} \times\frac{(\Xi S/Z;q)_\infty}{\prod_{i,j=1}^{x}(\xi_j s_j/z_i;q)_\infty}\left(\frac{\Xi S}{Z}\right)^l, \tag{4.42}$$

where $\Xi S/Z=\mathscr{d}(\prod_{i=2}^{x}\xi_i s_i)/(\prod_{i=1}^{n}z_i)$*,* m_x^q *is the* q*-Sklyanin measure* (3.2) *and the factor* Π *in the integrand is given by*

$$\Pi(\mathbf{z};\,\Xi^{-1}\mathbf{S},\mathbf{U})=\prod_{j=1}^{x}\left(\prod_{i=2}^{x}(z_j s_i/\xi_i;q)_\infty^{-1}\prod_{i=1}^{t}(z_j u_i;q)_J(qz_j/v;q)_K\right). \tag{4.43}$$

Proof The matching of q-moments reported in Lemma 4.12 implies that the q-Laplace transforms $\mathbb{E}_{q^{-K},v,\mathscr{d}}(1/(\zeta q^{\overline{\mathcal{H}}(x,t)+K};q)_\infty)$ and $\mathbb{E}_{\mathbb{W}_{\Xi,\mathbf{S},\hat{\mathbf{U}}}}(1/(\zeta q^{\lambda_x};q)_\infty)$ are equal. This implies that

$$\mathbb{P}_{q^{-K},v,\mathscr{d}}\left(\overline{\mathcal{H}}(x,t)+K=l\right)=\mathbb{W}_{\Xi,\mathbf{S},\hat{\mathbf{U}}}(\lambda_x=l) \tag{4.44}$$

for all l in $\mathbb{Z}$, from which (4.42) follows after specializing (3.10) according to (4.31), (4.32). ☐

Proof of Proposition 4.10 Lemma 4.12 established the claim of Proposition 4.10 for the choice (4.28) of parameters $\wp$, v, so in order to conclude our argument we need to extend such result to the region (4.27) as well. We will show that both sides of (4.39) are analytic functions of the variable v in a neighborhood of zero. Expanding in Taylor series the equality (4.39) can be written as

$$\sum_{n\geq 0}P_n(\wp)v^n=\sum_{n\geq 0}R_n(\wp)v^n. \tag{4.45}$$

where the radius of convergence of both series depends on the magnitude of $\wp$ and it is given by conditions

$$\max_i\left|\frac{v\wp}{\xi_i s_i}\right|<1,\quad \max_i\left|\frac{v}{\xi_i s_i}\right|<1. \tag{4.46}$$

In particular, for any compact set $\mathcal{C}\subset\mathbb{C}$, there exists a small enough neighborhood of $v=0$ such that both sides of (4.39) are well defined for all $\wp\in\mathcal{C}$. For all n, we will prove that P_n and R_n are polynomials in the variable $\wp$. We can therefore set

$$d_n=\max(\deg(P_n),\deg(R_n))$$

and take v small enough so that the Taylor expansions in (4.45) hold for all $\wp$ in a disk of radius greater than q^{-d_n-1} and centered at the origin. As a result of Lemma 4.12, P_n, R_n assume the same value when $\wp = q^{-1}, \dots, q^{-d_n-1}$, since for these particular choices (4.39) holds. This means that $P_n - R_n$ is a polynomial of degree d_n with $d_n + 1$ zeros and hence P_n and R_n are the same function. Since n is generic we can conclude that all Taylor coefficients of the expansion of left and right hand side of (4.45) coincide and this concludes our argument.

We come now to verify the claim that all expressions we deal with are analytic in v and that P_n, R_n are polynomials. We treat separately the left and the right hand side of (4.39).

lhs of (4.39): first we write down the probability of the event $\{\overline{\mathcal{H}}(x,t) = l\}$ as

$$\mathbb{P}_{\wp,v,d}\left(\overline{\mathcal{H}}(x,t) = l\right) = \sum_{M_1,\dots,M_x \geq 0} \prod_{i=1}^{x} \boldsymbol{\ell}^{(i)}_{\wp,v}\left(M_i; \sum_{j=1}^{i-1} M_j\right) \mathbb{P}_{M_1,\dots,M_x}\left(\overline{\mathcal{H}}(x,t) = l\right),$$

where the families of weights $\boldsymbol{\ell}^{(i)}_{\wp,v}$ have been defined in (4.26) and the notation $\mathbb{P}_{M_1,\dots,M_x}(E)$ is a shorthand for $\mathbb{P}_{\wp,v,d}(E|\mathsf{m} = M_1, \dots \mathsf{m}^0_x = M_x)$ for any event E. Naturally the probabilities $\mathbb{P}_{M_1,\dots,M_x}$ do not depend either on v or $\wp$ as these are probabilities of events in the Higher Spin Six Vertex Model with deterministic boundary conditions given by occupation numbers $M_1, \dots, M_x$ and v, $\wp$ only pertain factors $\boldsymbol{\ell}^{(i)}_{\wp,v}$. Set a small positive number ϵ such that

$$\left|\frac{v}{\xi_i s_i}\right| < 1 - \epsilon, \quad \left|\frac{\wp v s_i}{\xi_i}\right| < 1 - \epsilon, \quad \text{for all } i = 1, \dots, x.$$

With these conditions it is easy to see, from the definition (4.26) of the weight $\boldsymbol{\ell}^{(i)}_{\wp,v}$ that a bound as

$$\begin{aligned} \left|\boldsymbol{\ell}^{(i)}_{\wp,v}(M; \overline{M})\right| &< \left|\frac{v}{\xi_i s_i}\right|^{M} \frac{\left(-|s_i|^2, -|\wp q^{\overline{M}}|, q\right)_\infty}{\left(|v\wp q^{\overline{M}} s_i/\xi_i|, q; q\right)_\infty} \\ &\quad \times \frac{\left(-|v\wp q^{\overline{M}} s_i/\xi_i|, -|v/(\xi_i s_i)|; q\right)_\infty}{\left(|v\wp q^{\overline{M}}/(\xi_i s_i)|, |v s_i/\xi_i|; q\right)_\infty} < C' \left|\frac{v}{\xi_i s_i}\right|^{M_i} \end{aligned}$$

holds for a constant $C' = C'(\wp, \epsilon, \xi_i, s_i)$. The lhs of (4.39) is therefore an absolutely convergent series of analytic functions in v and the analyticity in the region (4.46) follows. The Taylor expansion at $v = 0$ of the weight $\boldsymbol{\ell}^{(i)}_{\wp,v}$ is easily seen to be of the form

$$\sum_{n \geq M} \widetilde{p}^{(i)}_n(\wp, M, \overline{M}) v^n, \tag{4.47}$$

where $\widetilde{p}_n^{(i)}(\wp, M, \overline{M})$ is a polynomial in $\wp$. Alternatively we can rewrite the right hand side of (4.47) as

$$\sum_{n\geq 0} p_{n-M}^{(i)}(\wp, M, \overline{M}) v^n,$$

where $p_L^{(i)}(\wp, M, \overline{M})$ is again a polynomial in $\wp$, which is zero when $L < 0$. With this notation the lhs of (4.39) admits the expansion

$$\begin{aligned}
&\sum_{M_1,\dots,M_x\geq 0} \prod_{i=1}^{x} \left(\sum_{n_i\geq 0} p_{n_i-M_i}^{(i)}(\wp, M_i, \sum_{j=1}^{i-1} M_j) v^{n_i} \right) \mathbb{P}_{M_1,\dots,M_x}\left(\overline{\mathcal{H}}(x,t)=l\right)\\
&= \sum_{M_1,\dots,M_x\geq 0} \left[\sum_{n\geq 0} \left(\sum_{n_1+\dots+n_x=n} \prod_{i=1}^{x} p_{n_i-M_i}^{(i)}(\wp, M_i, \sum_{j=1}^{i-1} M_j) \right) v^n \right]\\
&\quad \times \mathbb{P}_{M_1,\dots,M_x}\left(\overline{\mathcal{H}}(x,t)=l\right)\\
&= \sum_{n\geq 0} \left(\sum_{\substack{n_1+\dots+n_x=n\\ M_1,\dots,M_x\geq 0}} \mathbb{P}_{M_1,\dots,M_x}\left(\overline{\mathcal{H}}(x,t)=l\right) \prod_{i=1}^{x} p_{n_i-M_i}^{(i)}(\wp, M_i, \sum_{j=1}^{i-1} M_j) \right) v^n,
\end{aligned}$$

where one can see that in the last equality the coefficient of v^n is a polynomial in $\wp$ as the generic summation in the M_i terminates when $n_i - M_i < 0$.

rhs of (4.39): the analyticity in the variable v is evident, so we look at the Taylor expansion around zero. Using the q-binomial theorem (A.7), the term in the integrand depending on v and $\wp$ can be expanded as

$$\frac{\left(\frac{\wp v}{z_j}, \frac{v}{\xi_j s_j}; q\right)_\infty}{\left(\frac{\wp v}{\xi_j s_j}, \frac{v}{z_j}; q\right)_\infty} = \sum_{n\geq 0} v^n \left(\frac{1}{z_j^n} \sum_{m=0}^{n} \frac{\left(\frac{\xi_j s_j}{z_j}; q\right)_m \left(\frac{z_j}{\xi_j s_j}; q\right)_{n-m}}{(q;q)_m (q;q)_{n-m}} \left(\frac{\wp z_j}{\xi_j s_j}\right)^m \right) \tag{4.48}$$

By means of simple inequalities as

$$\left|1 - q^k \frac{\xi_j s_j}{z_j}\right| \leq 1 + \left|\frac{\xi_j s_j}{z_j}\right|, \quad \left|1 - q^k \frac{z_j}{\xi_j s_j}\right| \leq 1 + \left|\frac{z_j}{\xi_j s_j}\right|, \quad |1-q^k| \geq 1-q$$

we see that the coefficient of v^n in the right hand side of (4.48) is a polynomial in $\wp$ and can be bounded by nC''^n, where C'' does not depend on v. By taking v sufficiently small we can bring the summation outside of the integral in the rhs of (4.39) and obtain a summation of the form

$$\sum_{n\geq 0} R_n(\wp) v^n$$

with $R_n(\wp)$ polynomials as promised. □

The statement of Proposition 4.10 establishes the exact solvability of the coupled measure $\mathbb{P}_{\wp,\upsilon,\mathscr{d}}$. Naturally, $\mathbb{P}_{\wp,\upsilon,\mathscr{d}}$ isn't a particularly interesting object per se, but rather its specialization $\wp = 0$, which describes a double sided q-negative binomial Higher Spin Six Vertex Model coupled with an independent random variable $\mathsf{m} \sim q\mathrm{Poi}(\upsilon/\mathscr{d})$. Unfortunately, due to the presence of m, the measure $\mathbb{P}_{0,\upsilon,\mathscr{d}} = \mathbb{P}_{\mathrm{HS}(\upsilon,\mathscr{d})\otimes\mathsf{m}}$ is only well defined when $\upsilon < \mathscr{d}$ and this condition prevents us to study the stationary model directly from $\mathbb{P}_{\mathrm{HS}(\upsilon,\mathscr{d})\otimes\mathsf{m}}$. In order to consider the case $\upsilon = \mathscr{d}$, in Sect. 5 we will decouple the Higher Spin Six Vertex Model from m, expressing the probability distribution of $\mathcal{H}$ rather than that of $\overline{\mathcal{H}}$. This will allow us to consider a different family of double sided q-negative binomial boundary conditions, where parameters υ, $\mathscr{d}$ will be subjected to the less stringent bound (1.18) as explained below in the proof of Theorem 1.4.

5 Fredholm determinant formulas for double sided *q*-negative binomial boundary conditions

The main content of this section consists in the proofs of Theorems 1.3, 1.4 presented in the Introduction. We do this by first considering the coupled model $\mathbb{P}_{\wp,\upsilon,\mathscr{d}}$, which in Sect. 4.2 was proven to be integrable, and then considering its degeneration $\wp = 0$.

5.1 Fredholm determinants in the coupled model $\mathbb{P}_{\wp,\upsilon,\mathscr{d}}$

In this section we give a Fredholm determinant expression for the q-Laplace transform of the probability mass function of the shifted height function $\overline{\mathcal{H}}$ defined in (4.30). Results given in Proposition 5.1 hold for the coupled measure $\mathbb{P}_{\wp,\upsilon,\mathscr{d}}$, for a general coupling parameter $\wp$. In Sect. 5.2 we will consider the meaningful choice $\wp = 0$ and hence the model with double sided q-negative binomial boundary conditions. The proof of Proposition 5.1 is based on calculations involving an elliptic version of the Cauchy determinant that were developed in a previous work by two of the authors [39] and it is therefore omitted.

Proposition 5.1 *Assume conditions on parameters* $\Xi, \mathbf{S}, \mathbf{U}, \wp, \upsilon$ (1.16), (4.27)*, take* $\upsilon < \mathscr{d}$ *and* $\zeta \in \mathbb{C} \setminus q^{\mathbb{Z}}$*. Then we have*

$$\mathbb{E}_{\wp,\upsilon,\mathscr{d}}\left(\frac{1}{(\zeta q^{\overline{\mathcal{H}}(x,t)};q)_\infty}\right) = \det(\mathbf{1} - fK)_{l^2(\mathbb{Z})}, \tag{5.1}$$

where

$$f(n) = \frac{1}{1 - q^n/\zeta}, \tag{5.2}$$

$$K(n,m) = \sum_{l=1}^{x-1} \phi_l(m)\psi_l(n) + (\mathscr{d} - \upsilon)\Phi_x(m)\Psi_x(n), \tag{5.3}$$

$$\phi_l(n) = \tau(n) \int_D \frac{dw}{2\pi \mathrm{i}} \frac{1}{w^{x+n-l+1}} \prod_{k=1}^{l} \frac{1}{(w - \xi_{k+1}s_{k+1})} \frac{(qv/w;q)_\infty}{F(w)}, \tag{5.4}$$

$$\psi_l(n) = \frac{\xi_{l+1}s_{l+1}}{\tau(n)} \int_C \frac{dz}{2\pi \mathrm{i}} z^{n+x-l-1} \prod_{k=2}^{l} (z - \xi_k s_k) \frac{F(z)}{(qv/z;q)_\infty}, \tag{5.5}$$

$$\Phi_x(n) = \tau(n) \int_D \frac{dw}{2\pi \mathrm{i}} \frac{1}{w^{n+1}} \frac{1}{w - \mathscr{d}} \prod_{k=2}^{x} \frac{1}{w - \xi_k s_k} \frac{(qv/w;q)_\infty}{F(w)}, \tag{5.6}$$

$$\Psi_x(n) = \frac{1}{\tau(n)} \int_C \frac{dz}{2\pi \mathrm{i}} \frac{z^{n-1}}{(v/z;q)_\infty} \prod_{k=2}^{x} (z - \xi_k s_k) F(z). \tag{5.7}$$

The contour D encircles $\{\mathscr{d}, \xi_2 s_2, \ldots, \xi_x s_x\}$ *and no other singularity, whereas C contains 0 and* vq^k*, for any k in* $\mathbb{Z}_{\geq 0}$*. Finally,* $\tau(n)$ *is taken to be*

$$\tau(n) = \begin{cases} b^n & \textit{if } n \geq 0, \\ c^n & \textit{if } n < 0, \end{cases} \tag{5.8}$$

with

$$v < b < \mathscr{d} \leq \inf_{i \geq 2}\{\xi_i s_i\} \leq \sup_{i \geq 2}\{\xi_i s_i\} < c < \inf_{i \geq 2}\{\xi_i / s_i\},$$

and

$$F(z) = (v\wp/z, qz/\mathscr{d}; q)_\infty \prod_{j=1}^{t} (zu_j;q)_J \prod_{k=2}^{x} \frac{(qz/(\xi_k s_k);q)_\infty}{(zs_k/\xi_k;q)_\infty}. \tag{5.9}$$

Proof From Proposition 4.10 we can apply, with minor changes the same argument of [39], Theorem 4.3. More specifically, using the notation used in [39], we need to set

$$a_k = \begin{cases} s_{k+1}\xi_{k+1}, & \text{if } k \neq x \\ \mathscr{d}, & \text{if } k = x \end{cases}, \quad \alpha_k = \delta_{k,x} v,$$

and substitute e^{zt} with the expression

$$(v\wp/z;q)_\infty \frac{\prod_{k=1}^{t} (zu_k;q)_J}{\prod_{k=1}^{x} (zs_k/\xi_k;q)_\infty},$$

as a result of considering a more general specialization of the q-Whittaker measure as that presented in Proposition 3.6. □

An alternative expression for the kernel K is given defining an auxiliary kernel A as

$$A(n,m) = \sum_{l=1}^{x-1} \phi_l(n)\psi_l(m) \tag{5.10}$$

of which we report the explicit form.

Proposition 5.2 (Double integral kernel) *The discrete kernel A admits the following expression*

$$\begin{aligned} A(n,m) = \frac{\tau(n)}{\tau(m)} \frac{1}{(2\pi \mathrm{i})^2} \int_D dw \int_C dz \frac{z^m}{w^{n+1}} \prod_{j=1}^{t} \left(\frac{(u_j z; q)_J}{(u_j w; q)_J} \right) \\ \times \prod_{k=2}^{x} \left(\frac{(z/(\xi_k s_k), w s_k/\xi_k; q)_\infty}{(w/(\xi_k s_k), z s_k/\xi_k; q)_\infty} \right) \frac{(qv/w, v\wp/z, qz/\mathscr{d}; q)_\infty}{(qv/z, v\wp/w, qw/\mathscr{d}; q)_\infty} \frac{1}{z-w}. \end{aligned} \tag{5.11}$$

Proof All it takes to show (5.11) is to perform the summation

$$\sum_{l=1}^{x-1} \phi_l(n)\psi_l(m),$$

using the rather tricky identity

$$\frac{1}{z-w}\left[\frac{w^{x-1}}{z^{x-1}} \prod_{k=2}^{x} \frac{z-a_k}{w-a_k} - 1\right] = \sum_{l=1}^{x-1} \frac{a_{l+1}}{w-a_{l+1}} \frac{w^{l-1}}{z^l} \prod_{k=2}^{l} \frac{z-a_k}{w-a_k},$$

which can be proven by induction. We see that the addend $(z-w)^{-1}$ in the left hand side doesn't give any contribution to the integral as integrating over the variable z it only leaves an integral in w over a path containing no singularities. □

Before moving to the analysis of the stationary case we show that the Fredholm determinant expression (5.1) makes sense. We need the following.

Proposition 5.3 *The kernel fK defined by Eqs.* (5.2) *to* (5.7) *is trace class.*

Proof From fK being a finite sum of products of operators of rank one, it is enough to show that each one of these operators is of Hilbert-Schmidt class. This is essentially proven in "Appendix B". In fact the generic terms $f(n)\phi_l(n)\psi_l(m)$ and $f(n)\Phi_x(n)\Psi_x(m)$ are bounded in absolute value by quantities exponentially small in $|n| + |m|$, thanks to Proposition B.1, and therefore the double summation

$$\sum_{n,m\in\mathbb{Z}} |f(n)K(n,m)|^2$$

is indeed convergent. □

We close this subsection by offering the proof of Theorem 1.3, presented in Sect. 1.5.

Proof of Theorem 1.3 This is a trivial consequence of Proposition 5.1, after setting $\wp = 0$. □

5.2 The double sided *q*-negative binomial case and the stationary specialization

In this Section we give a proof of Theorem 1.4, that characterizes the probability distribution of the height function $\mathcal{H}$ in the Higher Spin Six Vertex Model with double sided q-negative binomial boundary conditions. Our starting point is the Fredholm determinant formula stated in Theorem 1.3. Removing the effect of the independent q-Poisson random variable m from expression (1.17) we find that determinantal expressions we obtain are well defined in the region (1.18). In Corollary 5.7 we specialize the result of Theorem 1.4 to the relevant case of the stationary Higher Spin Six Vertex Model.

Before we begin the proof of Theorem 1.4, we like to state some regularity properties of the integral kernel fA defined by (5.2), (5.10) that hold for parameters v, d in (1.18).

Proposition 5.4 *Let $\zeta < 0$ and take f, A as in (5.2), (5.10). Then $\mathbf{1} - fA$ is an invertible operator. Moreover both fA and $(\mathbf{1} - fA)^{-1}$ are well defined, bounded operators in the region (1.18).*

In the proof of Proposition 5.4 we use the following biorthogonality property.

Lemma 5.5 *Let v, d be such that $v < d$ or (1.18) holds. Then we have*

$$\sum_{n \in \mathbb{Z}} \phi_l(n) \psi_m(n) = \delta_{l,m}. \tag{5.12}$$

Proof This is a simple consequence of the contour integral expressions (5.4), (5.5). The generic term in the summation in the left hand side of (5.12) is

$$\phi_l(n)\psi_m(n) = \xi_{m+1} s_{m+1} \int_D \int_C \frac{dw dz}{(2\pi \mathrm{i})^2} \left(\frac{z}{w}\right)^n \frac{w^{l-1}}{z^{m+1}} \frac{\prod_{k=2}^{m}(z - \xi_k s_k)}{\prod_{k=2}^{l+1}(w - \xi_k s_k)} \frac{G(w)}{G(z)},$$

where, in the function G, we gathered together the factors independent on n or l, m as

$$G(u) = \frac{(qv/u; q)_\infty}{u^x F(u)}.$$

We see that in order to take the summation over all integers inside the integrals we need $|z/w|$ to be suitably defined depending on the positivity of n itself.

Consider the contour $\widetilde{C}$ being a circle of radius r such that $\max_i |\xi_i s_i| < r < \min_i |\xi_i / s_i|$. We can write

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \phi_l(n)\psi_m(n)\\
&\quad = \xi_{m+1}s_{m+1}\left[\sum_{n\geq 0}\int_D\int_C \frac{dwdz}{(2\pi\mathrm{i})^2}\left(\frac{z}{w}\right)^n \frac{w^{l-1}}{z^{m+1}} \frac{\prod_{k=2}^m (z-\xi_k s_k)}{\prod_{k=2}^{l+1}(w-\xi_k s_k)}\frac{G(w)}{G(z)}\right.\\
&\quad \left. + \sum_{n<0}\int_D\int_{\widetilde{C}} \frac{dwdz}{(2\pi\mathrm{i})^2}\left(\frac{z}{w}\right)^n \frac{w^{l-1}}{z^{m+1}} \frac{\prod_{k=2}^m (z-\xi_k s_k)}{\prod_{k=2}^{l+1}(w-\xi_k s_k)}\frac{G(w)}{G(z)}\right]\\
&\quad = \xi_{m+1}s_{m+1}\left[\int_D\left(\int_{\widetilde{C}}-\int_C\right)\frac{dwdz}{(2\pi\mathrm{i})^2}\frac{1}{z-w}\frac{w^l}{z^{m+1}}\frac{\prod_{k=2}^m (z-\xi_k s_k)}{\prod_{k=2}^{l+1}(w-\xi_k s_k)}\frac{G(w)}{G(z)}\right]\\
&\quad = \xi_{m+1}s_{m+1}\left[\int_D\int_{\widetilde{D}}\frac{dwdz}{(2\pi\mathrm{i})^2}\frac{1}{z-w}\frac{w^l}{z^{m+1}}\frac{\prod_{k=2}^m (z-\xi_k s_k)}{\prod_{k=2}^{l+1}(w-\xi_k s_k)}\frac{G(w)}{G(z)}\right],
\end{aligned}$$

where, for the last equality, we deformed $\widetilde{C}$ into an union or the two contours C and $\widetilde{D}$, with the latter being a curve encircling D and no other singularity for the z variable. Performing the z integral we get

$$\begin{aligned}
&\xi_{m+1}s_{m+1}\int_D \frac{dw}{2\pi\mathrm{i}} w^{l-m-1}\left(\mathbb{1}_{m=l+1} + \mathbb{1}_{m\leq l}\prod_{k=m+1}^{l+1}\frac{1}{w-\xi_k s_k}\right.\\
&\quad \left. +\mathbb{1}_{m\geq l+2}\prod_{k=l+2}^{m}(w-\xi_k s_k)\right).
\end{aligned}$$

Naturally, when $m > l$ there is no pole inside D and the integral vanishes. On the other hand, if $m \leq l$ we can evaluate the residue at infinity and obtain the result. □

Proof We want to show that $\|fA\|_{l^2(\mathbb{Z})} < 1$, so to define $(\mathbf{1} - fA)^{-1}$ through the geometric series

$$\sum_{k\geq 0}(fA)^k.$$

A first observation is that $\|A\| = 1$ and this follows from the biorthogonality relation (5.12). For this set $V = \mathrm{span}\{\psi_i \,|\, i = 1, \dots, x-1\}$ and notice that, by biorthogonality, we also have $V = \mathrm{span}\{\phi_i \,|\, i = 1, \dots, x-1\}$ (if $\widetilde{v} = \sum c_i\phi_i$ such that $\widetilde{v} \perp V$, then $\widetilde{v} = 0$). For any $h \in \ell^2(\mathbb{Z})$ write the orthogonal decomposition $h = h_V + h_{V^\perp}$ where $h_V \in V$ and $h_{V^\perp} \in V^\perp$. Then we have $Ah = Ah_V = h_V$ for all h since $A\phi_i = \phi_i$ for all $i = 1, \dots, x-1$ and we conclude that

$$\|A\| = \sup_{h_V\in V}\frac{\|Ah_V\|}{\|h_V\|} = 1.$$

To show that the operator norm of fA is strictly smaller than 1, consider the following bound for the function f, defined in (5.2),

$$f(n) \leq (1-\varepsilon) f(n+n_0) + \varepsilon P_{n_1}(n),$$

with $P_{n_1}(n) = \mathbb{1}_{n \geq n_1}$ and ε, n_0, n_1, suitably chosen. In particular, taking n_0 large enough and ε small we can let n_1 be an arbitrary big number. Moreover, the fact that f is a diagonal operator implies the simple estimate

$$\|fA\| \leq (1-\varepsilon)\|f\|\|A\| + \varepsilon\|P_{n_1}A\| \leq 1 - \varepsilon + \varepsilon\|P_{n_1}A\|.$$

Let's consider now an element η in the unitary sphere of $l^2(\mathbb{Z})$. By simply using the definition of the kernel A and the Schwartz inequality we have

$$\|P_{n_1}A\eta\| \leq \sum_{l=1}^{x-1} \left(\sum_{n\in\mathbb{Z}} \left| \mathbb{1}_{n\geq n_1} \phi_l(n) \sum_{m\in\mathbb{Z}} \psi_l(m)\eta(m) \right|^2 \right)^{1/2} \leq \sum_{l=1}^{x-1} \|\psi_l\| \left(\sum_{n\geq n_1} |\phi_l(n)|^2 \right)^{1/2},$$

which, thanks to the bound (B.1), can be shown to be geometrically small in n_1. Therefore $\|fA\| < 1$.

Finally, we remark that for sequences ϕ_l or ψ_l the bounds (B.1) hold true also in the region (1.18), so A is analytic in this domain, so are its powers and so is $(\mathbf{1} - fA)^{-1}$ as one can show the geometrical decay of derivatives of $(fA)^N$ as well. □

The following lemma offers a tool to decouple a generic process from the contribution of an independent q-Poisson random variable.

