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Rigidity and a mesoscopic central limit theorem for Dyson Brownian motion for general β and potentials

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Abstract
We study Dyson Brownian motion with general potential V and for general $\beta \geq 1$. For short times $t = o(1)$ and under suitable conditions on V we obtain a local law and corresponding rigidity estimates on the particle locations; that is, with overwhelming probability, the particles are close to their classical locations with an almost-optimal error estimate. Under the condition that the density of states of the initial data is bounded below and above down to the scale $\eta_* \ll t \ll 1$, we prove a mesoscopic central limit theorem for linear statistics at all scales η with $N^{-1} \ll \eta \ll t$.

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1 Introduction

In 1962, Dyson interpreted the $N \times N$ Gaussian ensemble (real, complex or quaternion) as the dynamical limit of matrix-valued Brownian motion $H(t)$, and observed that the eigenvalues of $H(t)$ form an interacting N-particle system with a logarithmic Coulomb interaction and quadratic potential. That is, the eigenvalue process $\{\lambda_i(t)\}_{1 \leq i \leq N}$ satisfies the following system of stochastic differential equations with quadratic $V = x^2/2$ and classical $\beta = 1$, 2 or 4 (depending on the symmetry class of the Gaussian ensemble)

$$\mathrm{d}\lambda_i(t) = \sqrt{\frac{2}{\beta N}}\mathrm{d}B_i(t) + \frac{1}{N}\sum_{j:j\neq i}\frac{\mathrm{d}t}{\lambda_i(t) - \lambda_j(t)} - \frac{1}{2}V'(\lambda_i(t))\mathrm{d}t, \quad i = 1, 2, \ldots, N, \quad (1.1)$$

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where $(B_1, \dots, B_N)$ is an N-dimensional Brownian motion defined on a probability space with a filtration $\mathscr{F} = \{\mathscr{F}_t, t \geq 0\}$. The initial data $\lambda(0) = (\lambda_1(0), \lambda_2(0) \dots, \lambda_N(0)) \in \overline{\Delta_N}$ is given by the eigenvalues of $H(0)$. Here, Δ_N denotes the Weyl chamber

$$\Delta_N = \{\{x_i\}_{1 \leq i \leq N} \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}. \tag{1.2}$$

The process $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_N(t))$ defined by the stochastic differential equation system (1.1) is called the β-Dyson Brownian motion (β-DBM) with potential V, which is an interacting particle system with Hamiltonian of the form

$$E(x_1, \dots, x_N) := -\frac{1}{2N} \sum_{1 \leq i \neq j \leq N} \log |x_i - x_j| + \frac{1}{2} \sum_{i=1}^{N} V(x_i). \tag{1.3}$$

For the special case $\beta = 2$ and $V = x^2/2$, at each fixed time t, the particles $\lambda(t)$ have the same distribution as the eigenvalues of

$$H(t) \stackrel{d}{=} e^{-t/2} H(0) + \sqrt{1 - e^{-t}} G, \tag{1.4}$$

where G is a matrix drawn from the Gaussian Unitary Ensemble (GUE). The global eigenvalue density of the GUE follows Wigner's semi-circle distribution [54], and the local eigenvalue statistics are given by the Sine kernel [18–20]. Clearly, $H(t) \to G$ as $t \to \infty$ for any choice of the initial data $H(0)$, and so the system reaches a global equilibrium for $t \gg 1$. One can also investigate the time to local equilibrium - that is, how long it takes for the local statistics to coincide with the GUE. Dyson conjectured [17] that the time to local equilibrium should be much faster than the order 1 global scale. It is expected that in the bulk, an eigenvalue statistic on the scale η should coincide with the GUE as long as $t \gg \eta$. To be more precise, one expects the convergence of the following three types of statistics on three types of scales.

1. On the *macroscopic* scale, the global eigenvalue density should converge to Wigner's semi-circle distribution provided $t \gg 1$.
2. The linear eigenvalue statistics of test functions on the *mesoscopic* scale $N^{-1} \ll \eta \ll 1$ should coincide with the GUE as long as $t \gg \eta$.
3. On the *microscopic* scale $\mathrm{O}(N^{-1})$, the local eigenvalue statistics should be given by the sine kernel as long as $t \gg N^{-1}$.

For the macroscopic scale, with quadratic potential V, it was proven by Rogers and Shi [48], that the global density converges to the semicircle distribution. For general potential V, it was proven by Li, Li and Xie [43,44], the global eigenvalue density converges to a V-dependent equilibrium measure (which may not be the semicircle distribution for non-quadratic V). We refer to [4] for a nice presentation on the dynamical approach to Wigner's semi-circle law.

The time to equilibrium at the microscopic scale was studied in a series of works [21–27,29,31,32], by Erdős, Yau and their collaborators. For classical $\beta = 1, 2, 4$, quadratic V and initial data a Wigner matrix, it was proven that after a short time $t \gg$

N^{-1} the local statistics coincide with the GβE. Later, the works [28,39] established single gap universality for classical DBM for a broad class of initial data, relying on the discrete di-Giorgi–Nash–Moser theorem developed in [29]. Fixed energy universality was established in [12,38] by developing a sophisticated homogenization theory for discrete parabolic systems. These results are a crucial component in proving bulk universality for various classes of random matrix ensembles. Another approach to universality, applicable in special cases was developed independently and in parallel by Tao and Vu [50].

A central and basic tool in the study of the local statistics of random matrices is the local law and the associated rigidity estimates. The local law is usually formulated in terms of concentration of the Stieltjes transform of the empirical eigenvalue density at short scales $\eta \gtrsim N^{-1}$. Rigidity estimates give high probability concentration estimates for the eigenvalue locations. These results were first established for Wigner matrices in a series of papers [25,26,31,32,51], then extended to other matrix models, i.e. general Wigner-type matrices [1,35], sparse random matrices [21], deformed Wigner ensembles [39,42], correlated random matrices [2,3]. Beyond matrix models, rigidity estimates have been established for one-cut and multi-cut β-ensembles [9–11,45], and two-dimensional Couloub gas [5,40].

For the special case of classical $\beta = 1, 2, 4$ and quadratic potential V, the solution of (1.1) is given by a matrix model and so the methods developed for deformed Wigner matrices [39,41,42] yield a local law for the Stieltjes transform of empirical eigenvalue density,

$$\tilde{m}_t(z) := \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i(t)-z} = \int \frac{\mathrm{d}\tilde{\mu}_t(x)}{x-z}, \quad \tilde{\mu}_t = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(t)}. \tag{1.5}$$

However for nonclassical β or non-quadratic V, the process (1.1) is not given by a matrix model and so a corresponding local law is not known.

Our first main result is to establish a local law for the Stieltjes transform $\tilde{m}_t(z)$ for short scales and all short times $t \ll 1$. This result is stated as Theorem 3.1 below. This implies a rigidity estimate for the particle locations $\lambda_i(t)$, i.e., that they are close to deterministic classical locations with high probability. Our methods are purely dynamical and do not rely on any matrix representation. Instead, our method is based on analyzing the stochastic differential equation of the Stieltjes transform $\tilde{m}_t$ along the characteristics of the limiting continuum equation. We remark that since the β-ensemble is the equilibrium measure of β-DBM, our results may be used to provide another proof for the rigidity of β-ensemble in the case $\beta \geq 1$, provided that one takes some large deviation estimates (such as [47]) as input. The method of characteristics was used in [36] to prove that on the global scale the empirical measure process of β-DBM with quadratic potential V converges to a Gaussian process. The explicit formulas of the means and the covariances of the limit Gaussian process was derived in [7]. Very recently, Unterberger proved that the Global fluctuations of β-DBM with general potential V are asymptotically Gaussian [52,53]. We also comment that the method of characteristics has recently been used, independently and in parallel, for the analysis of a different equation in [8].

Relying on our local law we then prove a mesoscopic central limit theorem for linear statistics of the particle process on scales $\eta \ll t$. This is stated as Theorem 4.2 below. In particular we see that equilibrium holds for the process (1.1) on mesoscopic scales $\eta \ll t$. Central limit theorems for mesoscopic linear statistics of Wigner matrices at all scales were established in a series of papers [13,14,34,46]. Analogous results for invariant ensembles were proved in [6,33,37]. Mesoscopic statistics for DBM with $\beta = 2$ and quadratic potential was established in [16]. It was proven that at mesoscopic scale η, the mesoscopic central limit theorem holds if and only if $t \gg \eta$. Recently, related results were proven for classical β and the quadratic potential in [38]. The analysis in [16] relied on the Brézin–Hikami formula special to the $\beta = 2$ case, and the analysis in [38] relied on the matrix model which exists only for classical β, i.e. $\beta = 1, 2, 4$, neither of which are applicable here. Our approach is based on a direct analysis of the stochastic differential equation of $\tilde{m}_t$, where the leading fluctuation term is an integral with respect to Brownian motions. The central limit theorem follows naturally for all $\beta \geq 1$ and general potential V.

We now outline the organization of the rest of the paper. In Sect. 2, we collect some properties of β-DBM (1.1), i.e., the existence and uniqueness of strong solutions and the existence and uniqueness of the hydrodynamic limit of the empirical density $\tilde{\mu}_t$, which is a measure valued process μ_t. For quadratic V, these statements were proved by Chan [15] and Rogers and Shi [48]. For general potentials (under Assumption 2.1 below), the β-DBM was studied by Li, Li and Xie [43,44]. In the second part of Sect. 2, we study the Stieltjes transform of the limit measure valued process μ_t by the method of characteristics, which are used throughout the rest of the paper.

Section 3 contains the main novelty of this paper, in which we prove the local law and rigidity estimate of the particles Theorem 3.1. We directly analyze the stochastic differential equation satisfied by $\tilde{m}_t$ using the method of characteristics. In Sect. 4, we prove that the linear statistics satisfy a central limit theorem at mesoscopic scales.

Finally we remark that by combining the rigidity results proven here and the methodology of [38] one can prove gap universality for the process (1.1), thus yielding equilibrium on the local scale $\eta = 1/N$. We state the gap universality theorem in Appendix A, and sketch the proof.

In the rest of this paper, we use C to represent large universal constant, and c a small universal constant, which may depend on other universal constants, i.e., the constants $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{K}$ in Assumptions 2.1 and 4.1, and may be different from line by line. We write that $X = O(Y)$ if there exists some universal constant such that $|X| \leq CY$. We write $X = o(Y)$, or $X \ll Y$ if the ratio $|X|/Y \to 0$ as N goes to infinity. We write $X \asymp Y$ if there exist universal constants such that $cY \leq |X| \leq CY$. We denote the set $\{1, 2, \ldots, N\}$ by $[\![1, N]\!]$. We say an event Ω holds with overwhelming probability, if for any $D > 0$, and $N \geq N_0(D)$ large enough, $\mathbb{P}(\Omega) \geq 1 - N^{-D}$.

2 Background on β-Dyson Brownian motion

In this section we collect several properties of β-DBM, required in the remainder of the paper. More precisely, we state the existence and uniqueness of the strong solution to (1.1) and a weak convergence result for the empirical particle density.

In the rest of the paper, we make the following assumption on the potential V.

Assumption 2.1 We assume that the potential V is a C^4 function, and that there exists a constant $\mathfrak{K} \geq 0$ such that $\inf_{x\in\mathbb{R}} V''(x) \geq -2\mathfrak{K}$.

We denote $M_1(\mathbb{R})$ the set of probability measures on $\mathbb{R}$ and equip this set with the weak topology. For $T > 0$ we denote by $C([0, T], M_1(\mathbb{R}))$ the set of continuous processes on $[0, T]$ taking values in $M_1(\mathbb{R})$. We have the following existence result from [43].

Theorem 2.2 *Suppose that V satisfies Assumption 2.1. For all $\beta \geq 1$ and initial data $\lambda(0) \in \overline{\Delta_N}$, there exists a strong solution $(\lambda(t))_{t\geq 0} \in C(\mathbb{R}_+, \overline{\Delta_N})$ to the stochastic differential equation (1.1). For any $t > 0$, $\lambda(t) \in \Delta_N$ and $\lambda(t)$ is a continuous function of $\lambda(0)$.*

Proof The existence of strong solution with initial data $\lambda(0) \in \Delta_N$ follows from [43, Theorem 1.2]. Following the same argument in [4, Proposition 4.3.5], we can extend the statement to $\lambda(0) \in \overline{\Delta_N}$ by the following comparison lemma (the special case with potential $V \equiv 0$ is proved in [4, Lemma 4.3.6] and the proof below is based on the proof given there) between strong solutions of (1.1) with initial data in Δ_N. □

Lemma 2.3 *Suppose that V satisfies the Assumption 2.1. Let $(\lambda(t))_{t\geq 0}$ and $(\eta(t))_{t\geq 0}$ be two strong solutions of (1.1) with initial data $\lambda(0) \in \Delta_N$ and $\eta(0) \in \Delta_N$. Assume that $\lambda_i(0) > \eta_i(0)$ for all $i \in [\![1, N]\!]$. Then, almost surely, for all $t \geq 0$ and $i \in [\![1, N]\!]$,*

$$0 \leq \lambda_i(t) - \eta_i(t) \leq e^{\mathfrak{K}t} \max_{j\in[\![1,N]\!]}\{\lambda_j(0) - \eta_j(0)\}. \tag{2.1}$$

Proof By taking difference of the stochastic differential equations satisfied by $(\lambda(t))_{t\geq 0}$ and $(\eta(t))_{t\geq 0}$, we have

$$\begin{aligned} \partial_t(\lambda_i(t) - \eta_i(t)) = \frac{1}{N}\sum_{j:j\neq i} \frac{(\lambda_j(t) - \eta_j(t)) - (\lambda_i(t) - \eta_i(t))}{(\lambda_i(t) - \lambda_j(t))(\eta_i(t) - \eta_j(t))} \\ -\frac{1}{2}\left(V'(\lambda_i(t)) - V'(\eta_i(t))\right). \end{aligned} \tag{2.2}$$

Let $i_0 = \operatorname{argmax}_{i\in[\![N]\!]}\{\lambda_i(t) - \eta_i(t)\}$. For $i = i_0$, the first term of (2.2) is non-positive, and

$$\partial_t(\lambda_{i_0}(t) - \eta_{i_0}(t)) \leq -\frac{1}{2}\left(V'(\lambda_{i_0}(t)) - V'(\eta_{i_0}(t))\right). \tag{2.3}$$

Either $\lambda_{i_0}(t) - \eta_{i_0}(t) < 0$, or using Assumption 2.1 the above equation implies $\partial_t(\lambda_{i_0}(t) - \eta_{i_0}(t)) \leq \mathfrak{K}(\lambda_{i_0}(t) - \eta_{i_0}(t))$. Hence,

$$\partial_t(\lambda_{i_0}(t) - \eta_{i_0}(t))_+ \leq \mathfrak{K}(\lambda_{i_0}(t) - \eta_{i_0}(t))_+. \tag{2.4}$$

Therefore, it follows from Gronwall's inequality,

$$\max_{i\in[\![N]\!]}\{\lambda_i(t)-\eta_i(t)\}\le e^{\mathfrak{K}t}\max_{i\in[\![N]\!]}\{\lambda_i(0)-\eta_i(0)\}. \tag{2.5}$$

Similarly, let $i_0 = \mathrm{argmin}_{i\in[\![N]\!]}\{\lambda_i(t)-\eta_i(t)\}$. Either $\lambda_{i_0}(t)-\eta_{i_0}(t) > 0$, or $\partial_t(\lambda_{i_0}(t)-\eta_{i_0}(t)) \ge \mathfrak{K}(\lambda_{i_0}(t)-\eta_{i_0}(t))$. Again by Gronwall's inequality we obtain that $\min_{i\in[\![N]\!]}\{\lambda_i(t)-\eta_i(t)\}\ge 0$. □

The following theorem is a consequence of [43, Theorem 1.1 and 1.3]. It establishes the existence of a solution to the limiting hydrodynamic equation of the empirical particle process. In its statement we distinguish the parameter L from N. This is due to the fact that we will compare the empirical measure $\tilde{\mu}_t$ to a solution of the equation (2.8) with initial data coming from the initial value of $\lambda(0)$ which is a finite N object. The existence of this solution is easily established using the theorem below by introducing an auxilliary process $\lambda^{(L)}$ which converges to $\tilde{\mu}_0$ (a fixed finite N object) as $L\to\infty$.

Theorem 2.4 *Suppose V satisfies the Assumption 2.1. Let $\beta\ge 1$. Let $\lambda^{(L)}(0) = (\lambda_1^{(L)}(0), \lambda_2^{(L)}(0)\dots,\lambda_L^{(L)}(0))\in\overline{\Delta_N}$ be a sequence of initial data satisfying*

$$\sup_{L>0}\frac{1}{L}\sum_{i=1}^{L}\log(\lambda_i^{(L)}(0)^2+1)<\infty. \tag{2.6}$$

Assume that the empirical measure $\tilde{\mu}_0^{(L)} = \frac{1}{L}\sum_{i=1}^{L}\delta_{\lambda_i^{(L)}(0)}$ converges weakly as L goes to infinity to $\mu_0\in M_1(\mathbb{R})$.

