Probability Theory and Related Fields (2020) 177:217–256
https://doi.org/10.1007/s00440-019-00947-9

Symmetry of minimizers of a Gaussian isoperimetric problem

Marco Barchiesi[1] · **Vesa Julin**[2]

Received: 23 November 2018 / Revised: 20 August 2019 / Published online: 9 October 2019

Abstract

We study an isoperimetric problem described by a functional that consists of the standard Gaussian perimeter and the norm of the barycenter. The second term is in competition with the perimeter, balancing the mass with respect to the origin, and because of that the solution is not always the half-space. We characterize all the minimizers of this functional, when the volume is close to one, by proving that the minimizer is either the half-space or the symmetric strip, depending on the strength of the barycenter term. As a corollary, we obtain that the symmetric strip is the solution of the Gaussian isoperimetric problem among symmetric sets when the volume is close to one. As another corollary we obtain the optimal constant in the quantitative Gaussian isoperimetric inequality.

Mathematics Subject Classification 49Q20 · 60E15

Contents

✉ Marco Barchiesi
barchies@gmail.com

Vesa Julin
vesa.julin@jyu.fi

[1] Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, Monte Sant'Angelo, 80126 Naples, Italy

[2] Department of Mathematics and Statistics, University of Jyväskylä, P.O.Box 35 (MaD), 40014 Jyväskylä, Finland

1 Introduction

The Gaussian isoperimetric inequality (proved by Borell [7] and Sudakov–Tsirelson [33]) states that among all sets with given Gaussian measure the half-space has the smallest Gaussian perimeter. Since the half-space is not symmetric with respect to the origin, a natural question is to restrict the problem among sets which are symmetric, i.e., either central symmetric ($E = -E$) or coordinate wise symmetric (n-symmetric). This problem turns out to be rather difficult as every known method that has been used to prove the Gaussian isoperimetric inequality, such as symmetrization [15] and the Ornstein-Uhlenbeck semigroup argument [1], seems to fail.

The Gaussian isoperimetric problem for symmetric sets or its generalization to Gaussian noise is stated as an open problem in [8,19]. A natural candidate for the solution is the symmetric strip or its complement as this is the only reasonable one-dimensional candidate. In [4] Barthe proves that if one replaces the standard Gaussian perimeter by a certain anisotropic perimeter, the solution of the isoperimetric problem among n-symmetric sets is the symmetric strip or its complement. A somewhat similar result is by Latala and Oleszkiewicz [27, Theorem 3] who proved that the symmetric strip minimizes the Gaussian perimeter weighted with the width of the set among convex and symmetric sets with volume constraint. This result is related to the so-called S-conjecture, also proved in [27] (while the complex version was proved by Tkocz [34]). For the standard perimeter the problem is more difficult as a simple energy comparison shows (see [23]) that when the volume is exactly one half, the n-dimensional ball in $\mathbb{R}^n$ has smaller Gaussian perimeter than the symmetric strip. Similar difficulty appears also in the isoperimetric problem on the sphere for symmetric sets, where it is known that the union of two spherical caps does not always have the smallest surface area (see [4]). On the other hand, when the volume is close to one the symmetric strip has smaller perimeter than the n-dimensional ball. This suggests that the shape of the minimizer of the symmetric problem depends on the volume. Indeed, the conjecture states (see [23, Conjecture 1.3]) that the minimizer of the problem is always a cylinder $B^k_r \times \mathbb{R}^{n-k}$, or its complement, for some k depending on the volume and on the dimension. Here B^k_r denotes the k-dimensional ball with radius r. In particular, when the volume is one half the conjecture states that the minimizer is the n-dimensional ball B^n_r and when the volume is close to one the minimizer is the symmetric strip $(-r, r) \times \mathbb{R}^{n-1}$. There is some numerical evidence to support this fact and the results by Heilman [22,23] and La Manna [26] seem to indicate this. Note that if this conjecture is true, then the solution of the problem depends on the dimension of the ambient space.

To the best of the authors knowledge there are no other results directly related to this problem. In [14] Colding and Minicozzi introduce the Gaussian entropy, which is defined for sets as

$$\Lambda(\partial E) := \sup_{x_0 \in \mathbb{R}^n, \lambda > 0} P_\gamma(\lambda E - \{x_0\}),$$

where P_γ is the Gaussian perimeter defined below in (2). The Gaussian entropy is important since it is decreasing under the mean curvature flow and for this reason

in [14] the authors studied sets which are stable for the Gaussian entropy. It was conjectured in [13] that the sphere minimizes the entropy among closed hypersurfaces (at least in low dimensions). This was proved by Bernstein and Wang [5] in low dimensions and more recently by Zhu [35] in every dimension. It is natural to guess that minimizing the Gaussian entropy is related to the Gaussian isoperimetric problem for symmetric sets when the volume is one half, as this gives the largest value for the symmetric problem as a function of volume. For instance, by the argument in [14] it follows that for every symmetric set E which is C^2-close to the n-dimensional ball B_R^n with volume $\gamma(B_R^n) = 1/2$ it holds

$$\Lambda(\partial E) = P_\gamma(\lambda E) \tag{1}$$

for some $\lambda \in (0, 1)$. Therefore it sounds plausible that the ball minimizes both the Gaussian entropy among compact sets and the Gaussian perimeter among symmetric sets with volume one half. However, the latter does not follow directly from the result by Zhu, since e.g. the Gaussian entropy for sets in (1) is always larger than their Gaussian perimeter.

In this paper we partially prove the previously mentioned conjecture by showing that the symmetric strip is indeed the solution of the Gaussian isoperimetric problem for symmetric sets when the volume is close to one. Similarly, its complement is the solution when the volume is close to zero. We have an explicit estimate on how close to one the volume has to be. Most importantly this bound is independent of the dimension.

In order to describe the main result more precisely, we introduce our setting. Given a Borel set $E \subset \mathbb{R}^n$, $\gamma(E)$ denotes its Gaussian measure, defined as

$$\gamma(E) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}} dx.$$

If E is an open set with Lipschitz boundary, $P_\gamma(E)$ denotes its *Gaussian perimeter*, defined as

$$P_\gamma(E) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x), \tag{2}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. We define the (non-renormalized) barycenter of a set E as

$$b(E) := \int_E x \, d\gamma(x)$$

and define the function $\phi : \mathbb{R} \to (0, 1)$ as

$$\phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt.$$

Moreover, given $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$, $H_{\omega,s}$ denotes the half-space of the form

$$H_{\omega,s} := \{x \in \mathbb{R}^n : \langle x, \omega \rangle < s\},$$

while $D_{\omega,s}$ denotes the symmetric strip

$$D_{\omega,s} := \{x \in \mathbb{R}^n : |\langle x, \omega \rangle| < a(s)\},$$

where $a(s) > 0$ is chosen such that $\gamma(H_{\omega,s}) = \gamma(D_{\omega,s})$.

We approach the problem by studying the minimizers of the functional

$$\mathcal{F}(E) := P_\gamma(E) + \varrho \sqrt{\pi/2}\, |b(E)|^2 \tag{3}$$

under the volume constraint $\gamma(E) = \phi(s)$. Note that the isoperimetric inequality implies that for $\varrho = 0$ the half-space is the only minimizer of (3), while it is easy to see that the quantity $|b(E)|$ is maximized by the half-space. Therefore the two terms in (3) are in competition and we call the barycenter term repulsive, as it prefers to balance the volume around the origin. It is proven in [2,16] that when ϱ is small, the half-space is still the only minimizer of (3). This result implies the quantitative Gaussian isoperimetric inequality (see also [3,11,30,31]). It is clear that when we keep increasing the value ϱ, there is a threshold, say ϱ_s, such that for $\varrho > \varrho_s$ the half-space $H_{\omega,s}$ is no longer the minimizer of (3). In this paper we are interested in characterizing the minimizers of (3) after this threshold. Our main result reads as follows.

Main Theorem *Let $s \geq 10^3$. There is a threshold ϱ_s such that for $\varrho \in [0, \varrho_s)$ the minimizer of (3) under the volume constraint $\gamma(E) = \phi(s)$ is the half-space $H_{\omega,s}$, while for $\varrho \in (\varrho_s, \infty)$ the minimizer is the symmetric strip $D_{\omega,s}$.*

The result is sharp in the sense that the corresponding statement for s close to zero is false by the earlier discussion. On the other hand, the bound $s \geq 10^3$ is most likely far from optimal.

As a corollary the above theorem provides the solution for the symmetric Gaussian problem, because symmetric sets have barycenter zero.

Corollary 1 *Let $s \geq 10^3$. For any symmetric set E with volume $\gamma(E) = \phi(s)$ it holds*

$$P_\gamma(E) \geq P_\gamma(D_{\omega,s}) = \Big(1 + \frac{\ln 2}{s^2} + o(1/s^2)\Big) e^{-\frac{s^2}{2}},$$

and the equality holds if and only if $E = D_{\omega,s}$ for some $\omega \in \mathbb{S}^{n-1}$.

We remark that the bound $s \geq 10^3$ is far from the conjectured value which is approximately $s \geq 0,5$ (see [23]). We could slightly improve the bound on s, but as our proof is rather long, we prefer to avoid heavy computations and to state the theorem without trying to optimize this bound.

Another corollary of the theorem is the optimal constant in the quantitative Gaussian isoperimetric inequality (see [2,16]) when the volume is close to one. Let us denote by $\beta(E)$ the strong asymmetry

$$\beta(E) := \min_{\omega \in \mathbb{S}^{n-1}} \left| b(E) - b(H_{\omega,s}) \right|,$$

which measures the distance between a set E and the family of half-spaces.

Corollary 2 *Let $s \geq 10^3$. For every set E with volume $\gamma(E) = \phi(s)$ it holds*

$$P_\gamma(E) - P_\gamma(H_{\omega,s}) \geq c_s \beta(E).$$

The optimal constant is given by

$$c_s = \sqrt{2\pi}\, e^{s^2/2} \left(P_\gamma(D_{\omega,s}) - P_\gamma(H_{\omega,s}) \right) = \sqrt{2\pi}\, \frac{\ln 2}{s^2} + o(1/s^2).$$

It would be interesting to obtain a result analogous to Corollary 2 in the Euclidean setting, where the minimization problem which corresponds to (3) is introduced in [18], and on the sphere [6]. The motivation for this is that, by the result of the second author [24], the optimal constant for the quantitative Euclidean isoperimetric inequality implies an estimate on the range of volume where the ball is the minimizer of the Gamov's liquid drop model [20]. This is a classical model used in nuclear physics and has gathered a lot of attention in mathematics in recent years [9,10,25]. We also refer to the survey paper [17] for the state-of-the-art in the quantitative isoperimetric and other functional inequalities.

The main idea of the proof is to study the functional (3) when the parameter ϱ is within a carefully chosen range (ϱ_l, ϱ_r), depending on s, and to prove that within this range the only local minimizers, are the half-space $H_{\omega,s}$ and the symmetric strip $D_{\omega,s}$. We have to choose the lower bound ϱ_l large enough so that the symmetric strip is a local minimum of (3). On the other hand, we have to choose the upper bound ϱ_r small enough so that no other local minimum than $H_{\omega,s}$ and $D_{\omega,s}$ exist. Naturally also the threshold value ϱ_s has to be within the range (ϱ_l, ϱ_r).

Our proof is based on reduction argument where we reduce the dimension of the problem from $\mathbb{R}^n$ to $\mathbb{R}$ when s is large enough. First, in Theorem 2 in Sect. 3 we develop further our ideas from [2] to reduce the problem from $\mathbb{R}^n$ to $\mathbb{R}^2$ by a rather short argument. In this step it is crucial that we are not constrained to keep the sets symmetric. In other words, minimizing (3) is more flexible than minimizing the Gaussian perimeter among symmetric sets, which makes it easier to reduce the dimension in the former problem than in the latter. The main challenge is thus to prove the theorem in $\mathbb{R}^2$ which we do in Theorem 3 in Sect. 4. We cannot apply the previous reduction argument anymore which makes the proof of Theorem 3 very involved. In some sense in this step we have to pay the price that we are minimizing the functional (3) which is much more difficult than solving the Gaussian symmetric problem in $\mathbb{R}^2$. Indeed, proving the main theorem in $\mathbb{R}^2$ is essentially the same as proving the quantitative isoperimetric problem with the sharp constant in $\mathbb{R}^2$ which, for instance, is not known in the Euclidean setting

(see [12] for the best known result). We use an ad-hoc argument to reduce the problem from $\mathbb{R}^2$ to $\mathbb{R}$ essentially by deriving PDE type estimates from the Euler equation and from the stability condition. We give an independent overview of this argument at the beginning of the proof of Theorem 3 in Sect. 4. Finally, in Sect. 5 we solve the problem in $\mathbb{R}$ by a straightforward (but nontrivial) argument.

2 Notation and set-up

In this section we briefly introduce our notation and discuss about preliminary results. *We remark that throughout the paper the parameter s, associated with the volume, is assumed to be larger than 10^3 even if not explicitly mentioned. In particular, our estimates are understood to hold when $s \geq 10^3$.*

We denote the $(n-1)$-dimensional Hausdorff measure with Gaussian weight by $\mathcal{H}^{n-1}_\gamma$, i.e., for every Borel set A we define

$$\mathcal{H}^{n-1}_\gamma(A) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_A e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}.$$

We minimize the functional (3) among sets with locally finite perimeter and have the existence of a minimizer for every ϱ by an argument similar to [2, Proposition 1]. If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter we denote its reduced boundary by $\partial^* E$ and define its Gaussian perimeter by

$$P_\gamma(E) := \mathcal{H}^{n-1}_\gamma(\partial^* E).$$

We denote the generalized exterior normal by ν^E which is defined on $\partial^* E$. As introduction to the theory of sets of finite perimeter and perimeter minimizers we refer to [29].

If the reduced boundary $\partial^* E$ is a smooth hypersurface we denote the second fundamental form by B_E and the mean curvature by $\mathscr{H}_E$, which for us is the sum of the principle curvatures. We adopt the notation from [21] and define the tangential gradient of a function f, defined in a neighborhood of $\partial^* E$, by $\nabla_\tau f := \nabla f - \langle \nabla f, \nu^E \rangle \nu^E$. Similarly, we define the tangential divergence of a vector field by $\mathrm{div}_\tau X := \mathrm{div} X - \langle DX \nu^E, \nu^E \rangle$ and the Laplace-Beltrami operator as $\Delta_\tau f := \mathrm{div}_\tau(\nabla_\tau f)$. The divergence theorem on $\partial^* E$ implies that for every vector field $X \in C^1_0(\partial^* E; \mathbb{R}^n)$ it holds

$$\int_{\partial^* E} \mathrm{div}_\tau X \, d\mathcal{H}^{n-1} = \int_{\partial^* E} \mathscr{H}_E \langle X, \nu^E \rangle \, d\mathcal{H}^{n-1}.$$

If $\partial^* E$ is a smooth hypersurface, we may extend any function $f \in C^1_0(\partial^* E)$ to a neighborhood of $\partial^* E$ by the distance function. For simplicity we will omit to indicate the dependence on the set E when this is clear, by simply writing $\nu = \nu^E$, $\mathscr{H} = \mathscr{H}_E$ etc...