Lemma 5.6 *Let* $\mathsf{m} \sim qPoi(p)$*. Then, for any bounded function* B*, we have*

$$B(z) = \frac{1}{(p;q)_\infty} \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} p^k \mathbb{E}_{\mathsf{m}}(B(z-\mathsf{m}-k)). \tag{5.13}$$

Proof To verify identity (5.13) we simply open up the average in the right hand side with respect to m, as

$$\mathbb{E}_{\mathsf{m}}(B(z-\mathsf{m}-k)) = \sum_{l\geq 0} p^l \frac{(p;q)_\infty}{(q;q)_l} B(z-l-k).$$

We can now rearrange the double summation in the indices k, l, naming $L = l + k$, as

$$\text{rhs of (5.13)} = \sum_{L\geq 0} B(z-L) \sum_{k=0}^{L} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k (q;q)_{L-k}},$$

which completes the proof, after recognizing, in the right hand side, the q-Pochhammer expansion (A.3) (with $z = 1$) that is one for $L = 0$ and zero otherwise. □

In the remaining part of the paper we will use the following decomposition of terms Φ_x, Ψ_x:

$$\Phi_x(n) = \Phi_x^{(1)}(n) + \Phi_x^{(2)}(n), \tag{5.14}$$

$$\Psi_x(n) = \Psi_x^{(1)}(n) + \Psi_x^{(2)}(n), \tag{5.15}$$

obtained separating from the integration (5.6) (resp. (5.7)) the contribution of pole $w = \mathscr{d}$ (resp. $z = v$) from that of other poles. The exact expressions are

$$\Phi_x^{(1)}(n) = \frac{\tau(n)}{\mathscr{d}^{n+1}} \prod_{k=2}^{x} \frac{1}{\mathscr{d} - \xi_k s_k} \frac{(qv/\mathscr{d};q)_\infty}{F(\mathscr{d})}, \tag{5.16}$$

$$\Phi_x^{(2)}(n) = \tau(n) \int_{D_1} \frac{dw}{2\pi \mathrm{i}} \frac{1}{w^{n+1}} \frac{1}{w - \mathscr{d}} \prod_{k=2}^{x} \frac{1}{w - \xi_k s_k} \frac{(qv/w;q)_\infty}{F(w)}, \tag{5.17}$$

$$\Psi_x^{(1)}(n) = \frac{v^n}{\tau(n)} \prod_{k=2}^{x} (v - \xi_k s_k) \frac{F(v)}{(q;q)_\infty}, \tag{5.18}$$

$$\Psi_x^{(2)}(n) = \frac{1}{\tau(n)} \int_{C_1} \frac{dz}{2\pi \mathrm{i}} \frac{z^{n-1}}{(v/z;q)_\infty} \prod_{k=2}^{x} (z - \xi_k s_k) F(z), \tag{5.19}$$

where F was given in (5.9), contour D_1 contains $\{\xi_i s_i\}_{i \geq 2}$ and no other singularity and C_1 contains $\{q^k v\}_{k \geq 1}$ and no other singularity.

Proof of Theorem 1.4 Using Lemma 5.6 setting $B : z \to \mathbb{E}_{\mathrm{HS}(v,\mathscr{d})}(1/(\zeta q^{\mathcal{H}+z};q)_\infty)$ and expressing the q-Laplace transform $\mathbb{E}_{\mathrm{HS}(v,\mathscr{d})}(1/(\zeta q^{\mathcal{H}-\mathsf{m}};q)_\infty)$ as in (1.17) we obtain formula (1.19). Therefore we only need to show that expression (1.19) is well posed in the region (1.18).

Using basic properties of Fredholm determinants, along with the regularity of the kernel fA proved in Proposition 5.4, we can write

$$\begin{aligned} \det(\mathbf{1} - fK) &= \det\left(\mathbf{1} - fA - (\mathscr{d} - v) f \Phi_x \Psi_x\right) \\ &= \det(\mathbf{1} - fA) \det\left(\mathbf{1} - (\mathscr{d} - v)(\varrho f \Phi_x)\Psi_x\right), \end{aligned}$$

where we called $\varrho = (\mathbf{1} - fA)^{-1}$. This allows us to express $V_{x;v,\mathscr{d}}(\zeta)$ as

$$V_{x;v,\mathscr{d}}(\zeta) = \det(\mathbf{1} - fA) \frac{1}{1 - v/\mathscr{d}} \left(1 - (\mathscr{d} - v) \sum_{n \in \mathbb{Z}} (\varrho f \Phi_x)(n) \Psi_x(n) \right), \tag{5.20}$$

We turn our attention to the term

$$\sum_{n \in \mathbb{Z}} (\varrho f \Phi_x)(n) \Psi_x(n) = \sum_{n \in \mathbb{Z}} f(n) \Phi_x(n) \Psi_x(n) + \sum_{n \in \mathbb{Z}} (fA\varrho \Phi_x)(n) \Psi_x(n), \tag{5.21}$$

that we rewrite, using Eqs. (5.16) to (5.19) as

$$(5.21) = \sum_{n\in\mathbb{Z}} f(n)\Phi_x^{(1)}(n)\Psi_x^{(1)}(n) + \sum_{\substack{i,j=1,2\\(i,j)\neq(1,1)}} \sum_{n\in\mathbb{Z}} f(n)\Phi_x^{(i)}(n)\Psi_x^{(j)}(n) + \sum_{n\in\mathbb{Z}} (f A_\varrho f \Phi_x)(n)\Psi_x(n). \tag{5.22}$$

We easily see that, thanks to bounds stated in "Appendix B", the second and the third terms in the right hand side of (5.22) are geometrically convergent summations in the region (1.18). On the other hand, the generic term of the summation in the first addend of the right hand side of (5.22) takes the form

$$\frac{1}{1-q^n/\zeta}\frac{1}{\mathscr{d}}\left(\frac{v}{\mathscr{d}}\right)^n \prod_{k=2}^{x} \frac{v-\xi_k s_k}{\mathscr{d}-\xi_k s_k}\frac{(qv/\mathscr{d};q)_\infty}{(q;q)_\infty}\frac{F(v)}{F(\mathscr{d})}.$$

We can perform the summation over n through the Ramanujan ${}_1\psi_1$ formula ([41], Theorem 12.3.1) as

$$\sum_{n\in\mathbb{Z}} \frac{1}{1-q^n/\zeta}\left(\frac{v}{\mathscr{d}}\right)^n = \frac{(v/(\mathscr{d}\zeta), q\zeta\mathscr{d}/v, q, q; q)_\infty}{(v/\mathscr{d}, q\mathscr{d}/v, 1/\zeta, q\zeta; q)_\infty}.$$

which itself leads to the expression

$$\sum_{n\in\mathbb{Z}} f(n)\Phi_x^{(1)}(n)\Psi_x^{(1)}(n) = \frac{1}{\mathscr{d}}\frac{(v/(\mathscr{d}\zeta), q\zeta\mathscr{d}/v, q; q)_\infty}{(v/\mathscr{d}, 1/\zeta, q\zeta; q)_\infty}\frac{F(v)}{F(\mathscr{d})}\prod_{k=2}^{x}\frac{v-\xi_k s_k}{\mathscr{d}-\xi_k s_k}. \tag{5.23}$$

Combining the explicit formula (5.23) with (5.22) and (5.20) we see that $V_{x;v,\mathscr{d}}$ does not in fact present any singularity in $v=\mathscr{d}$ by virtue of the Taylor expansion

$$\text{rhs of (5.23)} = \frac{1}{\mathscr{d}-v} + \frac{1}{\mathscr{d}}\left(\upsilon_0(1/\zeta) - \upsilon_0(q\zeta) + 2\upsilon_0(q) + x h_0(\mathscr{d}) - \sum_{j=1}^{t} a_0(\mathscr{d};j)\right) + \mathcal{O}(\mathscr{d}-v), \tag{5.24}$$

where we used the q-polygamma type functions υ_k defined in "Appendix A", their combinations a_0, h_0 presented in (1.20) and the notation $a_0(\mathscr{d};j)$ stresses the dependence on the spectral parameters u_j as

$$a_k(\mathscr{d};j) = \upsilon_k(q^J u_j \mathscr{d}) - \upsilon_k(u_j\mathscr{d}).$$

Therefore $V_{x;v,\mathscr{d}}$ is an analytic function of both parameters v, $\mathscr{d}$ in the region (1.18) and this concludes the proof. □

Calculations performed during the proof of Theorem 1.4 can be exploited to obtain determinantal formulas for for the stationary Higher Spin Six Vertex Model.

Corollary 5.7 *Consider stationary Higher Spin Six Vertex Model with parameters* $\mathscr{d}, \Xi, \mathbf{S}$ *as in* (1.16)*. Then we have*

$$\mathbb{E}_{\mathrm{HS}(\mathscr{d},\mathscr{d})}\left(\frac{1}{(\zeta q^{\mathcal{H}(x,t)};q)_\infty}\right) = \frac{1}{(q;q)_\infty}\sum_{k\geq 0}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} q^k \left(V_x(\zeta q^{-k}) - V_x(\zeta q^{-k-1})\right), \tag{5.25}$$

with the function $V_x = V_{x;\mathscr{d},\mathscr{d}}$ *being*

$$V_x(\zeta) = \det(\mathbf{1} - fA)\Bigg(- \upsilon_0(1/\zeta) + \upsilon_0(q\zeta) - 2\upsilon_0(q) - x h_0(\mathscr{d}) + \sum_{j=1}^{t} a_0(\mathscr{d};j) - \mathscr{d}\sum_{\substack{i,j=1,2\\(i,j)\neq(1,1)}}\sum_{n\in\mathbb{Z}} f(n)\Phi_x^{(i)}(n)\Psi_x^{(j)}(n) - \mathscr{d}\sum_{n\in\mathbb{Z}}(fA\varrho f\Phi_x)(n)\Psi_x(n)\Bigg). \tag{5.26}$$

Proof All it takes to show (5.25) is to take the limit $v \to \mathscr{d}$ of both sides of equality (1.17). We exchange the limit sign in $v \to \mathscr{d}$ and the summation sign in the right hand side of (1.17) and this can be done as, in the proof of Theorem 1.4, the function $V_{x;v,\mathscr{d}}$ was shown to be uniformly bounded in a neighborhood of $v = \mathscr{d}$ and its limit $V_{x;v,\mathscr{d}} \to V_{x;\mathscr{d},\mathscr{d}}$ is readily computed using (5.24). This is enough to prove that

$$\lim_{v\to\mathscr{d}}\mathbb{E}_{\mathrm{HS}(v,\mathscr{d})}\left(\frac{1}{(\zeta q^{\mathcal{H}(x,t)};q)_\infty}\right) = \frac{1}{(q;q)_\infty}\sum_{k\geq 0}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} V_x(\zeta q^{-k}). \tag{5.27}$$

Finally, substituting $V_x(\zeta q^{-k})$ with $q^k V_x(\zeta q^{-k}) + (1-q^k)V_x(\zeta q^{-k})$ and rearranging the summation in the right hand side of (5.27) we obtain (5.25). □

6 Asymptotics along the critical line

In this section we discuss the time asymptotics of the stationary Higher Spin Six Vertex Model. First we give some details on the general conjecture concerning scaling limits of models in the Kardar–Parisi–Zhang universality class and subsequently we confirm these conjectures in this particular case.

6.1 The KPZ scaling for the higher spin six vertex model

Before we enter the discussion it is appropriate to recall the definition of the Baik–Rains distribution [8], which will ultimately describe long time fluctuations of the stationary height function $\mathcal{H}$ under a suitable scaling. Rather than showing the original definition given by authors in [8], formula (2.16), we present an equivalent expression first found in [38] (see also [33]).

The building blocks for the construction of the Baik–Rains distributions are given in the following

Definition 6.1 (*Airy function and Airy kernel*) The Airy function Ai is given by

$$Ai(\nu) = \frac{1}{2\pi \mathrm{i}} \int_{e^{-\frac{\mathrm{i}}{3}\pi}\infty}^{e^{\frac{\mathrm{i}}{3}\pi}\infty} \exp\left\{\frac{z^3}{3} - z\nu\right\} dz,$$

where the integration contour is any open complex curve having the half lines $\{Re^{\frac{\mathrm{i}}{3}\pi} | R \geq 0\}$ and $\{Re^{-\frac{\mathrm{i}}{3}\pi} | R \geq 0\}$ as asymptotes.

The Airy kernel[5] K_{Airy} is defined as

$$K_{\mathrm{Airy}}(\nu, \theta) = \int_{e^{-\frac{2}{3}\pi \mathrm{i}}\infty}^{e^{\frac{2}{3}\pi \mathrm{i}}\infty} \frac{dz}{2\pi \mathrm{i}} \int_{e^{\frac{\pi}{3}\mathrm{i}}\infty}^{e^{-\frac{\pi}{3}\mathrm{i}}\infty} \frac{dw}{2\pi \mathrm{i}} \frac{1}{z - w} \exp\left\{\frac{w^3}{3} - \frac{z^3}{3} - w\nu + z\theta\right\}, \tag{6.1}$$

where again integration contours are non intersecting complex curves whose asymptotes are half lines $\{Re^{\pm\frac{\mathrm{i}}{3}\pi} | R \geq 0\}$ for w and $\{Re^{\mp \mathrm{i}\frac{2}{3}\pi} | R \geq 0\}$ for z.

We come to the next

Definition 6.2 (*Baik–Rains distribution*) Let $\varpi \in \mathbb{R}$ and define the one parameter family of functions

$$\begin{aligned} \chi_\varpi(r) = F_2(r)\Bigg(r - \varpi^2 - \sum_{\substack{i,j=1,2 \\ (i,j)\neq(1,1)}} \int_r^\infty \Upsilon_{-\varpi}^{(i)}(\nu)\Upsilon_{\varpi}^{(j)}(\nu) d\nu \\ - \int_r^\infty d\nu \Upsilon_\varpi(\nu) \int_r^\infty d\lambda_1 \int_r^\infty d\lambda_2 \varrho_{\mathrm{Airy};r}(\nu, \lambda_1) K_{Airy}(\lambda_1, \lambda_2) \Upsilon_{-\varpi}(\lambda_2)\Bigg) \end{aligned} \tag{6.2}$$

where terms $F_2, \varrho_{Airy;r}, \Upsilon_\varpi^{(i)}, \Upsilon_\varpi$ are given by:

- $F_2(r)$ is the GUE Tracy–Widom distribution [66]

$$F_2(r) = \det\left(\mathbf{1} - \mathbb{1}_{[r,\infty)} K_{Airy}\right)_{\mathcal{L}^2(\mathbb{R})}; \tag{6.3}$$

---

[5] Sometimes the equivalent expression $K_{\mathrm{Airy}}(\nu, \theta) = \int_0^\infty \mathrm{Ai}(\lambda + \nu)\mathrm{Ai}(\lambda + \theta) d\lambda$ is found in literature.

- $\varrho_{Airy;r}(\nu, \lambda)$ is the kernel

$$\varrho_{Airy;r}(\nu, \lambda) = \left(\mathbf{1} - \mathbb{1}_{[r,\infty)} K_{Airy}\right)^{-1}(\nu, \lambda); \tag{6.4}$$

- auxiliary functions $\Upsilon_{\varpi}^{(1)}, \Upsilon_{\varpi}^{(2)}$ are

$$\Upsilon_{\varpi}^{(1)}(\nu) = e^{\frac{\varpi^3}{3} - \nu\varpi}, \quad \Upsilon_{\varpi}^{(2)}(\nu) = \int_{e^{-\frac{2}{3}\mathrm{i}\pi}\infty}^{e^{\frac{2}{3}\mathrm{i}\pi}\infty} \frac{d\omega}{2\pi\mathrm{i}} \frac{\exp\{-\frac{\omega^3}{3} + \omega\nu\}}{\omega + \varpi}, \tag{6.5}$$

  where the integration contour passes to the left of $-\varpi$;
- lastly

$$\Upsilon_{\varpi}(\nu) = \Upsilon_{\varpi}^{(1)}(\nu) + \Upsilon_{\varpi}^{(2)}(\nu). \tag{6.6}$$

The Baik–Rains distribution F_{ϖ} is

$$F_{\varpi}(r) = \frac{\partial}{\partial r}\chi_{\varpi}(r). \tag{6.7}$$

Remark 6.3 The equivalence between F_{ϖ} of (6.7) and an analogous expression found in [33] is discussed in the Remark at the end of Section 5 of [39]. Further, in Appendix A of [33] the authors prove that their definition of the Baik–Rains distribution is equivalent to the original one introduced in [8].

We now possess all the ingredients to give a brief explanation of the KPZ scaling theory, which gives a precise conjecture to describe stationary (asymptotic) fluctuations of the height function of models in the KPZ universality class [63].

We start considering a properly rescaled version of our model, where, for convenience we interpret the vertical spatial coordinate as a time direction and where we regard space and time as continuous parameters. In this case the height function $\mathcal{H}$, defined for the Higher Spin Six Vertex Model in (1.12), still contains every information on the random dynamics thanks to relations

$$\mathcal{H}(x, t) - \mathcal{H}(x + dx, t) = \#\text{of paths in } [x, x + dx] \text{ at time t}, \tag{6.8}$$

$$\mathcal{H}(x, t + dt) - \mathcal{H}(x, t) = \#\text{of paths crossing } x \text{ during the time interval } [t, t + dt]. \tag{6.9}$$

For the sake of argument assume that the average of space and time infinitesimal increments of $\mathcal{H}$ are regular enough to define the deterministic density ρ and current j as

$$\mathbb{E}\left(\mathcal{H}(x + dx, t) - \mathcal{H}(x, t)\right) \approx -\rho(x, t)dx,$$
$$\mathbb{E}\left(\mathcal{H}(x, t + dt) - \mathcal{H}(x, t)\right) \approx j(x, t)dt.$$

The system is autonomous, or, in other words, its evolution depends on space and time only implicitly, therefore the current j must only be a function of ρ and the continuity equation linking these quantities reads

$$\partial_t \rho(x,t) + \partial_x j(\rho(x,t)) = 0. \tag{6.10}$$

At this stage the height $\mathcal{H}$ remains defined, through (6.8), (6.9), only up to a global constant and to remove this ambiguity we fix its value at the reference point $x = 0$, $t = 0$ to be $\mathcal{H}(0,0) = 0$. With this choice the average profile of $\mathcal{H}$ at the generic space time point (x,t) can be expressed as

$$\eta(x,t) = -\int_0^x \rho(y,0)dy + \int_0^t j(\rho(x,s))ds \tag{6.11}$$

and the study of fluctuations of the height is, by definition, the study of the random quantity

$$\mathcal{H}(x,t) - \eta(x,t). \tag{6.12}$$

Assume now that the system has reached its steady state, or equivalently assume that at time zero the measure is stationary. Qualitatively, the randomness of (6.12) is affected by two different contributions. One is coming from the stochastic evolution of the system and the other is given by initial conditions. For growth processes in the KPZ universality class, when initial conditions are deterministic and sufficiently regular, fluctuations in the long time scale are expected to present with size of order $t^{1/3}$. This conjecture goes back to the seminal paper [44], where authors argued such property to hold for the solution of the one dimensional KPZ equation itself. On the other hand, from our knowledge of the stationary measure of the Higher Spin Six Vertex Model displayed in Proposition 1.2, we certainly expect fluctuations in the space direction to have size of order $x^{1/2}$, as a result of the Central Limit Theorem applied to independent occupation numbers m_x^0 at each site. This means that the information we have about $\mathcal{H}$, which is the choice $\mathcal{H}(0,0) = 0$, will be transported by the random dynamics along the direction of growth of the surface and along this line we can observe the emergence of the $1/3$ exponent. Along all other lines, the distribution of (6.12) will be affected very little by the process, and asymptotic fluctuations remain of gaussian nature. An earlier evidence of this last fact was found in [31] (see also [6], Appendix D).

The direction along which nontrivial fluctuations are observed is given by the characteristic line of partial differential equation (6.10). This is the curve (x_t, t), where x_t is set to be the solution of the differential equation

$$\begin{cases} \dot{x}_t = j'(\rho(x_t,t)), \\ x_0 = 0. \end{cases} \tag{6.13}$$

When the system is in its stationary state, equation (6.13) loses its dependence on time and the $\dot{x_t}$ is only function of the stationary density profile ρ_{st}. In case the model does not present space inhomogeneities the characteristic curve is simply the line $(j'(\rho_{\mathrm{st}})t, t)$, but when the stationary density is not constant, this is no more true. We use the explicit parametrization (x, t_x), rather than (x_t, t) and by integration of (6.13), we obtain

$$t_x = \int_0^x \frac{dy}{j'(\rho_{\mathrm{st}}(y))}. \tag{6.14}$$

To reiterate what we just explained, consider diverging x and t. Assume first that

$$|t - t_x| = \mathcal{O}\Big(x^{2/3+\delta}\Big),$$

for some $\delta > 0$. Then, asymptotically (6.12) obeys the gaussian distributions and its size becomes of order $x^{1/2}$. When on the other hand, (x, t) is taken in the vicinity of the characteristic curve, say

$$|t - t_x| = \mathcal{O}\Big(x^{2/3}\Big),$$

then fluctuations become of size $x^{1/3}$ and their law is described by the Baik–Rains distribution.

We can be more precise. Take at first $t = t_x$. The convergence result, in this case, is

$$\frac{\mathcal{H}(x, t_x) - \eta(x, t_x)}{\gamma x^{1/3}} \xrightarrow[x\to\infty]{\mathcal{D}} F_0,$$

where, calling $\sigma_y^2 dy$ the variance of the number of paths lying at time 0 in the infinitesimal segment $[y, y + dy]$ and its mean

$$\overline{\sigma^2} = \lim_{x\to\infty} \frac{1}{x} \int_0^x \sigma_y^2 dy,$$

then the constant γ is given by

$$\gamma^3 = -\lim_{x\to\infty} \frac{1}{2} j''(\rho_{\mathrm{st}})(\overline{\sigma^2})^2 \frac{t_x}{x}. \tag{6.15}$$

The explicit parametrization of a fan of size $x^{2/3}$ around the characteristic line can be still expressed in terms of macroscopic quantities. Consider a perturbation of t_x of the form

$$t_{x,\varpi} = t_x - \varpi \frac{\overline{\sigma}^2 j''(\rho_{\mathrm{st}})}{\gamma j'(\rho_{\mathrm{st}})^2} x^{2/3}. \tag{6.16}$$

with ϖ being a real number. The resulting effect on the expression of η reads, up to order $x^{1/3}$, as

$$\eta_{x,\varpi} = \eta(x,t_x) - \varpi \frac{\overline{\sigma}^2 j''(\rho_{\mathrm{st}}) j(\rho_{\mathrm{st}})}{\gamma j'(\rho_{\mathrm{st}})^2} x^{2/3} - \frac{1}{2}\varpi^2 \frac{(\overline{\sigma^2})^2 j''(\rho_{\mathrm{st}})}{\gamma^2 j'(\rho_{\mathrm{st}})} x^{1/3}. \tag{6.17}$$

In this case, the convergence result for fluctuations along the line $(x, t_{x,\varpi})$ becomes

$$\frac{\mathcal{H}(x, t_{x,\varpi}) - \eta_{x,\varpi}}{\gamma x^{1/3}} \xrightarrow[x\to\infty]{\mathcal{D}} F_{\varpi}. \tag{6.18}$$

The same kind of results are conjectured for discrete time systems, where the characteristic curves can be again explicitely expressed through relation (6.14). In the next section we will establish result (6.18) for the stationary Higher Spin Six Vertex Model. For this model, the scaling parameters κ_ϖ, η_ϖ, γ were defined in Eqs. (1.21) to (1.23) and it is a simple exercise to verify that they match with expressions given in Eqs. (6.15) to (6.17).

6.2 The Baik–Rains limit

This subsection is devoted to the proof of Theorem 1.5, that characterizes the asymptotic fluctuations of the height function in the stationary Higher Spin Six Vertex Model. Throughout the proof we will assume that the model presents only spatial inhomogeneities and hence the spectral parameters $\mathbf{U}$ are taken as

$$\mathbf{U} = (u, u, u, \dots). \tag{6.19}$$

This simply implies that the transfer matrix $\mathfrak{X}^{(J)}$ stays the same at each time step.

Our strategy relies on taking the large x limit of the q-Laplace transform of the height functions $\mathcal{H}$ given in (5.25). The computation of such a limit for the right hand side of (5.25) might look complicated, but the expression will simplify after the right change of variables (discussed in detail below in Sect. 6.3). Motivated by the KPZ scaling theory discussed in Sect. 6.1, we fix parameters t and ζ as

$$t = \kappa x \quad \text{and} \quad \zeta = -q^{-\eta x + \gamma x^{1/3} r}. \tag{6.20}$$

Here and in the rest of the Section, for the sake of a cleaner notation, we set

$$\eta = \eta_\varpi, \quad \text{and} \quad \kappa = \kappa_\varpi,$$

dropping the explicit dependence on the real number ϖ from η_ϖ, κ_ϖ introduced in eqs. (1.22) and (1.23). The first choice in (6.20) means that we are considering the behaviour of the height along the critical line, while the choice for ζ reflects the fact that we study fluctuations of size $\gamma x^{1/3}$ around the expected value η of the height function. We assume the parameter r of (6.20) to be fixed throughout the entire section and it will ultimately represent the argument of the Baik–Rains distribution, as in (1.24).

The asymptotic analysis of expressions given in Corollary 5.7 will be performed via a rigorous steep descent method. Although we postpone the details of the analysis to Sects. 6.4 and 6.5, we now fix hypothesis on parameters q, Ξ, $\mathbf{S}$, that will hold true throughout the rest of the Section.

Definition 6.4 (*Conditions on parameters*) Take a, σ such that $a > \mathpzc{d}$ and $\sigma \in [0, 1)$. Parameters $q, \mathpzc{d}, \Xi, \mathbf{S}$ are assumed to satisfy (1.8),(1.16) and they are spaced so that there exist R_a, R_σ, R_q, with the properties that

$$a \le \xi_k s_k \le a + R_a, \quad \sigma \le s_k^2 \le \sigma + R_\sigma, \quad \text{for all } k, \quad 0 \le q \le R_q \tag{6.21}$$

and

$$a + R_a < \frac{2a}{1+\sigma} < \mathpzc{d}/q. \tag{6.22}$$

Numbers R_a, R_σ, R_q are strictly positive, yet small in the sense given by Proposition C.3.

The first and the second conditions in (6.21) are not too much prohibitive and they essentially say that we can consider perturbations of a general homogeneous model, since parameters a and σ can be chosen freely. The strongest assumption in Definition 6.4 is indeed the third one in (6.21) and it says that the parameter q has to remain reasonably close to 0. The reason for such restrictions in the choice of parameters lies in the perturbative approach we used to prove Proposition C.3. There, we showed the steep ascent property (for the function g defined below in (6.42)) of integration contour D, in the case where $q = 0, \xi_k s_k = a, s_k^2 = \sigma$ for each $k \ge 2$. Subsequently, through a continuity argument we concluded that the same property must hold also when parameters are taken in suitably small neighborhoods of our original choices, hence (6.21).

Although numerical checks show that a steep ascent contour D indeed exists also for q reasonably greater than zero (yet not too much close to 1), obtaining sharp bounds for parameters becomes difficult due to the complicated expressions we encounter setting $q > 0$. More precisely, to prove Theorem 1.5 when q is taken far from 0 would mean constructing an explicit closed contour D on which one would be able to show that the function g assumes a global minimum in a neighborhood of $\mathpzc{d}$. This is indeed possible in principle, but obtaining explicit bounds for parameters q, ξ_k, s_k becomes prohibitive.

The first inequality in (6.22) is also technical and not very restrictive. It is used in the construction of the explicit steep ascent contour D in Proposition C.3. On the other hand the assumption $2a/(1+\sigma) < \mathpzc{d}/q$, reported in (6.22), is used to ensure the exponential decay of rear tails of $f\Phi_x\Psi_x$ in Propositions 6.8, 6.9.

Remark 6.5 Conditions stated in Definition 6.4 are far from being optimal and they are essentially consequences of our choice for the representation of the integral kernels K, Φ_x, Ψ_x in (5.3),(5.6),(5.7). In particular these technical assumptions are consequence of the fact that D is a closed contour.

We will now present limiting expressions of terms entering the definition (5.26) of function $V_x(\zeta)$, for $x \to \infty$. The proof of the next Proposition is reported in Sect. 6.4.

Proposition 6.6 *We have*

$$\det\left(\mathbf{1} - fA\right)_{l^2(\mathbb{Z})} = F_2(r) + \frac{1}{\gamma x^{1/3}} R_x^{(1)}(r), \tag{6.23}$$

where F_2 is defined in (6.3) and the error term $R_x^{(1)}$ satisfies the following properties

1. *For each $r^* \in \mathbb{R}$, there exists $M_{r^*} > 0$ such that, for all x,*

$$\left| R_x^{(1)}(r^*) \right| < M_{r^*}; \tag{6.24}$$

2. *There exist $\epsilon > 0$, such that, for all $r^* \in [r - \epsilon, r]$, we have*

$$\lim_{x \to \infty} \left(R_x^{(1)}(r^*) - R_x^{(1)}(r^* - \frac{1}{\gamma x^{1/3}}) \right) = 0, \tag{6.25}$$

   uniformly.

Let's now see what is the asymptotic behavior of remaining terms of (5.26). The proofs of the following three Propositions are given in Sect. 6.5 below.