Let $\lambda^{(L)}(t) = (\lambda_1^{(L)}(t),\dots,\lambda_L^{(L)}(t))_{t\ge 0}$ be the solution of (1.1) with initial data $\lambda^{(L)}(0)$, and set

$$\tilde{\mu}_t^{(L)} = \frac{1}{L}\sum_{i=1}^{L}\delta_{\lambda_i^{(L)}(t)}. \tag{2.7}$$

Then for any fixed time T, $(\tilde{\mu}_t^{(L)})_{t\in[0,T]}$ converges almost surely in $C([0,T],M_1(\mathbb{R}))$. Its limit is the unique measure-valued process $(\mu_t)_{t\in[0,T]}$ characterized by the McKean–Vlasov equation, i.e., for all $f\in C_b^2(\mathbb{R})$, $t\in[0,T]$,

$$\begin{aligned}\partial_t\int_{\mathbb{R}} f(x)\mathrm{d}\mu_t(x) &= \frac{1}{2}\int\int_{\mathbb{R}^2}\frac{\partial_x f(x)-\partial_y f(y)}{x-y}\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y)\\ &\quad -\frac{1}{2}\int_{\mathbb{R}} V'(x)f'(x)\mathrm{d}\mu_t(x).\end{aligned} \tag{2.8}$$

Taking $f(x)=(x-z)^{-1}$ for $z\in\mathbb{C}\setminus\mathbb{R}$ in (2.8), we see that the Stieltjes transform of the limiting measure-valued process, which is defined by

$$m_t(z)=\int_{\mathbb{R}}(x-z)^{-1}\mathrm{d}\mu_t(x), \tag{2.9}$$

satisfies the equation

$$\partial_t m_t(z) = m_t(z)\partial_z m_t(z) + \frac{1}{2}\int_{\mathbb{R}} \frac{V'(x)}{(x-z)^2}\mathrm{d}\mu_t(x). \tag{2.10}$$

In a moment we will introduce a spatial cut-off of V. In order to do this we require the following exponential bound for $||\lambda_i(t)||_\infty$.

Proposition 2.5 *Suppose V satisfies Assumption 2.1. Let $\beta \geq 1$, and $\lambda(0) \in \overline{\Delta_N}$. Let $\mathfrak{a}$ be a constant such that the initial data $\|\lambda(0)\|_\infty \leq \mathfrak{a}$. Then for any fixed time T, there exists a finite constant $\mathfrak{b} = \mathfrak{b}(\mathfrak{a}, T)$, such that for any $0 \leq t \leq T$, the unique strong solution of (1.1) satisfies:*

$$\mathbb{P}(\max\{|\lambda_1(t)|, |\lambda_N(t)|\} \geq \mathfrak{b}) \leq e^{-N}. \tag{2.11}$$

Proof Let $(\eta(t))_{t\geq 0}$ be the strong solution of β-DBM with potential $V = 0$,

$$\mathrm{d}\eta_i(t) = \sqrt{\frac{2}{\beta N}}\mathrm{d}B_i(t) + \frac{1}{N}\sum_{j:j\neq i}\frac{\mathrm{d}t}{\eta_i(t)-\eta_j(t)}, \quad i = 1, 2, \dots, N. \tag{2.12}$$

We take the initial data as $\eta(0) = \lambda(0) \in \overline{\Delta_N}$. Thanks to [4, Lemma 4.3.17], there exists a finite constant $\mathfrak{b}_1 = \mathfrak{b}_1(\mathfrak{a}, T)$, such that

$$\Omega := \{\max\{|\eta_1(t)|, |\eta_N(t)|\} \leq \mathfrak{b}_1\}, \quad \mathbb{P}(\Omega) \geq 1 - e^{-N}. \tag{2.13}$$

By taking difference of the stochastic differential equations satisfied by $(\lambda(t))_{t\geq 0}$ and $(\eta(t))_{t\geq 0}$, we get

$$\partial_t(\lambda_i(t) - \eta_i(t)) = \frac{1}{N}\sum_{j:j\neq i}\frac{(\lambda_j(t)-\eta_j(t)) - (\lambda_i(t)-\eta_i(t))}{(\lambda_i(t)-\lambda_j(t))(\eta_i(t)-\eta_j(t))}\mathrm{d}t - \frac{1}{2}V'(\lambda_i(t))\mathrm{d}t. \tag{2.14}$$

Let $i_0 = \operatorname{argmax}_{i\in[\![N]\!]}\{\lambda_i(t)-\eta_i(t)\}$. For $i = i_0$, the first term of (2.14) is non-positive, and thus on the event Ω,

$$\begin{aligned}\partial_t(\lambda_{i_0}(t) - \eta_{i_0}(t)) &\leq -\frac{1}{2}\left(V'(\lambda_{i_0}(t)) - V'(\eta_{i_0}(t))\right) - \frac{1}{2}V'(\eta_{i_0}(t)) \\ &\leq -\frac{1}{2}\left(V'(\lambda_{i_0}(t)) - V'(\eta_{i_0}(t))\right) + C,\end{aligned} \tag{2.15}$$

where $C = \max_{x\in[-\mathfrak{b}_1,\mathfrak{b}_1]}|V'(x)|/2$. Then thanks to Assumption (2.1), either $\lambda_{i_0}(t) - \eta_{i_0}(t) < 0$, or $\partial_t(\lambda_{i_0}(t) - \eta_{i_0}(t)) \leq \mathfrak{K}(\lambda_{i_0}(t) - \eta_{i_0}(t)) + C$. Therefore, it follows from Gronwall's inequality,

$$\max_{i\in[\![N]\!]}\{\lambda_i(t) - \eta_i(t)\} \leq \frac{C(e^{\mathfrak{K}t}-1)}{\mathfrak{K}}. \tag{2.16}$$

And thus

$$\max_{i\in[\![N]\!]}\{\lambda_i(t)\} \le \mathfrak{b}_1 + \frac{C(e^{\mathfrak{K}t}-1)}{\mathfrak{K}}. \tag{2.17}$$

Similarly, let $i_0 = \operatorname{argmin}_{i\in[\![N]\!]}\{\lambda_i(t)-\eta_i(t)\}$, then either $\lambda_{i_0}(t)-\eta_{i_0}(t)>0$, or $\partial_t(\lambda_{i_0}(t)-\eta_{i_0}(t)) \ge \mathfrak{K}(\lambda_{i_0}(t)-\eta_{i_0}(t)) - C$. It follows from Gronwall's inequality that $\min_{i\in[\![N]\!]}\{\lambda_i(t)\} \ge -\mathfrak{b}_1 - C(e^{\mathfrak{K}t}-1)/\mathfrak{K}$. Proposition 2.5 follows by taking $\mathfrak{b} = \mathfrak{b}_1 + C(e^{\mathfrak{K}t}-1)/\mathfrak{K}$. □

Note that the constant $\mathfrak{b}$ in the previous proposition depends only on V through its C^1 norm on the interval $[-\mathfrak{b}_1, \mathfrak{b}_1]$ and $\mathfrak{K}$. Hence, if we replace $V'(x)$ by $V'(x)\chi(x)$ where χ is a smooth cut-off function on $[-2\mathfrak{b}, 2\mathfrak{b}]$ (we assume $\mathfrak{b}>1$), then by Proposition 2.5 the solutions of (1.1) with the original potential $V'(x)$ and the cut-off potential $V'(x)\chi(x)$ agree with exponentially high probability. Hence for the remainder of the paper it will suffice for our purposes to work with the cut-off potential $V'(x)\chi(x)$.

We introduce the following quasi-analytic extension of V' of order three,

$$V'(x+\mathrm{i}y) := \left(V'(x)\chi(x) + \mathrm{i}y\partial_x(V'(x)\chi(x)) - \frac{y^2}{2}\partial_x^2(V'(x)\chi(x))\right)\chi(y). \tag{2.18}$$

We denote,

$$\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y), \quad \partial_{\bar z} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y). \tag{2.19}$$

We rewrite (2.10) in the following

$$\partial_t m_t(z) = \partial_z m_t(z)\left(m_t(z) + \frac{V'(z)}{2}\right) + \frac{m_t(z)\partial_z V'(z)}{2} + \int_{\mathbb{R}} g(z,x)\mathrm{d}\mu_t(x), \tag{2.20}$$

where

$$g(z,x) := \frac{V'(x)-V'(z)-(x-z)\partial_z V'(z)}{2(x-z)^2}, \quad g(x,x) := \frac{V'''(x)}{4}. \tag{2.21}$$

By our definition (2.18), V' is quasi-analytic along the real axis. One can directly check the following properties of $g(z,x)$ and V.

Proposition 2.6 *Suppose V satisfies Assumption 2.1. Let $V'(z)$ and $g(z,x)$ be as defined in (2.18) and (2.21). There exists a universal constant C depending on V, such that*

1. $\|V'(z)\|_{C^1} \le C$, $|\operatorname{Im}[V'(z)]| \le C|\operatorname{Im}[z]|$ *and* $|\operatorname{Im}[\partial_z V'(z)]| \le C|\operatorname{Im}[z]|$.

2. *The following bounds hold uniformly over* $z \in \mathbb{C}$ *and* $x \in \mathbb{R}$*. We have* $|g(z,x)| + |\partial_x g(z,x)| \leq C$*. Furthermore,* $|\partial_x^2 g(z,x)| \leq C|z-x|^{-1}$ *and* $|\operatorname{Im}[g(z,x)]| \leq C|\operatorname{Im}[z]|$.
3. *If we further assume* V *is* C^5*, then* $\|V'(z)\|_{C^2} \leq C$*, and uniformly over* $z \in \mathbb{C}$ *and* $x \in \mathbb{R}$*,* $|\partial_z g(z,x)| + |\partial_{\bar{z}} g(z,x)| \leq C$.

We define the following quasi-analytic extension of $g(z,\cdot)$ of order two,

$$\tilde{g}(z, x+\mathrm{i}y) := (g(z,x) + \mathrm{i}y\partial_x g(z,x))\chi(y), \tag{2.22}$$

By the Helffer-Sjöstrand formula, see [30, Chapter 11.2],

$$\int_{\mathbb{R}} g(z,x)\mathrm{d}\mu_t(x) = \frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar{w}}\tilde{g}(z,w)m_t(w)\mathrm{d}^2 w, \tag{2.23}$$

and so we can rewrite (2.20) as an autonomous differential equation of $m_t(z)$:

$$\begin{aligned}\partial_t m_t(z) = \partial_z m_t(z)\left(m_t(z) + \frac{V'(z)}{2}\right) + \frac{m_t(z)\partial_z V'(z)}{2}\\ +\frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar{w}}\tilde{g}(z,w)m_t(w)\mathrm{d}^2 w.\end{aligned} \tag{2.24}$$

2.1 Stieltjes transform of the limit measure-valued process

In this subsection we analyze the differential equation of the Stieltjes transform of the limiting measure-valued process (2.24) with initial data μ_0 which we assume to have $\operatorname{supp}\mu_0 \in [-\mathfrak{a}, \mathfrak{a}]$. We fix a constant time T. By Theorem 2.4 and Proposition 2.5, there exists a finite constant $\mathfrak{b} = \mathfrak{b}(\mathfrak{a}, T)$ such that $\operatorname{supp}\mu_t \in [-\mathfrak{b}, \mathfrak{b}]$ for any $0 \leq t \leq T$.

We analyze (2.24) by the method of characteristics. Let

$$\partial_t z_t(u) = -m_t(z_t(u)) - \frac{V'(z_t(u))}{2}, \qquad z_0 = u \in \mathbb{C}_+. \tag{2.25}$$

If the context is clear, we omit the parameter u, i.e., we simply write z_t instead of $z_t(u)$.

For any $\varepsilon > 0$, let $\mathbb{C}_+^\varepsilon = \{z : \operatorname{Im}[z] > \varepsilon\}$. Since m_t is analytic, bounded and Lipschitz on the closed domain $\overline{\mathbb{C}_+^\varepsilon}$, we have that for any u with $u \in \mathbb{C}_+^\varepsilon$, the solution $z_t(u)$ exists, is unique, and is well defined before exiting the domain. Thanks to the local uniqueness of the solution curve, it follows, by taking $\varepsilon \to 0$, that for any u with $\operatorname{Im}[u] > 0$, the solution curve $z_t(u)$ is well defined before it exits the upper half plane. For any $u \in \mathbb{C}_+$, either the flow $z_t(u)$ stays in the upper half plane forever, or there exists some finite time t such that $\lim_{s\to t}\operatorname{Im}[z_s(u)] = 0$.

Plugging (2.25) into (2.24), and applying the chain rule we obtain

$$\partial_t m_t(z_t) = \frac{m_t(z_t)\partial_z V'(z_t)}{2} + \frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar{w}}\tilde{g}(z_t,w)m_t(w)\mathrm{d}^2 w. \tag{2.26}$$

The behaviors of z_s and $m_s(z_s)$ are governed by the system of equations (2.25) and (2.26). The following properties of the flows $\mathrm{Im}[z_s]$ and $\mathrm{Im}[m_s(z_s)]$ will be used throughout the paper.

Proposition 2.7 *Suppose V satisfies the Assumption 2.1. Fix a time $T > 0$. There exists a constant $C = C(V, \mathfrak{b})$ such that the following holds. For any $0 \le s \le t \le T$ with $\mathrm{Im}[z_t] > 0$, the following estimates hold uniformly for initial $u = z_0$ in compact subsets of $\mathbb{C}_+$.*

$$\mathrm{Im}[m_t(z)]\,\mathrm{Im}[z] \le 1, \quad |\partial_z m_t(z)| \le \frac{\mathrm{Im}[m_t(z)]}{\mathrm{Im}[z]}, \tag{2.27}$$

$$e^{-C(t-s)}\,\mathrm{Im}[z_t] \le \mathrm{Im}[z_s], \tag{2.28}$$

$$e^{-tC}\,\mathrm{Im}[m_0(z_0)] \le \mathrm{Im}[m_t(z_t)] \le e^{tC}\,\mathrm{Im}[m_0(z_0)], \tag{2.29}$$

$$\begin{aligned} &e^{-Ct}\left(\mathrm{Im}[z_0] - \frac{e^{Ct}-1}{C}\,\mathrm{Im}[m_0(z_0)]\right) \\ &\quad \le \mathrm{Im}[z_t] \le e^{Ct}\left(\mathrm{Im}[z_0] - \frac{1-e^{-Ct}}{C}\,\mathrm{Im}[m_0(z_0)]\right), \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} \int_s^t \frac{\mathrm{Im}[m_\tau(z_\tau)]\mathrm{d}\tau}{\mathrm{Im}[z_\tau]} &\le C(t-s) + \log\frac{\mathrm{Im}[z_s]}{\mathrm{Im}[z_t]}, \\ \int_s^t \frac{\mathrm{Im}[m_\tau(z_\tau)]\mathrm{d}\tau}{\mathrm{Im}[z_\tau]^p} &\le \frac{C}{\mathrm{Im}[z_t]^{p-1}}, \quad p > 1. \end{aligned} \tag{2.31}$$

Proof The estimates (2.27) are general and hold for any Stieltjes transform. First, we have

$$\mathrm{Im}[m_t(z)]\,\mathrm{Im}[z] = \int_{\mathbb{R}} \frac{\mathrm{Im}[z]^2\mathrm{d}\mu_t(x)}{|x-z|^2} \le \int_{\mathbb{R}} \mathrm{d}\mu_t(x) = 1,$$

and secondly we have,

$$|\partial_z m_t(z)| = \left|\int_{\mathbb{R}} \frac{\mathrm{d}\mu_t(x)}{(x-z)^2}\right| \le \frac{1}{\mathrm{Im}[z]}\int_{\mathbb{R}} \frac{\mathrm{Im}[z]\mathrm{d}\mu_t(x)}{|x-z|^2} = \frac{\mathrm{Im}[m_t(z)]}{\mathrm{Im}[z]}.$$

Since $\mathrm{Im}[m_s(z_s)] \ge 0$, it follows from (2.25) and the estimate $|\,\mathrm{Im}[V'(z_s)]| = \mathrm{O}(\mathrm{Im}[z_s])$ of Proposition 2.6 that there exists a constant C s.t.

$$\partial_s\,\mathrm{Im}[z_s] \le C\,\mathrm{Im}[z_s]. \tag{2.32}$$

The estimate (2.28) follows.