We denote the mean value of a function $f : \partial^* E \to \mathbb{R}$ by

$$\bar{f} := \fint_{\partial^* E} f \, \mathcal{H}^{n-1}_\gamma,$$

and its average over a subset $\Sigma \subset \partial^* E$ by

$$(f)_\Sigma := \fint_{\Sigma} f \, \mathcal{H}^{n-1}_\gamma.$$

We recall that for every number $a \in \mathbb{R}$ it holds

$$\int_\Sigma (f - (f)_\Sigma)^2 \, d\mathcal{H}^1_\gamma \le \int_\Sigma (f - a)^2 \, d\mathcal{H}^1_\gamma.$$

Recall that $H_{\omega,s}$ denotes the half-space $\{x \in \mathbb{R}^n : \langle x, \omega \rangle < s\}$ and $D_{\omega,s}$ denotes the symmetric strip $\{x \in \mathbb{R}^n : |\langle x, \omega \rangle| < a(s)\}$, where $a(s)$ is chosen such that $\gamma(D_{\omega,s}) = \gamma(H_{\omega,s}) = \phi(s)$. For future purpose it is important to estimate the asymptotic behavior of the quantities $a(s)$ and $P(D_{\omega,s})$. We claim that

$$s < a(s) < s + \frac{\ln 2}{s} \tag{4}$$

and

$$\left(1 + \frac{\ln 2}{s^2} - \frac{8}{s^4}\right) e^{-\frac{s^2}{2}} < P_\gamma(D_{\omega,s}) \le \left(1 + \frac{\ln 2}{s^2}\right) e^{-\frac{s^2}{2}}. \tag{5}$$

These bounds are probably well known but we sketch the proof in the Appendix for the reader's convenience. In particular, it follows from (5) and from our main theorem that the threshold value ϱ_s has the asymptotic behavior

$$\varrho_s = 2 \ln 2 \, \frac{\sqrt{2\pi}}{s^2} e^{\frac{s^2}{2}} \, (1 + o(1)) \, .$$

This follows from the fact that ϱ_s is the unique value of ϱ for which the functional (3) satisfies $\mathcal{F}(H_{\omega,s}) = \mathcal{F}(D_{\omega,s})$, i.e.,

$$P_\gamma(D_{\omega,s}) = e^{-\frac{s^2}{2}} + \frac{\varrho_s}{2\sqrt{2\pi}} e^{-s^2} \tag{6}$$

by taking into account that $|b(H_{\omega,s})| = e^{-s^2/2}/\sqrt{2\pi}$.

In order to simplify the upcoming technicalities we replace the volume constraint in the original functional (3) with a volume penalization. We redefine $\mathcal{F}$ for any set of locally finite perimeter as

$$\mathcal{F}(E) := P_\gamma(E) + \varrho\sqrt{\pi/2} \, |b(E)|^2 + \Lambda\sqrt{2\pi} \, \big|\gamma(E) - \phi(s)\big|, \tag{7}$$

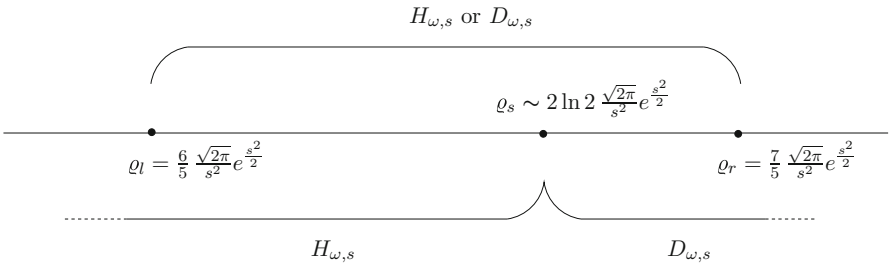

Fig. 1 The thresholds

where we choose

$$\Lambda = s + 1. \tag{8}$$

As with the original functional the existence of a minimizer of (7) follows from [2, Proposition 1]. It turns out that the minimizers of (7) are the same as the minimizers of (3) under the volume constraint $\gamma(E) = \phi(s)$, as proved in the last section. The advantage of having a volume penalization is that it helps us to bound the Lagrange multiplier in a simple way. The constants $\sqrt{\pi/2}$ and $\sqrt{2\pi}$ in front of the last two terms are chosen to simplify the formulas of the Euler equation and the second variation.

As we explained in the introduction, the idea is to restrict the parameter ϱ in (7) within a range, which contains the threshold value ϱ_s defined by (6) and such that the only local minimizers of (7) are the half-space and the symmetric strip. To this aim we assume from now on that ϱ is in the range

$$\varrho_l := \frac{6}{5}\,\frac{\sqrt{2\pi}}{s^2}e^{\frac{s^2}{2}} \le \varrho \le \frac{7}{5}\,\frac{\sqrt{2\pi}}{s^2}e^{\frac{s^2}{2}} =: \varrho_r. \tag{9}$$

Note that by (5) the threshold value ϱ_s defined by (6) is within this interval. If we are able to show that when ϱ satisfies (9) the only local minimizers of (7) are $H_{\omega,s}$ and $D_{\omega,s}$, we obtain the main result. Indeed, when $\varrho \in (\varrho_l, \varrho_s)$ it holds $\mathcal{F}(H_{\omega,s}) < \mathcal{F}(D_{\omega,s})$ by (6) and the minimizer is $H_{\omega,s}$. It is then not difficult to see that for every value ϱ smaller than ϱ_l, the minimizer is still $H_{\omega,s}$. Indeed, the half-space has the barycenter with the largest norm and it is favored by a smaller ϱ. Similarly, when $\varrho \in (\varrho_s, \varrho_r)$ in (9) it holds $\mathcal{F}(D_{\omega,s}) < \mathcal{F}(H_{\omega,s})$ again by (6) and the minimizer is $D_{\omega,s}$. Hence, for every value ϱ larger than ϱ_r, $D_{\omega,s}$ is still the minimizer of (7), since it has barycenter zero. In Fig. 1 we have sketched the situation.

We have also the following a priori perimeter bounds for minimizer E of (7) (see the Appendix for the proof),

$$\frac{5}{6}e^{-\frac{s^2}{2}} \le P_\gamma(E) \le \left(1 + \frac{\ln 2}{s^2}\right)e^{-\frac{s^2}{2}}. \tag{10}$$

For the reader's convenience we summarize the results concerning the regularity of minimizers and the first and the second variation of (7) contained in [2, Section 4] in the following theorem.

Theorem 1 *Let E be a minimizer of (7). Then the reduced boundary $\partial^* E$ is a relatively open, smooth hypersurface and satisfies the Euler equation*

$$\mathscr{H} - \langle x, \nu\rangle + \varrho\langle b, x\rangle = \lambda \quad \text{on } \partial^* E. \tag{11}$$

The Lagrange multiplier λ can be estimated by $|\lambda| \leq \Lambda$. The singular part of the boundary $\partial E \setminus \partial^* E$ is empty when $n < 8$, while for $n \geq 8$ its Hausdorff dimension can be estimated by $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \leq n-8$. Moreover, the quadratic form associated with the second variation is non-negative

$$\begin{aligned}\mathcal{F}[\varphi] := &\int_{\partial^* E} \left(|\nabla_\tau \varphi|^2 - |B_E|^2\varphi^2 + \varrho\langle b, \nu\rangle\varphi^2 - \varphi^2\right) d\mathcal{H}^{n-1}_\gamma \\ &+ \frac{\varrho}{\sqrt{2\pi}} \left| \int_{\partial^* E} x\, \varphi\, d\mathcal{H}^{n-1}_\gamma \right|^2 \geq 0\end{aligned} \tag{12}$$

for every $\varphi \in C^\infty_0(\partial^* E)$ which satisfies $\int_{\partial^* E} \varphi\, d\mathcal{H}^{n-1}_\gamma = 0$.

The Euler equation (11) yields important geometric equations for the position vector x and for the Gauss map ν. For arbitrary $\omega \in \mathbb{S}^{n-1}$ we write

$$x_\omega = \langle x, \omega\rangle \qquad \text{and} \qquad \nu_\omega = \langle \nu, \omega\rangle.$$

If $\{e^{(1)}, \dots, e^{(n)}\}$ is a canonical basis of $\mathbb{R}^n$ we simply write

$$x_i = \langle x, e_i\rangle \qquad \text{and} \qquad \nu_i = \langle \nu, e_i\rangle.$$

From (11) and from the fact $\Delta_\tau x_\omega = -\mathscr{H}\nu_\omega$ [28, Proposition 1] we have

$$\Delta_\tau x_\omega - \langle \nabla_\tau x_\omega, x\rangle = -x_\omega - \lambda\nu_\omega + \varrho\langle b, x\rangle\nu_\omega. \tag{13}$$

Moreover, from (11) and from the fact $\Delta_\tau \nu_\omega = -|B_E|^2\nu_\omega + \langle \nabla_\tau \mathscr{H}, \omega\rangle$ [21, Lemma 10.7] we get

$$\Delta_\tau \nu_\omega - \langle \nabla_\tau \nu_\omega, x\rangle = -|B_E|^2\nu_\omega + \varrho\langle b, \nu\rangle\nu_\omega - \varrho\langle b, \omega\rangle. \tag{14}$$

By the divergence theorem on $\partial^* E$ we have that for any functions $\varphi \in C^\infty_0(\partial^* E)$ and $\psi \in C^1(\partial^* E)$ it holds

$$\int_{\partial^* E} \operatorname{div}_\tau\left(e^{-\frac{|x|^2}{2}}\psi\nabla_\tau\varphi\right) d\mathcal{H}^{n-1} = \int_{\partial^* E} \mathscr{H}\langle e^{-\frac{|x|^2}{2}}\psi\nabla_\tau\varphi, \nu^E\rangle\, d\mathcal{H}^{n-1} = 0.$$

The previous equality implies the following integration by parts formula

$$\int_{\partial^* E} \psi(\Delta_\tau \varphi - \langle \nabla_\tau \varphi, x\rangle)\, d\mathcal{H}^{n-1}_\gamma = -\int_{\partial^* E} \langle \nabla_\tau \psi, \nabla_\tau \varphi\rangle\, d\mathcal{H}^{n-1}_\gamma.$$

We will use along the paper the above formula with $\varphi = x_\omega$ or $\varphi = \nu_\omega$. Also if they do not belong to $C^\infty_0(\partial^* E)$, we are allowed to do so by an approximation argument (see [2,32]).

Remark 1 We associate the following second order operator L with the first four terms in the quadratic form (12),

$$L[\varphi] := -\Delta_\tau \varphi + \langle \nabla_\tau \varphi, x\rangle - |B_E|^2 \varphi + \varrho \langle b, \nu\rangle \varphi - \varphi, \tag{15}$$

where $\varphi \in C^\infty_0(\partial^* E)$. By integration by parts the inequality (12) can be written as

$$\int_{\partial^* E} L[\varphi]\varphi\, d\mathcal{H}^{n-1}_\gamma + \frac{\varrho}{\sqrt{2\pi}} \left| \int_{\partial^* E} x\, \varphi\, d\mathcal{H}^{n-1}_\gamma \right|^2 \geq 0.$$

Note that when the vector ω is orthogonal to the barycenter, i.e., $\langle \omega, b\rangle = 0$, then by (14) the function ν_ω is an eigenfunction of L and satisfies

$$L[\nu_\omega] = -\nu_\omega.$$

For every $\omega \in \mathbb{S}^{n-1}$ it holds by the divergence theorem in $\mathbb{R}^n$ that

$$\begin{aligned}\int_{\partial^* E} \nu_\omega\, d\mathcal{H}^{n-1}_\gamma(x) &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_E \operatorname{div}(\omega e^{-\frac{|x|^2}{2}})\, dx \\ &= -\sqrt{2\pi} \int_E \langle x, \omega\rangle\, d\gamma(x) \\ &= -\sqrt{2\pi}\, \langle b, \omega\rangle.\end{aligned}$$

In particular, when $\langle \omega, b\rangle = 0$ the function $\varphi = \nu_\omega$ has zero average. Therefore by Remark 1 it is natural to use ν_ω with $\langle \omega, b\rangle = 0$ as a test function in the second variation condition (12).

The equality $\int_{\partial^* E} \nu_\omega\, d\mathcal{H}^{n-1}_\gamma = -\sqrt{2\pi}\, \langle b, \omega\rangle$ for every $\omega \in \mathbb{S}^{n-1}$ also implies

$$\bar{\nu} P_\gamma(E) = -\sqrt{2\pi}\, b. \tag{16}$$

In particular, we have by (9), (10)

$$\frac{1}{s^2}|\bar{\nu}| \leq \varrho |b| \leq \frac{3}{2s^2}|\bar{\nu}|. \tag{17}$$

We conclude this preliminary section by providing further "regularity" estimates from (13) for the minimizers of (7). We call the estimates in the following lemma

"Caccioppoli inequalities" since they follow from (13) by an argument which is similar to the classical proof of Caccioppoli inequality known in elliptic PDEs. This result is an improved version of [2, Proposition 1].

Lemma 1 (Caccioppoli inequalities) *Let $E \subset \mathbb{R}^n$ be a minimizer of (7). Then for any $\omega \in \mathbb{S}^{n-1}$ it holds*

$$\int_{\partial^* E} x_\omega^2 \, d\mathcal{H}_\gamma^{n-1} \leq (s+1)^2 \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}_\gamma^{n-1} + 8 P_\gamma(E) \tag{18}$$

and

$$\int_{\partial^* E} (x_\omega - \bar{x}_\omega)^2 \, d\mathcal{H}_\gamma^{n-1} \leq (s+1)^2 \int_{\partial^* E} (\nu_\omega - \bar{\nu}_\omega)^2 \, d\mathcal{H}_\gamma^{n-1} + 8 P_\gamma(E). \tag{19}$$

Proof Let us first prove (18). To simplify the notation we define

$$x_b := \begin{cases} \left\langle x, \frac{b}{|b|} \right\rangle & \text{if } b \neq 0, \\ 0 & \text{if } b = 0. \end{cases}$$

We multiply (13) by x_ω and integrate by parts over $\partial^* E$ to get

$$\begin{aligned} \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}_\gamma^{n-1} = -\lambda \int_{\partial^* E} \nu_\omega x_\omega \, d\mathcal{H}_\gamma^{n-1} + \int_{\partial^* E} |\nabla_\tau x_\omega|^2 \, d\mathcal{H}_\gamma^{n-1} \\ + \varrho |b| \int_{\partial^* E} x_b \nu_\omega x_\omega \, d\mathcal{H}_\gamma^{n-1}. \end{aligned} \tag{20}$$

We estimate the right-hand-side of (20) in the following way. We estimate the first term by Young's inequality

$$\begin{aligned} -\lambda \int_{\partial^* E} \nu_\omega x_\omega \, d\mathcal{H}_\gamma^{n-1} &\leq \frac{1}{2} \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}_\gamma^{n-1} + \frac{\lambda^2}{2} \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}_\gamma^{n-1} \\ &\leq \frac{1}{2} \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}_\gamma^{n-1} + \frac{(s+1)^2}{2} \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}_\gamma^{n-1}, \end{aligned}$$

where the last inequality follows from the bound on the Lagrange multiplier

$$|\lambda| \leq s + 1$$

given by Theorem 1 and by our choice of Λ in (8). Since $|\nabla_\tau x_\omega|^2 = 1 - \nu_\omega^2 \leq 1$, we may bound the second term simply by

$$\int_{\partial^* E} |\nabla_\tau x_\omega|^2 \, d\mathcal{H}_\gamma^{n-1} \leq P_\gamma(E).$$

Finally we bound the last term again by Young's inequality and by $\varrho|b| \le \frac{3}{2s^2}$ (given in (17))

$$\varrho|b| \int_{\partial^* E} x_b \nu_\omega x_\omega \, d\mathcal{H}^{n-1}_\gamma \le \frac{1}{s^2} \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + \frac{1}{s^2} \int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma.$$

By using these three estimates in (20) we obtain

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{s^2}\right) \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \le \frac{(s+1)^2}{2} \int_{\partial^* E} v_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + P_\gamma(E) \\ + \frac{1}{s^2} \int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma. \end{aligned} \tag{21}$$

If the barycenter is zero the claim follows immediately from (21). If $b \neq 0$, we first use (21) with $\omega = \frac{b}{|b|}$ and obtain

$$\begin{aligned}\left(\frac{1}{2} - \frac{2}{s^2}\right) \int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma &\le \frac{(s+1)^2}{2} \int_{\partial^* E} v_b^2 \, d\mathcal{H}^{n-1}_\gamma + P_\gamma(E) \\ &\le \left(\frac{(s+1)^2}{2} + 1\right) P_\gamma(E).\end{aligned}$$

This implies

$$\int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma \le \frac{3}{2} s^2 P_\gamma(E). \tag{22}$$

Therefore we have by (21)

$$\int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \le \left(2 + \frac{4}{s^2 - 2}\right) \left(\frac{(s+1)^2}{2} \int_{\partial^* E} v_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + \frac{5}{2} P_\gamma(E)\right),$$

which yields the claim.