Proposition 6.7 *Recall choice (6.20). Then we have*

$$\begin{aligned} &\frac{1}{\gamma x^{1/3}} \left(t a_0(d) - \upsilon_0(1/\zeta) - 2\upsilon_0(q) + \upsilon_0(q\zeta) - x h_0(d) \right) \\ &\quad = r - \varpi^2 + \frac{1}{\gamma x^{1/3}} R_x^{(2)}(r). \end{aligned} \tag{6.26}$$

The error term $R_x^{(2)}$ satisfies the following properties

1. *for each $r^* \in \mathbb{R}$, there exists $M_{r^*} > 0$ such that, for all x,*

$$\left| R_x^{(2)}(r^*) \right| < M_{r^*}; \tag{6.27}$$

2. *there exist $\epsilon > 0$, such that, foe all $r^* \in [r - \epsilon, r]$ we have*

$$\lim_{x \to \infty} \left(R_x^{(2)}(r^*) - R_x^{(2)}(r^* - \frac{1}{\gamma x^{1/3}}) \right) = 0. \tag{6.28}$$

   uniformly.

Lastly we state the convergence result for terms $\Phi_x^{(i)}$, Φ_x, $\Psi_x^{(j)}$, Ψ_x.

Proposition 6.8 *We have*

$$\begin{aligned}&\frac{d}{\gamma x^{1/3}}\sum_{n\in\mathbb{Z}}\sum_{\substack{i,j=1,2\\(i,j)\neq(1,1)}} f(n)\Phi_x^{(i)}(n)\Psi_x^{(j)}(n)\\&=\sum_{\substack{i,j=1,2\\(i,j)\neq(1,1)}}\int_r^\infty \Upsilon^{(i)}_{-\varpi}(\nu)\Upsilon^{(j)}_{\varpi}(\nu)d\nu+\frac{1}{\gamma x^{1/3}}R_x^{(3)}(r),\end{aligned} \tag{6.29}$$

where functions $\Upsilon^{(1)}, \Upsilon^{(2)}$ *are defined in* (6.5) *and the error term* $R_x^{(3)}$ *satisfies the following properties*

1. *for each* $r^* \in \mathbb{R}$ *there exists* $M_{r^*} > 0$ *such that, for all* x*,*

$$\left|R_x^{(3)}(r^*)\right| < M_{r^*}; \tag{6.30}$$

2. *there exists* $\epsilon > 0$*, such that, for all* $r^* \in [r-\epsilon, r]$ *we have*

$$\lim_{x\to\infty}\left(R_x^{(3)}(r^*+\frac{1}{\gamma x^{1/3}})-R_x^{(3)}(r^*)\right)=0. \tag{6.31}$$

   uniformly.

Proposition 6.9 *We have*

$$\begin{aligned}&\frac{d}{\gamma x^{1/3}}\sum_{n\in\mathbb{Z}}(fA\varrho f\ \Phi_x)(n)\Psi_x(n)\\&=\int_r^\infty d\nu\Upsilon_{\varpi}(\nu)\int_r^\infty d\lambda_1\int_r^\infty d\lambda_2\varrho_{\mathrm{Airy};r}(\nu,\lambda_1)\\&\times K_{\times Airy}(\lambda_1,\lambda_2)\Upsilon_{-\varpi}(\lambda_2)+\frac{1}{\gamma x^{1/3}}R_x^{(4)}(r),\end{aligned} \tag{6.32}$$

where the integral kernel $\varrho_{\mathrm{Airy};r}$ *and the function* Υ_{ϖ} *were defined in* (6.4)*,* (6.6) *and the error term* $R_x^{(4)}$ *satisfies the following properties*

1. *for each* $r^* \in \mathbb{R}$ *there exists* $M_{r^*} > 0$ *such that, for all* x

$$\left|R_x^{(4)}(r^*)\right| < M_{r^*}; \tag{6.33}$$

2. *there exists* $\epsilon > 0$ *such that, for all* $r^* \in [r-\epsilon, r]$*, we have*

$$\lim_{x\to\infty}\left(R_x^{(4)}(r^*)-R_x^{(4)}\left(r^*-\frac{1}{\gamma x^{1/3}}\right)\right)=0. \tag{6.34}$$

   uniformly.

Using convergence results reported in the Propositions 6.6, 6.7, 6.8, 6.9 we are now ready to prove our main Theorem.

Proof of Theorem 1.5 Using a rather elementary argument, detailed in Secion 5 of [34], it is possible to show that proving (1.24) is equivalent to showing that

$$\lim_{x\to\infty} \mathbb{E}_{\mathrm{HS}(\mathscr{d},\mathscr{d})}\left(\frac{1}{\left(-q^{\mathcal{H}(x,\kappa x)-\eta x+r\gamma x^{1/3}};q\right)_\infty}\right) = F_\varpi(r).$$

To do so we use formula (5.25) to express the q-Laplace transform on the left hand side. We want to evaluate

$$\lim_{x\to\infty}\frac{1}{(q;q)_\infty}\sum_{k\geq 0}\frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k}\left(V_x\left(\zeta q^{-k}\right)-V_x\left(\zeta q^{-k-1}\right)\right), \tag{6.35}$$

and to do so we aim to bring the limit inside the summation symbol. We start by fixing a small number ϵ and we split the summation in (6.35) into two different contributions. One comes from the sum over k ranging in the region $[0,\epsilon\gamma x^{1/3}]$ and the other is given by k in $(\epsilon\gamma x^{1/3},\infty)$. For each of these terms we can use different estimates.

We start with the latter, that is we take $k>\epsilon\gamma x^{1/3}$. A general inequality that can be deduced from the definition of $V_x=\lim_{v\to\mathscr{d}} V_{x;v,\mathscr{d}}$ and from Theorem 1.3, in case ζ is a negative number, is

$$\begin{aligned} V_x(\zeta q^{-k}) &= \lim_{v\uparrow\mathscr{d}}\frac{1}{1-v/\mathscr{d}}\mathbb{E}_{\mathrm{HS}(v,\mathscr{d})\otimes\mathsf{m}}\left(\frac{1}{\left(\zeta q^{-k}q^{\mathcal{H}-\mathsf{m}};q\right)_\infty}\right)\\ &\leq \lim_{v\uparrow\mathscr{d}}\frac{1}{1-v/\mathscr{d}}\mathbb{E}_{\mathrm{HS}(v,\mathscr{d})\otimes\mathsf{m}}\left(\frac{1}{\left(\zeta q^{-k^*}q^{\mathcal{H}-\mathsf{m}};q\right)_\infty}\right) = V_x\left(\zeta q^{-k^*}\right), \end{aligned}$$

which holds for every $k^*<k$. By taking $k^*=\epsilon\gamma x^{1/3}$ and $r^*=r-\epsilon$ we obtain the estimate

$$\begin{aligned} &q^{\binom{k+1}{2}}\left(V_x(\zeta q^{-k})-V_x(\zeta q^{-k-1})\right)\\ &\leq 2q^{\binom{k+1}{2}}V_x(\zeta q^{(r^*-r)\gamma x^{1/3}})\\ &= 2q^{\binom{k+1}{2}}\left(\gamma x^{1/3}\chi_\varpi(r^*)+\sum_{i=1}^4 S^{(i)}(r^*)R_x^{(i)}(r^*)+\mathcal{O}(x^{-1/3})\right). \end{aligned} \tag{6.36}$$

In the right hand side of (6.36) we used results of Propositions 6.6, 6.7, 6.8, 6.9 to provide the approximate expression of V_x. Function χ_ϖ was defined in (6.2) and terms $S^{(i)}$'s are explicit, bounded functions which for convenience we do not report explicitly. We can therefore write

$$\frac{1}{(q;q)_\infty} \sum_{k>\epsilon\gamma x^{1/3}} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k} \left(V_x(\zeta q^{-k}) - V_x(\zeta q^{-k-1}) \right) = \mathcal{O}(e^{-cx^{2/3}}), \tag{6.37}$$

for some positive constant c, since, from (6.36) we see that the right hand side is a quantity exponentially small in $x^{2/3}$, due to the presence of the term $q^{\binom{k+1}{2}}$.

We now consider the contribution of the summation in (6.35), when the index k is smaller than $\epsilon\gamma x^{1/3}$. Once again, using results of Propositions 6.6, 6.7, 6.8, 6.9 we have

$$\begin{aligned} & V_x(\zeta q^{-k}) - V_x(\zeta q^{-k-1}) \\ & \quad = \gamma x^{1/3} \left(\chi_\varpi\left(r - \frac{k}{\gamma x^{1/3}}\right) - \chi_\varpi\left(r - \frac{k+1}{\gamma x^{1/3}}\right) \right) \\ & \quad + \sum_{i=1}^{4} S^{(i)}\left(r - \frac{k}{\gamma x^{1/3}}\right) \left(R_x^{(i)}\left(r - \frac{k}{\gamma x^{1/3}}\right) - R_x^{(i)}\left(r - \frac{k+1}{\gamma x^{1/3}}\right) \right) \\ & \quad + \mathcal{O}(x^{-1/3}), \end{aligned} \tag{6.38}$$

which immediately implies

$$\left| V_x(\zeta q^{-k}) - V_x(\zeta q^{-k-1}) \right| \leq \text{const}, \tag{6.39}$$

after expanding χ_ϖ around $r - \frac{k}{\gamma x^{1/3}}$.

We can finally evaluate the limit (6.35). Using the bound (6.37), we write

$$\begin{aligned} (6.35) &= \lim_{x\to\infty} \left(\sum_{k=0}^{\epsilon\gamma x^{1/3}} + \sum_{k>\epsilon\gamma x^{1/3}} \right) \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_\infty (q;q)_k} \left(V_x(\zeta q^{-k}) - V_x(\zeta q^{-k-1}) \right) \\ &= \lim_{x\to\infty} \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_\infty (q;q)_k} \left(V_x(\zeta q^{-k}) - V_x(\zeta q^{-k-1}) \right) \mathbb{1}_{[0,\epsilon\gamma x^{1/3}]}(k) \end{aligned} \tag{6.40}$$

and following estimate (6.39), we can employ the bounded convergence theorem to exchange the limit and summation symbols in the right hand side of (6.40). Here the pointwise convergence

$$\lim_{x\to\infty} V_x(\zeta q^{-k}) - V_x(\zeta q^{-k-1}) = \frac{\partial}{\partial r} \chi_\varpi(r)$$

can be established through the expansion (6.38), using the fact that the difference between remainder terms $R_x^{(i)}$'s converges to zero, as reported in Propositions 6.6, 6.7, 6.8, 6.9. We can therefore write

$$(6.35) = \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_\infty (q;q)_k} \frac{\partial}{\partial r} \chi_\varpi(r) = \frac{\partial}{\partial r} \chi_\varpi(r),$$

which concludes the proof. □

6.3 Scaling form of determinantal formulas

In this Section we present expressions of kernels $f(n)$, $A(n,m)$, $\Phi^{(i)}(n)$, $\Psi^{(j)}(n)$, $\varrho(n,m)$ one finds in the definition of function V_x given in (5.26), that are amaenable to rigorous asymptotic analysis.

We fix a parametrization of integer indeces n, m as

$$n = n_\nu = -\eta x + \nu \gamma x^{1/3}, \quad m = m_\theta = -\eta x + \theta \gamma x^{1/3}. \tag{6.41}$$

here ν, θ belong to the set of rescaled integers

$$\widetilde{\mathbb{Z}} = \{\nu \in \mathbb{R} | -\eta x + \nu \gamma x^{1/3} \in \mathbb{Z}\}$$

and we will use the symbol $\widetilde{\sum}$ to denote a summation where the index ranges over $\widetilde{\mathbb{Z}}$ rather than $\mathbb{Z}$. We also introduce the scaling function

$$g(z) = -\eta \log(z) + \kappa a_{-1}(z) - h_{-1}(z), \tag{6.42}$$

where

$$a_{-1}(z) = \log(zu;q)_J \quad \text{and} \quad h_{-1}(z) = \frac{1}{x} \sum_{y=2}^{x} \log\left(\frac{(zs_y/\xi_y;q)_\infty}{(z/(\xi_y s_y);q)_\infty} \right).$$

Functions a_k, h_k, for $k \geq 0$, were defined in (1.20) and they satisfy the properties

$$z\frac{d}{dz} a_k(z) = a_{k+1}(z) \quad \text{and} \quad z\frac{d}{dz} h_k(z) = h_{k+1}(z),$$

for all $k \geq -1$.

The combination of the KPZ scaling (6.20) and of the change of variable (6.41) is summarized in the following:

Proposition 6.10 *Assume (6.20), (6.41) and fix a real number L such that*

$$-L < r. \tag{6.43}$$

Define the sequence

$$\widetilde{\tau}(\nu) = \begin{cases} \varsigma^{\nu\gamma x^{1/3}}, & \text{if } \nu > -L, \\ \tau(n_\nu), & \text{if } \nu \le -L, \end{cases} \tag{6.44}$$

where ς *is a number in a neighborhood of order* $x^{-1/3}$ *of* $\mathscr{d}$ *and its exact expression in given below in* (6.48)*. Define also*

$$\widetilde{f}(\nu) = f(n_\nu) = \frac{1}{1+q^{(\nu-r)\gamma x^{1/3}}}, \tag{6.45}$$

$$\begin{aligned} \widetilde{A}(\nu,\theta) &= \frac{\widetilde{\tau}(\nu)}{\widetilde{\tau}(\theta)} \int_D \frac{dw}{2\pi \mathrm{i} w} \int_C \frac{dz}{2\pi \mathrm{i}} \frac{z^{\theta\gamma x^{1/3}}}{w^{\nu\gamma x^{1/3}}} \frac{e^{xg(z)}}{e^{xg(w)}} \frac{(q\mathscr{d}/w, qz/\mathscr{d};q)_\infty}{(q\mathscr{d}/z, qw/\mathscr{d};q)_\infty} \frac{1}{z-w}, \\ \widetilde{\Phi}_x^{(1)}(\nu) &= \mathscr{d}^{-\nu\gamma x^{1/3}-1} e^{-xg(\mathscr{d})}, \quad \widetilde{\Phi}_x^{(2)}(\nu) \\ &= \int_{D_1} \frac{dw}{2\pi \mathrm{i} w} w^{-\nu\gamma x^{1/3}} e^{-xg(w)} \frac{(q\mathscr{d}/w;q)_\infty}{(qw/\mathscr{d};q)_\infty} \frac{1}{w-\mathscr{d}}, \\ \widetilde{\Psi}_x^{(1)}(\nu) &= \mathscr{d}^{\nu\gamma x^{1/3}} e^{xg(\mathscr{d})}, \quad \widetilde{\Psi}_x^{(2)}(\nu) = \int_{C_1} \frac{dz}{2\pi \mathrm{i}} z^{\nu\gamma x^{1/3}} e^{xg(z)} \frac{(qz/\mathscr{d};q)_\infty}{(q\mathscr{d}/z;q)_\infty} \frac{1}{z-\mathscr{d}}. \\ \widetilde{\Phi}_x(\nu) &= \widetilde{\Phi}_x^{(1)}(\nu) + \widetilde{\Phi}_x^{(2)}(\nu), \quad \widetilde{\Psi}_x(\nu) = \widetilde{\Psi}_x^{(1)}(\nu) + \widetilde{\Psi}_x^{(2)}(\nu) \end{aligned} \tag{6.46}$$

Then formula (5.25) *for the* q*-Laplace transform still holds if we substitute, in the expression of* V_x (5.26)*,* $f, A, \Phi_x^{(i)}, \Psi_x^{(j)}, \Phi_x, \Psi_x$ *with* $\widetilde{f}, \widetilde{A}, \widetilde{\Phi}_x^{(i)}, \widetilde{\Psi}_x^{(j)}, \widetilde{\Phi}_x, \widetilde{\Psi}_x$ *and we change the summation signs* $\sum$ *with* $\widetilde{\sum}$*.*

Proof We can easily see that the tilde notation corresponds to applying to functions in (5.26) the change of variables (6.41). The Fredholm determinant $\det(\mathbf{1}-fA)_{l^2(\mathbb{Z})}$ is clearly not affected by the multiplication of A with the gauge factor $\frac{\widetilde{\tau}(\nu)}{\widetilde{\tau}(\theta)} \frac{\tau(m_\theta)}{\tau(n_\nu)}$, nor by the change of variables and it is therefore equal to $\det(\mathbf{1}-\widetilde{f}\widetilde{A})_{l^2(\widetilde{\mathbb{Z}})}$. Similar considerations are true also for the remaining functions in (5.26). □

The function g was used to simplify the expression of integrands of quantities $A, \Phi^{(i)}, \Psi^{(j)}$. Its crucial feature is that, in a neighborhood of size $x^{-1/3}$ of $\mathscr{d}$, it admits the expansion

$$g(z) = g(\varsigma) + g'''(\varsigma)\frac{(z-\varsigma)^3}{6!} + \mathcal{O}\big(x^{-1/3}\big),$$

where ς is another point in a neighborhood of size $x^{-1/3}$ of $\mathscr{d}$. In other words g has a double critical point ς in the vicinity of $\mathscr{d}$ and this will enable us to analyze the asymptotic form of $A, \Phi^{(i)}, \Psi^{(j)}$ through saddle point method. This is precisely stated in the following:

Proposition 6.11 *With the choice* $\varpi = 0$*, the function* g *has a double critical point at* $\mathscr{d}$*, that is* $g'(\mathscr{d}) = g''(\mathscr{d}) = 0$*. When* $\varpi \neq 0$*, there exists a point* $\varsigma = \varsigma(\varpi)$ *such that*

$$g'(\varsigma) = g''(\varsigma) = \mathcal{O}(1/x), \quad g'''(\varsigma) = -2\frac{\gamma^3}{\varsigma^3} + \mathcal{O}(x^{1/3}) \tag{6.47}$$

and $g'''(\varsigma) < 0$ *for* x *large enough. Moreover, such* ς *admits the expansion*

$$\varsigma = \mathscr{d}\left(1 + \frac{\varpi}{\gamma x^{1/3}} + \frac{1}{2}\frac{\varpi^2}{\gamma^2 x^{2/3}}\left(1 + \frac{a_1^2 h_3 - a_1 a_2 h_2 + 2a_2^2 h_1 - a_1 a_3 h_1}{a_1(a_2 h_1 - a_1 h_2)}\right)\right) + \mathcal{O}\big(1/x^{2/3}\big), \tag{6.48}$$

where $a_k = a_k(\mathscr{d}), h_k = h_k(\mathscr{d})$ *are as in* (1.20).

Proof Equalities reported in (6.47) can be verified by direct inspection making use of the approximate form of ς (6.48). Therefore the only thing we are left to prove is that γ is a positive quantity. From expression (1.21) we write

$$\gamma^3 = \frac{1}{2}\left(h_2(\mathscr{d}) - \frac{h_1(\mathscr{d})}{a_1(\mathscr{d})}a_2(\mathscr{d})\right). \tag{6.49}$$

Functions a_1, a_2 have the explicit expressions

$$a_1(\mathscr{d}) = \sum_{j=0}^{J-1} \frac{-\mathscr{d}uq^j}{(1-\mathscr{d}uq^j)^2}, \quad a_2(\mathscr{d}) = \sum_{j=0}^{J-1} \frac{-\mathscr{d}uq^j(1+\mathscr{d}uq^j)}{(1-\mathscr{d}uq^j)^3},$$

that can be recovered using the form (A.13) to compute υ_1, υ_2. On the other hand, expressing the υ_k's in h_1, h_2 using (A.12) we can write, after some algebraic manipulations, the right hand side of (6.49) as

$$\frac{1}{2}\frac{1}{x}\sum_{y=2}^{x}\sum_{k\geq 1}\left(\frac{\mathscr{d}}{\xi_y s_y}\right)^k \frac{k(1-s_y^{2k})}{1-q^k}\sum_{j=0}^{J-1}\frac{-u\mathscr{d}q^j}{(1-u\mathscr{d}q^j)^2}\left(k - \frac{1+u\mathscr{d}q^j}{1-u\mathscr{d}q^j}\right),$$

that is a sum of positive terms since $u < 0$. □

6.4 Proof of Proposition 6.6

In this Section we present the proof of Proposition 6.6 that establishes the convergence of the Fredholm determinant $\det(\mathbf{1} - fA)$ to the GUE Tracy–Widom distribution. Rather than using the original expressions (5.2) and (5.10) for f and A we will use their riscaled forms $\widetilde{f}, \widetilde{A}$ discussed in Proposition 6.10. Before we move to the more rigorous part of the presentation we like to outline the strategy we will follow:

Step 1 We first establish the pointwise convergence of the kernel $\widetilde{A}(\nu,\theta)$ to the Airy kernel $K_{\mathrm{Airy}}(\nu,\theta)$ and this is done in Lemma 6.12. Our analysis relies on a saddle point method and because of this we can control the error term up to order $x^{-1/3}$ for (ν,θ) lying in an enlarging rectangle $[-L, x^{\delta/3}]^2$ with $\delta \in (0, 1/3)$. This corresponds to the red region in Fig. 13. The reason why we consider the convergence on rectangles that grow with x is that this will allow us to control the difference between $\det(\mathbf{1} - \widetilde{f}\widetilde{A})$ and F_2 as a function of x (see also Remark 6.17). (6.50)

Step 2 We estimate the decay of the kernel $\widetilde{A}(\nu,\theta)$ for (ν,θ) in the set $[-L,\infty]^2 \setminus [-L, x^{\delta/3}]^2$, corresponding to the yellow region in Fig. 13. This is also done through a saddle point analysis; see Lemma 6.13.

Step 3 Combining the exponential decay in x of $\widetilde{f}(\nu)$ for $\nu < r$ and the fact that terms $\widetilde{A}(\nu,\theta)$ are bounded in modulus by 1 (see Lemma 6.14), we can estimate $\det(\mathbf{1} - \widetilde{f}\widetilde{A})_{l^2(\widetilde{\mathbb{Z}})}$ with $\det(\mathbf{1} - \widetilde{f}\widetilde{A})_{l^2(\widetilde{\mathbb{Z}}_{\geq -L})}$ up to an error of order $e^{-\mathrm{const}.x^{1/3}}$. This is the result of Lemma 6.15.

Step 4 As a result of the exponential decay of front tails of the kernel $\widetilde{A}(\nu,\theta)$ we can further approximate $\det(\mathbf{1} - \widetilde{f}\widetilde{A})_{l^2(\widetilde{\mathbb{Z}}_{\geq -L})}$ with $\det(\mathbf{1} - \widetilde{f}\widetilde{A})_{l^2(\widetilde{\mathbb{Z}}\cap[-L,x^{\delta/3}])}$ up to an error of order $e^{-\mathrm{const}.x^{1/3}}$. This is the result of Lemma 6.16.

Step 5 We can finally evaluate the convergence of $\det(\mathbf{1} - \widetilde{f}\widetilde{A})_{l^2(\widetilde{\mathbb{Z}}\cap[-L,x^{\delta/3}])}$ to the Tracy–Widom distribution and we can verify the "continuity" properties of the remainder $R_x^{(1)}(r^*)$ in a neighborhood of r.

Let us now start.

Step 1 we have the following Lemma.

Lemma 6.12 (Convergence on moderately large sets) *Let δ be a number in the interval $(0, 1/3)$. Then for $(\nu,\theta) \in [-L, x^{\delta/3}]^2$ we have,*[6]

$$\widetilde{A}(\nu,\theta) = \frac{1}{\gamma x^{1/3}} K_{\mathrm{Airy}}(\nu,\theta) + \frac{1}{\gamma^2 x^{2/3}} Q(\nu,\theta) + \mathcal{O}\left(x^{2\delta/3-1}\right) \tag{6.51}$$

and the error term satisfies

$$x^{2/3}\mathcal{O}\big(x^{2\delta/3-1}\big) \xrightarrow[x\to\infty]{} 0, \tag{6.52}$$

uniformly in the sequence of sets $(\nu,\theta) \in [-L, x^{\delta/3}]^2$. Moreover the exponential estimates,

$$|K_{\mathrm{Airy}}(\nu,\theta)|, |Q(\nu,\theta)| < c_1 e^{-c_2(\nu+\theta)}, \tag{6.53}$$

hold for all $(\nu,\theta) \in [-L, x^{\delta/3}]^2$, for an opportune choice of positive constants c_1, c_2 which do not depend on x.

Proof The definition itself of scaling parameters is functional to perform a saddle point analysis. In particular we want to show that, when ν and θ are relatively small quantities, compared to $x^{1/3}$, the integrals in (6.46) are dominated by the value of the

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[6] For motivation on the choice of sets $[-L, x^{\delta/3}]$ see Remark 6.17.

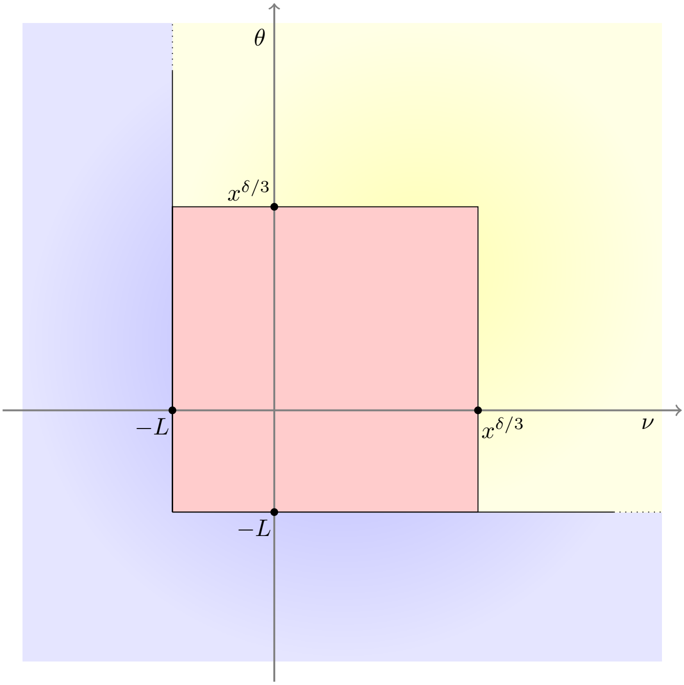

Fig. 13 We estimate the kernel $\widetilde{A}(\nu, \theta)$ in the red the region (see Lemma 6.12) and in the yellow region (see Lemma 6.13). Because $\widetilde{f}(\nu)$ converges to the indicator function $\mathbb{1}_{(r,\infty)}$ and $-L < r$, the contribution to the Fredholm determinant of $\widetilde{f}\widetilde{A}$ of integrations in the blue region is negligible (see Lemma 6.15) (color figure online)

integrands at the double critical point ς. To do so we suitably deform contours C, D in such a way that, for x large enough, the following properties hold:[7]

1. $\max_{z\in C} \mathfrak{Re}\{g(z)\} = g(\varsigma(1 - \frac{1}{2\gamma x^{1/3}}))$;
2. $\max_{z\in C} |z| = \varsigma(1 - \frac{1}{2\gamma x^{1/3}})$;
3. $\min_{w\in D} \mathfrak{Re}\{g(z)\} = g(\varsigma(1 + \frac{1}{2\gamma x^{1/3}}))$;
4. $\min_{w\in D} |w| = \varsigma(1 + \frac{1}{2\gamma x^{1/3}})$.

The idea is to take paths like those depicted in Fig. 14. Based on results of "Appendix C", we now construct the steep descent contour C. The same procedure can be applied to provide an exact expression for D as well and therefore we will omit this in the discussion.