By Proposition 2.6, $\mathrm{Im}[V'(z)] = \mathrm{O}(\mathrm{Im}[z])$. It follows from taking imaginary part of (2.25) that there exists some constant C depending on V, such that

$$|\partial_s\,\mathrm{Im}[z_s] + \mathrm{Im}[m_s(z_s)]| \le C\,\mathrm{Im}[z_s]. \tag{2.33}$$

By rearranging, (2.33) leads to the inequalities

$$- e^{Cs}\,\mathrm{Im}[m_s(z_s)] \le \partial_s \left(e^{Cs}\,\mathrm{Im}[z_s]\right), \quad \partial_s \left(e^{-Cs}\,\mathrm{Im}[z_s]\right) \le -e^{-Cs}\,\mathrm{Im}[m_s(z_s)]. \tag{2.34}$$

Similarly, by taking the imaginary part of (2.26), i.e.

$$\partial_s m_s(z_s) = \frac{m_s(z_s)\partial_z V'(z_s)}{2} + \int_{\mathbb{R}} g(z,x)\mathrm{d}\mu_s(x), \tag{2.35}$$

and using the estimates $|\,\mathrm{Im}[\partial_z V'(z)]|+|\,\mathrm{Im}[g(z,x)]| = \mathrm{O}(\mathrm{Im}[z])$ and $|\partial_z V'(z)| \le C$ in Proposition 2.6 we obtain,

$$|\partial_s\,\mathrm{Im}[m_s(z_s)]| \le \left|\mathrm{Im}\left[\frac{\partial_z V'(z_s)m_s(z_s)}{2}\right]\right| + C'\,\mathrm{Im}[z_s] \le C\,\mathrm{Im}[m_s(z_s)]. \tag{2.36}$$

In the last inequality we used $\mathrm{supp}\,\mu_s \in [-\mathfrak{b}, \mathfrak{b}]$, and so

$$\begin{aligned}
&|\,\mathrm{Im}[\partial_z V'(z_s)]\,\mathrm{Re}[m_s(z_s)]| = 1_{\{|\,\mathrm{Re}[z_s]|\le 2\mathfrak{b}\}}\,\mathrm{O}(\mathrm{Im}[z_s]|\,\mathrm{Re}[m_s(z_s)]|)\\
&\quad = 1_{\{|\,\mathrm{Re}[z_s]|\le 2\mathfrak{b}\}}\,\mathrm{O}\left(\int_{\mathbb{R}} \frac{\mathrm{Im}[z_s]\,\mathrm{Re}[x-z_s]\mathrm{d}\mu_s(x)}{|x-z_s|^2}\right)\\
&\quad = \mathrm{O}\left(\int_{\mathbb{R}} \frac{3\mathfrak{b}\,\mathrm{Im}[z_s]\mathrm{d}\mu_s(x)}{|x-z_s|^2}\right) = \mathrm{O}(\mathrm{Im}[m_s(z_s)]).
\end{aligned}$$

We also used $\mathrm{supp}\,\mu_s \in [-\mathfrak{b}, \mathfrak{b}]$, thus for $x \in \mathrm{supp}\,\mu_s$ and $|\,\mathrm{Re}[z_s]| \le 2\mathfrak{b}$, it holds $|\,\mathrm{Re}[x - z_s]| \le 3\mathfrak{b}$.

The estimate (2.29) then follows from (2.36) and Gronwall's inequality, and the estimate (2.30) follows from combining (2.34) and (2.29). For (2.31) we have by (2.34),

$$\int_s^t \frac{\mathrm{Im}[m_\tau(z_\tau)]\mathrm{d}\tau}{\mathrm{Im}[z_\tau]^p} \le \int_s^t \frac{-\partial_\tau\left(e^{-C\tau}\,\mathrm{Im}[z_\tau]\right)\mathrm{d}\tau}{\left(e^{-C\tau}\,\mathrm{Im}[z_\tau]\right)^p}. \tag{2.37}$$

The case $p = 1$ follows. For $p > 1$, we have

$$\begin{aligned}
\int_s^t \frac{\mathrm{Im}[m_\tau(z_\tau)]\mathrm{d}\tau}{\mathrm{Im}[z_\tau]^p} &\le \frac{1}{p-1}\left(\frac{1}{\left(e^{-Ct}\,\mathrm{Im}[z_t]\right)^{p-1}} - \frac{1}{\left(e^{-Cs}\,\mathrm{Im}[z_s]\right)^{p-1}}\right)\\
&\le \frac{C'}{\mathrm{Im}[z_t]^{p-1}}.
\end{aligned} \tag{2.38}$$

□

We have the following result for the flow map $u \to z_t(u)$.

Proposition 2.8 *Suppose that V satisfies Assumption 2.1. Fix a time T. For any $0 \leq t \leq T$, there exists an open domain $\Omega_t \subset \mathbb{C}_+$, such that the vector flow map $u \mapsto z_t(u)$ is a C^1 homeomorphism from Ω_t to $\mathbb{C}_+$.*

Proof We define

$$\Omega_t := \{u \in \mathbb{C}_+ : z_s(u) \in \mathbb{C}_+, 0 \leq s \leq t\}. \tag{2.39}$$

By Assumption 2.1, V is a C^4 function. From our construction of $V'(z)$ as in (2.18), $V'(z)$ is a C^1 function. Thus the vector flow map $u \mapsto z_t(u)$ is a C^1 map from Ω_t to $\mathbb{C}_+$. We need to show that it is invertible. Define the following flow map by

$$\partial_s y_s(v) = m_{t-s}(y_s(v)) + \frac{V'(y_s(v))}{2}, \quad y_0 = v \in \mathbb{C}_+. \tag{2.40}$$

for $0 \leq s \leq t$. Since $\operatorname{Im}[m_{t-s}(y_s(v))] \geq 0$, there exists some constant C depending on V,

$$\partial_s \left(e^{Cs} \operatorname{Im}[y_s]\right) \geq 0. \tag{2.41}$$

Therefore $y_s(v)$ is well defined for $0 \leq s \leq t$, and it will stay in $\mathbb{C}_+$. Furthermore, $v \mapsto y_t(v)$ is a C^1 map, and is the inverse of $u \mapsto z_t(u)$. □

3 Rigidity of β-DBM

In this section we prove the local law and optimal rigidity for β-DBM with general initial data. Let μ_t be the unique solution of (2.8) with initial data

$$\mu_0(x) := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(0)}(x). \tag{3.1}$$

Denote by m_t its Stieltjes transform. We introduce some notation used in the statement and proof of the local law. We fix a small parameter δ, a large constant $\mathfrak{c} \geq 1$, a large constant K, and control parameter $M = (\log N)^{2+2\delta}$. For any time $s \ll 1$, we define the spectral domain,

$$\mathcal{D}_s = \left\{ w \in \mathbb{C}_+ : \operatorname{Im}[w] \geq \frac{e^{Ks} M \log N}{N \operatorname{Im}[m_s(w)]} \vee \frac{e^{Ks}}{N^{\mathfrak{c}}}, \quad \operatorname{Im}[w] \leq 3\mathfrak{b} - s, \right.$$
$$\left. |\operatorname{Re}[w]| \leq 3\mathfrak{b} - s \right\}. \tag{3.2}$$

We take the spectral domain $\mathcal{D}_s$ depending on time s, such that for any characteristic flow $z_s(u)$ with initial value $u \in \mathcal{D}_0$, then it will remain in the spectral domain, $z_s(u) \in \mathcal{D}_s$.

The following is the local law for β-DBM.

Theorem 3.1 *Suppose V satisfies the Assumption 2.1. Fix $T = (\log N)^{-2}$. Let $\beta \geq 1$ and assume that the initial data satisfies $-\mathfrak{a} \leq \lambda_1(0) \leq \lambda_2(0) \cdots \leq \lambda_N(0) \leq \mathfrak{a}$ for a fixed $\mathfrak{a} > 0$. Uniformly for any $0 \leq t \ll T$, and $w \in \mathcal{D}_t$ the following estimate holds with overwhelming probability,*

$$|\tilde{m}_t(w) - m_t(w)| \leq \frac{M}{N \operatorname{Im}[w]}. \tag{3.3}$$

The following rigidity estimates are a consequence of the local law.

Corollary 3.2 *Under the assumptions of Theorem 3.1. Fix time $T = (\log N)^{-2}$. With overwhelming probability, uniformly for any $0 \leq t \leq T$ and $i \in [\![1, N]\!]$, we have*

$$\gamma_{i-CM\log N}(t) - N^{-\mathfrak{c}+1} \leq \lambda_i(t) \leq \gamma_{i+CM\log N}(t) + N^{-\mathfrak{c}+1}, \tag{3.4}$$

where $\mathfrak{c}$ is any large constant, $\gamma_i(t)$ is the classical particle location at time t,

$$\gamma_i(t) = \inf_x \left\{ \int_{-\infty}^{x} \mathrm{d}\mu_t(x) \geq \frac{i}{N} \right\}, \quad i \in [\![1, N]\!]. \tag{3.5}$$

We make the convention that $\gamma_i(t) = -\infty$ if $i < 0$, and $\gamma_i(t) = +\infty$ if $i > N$.

We will prove Theorem 3.1 at the end of Sect. 3.2. The proof of Corollary 3.2 is standard and is given in Sect. 3.3.

Remark 3.3 Notice that

$$\eta \mapsto \eta \operatorname{Im}[m_t(E + \mathrm{i}\eta)] = \int_{\mathbb{R}} \frac{\eta^2 \mathrm{d}\mu_t(x)}{(E-x)^2 + \eta^2},$$

is a monotonically increasing function. Similarly, $\eta \mapsto \eta \operatorname{Im}[\tilde{m}_t(E+\mathrm{i}\eta)]$ is monotonic. We now prove the following deterministic fact. Suppose that the estimate (3.3) holds on $\mathcal{D}_t$. We claim that under this assumption the estimate

$$|\operatorname{Im}[\tilde{m}_t(w)] - \operatorname{Im}[m_t(w)]| \leq \frac{3e^{Kt} M \log N}{N \operatorname{Im}[w]}, \tag{3.6}$$

holds on the larger domain $w = E + \mathrm{i}\eta$ with $|E| \leq 3\mathfrak{b} - t$ and $e^{Kt}N^{-\mathfrak{c}} \leq \eta \leq 3\mathfrak{b} - t$.
Let

$$\eta(E) = \inf_{\eta \geq 0} \{\eta \operatorname{Im}[m_t(E + \mathrm{i}\eta)] \geq e^{Kt} M \log N / N\}. \tag{3.7}$$

By the assumption (3.3) and the definition of $\mathcal{D}_t$ we only need to check the case that $\eta(E) > e^{Kt}N^{-\mathfrak{c}}$ and $\eta < \eta(E)$. In this case we have $\eta(E) \operatorname{Im}[m_t(E + \mathrm{i}\eta(E))] = e^{Kt} M \log N/N$, and $\operatorname{Im}[m_t(w)] \leq e^{Kt} M \log N / N\eta$, and so

$$
\begin{aligned}
&|\operatorname{Im}[\tilde{m}_t(w)] - \operatorname{Im}[m_t(w)]| \\
&\quad \leq \operatorname{Im}[\tilde{m}_t(w)] + \frac{e^{Kt} M \log N}{N\eta} \\
&\quad \leq \frac{\eta(E)}{\eta} \operatorname{Im}[\tilde{m}_t(E + \mathrm{i}\eta(E))] + \frac{e^{Kt} M \log N}{N\eta} \\
&\quad \leq \frac{\eta(E)}{\eta} |\operatorname{Im}[\tilde{m}_t(E + \mathrm{i}\eta(E))] - \operatorname{Im}[m_t(E + \mathrm{i}\eta(E)]| + \frac{2e^{Kt} M \log N}{N\eta} \\
&\quad \leq \frac{3e^{Kt} M \log N}{N\eta}.
\end{aligned}
$$

In the second inequality we used monotonicity of $\eta \operatorname{Im}[\tilde{m}_t(E + \mathrm{i}\eta)]$.

3.1 Properties of the spectral domain

In this section, we prove some properties of the spectral domain $\mathcal{D}_s$ as defined in (3.2), which will be used throughout the proof of Theorem 3.1.

Lemma 3.4 *Suppose V satisfies Assumption 2.1. Fix a time T. For any $0 \leq t \leq T$ such that $z_t \in \mathcal{D}_t$, we have*

$$\operatorname{Im}[u] \leq CN \operatorname{Im}[z_t(u)]. \tag{3.8}$$

Proof Notice that from (2.27), $\operatorname{Im}[u] \leq \operatorname{Im}[m_0(u)]^{-1}$, and by our assumption $z_t \in \mathcal{D}_t$ and (2.29),

$$\operatorname{Im}[z_t(u)] \geq (N \operatorname{Im}[m_t(z_t)])^{-1} \geq (CN \operatorname{Im}[m_0(z_0)])^{-1} \geq (CN)^{-1} \operatorname{Im}[u], \tag{3.9}$$

which yields (3.8). □

Proposition 3.5 *Suppose V satisfies the Assumption 2.1. Fix time T. If for some $t \in [0, T]$, $z_t \in \mathcal{D}_t$, then for any $s \in [0, t]$, $z_s \in \mathcal{D}_s$.*

Proof By (2.28), we have $\operatorname{Im}[z_s] \geq e^{-C(t-s)} \operatorname{Im}[z_t]$. Therefore if we have $\operatorname{Im}[z_t] \geq e^{Kt} N^{-\mathfrak{c}}$ then $\operatorname{Im}[z_s] \geq e^{Ks} N^{-\mathfrak{c}}$ as long as we take $K \geq C$.

Combining $\partial_s \operatorname{Im}[z_s] \leq C \operatorname{Im}[z_s]$ from (2.32), with (2.36), yields that there is a constant C' so that

$$\partial_s (\operatorname{Im}[z_s] \operatorname{Im}[m_s(z_s)]) \leq C' \operatorname{Im}[z_s] \operatorname{Im}[m_s(z_s)]. \tag{3.10}$$

Therefore, $\operatorname{Im}[z_s] \operatorname{Im}[m_s(z_s)] \geq e^{-C'(t-s)} \operatorname{Im}[z_t] \operatorname{Im}[m_t(z_t)]$ and so if $\operatorname{Im}[z_t] \operatorname{Im}[m_t(z_t)] \geq e^{Kt} M \log N/N$, then $\operatorname{Im}[z_s] \operatorname{Im}[m_s(z_s)] \geq e^{Ks} M \log N/N$, provided $K \geq C'$.

Finally, we must prove that if $\operatorname{Im}[z_t] \leq 3\mathfrak{b} - t$ and $|\operatorname{Re}[z_t]| \leq 3\mathfrak{b} - t$, then for any $s \in [0, t]$, we have $\operatorname{Im}[z_s] \leq 3\mathfrak{b} - s$ and $|\operatorname{Re}[z_s]| \leq 3\mathfrak{b} - s$. First, suppose for a contradiction that there exists s such that, say, $|\operatorname{Re}[z_s]| > 3\mathfrak{b} - s$. By symmetry, say

$\operatorname{Re}[z_s] > 3\mathfrak{b} - s$. Let $\tau = \inf_{\sigma \geq s}\{\operatorname{Re}[z_\sigma] \leq 3\mathfrak{b} - \sigma\}$; then $\tau \leq t$. For any $\sigma \in [s, \tau]$, $\operatorname{Re}[z_\sigma] \geq 3\mathfrak{b} - T \geq 2\mathfrak{b}$, we have $V'(z_\sigma) = 0$, and therefore $|\partial_\sigma z_\sigma| \leq |m_\sigma(z_\sigma)| \leq \operatorname{dist}(z_\sigma, \operatorname{supp}\mu_\sigma)^{-1}$. Recall that we have chosen $\mathfrak{b}$ large so that, $\operatorname{supp}\mu_t \in [-\mathfrak{b}, \mathfrak{b}]$. Therefore $|\partial_\sigma z_\sigma| \leq \mathfrak{b}^{-1}$ and $\operatorname{Re}[z_\tau] \geq \operatorname{Re}[z_s] - (\tau - s)/\mathfrak{b} > 3\mathfrak{b} - \tau$, as long as we take $\mathfrak{b} > 1$. Therefore we derive a contradiction. A similar argument applies to the case that $\operatorname{Im}[z_s] > 3\mathfrak{b} - s$. This finishes the proof of Proposition 3.5. □

We have the following weak control on the C^1 norm of the flow map $u \mapsto z_t(u)$. A much stronger version will be proved in Proposition 4.9.

Proposition 3.6 *Suppose V satisfies the Assumption 2.1. Fix time T. For any $0 \leq t \leq T$ with $z_t \in \mathcal{D}_t$, we have with $u = x + \mathrm{i}y$,*

$$|\partial_x z_t(u)| + |\partial_y z_t(u)| = \mathrm{O}(N), \tag{3.11}$$

where the implicit constant depends on T and V.