The proof of the second inequality is similar. We multiply the equation (13) by $(x_\omega - \bar{x}_\omega)$ and integrate by parts over $\partial^* E$ to get

$$\begin{aligned}\int_{\partial^* E} (x_\omega - \bar{x}_\omega)^2 \, d\mathcal{H}^{n-1}_\gamma &= -\lambda \int_{\partial^* E} (x_\omega - \bar{x}_\omega)(v_\omega - \bar{v}_\omega) \, d\mathcal{H}^{n-1}_\gamma \\ &+ \int_{\partial^* E} |\nabla_\tau x_\omega|^2 \, d\mathcal{H}^{n-1}_\gamma \\ &+ \varrho|b| \int_{\partial^* E} x_b \nu_\omega (x_\omega - \bar{x}_\omega) \, d\mathcal{H}^{n-1}_\gamma.\end{aligned}$$

By estimating the three terms on the right-hand-side precisely as before, we deduce

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{s^2}\right)\int_{\partial^* E}(x_\omega-\bar{x}_\omega)^2\,d\mathcal{H}_\gamma^{n-1} &\le \frac{(s+1)^2}{2}\int_{\partial^* E}(v_\omega-\bar{v}_\omega)^2\,d\mathcal{H}_\gamma^{n-1}\\ &\quad + P_\gamma(E)+\frac{1}{s^2}\int_{\partial^* E}x_b^2\,d\mathcal{H}_\gamma^{n-1}\\ &\le \frac{(s+1)^2}{2}\int_{\partial^* E}(v_\omega-\bar{v}_\omega)^2\,d\mathcal{H}_\gamma^{n-1}+\frac{5}{2}P_\gamma(E),\end{aligned}$$

where the last inequality follows from (22). This implies (19). □

3 Reduction to the two dimensional case

In this section we prove that it is enough to obtain the result in the two dimensional case. More precisely, we prove the following result.

Theorem 2 *Let E be a minimizer of (7). Then, up to a rotation, $E = F\times\mathbb{R}^{n-2}$ for some set $F\subset\mathbb{R}^2$.*

Proof Let $\{e^{(1)},\dots,e^{(n)}\}$ be an orthonormal basis of $\mathbb{R}^n$. We begin with a simple observation: if $i\neq j$ then by the divergence theorem

$$\int_{\partial^* E}x_i v_j\,d\mathcal{H}_\gamma^{n-1}=-\sqrt{2\pi}\int_E x_i x_j\,d\gamma.$$

In particular, the matrix $A_{ij}=\int_{\partial E}x_i v_j\,d\mathcal{H}_\gamma^{n-1}$ is symmetric. We may therefore assume that A_{ij} is diagonal by changing the basis of $\mathbb{R}^n$ if necessary. In particular, it holds

$$\int_{\partial^* E}x_i v_j\,d\mathcal{H}_\gamma^{n-1}=0\qquad\text{for } i\neq j. \tag{23}$$

By reordering the elements of the basis we may also assume that

$$\int_{\partial^* E}x_j^2\,d\mathcal{H}_\gamma^{n-1}\ge\int_{\partial^* E}x_{j+1}^2\,d\mathcal{H}_\gamma^{n-1} \tag{24}$$

for $j\in\{1,\dots,n-1\}$.

Since we assume $n\ge 3$, we may choose a direction $\omega\in\mathbb{S}^{n-1}$ which is orthogonal both to the barycenter b and to $e^{(1)}$. To be more precise, we choose ω such that $\langle\omega,b\rangle=0$ and $\omega\in\operatorname{span}\{e^{(2)},e^{(3)}\}$. Since $\langle\omega,b\rangle=0$, (16) yields $\bar{v}_\omega=0$. In other words, the function v_ω has zero average. We use $\varphi=v_\omega$ as a test function in the second variation condition (12). According to Remark 1 we may write the inequality (12) as

$$\int_{\partial^* E}L[v_\omega]v_\omega\,d\mathcal{H}_\gamma^{n-1}+\frac{\varrho}{\sqrt{2\pi}}\left|\int_{\partial^* E}x\,v_\omega\,d\mathcal{H}_\gamma^{n-1}\right|^2\ge 0,$$

where the operator L is defined in (15). Since ω is orthogonal to b we deduce by Remark 1 that v_ω is an eigenfunction of L and satisfies $L[v_\omega] = -v_\omega$. Therefore we get

$$-\int_{\partial^* E} v_\omega^2 \, d\mathcal{H}_\gamma^{n-1} + \frac{\varrho}{\sqrt{2\pi}} \left| \int_{\partial^* E} x \, v_\omega \, d\mathcal{H}_\gamma^{n-1} \right|^2 \geq 0. \tag{25}$$

The crucial step in the proof is to estimate the second term in (25), by showing that it is small enough. This is possible due to the fact that ω is orthogonal to $e^{(1)}$. Indeed, by using (23) and the fact that $\omega \in \mathrm{span}\{e^{(2)}, e^{(3)}\}$, and then Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left| \int_{\partial^* E} x \, v_\omega \, d\mathcal{H}_\gamma^{n-1} \right|^2 &= \left(\int_{\partial^* E} x_2 v_\omega \, d\mathcal{H}_\gamma^{n-1} \right)^2 + \left(\int_{\partial^* E} x_3 v_\omega \, d\mathcal{H}_\gamma^{n-1} \right)^2 \\ &\leq \left(\int_{\partial^* E} x_2^2 + x_3^2 \, d\mathcal{H}_\gamma^{n-1} \right) \left(\int_{\partial^* E} v_\omega^2 \, d\mathcal{H}_\gamma^{n-1} \right). \end{aligned}$$

We estimate the first term on the right-hand-side first by (24), then by the Caccioppoli estimate (18) and finally by (10)

$$\begin{aligned} \int_{\partial^* E} (x_2^2 + x_3^2) \, \mathcal{H}_\gamma^{n-1} &\leq \frac{2}{3} \int_{\partial^* E} (x_1^2 + x_2^2 + x_3^2) \, \mathcal{H}_\gamma^{n-1} \\ &\leq \frac{2}{3} \left[(s+1)^2 \int_{\partial^* E} (v_1^2 + v_2^2 + v_3^2) \, \mathcal{H}_\gamma^{n-1} + 24 P_\gamma(E) \right] \\ &\leq \frac{2}{3} \left[(s+1)^2 + 24 \right] P_\gamma(E) < \frac{5}{7} s^2 e^{-\frac{s^2}{2}}. \end{aligned} \tag{26}$$

Since we assume $\varrho \leq \frac{7\sqrt{2\pi}}{5s^2} e^{\frac{s^2}{2}}$ (see (9)), the previous two inequalities yield

$$\frac{\varrho}{\sqrt{2\pi}} \left| \int_{\partial^* E} x \, v_\omega \, d\mathcal{H}_\gamma^{n-1} \right|^2 \leq \mu \int_{\partial^* E} v_\omega^2 \, d\mathcal{H}_\gamma^{n-1} \tag{27}$$

for some $\mu < 1$. Then, by collecting (25) and (27) we obtain

$$-\int_{\partial^* E} v_\omega^2 \, d\mathcal{H}_\gamma^{n-1} \geq 0.$$

This implies $v_\omega = 0$. We have thus reduced the problem from n to $n-1$. By repeating the previous argument we reduce the problem to the planar case. □

Remark 2 We have to be careful in our choice of direction ω, and in general we may not simply choose any direction orthogonal to the barycenter b. Indeed, let $\omega \in \mathbb{S}^{n-1}$ be a vector such that $\langle b, \omega \rangle = 0$ and let $v \in \mathbb{S}^{n-1}$ be such that

$$\left| \int_{\partial^* E} x \, v_\omega \, d\mathcal{H}_\gamma^{n-1} \right| = \left\langle \int_{\partial^* E} x \, v_\omega \, d\mathcal{H}_\gamma^{n-1}, v \right\rangle = \int_{\partial^* E} \langle x, v \rangle v_\omega \, d\mathcal{H}_\gamma^{n-1}.$$

Then, by using Cauchy–Schwarz inequality, we may estimate the second term in (25) by

$$\frac{\varrho}{\sqrt{2\pi}}\left|\int_{\partial^* E} x\,\nu_\omega\,d\mathcal{H}^{n-1}_\gamma\right|^2 \le \frac{\varrho}{\sqrt{2\pi}}\left(\int_{\partial^* E} x_\upsilon^2\,d\mathcal{H}^{n-1}_\gamma\right)\left(\int_{\partial^* E}\nu_\omega^2\,d\mathcal{H}^{n-1}_\gamma\right).$$

We can estimate the term $\frac{\varrho}{\sqrt{2\pi}}\int_{\partial^* E} x_\upsilon^2\,d\mathcal{H}^{n-1}_\gamma$ at our best via (9), (10) and (18), obtaining

$$\frac{\varrho}{\sqrt{2\pi}}\left|\int_{\partial^* E} x\,\nu_\omega\,d\mathcal{H}^{n-1}_\gamma\right|^2 \le \frac{8}{5}\int_{\partial^* E}\nu_\omega^2\,d\mathcal{H}^{n-1}_\gamma.$$

Unlike (27), this estimate is not good enough. Note that we cannot shrink ϱ, since we have the constrain given by (9).

Remark 3 In the next section we will reduce the problem to the one dimensional case. In doing that, we can assume that it holds

$$⨍_{\partial E}|\nu-\bar\nu|^2\,d\mathcal{H}^{n-1}_\gamma \ge \frac{4}{7}, \tag{28}$$

and then in particular

$$|\bar\nu|^2 \le \frac{3}{7}. \tag{29}$$

Indeed, when $⨍_{\partial E}|\nu-\bar\nu|^2\,d\mathcal{H}^{n-1}_\gamma < \frac{4}{7}$, the Caccioppoli estimate (19) yields

$$⨍_{\partial E}|x-\bar x|^2\,d\mathcal{H}^{n-1}_\gamma \le \frac{4}{7}(s+1)^2+16.$$

With this estimate the further dimensional reduction can be done by simply using the argument in the proof of Theorem (2): given a vector $\omega\in\mathbb{S}^{n-1}$ such that $\langle b,\omega\rangle=0$, we have

$$\begin{aligned}\frac{\varrho}{\sqrt{2\pi}}\left|\int_{\partial E} x\,\nu_\omega\,d\mathcal{H}^1_\gamma\right|^2 &= \frac{\varrho}{\sqrt{2\pi}}\left|\int_{\partial E}(x-\bar x)\,\nu_\omega\,d\mathcal{H}^1_\gamma\right|^2\\ &\le \frac{\varrho}{\sqrt{2\pi}}\left(\int_{\partial^* E}|x-\bar x|^2\,d\mathcal{H}^{n-1}_\gamma\right)\left(\int_{\partial^* E}\nu_\omega^2\,d\mathcal{H}^{n-1}_\gamma\right)\\ &\le \mu\int_{\partial^* E}\nu_\omega^2\,d\mathcal{H}^{n-1}_\gamma\end{aligned}$$

for some $\mu<1$. In other words, the crucial estimate (27) in the proof of Theorem 2 holds and we can conclude that $\nu_\omega=0$.

Remark 4 We may reduce the problem to the one dimensional case also if $b = 0$, since we may use $\omega = e^{(2)}$ in the previous argument (ν_ω has zero average and $\int_{\partial^* E} x_2^2$ is small enough). However, this is a special case and a priori nothing guarantees that $b = 0$.

Remark 5 The result of this section holds when s is large enough and the reader may wonder what happens when the parameter s is close to zero. The conjecture in [23] states that the n-dimensional ball is the solution of the symmetric isoperimetric problem when $s = 0$. Since $\gamma(B^n_{\sqrt{n}}) \to 1/2$ as $n \to \infty$ and

$$P_\gamma(B^n_{\sqrt{n}}) = \Big(\frac{n}{2\pi}\Big)^{\frac{n-1}{2}} \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{e^{\frac{n}{2}}} \to \sqrt{2}$$

as $n \to \infty$ we should choose the value of ϱ in the functional $\mathcal{F}$ such that $\mathcal{F}(H_{\omega,0}) > \mathcal{F}(B^n_{\sqrt{n}})$, i.e.,

$$\varrho > 2\sqrt{2\pi}(\sqrt{2} - 1).$$

With this threshold we cannot apply our dimensional reduction argument for $s = 0$, which is in accordance with the conjecture. Indeed, when $s = 0$ the Caccioppoli estimate (18) becomes weaker, since the perimeter term in its right-hand side becomes dominant.

4 Reduction to the one dimensional case

In this section we will prove a further reduction of the problem, by showing that it is enough to obtain the result in the one dimensional case. This is technically more involved than Theorem 2 and requires more a priori information on the minimizers.

Theorem 3 *Let E be a minimizer of (7). Then, up to a rotation, $E = F \times \mathbb{R}^{n-1}$ for some set $F \subset \mathbb{R}$.*

Thanks to Theorem 2 we may assume from now on that $n = 2$. In particular, by Theorem 1 the boundary is regular and $\partial E = \partial^* E$. Moreover the Euler equation and (14) simply read as

$$k = \lambda + \langle x, \nu\rangle - \varrho\langle b, x\rangle, \tag{30}$$

$$\Delta_\tau \nu_\omega - \langle \nabla_\tau \nu_\omega, x\rangle = -k^2 \nu_\omega + \varrho\langle b, \nu\rangle \nu_\omega - \varrho\langle b, \omega\rangle, \tag{31}$$

where k is the curvature of ∂E.

The idea is to proceed by using the second variation argument once more, but this time in a direction that it is not necessarily orthogonal to the barycenter. This argument does not reduce the problem to $\mathbb{R}$, but gives us the following information on the minimizers.

Lemma 2 *Let $E \subset \mathbb{R}^2$ be a minimizer of (7). Then*

$$\fint_{\partial E} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{3}{2s^2}. \tag{32}$$

Moreover, there exists a direction $v \in \mathbb{S}^1$ such that

$$\fint_{\partial E} (\nu_v - \bar{\nu}_v)^2 \, d\mathcal{H}^1_\gamma \leq \frac{15}{2s^2} \, \bar{\nu}_v^2. \tag{33}$$

Observe that the estimate (33) implies that ν_v is close to a constant and thus ∂E is flat in shape. In particular, this estimate excludes the minimizers to be close to the disk.