Fix an arbitrarily small positive number ϵ and consider C to be the union of two curves $\widetilde{C}_1$, $\widetilde{C}_2$ such that

$$\widetilde{C}_1 = \partial \mathsf{D}\left(0, \varsigma(1-\epsilon)\right) \cap \left\{ z \in \mathbb{C} | \ \mathfrak{Re}(z) \leq \frac{\varsigma}{4}(3 + \sqrt{1 - 8\epsilon + 4\epsilon^2}) \right\}, \tag{6.54}$$

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[7] For the sake of the uniform convergence over compact sets conditions 2,4 are not necessary, but we still state them as they will become useful later in Lemma 6.64.

$$\widetilde{C}_2=\left\{\mathbb{1}_{[0,\frac{2}{\gamma x^{1/3}}]}(|\rho|)\varsigma\left(1-\frac{4+\gamma^2x^{2/3}\rho^2}{8\gamma x^{1/3}}+\mathrm{i}\frac{\sqrt{3}}{2}\rho\right)\right.$$
$$\left.+\mathbb{1}_{[\frac{2}{\gamma x^{1/3}},\infty)}(|\rho|)\varsigma\left(1-\frac{|\rho|}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\rho\right)\,:\,|\rho|\le\frac{1}{2}(1-\sqrt{1-8\epsilon+4\epsilon^2})\right\},\tag{6.55}$$

where $\partial\mathrm{D}(c,R)$ indicates a circumference of center c and radius R. To put it in simple terms C is a circle of radius $\varsigma(1-\epsilon)$ up until it intersects for the first time (from the left) the two complex lines exiting from ς with slope $\pm\frac{2\pi}{3}$ (as in Fig. 14b)). After C meets these intersection points, denoted with $p_\pm$, it becomes $\widetilde{C}_2$, a regular curve which coincides with such lines for a while and passes strictly to the left of ς.

We claim that the contribution of the integral in the z variable in (6.46) are given, up to an error which is exponentially small in x, by the integral along the contour $\widetilde{C}_2$. To show this, we first notice that from Proposition C.1, if ϵ is small enough we can assume that, along $\widetilde{C}_1$ the real part of $g(z)$ is a decreasing function. Therefore, the contribution of the term $e^{xg(z)}$ can be estimated by its values at the extremal points of $\widetilde{C}_1$,

$$p_\pm=\frac{\varsigma}{4}\left(3+\sqrt{1-8\epsilon+4\epsilon^2}\right)\pm\mathrm{i}\frac{\sqrt{3}}{4}\varsigma\left(1-\sqrt{1-8\epsilon+4\epsilon^2}\right)\approx\varsigma\left(1-\epsilon\pm\mathrm{i}\sqrt{3}\epsilon\right).$$

Let us evaluate the quantity $\mathfrak{Re}\{g(p_\pm)\}-g(\varsigma)$ through a Taylor expansion. By using (6.47), we have

$$\mathfrak{Re}\{g(p_\pm)\}-g(\varsigma)=\frac{8g'''(\varsigma)\varsigma^3}{3!}\epsilon^3+R(\epsilon)\epsilon^4,$$

where $R(\epsilon)$ is the Taylor remainder and it is a regular, bounded function in a neighborhood of zero. The factor $\varsigma^3g'''(\varsigma)$ is strictly negative, as stated in Proposition 6.11 and therefore we obtain the bound

$$e^{x(g(z)-g(\varsigma))}\le e^{-cx},\qquad \text{for each } z\in\widetilde{C}_1,$$

which holds for some positive constant c.

Through an analogous argument we can deform the D contour too and separate it in an union of two curves $\widetilde{D}_1$ and $\widetilde{D}_2$ (see Fig. 14). As for the C contour case, we can take $\widetilde{D}_2$ to be a curve that follows the two complex half lines $\{\varsigma+e^{\pm\mathrm{i}\frac{\pi}{3}}\rho\,:\,\rho\ge0\}$ in a neighborhood of size ϵ of ς and that passes strictly to the right of ς. For ϵ small enough, but still of order 1, the remaining contour $\widetilde{D}_1$ can be chosen so that the contribution of the w integral over $\widetilde{D}_1$ to the kernel $\widetilde{A}$ are exponentially small in x.

We also remark that curves $\widetilde{C}_2$, $\widetilde{D}_2$ are kept at a distance of size $x^{-1/3}$ from ς (and hence from each other) due to the presence in the integral expression of $\widetilde{A}$ of a singularity at $z=w$.

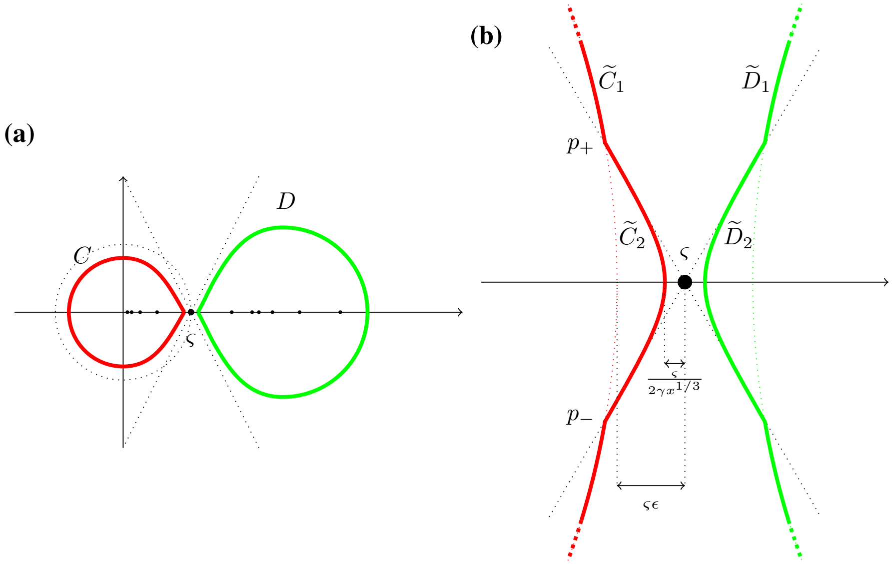


Fig. 14 **a** Choices of integration contours in Lemma 6.12. The red contour C encircles the sigularities $\{q^k đ\}_{k\geq 1}$, is contained inside a circle of radius $\varsigma(1-\epsilon)$ and joins the point $\varsigma(1-\frac{1}{2\gamma x^{1/3}})$ with slope $\frac{\pi}{3}$ from above (resp. $-\frac{\pi}{3}$ from below). The green contour D contains the singularities $\{\xi_k s_k\}_{k=2,\dots,x}$ and forms at the point $\varsigma(1+\frac{1}{2\gamma x^{1/3}})$ a cusp of width $\frac{2}{3}\pi$, symmetric to that of C. **b** A representation of contours C and D in the immediate vicinity of the critical point ς (color figure online)

We can summarize discussion made so far expressing the kernel $\widetilde{A}$ as

$$\widetilde{A}(\nu,\theta) = \frac{\varsigma^{\nu\gamma x^{1/3}}}{\varsigma^{\theta\gamma x^{1/3}}} \frac{1}{(2\pi \mathrm{i})^2} \int_{\widetilde{D}_2} \frac{dw}{w} \int_{\widetilde{C}_2} dz \frac{z^{\theta\gamma x^{1/3}}}{w^{\nu\gamma x^{1/3}}} \\ \times \frac{\exp\{xg(z)\}}{\exp\{xg(w)\}} \frac{(qđ/w, qz/đ; q)_\infty}{(qđ/z, qw/đ; q)_\infty} \frac{1}{z-w} + \mathcal{O}\big(e^{-cx}\big), \qquad (6.56)$$

where we notice that, with respect to (6.46), the integration contours have become $\widetilde{C}_2$ and $\widetilde{D}_2$ and the remainder is a quantity which decays as an exponential in x. We can now safely employ the saddle point method to give an estimate of the integral expression in (6.56). The only significant contribution to the double integral (6.46) is given when variables z, w are separated from ς by a distance of order $x^{-1/3}$. For this reason we like to apply the change of variables

$$z = \varsigma\left(1 - \frac{Z}{\gamma x^{1/3}}\right), \qquad w = \varsigma\left(1 - \frac{W}{\gamma x^{1/3}}\right),$$

and we write, through simple Taylor expansions, different terms of the integrand function in (6.56) as

$$\frac{z^{\theta\gamma x^{1/3}}}{w^{\nu\gamma x^{1/3}}} = \frac{\varsigma^{\theta\gamma x^{1/3}}}{\varsigma^{\nu\gamma x^{1/3}}} \frac{e^{-\theta Z}}{e^{-\nu W}} \left[1 + \frac{1}{\gamma x^{1/3}} \left(\frac{\nu W^2}{2} - \frac{\theta Z^2}{2} \right) \right. \\ \left. + \mathcal{O}\left(\frac{Z^3\theta}{x^{2/3}}, \frac{W^3\nu}{x^{2/3}}, \frac{Z^4\theta^2}{x^{2/3}}, \frac{W^4\nu^2}{x^{2/3}} \right) \right], \tag{6.57}$$

$$\frac{e^{xg(z)}}{e^{xg(w)}} = \frac{e^{Z^3/3}}{e^{W^3/3}} \left[1 + \frac{1}{\gamma x^{1/3}} \left(E_1 Z^4 - E_1 W^4 \right) + \mathcal{O}\left(\frac{Z^8}{x^{2/3}}, \frac{W^8}{x^{2/3}} \right) \right], \tag{6.58}$$

$$\frac{(q\mathpzc{d}/w, qz/\mathpzc{d}; q)_\infty}{(q\mathpzc{d}/z, qw/\mathpzc{d}; q)_\infty} = 1 + \frac{1}{\gamma x^{1/3}} (E_2 Z - E_2 W) + \mathcal{O}\left(\frac{Z^2}{x^{2/3}}, \frac{W^2}{x^{2/3}} \right). \tag{6.59}$$

In these expressions, coefficients E_1, E_2, naturally possess exact expressions, which we do not report as they are irrelevant for the computations.

Thanks to (6.56),(6.57), (6.58), (6.59) we obtain an expansion of $\widetilde{A}$ in the infinitesimal quantity $1/(\gamma x^{1/3})$. Collecting together terms of order $1/(\gamma x^{1/3})$ and $1/(\gamma x^{1/3})^2$ we obtain

$$\widetilde{A}(\nu, \theta) = \frac{1}{\gamma x^{1/3}} \int_{e^{-\frac{2}{3}\pi \mathrm{i}}\infty}^{e^{\frac{2}{3}\pi \mathrm{i}}\infty} \frac{dW}{2\pi \mathrm{i}} \int_{e^{\frac{\pi}{3}\mathrm{i}}\infty}^{e^{-\frac{\pi}{3}\mathrm{i}}\infty} \frac{dZ}{2\pi \mathrm{i}} \frac{e^{Z^3/3-\theta Z}}{e^{W^3/3-\nu W}} \frac{1}{W-Z} \\ + \frac{1}{\left(\gamma x^{1/3}\right)^2} Q(\nu, \theta) + \mathcal{O}\big(x^{2\delta/3-1}\big). \tag{6.60}$$

with the kernel Q being given by

$$Q(\nu, \theta) = \int_{e^{-\frac{2}{3}\pi \mathrm{i}}\infty}^{e^{\frac{2}{3}\pi \mathrm{i}}\infty} \frac{dW}{2\pi \mathrm{i}} \int_{e^{\frac{\pi}{3}\mathrm{i}}\infty}^{e^{-\frac{\pi}{3}\mathrm{i}}\infty} \frac{dZ}{2\pi \mathrm{i}} \frac{e^{Z^3/3-\theta Z}}{e^{W^3/3-\nu W}} \left(\frac{\nu W^2}{2} - \frac{\theta Z^2}{2} \right. \\ \left. + E_1(Z^4 - W^4) + E_2(Z - W) \right) \frac{1}{W-Z}. \tag{6.61}$$

By recognizing the expression of the Airy kernel (6.1) in (6.60) we write $\widetilde{A}$ as in (6.51).

All we are left to do is to prove the exponential bound (6.53) for Q, since the same type of estimate for K_{Airy} follows from well known decay properties of the Airy functions [1]. To do this consider the following parametrization of the integration variables

$$Z = \frac{\widetilde{b}}{2} + |\varrho_1| e^{-\operatorname{sign}(\varrho_1)\mathrm{i}\pi/3}, \quad W = -\frac{\widetilde{b}}{2} + |\varrho_2| e^{-\operatorname{sign}(\varrho_2)\mathrm{i}2\pi/3}, \tag{6.62}$$

for $\rho_1, \rho_2 \in \mathbb{R}$ and $\widetilde{b}$ being a positive real number. Applying the substitution (6.62) in (6.61), we straightforwardly obtain an inequality like

$$\begin{aligned} |Q(\nu, \theta)| < e^{-\frac{\widetilde{b}}{2}(\theta+\nu)} \frac{e^{\widetilde{b}^3/12}}{\widetilde{b}} \int_0^\infty d\varrho_1 \int_0^\infty d\varrho_2 \Big(|\theta| P_{\widetilde{b}}(\rho_1) \\ + |\nu| P_{\widetilde{b}}(\rho_2) + S_{\widetilde{b}}(\varrho_1) + S_{\widetilde{b}}(\varrho_2) \Big) \\ \times e^{-\frac{\widetilde{b}}{4}(\rho_1^2+\rho_2^2) - \left(\frac{\theta}{2} - \frac{\widetilde{b}^2}{8}\right)\rho_1 - \left(\frac{\nu}{2} - \frac{\widetilde{b}^2}{8}\right)\rho_2}, \end{aligned} \tag{6.63}$$

where $P_{\widetilde{b}}$ and $S_{\widetilde{b}}$ are polynomials and by making use of elementary estimates on the integrals on the right hand side of (6.63), we can finally show (6.53).

The error term $\mathcal{O}(x^{2\delta/3-1})$ in (6.51) is obtained taking into account quantities

$$\mathcal{O}\left(\frac{Z^3\theta}{x^{2/3}}, \frac{W^3\nu}{x^{2/3}}, \frac{Z^4\theta^2}{x^{2/3}}, \frac{W^4\nu^2}{x^{2/3}}\right), \mathcal{O}\left(\frac{Z^8}{x^{2/3}}, \frac{W^8}{x^{2/3}}\right), \mathcal{O}\left(\frac{Z^2}{x^{2/3}}, \frac{W^2}{x^{2/3}}\right), \mathcal{O}(e^{-cx})$$

from (6.56) and (6.57), (6.58), (6.59) in the saddle point integration. Due to the presence of the exponentially decaying term $e^{Z^3/3-W^3/3-Z\theta+W\nu}$ we can formulate bounds like (6.63) for these remainders as well, to finally show (6.52). This concludes our proof. □

Step 2 the following lemma establishes the exponential decay of $\widetilde{A}(\nu, \theta)$ in the yellow region in Fig. 13.

Lemma 6.13 (Exponential decay of front tails) *Let L' be an arbitrary large positive real numbers (possibly of order x raised to some power). Then there exists x_*, such that for all $x > x_*$ the bound*

$$\left| \gamma x^{1/3} \widetilde{A}(\nu, \theta) \right| < e^{-\nu-\theta} \tag{6.64}$$

holds for each $(\nu, \theta) \in [-L, \infty)^2 \setminus [-L, L']^2$.

Proof We use again suitable deformations of contours described in Lemma 6.12 to estimate, for large x, the contribution of the factor

$$\frac{z^{\theta\gamma x^{1/3}}}{w^{\nu\gamma x^{1/3}}}$$

to the double integral (6.46). Let's first prove (6.64) in the case $\theta \geq \nu$. When this is the case, we take the contour D exactly as in Lemma 6.12 and we modify $C = \widetilde{C}_1 \cup \widetilde{C}_2$, where

$$\begin{aligned} \widetilde{C}_1 &= \partial \mathrm{D}\left(0, \varsigma - \frac{2\varsigma}{\gamma x^{1/3}}\right) \cap \{z \in \mathbb{C} | \,\mathfrak{Re}(z) \leq \varsigma(1 - \frac{3}{\gamma x^{1/3}})\}, \\ \widetilde{C}_2 &= \varsigma(1 - \frac{3}{\gamma x^{1/3}}) + \mathrm{i}[-\varsigma\widetilde{a}, \varsigma\widetilde{a}] \end{aligned}$$

and $\widetilde{a}$ is given by the intersections of the vertical complex line $\{\varsigma(1-\frac{3}{\gamma x^{1/3}})+\mathrm{i}y \,|\, y \in \mathbb{R}\}$ with the circle $\partial \mathsf{D}\left(0, \varsigma - \frac{2\varsigma}{\gamma x^{1/3}}\right)$. We can also write down its exact expression as

$$\widetilde{a} = \sqrt{\frac{2}{\gamma x^{1/3}} - \frac{5}{\gamma^2 x^{2/3}}} \approx \sqrt{\frac{2}{\gamma}}\frac{1}{x^{1/6}} + \mathcal{O}(x^{-1/3}).$$

From Proposition C.1, $\partial \mathsf{D}\left(0, \varsigma(1 - \frac{2}{\gamma x^{1/3}})\right)$ is a steep descent contour for $\mathfrak{Re}(g)$ and we can assume that

$$\max_{z \in C_1} \mathfrak{Re}\{g(z)\} = \mathfrak{Re}\left\{g(\varsigma(1 - \frac{3}{\gamma x^{1/3}} + \mathrm{i}\widetilde{a}))\right\}.$$

To evaluate the real part of the function g on the complex segment $\widetilde{C}_2$ we use the parametrization

$$z = \varsigma\left(1 - \frac{3}{\gamma x^{1/3}} + \mathrm{i}\frac{Z}{\gamma x^{1/3}}\right). \tag{6.65}$$

In this case Z is a real number ranging in an interval which, up to corrections of order $x^{-1/3}$ is $[-\sqrt{2\gamma}x^{1/6}, \sqrt{2\gamma}x^{1/6}]$. Expanding g in Taylor series around ς and recalling (6.47), we have

$$\begin{aligned}\mathfrak{Re}\{g(z)\} - g(\varsigma) = &\frac{1}{x}\frac{\varsigma^3 g'''(\varsigma)}{3!\gamma^3}(-27 + 9Z^2) + \frac{1}{x^{4/3}}\frac{\varsigma^4 g^{(4)}(\varsigma)}{4!\gamma^4}Z^4 \\ &+\mathcal{O}\left(\frac{Z^2}{x^{4/3}}, \frac{Z^4}{x^{5/3}}, \frac{Z^6}{x^2}\right),\end{aligned} \tag{6.66}$$

where the presence of terms of order higher than three takes into account the fact that Z can be of order $x^{1/6}$. When $Z/(\gamma x^{1/3}) = \widetilde{a}$, (6.66) becomes

$$\mathfrak{Re}\left\{g(\varsigma(1 - \frac{3}{\gamma x^{1/3}} + \mathrm{i}\widetilde{a}))\right\} - g(\varsigma) = \frac{1}{\gamma^2 x^{2/3}}\left(3\varsigma^3 g'''(\varsigma) + \frac{1}{6}\varsigma^4 g^{(4)}(\varsigma)\right) + \mathcal{O}(x^{-1})$$

and the term on the right hand side of order $x^{-2/3}$ is negative. This can be shown either directly computing the derivatives of g or simply recalling that the point $\varsigma(1 - \frac{3}{\gamma x^{1/3}} + \mathrm{i}\widetilde{a})$ lies on a steep descent contour. These calculations imply the estimate

$$\left|e^{x(g(z)-g(\varsigma))}\right| \leq e^{-cx^{1/3}}, \quad \text{for each } z \in \widetilde{C}_1, \tag{6.67}$$

for some positive constant c. On the other hand, when z belongs to $\widetilde{C}_2$, (6.66) gives us that

$$\left|e^{x(g(z)-g(\varsigma))}\right| \leq e^{9-\widetilde{c}Z^2}, \quad \text{for each } |Z| \leq \gamma\widetilde{a}x^{1/3}, \tag{6.68}$$

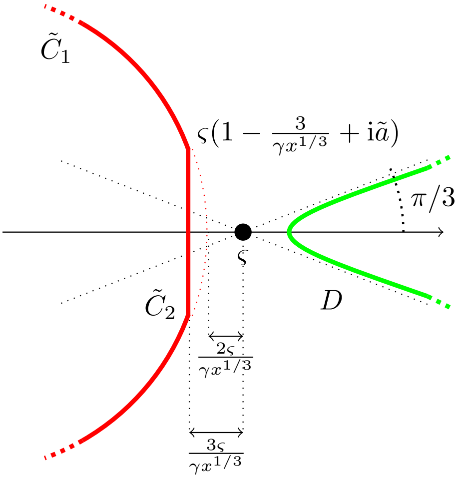


Fig. 15 Choice of integration contours of Lemma 6.13. The red contour C is the union of $\widetilde{C}_1$, an arc of the circle of center 0 and radius $\varsigma(1-\frac{2}{\gamma x^{1/3}})$, and $\widetilde{C}_2$, a vertical segment passing for the point $\varsigma(1-\frac{3}{\gamma x^{1/3}})$ on the real axis (Fig. 15). On the other hand D, in the vicinity of the critical point ς, stays close to the lines exiting from ς with slope $\pm\frac{\pi}{3}$ (dotted lines) (color figure online)

for some other positive constant $\widetilde{c}$.

To complete the list of preliminary estimates for terms depending on z in the integral formula (6.46) of the kernel $\widetilde{A}$, we need to address the factor $z^{\theta\gamma x^{1/3}}$. First we notice that, since the contour C lies inside the circle centered at 0 with radius $\varsigma(1-2/(\gamma x^{1/3}))$, we have

$$\left|\frac{z}{\varsigma}\right|^{\theta\gamma x^{1/3}} \leq \exp\left\{\theta\gamma x^{1/3}\log\left(1-\frac{2}{\gamma x^{1/3}}\right)\right\} \leq e^{-2\theta}, \quad \text{for each } z \in C, \tag{6.69}$$

as a result of the simple inequlity $\log(1+y) \leq y$, valid for all $y > -1$. Moreover, when z is on $\widetilde{C}_2$, using the parametrization (6.65), we have

$$\begin{aligned}\left|\frac{z}{\varsigma}\right|^{\theta\gamma x^{1/3}} &= \exp\left\{\theta\gamma x^{1/3}\log\left|1-\frac{3}{\gamma x^{1/3}}+\frac{\mathrm{i}Z}{\gamma x^{1/3}}\right|\right\}\\ &\leq \exp\left\{-\theta\left(3-\frac{9+Z^2}{2\gamma x^{1/3}}\right)\right\}.\end{aligned} \tag{6.70}$$

To evaluate the kernel $\widetilde{A}$ we also need to provide some estimates for quantities involving the variable w. The choice of contours C, D implies that

$$\frac{1}{z-w} \leq \frac{\varsigma}{\gamma x^{1/3}} \quad \text{and} \quad \left|\frac{(q\mathscr{d}/w, qz/\mathscr{d}; q)_\infty}{(q\mathscr{d}/z, qw/\mathscr{d}; q)_\infty}\right| \leq \Gamma_1, \tag{6.71}$$

for some constant Γ_1. In addition, since $\nu > -L$ and $|w| > \varsigma$, combined with the fact that D is steep ascent for the function $\mathfrak{Re}\{g\}$, as proved in Proposition C.3, we have that

$$\left|\frac{\varsigma}{w}\right|^{\nu\gamma x^{1/3}} \exp\{x(g(\varsigma)-g(w))\} \leq \left|\frac{\varsigma}{w}\right|^{-L\gamma x^{1/3}} \exp\{x(g(\varsigma)-g(w))\} \leq \Gamma_2, \tag{6.72}$$

for some other constant Γ_2. Combining together inequalities (6.70), (6.71), (6.72), we can write

$$\begin{aligned} |\widetilde{A}(\nu,\theta)| &= \frac{\varsigma^{\nu\gamma x^{1/3}}}{\varsigma^{\theta\gamma x^{1/3}}} \left| \int_C \frac{dz}{2\pi} \int_D \frac{dw}{2\pi w} \frac{z^{\theta\gamma x^{1/3}}}{w^{\nu\gamma x^{1/3}}} \frac{\exp\{xg(z)\}}{\exp\{xg(w)\}} \frac{(q\text{đ}/w, qz/\text{đ}; q)_\infty}{(q\text{đ}/z, qw/\text{đ}; q)_\infty} \frac{1}{z-w} \right| \\ &\leq \frac{\Gamma_1\Gamma_2 l(D)}{(2\pi)^2\gamma x^{1/3}} \int_C \left| dz \left(\frac{z}{\varsigma}\right)^{\theta\gamma x^{1/3}} \exp\{x(g(z)-g(\varsigma))\} \right|, \end{aligned} \tag{6.73}$$

where $l(D)$ is the length of the curve D. The integral over C is naturally split into different contributions coming from contours $\widetilde{C}_1$ and $\widetilde{C}_2$. On $\widetilde{C}_1$, utilizing (6.67) and (6.69) we have

$$\int_{\widetilde{C}_1} \left| dz \left(\frac{z}{\varsigma}\right)^{\theta\gamma x^{1/3}} \exp\{x(g(z)-g(\varsigma))\} \right| \leq e^{-2\theta} e^{-cx^{-1/3}} l(\widetilde{C}_1), \tag{6.74}$$

whereas on $\widetilde{C}_2$, from (6.68), (6.70) we obtain

$$\begin{aligned} &\int_{\widetilde{C}_2} \left| dz \left(\frac{z}{\varsigma}\right)^{\theta\gamma x^{1/3}} \exp\{x(g(z)-g(\varsigma))\} \right| \\ &\quad \leq \int_{-a\gamma x^{1/3}}^{a\gamma x^{1/3}} dZ \exp\left\{ -\theta\left(3 - \frac{9+Z^2}{2\gamma x^{1/3}}\right) + 9 - \widetilde{c}Z^2 \right\}. \end{aligned} \tag{6.75}$$

To estimate the integral on the right hand side of (6.75), set a large integer N and split the integration segmant into $|Z| < N$ and $N < |Z| < \widetilde{a}\gamma x^{1/3}$. When $|Z| < N$ the term $\frac{9+Z^2}{2\gamma x^{1/3}}$ is small and we can denote it with $\mathcal{O}(N^2/x^{1/3})$. On the other hand, when $N < |Z| < \widetilde{a}\gamma x^{1/3}$, since $(3 - \frac{9+Z^2}{2\gamma x^{1/3}}) > 2$, the integrand becomes very small due to the presence of the exponential of $-\widetilde{c}Z^2$. We can therefore write

$$\begin{aligned} \text{rhs (6.75)} &\leq e^{-2\theta} \left(e^{-\theta(1-\mathcal{O}(N^2/x^{1/3}))} \int_{-N}^{N} e^{9-\widetilde{c}Z^2} dZ + \int_{N<|Z|\widetilde{a}\gamma x^{1/3}} dZ e^{9-\widetilde{c}Z^2} \right) \\ &= e^{-2\theta} \left(e^{-\theta(1-\mathcal{O}(N^2/x^{1/3}))} \Gamma_3 + \mathcal{O}\big(e^{-\widetilde{c}N^2}\big) \right), \end{aligned} \tag{6.76}$$

with Γ_3 being a constant coming from the integration of the exponential.

We can now plug (6.74), (6.75), (6.76) into the right hand side of (6.73) to finally obtain

$$\left|\widetilde{A}(\nu,\theta)\right| < e^{-2\theta}\left(e^{-cx^{1/3}}l(C_1) + e^{-\theta(1-\mathcal{O}(N^2/x^{1/3}))}\Gamma_3 + \mathcal{O}\big(e^{-\tilde{c}N^2}\big)\right)\frac{\Gamma_1\Gamma_2 l(D)}{(2\pi)^2\gamma x^{1/3}}. \tag{6.77}$$

The term inside the parentheses can be made smaller than $\frac{(2\pi)^2}{\Gamma_1\Gamma_2 l(D)}$ taking $x \gg 0$ and $L' \gg 0$ (remember $L' < \theta$), so that (6.77) reduces to

$$\left|\widetilde{A}(\nu,\theta)\right| < e^{-2\theta}\frac{1}{\gamma x^{1/3}},$$

which implies (6.64) since $-2\theta < -\theta - \nu$.

The complementary case $\nu > \theta$ can be studied analogously, deforming the contour D, instead of C, symmetrically with respect to the critical point ς. □

Up to this point we estimated the kernel $\widetilde{A}$ in a region where both θ and ν are bounded from below. When this is not the case the saddle point method cannot be applied any longer as the contribution to the integral (6.46) of the term

$$\frac{z^{\gamma x^{1/3}\theta}}{w^{\gamma x^{1/3}\nu}}$$

is no more negligible. In the following Lemma we show how to control the rear tails of $\widetilde{A}$.