Proof From Proposition 2.8, we know that $u \mapsto z_t(u)$ is a C^1 map. By differentiating both sides of (2.25), we get

$$\partial_s \partial_x z_s(u) = -\partial_z m_s(z_s(u))\partial_x z_s(u) - \frac{\partial_z V'(z_s(u))\partial_x z_s(u) + \partial_{\bar{z}} V'(z_s(u))\partial_x \bar{z}_s(u)}{2}. \tag{3.12}$$

It follows that

$$\begin{aligned}\partial_s |\partial_x z_s(u)|^2 &= 2\operatorname{Re}[\partial_s \partial_x z_s(u)\partial_x \bar{z}_s(u)] \\ &= -2\operatorname{Re}[\partial_z m_s(z_s(u))]|\partial_x z_s(u)|^2 \\ &\quad - \operatorname{Re}[\partial_z V'(z_s(u))|\partial_x z_s(u)|^2 + \partial_{\bar{z}} V'(z_s(u))(\partial_x \bar{z}_s(u))^2].\end{aligned} \tag{3.13}$$

By Proposition 2.7 we have $|\partial_z m_s(z_s(u))| \leq \operatorname{Im}[m_s(z_s(u))]/\operatorname{Im}[z_s(u)]$, and by Proposition 2.6 we have $|\partial_z V'(z_s(u))|, |\partial_{\bar{z}} V'(z_s(u))| = \mathrm{O}(1)$. Therefore,

$$\partial_s |\partial_x z_s(u)|^2 \leq 2\left(\frac{\operatorname{Im}[m_s(z_s(u))]}{\operatorname{Im}[z_s(u)]} + C\right)|\partial_x z_s(u)|^2. \tag{3.14}$$

Since $z_0(u) = u$, by Gronwall's inequality and (2.31) of Proposition 2.7, we have

$$|\partial_x z_t(u)|^2 \leq \exp\left(2\int_0^t \left(\frac{\operatorname{Im}[m_s(z_s(u))]}{\operatorname{Im}[z_s(u)]} + C\right)\mathrm{d}s\right) \leq \left(\frac{e^{Ct}\operatorname{Im}[u]}{\operatorname{Im}[z_t(u)]}\right)^2 \leq CN^2, \tag{3.15}$$

where we used (3.8). It follows that $|\partial_x z_t(u)| = \mathrm{O}(N)$. The estimate for $|\partial_y z_t(u)|$ follows from the same argument. □

We define the following lattice on the upper half plane $\mathbb{C}_+$,

$$\mathcal{L} = \left\{ E + \mathrm{i}\eta \in \mathcal{D}_0 : E \in \mathbb{Z}/N^{(3\mathfrak{c}+1)}, \eta \in \mathbb{Z}/N^{(3\mathfrak{c}+1)} \right\}. \tag{3.16}$$

Thanks to Propositions 2.8 and 3.6, we have the following.

Proposition 3.7 *Suppose V satisfies Assumption 2.1. Fix a time T. For any $0 \le t \le T$ and $w \in \mathcal{D}_t$, there exists some lattice point $u \in \mathcal{L} \cap z_t^{-1}(\mathcal{D}_t)$, such that*

$$|z_t(u) - w| = \mathrm{O}(N^{-3\mathfrak{c}}), \tag{3.17}$$

where the implicit constant depends on T and V.

3.2 Proof of Theorem 3.1

In this section we prove (3.3). By Proposition 2.8, the flow map $u \mapsto z_t(u)$ is a surjection from Ω_t (as defined in Proposition 2.8) to the upper half plane $\mathbb{C}_+$. We want to prove the estimate

$$|\tilde{m}_t(z_t) - m_t(z_t)| \le \frac{M}{N\,\mathrm{Im}[z_t]}, \tag{3.18}$$

for $z_t \in \mathcal{D}_t$.

By Ito's formula, $\tilde{m}_s(z)$ satisfies the stochastic differential equation

$$\begin{aligned} \mathrm{d}\tilde{m}_s(z) = &-\sqrt{\frac{2}{\beta N^3}} \sum_{i=1}^N \frac{\mathrm{d}B_i(s)}{(\lambda_i(s) - z)^2} + \tilde{m}_s(z)\partial_z \tilde{m}_s(z)\mathrm{d}s \\ &+ \frac{1}{2N} \sum_{i=1}^N \frac{V'(\lambda_i(s))}{(\lambda_i(s) - z)^2}\mathrm{d}s + \frac{2-\beta}{\beta N^2} \sum_{i=1}^N \frac{\mathrm{d}s}{(\lambda_i(s) - z)^3}. \end{aligned} \tag{3.19}$$

We can rewrite (3.19) as

$$\begin{aligned} \mathrm{d}\tilde{m}_s(z) = &-\sqrt{\frac{2}{\beta N^3}} \sum_{i=1}^N \frac{\mathrm{d}B_i(s)}{(\lambda_i(s) - z)^2} + \partial_z \tilde{m}_s(z) \left(\tilde{m}_s(z) + \frac{V'(z)}{2} \right) \mathrm{d}s \\ &+ \frac{\tilde{m}_s(z)\partial_z V'(z)}{2}\mathrm{d}s \\ &+ \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{w}} \tilde{g}(z, w)\tilde{m}_s(w)\mathrm{d}^2 w \mathrm{d}s + \frac{2-\beta}{\beta N^2} \sum_{i=1}^N \frac{\mathrm{d}s}{(\lambda_i(s) - z)^3}, \end{aligned} \tag{3.20}$$

where $V'(z)$ and $\tilde{g}(z, w)$ are defined in (2.18) and (2.22) respectively. Plugging (2.25) into (3.20), and by the chain rule, we have

$$\begin{aligned} \mathrm{d}\tilde{m}_s(z_s) = -\sqrt{\frac{2}{\beta N^3}} \sum_{i=1}^{N} \frac{\mathrm{d}B_i(s)}{(\lambda_i(s) - z_s)^2} + \partial_z \tilde{m}_s(z_s)\, (\tilde{m}_s(z_s) \\ -m_s(z_s))\, \mathrm{d}s + \frac{\tilde{m}_s(z_s)\partial_z V'(z_s)}{2} \mathrm{d}s \\ + \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{w}} \tilde{g}(z_s, w) \tilde{m}_s(w) \mathrm{d}^2 w \mathrm{d}s + \frac{2-\beta}{\beta N^2} \sum_{i=1}^{N} \frac{\mathrm{d}s}{(\lambda_i(s) - z_s)^3}. \end{aligned} \tag{3.21}$$

It follows by taking the difference of (2.26) and (3.21) that,

$$\begin{aligned} \mathrm{d}(\tilde{m}_s(z_s) - m_s(z_s)) = -\sqrt{\frac{2}{\beta N^3}} \sum_{i=1}^{N} \frac{\mathrm{d}B_i(s)}{(\lambda_i(s) - z_s)^2} \\ + (\tilde{m}_s(z_s) - m_s(z_s))\, \partial_z \left(\tilde{m}_s(z_s) + \frac{V'(z_s)}{2} \right) \mathrm{d}s \\ + \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{w}} \tilde{g}(z_s, w) (\tilde{m}_s(w) - m_s(w)) \mathrm{d}^2 w \mathrm{d}s + \frac{2-\beta}{\beta N^2} \sum_{i=1}^{N} \frac{\mathrm{d}s}{(\lambda_i(s) - z_s)^3}. \end{aligned} \tag{3.22}$$

Using the fact that $m_0(z) = \tilde{m}_0(z)$, we can integrate both sides of (3.22) from 0 to t and obtain

$$\tilde{m}_t(z_t) - m_t(z_t) = \int_0^t (\mathcal{E}_1(s)\mathrm{d}s + \mathrm{d}\mathcal{E}_2(s))\,, \tag{3.23}$$

where the error terms are

$$\begin{aligned} \mathcal{E}_1(s) = (\tilde{m}_s(z_s) - m_s(z_s))\, \partial_z \left(\tilde{m}_s(z_s) + \frac{V'(z_s)}{2} \right) \\ + \frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar{w}} \tilde{g}(z_s, w) (\tilde{m}_s(w) - m_s(w)) \mathrm{d}^2 w, \end{aligned} \tag{3.24}$$

$$\mathrm{d}\mathcal{E}_2(t) = \frac{2-\beta}{\beta N^2} \frac{\mathrm{d}s}{(\lambda_i(s) - z_s)^3} - \sqrt{\frac{2}{\beta N^3}} \sum_{i=1}^{N} \frac{\mathrm{d}B_i(s)}{(\lambda_i(s) - z_s)^2}. \tag{3.25}$$

We remark that $\mathcal{E}_1$ and $\mathcal{E}_2$ implicitly depend on u, the initial value of the flow $z_s(u)$. The local law will eventually follow from an application of Gronwall's inequality to (3.23).

We define the stopping time

$$\sigma := \inf_{s \geq 0} \left\{ \exists w \in \mathcal{D}_s : |\tilde{m}_s(w) - m_s(w)| \geq \frac{M}{N \operatorname{Im}[w]} \right\} \wedge t. \tag{3.26}$$

In the rest of this section we prove that with overwhelming probability we have $\sigma = t$. Theorem 3.1 follows.

For any lattice point $u \in \mathcal{L}$ as in (3.16), we denote

$$t(u) = \sup_{s\geq 0}\{z_s(u) \in \mathcal{D}_s\} \wedge t. \tag{3.27}$$

By Proposition 3.5 we have that $z_s(u) \in \mathcal{D}_s$ for any $0 \leq s \leq t(u)$. We decompose the time interval $[0, t(u)]$ in the following way. First set $t_0 = 0$, and define

$$t_{i+1}(u) := \sup_{s\geq t_i(u)} \left\{ \mathrm{Im}[z_s(u)] \geq \frac{\mathrm{Im}[z_{t_i}(u)]}{2} \right\} \wedge t(u), \quad i = 0, 1, 2, \dots. \tag{3.28}$$

By (3.8), there exists some constant C depending on V, such that $\mathrm{Im}[z_0(u)] \leq CN\,\mathrm{Im}[z_{t(u)}(u)]$, and thus the above sequence will terminate at some $t_k(u) = t(u)$ for $k = \mathrm{O}(\log N)$ depending on u, the initial value of z_s. Moreover, by (2.28), for any $t_i(u) \leq s_1 \leq s_2 \leq t_{i+1}(u)$,

$$e^{-CT} \leq e^{-C(s_2-s_1)} \leq \frac{\mathrm{Im}[z_{s_1}(u)]}{\mathrm{Im}[z_{s_2}(u)]} \leq \frac{e^{C(s_1-t_i)}\,\mathrm{Im}[z_{t_i}(u)]}{e^{C(s_2-t_{i+1})}\,\mathrm{Im}[z_{t_{i+1}}(u)]} \leq 2e^{CT}. \tag{3.29}$$

We first derive an estimate of $\int \mathrm{d}\mathcal{E}_2(s)$ in terms of $\{m_s(z_s(u)), 0 \leq s \leq t(u)\}$.

Proposition 3.8 *Under the assumptions of Theorem 3.1. There exists an event Ω that holds with overwhelming probability on which we have for every $0 \leq \tau \leq t(u)$ and $u \in \mathcal{L}$,*

$$\left| \int_0^{\tau\wedge\sigma} \mathrm{d}\mathcal{E}_2(s) \right| \leq \frac{C(\log N)^{1+\delta}}{N\,\mathrm{Im}[z_{\tau\wedge\sigma}(u)]}. \tag{3.30}$$

Proof For simplicity of notation, we write $t_i = t_i(u)$ and $z_s = z_s(u)$. For any $s \leq t_i$, by our choice of the stopping time σ (as in (3.26)), and the definition of domain $\mathcal{D}_s$ (as in (3.2)), we have

$$\mathrm{Im}[\tilde{m}_{s\wedge\sigma}(z_{s\wedge\sigma})] \leq 2\,\mathrm{Im}[m_{s\wedge\sigma}(z_{s\wedge\sigma})]. \tag{3.31}$$

For the first term in (3.25) we have

$$\sup_{0\leq\tau\leq t_i} \left| \frac{2-\beta}{\beta N^2} \int_0^{\tau\wedge\sigma} \sum_{i=1}^N \frac{\mathrm{d}s}{(\lambda_i(s)-z_s)^3} \right|$$
$$\leq \frac{2-\beta}{\beta N^2} \int_0^{t_i\wedge\sigma} \sum_{i=1}^N \frac{\mathrm{d}s}{|\lambda_i(s)-z_s|^3}$$

$$
\begin{aligned}
&\leq \frac{C}{N^2}\int_0^{t_i\wedge\sigma}\sum_{i=1}^N \frac{\mathrm{d}s}{\operatorname{Im}[z_s]|\lambda_i(s)-z_s|^2} = C\int_0^{t_i\wedge\sigma}\frac{\operatorname{Im}[\tilde{m}_s(z_s)]\mathrm{d}s}{N\operatorname{Im}[z_s]^2}\\
&\leq C\int_0^{t_i\wedge\sigma}\frac{2\operatorname{Im}[m_s(z_s)]\mathrm{d}s}{N\operatorname{Im}[z_s]^2}\leq\frac{C'}{N\operatorname{Im}[z_{t_i\wedge\sigma}]},
\end{aligned}
\tag{3.32}
$$

where we used (3.31) and (2.31).

For the second term in (3.25) we have

$$
\begin{aligned}
&\left\langle\sqrt{\frac{2}{\beta N^3}}\int_0^{\cdot\wedge\sigma}\sum_{i=1}^N\frac{\mathrm{d}B_i(s)}{(\lambda_i(s)-z_s)^2}\right\rangle_{t_i} = \frac{2}{\beta N^{3/2}}\int_0^{t_i\wedge\sigma}\sum_{i=1}^N\frac{\mathrm{d}s}{|\lambda_i(s)-z_s|^4}\\
&\leq \frac{2}{\beta N^{3/2}}\int_0^{t_i\wedge\sigma}\sum_{i=1}^N\frac{\mathrm{d}s}{\operatorname{Im}[z_s]^2|\lambda_i(s)-z_s|^2}=\frac{2}{\beta}\int_0^{t_i\wedge\sigma}\frac{\operatorname{Im}[\tilde{m}_s(z_s)]\mathrm{d}s}{N^2\operatorname{Im}[z_s]^3}\\
&\leq\frac{2}{\beta}\int_0^{t_i\wedge\sigma}\frac{2\operatorname{Im}[m_s(z_s)]\mathrm{d}s}{N^2\operatorname{Im}[z_s]^3}\leq\frac{C}{N^2\operatorname{Im}[z_{t_i\wedge\sigma}]^2},
\end{aligned}
\tag{3.33}
$$

again we used (3.31) and (2.31). Therefore, by Burkholder–Davis–Gundy inequality, for any $u\in\mathcal{L}$ and t_i, the following holds with overwhelming probability, i.e., $1-C\exp\{-c(\log N)^{1+\delta}\}$,

$$
\sup_{0\leq\tau\leq t_i}\left|\sqrt{\frac{2}{\beta N^3}}\int_0^{\tau\wedge\sigma}\sum_{i=1}^N\frac{\mathrm{d}B_i(s)}{(\lambda_i(s)-z_s)^2}\right|\leq\frac{C(\log N)^{1+\delta}}{N\operatorname{Im}[z_{t_i\wedge\sigma}]}.
\tag{3.34}
$$

We define Ω to be the set of Brownian paths $\{B_1(s),\dots,B_N(s)\}_{0\leq s\leq t}$ on which the following two estimates hold.

1. First we have, $-\mathfrak{b}\leq\lambda_1(s)\leq\lambda_2(s)\leq\cdots\leq\lambda_N(s)\leq\mathfrak{b}$ uniformly for all $s\in[0,T]$.
2. Second for any $u\in\mathcal{L}$ and $i=0,1,2,\dots,k$, (3.34) holds. We recall that $t_0,t_1,\dots,t_k$ are recursively defined in (3.28), and $k=\mathrm{O}(\log N)$.

It follows from Proposition 2.5 and the discussion above, Ω holds with overwhelming probability, i.e., $\mathbb{P}(\Omega)\geq 1-C|\mathcal{L}|(\log N)\exp\{-c(\log N)^{1+\delta}\}$.

Therefore, for any $\tau\in[t_{i-1},t_i]$, the bounds (3.32) and (3.34) yield

$$
\left|\int_0^{\tau\wedge\sigma}\mathrm{d}\mathcal{E}_2(s)\right|\leq\frac{C'(\log N)^{1+\delta}}{N\operatorname{Im}[z_{t_i\wedge\sigma}]}\leq\frac{C(\log N)^{1+\delta}}{N\operatorname{Im}[z_{\tau\wedge\sigma}]},
\tag{3.35}
$$

where we used our choice of t_i's, i.e. (3.29). □

We now bound the second term of (3.24).