Proof We begin by showing that for any $\omega \in \mathbb{S}^1$ it holds

$$\begin{aligned} &\bar{\nu}_\omega^2 \int_{\partial E} k^2 \, d\mathcal{H}^1_\gamma + \int_{\partial E} |\nu_\omega - \bar{\nu}_\omega|^2 \, d\mathcal{H}^1_\gamma \\ &\quad \leq \frac{3}{2s^2} \, \bar{\nu}_\omega^2 P_\gamma(E) + \frac{\varrho}{\sqrt{2\pi}} \left| \int_{\partial E} x \, (\nu_\omega - \bar{\nu}_\omega) \, d\mathcal{H}^1_\gamma \right|^2. \end{aligned} \tag{34}$$

To this aim we choose $\varphi = \nu_\omega - \bar{\nu}_\omega$ as a test function in the second variation condition (12). We remark that because ω might not be orthogonal to the barycenter b, neither ν_ω nor $\nu_\omega - \bar{\nu}_\omega$ is an eigenfunction of the operator L associated with the second variation defined in Remark 1. We multiply the equation (31) by ν_ω and integrate by parts to obtain

$$\int_{\partial E} \left(|\nabla_\tau \nu_\omega|^2 - k^2 \nu_\omega^2 + \varrho \langle b, \nu \rangle \nu_\omega^2 \right) d\mathcal{H}^1_\gamma = \varrho \langle b, \omega \rangle \bar{\nu}_\omega P_\gamma(E), \tag{35}$$

and simply integrate (14) over ∂E to get

$$\int_{\partial E} \left(k^2 \nu_\omega - \varrho \langle b, \nu \rangle \nu_\omega \right) d\mathcal{H}^1_\gamma = -\varrho \langle b, \omega \rangle P_\gamma(E). \tag{36}$$

Hence, by also using $\bar{\nu} P_\gamma(E) = -\sqrt{2\pi}\, b$ (see (16)), we may write

$$\begin{aligned} &\int_{\partial E} \left(|\nabla_\tau \nu_\omega|^2 - k^2 (\nu_\omega - \bar{\nu}_\omega)^2 + \varrho \langle b, \nu \rangle (\nu_\omega - \bar{\nu}_\omega)^2 \right) d\mathcal{H}^1_\gamma \\ &\quad = -\bar{\nu}_\omega^2 \int_{\partial E} k^2 \, d\mathcal{H}^1_\gamma + \varrho \langle b, \bar{\nu} \rangle \bar{\nu}_\omega^2 P_\gamma(E) - \varrho \langle b, \omega \rangle \, \bar{\nu}_\omega P_\gamma(E) \\ &\quad = -\bar{\nu}_\omega^2 \int_{\partial E} k^2 \, d\mathcal{H}^1_\gamma + \frac{\varrho}{\sqrt{2\pi}} (1 - |\bar{\nu}|^2) \bar{\nu}_\omega^2 P_\gamma^2(E) \\ &\quad \leq -\bar{\nu}_\omega^2 \int_{\partial E} k^2 \, d\mathcal{H}^1_\gamma + \frac{3}{2s^2} \bar{\nu}_\omega^2 P_\gamma(E), \end{aligned}$$

where in the last inequality we have used (9) and (10). The above inequality and the second variation condition (12) with $\varphi = \nu_\omega - \bar\nu_\omega$ imply (34).

Let us consider an orthonormal basis $\{e^{(1)}, e^{(2)}\}$ of $\mathbb{R}^2$ and assume $\int_{\partial E} x_1^2\, d\mathcal{H}^1_\gamma \geq \int_{\partial E} x_2^2\, d\mathcal{H}^1_\gamma$. As in (26), we use the Caccioppoli estimate (18) and (10) to get

$$\begin{aligned}\int_{\partial E} x_2^2\, d\mathcal{H}^1_\gamma &\leq \frac{1}{2}\int_{\partial E}(x_1^2+x_2^2)\,\mathcal{H}^1_\gamma \\ &\leq \frac{1}{2}\big[(s+1)^2+16\big]P_\gamma(E) \leq \frac{1}{2}\big[(s+2)^2\big]e^{-\frac{s^2}{2}}.\end{aligned} \tag{37}$$

We choose a direction $v \in \mathbb{S}^1$ which is orthogonal to the vector $\int_{\partial E} x_1(\nu - \bar\nu)\, d\mathcal{H}^1_\gamma$. Since $⨍_{\partial E} x_1(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma = \langle ⨍_{\partial E} x_1(\nu - \bar\nu)\, d\mathcal{H}^1_\gamma, v\rangle = 0$, we have

$$\Big|⨍_{\partial E} x(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma\Big| = \Big|⨍_{\partial E} x_2(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma\Big|.$$

Then, by the above equality, by Cauchy–Schwarz inequality and by (37) we have

$$\begin{aligned}\Big|\int_{\partial E} x(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma\Big|^2 &= \left(\int_{\partial E} x_2(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma\right)^2 \\ &\leq \left(\int_{\partial E} x_2^2\, d\mathcal{H}^1_\gamma\right)\left(\int_{\partial E}(\nu_v - \bar\nu_v)^2\, d\mathcal{H}^1_\gamma\right) \\ &\leq \frac{(s+1)^2}{2}e^{-\frac{s^2}{2}}\left(\int_{\partial E}(\nu_v - \bar\nu_v)^2\, d\mathcal{H}^1_\gamma\right).\end{aligned}$$

With the bound $\varrho \leq \frac{7\sqrt{2\pi}}{5s^2}e^{\frac{s^2}{2}}$ (see 9), the previous inequality yields

$$\frac{\varrho}{\sqrt{2\pi}}\Big|\int_{\partial E}(\nu_v - \bar\nu_v)\, x\, d\mathcal{H}^1_\gamma\Big|^2 \leq \frac{4}{5}\int_{\partial E}(\nu_v - \bar\nu_v)^2\, d\mathcal{H}^1_\gamma.$$

Hence, the inequality (34) implies

$$\bar\nu_v^2\int_{\partial E} k^2\, d\mathcal{H}^1_\gamma + \frac{1}{5}\int_{\partial E}(\nu_v - \bar\nu_v)^2\, d\mathcal{H}^1_\gamma \leq \frac{3}{2s^2}\,\bar\nu_v^2 P_\gamma(E).$$

From this inequality we have immediately (33), and also (32), if $\bar\nu_v$ is not zero. If instead $\bar\nu_v = 0$, then also $\nu_v = 0$ by (33). Thus ∂E is flat, $k = 0$ and (32) holds again. □

We will also need the following auxiliary result.

Lemma 3 *Let $E \subset \mathbb{R}^2$ be a minimizer of (7). Then, for every $x \in \partial E$ it holds*

$$|x| \geq s - 1. \tag{38}$$

Proof We argue by contradiction by assuming that there exists $\tilde{x} \in \partial E$ such that $|\tilde{x}| < s - 1$. For this $\tilde{x}$ we will show that

$$\mathcal{H}^1(\partial E \cap B_{1/2}(\tilde{x})) \geq \frac{1}{s}. \tag{39}$$

We remark that $\mathcal{H}^1$ is the standard Hausdorff measure, i.e., $\mathcal{H}^1(\partial E \cap B_{1/2}(\tilde{x}))$ denotes the length of the curve. We divide the proof of (39) in two cases.

Assume first that there is a component of ∂E, say $\tilde{\Gamma}$, which is contained in the disk $B_{1/2}(\tilde{x})$. By regularity, $\tilde{\Gamma}$ is a smooth Jordan curve which encloses a bounded set $\tilde{E}$, i.e., $\tilde{\Gamma} = \partial\tilde{E}$. Note that then it holds $\tilde{E} \subset B_R$ for $R = s - 1/2$. We integrate the Euler equation (30) over $\partial\tilde{E}$ with respect to the standard Hausdorff measure and obtain by the Gauss-Bonnet formula and by the divergence theorem that

$$\begin{aligned} 2\pi = \int_{\tilde{\Gamma}} k \, d\mathcal{H}^1 &= \int_{\tilde{\Gamma}} (\langle x, \nu\rangle + \lambda - \varrho\langle b, x\rangle) \, d\mathcal{H}^1 \\ &\leq 2|\tilde{E}| + \left(|\lambda| + \frac{3}{2s}\right)\mathcal{H}^1(\tilde{\Gamma}), \end{aligned} \tag{40}$$

where in the last inequality we have used $\varrho|b| \leq \frac{3}{2s^2}$ (given in (17)) and the fact that for all $x \in \tilde{E}$ it holds $|x| \leq s - 1/2$. The isoperimetric inequality in $\mathbb{R}^2$ implies

$$|\tilde{E}| \leq \frac{1}{4\pi}\mathcal{H}^1(\tilde{\Gamma})^2.$$

Therefore since $|\lambda| \leq s + 1$ we obtain from (40) that

$$2\pi \leq \frac{1}{2\pi}\mathcal{H}^1(\tilde{\Gamma})^2 + (s+2)\mathcal{H}^1(\tilde{\Gamma}).$$

This implies $\mathcal{H}^1(\tilde{\Gamma}) \geq \frac{1}{s}$ and the claim (39) follows.

Let us then assume that no component of ∂E is contained in $B_{1/2}(\tilde{x})$. In this case the boundary curve passes $\tilde{x}$ and exits the disk $B(\tilde{x}, \frac{1}{2})$. In particular, it holds $\mathcal{H}^1(\partial E \cap B_{1/2}(\tilde{x})) \geq 1/2$ which implies (39).

Since for all $x \in \partial E \cap B_{1/2}(\tilde{x})$ it holds $|x| \leq s - 1/2$, the estimate (39) implies

$$\begin{aligned} P_\gamma(E) &\geq \frac{1}{\sqrt{2\pi}} \int_{\partial E \cap B_{1/2}(\tilde{x})} e^{-\frac{|x|^2}{2}} \, d\mathcal{H}^1 \\ &\geq \frac{1}{\sqrt{2\pi}} e^{-\frac{(s-1/2)^2}{2}} \mathcal{H}^1(\partial E \cap B_{1/2}(\tilde{x})) \geq 2e^{-\frac{s^2}{2}}. \end{aligned}$$

This contradicts (10). □

For the remaining part of this section we choose a basis $\{e^{(1)}, e^{(2)}\}$ for $\mathbb{R}^2$ such that $e^{(1)} = v$, where v is the direction in Lemma 2 and $e^{(2)}$ is an orthogonal direction to that. Let us define

$$\Sigma_+ = \{x \in \partial E : x_2 \geq 0\} \qquad \text{and} \qquad \Sigma_- = \{x \in \partial E : x_2 \leq 0\}.$$

In the next lemma we use (33) to obtain that the Gaussian measure of $\{x \in \partial E : |x_2| \leq \frac{s}{3}\}$ is small. This implies, from the measure point of view, that Σ_+ and Σ_- are almost disconnected. This enables us to variate Σ_+ and Σ_- separately, which will be crucial in the proof of Theorem 3.

Lemma 4 *Let $E \subset \mathbb{R}^2$ be a minimizer of (7) and assume (28) holds. Then there is a number $a_+ \in (0, s+1]$ such that*

$$\int_{\Sigma_+} (x_2 - a_+)^2 \, d\mathcal{H}^1_\gamma \leq 64 \, P_\gamma(E). \tag{41}$$

Moreover, it holds

$$\mathcal{H}^1_\gamma \left(\partial E \cap \{|x_2| \leq \tfrac{s}{3}\}\right) \leq \frac{1}{s} \bar{\nu}_1^2 \, P_\gamma(E). \tag{42}$$

Proof **Inequality** (41). We first show that

$$\int_{\Sigma_+} \left(|\nu_2| - (|\nu_2|)_{\Sigma_+}\right)^2 d\mathcal{H}^1_\gamma \leq \frac{23}{s^2} P_\gamma(E), \tag{43}$$

where the number $(|\nu_2|)_{\Sigma_+}$ is the average of $|\nu_2|$ on Σ_+. By (29) and (33) we obtain

$$\begin{aligned}
\int_{\Sigma_+} \left(|\nu_2| - \sqrt{1-\bar{\nu}_1^2}\right)^2 d\mathcal{H}^1_\gamma &= \int_{\Sigma_+} \frac{\left(\nu_2^2 - (1-\bar{\nu}_1^2)\right)^2}{\left(|\nu_2| + \sqrt{1-\bar{\nu}_1^2}\right)^2} \, d\mathcal{H}^1_\gamma \\
&\leq \frac{7}{4} \int_{\partial E} (\nu_1^2 - \bar{\nu}_1^2)^2 \, d\mathcal{H}^1_\gamma \\
&\leq 7 \int_{\partial E} (\nu_1 - \bar{\nu}_1)^2 \, d\mathcal{H}^1_\gamma \\
&\leq \frac{23}{s^2} P_\gamma(E).
\end{aligned}$$

Since

$$\int_{\Sigma_+} \left(|\nu_2| - (|\nu_2|)_{\Sigma_+}\right)^2 d\mathcal{H}^1_\gamma \leq \int_{\Sigma_+} \left(|\nu_2| - \sqrt{1-\bar{\nu}_1^2}\right)^2 d\mathcal{H}^1_\gamma$$

we have (43).

To prove the inequality (41) we multiply the equation (13), with $\omega = e_2$, by $(x_2 + \lambda\nu_2)$ and integrate by parts

$$\int_{\partial E} (x_2 + \lambda\nu_2)^2 \, d\mathcal{H}^1_\gamma \le \int_{\partial E} \big(\langle \nabla_\tau (x_2 + \lambda\nu_2), \nabla_\tau x_2 \rangle - \varrho \langle b, x \rangle \nu_2 (x_2 + \lambda\nu_2) \big) \, d\mathcal{H}^1_\gamma .$$

We estimate the first term on the right-hand-side by Young's inequality and by $|\lambda| \le s + 1$

$$\langle \nabla_\tau (x_2 + \lambda\nu_2), \nabla_\tau x_2 \rangle \le 2|\nabla_\tau x_2|^2 + \lambda^2 |\nabla_\tau \nu_2|^2 \le 2 + (s+1)^2 k^2$$

and the second as

$$\varrho \langle b, x \rangle \nu_2 (x_2 + \lambda\nu_2) \le 2\varrho |b| \left(|x|^2 + (s+1)^2 \nu_2^2 \right).$$

Hence, we have by $\varrho|b| \le \frac{1}{s^2}$ (from (17) and (29)), (18) and (32) that

$$\int_{\partial E} (x_2 + \lambda\nu_2)^2 \, d\mathcal{H}^1_\gamma \le \int_{\partial E} \left(2 + (s+1)^2 k^2 + \frac{2}{s^2} (|x|^2 + (s+1)^2 \nu_2^2) \right) d\mathcal{H}^1_\gamma \le 8 P_\gamma(E).$$

Therefore it holds (recall that $x_2 \ge 0$ on Σ_+)

$$\begin{aligned} 8 P_\gamma(E) &\ge \int_{\partial E} (x_2 + \lambda\nu_2)^2 \, d\mathcal{H}^1_\gamma \ge \int_{\Sigma_+} (x_2 + \lambda\nu_2)^2 \, d\mathcal{H}^1_\gamma \\ &\ge \int_{\Sigma_+} (|x_2| - |\lambda||\nu_2|)^2 \, d\mathcal{H}^1_\gamma = \int_{\Sigma_+} (x_2 - |\lambda||\nu_2|)^2 \, d\mathcal{H}^1_\gamma \\ &\ge \frac{1}{2} \int_{\Sigma_+} (x_2 - |\lambda|(|\nu_2|)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma - \lambda^2 \int_{\Sigma_+} (|\nu_2| - (|\nu_2|)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma . \end{aligned}$$

Hence, by (43) and $|\lambda| \le s + 1$ we deduce

$$\int_{\Sigma_+} (x_2 - |\lambda|(|\nu_2|)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \le 64 \, P_\gamma(E).$$

The claim then follows from $|\lambda| \le s + 1$.