Lemma 6.14 *The kernel $\widetilde{A}$ defines a trace class operator on $l^2(\widetilde{\mathbb{Z}})$ with $\|\widetilde{A}\| = 1$. In particular we have the bound*

$$|\widetilde{A}(\nu,\theta)| \leq 1. \tag{6.78}$$

Proof Since $\widetilde{A}$ is obtained from the kernel A through a simple change of variable and a multiplication by a gauge factor $\frac{\widetilde{\tau}(\nu)}{\widetilde{\tau}(\theta)}\frac{\tau(m_\theta)}{\tau(n_\nu)}$, we can still write the expansion

$$\widetilde{A}(\nu,\theta) = \sum_{l=1}^{x-1}\frac{\widetilde{\tau}(\nu)}{\tau(n_\nu)}\phi_l(n_\nu)\frac{\tau(m_\theta)}{\widetilde{\tau}(\theta)}\psi_l(m_\theta) = \sum_{l=1}^{x-1}\widetilde{\phi}_l(\nu)\widetilde{\psi}_l(\theta),$$

where $\widetilde{\phi}_l(\nu) = \frac{\widetilde{\tau}(\nu)}{\tau(n_\nu)}\phi_l(n_\nu)$ and $\widetilde{\psi}_l(\theta) = \frac{\tau(m_\theta)}{\widetilde{\tau}(\theta)}\psi_l(m_\theta)$. Functions $\widetilde{\phi}_l$, $\widetilde{\psi}_l$, like ϕ_l, ψ_l, are still a biorthogonal family and to prove this we only have to check that the summation $\sum_\nu \widetilde{\phi}_l(\nu)\widetilde{\psi}_k(\nu)$ is absolutely convergent. This can be done establishing exponential decay of tails of $\widetilde{\phi}_l$, $\widetilde{\psi}_k$. For the rear tails this is done as in "Appendix B" for functions ϕ_l, ψ_l (for the real tails), while front tails are estimated using a saddle point analysis as that performed for the kernel $\widetilde{A}$ in Lemmas 6.12, 6.13. Such exponential bounds imply that $\widetilde{A}$ is trace class for each x, following the argument in the proof of Proposition 5.3. Also, the biorthogonality of $\widetilde{\phi}_l$, $\widetilde{\psi}_k$ implies that $\|\widetilde{A}\|= 1$ as shown in the proof of Proposition 5.4. □

Step 3 we can now start the evaluation of the Fredholm determinant of the kernel $\widetilde{f}\widetilde{A}$.

Lemma 6.15 *There exist constants c, x^* such that, for each $x > x^*$, we have*

$$\left|\det\left(\mathbf{1}-\widetilde{f}\widetilde{A}\right)_{l^2(\widetilde{\mathbb{Z}})}-\det\left(\mathbf{1}-\widetilde{f}\widetilde{A}\right)_{l^2(\widetilde{\mathbb{Z}}\cap[-L,\infty))}\right|<e^{-cL\gamma x^{1/3}}. \tag{6.79}$$

Proof First we set constants c_1, c_2 such that the estimate

$$\left|\sqrt{\widetilde{f}(\nu)\widetilde{f}(\theta)}\widetilde{A}(\nu,\theta)\right|<\begin{cases}\frac{1}{\gamma x^{1/3}}c_1 e^{-\theta-\nu}, & \text{if } \theta, \nu \in [-L,\infty),\\ \frac{1}{\gamma x^{1/3}}e^{c_2(\min(\theta,-L)+\min(\nu,-L))\gamma x^{1/3}}, & \text{else,}\end{cases}$$

holds, for each x sufficiently large. This is always possible as a result of Lemmas 6.12, 6.13, 6.14 and from the fact that $\widetilde{f}(\nu)$, given in (6.45), decays exponentially in $x^{1/3}$ when $\nu \ll 0$. In particular we can easily deduce the additional bound

$$\left|\sqrt{\widetilde{f}(\nu)\widetilde{f}(\theta)}\widetilde{A}(\nu,\theta)\right|<c_3\frac{1}{\gamma x^{1/3}}e^{-|\theta|-|\nu|},$$

true for any θ, ν, for some constant c_3. We have

$$\begin{aligned}\text{lhs of (6.79)}&=\left|\sum_{k\geq 1}\frac{(-1)^k}{k!}\widetilde{\sum}_{(\nu_1,\dots,\nu_k)\notin[-L,\infty)^k}\det_{i,j=1}^{k}\left(\sqrt{\widetilde{f}(\nu_i)\widetilde{f}(\nu_j)}\widetilde{A}(\nu_i,\nu_j)\right)\right|\\&\leq\sum_{k\geq 1}\frac{(-1)^k}{(k-1)!}\widetilde{\sum}_{\nu_1\leq -L}\sum_{\nu_2,\dots,\nu_k}\left|\det_{i,j=1}^{k}\left(\sqrt{\widetilde{f}(\nu_i)\widetilde{f}(\nu_j)}\widetilde{A}(\nu_i,\nu_j)\right)\right|.\end{aligned} \tag{6.80}$$

Thanks to the Hadamard's inequality we can estimate the determinantal term in the sum as

$$\begin{aligned}\left|\det_{i,j=1}^{k}\left(\sqrt{\widetilde{f}(\nu_i)\widetilde{f}(\nu_j)}\widetilde{A}(\nu_i,\nu_j)\right)\right|&\leq\prod_{i=1}^{k}\left(\sum_{j=1}^{k}\widetilde{f}(\nu_i)\widetilde{f}(\nu_j)\left|\widetilde{A}(\nu_i,\nu_j)\right|^2\right)^{1/2}\\&\leq\frac{k^{k/2}}{(\gamma x^{1/3})^k}e^{c_2\nu_1\gamma x^{1/3}}c_3^{k-1}\prod_{j=2}^{k}e^{-|\nu_j|},\end{aligned}$$

so that, using this bound in (6.80), we obtain our result. □

Step 4

Lemma 6.16 *Take constants L, δ such that $-L < r$ and $\delta \in (0, 1/3)$. Then there exist constants C, x^* such that, for each $x > x^*$, we have*

$$\left|\det\left(\mathbf{1}-\widetilde{f}\widetilde{A}\right)_{l^2(\widetilde{\mathbb{Z}}_{\geq -L})}-\det\left(\mathbf{1}-\widetilde{f}\widetilde{A}\right)_{l^2(\widetilde{\mathbb{Z}}\cap[-L,x^{\delta/3}))}\right|<Ce^{-x^{\delta/3}}. \tag{6.81}$$

Proof The proof of (6.81) makes use of the exponential bound (6.64) choosing $L' = x^{\delta/3}$ and it is similar to that of Lemma 6.15, therefore we omit it. □

Step 5 we can finally combine all previous preliminary Lemmas and give the proof of Proposition 6.6.

Proof of Proposition 6.6 To prove this result we first use Lemma 6.15 and Lemma 6.16 to restrict our attention to the Fredholm determinant of $\widetilde{f}\widetilde{A}$ in $l^2(\widetilde{\mathbb{Z}} \cap [-L, x^{\delta/3}])$. The error we make while considering this restriction is exponentially small in x and hence it is irrelevant when it comes to a decomposition like (6.23). Using results of Lemma 6.12 we have

$$\widetilde{A}(\nu,\theta) = \frac{1}{\gamma x^{1/3}} K_{\mathrm{Airy}}(\nu,\theta) + \frac{1}{(\gamma x^{1/3})^2} Q(\nu,\theta) + \mathcal{O}\big(x^{2\delta/3-1}\big),$$
$$\widetilde{f}(\nu) = \mathbb{1}_{[r,\infty)}(\nu) + \Delta_r(\nu),$$

where the term Δ_r is simply expressed as

$$\Delta_r(\nu) = \begin{cases} \frac{1}{1+q^{(\nu-r)\gamma x^{1/3}}}, & \text{if } \nu < r, \\ \frac{-q^{(\nu-r)\gamma x^{1/3}}}{1+q^{(\nu-r)\gamma x^{1/3}}}, & \text{if } \nu \geq r. \end{cases}$$

We can saparate the terms of the product $\widetilde{f}(\nu)\widetilde{A}(\nu,\theta)$ based on their order in $x^{-1/3}$ as

$$B^{(1)}(\nu,\theta) = \mathbb{1}_{[r,\infty)}(\nu) K_{\mathrm{Airy}}(\nu,\theta), \qquad B^{(2)}(\nu,\theta) = \frac{1}{\gamma x^{1/3}} \mathbb{1}_{[r,\infty)}(\nu) Q(\nu,\theta),$$
$$B^{(3)}(\nu,\theta) = \Delta_r(\nu) K_{\mathrm{Airy}}(\nu,\theta), \qquad B^{(4)}(\nu,\theta) = \frac{1}{\gamma x^{1/3}} \Delta_r(\nu) Q(\nu,\theta),$$
$$B^{(5)}_{i,j}(\nu,\theta) = \mathcal{O}\big(x^{2(\delta-1)/3}\big).$$

In this notation we write the Fredholm determinant of $\widetilde{f}\widetilde{A}$ as

$$\det\big(\mathbf{1} - \widetilde{f}\widetilde{A}\big)_{l^2(\widetilde{\mathbb{Z}}\cap[-L,x^{\delta/3}])} = 1 + \sum_{k\geq 1} \frac{(-1)^k}{k!} \widetilde{\sum}_{\substack{\nu_l \in [-L,x^{\delta/3}] \\ l=1,\dots,k}} \left(\frac{1}{\gamma x^{1/3}}\right)^k \det_{i,j=1}^{k} \Big(B^{(1)}(\nu_i,\nu_j) + \dots + B^{(5)}(\nu_i,\nu_j)\Big) \tag{6.82}$$

and our goal is to separate the contribution of higher order terms $B^{(2)}, \dots, B^{(5)}$ from that of $B^{(1)}$. To do so we use a formula that expresses the determinant of a sum of matrices $B^{(1)} + \dots + B^{(N)}$ in terms of sums of determinants of matrices having for column i, the i-th column of exactly one of the $B^{(1)}, \dots, B^{(N)}$. More precisely we have

$$\det\left(B^{(1)}+\cdots+B^{(N)}\right)=\sum_{\substack{\cup_{i=1}^{N} I_i=\{1,\ldots,k\}\\ I_i\cap I_j=\emptyset \text{ if } i\neq j}}\det\left(B^{(I_1,\ldots,I_N)}\right), \tag{6.83}$$

where

$$B_{i,j}^{(I_1,\ldots,I_N)}=B_{i,j}^{(l)} \quad \text{if } i\in I_l.$$

This expansion holds for generic $k\times k$ matrices $B^{(1)},\ldots,B^{(N)}$ and it follows directly from the multi linearity of the determinant. Using (6.83) can rewrite the determinant in the right hand side of (6.82) as

$$\det\left(B^{(1)}+\cdots+B^{(5)}\right)=\det\left(B^{(1)}\right)+\sum_{\substack{\cup_{i=1}^{5} I_i=\{1,\ldots,k\}\\ I_i\cap I_j=\emptyset \text{ if } i\neq j\\ I_1\neq\varnothing}}\det\left(B^{(I_1,I_2,I_3,I_4,I_5)}\right). \tag{6.84}$$

The Hadamard inequality provides the bound

$$\left|\det(B^{(I_1,I_2,I_3,I_4,I_5)})\right|\leq\prod_{l=1}^{5}\prod_{i\in I_l}\left(\sum_{j=1}^{k}|B^{(l)}(\nu_i,\nu_j)|^2\right)^{1/2},$$

while exponential inequality (6.53) allows us to write

$$\begin{aligned}&B^{(1)}(\nu_i,\nu_j)<c_1e^{-c_2\nu_i},\quad B^{(2)}(\nu_i,\nu_j)<\frac{1}{\gamma x^{1/3}}c_1e^{-c_2\nu_i},\\&\quad B^{(3)}(\nu_i,\nu_j)<c_1q^{|\nu_i-r|\gamma x^{1/3}},\\&B^{(4)}(\nu_i,\nu_j)<\frac{c_1q^{|\nu_i-r|\gamma x^{1/3}}}{\gamma x^{1/3}},\quad B^{(5)}(\nu_i,\nu_j)=\mathcal{O}(x^{2(\delta-1)/3}).\end{aligned}$$

Integrating the generic term of the summation in (6.84) we obtain

$$\begin{aligned}&\widetilde{\sum}_{\substack{\nu_l\in[-L,x^{\delta/3}]\\ l=1,\ldots,k}}\left(\frac{1}{\gamma x^{1/3}}\right)^k\left|\det(B^{(I_1,I_2,I_3,I_4,I_5)})\right|\\&\quad=k^{k/2}\mathcal{O}\left(x^{-|I_2|/3-|I_3|/3-2|I_4|/3-|I_5|(2/3-\delta)}\right),\end{aligned} \tag{6.85}$$

where the exponents $-|I_3|/3$ and $-2|I_4|/3$ appear due to the fact that the function $q^{|\nu-r|\gamma x^{1/3}}$ is exponentially small in x outside of a neighborhood of size $x^{-1/3}$ of r.

Because at least one among $I_2, \dots, I_5$ is not empty, we have proven that

$$\begin{aligned} &\det\left(\mathbf{1} - \widetilde{f}\widetilde{A}\right)_{l^2(\widetilde{\mathbb{Z}}\cap[-L,x^{\delta/3}])} \\ &\quad = \det\left(\mathbf{1} - \mathbb{1}_{[r,\infty)}K_{\mathrm{Airy}}\right)_{l^2(\widetilde{\mathbb{Z}}\cap[-L,x^{\delta/3}])} + \frac{1}{\gamma x^{1/3}} S_x^{(1)}(r) \\ &\quad = \det\left(\mathbf{1} - \mathbb{1}_{[r,\infty)}K_{\mathrm{Airy}}\right)_{\mathcal{L}^2(\mathbb{R})} + \frac{1}{\gamma x^{1/3}}\left(S_x^{(1)}(r) + S_x^{(2)}(r)\right). \end{aligned} \tag{6.86}$$

In the right hand side the error $S^{(1)}$ comes from contribution to the Fredholm determinant of matrices $B^{(2)}, \dots, B^{(5)}$, while $S^{(2)}$ comes from substituting discrete integrations $\widetilde{\sum}$ with integral symbols. Both these quantities are explicit and clearly bounded due to the exponential estimates (6.53) and this proves (6.23), (6.24).

The continuity of the remainder term (6.25) can be proven following the same strategy used to prove its boundedness. In fact for any r^*, $S_x^{(1)}(r^*)$, $S_x^{(2)}(r^*)$ are sums of determinants of kernels depending on r^* and to evaluate the differences

$$S_x^{(i)}(r^*) - S_x^{(i)}\left(r^* - 1/\gamma x^{1/3}\right)$$

we can first expand these kernels around r^* and subsequently analyse the contributions of terms of order zero and one in $x^{-1/3}$ using an expansion of the form (6.84). We don't discuss these details any further. ☐

Remark 6.17 The statement of Proposition 6.6 not only tells us that

$$\det(\mathbf{1} - \widetilde{f}\widetilde{A}) \xrightarrow[x\to\infty]{} F_2(r),$$

but also it gives us an estimate of the error depending on x and this will be essential in the proof of Theorem 1.5. To measure such error term, namely $\frac{1}{\gamma x^{1/3}} R_x^{(1)}$ in (6.23), we approximated the kernel $\widetilde{f}\widetilde{A}$ on $l^2(\widetilde{\mathbb{Z}}^2)$ with its truncated version defined only on $(\widetilde{\mathbb{Z}} \cap [-L, x^{\delta/3}])^2$. The choice of the supremum of the segment $[-L, x^{\delta/3}]$ is actually very relevant and possibly differentiate our analysis of the Fredholm determinant from that of earlier works, such as [16]. Had we considered the convergence of $\widetilde{f}\widetilde{A}$ only on compact sets like $[-L, L']$, with L' being some finite constant, we would have ended up, in Lemma 6.16 (replacing every $x^{\delta/3}$ with L'), with a bound like

$$\text{lhs of (6.81)} < Ce^{-L'}. \tag{6.87}$$

This clearly would have not been enough for our purposes, as the right hand side of (6.87) has no dependence on x and in particular does not decay when x becomes infinite.

6.5 Proof of Propositions 6.7, 6.8, 6.9

In this Section we carry out proofs of Propositions 6.7, 6.8 and 6.9, that were stated in Sect. 6.2 and used in the proof of our main result Theorem 1.5. As in Sect. 6.4, rather than the original expressions of f, A, $\Phi^{(i)}$, $\Psi^{(j)}$ we will make use of their rescaled forms $\widetilde{f}$, $\widetilde{A}$, $\widetilde{\Phi}^{(i)}$, $\widetilde{\Psi}^{(j)}$ discussed in Proposition 6.10. First we present the proof of Proposition 6.7 and the argument we follow traces that given in Lemma 5.12 of [39].

Proof of Proposition 6.7 First we see that the term

$$\upsilon_0(1/\zeta) + 2\,\upsilon_0(q) = \sum_{n\geq 0}\left(\frac{q^n/\zeta}{1-q^n/\zeta} + 2\frac{q^{n+1}}{1-q^{n+1}}\right)$$

plays no role in the limit as it is a bounded quantity in x for each fixed r.

Less trivial is to calculate the limiting form of $\upsilon_0(q\zeta)$, which is a summation like

$$\sum_{n\geq 0}\frac{q^{n+1-X}}{1+q^{n+1-X}},$$

for X being large. The kicker here is understanding that the main contribution to the sum is given by terms where m runs between 0 and $2\lceil X\rceil$.[8] Coupling the $(k-1)$th and the $(2\lceil X\rceil - k - 1)$th addends and using the simple inequality

$$1 - \frac{1-q^2}{1+q^2+q^{k-X}+q^{X-k+2}} \leq \frac{1}{1+q^{X-k}} + \frac{1}{1+q^{X-2\lceil X\rceil+k}} \leq 1,$$

we see that

$$\sum_{n\geq 0}\frac{1}{1+q^{X-n-1}} = X + \mathcal{O}(1).$$

We are interested in the case when $X = \eta x - \gamma x^{1/3} r$, so that, plugging this result into (6.26) we are left to calculate

$$\begin{aligned}&\lim_{x\to\infty}\frac{1}{\gamma x^{1/3}}\left(\kappa x a_0(\mathscr{d}) - x h_0(\mathscr{d}) - \eta x + r\gamma x^{1/3}\right)\\ &= r + \lim_{x\to\infty}\frac{x^{2/3}}{\gamma}\mathscr{d}g'(\mathscr{d}),\end{aligned}$$

which gives (6.26) and (6.27) after expanding g' around its critical point ς as

$$g'(\mathscr{d}) \approx \frac{1}{2}g'''(\varsigma)(\mathscr{d}-\varsigma)^2 = \frac{\varsigma^2}{2\gamma^2}g'''(\varsigma)\frac{\varpi^2}{x^{2/3}} + \mathcal{O}\big(x^{-2/3}\big).$$

This procedure also proves the boundedness of the remainder $R_x^{(3)}$ due to the generality of r.

---

[8] Here $\lceil\rceil$ is the ceiling function.

Result (6.28) follows from expression (6.26). We have

$$R_x^{(2)}(r^*) = -\gamma x^{1/3} r^* - \sum_{n \geq 0} \left(\frac{q^n/\zeta}{1 - q^n/\zeta} - \frac{q^{n+1}\zeta}{1 - q^{n+1}\zeta} \right) + (\text{terms independent of } r^*),$$

where $\zeta = -q^{-\eta x + \gamma x^{1/3} r^*}$. In this way the difference $R_x^{(2)}(r^*) - R_x^{(2)}(r^* - 1/(\gamma x^{1/3}))$ becomes

$$-1 + \frac{q^{\eta x - \gamma x^{1/3} r^*}}{1 + q^{\eta x - \gamma x^{1/3} r^*}} + \frac{q^{-\eta x + \gamma x^{1/3} r^*}}{1 + q^{-\eta x + \gamma x^{1/3} r^*}},$$

which converges to zero exponentially as x goes to infinity. □

Next we present the proofs of Propositions 6.8, 6.9. Our approach follows a saddle point analysis analogous to that showed in Sect. 6.4. The only difference between the argument we present next and that used for the evaluation of the Fredholm determinant of $\widetilde{f}\widetilde{A}$ consists in the proof of the exponential decay of rear tails of $\widetilde{f}(\nu)\widetilde{\Phi}_x^{(i)}(\nu)\widetilde{\Psi}_x^{(j)}(\nu)$. In fact in Sect. 6.4 the decay of rear tails of $\widetilde{f}\widetilde{A}$ was implied by the bound (6.78) and by the convergence $\widetilde{f} \to \mathbb{1}_{(r,\infty)}$. Below to obtain similar estimates we use more direct computations and hypothesis (6.22).

Proof of Proposition 6.8 By making use of the saddle point method it is easy, at this stage, to obtain a convergence result as

$$\mathscr{d} \sum_{\substack{i,j=1,2 \\ (i,j)\neq(1,1)}} \widetilde{f}(\nu)\widetilde{\Phi}_x^{(i)}(\nu)\widetilde{\Psi}_x^{(j)}(\nu) \xrightarrow[x\to\infty]{} \sum_{\substack{i,j=1,2 \\ (i,j)\neq(1,1)}} \mathbb{1}_{[r,\infty)}(\nu)\Upsilon_{-\varpi}^{(i)}(\nu)\Upsilon_{\varpi}^{(j)}(\nu) \tag{6.88}$$

and to estimate the error term depending on x and r. This holds for ν in relatively large sets of the form $[-L, x^{\delta/3}]$ for some fixed $L > 0$ and $\delta \in (0, 1/3)$. Also, assuming a suitably strong decay of tails of summands in the left hand side of (6.88), this easily leads to an expansion of type (6.29).

Using suitable deformations of contours in the integral expressions of $\widetilde{\Phi}_x^{(2)}$, $\widetilde{\Psi}_x^{(2)}$, such as those seen in Lemma 6.13, one can also establish an exponential type decay for the front tail ($\nu \gg 0$) of (6.88).

The exponential decay we have in the left hand side of (6.88), when ν goes to $-\infty$ is slightly different from what seen previously and in particular, here we make use of the hypothesis $\frac{2a}{1+\sigma} < q^{-1}\mathscr{d}$ stated in (6.22). We evaluate separately each one of the three summands in the left hand side of (6.29), when (i, j) is either equal to $(1, 2)$, $(2, 1)$ or $(2, 2)$.

We start with the $(i, j) = (1, 2)$ term. From expressions reported in Proposition 6.10, we write

$$\begin{aligned}
&\mathscr{d}\widetilde{f}(\nu)\widetilde{\Phi}_x^{(1)}(\nu)\widetilde{\Psi}_x^{(2)}(\nu)\\
&\quad= \frac{1}{1+q^{\gamma x^{1/3}(\nu-r)}} \int_{C_1} \frac{dz}{2\pi \mathrm{i} z} \left(\frac{z}{\mathscr{d}}\right)^{\gamma x^{1/3}\nu} e^{x(g(z)-g(\mathscr{d}))} \frac{(qz/\mathscr{d};q)_\infty}{(q;q)_\infty}. \qquad (6.89)
\end{aligned}$$

We take C_1 to be a circle of center in 0 and radius $\varsigma(1-h/(\gamma x^{1/3}))$, where h is chosen so that $|z| < \mathscr{d}$ for all z in C_1 (e.g. take $h > \varpi$). When x is large enough, such C_1 is a steep descent contour for $\mathfrak{Re}\{g\}$, as proven in Proposition C.1. This, along with the fact that ς is a double critical point for g allows us to state the bound

$$|\exp\{x(g(z)-g(\mathscr{d}))\}| < \text{const} \quad \text{for all } z \in C_1. \qquad (6.90)$$

Moreover, the choice of C_1 also allows us to write

$$\begin{aligned}
\left|\frac{z}{\mathscr{d}}\right|^{\gamma x^{1/3}\nu} &= \exp\left\{\gamma x^{1/3}\nu\left(\log\left(1-\frac{h}{\gamma x^{1/3}}\right) - \log\left(1-\frac{w}{\gamma x^{1/3}}\right)\right) + \mathcal{O}(1)\right\}\\
&\le \exp\left\{\nu(\varpi - h) + \mathcal{O}(1)\right\}, \qquad (6.91)
\end{aligned}$$

having used the simple logarithmic inequality $\frac{y}{1+y} \le \log(1+y) \le y$, valid for all $y > -1$. Despite the right hand side of (6.91) is a quantity which diverges exponentially when $\nu \to -\infty$, its contribution is easily balanced by the term $1/(1+q^{\gamma x^{1/3}(\nu-r)})$ in (6.89), which, for $\nu < -L$, decays as $q^{-(1-r/L)\gamma x^{1/3}\nu}$. Following (6.90), (6.91) we come to the estimate

$$|(6.89)| < \text{const}\frac{l(C_1)}{2\pi} e^{(h-\varpi)\nu} q^{-(1-r/L)\gamma x^{1/3}\nu}$$

where in the right hand side the constant term also includes a trivial bound for the factor $\frac{(qz/\mathscr{d};q)_\infty}{(q;q)_\infty}$. This is enough to show that for L large enough, we have

$$|(6.89)| < c_1 e^{c_2 \gamma x^{1/3}\nu}, \quad \text{for all } \nu < -L, \qquad (6.92)$$

where c_1 and c_2 are two suitably chosen positive constants.

We now want to establish a type of bound similar to (6.92) for the term $(i, j) = (2, 1)$ of the left hand side of (6.29). Again, from (5.17), (5.18) we write

$$\begin{aligned}
\mathscr{d}\widetilde{f}(\nu)\widetilde{\Phi}_x^{(2)}(\nu)\widetilde{\Psi}_x^{(1)}(\nu) &= \frac{1}{1+q^{\gamma x^{1/3}(\nu-r)}} \int_{D_1} \frac{dw}{2\pi \mathrm{i} w}\left(\frac{\mathscr{d}}{w}\right)^{\gamma x^{1/3}\nu}\\
&\quad\times e^{x(g(\mathscr{d})-g(w))} \frac{(q\mathscr{d}/w;q)_\infty}{(qw/\mathscr{d};q)_\infty} \frac{\mathscr{d}}{w-\mathscr{d}}. \qquad (6.93)
\end{aligned}$$

As a contour D_1 we can simply take the contour D described in Proposition C.3. Since we can always deform the integration contour in a neighborhood of size $x^{-1/3}$ of ς, without loss of generality, we assume that $\mathscr{d}$ lies strictly at the left of D_1. With this choice, we know that D_1 is a steep ascent contour for $\mathfrak{Re}\{g\}$ and this, along with the fact that ς is double critical point for g implies the bound

$$|\exp\{x(g(\mathfrak{d}) - g(w))\}| < \text{const} \quad \text{for all } w \in D_1.$$

Another consequence of the choice of contour D_1 is that

$$\max_{w \in D_1} |w| \leq \frac{2a}{1+\sigma},$$

as reported in (C.4). This immediately gives us the estimate

$$\left|\frac{\mathfrak{d}}{w}\right|^{\gamma x^{1/3}\nu} \leq \left|\frac{(1+\sigma)\mathfrak{d}}{2a}\right|^{\gamma x^{1/3}\nu} \qquad \text{for all } w \in D_1,$$

since in this case ν is taken to be negative. In expression (6.93), the contribution of the factor $\mathfrak{d}/(w - \mathfrak{d})$ is bounded, in absolute value, by a quantity of order $x^{1/3}$ and therefore we come to write

$$|(6.93)| < \text{const}\frac{l(D_1)}{2\pi}x^{1/3}\left|\frac{(1+\sigma)}{2a}q^{-(1-r/L)}\mathfrak{d}\right|^{\gamma x^{1/3}\nu}. \tag{6.94}$$

When L is large enough, the assumption $2a/(1+\sigma) < q^{-1}\mathfrak{d}$ of (6.22) guarantees that the right hand side of (6.94) is bounded by an exponential function in $\gamma x^{1/3}\nu$ whenever $\nu < -L$ and this concludes our analysis of the rear tail of the term $(i, j) = (2, 1)$.

To obtain the same type of result also for the case when $(i, j) = (2, 2)$ one can reproduce, with minor adjustments, the same argument we used for $(i, j) = (2, 1)$ and therefore we omit details on this part.

We have, at this point proved a bound for the summands in expression (6.29) of the form

$$\left|\sum_{\substack{i,j=1,2\\(i,j)\neq(1,1)}} \widetilde{f}(\nu)\widetilde{\Phi}_x^{(i)}(\nu)\widetilde{\Psi}_x^{(j)}(\nu)\right| < c_1 e^{c_2 \gamma x^{1/3}\nu} \quad \text{for all } \nu < -L,$$

for suitably chosen positive constants c_1, c_2. This concludes our argument. □

We conclude this Section presenting the proof of Proposition 6.9.