Proposition 3.9 *Under the assumptions of Theorem 3.1, for any $u\in\mathcal{L}$ and $s\in[0,t(u)]$ (as in (3.27)), with probability $1-e^{-N}$, we have*

$$\left|\frac{1}{\pi}\int_{\mathbb{C}}\partial_{\bar{w}}\tilde{g}(z_{s\wedge\sigma}(u),w)(\tilde{m}_{s\wedge\sigma}(w)-m_{s\wedge\sigma}(w))\mathrm{d}^2w\right|\leq\frac{CM(\log N)^2}{N},\quad(3.36)$$

where the constant C depends on V.

Proof First, we note that by Proposition 2.5, we can restrict to the case such that $\tilde{\mu}_t,\mu_t$ are supported on $[-\mathfrak{b},\mathfrak{b}]$, and replace g by $g_1(z,x):=g(z,x)\chi(x)$ and the quasi-analytic extension by $\tilde{g}_1(z,x+\mathrm{i}y):=(g_1(z,x)+\mathrm{i}y\partial_x g_1(z,x))\chi(y)$.

The proof follows the same argument as [24, Lemma B.1]. Let $S(x+\mathrm{i}y)=\tilde{m}_{s\wedge\sigma}(x+\mathrm{i}y)-m_{s\wedge\sigma}(x+\mathrm{i}y)$. we have

$$\begin{aligned}&\left|\frac{1}{\pi}\int_{\mathbb{C}}\partial_{\bar{w}}\tilde{g}_1(z_{s\wedge\sigma}(u),w)(\tilde{m}_{s\wedge\sigma}(w)-m_{s\wedge\sigma}(w))\mathrm{d}^2w\right|\\&\quad=\left|\int_{\mathbb{R}}g_1(z_{s\wedge\sigma}(u),x)(\mathrm{d}\tilde{\mu}_{s\wedge\sigma}(x)-\mathrm{d}\mu_{s\wedge\sigma}(x))\right|\\&\quad\leq C\int_{\mathbb{C}_+}(|g_1(z_{s\wedge\sigma},x)|+y|\partial_x g_1(z_{s\wedge\sigma}(u),x)|)|\chi'(y)||S(x+\mathrm{i}y)|\mathrm{d}x\mathrm{d}y\\&\qquad+C\int_{\mathbb{C}_+}y\chi(y)|\partial_x^2g_1(z_{s\wedge\sigma}(u),x)||\operatorname{Im}[S(x+\mathrm{i}y)]|\mathrm{d}x\mathrm{d}y.\end{aligned}\quad(3.37)$$

We start by handling the first term on the RHS of (3.37). The integrand is supported in $\{x+\mathrm{i}y:|x|\leq 2\mathfrak{b},\mathfrak{b}\leq|y|\leq 2\mathfrak{b}\}\subseteq\mathcal{D}_t$ for every t. In this region we have from Proposition 2.6 that g_1 and $\partial_x g_1$ are bounded, and by the definition of σ we have that $|S(x+\mathrm{i}y)|\leq M/N$ in this region and so

$$\int_{\mathbb{C}_+}(|g_1(z_{s\wedge\sigma},x)|+y|\partial_x g_1(z_{s\wedge\sigma}(u),x)|)|\chi'(y)|S(x+\mathrm{i}y)|\mathrm{d}x\mathrm{d}y\leq\frac{CM}{N}.\quad(3.38)$$

We now handle the second term on the RHS of (3.37). By the definition of $t(u)$, $z_{s\wedge\sigma}(u)\in\mathcal{D}_0$, and so $\operatorname{Im}[z_{s\wedge\sigma}(u)]\geq N^{-\mathfrak{c}}$. From Proposition 2.6 we have $|\partial_x^2 g(z,x)|\leq C|z-x|^{-1}$, and $|\partial_x^2 g_1(z,x)|\leq C\chi(x)|z-x|^{-1}$.

We split the second integral on the righthand side of (3.37) into the two regions $\Lambda:=\{E+\mathrm{i}\eta:0\leq\eta\leq e^{K(s\wedge\sigma)}N^{-\mathfrak{c}}\}$ and $\mathbb{C}_+\setminus\Lambda$. On Λ, we use the trivial bound $\operatorname{Im}[S(x+\mathrm{i}y)]\leq|\tilde{m}_{s\wedge\sigma}(x+\mathrm{i}y)|+|m_{s\wedge\sigma}(x+\mathrm{i}y)|\leq 2/y$, and obtain

$$\begin{aligned}&\int_{\Lambda}y\chi(y)|\partial_x^2g_1(z_{s\wedge\sigma}(u),x)||\operatorname{Im}[S(x+\mathrm{i}y)]|\mathrm{d}x\mathrm{d}y\\&\quad\leq C\int_{0\leq y\leq e^{K(s\wedge\sigma)}N^{-\mathfrak{c}}}\frac{\chi(x)}{|z_{s\wedge\sigma}(u)-x|}\mathrm{d}x\mathrm{d}y\leq\frac{C\log N}{N^{\mathfrak{c}}}.\end{aligned}\quad(3.39)$$

On $\mathbb{C}_+\setminus\Lambda$, by Remark 3.3, and the definition of σ that (3.6) holds for time $0\leq t\leq s\wedge\sigma$, we have $|\operatorname{Im}[S(x+\mathrm{i}y)]|\leq 3e^{K(s\wedge\sigma)}M\log N/(Ny)$, and therefore,

$$\begin{aligned}&\int_{\mathbb{C}_+\setminus\Lambda} y\chi(y)|\partial_x^2 g_1(z_{s\wedge\sigma}(u),x)||\operatorname{Im}[S(x+\mathrm{i}y)]|\mathrm{d}x\mathrm{d}y\\&\quad\leq \frac{CM\log N}{N}\int_{\mathbb{C}_+}\frac{\chi(x)\chi(y)}{|z_{s\wedge\sigma}(u)-x|}\mathrm{d}x\mathrm{d}y\leq \frac{CM(\log N)^2}{N}.\end{aligned}\tag{3.40}$$

This completes the proof of (3.36). □

Proof of Theorem 3.1 We can now start analyzing (3.23). For any lattice point $u\in\mathcal{L}$ and $\tau\in[0,t(u)]$ (as in (3.27)), by Proposition 3.8 and 3.9, we have

$$\begin{aligned}|\tilde m_{\tau\wedge\sigma}(z_{\tau\wedge\sigma}(u))-m_{\tau\wedge\sigma}(z_{\tau\wedge\sigma}(u))|\leq\int_0^{\tau\wedge\sigma}\\|\tilde m_s(z_s(u))-m_s(z_s(u))|\left|\partial_z\left(\tilde m_s(z_s(u))+\frac{V'(z_s(u))}{2}\right)\right|\mathrm{d}s\\+\frac{C(\tau\wedge\sigma)M(\log N)^2}{N}+\frac{C(\log N)^{1+\delta}}{N\operatorname{Im}[z_{\tau\wedge\sigma}(u)]}.\end{aligned}$$

Notice that for $s\leq\tau\wedge\sigma$,

$$\left|\partial_z\left(\tilde m_s(z_s(u))+\frac{V'(z_s(u))}{2}\right)\right|\leq\frac{\operatorname{Im}[\tilde m_s(z_s(u))]}{\operatorname{Im}[z_s(u)]}+C,\tag{3.41}$$

where we used (2.27), for the Stieltjes transform $\tilde m_s$, and $\partial_z V'(z_s(u))=\mathrm{O}(1)$ from Proposition 2.6. Since $z_s(u)\in\mathcal{D}_s$, by the definition of $\mathcal{D}_s$, we have $\operatorname{Im}[m_s(z_s(u))]\geq M\log N/(N\operatorname{Im}[z_s(u)])$. Moreover, since $s\leq\sigma$, we have $|\tilde m_s(z_s(u))-m_s(z_s(u))|\leq M/(N\operatorname{Im}[z_s(u)])\leq\operatorname{Im}[m_s(z_s(u))]/\log N$. Therefore,

$$\begin{aligned}\left|\partial_z\left(\tilde m_s(z_s(u))+\frac{V'(z_s(u))}{2}\right)\right|&\leq\frac{\operatorname{Im}[\tilde m_s(z_s(u))]}{\operatorname{Im}[z_s(u)]}+C\\&\leq\left(1+\frac{1}{\log N}\right)\frac{\operatorname{Im}[m_s(z_s(u))]}{\operatorname{Im}[z_s(u)]}+C.\end{aligned}\tag{3.42}$$

We denote

$$\beta_s(u):=\left(1+\frac{1}{\log N}\right)\frac{\operatorname{Im}[m_s(z_s(u))]}{\operatorname{Im}[z_s(u)]}+C=\mathrm{O}\left(\frac{\operatorname{Im}[m_s(z_s(u))]}{\operatorname{Im}[z_s(u)]}\right).\tag{3.43}$$

We have derived the inequality,

$$\begin{aligned}|\tilde m_{\tau\wedge\sigma}(z_{\tau\wedge\sigma}(u))-m_{\tau\wedge\sigma}(z_{\tau\wedge\sigma}(u))|&\leq\int_0^{\tau\wedge\sigma}\beta_s(u)\,|\tilde m_s(z_s(u))-m_s(z_s(u))|\,\mathrm{d}s\\&\quad+\frac{C(\tau\wedge\sigma)M(\log N)^2}{N}+\frac{C(\log N)^{1+\delta}}{N\operatorname{Im}[z_{\tau\wedge\sigma}(u)]}.\end{aligned}\tag{3.44}$$

By Gronwall's inequality, this implies the estimate

$$|\tilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma}(u)) - m_{t\wedge\sigma}(z_{t\wedge\sigma}(u))| \leq \frac{C(t\wedge\sigma)M(\log N)^2}{N} + \frac{C(\log N)^{1+\delta}}{N\,\mathrm{Im}[z_{t\wedge\sigma}(u)]} + \int_0^{t\wedge\sigma} \beta_s(u)\left(\frac{sM(\log N)^2}{N} + \frac{C(\log N)^{1+\delta}}{N\,\mathrm{Im}[z_s(u)]}\right) e^{\int_s^{t\wedge\sigma}\beta_\tau(u)\mathrm{d}\tau}\mathrm{d}s. \tag{3.45}$$

By (2.31) of Proposition 2.7, and (3.43), we have

$$e^{\int_s^{t\wedge\sigma}\beta_\tau(u)\mathrm{d}\tau} \leq e^{C(t-s)} e^{\left(1+\frac{1}{\log N}\right)\log\left(\frac{\mathrm{Im}[z_s(u)]}{\mathrm{Im}[z_{t\wedge\sigma}(u)]}\right)} = e^{C(t-s)}\left(\frac{\mathrm{Im}[z_s(u)]}{\mathrm{Im}[z_{t\wedge\sigma}(u)]}\right)^{1+\frac{1}{\log N}} \leq C\frac{\mathrm{Im}[z_s(u)]}{\mathrm{Im}[z_{t\wedge\sigma}(u)]}.$$

In the last equality, we used the estimate (3.8) which shows that $\mathrm{Im}[z_s(u)]/\,\mathrm{Im}[z_{t\wedge\sigma}(u)] \leq CN$. Combining the above inequality with (3.43) we can bound the last term in (3.45) by

$$C\int_0^{t\wedge\sigma}\frac{\mathrm{Im}[m_s(z_s(u))]}{\mathrm{Im}[z_{t\wedge\sigma}(u)]}\left(\frac{sM(\log N)^2}{N} + \frac{C(\log N)^{1+\delta}}{N\,\mathrm{Im}[z_s(u)]}\right)\mathrm{d}s \leq \frac{CM(\log N)^2}{N\,\mathrm{Im}[z_{t\wedge\sigma}(u)]}\int_0^{t\wedge\sigma} s\,\mathrm{Im}[m_s(z_s(u))]\mathrm{d}s + \frac{C(\log N)^{2+\delta}}{N\,\mathrm{Im}[z_{t\wedge\sigma(u)}]}, \tag{3.46}$$

where we used (2.34) and that $\log(\mathrm{Im}[z_0(u)/z_{t\wedge\sigma}(u)]) = \mathrm{O}(\log N)$ from (3.8). Since $|V'(z)| \leq C$, it follows from (2.25) that $\mathrm{Im}[m_s(z_s(u))] = -\partial_s\,\mathrm{Im}[z_s(u)] + \mathrm{O}(1)$. Therefore we can bound the integral term in (3.46) by,

$$\int_0^{t\wedge\sigma} s\,\mathrm{Im}[m_s(z_s(u))]\mathrm{d}s = \int_0^{t\wedge\sigma}(-\partial_s\,\mathrm{Im}[z_s(u)])s\mathrm{d}s + \mathrm{O}((t\wedge\sigma)^2) = \mathrm{O}(t\wedge\sigma). \tag{3.47}$$

It follows by combining (3.45), (3.46) and (3.47) that

$$|\tilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma}(u)) - m_{t\wedge\sigma}(z_{t\wedge\sigma}(u))| \leq C\left(\frac{(t\wedge\sigma)M(\log N)^2 + (\log N)^{2+\delta}}{N\,\mathrm{Im}[z_{t\wedge\sigma}(u)]}\right). \tag{3.48}$$

Therefore on the event Ω as defined in Proposition 3.8,

$$|\tilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma}(u)) - m_{t\wedge\sigma}(z_{t\wedge\sigma}(u))| = o\left(\frac{M}{N\,\mathrm{Im}[z_{t\wedge\sigma}(u)]}\right), \tag{3.49}$$

provided $t \ll T = (\log N)^{-2}$, and $M = (\log N)^{2+2\delta}$. By Proposition 3.7, for any $w \in \mathcal{D}_{t\wedge\sigma}$, there exists some $u \in \mathcal{L}$ such that $z_{t\wedge\sigma}(u) \in \mathcal{D}_{t\wedge\sigma}$, and

$$|z_{t\wedge\sigma}(u) - w| = \mathrm{O}(N^{-3\mathfrak{c}}). \tag{3.50}$$

Moreover, on the domain $\mathcal{D}_{t\wedge\sigma}$, both $\tilde{m}_{t\wedge\sigma}$ and $m_{t\wedge\sigma}$ are Lipschitz with constant $N^{2\mathfrak{c}}$. Therefore

$$\begin{aligned}
|\tilde{m}_{t\wedge\sigma}(w) - m_{t\wedge\sigma}(w)| &\le |\tilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma}(u)) - m_{t\wedge\sigma}(z_{t\wedge\sigma}(u))| \\
&\quad + |\tilde{m}_{t\wedge\sigma}(w) - \tilde{m}_{t\wedge\sigma}(z_{t\wedge\sigma}(u))| + |m_{t\wedge\sigma}(w) - m_{t\wedge\sigma}(z_{t\wedge\sigma}(u))| \\
&= o\left(\frac{M}{N\,\mathrm{Im}[z_{t\wedge\sigma}(u)]}\right) + \mathrm{O}\left(\frac{|z_{t\wedge\sigma}(u) - w|}{N^{-2\mathfrak{c}}}\right) = o\left(\frac{M}{N\,\mathrm{Im}[z_{t\wedge\sigma}(u)]}\right).
\end{aligned} \tag{3.51}$$

If $\sigma < t$ somewhere on the event Ω then by continuity there must be a point $z \in \mathcal{D}_\sigma$ s.t.

$$|\tilde{m}_\sigma(z) - m_\sigma(z)| = \frac{M}{N\,\mathrm{Im}[z]}. \tag{3.52}$$

This contradicts (3.51), and so we see that on Ω, $\sigma = t$. This completes the proof of (3.3). □

3.3 Proof of Corollary 3.2

Proof of Corollary 3.2 The proof follows a similar argument to [24, Lemma B.1]. Recall the function $\eta(x)$ from Remark 3.3. Let $S(x+\mathrm{i}y) = \tilde{m}_t(x+\mathrm{i}y) - m_t(x+\mathrm{i}y)$. Fix some $E_0 \in [-\mathfrak{b}, \mathfrak{b}]$. Define

$$\tilde{\eta} := \inf_{\eta \ge e^{Kt} N^{1-\mathfrak{c}}} \left\{\eta : \max_{E_0 \le x \le E_0+\eta} \eta(x) \le \eta\right\}. \tag{3.53}$$