Inequality (42). We have by (18), (19), (29), and (33) that

$$\begin{aligned} \bar{x}_1^2 \le \fint_{\partial E} x_1^2 \, d\mathcal{H}^1_\gamma \le (s+1)^2 \fint_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma + 8 \le \frac{s^2}{2}, \\ \fint_{\partial E} (x_1 - \bar{x}_1)^2 \, d\mathcal{H}^1_\gamma \le (s+1)^2 \fint_{\partial E} (\nu_1 - \bar{\nu}_1)^2 \, d\mathcal{H}^1_\gamma + 8 \le 12. \end{aligned} \tag{44}$$

We also have from (38) that for every $x \in \{x \in \partial E : |x_2| \le \frac{s}{3}\}$ it holds

$$x_1^2 = |x|^2 - x_2^2 \ge (s-1)^2 - \frac{s^2}{9} \ge \frac{7}{8} s^2.$$

These three inequalities imply

$$\begin{aligned} 12\, P_\gamma(E) &\geq \int_{\partial E} (x_1 - \bar{x}_1)^2 \, d\mathcal{H}^1_\gamma \geq \int_{\partial E \cap \{|x_2| \leq \frac{s}{3}\}} (|x_1| - |\bar{x}_1|)^2 \, d\mathcal{H}^1_\gamma \\ &\geq s^2 \int_{\partial E \cap \{|x_2| \leq \frac{s}{3}\}} \left(\frac{\sqrt{7}}{\sqrt{8}} - \frac{1}{\sqrt{2}} \right)^2 d\mathcal{H}^1_\gamma \\ &\geq \frac{s^2}{20} \mathcal{H}^1_\gamma \big(\partial E \cap \{|x_2| \leq \tfrac{s}{3}\} \big). \end{aligned}$$

This yields the inequality (42) when $|\bar{\nu}_1| \geq \frac{1}{2}$. Let us then assume that it holds

$$|\bar{\nu}_1| < \frac{1}{2}.$$

We choose the Lipshitz continuous cut-off function $\zeta : \mathbb{R} \to [0, 1]$ such that

$$\zeta(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{s}{3} \\ 0 & \text{for } |t| \geq \frac{5}{12} s \end{cases}$$

and

$$|\zeta'(t)| \leq \frac{12}{s} \qquad \text{for } t \in \mathbb{R}.$$

We multiply the equation (13), with $\omega = e_1$, by $x_1 \zeta^2(x_2)$ and integrate by parts

$$\begin{aligned} \int_{\partial E} x_1^2 \zeta^2(x_2) \, d\mathcal{H}^1_\gamma = \int_{\partial E} \big(&- \lambda x_1 \nu_1 \zeta^2(x_2) + \langle \nabla_\tau x_1, \nabla_\tau (x_1 \zeta^2(x_2)) \rangle \\ &+ \varrho \langle b, x \rangle \nu_1 x_1 \zeta^2(x_2) \big) \, d\mathcal{H}^1_\gamma. \end{aligned} \tag{45}$$

We estimate the first term on the right-hand-side by Young's inequality and by $|\lambda| \leq s + 1$

$$-\lambda x_1 \nu_1 \zeta^2 \leq \frac{1}{2} x_1^2 \zeta^2 + \frac{(s+1)^2}{2} \nu_1^2 \zeta^2,$$

where we have written $\zeta = \zeta(x_2)$ for short. We estimate the second term by using $|\nabla_\tau \zeta(x_2)| = |\zeta'(x_2)||\nabla x_2| \leq \frac{12}{s} |\nu_1|$ as follows

$$\begin{aligned} \langle \nabla_\tau x_1, \nabla_\tau (x_1 \zeta^2(x_2)) \rangle &\leq |\nabla_\tau x_1|^2 \zeta^2 + \frac{12}{s} \zeta |x_1| |\nabla_\tau x_1| |\nu_1| \\ &\leq \zeta^2 + \frac{1}{200} x_1^2 \zeta^2 + \frac{\tilde{C}}{s^2} \nu_1^2 \end{aligned}$$

for $\tilde{C} = 7200$. We estimate the third term simply by using $\varrho|b| \le \frac{1}{s^2}$

$$\varrho\langle b, x\rangle \nu_1 x_1 \zeta^2 \le \frac{1}{s^2}|x|^2\zeta^2.$$

Hence, we deduce from (45) and from the three above inequalities that

$$\int_{\partial E}\left(x_1^2 - \frac{1}{100}x_1^2 - \frac{2}{s^2}|x|^2 - 2\right)\zeta^2\, d\mathcal{H}^1_\gamma \le \int_{\partial E}\left((s+1)^2\nu_1^2\zeta^2 + \frac{2\tilde{C}}{s^2}\nu_1^2\right) d\mathcal{H}^1_\gamma.$$

Recall that $\zeta = 0$ when $|x_2| \ge \frac{5s}{12}$ and that by (38) we have $|x|^2 \ge (s-1)^2$ on ∂E. In particular, for every $x \in \{x \in \partial E : |x_2| \le \frac{5s}{12}\}$ it holds

$$x_1^2 = |x|^2 - x_2^2 \ge \frac{4}{5}(s+1)^2$$

and $|x|^2 \le \frac{3}{2}x_1^2$. Therefore the two previous inequalities yield

$$\tilde{c}\,(s+1)^2\int_{\partial E}\zeta^2\, d\mathcal{H}^1_\gamma \le (s+1)^2\int_{\partial E}\nu_1^2\zeta^2\, d\mathcal{H}^1_\gamma + \frac{2\tilde{C}}{s^2}\int_{\partial E}\nu_1^2\, d\mathcal{H}^1_\gamma \tag{46}$$

with $\tilde{c} = \frac{49}{62}$.

We write the first term on the right-hand-side of (46) as

$$\begin{aligned}&(s+1)^2\int_{\partial E\cap\{\nu_1^2\le\frac{2}{3}\}}\nu_1^2\zeta^2\, d\mathcal{H}^1_\gamma + (s+1)^2\int_{\partial E\cap\{\nu_1^2>\frac{2}{3}\}}\nu_1^2\zeta^2\, d\mathcal{H}^1_\gamma\\ &\quad\le \frac{2}{3}(s+1)^2\int_{\partial E}\zeta^2\, d\mathcal{H}^1_\gamma + (s+1)^2\int_{\partial E\cap\{\nu_1^2>\frac{2}{3}\}}\nu_1^2\, d\mathcal{H}^1_\gamma.\end{aligned}$$

Therefore (46) implies

$$\frac{3}{25}\int_{\partial E}\zeta^2\, d\mathcal{H}^1_\gamma \le \int_{\partial E\cap\{\nu_1^2>\frac{2}{3}\}}\nu_1^2\, d\mathcal{H}^1_\gamma + \frac{2\tilde{C}}{s^4}\int_{\partial E}\nu_1^2\, d\mathcal{H}^1_\gamma. \tag{47}$$

The first term on the right-hand-side of (47) can be estimate by (33) and the assumption $|\bar{\nu}_1| < \frac{1}{2}$ as

$$\begin{aligned}\int_{\partial E\cap\{\nu_1^2>\frac{2}{3}\}}\nu_1^2\, d\mathcal{H}^1_\gamma &\le \mathcal{H}^1_\gamma\big(\partial E\cap\{\nu_1^2 > \tfrac{2}{3}\}\big)\\ &\le 10\int_{\partial E\cap\{\nu_1^2>\frac{2}{3}\}}\left(\frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{2}\right)^2 d\mathcal{H}^1_\gamma\\ &\le 10\int_{\partial E}\big||\nu_1| - |\bar{\nu}_1|\big|^2\, d\mathcal{H}^1_\gamma \le \frac{75}{s^2}P_\gamma(E)\,\bar{\nu}_1^2.\end{aligned}$$

Then, noted that $\zeta(x_2) = 1$ for $|x_2| \le \frac{s}{3}$ and that $\int_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \le (1 + \frac{15}{2s^2})\,\bar{\nu}_1^2 \le 2\bar{\nu}_1^2$ by (33), we have

$$\mathcal{H}^1_\gamma\big(\partial E \cap \{|x_2| \le \tfrac{s}{3}\}\big) \le \int_{\partial E} \zeta^2 \, d\mathcal{H}^1_\gamma \le \frac{626}{s^2} P_\gamma(E)\,\bar{\nu}_1^2$$

and (42) follows since we assume $s \ge 10^3$. □

We are now ready to prove the reduction to the one dimensional case.

Proof of Theorem 3 We recall that

$$\Sigma_+ = \{x \in \partial E : x_2 \ge 0\} \quad \text{and} \quad \Sigma_- = \{x \in \partial E : x_2 \le 0\}.$$

As we mentioned in Remark 2, using $\varphi = \nu_e$ with $e \in \mathbb{S}^1$ orthogonal to the barycenter as a test function in the second variation inequality (12), does not provide any information on the minimizer since the term $|\int_{\partial E} x\,\nu_e \, d\mathcal{H}^1_\gamma|$ can be too large and thus (25) becomes trivial inequality. We overcome this problem by essentially variating only Σ_+ while keeping Σ_- unchanged, and vice-versa (see Fig. 2). To be more precise, we restrict the class of test function to $\varphi \in C^\infty(\partial E)$ with zero average and which satisfy $\varphi(x) = 0$ for every $x \in \partial E \cap \{x_2 \le -\frac{s}{3}\}$ (or $\varphi(x) = 0$ for every $x \in \partial E \cap \{x_2 \ge \frac{s}{3}\}$). The point is that for these test function an estimate similar to (27) holds,

$$\frac{\varrho}{\sqrt{2\pi}}\Big|\int_{\partial E} x\,\varphi \, d\mathcal{H}^1_\gamma\Big|^2 \le \frac{1}{5}\int_{\partial E} \varphi^2 \, d\mathcal{H}^1_\gamma. \tag{48}$$

Indeed, we first write

$$\begin{aligned}
\Big|\int_{\partial E} \varphi\, x \, d\mathcal{H}^1_\gamma\Big|^2 &= \left(\int_{\partial E} x_1 \varphi \, d\mathcal{H}^1_\gamma\right)^2 + \left(\int_{\partial E} x_2 \varphi \, d\mathcal{H}^1_\gamma\right)^2 \\
&= \left(\int_{\partial E} (x_1 - \bar{x}_1)\varphi \, d\mathcal{H}^1_\gamma\right)^2 + \left(\int_{\partial E} (x_2 - a_+)\varphi \, d\mathcal{H}^1_\gamma\right)^2 \\
&\le \left(\int_{\partial E} (x_1 - \bar{x}_1)^2 \, d\mathcal{H}^1_\gamma + \int_{\partial E \cap \{x_2 \ge -\frac{s}{3}\}} (x_2 - a_+)^2 \, d\mathcal{H}^1_\gamma\right) \\
&\quad \left(\int_{\partial E} \varphi^2 \, d\mathcal{H}^1_\gamma\right),
\end{aligned}$$

where a_+ is from (41). We estimate the first term in the last line by (44)

$$\int_{\partial E} (x_1 - \bar{x}_1)^2 \, d\mathcal{H}^1_\gamma \le 12\, P_\gamma(E),$$

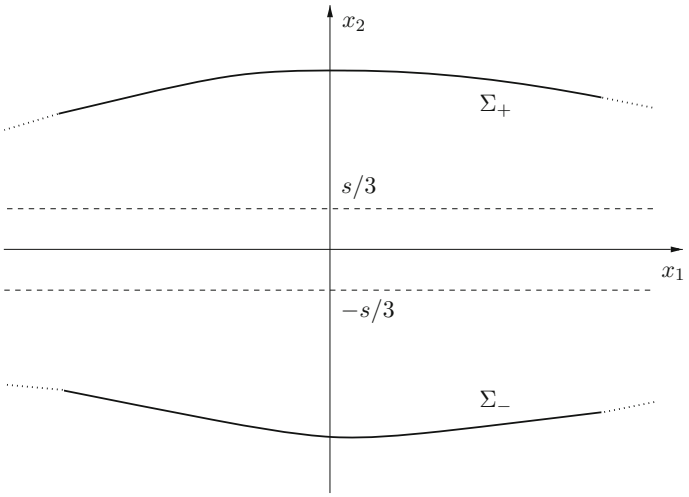

Fig. 2 The sets Σ_+ and Σ_-

while for the second term we use (41) and (42) (we recall that Remark 3 allows us to assume (28))

$$\begin{aligned}
&\int_{\partial E\cap\{x_2\geq -\frac{s}{3}\}} (x_2-a_+)^2\,d\mathcal{H}^1_\gamma \\
&\quad\leq \int_{\Sigma_+} (x_2-a_+)^2\,d\mathcal{H}^1_\gamma + \int_{\partial E\cap\{|x_2|\leq\frac{s}{3}\}} (|x_2|+a_+)^2\,d\mathcal{H}^1_\gamma \\
&\quad\leq \int_{\Sigma_+} (x_2-a_+)^2\,d\mathcal{H}^1_\gamma + \int_{\partial E\cap\{|x_2|\leq\frac{s}{3}\}} \left(\frac{s}{3}+(s+1)\right)^2\,d\mathcal{H}^1_\gamma \\
&\quad\leq \int_{\Sigma_+} (x_2-a_+)^2\,d\mathcal{H}^1_\gamma + \frac{16}{9}(s+1)^2\mathcal{H}^1_\gamma(\partial E\cap\{|x_2|\leq\tfrac{s}{3}\}) \\
&\quad\leq 3s\,P_\gamma(E).
\end{aligned}$$

Hence, we get (48) thanks to (10) and $\varrho\leq\frac{7\sqrt{2\pi}}{5s^2}e^{\frac{s^2}{2}}$ from (9).

In order to explain the idea of the proof, we assume first that Σ_+ and Σ_- are different components of ∂E. This is of course a major simplification but it will hopefully help the reader to follow the actual proof below. In this case we may use the following test functions in the second variation condition,

$$\varphi_i := \begin{cases} \nu_i - (\nu_i)_{\Sigma_+} & \text{on } \Sigma_+ \\ 0 & \text{on } \Sigma_- \end{cases} \tag{49}$$

for $i = 1, 2$, where $(\nu_i)_{\Sigma_+}$ is the average of ν_i on Σ_+. We use φ_i as a test function in the second variation condition (12) and use (48) to obtain

$$\int_{\partial E} \left(|\nabla_\tau \varphi_i|^2 - k^2 \varphi_i^2 + \varrho \langle b, \nu \rangle \varphi_i^2 - \frac{4}{5} \varphi_i^2 \right) d\mathcal{H}^1_\gamma(x) \geq 0.$$

By using the equalities (35) and (36), rewritten on Σ_+, we get after straightforward calculations

$$\begin{aligned} &(\nu_i)^2_{\Sigma_+} \int_{\Sigma_+} (k^2 - \varrho \langle b, \nu \rangle)\, d\mathcal{H}^1_\gamma + \frac{4}{5} \int_{\Sigma_+} (\nu_i - (\nu_i)_{\Sigma_+})^2\, d\mathcal{H}^1_\gamma \\ &\quad \leq -\varrho \langle b, e_i \rangle \int_{\Sigma_+} \nu_i\, d\mathcal{H}^1_\gamma. \end{aligned} \tag{50}$$

for $i = 1, 2$. By adding up the previous inequality for $i = 1, 2$ we get

$$\begin{aligned} &\Big[(\nu_1)^2_{\Sigma_+} + (\nu_2)^2_{\Sigma_+} \Big] \int_{\Sigma_+} (k^2 - \varrho \langle b, \nu \rangle)\, d\mathcal{H}^1_\gamma + \frac{4}{5} \int_{\Sigma_+} \Big[1 - (\nu_1)^2_{\Sigma_+} - (\nu_1)^2_{\Sigma_+} \Big] d\mathcal{H}^1_\gamma \\ &\quad \leq -\varrho \int_{\Sigma_+} \langle b, \nu \rangle\, d\mathcal{H}^1_\gamma. \end{aligned}$$

This can be rewritten as

$$\Big[1 - (\nu_1)^2_{\Sigma_+} - (\nu_2)^2_{\Sigma_+} \Big] \int_{\Sigma_+} \Big(\varrho \langle b, \nu \rangle + \frac{4}{5} \Big) d\mathcal{H}^1_\gamma + \Big[(\nu_1)^2_{\Sigma_+} + (\nu_2)^2_{\Sigma_+} \Big] \int_{\Sigma_+} k^2\, d\mathcal{H}^1_\gamma \leq 0.$$

By Jensen's inequality $1 - (\nu_1)^2_{\Sigma_+} - (\nu_2)^2_{\Sigma_+} \geq 0$, while $|\varrho \langle b, \nu \rangle| \leq \frac{3}{2s^2}$ which follows from (17). Therefore $k = 0$ and Σ_+ is a line. It is clear that a similar conclusion holds also for in Σ_-.