Proof of Proposition 6.9 First we expand the expression in the left hand side of (6.32) as

$$\frac{\mathfrak{d}}{\gamma x^{1/3}}\widetilde{\sum}_{\nu,\lambda_1,\lambda_2 \in \widetilde{\mathbb{Z}}} \widetilde{f}(\nu)\widetilde{A}(\nu,\lambda_1)\widetilde{\mathcal{Q}}(\lambda_1,\lambda_2)\widetilde{f}(\lambda_2)\widetilde{\Phi}_x(\lambda_2)\widetilde{\Psi}_x(\nu). \tag{6.95}$$

We can split the summation (6.95) as

$$\frac{d}{\gamma x^{1/3}} \Big(\widetilde{\sum}_{(\nu,\lambda_1,\lambda_2)\in[-L,x^{\delta/3}]^3} + \widetilde{\sum}_{\nu,\lambda_1,\lambda_2\notin[-L,x^{\delta/3}]^3} \Big)$$
$$\big(\widetilde{f}(\nu) \widetilde{A}(\nu, \lambda_1) \widetilde{\varrho}(\lambda_1, \lambda_2) \widetilde{f}(\lambda_2) \widetilde{\Phi}_x(\lambda_2) \widetilde{\Psi}_x(\nu) \big) .$$

Using estimates already encountered in the proofs of Propositions 6.6, 6.8, we know that the contribution of summation where indices do not belong to $[-L, x^{\delta/3}]^3$ is exponentially small in some power of x. On the other hand, when all ν, λ_1, λ_2 belong to $[-L, x^{\delta/3}]$, we can safely employ the saddle point method to estimate the summand terms in and obtain their expansion in power of $x^{-1/3}$, as done in (6.51) for $\widetilde{A}$. This would ultimately lead to the convergence result (6.32) and to a verification of properties (6.33), (6.34) for the remainder term. The procedure is analogous to what explained throughout the rest of the section and therefore we do not describe its details any further. ◻

7 Specializations of the higher spin six vertex model

In this section we take a look at the most relevant degenerations of the Higher Spin Six Vertex Model. Letting parameters vary and considering different scalings we can study models which could be discrete or continuous both in time or space.

7.1 Stationary q-Hahn particle process

First we will consider the q-Hahn TASEP, a space-time discrete particle process introduced in [56] as a dual counterpart of a general chipping model solvable by coordinate Bethe Ansatz. As a consequence of exact results obtained in Sect. 5 we will establish here determinantal formulas describing the position of a tagged particle for the model in the stationary regime and under certain assumptions on parameters we establish Baik–Rains fluctuations.

The q-Hahn TASEP is a three parameters dependent simple exclusion process where particles, at each time step, move in a predetermined direction with jumps distributed according to a q-deformed Beta binomial law. This means that, recording the position of particles in the lattice at a specific time t in a strictly decreasing sequence $\mathbf{y}(t) = \{y_k(t)\}_{k\in\mathbb{Z}}$, then after a time unit, $\mathbf{y}(t)$ is updated to a new sequence

$$\mathbf{y}(t+1) = \{y_k(t) + \mathsf{J}_k^{t+1}\}_{k\in\mathbb{Z}},$$

where the values of jumps J_k^{t+1} are chosen with probabilities $\mathbb{P}(\mathsf{J}_k^{t+1} = j|\mathbf{y}(t))$, given by

$$\varphi_{q,\mu,\nu}(j\,|g_k) = \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{g_k-j}}{(\nu;q)_{g_k}} \frac{(q;q)_{g_k}}{(q;q)_j(q;q)_{g_k-j}} \quad \text{for } j = 0, \ldots, g_k \tag{7.1}$$

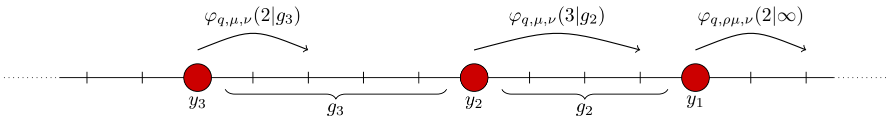


Fig. 16 A visualization of the dynamics of the q-Hahn TASEP

and $g_k = y_{k-1}(t) - y_k(t) - 1$ is the gap between the $(k-1)$-th and the k-th particle (Fig. 16). The fact that, provided

$$0 \le \nu < \mu < 1 \quad \text{and} \quad 0 \le q < 1,$$

$\varphi_{q,\mu,\nu}$ is a probability distribution is is consequence of the q-Gauss summation (A.8).

The case when the system possesses a rightmost particle, say the one labeled with 1, can be considered ideally placing particles with labels 0, −1, −2,…infinitely far away. Here, when we are interested in the evolution of $y_1(t), \dots, y_x(t)$, we can reduce to study a model with only x particles. This is due to the fact that the dynamics of particles $x+1, x+2, \dots$ cannot influence the motion of the ones to their right. In this case, Bethe Ansatz techniques are available (see [23,29]) and, for the special initial conditions

$$y_k(0) = -k \quad \text{a.s.} \quad \text{for } k = 1, \dots, x$$

the distribution of the single particle $y_x(t)$ exhibits a determinantal structure. This particular property was used in [68] to establish Tracy–Widom fluctuations for the integrated current.

Our goal is to study a different class of initial conditions, where particles fill, with different densities, expressed in terms of two parameters $0 < d_-, d_+ < 1$, the regions respectively at the left and at the right of the origin. More specifically, these are given setting

$$\begin{aligned} &y_1(0) = -1 \quad \text{a.s.}, \\ &y_{k-1}(0) - y_k(0) - 1 \sim \begin{cases} q\mathrm{NB}(\nu, d_+) & \text{if} \quad k \le 1 \\ q\mathrm{NB}(\nu, d_-) & \text{if} \quad k > 1, \end{cases} \end{aligned} \tag{7.2}$$

and we refer to these as *double sided q-negative binomial initial conditions*. In words, at time $t = 0$, consecutive particles occupying the negative half line $\mathbb{Z}_{\le -1}$ (those with labels greater or equal than 1) are spaced with q-negative binomial distribution of parameters (ν, d_-) and those in the portion of the lattice $\mathbb{Z}_{\ge -1}$ are spaced with q-negative binomial law of parameters (ν, d_+). An important particular case of initial conditions (7.2) is given setting

$$d_- = d_+ = d. \tag{7.3}$$

As proven in Proposition 7.1, with this particular choice, the dynamics of the q-Hahn TASEP preserves the distribution of gaps between consecutive particles and therefore (7.2), (7.3) are regarded as *stationary initial conditions*.

The reason behind the exact solvability of the model with initial conditions (7.2) is that the study of evolution of coordinates $y_1(t), \dots, y_x(t)$ (with $x \geq 1$), can be reduced to the study of the same quantities in a system including only finitely many particles. Indeed the presence of infinitely many particles (spaced with distribution $q\mathrm{NB}(\nu, \mathscr{d}_+)$) at the right of the first one can be mimicked by simply slowing down y_1 by a quantity depending on $\mathscr{d}_+$. This is a consequence of the fact that the dynamics of the q-Hahn TASEP preserves the spacing between $y_1, y_0, y_{-1}, \dots$ and of the simple identity

$$\mathbb{P}(\mathsf{J}_1^t = j) = \sum_{g \geq 0} \varphi_{q,\mu,\nu}(j|g)\pi_g = \varphi_{q,\mu\mathscr{d},\nu\mathscr{d}}(j|\infty), \tag{7.4}$$

where

$$\pi_M = \mathscr{d}^M \frac{(\nu;q)_M}{(q;q)_M} \frac{(\mathscr{d},q)_\infty}{(\nu\mathscr{d};q)_\infty}. \tag{7.5}$$

This can be proven expanding terms $\varphi_{q,\mu,\nu}(j|g)$, π_g and using the q-binomial theorem (A.7). Equality (7.4) shows that, when $y_0(t) - y_1(t) - 1$ is distributed according to $q\mathrm{NB}(\nu, \mathscr{d}_+)$ at each time, then the effective distribution of jumps of y_1 is given by $\varphi_{q,\mu\mathscr{d}_+,\nu\mathscr{d}_+}(\bullet|\infty)$.

We now come to prove the claim that, with initial conditions (7.2), (7.3), the dynamics preserves the distribution of gaps.

Proposition 7.1 *Let $\mathbf{y}(t) = \{y_k(t)\}_{k\in\mathbb{Z}}$ be the array of positions of particles of a q-Hahn TASEP having initial conditions*

$$y_{k-1}(0) - y_k(0) - 1 \sim qNB(\nu, \mathscr{d}), \quad \textit{for all } k \in \mathbb{Z},$$

and $\mathscr{d}$ is a fixed parameter in the interval $(0,1)$. Then, for each $t \geq 0$

$$y_{k-1}(t) - y_k(t) - 1 \sim qNB(\nu, \mathscr{d}), \quad \textit{for all } k \in \mathbb{Z}.$$

We start stating a simple summation identity

Lemma 7.2 *For any complex numbers a, b, c and integer $M \geq 0$, we have*

$$\sum_{k\geq 0} a^k \frac{(b;q)_k}{(q;q)_k} \frac{(c;q)_{M-k}}{(q;q)_{M-k}} = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{(zab;q)_\infty}{(za;q)_\infty} \frac{(zc;q)_\infty}{(z;q)_\infty} \frac{dz}{z^{M+1}}, \tag{7.6}$$

where C_0 is a sufficiently small contour encircling 0 and no other poles.

Proof For z sufficiently close to 0 we define the functions

$$F(z) = \sum_{l\geq 0} (za)^l \frac{(b;q)_l}{(q;q)_l} = \frac{(zab;q)_\infty}{(za;q)_\infty}, \quad G(z) = \sum_{l \geq 0} z^l \frac{(c;q)_l}{(q;q)_l}$$

and we see that their product can be written as

$$F(z)G(z) = \sum_{M\geq 0} \Big(\sum_{k\geq 0} a^k \frac{(b;q)_k}{(q;q)_k} \frac{(c;q)_{M-k}}{(q;q)_{M-k}} \Big) z^M,$$

so that

$$\sum_{k\geq 0} a^k \frac{(b;q)_k}{(q;q)_k} \frac{(c;q)_{M-k}}{(q;q)_{M-k}} = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} F(z)G(z) \frac{dz}{z^{M+1}}$$

and we have our result. □

Proof of Proposition 7.1 We proceed with a checking style argument. We will show that

$$\pi_M = \sum_{k\geq 0} \pi_k \sum_{l\geq 0} \mathbb{P}(\mathsf{J}_{x-1}^{t+1} = M - k + l) \varphi_{q,\mu,\nu}(l|\,k),$$

or, equivalently, expanding all terms, that

$$\begin{aligned} \mathscr{d}^M \frac{(\nu,q)_M}{(q;q)_M} = \sum_{k\geq 0} \mathscr{d}^k \sum_{l\geq 0} (\mu \mathscr{d})^{M-k+l} \frac{(\nu/\mu;q)_{M-k+l}}{(q;q)_{M-k+l}} \\ \times \frac{(\mathscr{d}\mu;q)_\infty}{(\mathscr{d}\nu;q)_\infty} \mu^l \frac{(\nu/\mu;q)_l (\mu;q)_{k-l}}{(q;q)_l (q;q)_{k-l}}, \end{aligned} \tag{7.7}$$

where we made use of a summation like (7.4) to express the probability of the $(x-1)$-th particle making a jump of $M-k+l$ steps. In the right hand side of (7.7) we can exchange the summation order noticing that the sum in the l index is nontrivial only for $l \leq k$. Therefore this can be written as

$$\begin{aligned} &\frac{(\mathscr{d}\mu;q)_\infty}{(\mathscr{d}\nu;q)_\infty} \sum_{l\geq 0} \mu^l \mathscr{d}^l \frac{(\nu/\mu;q)_l}{(q;q)_l} \sum_{k\geq l} \mathscr{d}^{k-l} (\mu \mathscr{d})^{M-(k-l)} \frac{(\nu/\mu;q)_{M-(k-l)} (\mu;q)_{k-l}}{(q;q)_{M-(k-l)} (q;q)_{k-l}} \\ &\quad = \mu^M \mathscr{d}^M \sum_{k'\geq 0} \mu^{-k'} \frac{(\nu/\mu;q)_{M-k'} (\mu;q)_{k'}}{(q;q)_{M-k'} (q;q)_{k'}}. \end{aligned}$$

The summation can be evaluated with (7.6) setting $a = 1/\mu$, $b = \mu$ and $c = \nu/\mu$. We get

$$\sum_{k'\geq 0} \mu^{-k'} \frac{(\nu/\mu;q)_{M-k'} (\mu;q)_{k'}}{(q;q)_{M-k'} (q;q)_{k'}} = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{(z\nu/\mu;q)_\infty}{(z/\mu;q)_\infty} \frac{dz}{z^{M+1}} = \mu^{-M} \frac{(\nu;q)_M}{(q;q)_M},$$

which combined with the previous identities completes the proof. □

The fact that (7.2), (7.3) constitute a family of translation invariant initial conditions was originally argued in [29]. There the author speculated the stationarity property starting from the fact that they are an infinite volume analog of the factorized steady state measures of the q-Hahn zero range process in the ring geometry [56]. Our proof is of some interest as it is elementary, in the sense that it only makes use of notable q-binomial identities.

Although the q-Hahn TASEP was introduced in [56] with no reference to stochastic vertex models, it is indeed possible to obtain it as a degeneration of the Higher Spin Six Vertex Model, as it was observed first in [30]. The natural way to construct a simple exclusion process from the Higher Spin Six Vertex Model is to interpret the vertical axis as a time direction and to read the number of paths vertically crossing vertices as the evolution of gaps between consecutive particles. More specifically, given occupation random variables $\mathsf{j}_1^1, \mathsf{j}_1^2, \dots, \mathsf{j}_1^t$ and $\mathsf{m}_2^t, \mathsf{m}_3^t, \dots, \mathsf{m}_x^t$ defined in (1.2), (1.3), we construct a configuration of particles $\mathbf{y}(t) = \{y_k(t)\}_{k\geq 1}$ such that

$$y_1(t) = -1 + \mathsf{j}_1^1 + \cdots + \mathsf{j}_1^t \quad \text{and} \quad y_{k-1}(t) - y_k(t) - 1 = \mathsf{m}_k^t \quad \text{for all } k \geq 2.$$

In this way, horizontal occupation numbers $\mathsf{j}_1^t, \mathsf{j}_2^t, \dots$ are interpreted as jumping distances of particles during the update at time t and the Markov operator describing the stochastic dynamics is given in general by the transfer operator $\mathfrak{X}_{u_t}^{(J)}$, as in (2.15). Although the exact form of the fused weights $\mathsf{L}^{(J)}$, reported in (2.17), appearing in the definition of $\mathfrak{X}_{u_t}^{(J)}$, looks rather complicated it is possible to degenerate it and match it with an instance of the q deformed beta binomial distribution (7.1). This fact was first observed in [12] and in our notation, Proposition 6.7 of the same article implies that

$$\mathsf{L}^{(J)}_{u_t\xi_k, s_k}(i_1, j_1 | \, i_2, j_2) \xrightarrow[\substack{s_k=s \\ \xi_k=1/s \\ u_t=s^2}]{} \mathbb{1}_{i_1+j_1=i_2+j_2} \mathbb{1}_{j_2 \leq i_1} \varphi_{q, q^J s^2, s^2}(j_2 | i_1). \tag{7.8}$$

Expression (7.8) suggests us the right specialization to turn the transfer operator $\mathfrak{X}_{u_t}^{(J)}$ into the Markov generator of the q-Hahn TASEP. On the other hand, thanks to arguments carried in Sect. 4.2, we also know how to employ analytic continuation techniques to describe the probability distribution of the model for certain random initial conditions, which indeed would correspond to (7.2).

We like to summarize this discussion concerning the matching between q-Hahn particle processes and Higher Spin Six Vertex Model in the following.

Proposition 7.3 *Consider the (non stochastic) Higher Spin Six Vertex Model on* $\Lambda_{0,-1}$ *with boundary conditions*

$$\mathsf{m}_x^{-1} = 0 \quad a.s., \quad \textit{for all } x \geq 1, \quad \mathsf{j}_0^0 = K \quad a.s, \quad \mathsf{j}_0^t = J \quad a.s, \quad \textit{for all } t \geq 1, \tag{7.9}$$

transfer operators $\mathfrak{X}^{(K)}_{q/\mathsf{d}_-}, \mathfrak{X}^{(J)}_{s^2}, \mathfrak{X}^{(J)}_{s^2}, \dots$ *and parameters*

$$\Xi = (\xi_1, s^{-1}, s^{-1}, \dots), \quad \mathbf{S} = (s_1, s, s, \dots). \tag{7.10}$$

Then, for each l, the signed measure $\mathbb{P}(\mathcal{H}(x,t) - \mathsf{m}_1^0 = l)$ is an analytic function of $\mu = q^J s^2$ and $\wp = q^{-K}$. Moreover, setting

$$s_1 = 1/N, \quad \xi_1 = \text{đ}_+ N, \quad \textit{and taking the limit } N \to \infty, \tag{7.11}$$

$$0 < \text{đ}_- < \text{đ}_+ < 1, \quad s^2 = \nu, \quad 0 \le \nu < \mu < 1, \quad \wp = 0, \tag{7.12}$$

we obtain

$$\mathbb{P}(\mathcal{H}(x,t) - \mathsf{m}_1^0 = l) = \mathbb{P}_{q\mathrm{H}(\text{đ}_-,\text{đ}_+)\otimes \mathsf{m}}(y_x(t) + x - \mathsf{m} = l),$$

for each $l \in \mathbb{Z}$, $x \ge 1$ and $t \ge 0$. In the last equality, both sides are probability measures, $\mathbb{P}_{q\mathrm{H}(\text{đ}_-,\text{đ}_+)\otimes \mathsf{m}}$ refers to a product measure of a q-Hahn TASEP with initial conditions (7.2) and of a qPoisson($\text{đ}_-/\text{đ}_+$) random variable m (independent of y_x).

Proof We start considering expression (4.39), which is stated for a model with boundary conditions (7.9), $J = 1$, transfer operator $\mathfrak{X}^{(K)}_{q/v}, \mathfrak{X}_{u_1}, \mathfrak{X}_{u_1}, \dots$ and generic parameters u_t, ξ_x, s_x. The fusion of rows procedure, as explained in Sect. 2.3, allows us to substitute $\mathfrak{X}$ with $\mathfrak{X}^{(J)}$ and it simply consists in specializing spectral parameters in geometric progressions of ratio q. We therefore operate the substitution

$$(u_{Jm+1}, u_{Jm+2}, \dots, u_{J(m+1)}) \to (s^2, qs^2, q^2s^2, \dots, q^{J-1}s^2) \quad \text{for } m \ge 0 \tag{7.13}$$

and, as a result, in (4.39), we change the factor $\widetilde{\Pi}$ in the integrand into

$$\widetilde{\Pi}(\mathbf{z}; \Xi^{-1}\mathbf{S}, \mathbf{U}) \to \prod_{j=1}^{x} \left(\left(\frac{1}{(z_j s^2; q)_\infty} \right)^{x-1} \left(\frac{(s^2 z_j; q)_\infty}{(q^J s^2 z_j; q)_\infty} \right)^t \right).$$

As long as the quantity $q^J s^2$ is smaller than 1 in absolute value, no new pole is created for the integration in $z_1, \dots, z_x$ on the torus $\mathbb{T}^x$ and therefore we can analytically prolong $\mathbb{P}(\mathcal{H}(x,t) - \mathsf{m}_1^0 = \bullet)$ to the region $s^2 = \nu, \mu = q^J s^2, 0 \le \nu < \mu < 1$. Choice of parameters (7.11) implies that $\mathsf{j}_1^t \sim \varphi_{q,\text{đ}_+\mu,\text{đ}_+\nu}(\bullet|\infty)$ as it was observed in (4.36) and together with conditions on ν, μ, it turns the transfer operator $\mathfrak{X}^{(J)}_{s^2}$ into a Markov generator describing a q-Hahn TASEP where the rightmost particle is slower of a factor đ_+ compared to the others. This is a basic consequence of (7.8).

The analytic continuation in parameter $\wp = q^{-K}$ is treated as in Proposition 4.10. As a result of choice $\wp = 0$, random variables $\mathsf{m}_1^0, \mathsf{m}_2^0, \dots$ become independently distributed as

$$\mathsf{m}_1^0 \sim q\mathrm{Poi}(\text{đ}_-/\text{đ}_+), \quad \text{and} \quad \mathsf{m}_x^0 \sim q\mathrm{NB}(\nu, \text{đ}_-).$$

This passage is explained more extensively in Sect. 4.2. Recalling the definition of $\mathcal{H}$, given in (4.30), interpreting m_k^t as the gap between the $(k-1)$-th and the k-th particle and j_k^t as the jumps made by the k-th particle during the update at time t we realize that

$$y_x(t) + x - \mathsf{m} \overset{\mathcal{D}}{=} \mathcal{H}(x,t) - \mathsf{m}_1^0 \quad \text{for all } x \geq 1, t \geq 0,$$

where the equality holds in distribution. So far y_x is the position of the x-th particle of a q-Hahn TASEP with a slower particle, but as a consequence of identity (7.4), this is equivalent, in distribution, to the position of the x-th particle in a model with infinitely many particles at the right of y_1 spaced with q-negative binomial distribution of parameters (ν, d_+). This concludes the proof. ◻

We come now to state our main results on the double sided q-negative binomial q-Hahn TASEP.

Proposition 7.4 *For $d_- < d_+$, we have*

$$\mathbb{E}_{q\mathrm{H}(d_-,d_+)\otimes m}\left(\frac{1}{(\zeta q^{y_x(t)+x-m};q)_\infty}\right) = \det(\mathbf{1} - fK_{q\mathrm{H}})_{l^2(\mathbb{Z})}, \tag{7.14}$$

where

$$f(n) = \frac{1}{1 - q^n/\zeta}, \tag{7.15}$$

$$K_{q\mathrm{H}}(n,m) = A_{q\mathrm{H}}(n,m) + (d_+ - d_-)\Phi_{q\mathrm{H},x}(m)\Psi_{q\mathrm{H},x}(n), \tag{7.16}$$

$$A_{q\mathrm{H}}(n,m) = \frac{\tau(n)}{\tau(m)} \int_D \frac{dw}{2\pi\mathrm{i}} \int_C \frac{dz}{2\pi\mathrm{i}} \frac{z^m}{w^{n+1}} \frac{F(z)}{F(w)} \frac{1}{z-w}, \tag{7.17}$$

$$\Phi_{q\mathrm{H},x}(n) = \tau(n) \int_D \frac{dw}{2\pi\mathrm{i}} \frac{1}{w^{n+2}} \frac{1}{(d_+/w;q)_\infty} \frac{1}{F(w)}, \tag{7.18}$$

$$\Psi_{q\mathrm{H},x}(n) = \frac{1}{\tau(n)} \int_C \frac{dz}{2\pi\mathrm{i}} z^n \frac{1}{z - d_-} (qz/d_+;q)_\infty F(z). \tag{7.19}$$

The contour D encircles $1, d_+$ and no other singularity, whereas C contains 0 and $q^k d_-$, for any k in $\mathbb{Z}_{\geq 0}$. Moreover, $\tau(n)$ is taken to be

$$\tau(n) = \begin{cases} b^n, & \text{if } n \geq 0 \\ c^n, & \text{if } n < 0, \end{cases} \tag{7.20}$$

with $d_- < b < d_+ < 1 < c < 1/\nu$, and

$$F(z) = \left(\frac{(\nu z;q)_\infty}{(\mu z;q)_\infty}\right)^t \left(\frac{(z;q)_\infty}{(\nu z;q)_\infty}\right)^{x-1} \frac{(qz/d_+;q)_\infty}{(qd_-/z;q)_\infty}. \tag{7.21}$$

Finally, m is a $q\mathrm{Poi}(d_-/d_+)$ random variable independent of y_x.

Proof We only need to specialize results of Theorem 1.3 to the same choice of parameters adopted in Proposition 7.3 and ultimately to perform the analytic continuation in parameter $\mu = q^J s^2$. ◻

It is now safe to apply techniques developed for the Higher Spin Six Vertex Model to describe asymptotic fluctuations of the position of a tagged particle in the stationary q-Hahn TASEP. In order to fix the parameters describing the scaling of the q-Hahn TASEP we introduce the families of functions

$$a_{-1}(z; p, \widetilde{p}) = \log\left(\frac{(zp; q)_\infty}{(z\widetilde{p}; q)_\infty}\right) \quad \text{and} \quad a_{k+1}(z; p, \widetilde{p}) = z\frac{d}{dz}a_k(z; p, \widetilde{p}),$$

for all $k \geq 0$. When $k \geq 0$, $a_k(z; p, \widetilde{p})$ is expressed in terms of q-polygamma like functions (A.12) as $\upsilon_k(\widetilde{p}z) - \upsilon_k(pz)$.

Definition 7.5 (Scalings for the stationary q-Hahn TASEP) For numbers $d \in (0, 1)$ and $\varpi \in \mathbb{R}$, we set

$$\begin{aligned}
\gamma_{qH} &= -\frac{1}{2^{1/3}}\left(\frac{a_1(d; \nu, 1)}{a_1(d; \nu, \mu)}a_2(d; \nu, \mu) - a_2(d; \nu, 1)\right)^{1/3}, \\
\kappa_{qH;\varpi} &= \frac{a_1(d; \nu, 1)}{a_1(d; \nu, \mu)} + \frac{a_2(d; \nu, 1)a_1(d; \nu, \mu) - a_1(d; \nu, 1)a_2(d; \nu, \mu)}{a_1(d; \nu, \mu)^2}\frac{\varpi}{\gamma_{qH}x^{1/3}}, \\
\eta_{qH;\varpi} &= \kappa_{qH;\varpi}a_0(d; \nu, \mu) - a_0(d; \nu, 1) \\
&\quad + \frac{a_2(d; \nu, 1)a_1(d; \nu, \mu) - a_1(d; \nu, 1)a_2(d; \nu, \mu)}{a_1(d; \nu, \mu)}\frac{\varpi^2}{\gamma_{qH}^2x^{2/3}}
\end{aligned}$$

By means of quantities $\kappa_{q\mathrm{H}}, \eta_{q\mathrm{H}}, \gamma_{q\mathrm{H}}$ we are now going to confirm the KPZ-scaling conjecture for the stationary q-Hahn TASEP, result that in our notation reads

$$\frac{y_x(\kappa_{q\mathrm{H};\varpi}x) - (\eta_{q\mathrm{H},\varpi} - 1)x}{\gamma_{q\mathrm{H}}x^{1/3}} \xrightarrow[x\to\infty]{\mathcal{D}} F_\varpi,$$

where F_ϖ is the Baik–Rains distribution introduced in Definition 6.2. For the sake of a rigorous procedure in the asymptotics we need to establish technical conditions on parameters defining the model. As for the Higher Spin Six Vertex Model, the main technical issue is to guarantee the existence of steep descent/ascent contours C/D for the real part of a function $g_{q\mathrm{H}}$, which in this case is given by

$$g_{q\mathrm{H}}(z) = -\eta_{q\mathrm{H},\varpi}\log(z) + \kappa_{q\mathrm{H},\varpi}a_{-1}(z; \nu, \mu) - a_{-1}(z; \nu, 1). \tag{7.22}$$

Function $g_{q\mathrm{H}}$ possesses a double critical point ς in a neighborhood of order $\varpi/x^{1/3}$ of d and the construction of contours C, D enables us to perform a saddle point analysis to evaluate the Fredholm determinant of the kernel $K_{q\mathrm{H}}$. The expression of the critical point ς can be given explicitely and it is identical to (6.48) once we substitute $\gamma = \gamma_{q\mathrm{H}}$, $a_k = a_k(d, \nu, \mu)$ and $h_k = a_k(d, \nu, 1)$.

We find that, in the q-Hahn TASEP case, the analysis of $g_{q\mathrm{H}}$ slighlty differs from that of the homologous function g for the general Higher Spin Six Vertex Model. In particular, the problem of the existence of steep contours was already considered in [68] and we can take advantage of results obtained by the author in the same paper, which we summarize in the following Proposition.