For later use we define

$$\tilde{E} := \mathrm{argmax}_{E_0 \le x \le E_0+\tilde{\eta}}\, \eta(x), \tag{3.54}$$

so that

$$\eta(\tilde{E}) = \tilde{\eta}. \tag{3.55}$$

We define a test function $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = 1$ on $x \in [-2\mathfrak{b}, E_0]$, and so that $f(x)$ vanishes outside $[-2\mathfrak{b}-1, E_0+\tilde{\eta}]$. We take f so that $f'(x) = \mathrm{O}(1)$ and $f''(x) = \mathrm{O}(1)$ on $[-2\mathfrak{b}-1, -2\mathfrak{b}]$ and $f'(x) = \mathrm{O}(1/\tilde{\eta})$ and $f''(x) = \mathrm{O}(1/\tilde{\eta}^2)$ on $[E_0, E_0+\tilde{\eta}]$. By the Helffer-Sjöstrand formula, see [30, Chapter 11.2], we have,

$$\left|\int_{-\infty}^{\infty} f(x)(\mathrm{d}\tilde{\mu}_t(x) - \mathrm{d}\mu_t(x))\right| \leq C \int_{\mathbb{C}_+} (|f(x)| + |y||f'(x)|)|\chi'(y)||S(x+\mathrm{i}y)|\mathrm{d}x\mathrm{d}y + C\left|\int_{\mathbb{C}_+} y\chi(y) f''(x) \operatorname{Im}[S(x+\mathrm{i}y)]\mathrm{d}x\mathrm{d}y\right|. \tag{3.56}$$

On the event such that (3.3) holds, the first term is easily bounded by

$$\int_{\mathbb{C}_+} (|f(x)| + |y||f'(x)|)|\chi'(y)||S(x+\mathrm{i}y)|\mathrm{d}x\mathrm{d}y \leq \frac{CM}{N}. \tag{3.57}$$

For the second term, recall that $f''(x) = 0$ unless $x \in [E_0, E_0+\tilde{\eta}] \cup [-2\mathfrak{b}-1, -2\mathfrak{b}]$. By Remark 3.3 we have the estimate

$$|\operatorname{Im}[S(x+\mathrm{i}y)]| \leq \frac{CM\log(N)}{Ny}, \qquad y \geq \frac{e^{Kt}}{N^{\mathfrak{c}}} =: \eta_{\mathfrak{c}}. \tag{3.58}$$

Hence,

$$\begin{aligned}
&\left|\int_{-2\mathfrak{b}-1\leq x\leq -2\mathfrak{b}} y\chi(y) f''(x) \operatorname{Im}[S(x+\mathrm{i}y)]\mathrm{d}x\mathrm{d}y\right| \\
&\quad\leq \left|\int_{-2\mathfrak{b}-1\leq x\leq -2\mathfrak{b}, |y|\geq\eta_{\mathfrak{c}}} y\chi(y) f''(x) \operatorname{Im}[S(x+\mathrm{i}y)]\mathrm{d}x\mathrm{d}y\right| \\
&\qquad+ \left|\int_{-2\mathfrak{b}-1\leq x\leq -2\mathfrak{b}, |y|\leq\eta_{\mathfrak{c}}} |f''(x)|\mathrm{d}x\mathrm{d}y\right| \\
&\quad\leq \frac{CM\log N}{N} + \frac{C}{N^{\mathfrak{c}}} \leq \frac{CM\log N}{N}.
\end{aligned} \tag{3.59}$$

In the first integral we used the estimate (3.58) and in the second we used $|y\operatorname{Im}[S(x+\mathrm{i}y)]| \leq 2$. For the region $x \in [E_0, E_0+\tilde{\eta}]$ we do a similar decomposition. First we bound the region $|y| \leq \tilde{\eta}$. We have,

$$\begin{aligned}
&\left|\int_{E_0\leq x\leq E_0+\tilde{\eta}} \int_{y\leq\tilde{\eta}} y\chi(y) f''(x) \operatorname{Im}[S(x+\mathrm{i}y)]\mathrm{d}x\mathrm{d}y\right| \\
&\quad\leq \left|\int_{E_0\leq x\leq E_0+\tilde{\eta}} \int_{y\leq\eta_{\mathfrak{c}}} y\chi(y) f''(x) \operatorname{Im}[S(x+\mathrm{i}y)]\mathrm{d}x\mathrm{d}y\right| \\
&\qquad+ \left|\int_{E_0\leq x\leq E_0+\tilde{\eta}} \int_{\eta_{\mathfrak{c}}\leq y\leq\tilde{\eta}} y\chi(y) f''(x) \operatorname{Im}[S(x+\mathrm{i}y)]\mathrm{d}x\mathrm{d}y\right| \\
&\quad\leq \frac{C\eta_{\mathfrak{c}}}{\tilde{\eta}} + \frac{C\log(N)M}{N} \leq \frac{C\log(N)M}{N}.
\end{aligned} \tag{3.60}$$

For the first integral we used $y|\operatorname{Im}[S(x+\mathrm{i}y)]| \le 2$ and in the second region we used (3.58). For the other region we integrate by parts,

$$\begin{aligned}\int_{E_0 \le x \le E_0+\tilde{\eta}} \int_{y \ge \tilde{\eta}} & y\chi(y) f''(x) \operatorname{Im}[S(x+\mathrm{i}y)]\mathrm{d}x\mathrm{d}y \\ = & -\int_{E_0 \le x \le E_0+\tilde{\eta}} f'(x)\tilde{\eta} \operatorname{Re}[S(x+\mathrm{i}\tilde{\eta})]\mathrm{d}x \\ & - \int_{E_0 \le x \le E_0+\tilde{\eta}} \int_{y \ge \tilde{\eta}} f'(x)\partial_y(y\chi(y)) \operatorname{Re}[S(x+\mathrm{i}y)]\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{3.61}$$

By the definition of $\tilde{\eta}$, the estimate (3.3) holds in the region $x+\mathrm{i}y$ for $x \in [E_0, E_0+\mathrm{i}\tilde{\eta}]$ and $y \ge \tilde{\eta}$. Hence, both terms are easily estimated by $C\log(N)M/N$.

The above estimates imply

$$\left|\int_{-\infty}^{\infty} f(x)(\mathrm{d}\tilde{\mu}_t(x) - \mathrm{d}\mu_t(x))\right| \le \frac{CM\log N}{N}. \tag{3.62}$$

We can now prove the lower bound of (3.4). We have,

$$|\{i : \lambda_i(t) \le E_0\}| \le N\int_{-\infty}^{\infty} f(x)\mathrm{d}\tilde{\mu}_t(x) \le N\int_{-\infty}^{E_0+\tilde{\eta}} \mathrm{d}\mu_t(x) + CM\log N. \tag{3.63}$$

If $\tilde{\eta} = e^{Kt}/N^{\mathfrak{c}}$ then the lower bound of (3.4) follows by taking $E_0 = \gamma_{i-CM\log N}(t) - N^{-\mathfrak{c}+1}$. If $\tilde{\eta} = \eta(\tilde{E}) > e^{Kt}/N^{\mathfrak{c}}$, then by the defining relation of the function $\eta(E)$ as in (3.7), we have $\tilde{\eta}\operatorname{Im}[m_t(\tilde{E}+\mathrm{i}\tilde{\eta})] = e^{Kt}M\log N/N$. We calculate

$$\begin{aligned}\int_E^{E+\tilde{\eta}} \mathrm{d}\mu_t(x) &\le \int_E^{E+\tilde{\eta}} \frac{2\tilde{\eta}^2}{(x-\tilde{E})^2+\tilde{\eta}^2}\mathrm{d}\mu_t(x) \\ &\le 2\tilde{\eta}\operatorname{Im}[m_t(\tilde{E}+\mathrm{i}\tilde{\eta})] = \frac{2e^{Kt}M\log N}{N}.\end{aligned} \tag{3.64}$$

Hence,

$$|\{i : \lambda_i(t) \le E_0\}| \le N\int_{-\infty}^{E_0} \mathrm{d}\mu_t(x) + CM\log(N). \tag{3.65}$$

The lower bound then follows by taking $E_0 = \gamma_{i-CM\log N}(t)$. The upper bound of (3.4) is proven similarly. □

4 Mesoscopic central limit theorem

In this section we prove a mesoscopic central limit theorem for β-DBM (1.1). We recall the parameters δ and M defined at the beginning of Sect. 3. In this section, we fix scale parameters, η_* and r such that $N^{-1} \leq \eta_* \ll r \leq 1$. If we assume that the initial data $\lambda(0)$ is regular down to the scale η_*, on the interval $[E_0 - r, E_0 + r]$, we can prove that after time $t \gg \eta_*$, the linear statistics satisfy a central limit theorem on the scale $\eta \ll t$. The precise definition of regularity is the following assumption.

Assumption 4.1 We assume that the initial data satisfies the following two conditions.

1. There exists some finite constant $\mathfrak{a}$, such that $-\mathfrak{a} \leq \lambda_1(0) \leq \lambda_2(0) \cdots \leq \lambda_N(0) \leq \mathfrak{a}$;
2. There exists some finite constant $\mathfrak{d}$, such that

$$\mathfrak{d}^{-1} \leq \operatorname{Im}[m_0(z)] \leq \mathfrak{d}, \tag{4.1}$$

uniformly for any $z \in \{E + \mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta_* \leq \eta \leq 1\}$.

Under the above assumption we can prove the following mesoscopic central limit theorem for the Stieltjes transform.

Theorem 4.2 *Suppose V satisfies Assumption 2.1, and moreover that V is C^5. Fix small constant $\delta > 0$, $M = (\log N)^{2+2\delta}$, and $N^{-1} \leq \eta_* \ll r \leq 1$, and assume that the initial data $\lambda(0)$ satisfies Assumption 4.1. For any time t with $\eta_* \ll t \ll (\log N)^{-1} r \wedge (\log N)^{-2}$, the normalized Stieltjes transform $\Gamma_t(z) := N \operatorname{Im}[z]\,(\tilde{m}_t(z) - m_t(z))$ is asymptotically a Gaussian field on $\{E + \mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], M^2/N \ll \eta \ll t/(M \log N)\}$. We have for any $z_1, z_2, \dots, z_k \in \{E + \mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], M^2/N \ll \eta \ll t/(M \log N)\}$, the joint characteristic function of $\Gamma_t(z_1), \Gamma_t(z_2), \dots, \Gamma_t(z_k)$ is given by*

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{\mathrm{i}\sum_{j=1}^{k} a_j \operatorname{Re}[\Gamma_t(z_j)] + b_j \operatorname{Im}[\Gamma_t(z_j)]\right\}\right] \\ &= \exp\left\{\sum_{1\leq j,\ell\leq k} \operatorname{Re}\left[\frac{(a_j - \mathrm{i}b_j)(a_\ell + \mathrm{i}b_\ell)\operatorname{Im}[z_j]\operatorname{Im}[z_\ell]}{2\beta(z_j - \bar{z}_\ell)^2}\right]\right\} \\ &\quad + \mathrm{O}\left(\frac{M^2}{N \min_j\{\operatorname{Im}[z_j]\}} + \frac{M \log N \max_j\{\operatorname{Im}[z_j]\}}{t}\right). \end{aligned} \tag{4.2}$$

Remark 4.3 The above covariance structure is universal, independent of the potential V. By a slightly more elaborate analysis, we can compute the joint characteristic function of $\Gamma_{t_1}(z_1), \Gamma_{t_2}(z_2), \dots, \Gamma_{t_k}(z_k)$, where the times $\eta_* \ll t_1, t_2, \dots, t_k \ll (\log N)^{-1} r \wedge (\log N)^{-2}$ and $z_j \in \{E + \mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], M^2/N \ll \eta \ll t_i/(M \log N)\}$ for $j = 1, 2, \dots, k$,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\mathrm{i}\sum_{j=1}^k a_j \operatorname{Re}[\Gamma_{t_j}(z_j)] + b_j \operatorname{Im}[\Gamma_{t_j}(z_j)]\right\}\right] \\
&= \exp\left\{\sum_{1\le j,\ell\le k} \operatorname{Re}\left[\frac{(a_j - \mathrm{i}b_j)(a_\ell + \mathrm{i}b_\ell)\operatorname{Im}[z_j]\operatorname{Im}[z_\ell]}{2\beta(z_{t_j\wedge t_\ell}\circ z_{t_j}^{-1}(z_j) - \overline{z_{t_j\wedge t_\ell}\circ z_{t_\ell}^{-1}(z_\ell)})^2}\right]\right\} \\
&\quad + \mathrm{O}\left(\frac{M^2}{N\min_j\{\operatorname{Im}[z_j]\}} + M\log N \max_j\left\{\frac{\operatorname{Im}[z_j]}{t_j}\right\}\right).
\end{aligned} \tag{4.3}$$

By the Littlewood–Paley type decomposition argument developed in [49], the above theorem implies the following central limit theorem for mesoscopic linear statistics.

Corollary 4.4 *Under the assumptions of Theorem 4.2, the following holds for any compactly supported test function ψ in the Sobolev space H^s with $s > 1$. Let $M^2/N \ll \eta \ll t$, $E \in [E_0 - r, E_0 + r]$, and define*

$$\psi_{\eta,E}(x) = \psi\left(\frac{x-E}{\eta}\right). \tag{4.4}$$

The normalized linear statistics converges to a Gaussian

$$\mathcal{L}(\psi_{\eta,E}) := \sum_{i=1}^N \psi_{\eta,E}(\lambda_i(t)) - N\int_{\mathbb{R}} \psi_{\eta,E}(x)\mathrm{d}\mu_t(x) \to N(0,\sigma_\psi^2), \tag{4.5}$$

in distribution as $N \to \infty$, where

$$\sigma_\psi^2 := \frac{1}{2\beta\pi^2}\int_{\mathbb{R}^2}\left(\frac{\psi(x)-\psi(y)}{x-y}\right)^2 \mathrm{d}x\mathrm{d}y. \tag{4.6}$$

4.1 Regularity of the Stieltjes transform of the limit measure-valued process

In this subsection we analyze the differential equation of the Stieltjes transform of the limit measure-valued process (2.26) under the assumptions of Theorem 4.2. We will need some regularity results for m_t. First we prove some preliminary estimates. The following two estimates are standard.

Lemma 4.5 *Under the assumptions of Theorem 4.2, we have, for any interval $I = [E-\eta, E+\eta]$ with $E \in [E_0 - r, E_0 + r]$ and $\eta \in [4\mathfrak{d}^2\eta_*, 1]$, the estimate*

$$\frac{|I|N}{16\mathfrak{d}^3} \le |\{i : \lambda_i(0) \in I\}| \le \mathfrak{d}|I|N. \tag{4.7}$$

Proof For the upper bound, by taking $z = E + \mathrm{i}\eta$, we have

$$\mathfrak{d} \ge \operatorname{Im}[m_0(E+\mathrm{i}\eta)] \ge \frac{1}{N}\sum_{i:\lambda_i\in I}\frac{\eta}{(\lambda_i - E)^2 + \eta^2} \ge \frac{|\{i : \lambda_i \in I\}|}{2N\eta}. \tag{4.8}$$

For the lower bound, let $\eta_1 = \eta/(4\mathfrak{d}^2) \geq \eta_*$, we have

$$\begin{aligned}
\mathfrak{d}^{-1} \leq \operatorname{Im}[m_0(E+\mathrm{i}\eta_1)] &= \frac{1}{N}\sum_{i:\lambda_i \in I} \frac{\eta_1}{(\lambda_i - E)^2 + \eta_1^2} + \frac{1}{N}\sum_{i:\lambda_i \notin I} \frac{\eta_1}{(\lambda_i - E)^2 + \eta_1^2} \\
&\leq \frac{|\{i : \lambda_i \in I\}|}{N\eta_1} + \frac{\eta_1}{\eta}\frac{1}{N}\sum_{i=1}^{N} \frac{2\eta}{(\lambda_i - E)^2 + \eta^2} \\
&\leq \frac{|\{i : \lambda_i \in I\}|}{N\eta_1} + \frac{2\eta_1}{\eta}\operatorname{Im}[m_0(E+\mathrm{i}\eta)] \\
&\leq \frac{4\mathfrak{d}^2|\{i : \lambda_i \in I\}|}{N\eta} + \frac{1}{2\mathfrak{d}},
\end{aligned} \tag{4.9}$$

and the lower bound follows by rearranging. □

Corollary 4.6 *Assume the conditions of Theorem 4.2. Let $u = E + \mathrm{i}\eta$ with $E \in [E_0 - r, E_0 + r]$ and $\eta \in [\eta_*, 1]$. There exists a constant $C > 0$ so that if $\operatorname{Im}[z_t(u)] > 0$, then*

$$|m_t(z_t(u))| \leq C \log N, \tag{4.10}$$

and

$$|\partial_t m_t(z_t(u))| \leq C \log N. \tag{4.11}$$

Proof For $t = 0$, let $\eta_1 = 4\mathfrak{d}^2\eta$. By a dyadic decomposition we have

$$\begin{aligned}
|m_0(u)| &\leq \frac{1}{N}\left(\sum_{|\lambda_i - E| \leq \eta_1} \frac{1}{\eta} + \sum_{k=1}^{\lfloor -\log_2(\eta_1)\rfloor} \sum_{2^k\eta_1 \geq |\lambda_i - E| \geq 2^{k-1}\eta_1} \frac{1}{|\lambda_i - E|} + \sum_{|\lambda_i - E| \geq 1/2} \frac{1}{|\lambda_i - E|}\right) \\
&\leq 2\mathfrak{d}\eta_1/\eta - 4\mathfrak{d}\log_2 \eta_1 + 2 \leq C \log N.
\end{aligned} \tag{4.12}$$

By Proposition 2.6 we have that $|\partial_z V'(z)| \leq C$ and $|g(z,x)| \leq C$ and so

$$|\partial_s m_s(z_s(u)| \leq C(|m_s(z_s)| + 1), \tag{4.13}$$

and therefore,

$$|m_t(z_t(u))| \leq e^{Ct}(|m_0(z_0(u))| + 1) = \mathrm{O}(\log N). \tag{4.14}$$

The claim follows. □

We now derive estimates on quantities appearing in our analysis of m_t.