When Σ_+ and Σ_- are connected the argument is more involved, since we need a cut-off argument in order to "separate" Σ_+ and Σ_-. This is possible due to (42), which implies that the perimeter of the minimizer in the strip $\{|x_2| \leq s/3\}$ is small. Therefore the cut-off argument produces an error term, which by (42) is small enough so that we may apply the previous argument. However, the presence of the cut-off function makes the estimates more complicated and since the argument is technically involved we split the rest of the proof in two steps.

Step 1. Without loss of generality we may assume that $\mathcal{H}^1_\gamma(\Sigma_+) \geq \mathcal{H}^1_\gamma(\Sigma_-)$. Let us denote

$$C_+ := \frac{1}{s^2}\, \mathcal{H}^1_\gamma\big(\partial E \cap \{-\tfrac{s}{3} < x_2 < 0\}\big).$$

In the first step we prove

$$\Big((\nu_1)^2_{\Sigma_+} + (\nu_2)^2_{\Sigma_+} \Big) \int_{\Sigma_+} k^2\, d\mathcal{H}^1_\gamma + \int_{\Sigma_+} \Big(1 - (\nu_1)^2_{\Sigma_+} - (\nu_2)^2_{\Sigma_+} \Big) d\mathcal{H}^1_\gamma \leq 330 C_+. \tag{51}$$

We do this by proving the counterpart of (50), which now reads as

$$\begin{aligned}(\nu_i)^2_{\Sigma_+}\int_{\Sigma_+}\left(\frac{3}{4}k^2-\varrho\langle b,\nu\rangle\right)d\mathcal{H}^1_\gamma+\frac{4}{5}\int_{\Sigma_+}(\nu_i-(\nu_i)_{\Sigma_+})^2\,d\mathcal{H}^1_\gamma\\ \le-\varrho\langle b,e_i\rangle\int_{\Sigma_+}\nu_i\,d\mathcal{H}^1_\gamma+122C_+.\end{aligned}\tag{52}$$

for $i=1,2$. Let us show first how (51) follows from (52).

Indeed, by $\varrho|b|\le\frac{3}{2s^2}$ we have

$$\begin{aligned}-(\nu_i)^2_{\Sigma_+}\int_{\Sigma_+}\varrho\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma\ge&-\left(\fint_{\Sigma_+}\nu_i^2\,d\mathcal{H}^1_\gamma\right)\int_{\Sigma_+}\varrho\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma\\&-\frac{3}{2s^2}\int_{\Sigma_+}(\nu_i-(\nu_i)_{\Sigma_+})^2\,d\mathcal{H}^1_\gamma.\end{aligned}$$

Therefore we have

$$\begin{aligned}&-(\nu_i)^2_{\Sigma_+}\int_{\Sigma_+}\varrho\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma+\frac{1}{20}\int_{\Sigma_+}(\nu_i-(\nu_i)_{\Sigma_+})^2\,d\mathcal{H}^1_\gamma\\&\ge-\left(\fint_{\Sigma_+}\nu_i^2\,d\mathcal{H}^1_\gamma\right)\int_{\Sigma_+}\varrho\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma.\end{aligned}$$

Thus we obtain from (52)

$$\begin{aligned}&(\nu_i)^2_{\Sigma_+}\int_{\Sigma_+}\frac{3}{4}k^2\,d\mathcal{H}^1_\gamma-\left(\fint_{\Sigma_+}\nu_i^2\,d\mathcal{H}^1_\gamma\right)\int_{\Sigma_+}\varrho\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma+\frac{3}{4}\int_{\Sigma_+}(\nu_i-(\nu_i)_{\Sigma_+})^2\,d\mathcal{H}^1_\gamma\\&\le-\varrho\langle b,e_i\rangle\int_{\Sigma_+}\nu_i\,d\mathcal{H}^1_\gamma+122C_+.\end{aligned}$$

Note that $\sum_{i=1}^2\int_{\Sigma_+}\langle b,e_i\rangle\nu_i\,d\mathcal{H}^1_\gamma=\int_{\Sigma_+}\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma$. Therefore, by adding the above inequality with $i=1,2$ we obtain

$$\begin{aligned}&\left((\nu_1)^2_{\Sigma_+}+(\nu_2)^2_{\Sigma_+}\right)\int_{\Sigma_+}\frac{3}{4}k^2\,d\mathcal{H}^1_\gamma-\int_{\Sigma_+}\varrho\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma\\&\quad+\frac{3}{4}\int_{\Sigma_+}\left(\nu_1^2-(\nu_1)^2_{\Sigma_+}+\nu_2^2-(\nu_2)^2_{\Sigma_+}\right)d\mathcal{H}^1_\gamma\\&\le-\int_{\Sigma_+}\varrho\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma+244C_+,\end{aligned}$$

which implies (51) (since $\frac{4}{3}\cdot244<330$).

We are left to prove (52). We will use the second variation condition (12) with test function

$$\varphi_i:=(\nu_i-\alpha_i)\zeta(x_2)$$

for $i = 1, 2$. Here $\zeta : \mathbb{R} \to [0, 1]$ is a smooth cut-off function such that

$$\zeta(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t \leq -s/3, \end{cases} \qquad \text{and} \qquad |\zeta'(t)| \leq \frac{c}{s} \quad \text{for all } t \in \mathbb{R}$$

and α_i is chosen so that φ_i has zero average and c is a number such that $c > 3$. This choice is the counterpart of (49) in the case when ∂E is connected. In particular, the cut-off function ζ guarantees that $\varphi_i(x) = 0$, for $x \in \partial E \cap \{x_2 \leq -\frac{s}{3}\}$. Therefore the estimate (48) holds and the second variation condition (12) yields

$$\int_{\partial E} \left(|\nabla_\tau \varphi_i|^2 - k^2 \varphi_i^2 + \varrho \langle b, \nu \rangle \varphi_i^2 - \frac{4}{5} \varphi_i^2 \right) d\mathcal{H}^1_\gamma \geq 0. \tag{53}$$

Let us simplify the above expression. Recall that the test function is $\varphi = (\nu_i - \alpha_i)\zeta$, where $\zeta = \zeta(x_2)$. By straightforward calculation

$$\begin{aligned} \int_{\partial E} |\nabla_\tau \varphi_i|^2 \, d\mathcal{H}^1_\gamma &= \int_{\partial E} \Big(\varphi_i (-\Delta_\tau \varphi_i + \langle \nabla \varphi_i, x \rangle \Big) \, d\mathcal{H}^1_\gamma \\ &= \int_{\partial E} \Big(\varphi_i \zeta (-\Delta_\tau \nu_i + \langle \nabla_\tau \nu_i, x \rangle) + (\nu_i - \alpha_i)^2 |\nabla_\tau \zeta|^2 \Big) \, d\mathcal{H}^1_\gamma. \end{aligned}$$

Therefore we have by the above equality and by multiplying the equation (31) with $\varphi_i \zeta$ that

$$\int_{\partial E} |\nabla_\tau \varphi_i|^2 \, d\mathcal{H}^1_\gamma = \int_{\partial E} (k^2 - \varrho \langle b, \nu \rangle) \zeta^2 \nu_i (\nu_i - \alpha_i) \, d\mathcal{H}^1_\gamma + R_1, \tag{54}$$

where the remainder term is

$$R_1 = \int_{\partial E} \Big(\varrho \langle b, e_i \rangle \, \varphi_i \zeta + (\nu_i - \alpha_i)^2 |\nabla_\tau \zeta|^2 \Big) \, d\mathcal{H}^1_\gamma. \tag{55}$$

On the other hand, multiplying (31) with ζ^2 and integrating by parts yields

$$\begin{aligned} &\alpha_i \int_{\partial E} \Big((k^2 - \varrho \langle b, \nu \rangle) \nu_i \zeta^2 \Big) \, d\mathcal{H}^1_\gamma \\ &= \alpha_i \int_{\partial E} \Big((-\Delta_\tau \nu_i + \langle \nabla_\tau \nu_i, x \rangle) \zeta^2 - \varrho \langle b, e_i \rangle \zeta^2 \Big) \, d\mathcal{H}^1_\gamma \\ &= -\alpha_i \int_{\partial E} \varrho \langle b, e_i \rangle \zeta^2 \, d\mathcal{H}^1_\gamma + R_2, \end{aligned} \tag{56}$$

where the remainder term is

$$R_2 = 2\alpha_i \int_{\partial E} \zeta \, \langle \nabla_\tau \nu_i, \nabla_\tau \zeta \rangle \, d\mathcal{H}^1_\gamma. \tag{57}$$

Collecting (53), (54) and (56) yields

$$\int_{\partial E}\left(\alpha_i^2\big(k^2-\varrho\langle b,\nu\rangle\big)+\frac{4}{5}|\nu_i-\alpha_i|^2\right)\zeta^2\,d\mathcal{H}^1_\gamma \\ \le -\alpha_i\int_{\partial E}\varrho\langle b,e_i\rangle\zeta^2\,d\mathcal{H}^1_\gamma+R_1+R_2, \tag{58}$$

where the remainder terms R_1 and R_2 are given by (55) and (57) respectively.

Let us next estimate the remainder terms in (58). Since $|\nabla_\tau\zeta(x)|\le c/s$, for $-s/3<x_2<0$ and $\nabla_\tau\zeta(x)=0$ otherwise, it holds

$$\int_{\partial E}|\nabla\zeta|^2\,d\mathcal{H}^1_\gamma\le\frac{c^2}{s^2}\,\mathcal{H}^1_\gamma\big(\partial E\cap\{-\tfrac{s}{3}<x_2<0\}\big).$$

We may therefore estimate R_2 (given by (57)) by Young's inequality as

$$R_2\le\frac{\alpha_i^2}{4}\int_{\partial E}|\nabla_\tau\nu_i|^2\zeta^2\,d\mathcal{H}^1_\gamma+4\int_{\partial E}|\nabla_\tau\zeta|^2\,d\mathcal{H}^1_\gamma \\ \le\frac{\alpha_i^2}{4}\int_{\partial E}k^2\zeta^2\,d\mathcal{H}^1_\gamma+\frac{4c^2}{s^2}\,\mathcal{H}^1_\gamma\big(\partial E\cap\{-\tfrac{s}{3}<x_2<0\}\big).$$

Similarly we may estimate R_1, given by (55), as

$$R_1\le\varrho\langle b,e_i\rangle\int_{\partial E}\varphi_i\zeta\,d\mathcal{H}^1_\gamma+\frac{4c^2}{s^2}\,\mathcal{H}^1_\gamma\big(\partial E\cap\{-\tfrac{s}{3}<x_2<0\}\big).$$

To estimate the first term in the right we recall that $\int_{\partial E}\varphi_i\,d\mathcal{H}^1_\gamma=0$ and therefore $\int_{\partial E}\varphi_i\zeta\,d\mathcal{H}^1_\gamma=\int_{\partial E}\varphi_i(\zeta-1)\,d\mathcal{H}^1_\gamma$. Since $\varphi_i(\zeta-1)=0$ on $\partial E\cap\{-\frac{s}{3}<x_2<0\}$, we deduce by $\varrho|b|\le\frac{3}{2s^2}$ that

$$\varrho\langle b,e_i\rangle\int_{\partial E}\varphi_i\zeta\,d\mathcal{H}^1_\gamma\le\frac{3}{2s^2}\,\mathcal{H}^1_\gamma\big(\partial E\cap\{-\tfrac{s}{3}<x_2<0\}\big).$$

Hence, we choose c close to 3 in the definition of ζ and deduce from (58)

$$\int_{\partial E}\left(\alpha_i^2\Big(\frac{3}{4}k^2-\varrho\langle b,\nu\rangle\Big)+\frac{4}{5}|\nu_i-\alpha_i|^2\right)\zeta^2\,d\mathcal{H}^1_\gamma \\ \le-\alpha_i\int_{\partial E}\varrho\langle b,e_i\rangle\zeta^2\,d\mathcal{H}^1_\gamma+74C_+. \tag{59}$$

Recall that $C_+=\frac{1}{s^2}\mathcal{H}^1_\gamma\big(\partial E\cap\{-\frac{s}{3}<x_2<0\}\big)$.

By a similar argument we may also get rid of the cut-off function ζ in (59). Indeed by $\varrho|b|\le\frac{3}{2s^2}$ we have $-\int_{\partial E}\varrho\langle b,\nu\rangle\zeta^2\,d\mathcal{H}^1_\gamma\ge-\int_{\Sigma_+}\varrho\langle b,\nu\rangle\,d\mathcal{H}^1_\gamma-\frac{3}{2}C_+$. Similarly

we get $-\alpha_i \int_{\partial E} \varrho\langle b, e_i\rangle \zeta^2 \, d\mathcal{H}^1_\gamma \le -\alpha_i \int_{\Sigma_+} \varrho\langle b, e_i\rangle \, d\mathcal{H}^1_\gamma + \frac{3}{2} C_+$. Therefore we obtain from (59) that

$$\int_{\Sigma_+} \left(\alpha_i^2 \Big(\frac{3}{4} k^2 - \varrho\langle b, \nu\rangle \Big) + \frac{4}{5} |\nu_i - \alpha_i|^2 \right) d\mathcal{H}^1_\gamma \le -\alpha_i \int_{\Sigma_+} \varrho\langle b, e_i\rangle \, d\mathcal{H}^1_\gamma + 77 C_+. \tag{60}$$

We need yet to replace α_i by $(\nu_i)_{\Sigma_+}$ in order to obtain (52). We do this by showing that α_i is close to the average $(\nu_i)_{\Sigma_+}$. To be more precise we claim that

$$|\alpha_i - (\nu_i)_{\Sigma_+}| \le \frac{2}{\mathcal{H}^1_\gamma(\Sigma_+)} \mathcal{H}^1_\gamma\big(\partial E \cap \{-\tfrac{s}{3} < x_2 < 0\}\big). \tag{61}$$

Indeed, since $\zeta = 1$ on Σ_+ we may write

$$\mathcal{H}^1_\gamma(\Sigma_+)(\alpha_i - (\nu_i)_{\Sigma_+}) = \int_{\Sigma_+} (\alpha_i - \nu_i)\zeta \, d\mathcal{H}^1_\gamma.$$

Since $\zeta = 0$ when $x_2 \le -s/3$ we may estimate

$$\mathcal{H}^1_\gamma(\Sigma_+)\big|\alpha_i - (\nu_i)_{\Sigma_+}\big| \le \Big| \int_{\partial E} (\alpha_i - \nu_i)\zeta \, d\mathcal{H}^1_\gamma \Big| + 2\,\mathcal{H}^1_\gamma\big(\partial E \cap \{-\tfrac{s}{3} < x_2 < 0\}\big).$$

The inequality (61) then follows from $\int_{\partial E} (\alpha_i - \nu_i)\zeta \, d\mathcal{H}^1_\gamma = -\int_{\partial E} \varphi_i \, d\mathcal{H}^1_\gamma = 0$.