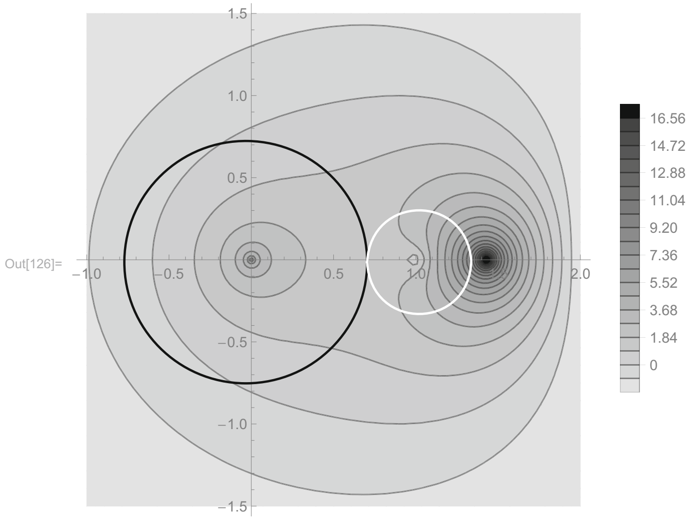


Fig. 17 In the picture we see the contour plot $\Re\mathfrak{e}\{g(z)\}$ for the particular choice of parameters $\varsigma = 0.7$, $\mu = 0.7$, $\nu = 0.4$, $q = 0.3$, $x = 20$. Here z lies the complex rectangle $[-1, 2] + \mathrm{i}[-1.5, 1.5]$ and, as the legend shows, to darker shades correspond greater values of $\Re\mathfrak{e}\{g(z)\}$. The black and the white circle are respectively C and D and they intersect in the critical point ς

Proposition 7.6 ([68], Prop. 6.2, 6.3) *Define the curves*

$$C = \left\{ \varsigma e^{\mathrm{i}\vartheta} \mid \vartheta \in [0, 2\pi) \right\}, \quad D = \left\{ 1 - (1 - \varsigma) e^{\mathrm{i}\vartheta} \mid \vartheta \in [0, 2\pi) \right\}. \tag{7.23}$$

Then, assuming

$$0 \leq q \leq \nu < \mu \leq 1/2, \tag{7.24}$$

we have, for x large enough

1. $\Re\mathfrak{e}\{g_{q\mathrm{H}}\}$ *assumes, on the contour C, a unique global maximum in ς;*
2. $\Re\mathfrak{e}\{g_{q\mathrm{H}}\}$ *assumes, on the contour D, a unique global minimum in ς.*

Condition (7.24) appears to be technical, as it could be argued through simple numerical tests. As an example we report in Fig. 17 the plot of the real part of $g_{q\mathrm{H}}$ for a choice of parameters q, ν, μ, *đ* not included in (7.24) and from where it appears evident that, also in that case steep contours C, D can be constructed. We do not attempt here to loosen hypothesis on Proposition 7.6 and we simply use such results to adapt our asymptotic analysis of the q-Laplace transform in the stationary q-Hahn TASEP setting.

We come to the following

Theorem 7.7 *Consider the q-Hahn TASEP with parameters q, ν, μ as in (7.24) and stationary initial conditions, where $\mathscr{d}$ satisfies*

$$\mathscr{d} > \frac{2q}{1+q}. \tag{7.25}$$

Then we have

$$\lim_{x\to\infty} \mathbb{P}_{q\mathrm{H}(\mathscr{d},\mathscr{d})} \left(\frac{y_x(\kappa_{q\mathrm{H};\varpi} x) - (\eta_{q\mathrm{H},\varpi} - 1)x}{\gamma_{q\mathrm{H}} x^{1/3}} > -r \right) = F_{\varpi}(r). \tag{7.26}$$

Proof We see that from the Fredholm determinant identity (7.14), employing the same procedure detailed in the proofs of Theorem 1.4, we can decouple the quantity y_x from the random shift m. This leads us to an exact expression for the q-Laplace transform $\mathbb{E}_{q\mathrm{H}(\mathscr{d},\mathscr{d})}\left((\zeta q^{y_x(t)+x}; q)_\infty^{-1}\right)$ as

$$\frac{1}{(q;q)_\infty} \sum_{k\geq 0} \frac{(-1)^k q^{\binom{k}{2}+k}}{(q;q)_k} \left(V_x(\zeta q^{-k}) - V_x(\zeta q^{-k-1}) \right), \tag{7.27}$$

where the function V_x takes the form

$$\begin{aligned} V_x(\zeta) = \det(\mathbf{1} - f A_{q\mathrm{H}}) \Bigg(& -\upsilon_0(1/\zeta) \\ & + \upsilon_0(q\zeta) - 2\upsilon_0(q) - x\mathscr{a}_0(\mathscr{d};\nu,1) + t\mathscr{a}_0(\mathscr{d};\nu,\mu) \\ & - \mathscr{d} \sum_{\substack{i,j=1,2 \\ (i,j)\neq(1,1)}} \sum_{n\in\mathbb{Z}} f(n) \Phi^{(i)}_{q\mathrm{H},x}(n) \Psi^{(j)}_{q\mathrm{H},x}(n) \\ & - \mathscr{d} \sum_{n\in\mathbb{Z}} (f A_{q\mathrm{H}} \varrho_{q\mathrm{H}} f \Phi_{q\mathrm{H},x})(n) \Psi_{q\mathrm{H},x}(n) \Bigg). \end{aligned} \tag{7.28}$$

Here $\varrho_{q\mathrm{H}} = (\mathbf{1} - f A_{q\mathrm{H}})^{-1}$ and terms $\Phi^{(i)}_{q\mathrm{H},x}$, $\Psi^{(j)}_{q\mathrm{H},x}$ are obtained from $\Phi_{q\mathrm{H},x}$, $\Psi_{q\mathrm{H},x}$ separating the contribution of pole $\mathscr{d}$ on integral expressions (7.18), (7.19), analogously to Eqs. (5.16) to (5.19).

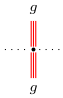
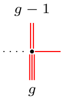
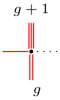
Result (7.26) now follows evaluating the large x limit of (7.27), (7.28) after setting

$$\zeta = -q^{-\eta_{q\mathrm{H};\varpi} x + \gamma_{q\mathrm{H}} x^{1/3} r}, \quad t = \kappa_{q\mathrm{H};\varpi} x$$

and this can be done through methods developed throughout Sect. 6. We want to remark that the main tool used to compute the asymptotic behavior of the q-Laplace transform is the saddle point method, applied to the complex integral expression of the kernels $A_{q\mathrm{H}}$, $\Phi_{q\mathrm{H},x} \otimes \Psi_{q\mathrm{H},x}$. This procedure is rigorously justified by the statement of Proposition 7.6 which guarantees the existence of steep integration contours C, D.

The additional hypothesis (7.25), we made on the density parameter $\mathscr{d}$ is analogous to condition $2a/(1+\sigma) < q^{-1}\mathscr{d}$ stated in (6.22) for the Higher Spin Six Vertex Model.

Table 2 Weights $\mathsf{L}_{\xi_x \varepsilon, s_x}(i_1, j_1 | i_2, j_2)$ in the continuous time scaling $u = \varepsilon \sim 0$

				
$\mathsf{L}_{\xi_x \varepsilon, s_x}$	$1 - \xi_x s_x (1 - q^g)\varepsilon$	$\xi_x s_x (1 - q^g)\varepsilon$	$s_x^2 q^g + \xi_x s_x (1 - s_x^2 q^g)\varepsilon$	$1 - s_x^2 q^g - \xi_x s_x (1 - s_x^2 q^g)\varepsilon$

In particular (7.25) implies that, for x large enough,

$$\frac{q}{\mathfrak{d}} \times \max_{w \in D} |w| < 1,$$

where the contour D is defined in (7.23). This fact can be used to establish the exponential decay of rear tails of terms

$$- \sum_{\substack{i,j=1,2 \\ (i,j)\neq(1,1)}} f(n) \Phi^{(i)}_{q\mathrm{H},x}(n) \Psi^{(j)}_{q\mathrm{H},x}(n) - (f A_{q\mathrm{H}} \varrho_{q\mathrm{H}} f \Phi_{q\mathrm{H},x})(n) \Psi_{q\mathrm{H},x}(n),$$

in the expression of V_x analogously to what is explained in the proof of Proposition 6.8.
□

7.2 Continuous time processes

There are mainly two possible scalings giving rise to meaningful continuous time versions of the Higher Spin Six Vertex Model (here we only treat the unfused model, therefore $J = 1$) and the aim of this paragraph is to briefly define and make a few comments on them.

Possibly the naivest way to proceed is to simply scale the spectral parameter $u = -\varepsilon$ along with the discrete time $t = \lceil \varepsilon^{-1} \mathfrak{t} \rceil$ and then let ε go to zero. In this limit the vertex weights $\mathsf{L}_{\xi_x u, s_x}$ become, up to order ε, as shown in Table 2.

Using standard arguments one can rigorously show the convergence to a Markov process $\mathfrak{X}^{\mathrm{hc}}$ which evolves according to the following rules

- paths move on the quadrant $\mathbb{Z}_{\geq 2} \times \mathbb{R}_{\geq 0}$, where at each discrete x-coordinate is associated a Poisson clock with rate $\xi_x s_x (1 - q^{\#\{\text{paths travelling the } x\text{-th lane}\}})$
- each path travels vertically with unitary speed, possibly temporarily sharing with others the same route. When the clock at the generic position x rings, one of the paths occupying this lane is immediately diverted to its right and placed at the random location $x + k$ with probability

$$s_{x+1}^2 \cdots s_{x+k-1}^2 q^{\mathfrak{h}(x+1) - \mathfrak{h}(x+k)} \left(1 - s_{x+k}^2 q^{\mathfrak{h}(x+k) - \mathfrak{h}(x+k+1)}\right) \qquad (7.29)$$

  and from there it continues its upward movement. If at the moment the x-th clock rings no path is occupying position x, nothing happens.

- paths randomly emanate from the boundary $\{1\} \times \mathbb{R}_{\geq 0}$ with exponential law

  $$\mathbb{P}(\text{a path is generated from the segment } \{1\} \times [t, t+\varepsilon]) = \xi_1 s_1 \varepsilon + o(\varepsilon). \tag{7.30}$$

  If a generation happens at ordinate t the path travels horizontally to the random location (k, t) with probability

  $$s_2^2 \cdots s_{k-1}^2 q^{\mathfrak{h}(2)-\mathfrak{h}(k)} (1 - s_k^2 q^{\mathfrak{h}(k)-\mathfrak{h}(k+1)}) \tag{7.31}$$

  and subsequently proceeds turning upward.

In our description we assumed, as before, the definition of the height function $\mathfrak{h}(x)$ at a specific ordinate t to be the number of paths strictly to the right of $x - 1$.

A possible relevant degenerations of this model is the q-TASEP. This is obtained setting $s_k^2 = 0$ at each location while keeping the $\xi_k s_k$'s finite positive quantities, to be interpreted as speeds of particles.

Other than the procedure we just described, one can possibly consider the ASEP scaling of the Stochastic Six Vertex Model. In this case we set

$$\begin{aligned} s_1^2 = 0, \quad -\xi_1 s_1 > 0, \quad \xi_x = 1, \quad s_x = q^{-\frac{1}{2}} \quad \text{for } x > 1, \\ u = q^{-\frac{1}{2}} \left(1 + (1-q)\varepsilon\right), \quad t = \lceil \varepsilon^{-1} \mathfrak{t} \rceil, \end{aligned} \tag{7.32}$$

while, at the same time we shift the position x to $x + t$. With the choice (7.32), when ε becomes small, we see that paths tend to have diagonal trajectories and the displacements from these diagonals have to be read as the movement of particles in an ASEP dynamics. The coefficient $-\xi_1 s_1$, which previously determined the rate at which paths entered the system now has to be interpreted as a density parameter. More specifically, the initial conditions given by this specializations are half-Bernoulli, in the sense that they describe an ASEP having, at time $\mathfrak{t} = 0$, the positive half line empty and each remaining location independently filled with a particle with probability

$$\frac{-\xi_1 s_1 q^{-\frac{1}{2}}}{1 - \xi_1 s_1 q^{-\frac{1}{2}}}.$$

The asymmetry here is governed by q and one interprets the height function of the Higher Spin Six Vertex Model as the integrated current of particles through a specific location.

By making use of analytic continuation techniques as those considered above, one can extend these initial conditions to the so called double sided Bernoulli initial conditions, where particles fill locations also in the positive half line independently with Bernoulli law. In this setting, in [2], the author was able to study asymptotic properties of the integrated current $\mathfrak{J}$ of the stationary ASEP. As one could expect, also in this case determinantal structures were found considering the q-Laplace transform

$$\left\langle \frac{1}{(\zeta q^{\mathfrak{Z}}; q)_\infty} \right\rangle. \tag{7.33}$$

An interesting observation is that determinantal expressions for the q-Laplace transform (7.33) obtained in [2] are similar yet different from the ones we would get employing elliptic determinantal techniques utilized in [39] and in this paper. In a future work we plan to shed light on relationships between these two different determinantal structures and there we will provide a more detailed analysis of the Stochastic Six Vertex Model, which therefore is here omitted.

7.3 Inhomogeneous exponential jump model

This continuous time/continuous space degeneration of the ($J = 1$) Higher Spin Six Vertex Model was recently introduced in [19], where authors were able to study asymptotics and phase transitions of the model with step initial conditions. Here we will apply our results to take into account its stationary state.

The emergence of a continuous space structure in the Higher Spin Six Vertex Model can be recovered considering a particular scaling of the Markov process $\mathfrak{X}^{\mathrm{hc}}$ defined in the Sect. 7.2. For its description we need the following

Definition 7.8 In this Section we denote with $\mathbf{B} \subset \mathbb{R}_{\geq 0}$ (set of roadblocks) a fixed discrete set, with no accumulation point and for any arbitrary small positive number ε we set

$$\mathbf{B}^\varepsilon = \{\lfloor \varepsilon^{-1} b \rfloor | b \in \mathbf{B}\}.$$

To the set $\mathbf{B}$ we associate a weight function

$$\mathsf{p} : \mathbf{B} \to [0, 1].$$

Moreover we set v, k to be positive functions and we refer to them respectively as speed and jumping distance function.

In light of Definition 7.8 we now specialize the continuous time process $\mathfrak{X}^{\mathrm{hc}}$ setting

$$\begin{gathered} s_i^2 = \begin{cases} e^{-\varepsilon \mathsf{k}(i\varepsilon)}, & \text{if } i \in \mathbb{Z}_{\geq 2} \setminus \mathbf{B}^\varepsilon \\ \mathsf{p}(b), & \text{if } i \in \mathbb{Z}_{\geq 2} \cap \mathbf{B}^\varepsilon \end{cases}, \\ \xi_i s_i = \mathsf{v}(i\varepsilon) \quad \text{for } i \geq 2, \quad x = \lceil \epsilon^{-1} \mathbf{x} \rceil. \end{gathered} \tag{7.34}$$

When ε goes to zero the half continuous Higher Spin Six Vertex Model $\mathfrak{X}^{\mathrm{hc}}$ converges to a process $\mathfrak{X}^{\mathrm{EJ}}$ which we are yet to describe. To do so we need to degenerate expressions (7.29), (7.30), (7.31) according to the scaling detailed in 7.34 and take the limit $\varepsilon \to 0$. In this case we make use of the zero range process language, where the paths at location (x, t) are interpreted as a stack of particles at location x and time t. As particles randomly move on $\mathbb{R}_{>0}$, we describe the process through the quantity

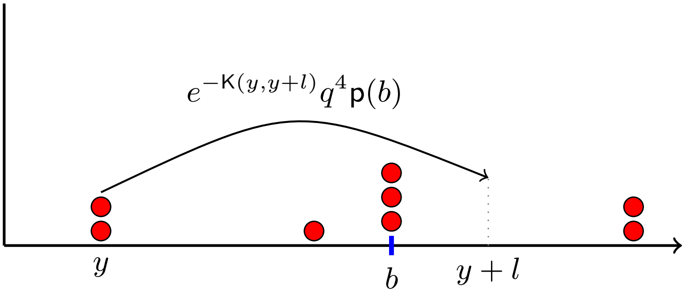


Fig. 18 A graphic visualization of the dynamics of the Exponential Jump Model. In the example exactly one particle at location y is attempting a jump of length greater than l. The probability of this event involves the presence of other particles in the term q^4 and the presence of a roadblock at location b in the term $\mathsf{p}(b)$ other than the exponential factor $e^{-\mathsf{K}(y,y+l)}$

$$\mathfrak{H}(\mathbf{x},\mathfrak{t}) = -\#\left\{\begin{array}{l}\text{particle in the interval}\\ (0,\mathbf{x}]\text{ at time } 0\end{array}\right\} + \#\left\{\begin{array}{l}\text{particle moving to the right}\\ \text{of } \mathbf{x} \text{ during the interval } (0,\mathfrak{t}]\end{array}\right\},$$

that is clearly the analogous of the height function $\mathcal{H}$ (1.12).

Given a locally finite configuration of stacks of particles on $\mathbb{R}_{>0}$, they evolve according to $\mathfrak{X}^{\mathrm{EJ}}$ as follows:

- at any location y hosting a stack of particles, independently of the rest of the system, a Poisson clock rings with rate $\mathsf{v}(y)(1-q^{\#\{\text{number of particles at } y\}})$. As the clock rings, exactly one particle of the stack becomes active
- an active particle at location y and time $\mathfrak{t}$ performs a random jump to its right of length Δy taken with law

$$\mathbb{P}\left(\Delta y \geq l | \text{ the jump started at } y\right) = e^{-\mathsf{K}(y,y+l)} q^{\mathfrak{H}(y_+,\mathfrak{t})-\mathfrak{H}(y+l,\mathfrak{t})} \prod_{b\in\mathbf{B}: y<b<y+l} \mathsf{p}(b).$$

  Here $\mathsf{K}(y,y+l) = \int_y^{y+l}\mathsf{k}(t)dt$ and the difference $\mathfrak{H}(y_+,\mathfrak{t}) - \mathfrak{H}(y+l,\mathfrak{t})$ is the number of particles lying within the interval $(y,y+l]$ at time $\mathfrak{t}$.
- active particles are injected at position $y=0$ according to a Poisson process with intensity $\mathsf{v}(0)$.

The mechanism is clear. When a particle decides to jump, it chooses a distance Δy with exponential distribution and, as it flies to reach the targeted destination, it might get captured by a stack of other particles with probability $1-q^{\#\{\text{ particles in the stack}\}}$ or blocked by a roadblock b with probability $1-\mathsf{p}(b)$ (see Fig. 18).

To discuss continuous degenerations of the Higher Spin Six Vertex Model with random boundary conditions considered above we give the following.

Definition 7.9 Assume we have positive piecewise continuous function $\mathfrak{L}$ possessing left and right limit at each point, a family of probability distributions $\{\boldsymbol{\varphi}_y\}_{y\in\mathbb{R}}$ on $\mathbb{Z}_{\geq 1}$ $\Omega\subset\mathbb{R}$. We define the marked Poisson process $\mathfrak{m}_{\mathfrak{L},\boldsymbol{\varphi}}$ as the process which picks a set of points $\{y_r\}_r$ on Ω according to an inhomogeneous Poisson process with rate given by $\mathfrak{L}$ and assigns to each one of the y_r's, independently, a mark chosen with law $\boldsymbol{\varphi}_{y_r}$.

The definition of the marked Poisson process comes in handy when we take the scaling form of the double sided q-negative binomial Higher Spin Six Vertex Model. The basic limit

$$\rho^M \frac{(e^{-\varepsilon\alpha};q)_M}{(q;q)_M}\frac{(\rho;q)_\infty}{(e^{-\varepsilon\alpha}\rho;q)_\infty} \xrightarrow[\varepsilon\sim 0]{} \left(\delta_{0,M}+\varepsilon\alpha\frac{\rho^M}{1-q^M}\right)(1+\varepsilon\alpha\upsilon_0(\rho))^{-1}+\mathcal{O}(\varepsilon),$$

implies that, with the scaling (7.34), the half continuous Higher Spin Six Vertex model with q-NB$(s_i^2, v/(\xi_i s_i))$ entries at location i in the horizontal boundary becomes the Exponential Jump Model with initial conditions described as:

- on $\mathbb{R}_{>0}\setminus\mathbf{B}$ places stacks of particles according to an inhomogeneous marked Poisson process $\mathfrak{m}_{\mathfrak{L},\boldsymbol{\varphi}}$, where

$$\boldsymbol{\varphi}_y(k)=\frac{(v/\mathsf{v}(y))^k}{1-q^k}\upsilon_0(v/\mathsf{v}(y))^{-1} \quad \text{and} \quad \mathfrak{L}(y)=\mathsf{k}(y)\upsilon_0(v/\mathsf{v}(y));$$

- on each $b\in\mathbf{B}$ places a stack of M_b particles with probability

$$(v/\mathsf{v}(b))^{M_b}\frac{(\mathsf{p}(b);q)_{M_b}}{(q;q)_{M_b}}\frac{(v/\mathsf{v}(b);q)_\infty}{(\mathsf{p}(b)v/\mathsf{v}(b);q)_\infty}. \tag{7.35}$$

We refer this process with the symbol $\mathcal{P}(v,\mathsf{v},\mathsf{k},\mathsf{p})$. We report a simple property of a general Marked Poisson process on the line.

Proposition 7.10 *Consider a marked Poisson process* $\mathfrak{m}_{\mathfrak{L},\boldsymbol{\varphi}}$*, as in Definition 7.9 and consider the random variable* $\mathcal{M}(a,b)$ *to be the sum of marks contained within the interval* (a,b)*. Then, we have*

$$\mathbb{E}\left(z^{\mathcal{M}(a,b)}\right)=\exp\left\{\int_a^b \mathfrak{L}(y)\left(\mathbb{E}_{\boldsymbol{\varphi}_y}(z^{\mathcal{M}})-1\right)dy\right\}, \tag{7.36}$$

where $\mathbb{E}_{\boldsymbol{\varphi}_y}(z^{\mathcal{M}})$ *is the generating function of the random variable* $\mathcal{M}$ *counting the marks at the generic location* y*.*

Proof First we see that, from the definition itself of the inhomogeneous marked Poisson process we can write the probability distribution of $\mathcal{M}(a,b)$ as

$$\mathbb{P}(\mathcal{M}(a,b)=N)=e^{-\int_a^b\mathfrak{L}(y)dy}\sum_{k\geq 0}\sum_{\substack{\mu\vdash N\\ l(\mu)=k}}\sum_{\nu\sim\mu}\int_{a\leq y_1<y_2<\cdots<y_k\leq b}\prod_{j=1}^k\mathfrak{L}(y_j)\varphi_{y_j}(\eta_j)dy_j$$

where the notation $\eta \sim \mu$ means that η is a permutation of the partition μ. The generating function can now be evaluated as

$$\mathbb{E}\left(z^{\mathcal{M}(a,b)}\right) = e^{-\int_a^b \mathfrak{L}(y)dy} \sum_{k\geq 0} \sum_{\mu:l(\mu)=k} \sum_{\nu\sim\mu} \int_{a\leq y_1<y_2<\cdots<y_k\leq b} \prod_{j=1}^{k} \mathfrak{L}(y_j) z^{\eta_j} \varphi_{y_j}(\eta_j) dy_j$$

$$= e^{-\int_a^b \mathfrak{L}(y)dy} \sum_{k\geq 0} \frac{1}{k!} \left(\sum_{\eta\geq 1} \int_a^b \mathfrak{L}(y) z^{\eta} \varphi_y(\eta) dy \right)^k,$$

which gives (7.36). ☐

Below we report a proof that the process $\mathcal{P}(v, \mathsf{v}, \mathsf{k}, \mathsf{p})$ indeed admit the stationary measure as a particular case.

Proposition 7.11 *Assume that* $\mathsf{v}(0) < \mathsf{v}(y)$ *for all* $y > 0$*. Then, the process* $\mathcal{P}(\mathsf{v}(0), \mathsf{v}, \mathsf{k}, \mathsf{p})$ *is stationary for the Exponential Jump Model.*

Proof The proof of this fact, in the homogeneous case, was already given in [19]. One could simply regard it as a continuous space modification of the argument we used in Sect. 4. Nonetheless it might still be interesting to explicitly work out the calculations in this particular case as well.

We aim to prove that at any location L the process which counts particles jumping from the region $[0, L]$ to (L, ∞) is a Poisson process with rate $\mathsf{v}(0)$, and hence the current is constant and the density is stationary.

Since the set $\mathbf{B}$ has no accumlation points we can write $\mathbf{B} \cap [0, L] = \{b_1, \ldots, b_n\}$, for some finite n and subsequently we partition $[0, L]$ as a disjoint union of intervals

$$[0, L] = I_0 \cup I_1 \cup \cdots \cup I_n,$$

where $I_0 = [0, b_1]$, $I_1 = (b_1, b_2], \ldots, I_n = (b_n, L]$. In the infinitesimal time interval $(0, \Delta\mathrm{t})$ we assume that, up to the terms quadratically small in $\Delta\mathrm{t}$, we can write

$$\mathbb{P}\left\{ \begin{array}{l} \text{a particle crosses } L \\ \text{during } (0, \Delta\mathrm{t}) \end{array} \right\} = \sum_{k=0}^{n} \mathbb{P}\left\{ \begin{array}{l} \text{a particle crosses } L \text{ during} \\ (0, \Delta\mathrm{t}) \text{ jumping from } I_k \end{array} \right\} + o(\Delta\mathrm{t}). \tag{7.37}$$

Now, let's consider the single term in the summation in the right hand side of (7.37) and after cleverly using the definition of the model we can easily see that

$$\mathbb{P}\left\{ \begin{array}{l} \text{a particle crosses } L \text{ during} \\ (0, \Delta\mathrm{t}) \text{ jumping from } I_k \end{array} \right\}$$

$$= \mathsf{v}(0)\Delta\mathrm{t} \int_{b_k}^{b_{k+1}} dy \frac{\mathsf{k}(y)}{1 - \mathsf{v}(0)/\mathsf{v}(y)} e^{-\mathsf{K}(y,L)} \mathbb{E}(q^{\mathfrak{H}(y,0)-\mathfrak{H}(L,0)}) \prod_{j=k+1}^{n} \mathsf{p}(b_j)$$

$$+\,\mathsf{v}(0)\Delta\mathsf{t}\left(1-\frac{\mathsf{p}(b_{k+1})(1-\mathsf{v}(0)/\mathsf{v}(b_{k+1}))}{1-\mathsf{p}(b_{k+1})\mathsf{v}(0)/\mathsf{v}(b_{k+1})}\right)\mathbb{E}(q^{\mathfrak{H}(y,0)-\mathfrak{H}(L,0)})\prod_{j=k+2}^{n}\mathsf{p}(b_j). \tag{7.38}$$

At this point we can split the difference $\mathfrak{H}(y,0)-\mathfrak{H}(L,0)$ in two independent contribution: one coming from the marked Poisson process which we baptize as $\mathcal{M}(y,L)$ and the other coming form particles encountered at roadblocks which we call $\mathcal{M}_{\mathbf{B}}(y,L)$. Using expression (7.36) and q summation identities concerning the measure (7.35), we obtain

$$\begin{aligned}&\mathbb{E}(q^{\mathcal{H}(y,0)-\mathcal{H}(L,0)})\\&\quad=\mathbb{E}(q^{\mathcal{M}(y,L)})\mathbb{E}(q^{\mathcal{M}_{\mathbf{B}}(y,L)})\\&\quad=\exp\left\{-\int_y^L\mathsf{k}(w)\frac{\mathsf{v}(0)/\mathsf{v}(w)}{1-\mathsf{v}(0)/\mathsf{v}(w)}dw\right\}\prod_{b:y<b<L}\frac{1-\mathsf{v}(0)/\mathsf{v}(b_j)}{1-\boldsymbol{p}(b_j)\mathsf{v}(0)/\mathsf{v}(b_j)}.\end{aligned}$$

Substituting this last identity in the left hand side of (7.38) we get

$$\begin{aligned}&\mathsf{v}(0)\Delta\mathsf{t}\Bigg[\prod_{j=k+2}^{n}\frac{\mathsf{p}(b_j)(1-\mathsf{v}(0)/\mathsf{v}(b_j))}{1-\mathsf{p}(b_j)\mathsf{v}(0)/\mathsf{v}(b_j)}\exp\left\{-\int_{b_{k+1}}^{L}\mathsf{k}(w)\frac{\mathsf{v}(0)/\mathsf{v}(w)}{1-\mathsf{v}(0)/\mathsf{v}(w)}dw\right\}\\&\quad-\prod_{j=k+1}^{n}\frac{\mathsf{p}(b_j)(1-\mathsf{v}(0)/\mathsf{v}(b_j))}{1-\mathsf{p}(b_j)\mathsf{v}(0)/\mathsf{v}(b_j)}\exp\left\{-\int_{b_k}^{L}\mathsf{k}(w)\frac{\mathsf{v}(0)/\mathsf{v}(w)}{1-\mathsf{v}(0)/\mathsf{v}(w)}dw\right\}\Bigg],\end{aligned}$$

from which we deduce that the sum on the right hand side of (7.37) telescopes to $\mathsf{v}(0)\Delta\mathsf{t}$. □

We now state a result analogous to that of Theorem 1.3 in order to characterize the distribution of $\mathfrak{H}$ in the Exponential Jump Model with initial conditions given by $\mathcal{P}(v,\mathsf{v},\mathsf{k},\mathsf{p})$. In order to apply techniques developed throughout Sect. 5 we will assume that the speed function v is of the form

$$\mathsf{v}(y)=\begin{cases}\mathsf{v}_0, & \text{if } y=0\\ 1, & \text{if } y>0,\end{cases} \tag{7.39}$$

where $\mathsf{v}_0<1$ and that the system presents no roadblocks. This means that the spatial inhomogeneity is all encoded in the jumping distance function k, on which we do not make any particular assumption. We will refer to an Exponential Jump Model with initial conditions $\mathcal{P}(v,\mathsf{v},\mathsf{k},\mathsf{p}=0)$ with v as in (7.39) with the shorthand $\mathrm{EJ}(v,\mathsf{v}_0;\mathsf{k})$.