Lemma 4.7 *Assume that Assumption 4.1 holds. Let* $t \ll (\log N)^{-1} r \wedge (\log N)^{-2}$, *and* $u \in \{E + \mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta \in [\eta_*, 1]\}$. *If* $z_t(u) \in \{E + \mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], M^2/N \ll \eta \ll t\}$, *then for* $0 \le s \le t$, *we have that* $z_s \in \mathcal{D}_s$ *as defined in (3.2), and moreover,*

$$\int_0^t \frac{\mathrm{d}s}{\mathrm{Im}[z_s]^p} \le 2\mathfrak{d} \int_0^t \frac{\mathrm{Im}[m_s(z_s)]}{\mathrm{Im}[z_s]^p} \mathrm{d}s \le \begin{cases} C \log N, & p = 1, \\ \frac{C}{\mathrm{Im}[z_t]^{p-1}}, & p > 1. \end{cases} \tag{4.15}$$

Proof Let u be as in the statement of the lemma and denote $z_s = z_s(u)$. By (4.10), we have $|\mathrm{Re}[z_s]| \le r + Ct \log N \le 3\mathfrak{b} - s$ and $\mathrm{Im}[z_s] \le 1 + Ct \log N \le 3\mathfrak{b} - s$, since $t \le (\log N)^{-2}$. By the assumption $\mathfrak{d}^{-1} \le \mathrm{Im}[m_0(u)] \le \mathfrak{d}$ and the estimate (2.29), we have uniformly for any $0 \le s \le t$,

$$(2\mathfrak{d})^{-1} \le \mathrm{Im}[m_s(z_s(u))] \le 2\mathfrak{d}, \tag{4.16}$$

since $t \ll 1$. Moreover, by (2.28), we have $\mathrm{Im}[z_s] \ge c\, \mathrm{Im}[z_t] \gg M^2/N$. Therefore,

$$\frac{e^{Ks} M \log N}{N\, \mathrm{Im}[m_s(z_s)]} \vee \frac{e^{Ks}}{N^{\mathfrak{c}}} \ll \frac{M^2}{N} \ll \mathrm{Im}[z_s].$$

It follows that $z_s(u) \in \mathcal{D}_s$. Since $\mathrm{Im}[m_s(z_s)] \ge (2\mathfrak{d})^{-1}$, we have

$$\int_0^t \frac{\mathrm{d}s}{\mathrm{Im}[z_s]^p} \le 2\mathfrak{d} \int_0^t \frac{\mathrm{Im}[m_s(z_s)]}{\mathrm{Im}[z_s]^p} \mathrm{d}s. \tag{4.17}$$

The case $p = 1$ estimate of (4.15) follows from (2.31) by using the estimate $\mathrm{Im}[u]/\mathrm{Im}[z_t(u)] \le CN$ of (3.8). The case $p > 1$ follows from (2.31). □

Lemma 4.8 *The following holds under the assumptions of Theorem 4.2. Let* $u = E + \mathrm{i}\eta$ *with* $E \in [E_0 - r, E_0 + r]$ *and* $\eta \in [\eta_*, 1]$. *There exists a uniform constant* $c > 0$ *so that If* $\mathrm{Im}[z_t(u)] > 0$, *then*

$$1 - t\, \mathrm{Re}[\partial_z m_0(u)] \ge c. \tag{4.18}$$

Proof By the upper bound in (2.30), since $\mathrm{Im}[z_t(u)] \ge 0$, we have

$$\eta = \mathrm{Im}[u] \ge \frac{1 - e^{-Ct}}{C} \mathrm{Im}[m_0(u)] \ge \left(t - \frac{Ct^2}{2}\right) \mathrm{Im}[m_0(u)]. \tag{4.19}$$

We write the LHS of (4.18) as

$$1 - t\, \mathrm{Re}[\partial_z m_0(u)] = 1 - \frac{t}{\eta} \mathrm{Im}[m_0(u)] + \frac{t}{N} \sum_{i=1}^{N} \frac{2\eta^2}{|\lambda_i(0) - u|^4}. \tag{4.20}$$

We consider the following two cases:

1. If $\eta \geq 2\mathfrak{d}t$, then by (4.20) and assumption (4.1), $1 - t\,\mathrm{Re}[\partial_z m_0(u)] \geq 1/2$.
2. If $\eta < 2\mathfrak{d}t$, let $\eta_1 = \eta \vee 4\mathfrak{d}^2\eta_* \leq 4\mathfrak{d}^2\eta$. By combining (4.19), (4.20) and (4.7), we have

$$1 - t\,\mathrm{Re}[\partial_z m_0(u)] \geq -\frac{Ct^2}{2\eta}\,\mathrm{Im}[m_0(u)] + \frac{t}{N}\sum_{i:|\lambda_i(0)-E|\leq \eta_1} \frac{2\eta^2}{(2\eta_1^2)^2}$$
$$\geq -\frac{C\mathfrak{d}t^2}{2\eta} + \frac{t}{2^{10}\mathfrak{d}^9\eta} = \frac{t}{2^{10}\mathfrak{d}^9\eta}\left(1 - 2^9 C\mathfrak{d}^{10}t\right) \geq \frac{1}{2^{12}\mathfrak{d}^{10}}, \tag{4.21}$$

where we used $t \ll 1$.

□

In the following we derive the regularity of the Stieltjes transform of the limiting measure-valued process (2.26). As a preliminary we study the flow map $u \to z_s(u)$, and prove that it is Lipschitz.

Proposition 4.9 *Under the assumptions of Theorem 4.2 we have the following. Let $u = E + \mathrm{i}\eta$, such that $E \in [E_0 - r, E_0 + r]$ and $\eta \in [\eta_*, 1]$. If $\mathrm{Im}[z_t(u)] > 0$, then for $0 \leq s \leq t$,*

$$c \leq |\partial_x z_s(u)|, |\partial_y z_s(u)| \leq C, \tag{4.22}$$
$$|\partial_z m_s(z_s(u))| = \mathrm{O}\left(t^{-1}\right). \tag{4.23}$$

where the constants depend on V' and $\mathfrak{d}$.

Proof For $s = 0$, by (4.19) we have

$$|\partial_z m_0(u)| \leq \frac{1}{N}\sum_{i=1}^{N} \frac{1}{|\lambda_i(0) - u|^2} = \frac{\mathrm{Im}[m_0(u)]}{\mathrm{Im}[u]} = \mathrm{O}\left(t^{-1}\right). \tag{4.24}$$

By taking derivative with respect to x on both sides of (2.25), we get

$$\partial_s \partial_x z_s(u) = -\partial_z m_s(z_s(u))\partial_x z_s(u) + \frac{\partial_x V'(z_s(u))}{2}, \quad \partial_x z_0(u) = 1, \tag{4.25}$$

where $\partial_x V'(z_s(u)) = \partial_z V'(z_s(u))\partial_x z_s(u) + \partial_{\bar{z}} V'(z_s(u))\partial_x \bar{z}_s(u)$. By taking derivative with respect to x on both sides of (2.26), we have

$$\begin{aligned}&\partial_s\left(\partial_z m_s(z_s(u))\right)\partial_x z_s(u) + \partial_z m_s(z_s(u))\partial_s\partial_x z_s(u)\\ &\quad = \frac{\partial_z m_s(z_s(u))\partial_z V'(z_s(u))\partial_x z_s(u) + m_s(z_s(u))\partial_x\partial_z V'(z_s(u))}{2}\\ &\qquad + \int_{\mathbb{R}} \partial_x g(z_s(u), w)\mathrm{d}\mu_s(w),\end{aligned} \tag{4.26}$$

where $\partial_x g(z_s(u), w) = \partial_z g(z_s(u), w)\partial_x z_s(u) + \partial_{\bar z} g(z_s(u), w)\partial_x \bar z_s(u)$. Note that $\partial_x z_0(u) = \partial_x(x + \mathrm{i}y) = 1$. We define

$$\sigma = t \wedge \inf_{s\geq 0}\{\partial_x z_s(u) = 0\}. \tag{4.27}$$

Then $0 < \sigma \leq t \ll (\log N)^{-2}$, and for any $0 \leq s < \sigma$ we have $|\partial_x \bar z_s(u)| = |\partial_x z_s(u)|$.

By combining (4.25) and (4.26), and rearranging we have

$$\partial_s\left[\partial_z m_s(z_s(u))\right] = (\partial_z m_s(z_s(u)))^2 + 2\partial_z m_s(z_s(u))b_s + c_s, \tag{4.28}$$

where

$$\begin{aligned} b_s &= \frac{\partial_x V'(z_s(u))}{4\partial_x z_s(u)} + \frac{\partial_z V'(z_s(u))}{4}, \quad c_s = \frac{m_s(z_s(u))\partial_x\partial_z V'(z_s(u))}{2\partial_x z_s(u)} \\ &+ \frac{\int_{\mathbb{R}} \partial_x g(z_s(u), w)\mathrm{d}\mu_s(w)}{\partial_x z_s(u)}. \end{aligned} \tag{4.29}$$

Under the assumptions of Theorem 4.2 we have $\|V'(z)\|_{C^2} \leq C$ and $|\partial_z g(z, w)| + |\partial_{\bar z} g(z, w)| \leq C$ by Proposition 2.6. Combining this with Corollary 4.6 we have $|b_s| + |c_s| \leq C$ for $0 \leq s \leq \sigma$.

First we derive an upper bound for the real part of $\partial_z m_s(z_s(u))$. It follows from taking real part on both sides of (4.28) that

$$\begin{aligned} &\partial_s \operatorname{Re}[\partial_z m_s(z_s(u))] \\ &\quad= (\operatorname{Re}[\partial_z m_s(z_s(u)])^2 - (\operatorname{Im}[\partial_z m_s(z_s(u))])^2 + 2\operatorname{Re}[\partial_z m_s(z_s(u))]\operatorname{Re}[b_s] \\ &\qquad - 2\operatorname{Im}[\partial_z m_s(z_s(u))]\operatorname{Im}[b_s] + \operatorname{Re}[c_s] \\ &\quad\leq (\operatorname{Re}[\partial_z m_s(z_s(u))])^2 + 2\operatorname{Re}[\partial_z m_s(z_s(u))]\operatorname{Re}[b_s] + \operatorname{Im}[b_s]^2 + \operatorname{Re}[c_s] \\ &\quad= (\operatorname{Re}[\partial_z m_s(z_s(u)) + b_s])^2 + \operatorname{Re}[c_s - b_s^2]. \end{aligned}$$

Therefore, we derive

$$\partial_s(\operatorname{Re}[\partial_z m_s(z_s(u))])_+ \leq ((\operatorname{Re}[\partial_z m_s(z_s(u))])_+ + C])^2 + C\log N, \tag{4.30}$$

with initial data $(\operatorname{Re}[\partial_z m_0(z_0(u))])_+ \leq (1-c)/t$ from (4.18). The above ODE is separable and by solving it explicitly and using the fact that $\sqrt{\log N}t \ll 1$, we get

$$\begin{aligned} \operatorname{Re}[\partial_z m_s(z_s(u))] \leq& \sqrt{C\log N}\tan\left(\arctan\left(\frac{(1-c)/t + C}{\sqrt{C\log N}}\right) + \sqrt{C\log N}s\right) \\ \asymp& \sqrt{C\log N}\tan\left(\frac{\pi}{2} - \frac{(c - Ct)\sqrt{C\log N}t}{1 - c + Ct}\right) \asymp \frac{1-c}{ct}, \end{aligned} \tag{4.31}$$

uniformly for $0 \leq s \leq \sigma$. Therefore, there exists some constant C, so that $\operatorname{Re}[\partial_z m_s(z_s(u))] \leq C/t$, uniformly for any $0 \leq s \leq \sigma$.

Using this we derive from (4.28) that,

$$\begin{aligned}\partial_s|\partial_z m_s(z_s(u))|^2 &= 2\,\mathrm{Re}[\partial_s\partial_z m_s(z_s(u))\partial_z\bar{m}_s(z_s(u))]\\ &= 2\,\mathrm{Re}[\partial_z m_s(z_s(u))]|\partial_z m_s(z_s(u))|^2+4\,\mathrm{Re}[b_s]|\partial_z m_s(z_s(u))|^2\\ &\quad+2\,\mathrm{Re}[c_s\partial_z\bar{m}_s(z_s(u))]\\ &\le \frac{C}{t}|\partial_z m_s(z_s(u))|^2+C(t\log N)^2.\end{aligned} \tag{4.32}$$

It follows by Gronwall's inequality that $|\partial_z m_s(z_s(u))| = \mathrm{O}(1/t)$ uniformly for $0 \le s \le \sigma$. Notice that $\partial_x z_0(u) = 1$, and that (4.25) implies

$$\partial_x z_s(u) = \exp\left\{\int_0^s -\partial_z m_s(z_s(u)) + \frac{\partial_z V'(z_s(u))}{2} + \frac{\partial_{\bar{z}} V'(z_s(u))\partial_x \bar{z}_s}{2\partial_x z_s(u)}\mathrm{d}\tau\right\} \asymp 1. \tag{4.33}$$

uniformly for $0 \le s \le \sigma$. Therefore, $\sigma = t$ and the estimates (4.23) and $|\partial_x z_t(u)| \asymp 1$ are immediate consequences. The estimate $|\partial_y z_t(u)| \asymp 1$ follows from the same argument. □

Finally, we have the following results for the regularity of $m_t(w)$.

Corollary 4.10 *Suppose that the assumptions of Theorem 4.2 hold, and let $\eta_* \ll t \ll (\log N)^{-1} r \wedge (\log N)^{-2}$. We have,*

i) For any $w \in \{E + \mathrm{i}\eta : E \in [E_0 - 3r/4, E_0 + 3r/4], 0 < \eta \le 3/4\}$, we have that $z_t^{-1}(w) \subset \{E + \mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta_* \le \eta \le 1\}$, and $\partial_z m_t(w) = \mathrm{O}(1/t)$.
ii) Fix $u \in \{E + \mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta \in [\eta_*, 1]\}$. If $z_t(u) \in \{E + \mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], 0 < \eta \ll t\}$, then for $0 \le s \le t$, and any $w \in \mathbb{C}_+$ such that $|w - z_s(u)| \le \mathrm{Im}[z_s(u)]/2$, we have $|\partial_z m_s(w)| = \mathrm{O}(1/t)$.

In both statements, the implicit constants depend on V and $\mathfrak{d}$.

Proof We first consider the first statement in i). Uniformly for any $u \in \{E + \mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta_* < \eta \le 1\} \cap \Omega_t$ (with Ω_t as in Proposition 2.8), we have by (2.30), (4.10) and (2.25), that there exists a constant C depending on V and $\mathfrak{d}$, such that

$$\begin{aligned}\max\{0, \mathrm{Im}[u] - 2tC\,\mathrm{Im}[m_0(u)]\} \le \mathrm{Im}[z_t(u)] \le e^{Ct}\left(\mathrm{Im}[u] - \frac{1-e^{-Ct}}{C}\mathrm{Im}[m_0(u)]\right),\\ \mathrm{Re}[u] - Ct\log N \le \mathrm{Re}[z_t(u)] \le \mathrm{Re}[u] + Ct\log N.\end{aligned} \tag{4.34}$$

By Proposition 2.8, z_t is surjective from Ω_t onto $\mathbb{C}_+$. The first statement in i) follows from the assumptions $t \gg \eta_*$ and $r \gg t\log N$. The second statement in i), is then a consequence of (4.23) and the equality $\partial_z m_t(w) = \partial_z m_t(z_t(z_t^{-1}(w)))$.