Recall that we assume $\mathcal{H}^1_\gamma(\Sigma_+) \ge \mathcal{H}^1_\gamma(\Sigma_-)$. In particular, this implies

$$\mathcal{H}^1_\gamma(\Sigma_+) \ge \frac{1}{8} P_\gamma(E). \tag{62}$$

We use (62) and (32) to conclude that

$$\int_{\Sigma_+} k^2 d\mathcal{H}^1_\gamma \le \frac{12}{s^2} \mathcal{H}^1_\gamma(\Sigma_+). \tag{63}$$

We estimate (60) using $\varrho|b| \le \frac{3}{2s^2}$, (61) and (63), and get

$$\begin{aligned} &\int_{\Sigma_+} \left((\nu_i)^2_{\Sigma_+} \Big(\frac{3}{4} k^2 - \varrho\langle b, \nu\rangle \Big) + \frac{4}{5} |\nu_i - \alpha_i|^2 \right) d\mathcal{H}^1_\gamma \\ &\quad \le -(\nu_i)_{\Sigma_+} \int_{\Sigma_+} \varrho\langle b, e_i\rangle \, d\mathcal{H}^1_\gamma + 122 C_+. \end{aligned}$$

Finally the inequality (52) follows from

$$\int_{\Sigma_+} |\nu_i - (\nu_i)_{\Sigma_+}|^2 \, d\mathcal{H}^1_\gamma \le \int_{\Sigma_+} |\nu_i - \alpha_i|^2 \, d\mathcal{H}^1_\gamma.$$

Step 2. Recall that we assume $\mathcal{H}^1_\gamma(\Sigma_+) \geq \mathcal{H}^1_\gamma(\Sigma_-)$. Let us show that also Σ_- satisfies (62), i.e., $\mathcal{H}^1_\gamma(\Sigma_-) \geq \frac{1}{8} P_\gamma(E)$. To this aim we deduce from (51) that

$$\int_{\Sigma_+} (\nu_2 - (\nu_2)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \leq \frac{330}{s^2} P_\gamma(E).$$

We use (28) and the inequality above to obtain

$$\begin{aligned}\frac{4}{7} P_\gamma(E) - \int_{\partial E} (\nu_1 - \bar{\nu}_1)^2 \, d\mathcal{H}^1_\gamma &\leq \int_{\partial E} (\nu_2 - \bar{\nu}_2)^2 \, d\mathcal{H}^1_\gamma \leq \int_{\partial E} (\nu_2 - (\nu_2)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \\ &= \int_{\Sigma_+} (\nu_2 - (\nu_2)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma + \int_{\Sigma_-} (\nu_2 - (\nu_2)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \\ &\leq \frac{330}{s^2} P_\gamma(E) + 4\mathcal{H}^1_\gamma(\Sigma_-).\end{aligned}$$

Hence, by (33), we have $\mathcal{H}^1_\gamma(\Sigma_-) \geq \frac{1}{8} P_\gamma(E)$.

We may thus use precisely the same argument as in the first step to prove the estimate (51) also for Σ_-, i.e.,

$$\left((\nu_1)^2_{\Sigma_-} + (\nu_2)^2_{\Sigma_-}\right) \int_{\Sigma_-} k^2 \, d\mathcal{H}^1_\gamma + \int_{\Sigma_-} \left(1 - (\nu_1)^2_{\Sigma_-} - (\nu_2)^2_{\Sigma_-}\right) d\mathcal{H}^1_\gamma \leq 330 C_-, \tag{64}$$

where

$$C_- := \frac{1}{s^2} \mathcal{H}^1_\gamma\big(\partial E \cap \{0 < x_2 < \tfrac{s}{3}\}\big).$$

We have by (63) $⨍_{\Sigma_+} k^2 \, \mathcal{H}^1_\gamma \leq 1$. Therefore we obtain from (51)

$$\int_{\Sigma_+} k^2 \, d\mathcal{H}^1_\gamma \leq 330 C_+ = \frac{330}{s^2} \mathcal{H}^1_\gamma\big(\partial E \cap \{-\tfrac{s}{3} < x_2 < 0\}\big).$$

Similarly (64) (and an estimate analogous to (63) with Σ_- in place of Σ_+) implies

$$\int_{\Sigma_-} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{330}{s^2} \mathcal{H}^1_\gamma\big(\partial E \cap \{0 < x_2 < \tfrac{s}{3}\}\big).$$

By adding these together and using (42) we obtain

$$⨍_{\partial E} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{330}{P_\gamma(E) s^2} \mathcal{H}^1_\gamma\big(\partial E \cap \{|x_2| \leq \tfrac{s}{3}\}\big) \leq \frac{330}{s^3} \bar{\nu}_1^2. \tag{65}$$

We proceed by recalling the equation (31) for ν_1, i.e.,

$$\Delta \nu_1 - \langle \nabla \nu_1, x \rangle = -k^2 \nu_1 + \varrho \langle b, \nu \rangle \nu_1 - \varrho \langle b, e^{(1)} \rangle.$$

We integrate this over ∂E and use (65) to get

$$\left| -\varrho \langle b, e^{(1)} \rangle + \varrho \fint_{\partial E} \langle b, \nu \rangle \nu_1 \, d\mathcal{H}^1_\gamma \right| \leq \fint_{\partial E} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{330}{s^3} \bar{\nu}_1^2.$$

Note that by $\bar{\nu} P_\gamma(E) = -\sqrt{2\pi}\, b$ (given in (16)) we have $|\bar{\nu}| \, \langle b, e^{(1)} \rangle = -|b| \bar{\nu}_1$. Thus we deduce from the above inequality that

$$\varrho |b| \, |\bar{\nu}_1| \leq \varrho |b| \, |\bar{\nu}| \fint_{\partial E} |\nu_1| \, d\mathcal{H}^1_\gamma + \frac{330}{s^3} \bar{\nu}_1^2 \, |\bar{\nu}|. \tag{66}$$

Using (29) and the inequality $\fint_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \leq \left(1 + \frac{15}{2s^2}\right) \bar{\nu}_1^2$ (given by (33)) we estimate

$$\varrho |b| \, |\bar{\nu}| \fint_{\partial E} |\nu_1| \, d\mathcal{H}^1_\gamma \leq \frac{\sqrt{3}}{\sqrt{7}} \varrho |b| \left(\fint_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \right)^{1/2} \leq \frac{2}{3} \varrho |b| \, |\bar{\nu}_1|.$$

Therefore we deduce from (66)

$$\frac{1}{3} \varrho |b| |\bar{\nu}_1| \leq \frac{330}{s^3} \bar{\nu}_1^2 \, |\bar{\nu}|.$$

We use $\varrho |b| \geq \frac{1}{s^2} |\bar{\nu}|$ (from (17)) to conclude

$$\frac{1}{s^2} |\bar{\nu}_1| |\bar{\nu}| \leq \frac{990}{s^3} \bar{\nu}_1^2 \, |\bar{\nu}| \leq \frac{990}{s^3} |\bar{\nu}_1| \, |\bar{\nu}|$$

and thus $\bar{\nu}_1 = 0$ since $s \geq 10^3$. But then (33) implies

$$\nu_1 = 0$$

(recall that we chose $e_1 = v$) and we have reduced the problem to the one dimensional case. □

5 The one dimensional case

In this short section we finish the proof of the main theorem which states that the minimizer of (7) is either the half-space $H_{\omega,s}$ or the symmetric strip $D_{\omega,s}$. By the previous results it is enough to solve the problem in the one-dimensional case. Surprisingly even this result does not seem to be trivial, even if its proof is based on elementary one-dimensional analysis.

Theorem 4 *The minimizer $E \subset \mathbb{R}$ of (7) is either $(-\infty, s)$, $(-s, \infty)$ or $(-a(s), a(s))$.*

Proof As we explained in Sect. 2, we have to prove that, when ϱ is in the interval (9), the only local minimizers of (7) are $(-\infty, s)$, $(-s, \infty)$ and $(-a(s), a(s))$.

Let us first show that the minimizer E is an interval. Recall that since $E \subset \mathbb{R}$ is a set of locally finite perimeter it has locally finite number of boundary points. Moreover, since there is no curvature in dimension one the Euler equation (11) reads as

$$-x\nu(x) + \varrho bx = \lambda. \tag{67}$$

By (38) we have that $(-s+1, s-1) \subset E$. It is therefore enough to prove that the boundary ∂E has at most one positive and one negative point. Assume by contradiction that ∂E has at least two positive points (the case of two negative points is similar).

If x is a positive point which is closest to the origin on ∂E then $\nu(x) = 1$. On the other hand, if y is the next boundary point, then $\nu(y) = -1$. Then the Euler equation yields

$$-x + \varrho bx = y + \varrho by.$$

By $\varrho|b| \leq \frac{3}{2s^2}$ (given in (17)) we conclude that

$$\left(1 - \frac{3}{2s^2}\right) y \leq -\left(1 - \frac{3}{2s^2}\right) x,$$

which is a contradiction since $x, y > 0$.

The minimizer of (7) is thus an interval of the form

$$E = (-x, y),$$

where $s - 1 \leq x, y \leq \infty$. Without loss of generality we may assume that $x \leq y$. In particular, we have

$$P_\gamma(E) \leq 2e^{-\frac{x^2}{2}}.$$

Using the bound on the perimeter (10) we conclude that $x < s + 1/s$. To bound y from above we use the Euler equation (67)

$$x + \varrho bx = y - \varrho by.$$

Thus we conclude from $\varrho|b| \leq \frac{3}{2s^2}$ that

$$y < s + \frac{5}{s}. \tag{68}$$

Let us next prove that the minimizer has the volume $\gamma(E) = \phi(s)$. Indeed, it is not possible that $\gamma(E) < \phi(s)$, because by enlarging E we can decrease its perimeter,

barycenter and the volume penalization term in (7). Also $\gamma(E) > \phi(s)$ is not possible. If this was the case we may perturb the set E by

$$E_t = (-x + t, y), \qquad t > 0.$$

Then $\phi(s) \le \gamma(E_t) < \gamma(E)$ and

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(E_t)\big|_{t=0} &= xe^{-\frac{x^2}{2}} + \varrho\, b(E)\, xe^{-\frac{x^2}{2}} - (s+1)e^{-\frac{x^2}{2}} \\ &\le \left(1 + \frac{3}{2s^2}\right) xe^{-\frac{x^2}{2}} - (s+1)e^{-\frac{x^2}{2}},\end{aligned}$$

taking again into account that $\varrho|b(E)| \le \frac{3}{2s^2}$. But since $x < s + 1/s$ the above inequality yields $\frac{d}{dt}\mathcal{F}(E_t)\big|_{t=0} < 0$, which contradicts the minimality of E. Note that since $E = (-x, y)$ has the volume $\phi(s)$ and $x \le y$, it holds $s \le x \le a(s) \le y$.

Let us finally show that if a local minimizer is a bounded interval $E = (-x, y)$ for $s \le x \le y < \infty$, then necessarily $x = y = a(s)$. We study the value of the functional (7) for intervals $E_t = (-\alpha(t), t)$ with $s \le \alpha(t) \le a(s) \le t$, which have the volume $\gamma(E_t) = \phi(s)$. By the inequality (68) we need only to study the case when $a(s) \le t < s + \frac{5}{s}$. This leads us to study the function $f : [a(s), s + \frac{5}{s}) \to \mathbb{R}$,

$$f(t) := \mathcal{F}(E_t) = e^{-\frac{t^2}{2}} + e^{-\frac{\alpha^2(t)}{2}} + \frac{\varrho}{2\sqrt{2\pi}}\left(e^{-\frac{\alpha^2(t)}{2}} - e^{-\frac{t^2}{2}}\right)^2.$$

The volume constraint reads as $\int_{-\alpha(t)}^{t} e^{-\frac{u^2}{2}}\, du = \sqrt{2\pi}\,\phi(s)$. By differentiating this we obtain

$$\alpha'(t)e^{-\frac{\alpha^2(t)}{2}} = -e^{-\frac{t^2}{2}}. \tag{69}$$

From (69) we conclude that for $t \ge \alpha(t)$ it holds $0 > \alpha'(t) > -1$.

Our goal is to show that f has at most two critical points. Moreover, it holds $\alpha(a(s)) = a(s)$ and therefore by symmetry $f'(a(s)) = 0$. We will also show that this point $t = a(s)$ is a strict local minimum. Therefore if the function f would have another local minimum say at $\tilde{t}$, there would be a third critical point in $(a(s), \tilde{t})$. This is a contradiction and we conclude the proof.

To this aim we differentiate f once and use (69) to get

$$f'(t) = \left(-t + \alpha(t) + \frac{\varrho}{\sqrt{2\pi}}(t + \alpha(t))\left(e^{-\frac{\alpha^2(t)}{2}} - e^{-\frac{t^2}{2}}\right)\right)e^{-\frac{t^2}{2}}.$$

Therefore at a critical point it holds

$$\frac{\varrho}{\sqrt{2\pi}}(t + \alpha(t))\left(e^{-\frac{\alpha^2(t)}{2}} - e^{-\frac{t^2}{2}}\right) = t - \alpha(t). \tag{70}$$

We are interested in the sign of $f''(t)$ at critical points in the interval $[a(s), s+\frac{5}{s})$. To simplify the notation we denote the barycenter of $E_t = (-\alpha(t), t)$ by

$$b_t := b(E_t) = \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{\alpha^2(t)}{2}} - e^{-\frac{t^2}{2}}\right). \tag{71}$$

By differentiating f twice and by using (69) and (70) we obtain

$$f''(t) = \left(-(1-\varrho b_t) + \alpha'(t)(1+\varrho b_t) + \frac{\varrho}{\sqrt{2\pi}}(t+\alpha(t))^2 e^{-\frac{t^2}{2}}\right) e^{-\frac{t^2}{2}}$$

at a critical point t. Let us write $\varrho = \frac{\varrho_0\sqrt{2\pi}}{s^2} e^{\frac{s^2}{2}}$, where $\frac{6}{5} \le \varrho_0 \le \frac{7}{5}$. In order to analyze the sign of $f''(t)$ at critical points we define $g : [a(s), s+\frac{5}{s}) \to \mathbb{R}$ as

$$g(t) := -(1-\varrho b_t) + \alpha'(t)(1+\varrho b_t) + \frac{\varrho_0}{s^2}(t+\alpha(t))^2 e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}}.$$

As we mentioned, the end point $t = \alpha(t) = a(s) > s$ is of course a critical point of f. Let us check that it is a local minimum. We have for the barycenter $b_{a(s)} = 0$, $\alpha'(a(s)) = -1$ by (69) and $2e^{-\frac{a(s)^2}{2}} > e^{-\frac{s^2}{2}}$ by (5). Therefore it holds

$$g(a(s)) \ge -2 + 2\varrho_0 > 0.$$

In particular, we deduce that $t = a(s)$ is a strict local minimum of f.

Let us next show that g is strictly decreasing. We first obtain by differentiating (69) that

$$\alpha'' = \alpha'(\alpha\alpha' - t).$$

By recalling that $|\alpha'(t)| \le 1$ and $\alpha(t) \le a(s) \le s + \ln 2/s$ by (4), we get that $|\alpha''(t)| \le 2s\,|\alpha'(t)| + 6/s$ for $t \in [a(s), s+\frac{5}{s})$. By $\alpha(t) \ge s$ we have $\varrho|b_t| \le \frac{\varrho_0}{s^2} e^{-\frac{\alpha^2(t)}{2}} e^{\frac{s^2}{2}} \le \frac{2}{s^2}$. Moreover, by differentiating the barycenter (71) and using (69) we get $\varrho\,|b_t'| \le 4/s$. We may then estimate the derivative of g as

$$\begin{aligned} g'(t) &\le \alpha''(t)(1+\varrho b_t) + (1+\alpha'(t))\,\varrho\, b_t' \\ &\quad - \frac{\varrho_0}{s^2}\, t(t+\alpha(t))^2 e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}} + \frac{2\varrho_0}{s^2}(t+\alpha(t))(1+\alpha'(t)) e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}} \\ &\le 2s\,|\alpha'(t)| - 4\varrho_0\, s\, e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}} + \frac{50}{s}. \end{aligned}$$

Next we observe that (69) and $\alpha(t) \le a(s) \le s + \ln 2/s$ imply

$$|\alpha'(t)| = e^{-\frac{t^2}{2}} e^{\frac{\alpha(t)^2}{2}} \le 2 \cdot \frac{8}{7} e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}}.$$

Therefore we deduce from the above two inequalities and from $\varrho_0 \geq \frac{6}{5} > \frac{8}{7}$ that g is strictly decreasing on $[a(s), s + \frac{5}{s})$.