Proposition 7.12 *For* $0<v<\mathsf{v}_0<1$, *we have*

$$\mathbb{E}_{\mathrm{EJ}(v,\mathsf{v}_0;\mathsf{k})\otimes m}\left(\frac{1}{(\zeta q^{\mathfrak{H}(\mathsf{x},\mathsf{t})-m};q)_\infty}\right)=\det(\mathbf{1}-fK_{\mathrm{EJ}})_{l^2(\mathbb{Z})}, \tag{7.40}$$

where

$$
\begin{aligned}
f(n) &= \frac{1}{1-q^n/\zeta},\\
K_{\mathrm{EJ}}(n,m) &= A_{\mathrm{EJ}}(n,m) + (\mathsf{v}_0 - v)\Phi_{\mathrm{EJ,x}}(m)\Psi_{\mathrm{EJ,x}}(n),\\
A_{\mathrm{EJ}}(n,m) &= \int_D \frac{dw}{2\pi\mathrm{i}} \int_C \frac{dz}{2\pi\mathrm{i}} \frac{z^m}{w^{n+1}} \frac{\exp\left\{\mathfrak{t}z - \upsilon_0(z)\int_0^{\mathrm{x}} \mathsf{k}(y)dy\right\}}{\exp\left\{\mathfrak{t}w - \upsilon_0(w)\int_0^{\mathrm{x}} \mathsf{k}(y)dy\right\}}\\
&\quad\times \frac{(qv/w, qz/\mathsf{v}_0;q)_\infty}{(qv/z, qw/\mathsf{v}_0;q)_\infty}\frac{1}{z-w},\\
\Phi_{\mathrm{EJ,x}}(n) &= \int_D \frac{dw}{2\pi\mathrm{i}} \frac{1}{w^{n+1}} \exp\left\{-\mathfrak{t}w + \upsilon_0(w)\int_0^{\mathrm{x}} \mathsf{k}(y)dy\right\} \frac{(qv/w;q)_\infty}{(qw/\mathsf{v}_0;q)_\infty}\frac{1}{w-\mathsf{v}_0},\\
\Psi_{\mathrm{EJ,x}}(n) &= \int_C \frac{dw}{2\pi\mathrm{i}} z^n \exp\left\{\mathfrak{t}z - \upsilon_0(z)\int_0^{\mathrm{x}} \mathsf{k}(y)dy\right\} \frac{(qz/\mathsf{v}_0;q)_\infty}{(qv/z;q)_\infty}\frac{1}{z-v}.
\end{aligned}
$$

The contour D encircles $\mathsf{v}_0, 1$ *and no other singularity, whereas C contains 0 and* $q^k v$*, for any k in* $\mathbb{Z}_{\geq 0}$*. Finally* m *is a qPoisson random variable with parameter* v/v_0 *independent of the particle process.*

As usual the way to obtain formulas useful for the analysis of the stationary state of the exponential jump model is to set $v = \mathsf{v}_0$. This degenerates the quantity $\mathfrak{H} - \mathsf{m}$ and subsequently the right hand side of (7.40). Therefore we might proceed with removing the dependence on the independent random quantity m with an argument equal to that of Lemma 5.6, obtaining an expression as

$$
\begin{aligned}
\mathbb{E}_{\mathrm{EJ}(v,\mathsf{v}_0;\mathsf{k})}\left(\frac{1}{(\zeta q^{\mathfrak{H}(\mathrm{x},\mathfrak{t})};q)_\infty}\right) &= \frac{1}{(v/\mathsf{v}_0;q)_\infty}\sum_{k\geq 0}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}\left(\frac{v}{\mathsf{v}_0}\right)^k\\
&\quad\times \mathbb{E}_{\mathrm{EJ}(v,\mathsf{v}_0;\mathsf{k})\otimes\mathsf{m}}\left(\frac{1}{(\zeta q^{\mathfrak{H}(\mathrm{x},\mathfrak{t})-\mathsf{m}-k};q)_\infty}\right).
\end{aligned}
\tag{7.41}
$$

Both left and right hand side of (7.41) can be proven to be analytic functions of both v_0 and v in a neighborhood of $v = \mathsf{v}_0$ which unlocks the mechanisms developed in Sects. 5.2 and 6 to study asymptotics. In this case, instead of considering large time/space asymptotics, we let the jumping parameter k grow along with the time. This corresponds to watching the system evolve with particles moving at a slow speed for long period of time. When this is the case, the scaling we adopt is

$$
\mathfrak{t} = \kappa_{\mathrm{EJ};\varpi}\mathcal{T} \quad \text{and} \quad \mathsf{k}(y) = \mathcal{k}(y)\mathcal{T},
$$

where $\kappa_{EJ;\varpi}$, along with other scaling parameters is fixed in the following

Definition 7.13 (*Scaling parameters Exponential Jump Model*) Set $0 < \mathsf{v}_0 < 1$ and $\mathcal{K}(x) = \int_0^{\mathbf{x}} \mathscr{k}(y)dy$. Then, we set

$$\begin{aligned}
\gamma_{EJ} &= \frac{1}{2^{1/3}} \left((\upsilon_3(\mathsf{v}_0) - \upsilon_2(\mathsf{v}_0)) \mathcal{K}(x) \right)^{1/3}, \\
\kappa_{EJ;\varpi} &= \frac{1}{\mathsf{v}_0} \upsilon_2(\mathsf{v}_0) \mathcal{K}(x) + \frac{1}{\mathsf{v}_0} (\upsilon_3(\mathsf{v}_0) - \upsilon_2(\mathsf{v}_0)) \mathcal{K}(x) \frac{\varpi}{\gamma x^{1/3}}, \\
\eta_{EJ;\varpi} &= \kappa_{EJ;\varpi} \mathsf{v}_0 - \upsilon_1(\mathsf{v}_0) \mathcal{K}(x) + (\upsilon_3(\mathsf{v}_0) - \upsilon_2(\mathsf{v}_0)) \mathcal{K}(x) \frac{\varpi^2}{\gamma^2 x^{2/3}}.
\end{aligned}$$

As a last result we can establish Baik–Rains fluctuations of $\mathfrak{h}(\mathrm{x}, \eta_{\mathrm{EJ};\varpi} \mathcal{T})$ around $\eta_{\mathrm{EJ};\varpi} \mathcal{T}$.

Theorem 7.14 *Consider the stationary state of the inhomogeneous exponential jump model and let q be in a sufficiently small neighborhood of zero. Then, we have*

$$\lim_{\mathcal{T} \to \infty} \mathbb{P}_{\mathrm{EJ}(v_0, v_0, \mathcal{T}\mathscr{k})} \left(\frac{\mathfrak{h}(\mathrm{x}, \kappa_{\mathrm{EJ};\varpi} \mathcal{T}) - \eta_{\mathrm{EJ};\varpi} \mathcal{T}}{\gamma_{\mathrm{EJ}} \mathcal{T}^{1/3}} > -r \right) = F_{\varpi}(r).$$

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A Preliminaries on *q*-deformed quantities

Along the course of the paper we largely made use of q-deformed quantities, such as q-Pochhammer symbols and q-hypergeometric series. The reader might consider these as fairly common and established notions, but, for the sake of completeness, we still like to dedicate this appendix to recall their definitions.

Assuming q is a parameter in the interval $[0, 1)$, we define the q-Pochhammer symbol

$$(z; q)_n = \begin{cases} (1 - z)(1 - zq) \cdots (1 - zq^{n-1}), & \text{if } n \in \mathbb{Z}_{>0}, \\ 1, & \text{if } n = 0, \\ (1 - zq^n)^{-1}(1 - zq^{n-1})^{-1} \cdots (1 - zq)^{-1}, & \text{if } n \in \mathbb{Z}_{<0}, \end{cases} \tag{A.1}$$

for every meaningful $z \in \mathbb{C}$. We also denote the product of multiple q-Pochhammer symbols of the same order in the compact notation

$$(z_1; q)_n \cdots (z_k; q)_n = (z_1, \dots, z_k; q)_n. \tag{A.2}$$

When n is positive, the q-Pochhammer symbol (A.1) is a polynomial in z and it admits the expansion

$$(z;q)_n = \sum_{k=0}^{n} (-z)^k q^{\binom{k}{2}} \binom{n}{k}_q, \tag{A.3}$$

where we introduced the q-binomial

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{A.4}$$

The the q-binomial admits the combinatorial expansion

$$\binom{n}{k}_q = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=k}} q^{\|I\| - \binom{k+1}{2}}, \tag{A.5}$$

with $I = \{i_1, \dots, i_k\}$ and $\|I\| = i_1 + \cdots + i_k$.

When we let the integer n grow to $+\infty$, we see that the product in the left hand side of (A.1) is convergent and hence we can define

$$(z;q)_\infty = \prod_{j \geq 0} (1 - zq^j). \tag{A.6}$$

An important result concerning q-Pochhammer symbols is the summation identity

$$\sum_{k \geq 0} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(za;q)_\infty}{(z;q)_\infty} \qquad \text{for } a \in \mathbb{C}, |z| < 1, \tag{A.7}$$

which can be found in [41], Theorem 12.2.5. and it is usually called q-binomial theorem. A slightly more general version of summation (A.7) is the so called q-Gauss summation ([41], Theorem 12.2.4)

$$\sum_{n \geq 0} \left(\frac{c}{ab} \right)^n \frac{(a,b;q)_n}{(c,q;q)_n} = \frac{(c/a, c/b; q)_\infty}{(c, c/(ab); q)_\infty}, \quad \text{for } |c/(ab)| < 1, \quad \text{or } b \in q^{\mathbb{Z}_{<0}}. \tag{A.8}$$

The q-hypergeometric series

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} \middle| q, z \right) = \sum_{k \geq 0} \frac{(a_1, \dots, a_{r+1}; q)_k}{(b_1, \dots, b_r, q; q)_k} z^k, \tag{A.9}$$

is defined for generic parameters $a_1, \dots, a_{r+1} \in \mathbb{C}$, $b_1, \dots, b_r \in \mathbb{C} \backslash q^{\mathbb{Z}_{<0}}$ and $|z| < 1$. In the case when at least one of the a_j is of the form q^{-k}, for some non-negative integer

k, the q-hypergeometric series (A.9) becomes a finite sum and its definition holds also for more general complex numbers z. The regularized terminating q-hypergeometric function is also defined as

$$
{}_{r+1}\bar{\phi}_r \left(\begin{matrix} q^{-n}, & a_1, & \dots, & a_r \\ b_1, & b_2, & \cdots, & b_r \end{matrix} \middle| q, z \right) = \sum_{k=0}^{n} z^k \frac{(q^{-n}; q)_k}{(q; q)_k} \prod_{j=1}^{r} (a_j; q)_k (q^k b_j; q)_{n-k}. \tag{A.10}
$$

In Sect. 4 we used the q-analog of the Chu–Vandermonde identity ([41], (12.2.17)) that we report as

$$
{}_2\bar{\phi}_1 \left(\begin{matrix} q^{-n}, a \\ c \end{matrix} \middle| q, q \right) = \frac{(c/a; q)_n}{(c; q)_n} a^n. \tag{A.11}
$$

In the paper we also made use of functions υ_j, defined as

$$
\upsilon_j(z) = \sum_{k \geq 1} \frac{k^j z^k}{1 - q^k}. \tag{A.12}
$$

They are related to the more classical q-polygamma function [69]

$$
\boldsymbol{\psi}_q(\theta) = -\log(1-q) + \log(q) \sum_{n \geq 0} \frac{q^{n+\theta}}{1 - q^{n+\theta}},
$$

since

$$
\upsilon_0(z) = \frac{1}{\log(q)} \left[\log(1-q) + \boldsymbol{\psi}_q \left(\frac{\log(z)}{\log(q)} \right) \right] \tag{A.13}
$$

and

$$
\frac{d}{dz} \upsilon_j(z) = \frac{1}{z} \upsilon_{j+1}(z). \tag{A.14}
$$

The inverse of the infinite q-Pochhammer symbol (A.6) is often called q-exponential and through it one can define a q-deformed notion of the common Laplace transform. For a given $f \in \ell^1(\mathbb{Z})$ the function

$$
\underset{\sim}{f}(\zeta) = \sum_{n \in \mathbb{Z}} \frac{f(n)}{(q^n \zeta; q)_\infty} \qquad \text{for } \zeta \in \mathbb{C} \setminus q^{\mathbb{Z}} \tag{A.15}
$$

is the $\underset{\sim}{q}$-Laplace transform of f. As for the usual Laplace transform, the operation $f \mapsto \underset{\sim}{f}$ admits an inverse. This is discussed, for example, in [39] and we do not report the exact form of the inverse q-Laplace transform as we do not explicitly make use of it during this paper.

B Bounds for ϕ_l, ψ_l, Φ_x, Ψ_x

We collect here some useful bounds for the quantities ϕ_l, ψ_l, Φ_x, Ψ_x defined in eqs. (5.4) to (5.7). Terms Φ_x, Ψ_x can be further decomposed as

$$\begin{aligned}\Phi_x(n) &= \Phi_x^{(1)}(n) + \Phi_x^{(2)}(n),\\ \Psi_x(n) &= \Psi_x^{(1)}(n) + \Psi_x^{(2)}(n),\end{aligned}$$

obtained separating from the integration (5.6) (resp. (5.7)) the contribution of pole $w = \mathscr{d}$ (resp. $z = v$) from that of other poles. Their exact expression was given in eqs. (5.16) to (5.19).

Proposition B.1 *Let $v < \mathscr{d}$. Then, for all fixed x, there exist constants $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 > 0$, such that*

$$|\phi_l(n)|, |\psi_l(n)|, |\Phi_x(n)|, |\Phi_x^{(2)}(n)| < \Gamma_1 e^{-\Gamma_2 |n|} \qquad \textit{for all } n \in \mathbb{Z} \tag{B.1}$$

and

$$|\Psi_x(n)| < \begin{cases} \Gamma_1 e^{-\Gamma_2 |n|} & \textit{if } n \in \mathbb{Z}_{<0} \\ \Gamma_3 e^{-\Gamma_4 |n|} & \textit{if } n \in \mathbb{Z}_{\geq 0}. \end{cases} \tag{B.2}$$

Moreover Γ_1, Γ_2 can be chosen so that their relative bounds also hold for $v, \mathscr{d}$ in the region (1.18) (in this case the parameter b appearing in the definition of τ (5.8) satisfies $qv < b < \mathscr{d}$).

Proof We start with the terms ϕ_l, $\Phi_x(n)$, $\Phi_x^{(2)}$. Evaluating the complex integrals as sums of residues it is straightforward to get the inequalities

$$\begin{aligned}|\phi_l(n)|, |\Phi_x(n)|, |\Phi_x^{(2)}(n)| &\leq \text{const.}\tau(n) \left(\frac{\mathbb{1}_{n\geq 0}}{|\mathscr{d}|^{|n|}} + \mathbb{1}_{n<0} \max_{i\geq 2}(|\xi_i s_i|)^{|n|} \right) \\ &\leq \Gamma_1 e^{-\Gamma_2 |n|},\end{aligned}$$

for some constants Γ_1, Γ_2 depending on the integrand functions but not on n.

To obtain a similar bound for the term $\psi_l(n)$ we distinguish two cases. When n is positive we take the contour C to be a circle of radius r_+ so that $qv < r_+ < b$. On the other hand, when n is negative we take C to be a circle of radius r_- strictly bigger than c, not containing any of the numbers ξ_i/s_i (we remark that the definition itself of $\tau(n)$ and of numbers b, c is tailor-made for these conditions to be possible). With this choices we easily get

$$|\psi_l(n)| \leq \text{const.} \frac{1}{\tau(n)} \left(\mathbb{1}_{n\geq 0} r_+^{|n|} + \mathbb{1}_{n<0} \frac{1}{r_-^{|n|}} \right) \leq \Gamma_1 e^{-\Gamma_2 |n|}.$$

An argument equivalent to that used for ψ_l can be carried to show (B.2). The only difference here is that the radius r_+ has to be chosen so that $v < r_+ < b$ and hence we cannot extend this bound to the region $qv < b \leq v$. $\square$

Proposition B.2 *Let* v *satisfy* (1.11) *and* $v < đ$ *or possibly* (1.18)*. Then, for each* x*, there exist constants* $\Gamma_1, \Gamma_2 > 0$ *such that*

$$|\phi_l(n)\Psi_x(n)|, |\Phi_x^{(2)}(n)\Psi_x(n)| < \Gamma_1 e^{-\Gamma_2 |n|}. \tag{B.3}$$

Proof From Proposition B.1 we see that we only have to prove (B.3) for positive n's. When this is the case we see directly from the integral expression (5.7) and (5.4) that we can bound both $|\phi_l(n)\Psi_x(n)|$ and $|\Phi_x^{(2)}(n)\Psi_x(n)|$ with some quantity proportional to

$$\frac{|v+\epsilon|^n}{\min_{k\geq 2}|\xi_k s_k|^n}, \tag{B.4}$$

by simply taking the C contour as a circle of radius $v+\epsilon$, for ϵ being sufficiently small. Due to the condition

$$v < \min_{k\geq 2} |\xi_k s_k|,$$

we see that ϵ can be chosen so that (B.4) decays to zero and this completes the proof. $\square$

Proposition B.3 *Let* v, $đ$ *satisfy* (1.18)*. Then, for each fixed* x*, there exist constants* $\Gamma_1, \Gamma_2 > 0$ *such that*

$$\left| f(n)\Phi_x^{(1)}(n)\Psi_x^{(2)}(n) \right| < \Gamma_1 e^{-\Gamma_2 |n|}.$$

Proof We use the integral expression (5.19). When n is positive we take the integration contour C_1 to be a circle of radius $qv+\epsilon$. A bound we can easily obtain is

$$\left| f(n)\Phi_x^{(1)}(n)\Psi_x^{(2)}(n) \right| < \text{const.}\frac{(qv+\epsilon)^n}{đ^n} \qquad \text{for } n \geq 0.$$

On the other hand, when n is negative we chose the contour C_1 as a circle of radius $v-\epsilon$ to get a bound like

$$\left| f(n)\Phi_x^{(1)}(n)\Psi_x^{(2)}(n) \right| < \text{const.}\frac{1}{1-q^n/\zeta}\frac{(v-\epsilon)^n}{đ^n} \qquad \text{for } n < 0.$$

In both cases condition (1.18) allows us to select ϵ small enough to guarantee exponential decay in $|n|$. $\square$

C Construction of contours

Here we discuss the construction of the steep descent contour C and that of the steep ascent contour D which were used in the asymptotic analysis of the Stationary Higher Spin Six Vertex Model in Sect. 6.

Proposition C.1 *Consider fixed real numbers*

$$0 < v < \varsigma, \qquad 0 < q < 1$$

and assume that

$$\varsigma < \inf_{k\geq 2}\{\xi_k s_k\} \leq \sup_{k\geq 2}\{\xi_k s_k\} < \infty \quad \textit{and} \quad 0 \leq s_k^2 < 1, \qquad \textit{for all } k \geq 2.$$

Take also a number $\rho < \varsigma$ and define the contour

$$C_\rho = \{\rho e^{\mathrm{i}\vartheta} | \vartheta \in [0, 2\pi)\}.$$

Then, for ρ sufficiently close to ς we have

$$\frac{d}{d\vartheta}\mathfrak{Re}\{g(\rho e^{\mathrm{i}\vartheta})\} < 0 \quad \textit{for } 0 < \vartheta < \pi, \tag{C.1}$$

where g is given in (6.42).

Remark C.2 The result of Proposition C.1 implies that C_ρ is a steep descent contour for $\mathfrak{Re}(g)$ and in particular

1. $\max_{z\in C_\rho} \mathfrak{Re}\{g(z)\} = g(\rho)$;
2. $\max_{z\in C_\rho} |z| = \rho$.

This easily follows from (C.1) and from the fact that $g(\overline{z}) = \overline{g(z)}$, which implies that $\mathfrak{Re}(g)$ is symmetric with respect to the real axis.

Proof Evaluating the derivative we have

$$\begin{aligned}
&\frac{d}{d\vartheta}\mathfrak{Re}\{g(\rho e^{\mathrm{i}\vartheta})\} \\
&\quad = \sin\vartheta\,\kappa\left(\sum_{i=0}^{J-1}\frac{q^i u\rho}{1+q^{2i}u^2\rho^2-2q^i u\rho\cos\vartheta}\right) \\
&\qquad + \sin\vartheta\,\frac{1}{x}\sum_{k=2}^{x}\sum_{j\geq 0}\left(\frac{\frac{q^j\rho}{\xi_k s_k}}{1+\left(\frac{q^j\rho}{\xi_k s_k}\right)^2-2\frac{q^j\rho}{\xi_k s_k}\cos\vartheta} - \frac{\frac{q^j s_k^2\rho}{\xi_k s_k}}{1+\left(\frac{q^j s_k^2\rho}{\xi_k s_k}\right)^2-2\frac{q^j s_k^2\rho}{\xi_k s_k}\cos\vartheta}\right).
\end{aligned} \tag{C.2}$$

Each term

$$\frac{q^i u\rho}{1+q^{2i}u^2\rho^2-2q^i u\rho\cos\vartheta},$$

has a maximum in $\vartheta=0$ due to the fact that u and ρ have opposite sign, and so does each single one of the summands in the double summation in (C.2), since the generic function

$$\frac{a}{1+a^2-2a\cos\vartheta}-\frac{a\sigma}{1+a^2\sigma^2-2a\sigma\cos\vartheta}$$

is decreasing in $0<\vartheta<\pi$, provided that $0<a,\sigma<1$. Now, if ρ is taken sufficiently close to the critical point ς, in a neighborhood of $\vartheta=0$, the derivative of $\mathfrak{Re}\{g(\rho e^{\mathrm{i}\vartheta})\}$ is negative by construction and, thanks to considerations we just made, it stays negative along the whole half circle. □

The construction of an explicit steepest descent contour D for a general choice of parameters q, Ξ, $\mathbf{S}$ becomes more complicated. Therefore we use the next Proposition both to exhibit a contour in a rather simple setting and to implicitly deduce conditions on q, Ξ, $\mathbf{S}$ under which our arguments of Sect. 6 are perfectly well posed.

Proposition C.3 *For each choice of*

$$0<\varsigma<a, \qquad u<0, \qquad 0<\sigma<1,$$

there exist constants $R_a, R_\sigma, R_q>0$, such that for each choice of parameters $\{\xi_k\}_{k\geq2}$, $\{s_k\}_{k\geq2}$, q satisfying

$$\varsigma<\inf_{k\geq2}\{\xi_k s_k\}, \qquad |\xi_k s_k-a|<R_a, \qquad |s_k^2-\sigma|<R_\sigma, \qquad q<R_q,$$

we are able to construct a complex contour D encircling the set $\{\xi_k s_k\}_{k\geq2}$, for which

1. $\min_{z\in D}\mathfrak{Re}\{g(z)\}=g(\varsigma)$;
2. $\min_{z\in D}|z|=\varsigma$,

where g is given in (6.42).

Proof To show this result we essentially make use of a continuity argument. We start studying the case when

$$q=0, \qquad \xi_k, s_k=a \qquad s_k^2=\sigma \qquad \text{for all } k\geq2.$$

With this choice of parameters the function g becomes

$$g(z)=-\eta\log(z)+\kappa\log(1-uz)+\log\left(\frac{a-z}{a-\sigma z}\right)+\mathcal{O}(x^{-1}), \tag{C.3}$$

where we can neglect the contribution of the $\mathcal{O}(x^{-1})$ term as we are interested in this result only in the limiting case of $x \to \infty$. We define the contour D to be the level curve

$$D = \left\{ z : \ \mathfrak{Re} \left\{ \log \left(\frac{a-z}{a-\sigma z} \right) \right\} = \log \left(\frac{a-\varsigma}{a-\sigma\varsigma} \right) \right\},$$

which is a circle and admit the parametrization

$$\left\{ \varsigma + \rho + \rho e^{\mathrm{i}\vartheta} \,|\, \vartheta \in [0, 2\pi) \right\},$$

with the radius ρ being

$$\rho = \frac{a^2 - a\varsigma - a\varsigma\sigma + \varsigma^2\sigma}{a + a\sigma - 2\varsigma\sigma}.$$

We also report that the leftmost and rightmost extremes of the contour D are respectively ς and $\varsigma + 2\rho$ and one can easily find that the latter satisfies the inequality

$$\varsigma + 2\rho \leq \frac{2a}{1+\sigma}. \tag{C.4}$$

Along the curve D we are able to calculate

$$\frac{d}{d\vartheta} \mathfrak{Re} \left\{ g(\varsigma + \rho + \rho e^{\mathrm{i}\vartheta}) \right\}$$

and to analytically show that its only critical points are $\theta \in \mathbb{Z}\pi$. More specifically, substituting in (C.3) the correct expressions of coefficients η, κ given in (1.21)

$$\eta = \frac{a\varsigma^2(1-\sigma)(a - a^2u + a\sigma - 2\varsigma\sigma + \varsigma^2 u\sigma)}{(a-\varsigma)^2(a-\varsigma\sigma)^2}, \tag{C.5}$$

$$\kappa = -\frac{a(1-\varsigma u)^2(1-\sigma)(a^2 - \varsigma^2\sigma)}{u(a-\varsigma)^2(a-\varsigma\sigma)^2}, \tag{C.6}$$

we get

$$\frac{d}{d\vartheta} \mathfrak{Re} \left\{ g(\varsigma + \rho + \rho e^{\mathrm{i}\vartheta}) \right\} = \sin\vartheta (1 + \cos\vartheta) \frac{1}{P(\cos\vartheta)}.$$

In the last expression P is a polynomial of degree two in the argument and we see that zeros are only achieved on the real axis for $\theta = k\pi$ for $k \in \mathbb{Z}$. We can at this point readily verify that, along D the real part of g assumes a minimum at $z = \varsigma$ and a maximum at $z = \varsigma + 2\rho$ as the function

$$-\eta \log(y) + \kappa \log(1 - uy)$$

is increasing for $y > \varsigma$, and one can check this by direct inspection of its first derivative, by making use of expressions (1.21) for η and κ.

We can now use the fact that g is continuous in the parameters Ξ, $\mathbf{S}$, q for z belonging to D and the fact that, by construction, it will always have a critical point in $z = \varsigma$, to state the existence of neighborhoods respectively of a, σ and 0 in which every choice of $\xi_k s_k$, s_k^2 and q will preserve the steepest descent properties 1 and 2. $\square$

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