Recall that $g(a(s)) > 0$. Since g is strictly decreasing, there is $t_0 \in (a(s), s + \frac{5}{s})$ such that $g(t) > 0$ for $t \in [a(s), t_0)$ and $g(t) < 0$ for $t \in (t_0, s + \frac{5}{s})$. Therefore the function f has no other local minimum on $[a(s), s + \frac{5}{s})$ than the end point $t = a(s)$. Indeed, if there were another local minimum on $(a(s), t_0]$ there would be at least one local maximum on $(a(s), t_0)$. This is impossible as the previous argument shows that $f''(t) > 0$ at every critical point on $(a(s), t_0)$. Moreover, from $g(t) < 0$ for $t \in (t_0, s + \frac{5}{s})$ we conclude that there are no local minimum points on $(t_0, s + \frac{5}{s}]$. This completes the proof. □

Acknowledgements The first author was supported by INdAM and by the project VATEXMATE. The second author was supported by the Academy of Finland Grant 314227.

Appendix

We first prove the inequalities (4) and (5). In fact, the proof gives us a slightly stronger estimate than (4). We recal that we are assuming $s \geq 10^3$.

Lemma 5 *The following estimates hold:*

$$s + \frac{1}{s} \ln\left(2 - 2/s^2\right) < a(s) < s + \frac{\ln 2}{s} \tag{72}$$

and

$$\left(1 + \frac{\ln 2}{s^2} - \frac{8}{s^4}\right) e^{-\frac{s^2}{2}} < P_\gamma(D_{\omega,s}) < \left(1 + \frac{\ln 2}{s^2}\right) e^{-\frac{s^2}{2}}. \tag{73}$$

Proof The right-hand inequality in (72) follows from the isoperimetric inequality $P_\gamma(D_{\omega,s}) > P_\gamma(H_{\omega,s})$ which we may write as $2e^{-\frac{a(s)^2}{2}} > e^{-\frac{s^2}{2}}$. This implies

$$a(s) < \sqrt{s^2 + 2\ln 2} < s + \frac{\ln 2}{s}.$$

In order to show the right-hand inequality in (73) we first note that the function $\psi : [0, \infty) \to \mathbb{R}$,

$$\psi(x) = x e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}}\, dt$$

is increasing. Indeed, its first derivative is

$$\psi'(x) = -x + (1 + x^2) e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}}\, dt,$$

and by a second order analysis it is easy to show that the quantity $\psi'(x)e^{-\frac{x^2}{2}}$ is positive. The volume condition $\gamma(D_{\omega,s}) = \phi(s)$ can be written as

$$\frac{2}{a(s)}e^{-\frac{a(s)^2}{2}}\psi(a(s)) = \frac{1}{s}e^{-\frac{s^2}{2}}\psi(s).$$

Since ψ is increasing and $a(s) > s$ we deduce by the upper bound for $a(s)$ that

$$2e^{-\frac{a(s)^2}{2}} < \frac{a(s)}{s}e^{-\frac{s^2}{2}} < \left(1 + \frac{\ln 2}{s^2}\right)e^{-\frac{s^2}{2}}.$$

Hence we have the right-hand inequality in (73).

To prove the left-hand inequality in (72) we use the above estimate to obtain

$$a(s)^2 > s^2 + 2\ln\left(\frac{2}{1+\ln 2/s^2}\right) \geq s^2 + 2\ln\left(2(1-\ln 2/s^2)\right).$$

In order to prove the inequality we need to show that

$$\sqrt{s^2 + 2\ln\left(2(1-\ln 2/s^2)\right)} > s + \frac{1}{s}\ln\left(2 - 2/s^2\right).$$

This is equivalent to

$$2\ln\left(\frac{1-\ln 2/s^2}{1-1/s^2}\right) > \frac{1}{s^2}\ln^2\left(2-2/s^2\right).$$

Use the fact that for $0 < y < 1/9$ it holds $\ln(1+y) \geq 9y/10$ to estimate

$$2\ln\left(\frac{1-\ln 2/s^2}{1-1/s^2}\right) = 2\ln\left(1 + \frac{(1-\ln 2)}{s^2-1}\right) \geq \frac{9}{5}\frac{1-\ln 2}{s^2-1} \geq \frac{9}{5}\frac{1-\ln 2}{s^2}.$$

The claim follows from the fact that $\ln^2\left(2-2/s^2\right) < 9(1-\ln 2)/5$.

In order prove the left-hand inequality in (73) we first obtain, by integrating by parts twice, that

$$\int_x^\infty e^{-\frac{t^2}{2}}\,dt = \left(\frac{1}{x} - \frac{1}{x^3}\right)e^{-\frac{x^2}{2}} + 3\int_x^\infty \frac{1}{t^4}e^{-\frac{t^2}{2}}\,dt. \tag{74}$$

This implies

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)e^{-\frac{x^2}{2}} < \int_x^\infty e^{-\frac{t^2}{2}}\,dt < \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}\right)e^{-\frac{x^2}{2}}.$$

Then the volume condition $\gamma(D_{\omega,s}) = \phi(s)$ yields

$$\left(\frac{1}{a(s)} - \frac{1}{a(s)^3} + \frac{3}{a(s)^5}\right)2e^{-\frac{a(s)^2}{2}} > 2\int_{a(s)}^\infty e^{-\frac{t^2}{2}}\,dt = \int_s^\infty e^{-\frac{t^2}{2}}\,dt > \left(\frac{1}{s} - \frac{1}{s^3}\right)e^{-\frac{s^2}{2}}$$

and therefore we have by (72) that

$$\begin{aligned} P_\gamma(D_{\omega,s}) = 2e^{-\frac{a(s)^2}{2}} &> \frac{a(s)}{s} e^{-\frac{s^2}{2}} \left(1 - \frac{1}{s^2}\right) \cdot \left(1 - \frac{1}{a(s)^2} + \frac{3}{a(s)^4}\right)^{-1} \\ &\geq \frac{a(s)}{s} e^{-\frac{s^2}{2}} \left(1 - \frac{1}{s^2}\right) \cdot \left(1 + \frac{1}{a(s)^2} - \frac{3}{a(s)^4}\right) \\ &\geq \left(1 + \frac{\ln 2}{s^2} - \frac{8}{s^4}\right) e^{-\frac{s^2}{2}}. \end{aligned}$$

□

Finally we prove the perimeter bounds in (10).

Lemma 6 *Let E be a minimizer of (7). Then it holds*

$$\frac{5}{6} e^{-\frac{s^2}{2}} \leq P_\gamma(E) \leq \left(1 + \frac{\ln 2}{s^2}\right) e^{-\frac{s^2}{2}}.$$

Proof The bound from above follows by the minimality and by (73):

$$P_\gamma(E) \leq \mathcal{F}(E) \leq \mathcal{F}(D_{\omega,s}) = P_\gamma(D_{\omega,s}) \leq \left(1 + \frac{\ln 2}{s^2}\right) e^{-\frac{s^2}{2}}.$$

The proof of the lower bound is slightly more difficult. Let $\bar{s}$ be such that $\gamma(E) = \phi(\bar{s})$. The value $\bar{s}$ has to be non-negative, otherwise $\mathcal{F}(E) > \mathcal{F}(\mathbb{R}^n \setminus E)$. If $\bar{s} \leq s$, then the claim follows easily by the Gaussian isoperimetric inequality. If instead $\bar{s} > s$, then again by the isoperimetric inequality we have

$$\mathcal{F}(E) \geq P_\gamma(E) + \Lambda\sqrt{2\pi}\,(\phi(\bar{s}) - \phi(s)) \geq e^{-\frac{\bar{s}^2}{2}} + (s+1) \int_s^{\bar{s}} e^{-\frac{t^2}{2}}\, dt.$$

Define function $f : [s, \infty) \to \mathbb{R}$, $f(x) := e^{-\frac{x^2}{2}} + (s+1) \int_s^x e^{-\frac{t^2}{2}}\, dt$. By differentiating we get

$$f'(x) = (-x + s + 1) e^{-\frac{x^2}{2}}.$$

The function is thus increasing up to $x = s+1$ and then decreasing. Denote $\hat{s} = s + \frac{1}{6s}$. Let us show that $f(x) > \mathcal{F}(D_{\omega,s})$ for every $x \geq \hat{s}$.

Note that $f'(x) \geq \frac{1}{2} e^{-\frac{s^2}{2}}$ for every $x \in (s, \hat{s})$. Therefore since $f(s) = e^{-\frac{s^2}{2}}$ we get

$$f(\hat{s}) \geq \left(1 + \frac{1}{12s}\right) e^{-\frac{s^2}{2}}.$$

Moreover we have by (74) that

$$\lim_{x\to\infty} f(x) = (s+1)\int_s^\infty e^{-\frac{t^2}{2}}\,dt \geq \left(1+\frac{1}{2s}\right)e^{-\frac{s^2}{2}}.$$

By the earlier analysis we deduce that for every $x \geq \hat{s}$ it holds

$$f(x) \geq \min\{f(\hat{s}), \lim_{x\to\infty} f(x)\} > \left(1+\frac{\ln 2}{s^2}\right)e^{-\frac{s^2}{2}}.$$

Hence we conclude by (73) that $f(x) > P_\gamma(D_{\omega,s}) = \mathcal{F}(D_{\omega,s})$ for every $x \geq \hat{s}$. This in turn implies that necessarily $\bar{s} < \hat{s}$. By the isoperimetric inequality we then have that

$$P_\gamma(E) \geq e^{-\frac{\hat{s}^2}{2}} \geq \frac{5}{6}e^{-\frac{s^2}{2}}.$$

□

References

1. Bakry, D., Ledoux, M.: Lévy-Gromov isoperimetric inequality for an infinite dimensional diffusion generator. Invent. Math. **123**, 259–281 (1995)
2. Barchiesi, M., Brancolini, A., Julin, V.: Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality. Ann. Probab. **45**, 668–697 (2017)
3. Barchiesi, M., Julin, V.: Robustness of the Gaussian concentration inequality and the Brunn–Minkowski inequality. Calc. Var. Partial Differ. Equ. **56**, Art. n.80 (2017)
4. Barthe, F.: An isoperimetric result for the Gaussian measure and unconditional sets. Bull. Lond. Math. Soc. **33**, 408–416 (2001)
5. Bernstein, J., Wang, L.: A sharp lower bound for the entropy of closed hypersurfaces up to dimension six. Invent. Math. **206**, 601–627 (2016)
6. Bögelein, V., Duzaar, F., Fusco, N.: A quantitative isoperimetric inequality on the sphere. Adv. Calc. Var. **10**, 223–265 (2017)
7. Borell, C.: The Brunn–Minkowski inequality in Gauss space. Invent. Math. **30**, 207–216 (1975)
8. Chakrabarti, A., Regev, O.: An optimal lower bound on the communication complexity of gap-Hamming-distance. In: STOC'11—Proceedings of the 43rd ACM Symposium on Theory of Computing, pp. 51–60. ACM, New York (2011)
9. Choksi, R., Muratov, C.B., Topaloglu, I.: An old problem resurfaces nonlocally: Gamow's liquid drops inspire today's research and applications. Not. Am. Math. Soc. **64**, 1275–1283 (2017)
10. Choksi, R., Peletier, M.: Small volume fraction limit of the diblock copolymer problem: I. Sharp-interface functional. SIAM J. Math. Anal. **42**, 1334–1370 (2012)
11. Cianchi, A., Fusco, N., Maggi, F., Pratelli, A.: On the isoperimetric deficit in Gauss space. Am. J. Math. **133**, 131–186 (2011)
12. Cicalese, C., Leonardi, G.P.: Best constants for the isoperimetric inequality in quantitative form. J. Eur. Math. Soc. **15**, 1101–1129 (2013)
13. Colding, T.H., Ilmanen, T., Minicozzi, W.P., William, P., White, B.: The round sphere minimizes entropy among closed self-shrinkers. J. Differ. Geom. **95**, 53–69 (2013)
14. Colding, T.H., Minicozzi, W.P.: Generic mean curvature flow I: generic singularities. Ann. Math. **175**, 755–833 (2012)
15. Ehrhard, A.: Symétrisation dans l'espace de Gauss. Math. Scand. **53**, 281–301 (1983)
16. Eldan, R.: A two-sided estimate for the Gaussian noise stability deficit. Invent. Math. **201**, 561–624 (2015)

17. Fusco, N.: The quantitative isoperimetric inequality and related topics. Bull. Math. Sci. **5**, 517–607 (2015)
18. Fusco, N., Julin, V.: A strong form of the quantitative isoperimetric inequality. Calc. Var. Partial Differ. Equ. **50**, 925–937 (2014)
19. Filmus, Y., Hatami, H., Heilman, S., Mossel, E., O'Donnell, R., Sachdeva, S., Wan, A., Wimmer, K.: Real Analysis in Computer Science: A collection of Open Problems. Available online (2014)
20. Gamow, G.: Mass defect curve and nuclear constitution. Proc. R. Soc. Lond. A **126**, 632–644 (1930)
21. Giusti, E.: Minimal Surfaces and Functions of Bounded Variations. Birkhäuser, Boston (1994)
22. Heilman, S.: Low Correlation Noise Stability of Symmetric Sets. Preprint (2015)
23. Heilman, S.: Symmetric convex sets with minimal Gaussian surface area. Preprint (2017)
24. Julin, V.: Isoperimetric problem with a Coulombic repulsive term. Indiana Univ. Math. J. **63**, 77–89 (2014)
25. Knüpfer, H., Muratov, C.B.: On an isoperimetric problem with a competing nonlocal term II: The general case. Commun. Pure Appl. Math. **67**, 1974–1994 (2014)
26. La Manna, D.A.: Local minimality of the ball for the Gaussian perimeter. Adv. Calc. Var. **12**, 193–210 (2019)
27. Latala, R., Oleszkiewicz, K.: Gaussian measures of dilatations of convex symmetric sets. Ann. Probab. **27**, 1922–1938 (1999)
28. Lawson, H.B.: Lectures on Minimal Submanifolds, Vol. I. Mathematics Lecture Series, vol. 9, 2nd edn. Publish or Perish Inc, Wilmington (1980)
29. Maggi, F.: Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory. Cambridge Studies in Advanced Mathematics, vol. 135. Cambridge University Press, Cambridge (2012)
30. Mossel, E., Neeman, J.: Robust dimension free isoperimetry in Gaussian space. Ann. Probab. **43**, 971–991 (2015)
31. Mossel, E., Neeman, J.: Robust optimality of Gaussian noise stability. J. Eur. Math. Soc. **17**, 433–482 (2015)
32. Rosales, C.: Isoperimetric and stable sets for log-concave perturbatios of Gaussian measures. Anal. Geom. Metr. Spaces **2**, 328–358 (2014)
33. Sudakov, V.N., Tsirelson, B.S.: Extremal properties of half-spaces for spherically invariant measures. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **41**, 14–24, 165 (1974)
34. Tkocz, T.: Gaussian measures of dilations of convex rotationally symmetric sets in $\mathbb{C}^N$. Electron. Commun. Probab. **16**, 38–49 (2011)
35. Zhu, J.: On the entropy of closed hypersurfaces and singular self-shrinkers. J. Differ. Geom. (**to appear**)