For ii), since $\mathrm{Im}[m_0(u)] = \mathrm{O}(1)$, it follows from (2.30) that $\mathrm{Im}[u] = t\,\mathrm{Im}[m_0(u)] + o(t)$. If $s \le t/2$, then we see that by (2.30) that $t/C \le \mathrm{Im}[z_s(u)] \le Ct$ for some $C > 0$.

Furthermore, by (4.10) and (2.25) we see that $\mathrm{Re}[u] - Ct\log(N) \le \mathrm{Re}[z_s(u)] \le \mathrm{Re}[u] + Ct\log(N)$. We also observe that $\mathrm{Im}[w] \ge t/2C$. It follows from the same argument as in i) that $\{w \in \mathbb{C}_+ : |w - z_s(u)| \le \mathrm{Im}[z_s(u)]/2\} \subseteq z_s(\{E + \mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta \in [\eta_*, 1]\} \cap \Omega_s)$. Therefore, by (2.29), uniformly for $\{w \in \mathbb{C}_+ : |w - z_s(u)| \le \mathrm{Im}[z_s(u)]/2\}$, $\mathrm{Im}[m_s(w)] = \mathrm{O}(\mathrm{Im}[m_0(z_s^{-1}(w))]) = \mathrm{O}(1)$, and therefore

$$|\partial_z m_s(w)| \le \frac{\mathrm{Im}[m_s(w)]}{\mathrm{Im}[w]} = \mathrm{O}\left(\frac{1}{t}\right). \tag{4.35}$$

If $s \ge t/2$, from i), uniformly for any $w \in \{E + \mathrm{i}\eta : E \in [E_0 - 3r/4, E_0 + 3r/4], 0 < \eta \le 3/4\}$, $\partial_z m_s(w) = \mathrm{O}(1/s) = \mathrm{O}(1/t)$. Moreover, we have $\{w \in \mathbb{C}_+ : |w - z_s(u)| \le \mathrm{Im}[z_s(u)]/2\} \subseteq \{E + \mathrm{i}\eta : E \in [E_0 - 3r/4, E_0 + 3r/4], 0 < \eta \le 3/4\}$. The statement follows. □

4.2 Proof of Theorem 4.2

Using regularity of m_t and the local law we infer the following regularity for the empirical Stieltjes transform $\tilde{m}_t$.

Lemma 4.11 *Suppose that the assumptions of Theorem 4.2 hold. Let $\eta_* \ll t \ll (\log N)^{-1} r \wedge (\log N)^{-2}$. Fix $u \in \{E + \mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta \in [\eta_*, 1]\}$. If $z_t(u) \in \{E + \mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], 0 < \eta \ll t\}$, then on the event Ω (as defined in the proof of Proposition 3.8), we have the following estimate uniformly for $0 \le s \le t$,*

$$\partial_z^p \tilde{m}_s(z_s(u)) = \mathrm{O}\left(\frac{M}{N\,\mathrm{Im}[z_s(u)]^{p+1}} + \frac{1}{t\,\mathrm{Im}[z_s(u)]^{p-1}}\right). \tag{4.36}$$

Proof The estimate (4.36) is a consequence of the following two statements.

$$\partial_z^p\left(\tilde{m}_s(z_s(u)) - m_s(z_s(u))\right) = \mathrm{O}\left(\frac{M}{N\,\mathrm{Im}[z_s(u)]^{p+1}}\right), \tag{4.37}$$

$$\partial_z^p m_s(z_s(u)) = \mathrm{O}\left(\frac{1}{t\,\mathrm{Im}[z_s(u)]^{p-1}}\right). \tag{4.38}$$

For (4.37), since both $\tilde{m}_s$ and m_s are analytic on the upper half plane, by Cauchy's integral formula

$$\partial_z^p\left(\tilde{m}_s(z_s(u)) - m_s(z_s(u))\right) = \frac{p!}{2\pi\mathrm{i}} \oint_{\mathcal{C}} \frac{\tilde{m}_s(w) - m_s(w)}{(w - z_s(u))^{p+1}}\mathrm{d}w, \tag{4.39}$$

where $\mathcal{C}$ is a small contour in the upper half plane centering at $z_s(u)$ with radius $\mathrm{Im}[z_s(u)]/2$. On the event Ω, we use (3.3) in Theorem 3.1 to bound the integral by

$$\left|\frac{p!}{2\pi \mathrm{i}}\oint_{\mathcal{C}}\frac{\tilde{m}_s(w)-m_s(w)}{(w-z_s(u))^{p+1}}\mathrm{d}w\right| \leq \frac{p!}{2\pi}\oint_{\mathcal{C}}\frac{|\tilde{m}_s(w)-m_s(w)|}{|w-z_s(u)|^{p+1}}\mathrm{d}w$$
$$= \mathrm{O}\left(\frac{M}{N\,\mathrm{Im}[z_s(u)]^{p+1}}\right). \tag{4.40}$$

For (4.38), Cauchy's integral formula leads to

$$\left|\partial_z^p m_s(z_s(u))\mathrm{d}s\right| \leq \frac{(p-1)!}{2\pi}\oint_{\mathcal{C}}\frac{|\partial_z m_s(w)|}{|w-z_s(u)|^p}\mathrm{d}w = \mathrm{O}\left(\frac{1}{t\,\mathrm{Im}[z_s(u)]^{p-1}}\right). \tag{4.41}$$

where we used ii) in Corollary 4.10 which states that $|\partial_z m_s(w)| = \mathrm{O}(1/t)$. □

By i) in Corollary 4.10, $\{E+\mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], M^2/N \ll \eta \ll t\} \subseteq z_t(\{E+\mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta \in [\eta_*, 1]\} \cap \Omega_t)$. In the following, we fix some $u \in \{E+\mathrm{i}\eta : E \in [E_0 - r, E_0 + r], \eta \in [\eta_*, 1]\}$, such that $z_t(u) \in \{E+\mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], M^2/N \ll \eta \ll t\}$. By Lemma 4.7, $z_t \in \mathcal{D}_t$, and the local law of Theorem 3.1 holds.

We integrate both sides of (3.22), and get the following integral expression for $\tilde{m}_t(z_t)$,

$$\begin{aligned}\tilde{m}_t(z_t) - m_t(z_t) &= \int_0^t (\tilde{m}_s(z_s) - m_s(z_s))\,\partial_z\left(\tilde{m}_s(z_s) + \frac{V'(z_s)}{2}\right)\mathrm{d}s\\ &+ \frac{1}{\pi}\int_0^t\int_{\mathbb{C}}\partial_{\bar{w}}\tilde{g}(z_s, w)(\tilde{m}_s(w) - m_s(w))\mathrm{d}^2w\mathrm{d}s + \frac{2-\beta}{\beta N^2}\int_0^t\sum_{i=1}^N\frac{\mathrm{d}s}{(\lambda_i(s)-z_s)^3}\\ &- \sqrt{\frac{2}{\beta N^3}}\int_0^t\sum_{i=1}^N\frac{\mathrm{d}B_i(s)}{(\lambda_i(s)-z_s)^2}.\end{aligned} \tag{4.42}$$

For the proof of the mesoscopic central limit theorem, we will show that the first three terms on the righthand side of (4.42) are negligible, and the Gaussian fluctuation is from the last term, i.e. the integral with respect to Brownian motion. In the following Proposition, we calculate the quadratic variance of the Brownian integrals.

Proposition 4.12 *Suppose that the assumptions of Theorem 4.1 hold. Fix* $u, u' \in \{E+\mathrm{i}\eta : E_0 - r \leq E \leq E_0 + r, \eta_* \leq \eta \leq 1\}$*. Let* $z_t := z_t(u)$ *and* $z_t' := z_t(u')$*. If*

$$z_t, z_t' \in \{E+\mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], M^2/N \ll \eta \ll t\}, \tag{4.43}$$

and $\mathrm{Im}[z_t] \geq \mathrm{Im}[z_t']$*, then*

$$\frac{1}{N^3}\int_0^t\sum_{i=1}^N\frac{\mathrm{d}s}{(\lambda_i(s)-z_s)^4} = \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_t]^3} + \frac{1}{N^2 t\,\mathrm{Im}[z_t]}\right), \tag{4.44}$$

$$\frac{1}{N^3}\int_0^t\sum_{i=1}^N\frac{\mathrm{d}s}{(\lambda_i(s)-z_s)^2(\lambda_i(s)-z_s')^2} = \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_t]^2\,\mathrm{Im}[z_t']} + \frac{1}{N^2 t\,\mathrm{Im}[z_t]}\right), \tag{4.45}$$

$$\frac{1}{N^3}\int_0^t \sum_{i=1}^N \frac{\mathrm{d}s}{(\lambda_i(s)-\bar{z}_s)^2(\lambda_i(s)-z_s')^2}$$
$$= -\frac{1}{N^2(\bar{z}_t - z_t')^2} + \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_t]^2\,\mathrm{Im}[z_t']} + \frac{1}{N^2 t\,\mathrm{Im}[z_t]}\right). \tag{4.46}$$

Proof Since $\mathrm{Im}[m_0(z_0)] = \mathrm{Im}[m_0(u)] \asymp 1$, by (2.30) and (2.29), we have $\mathrm{Im}[z_s] \asymp \mathrm{Im}[z_t] + (t-s)$ and $\mathrm{Im}[z_s'] \asymp \mathrm{Im}[z_t'] + (t-s)$. Since $\mathrm{Im}[z_t] \geq \mathrm{Im}[z_t']$, there exists a constant c depending on V and $\mathfrak{d}$, such that uniformly for $0 \leq s \leq t$, $\mathrm{Im}[z_s] \geq c\,\mathrm{Im}[z_s']$.

For (4.44), the lefthand side can be written as the derivative of the Stieltjes transform $\tilde{m}_s$ at z_s, and so

$$\left|\frac{1}{6N^2}\int_0^t \partial_z^3 \tilde{m}_s(z_s)\mathrm{d}s\right| \leq \frac{C}{6N^2}\int_0^t \left(\frac{M}{N\,\mathrm{Im}[z_s]^4} + \frac{1}{t\,\mathrm{Im}[z_s]^2}\right)\mathrm{d}s$$
$$= \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_t]^3} + \frac{1}{N^2 t\,\mathrm{Im}[z_t]}\right), \tag{4.47}$$

where we used Lemma 4.11 and (4.15).

We write the LHS of (4.45), as a contour integral of $\tilde{m}_s$:

$$\frac{1}{N^3}\sum_{i=1}^N \frac{1}{(\lambda_i(s)-z_s)^2(\lambda_i(s)-z_s')^2} = \frac{1}{2\pi \mathrm{i} N^2}\oint_{\mathcal{C}} \frac{\tilde{m}_s(w)}{(w-z_s)^2(w-z_s')^2}\mathrm{d}w, \tag{4.48}$$

where if $\mathrm{Im}[z_s]/3 \geq |z_s - z_s'|$, then $\mathcal{C}$ is a contour centered at z_s with radius $\mathrm{Im}[z_s]/2$. In this case we have $\mathrm{dist}(\mathcal{C}, \{z_s, z_s'\}) \geq \mathrm{Im}[z_s]/6$. In the case that $|z_s - z_s'| \geq \mathrm{Im}[z_s]/3$, we let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ consist of two contours, where $\mathcal{C}_1$ is centered at z_s with radius $\min\{\mathrm{Im}[z_s'], \mathrm{Im}[z_s]\}/6$, and $\mathcal{C}_2$ is centered at z_s' with radius $\min\{\mathrm{Im}[z_s'], \mathrm{Im}[z_s]\}/6$. Then in this case we have $\mathrm{dist}(\mathcal{C}_1, z_s') \geq \mathrm{Im}[z_s]/6$ and $\mathrm{dist}(\mathcal{C}_2, z_s) \geq \mathrm{Im}[z_s]/6$. In the first case, thanks to Lemma 4.11 and ii) in Corollary 4.10, for $w \in \mathcal{C}$ we have

$$\tilde{m}_s(w) = \tilde{m}_s(z_s) + (w-z_s)\partial_z \tilde{m}_s(z_s) + (w-z_s)^2\,\mathrm{O}\left(\frac{M}{N\,\mathrm{Im}[z_s]^3} + \frac{1}{t\,\mathrm{Im}[z_s]}\right). \tag{4.49}$$

Plugging (4.49) into (4.48), we see that t he first two terms vanish and

$$|(4.48)| \leq \frac{C}{N^2}\int_{\mathcal{C}} \left(\frac{M}{N\,\mathrm{Im}[z_s]^5} + \frac{1}{t\,\mathrm{Im}[z_s]^3}\right)\mathrm{d}w$$
$$= \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_s]^4} + \frac{1}{N^2 t\,\mathrm{Im}[z_s]^2}\right), \tag{4.50}$$

where we used that $|\mathcal{C}| \asymp \mathrm{Im}[z_s]$. In the second case, (4.49) holds on $\mathcal{C}_1$. Similarly, for $w \in \mathcal{C}_2$ we have

$$\tilde{m}_s(w) = \tilde{m}_s(z_s') + (w - z_s')\partial_z \tilde{m}_s(z_s') + (w - z_s')^2 \, \mathrm{O}\left(\frac{M}{N\,\mathrm{Im}[z_s']^3} + \frac{1}{t\,\mathrm{Im}[z_s']}\right). \tag{4.51}$$

It follows by plugging (4.49) and (4.51) into (4.48), that we can bound (4.48) by

$$\begin{aligned} &\frac{C}{N^2}\left(\int_{\mathcal{C}_1}\left(\frac{M}{N\,\mathrm{Im}[z_s]^5} + \frac{1}{t\,\mathrm{Im}[z_s]^3}\right)\mathrm{d}w \right.\\ &\quad \left. + \int_{\mathcal{C}_2}\left(\frac{M}{N\,\mathrm{Im}[z_s]^2\,\mathrm{Im}[z_s']^3} + \frac{1}{t\,\mathrm{Im}[z_s]^2\,\mathrm{Im}[z_s']}\right)\mathrm{d}w\right)\\ &= \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_s]^2\,\mathrm{Im}[z_s']^2} + \frac{1}{N^2 t\,\mathrm{Im}[z_s]^2}\right), \end{aligned} \tag{4.52}$$

where we used $\mathrm{Im}[z_s] \geq c\,\mathrm{Im}[z_s']$ and $|\mathcal{C}_1|, |\mathcal{C}_2| = \mathrm{O}(\mathrm{Im}[z_s'])$. The estimate of the LHS of (4.45) follows by combining (4.50) and (4.52),

$$\begin{aligned} |(4.45)| &\leq \frac{C}{N^2}\int_0^t \frac{M}{N^3\,\mathrm{Im}[z_s]^2\,\mathrm{Im}[z_s']^2} + \frac{1}{N^2 t\,\mathrm{Im}[z_s]^2}\mathrm{d}s\\ &= \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_t]^2}\int_0^t \frac{\mathrm{d}s}{\mathrm{Im}[z_s']^2} + \frac{1}{N^2 t\,\mathrm{Im}[z_t]}\right)\\ &= \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_t]^2\,\mathrm{Im}[z_t']} + \frac{1}{N^2 t\,\mathrm{Im}[z_t]}\right), \end{aligned} \tag{4.53}$$

where we used that $\mathrm{Im}[z_s] \geq c\,\mathrm{Im}[z_t]$ in the second line, and (4.15) for the last line.

Finally, for (4.46),

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^N \frac{1}{(\lambda_i(s) - \bar{z}_s)^2(\lambda_i(s) - z_s')^2} &= \frac{2(\overline{-\tilde{m}_s(z_s)} + \tilde{m}_s(z_s'))}{(\bar{z}_s - z_s')^3}\\ &\quad + \frac{\overline{\partial_z \tilde{m}_s(z_s)} + \partial_z \tilde{m}_s(z_s')}{(\bar{z}_s - z_s')^2}. \end{aligned} \tag{4.54}$$

Note that $|\bar{z}_s - z_s'| \geq \mathrm{Im}[z_s] + \mathrm{Im}[z_s'] \asymp \mathrm{Im}[z_s]$. For the second term in (4.54), we have by (4.36),

$$\begin{aligned} \left|\frac{1}{N^2}\int_0^t \frac{\overline{\partial_z \tilde{m}_s(z_s)} + \partial_z \tilde{m}_s(z_s')}{(\bar{z}_s - z_s')^2}\right| &\leq \frac{C}{N^2}\int_0^t \frac{1}{\mathrm{Im}[z_s]^2}\left(\frac{M}{N\,\mathrm{Im}[z_s']^2} + \frac{1}{t}\right)\mathrm{d}s\\ &= \mathrm{O}\left(\frac{M}{N^3\,\mathrm{Im}[z_t]^2\,\mathrm{Im}[z_t']} + \frac{1}{N^2 t\,\mathrm{Im}[z_t]}\right). \end{aligned} \tag{4.55}$$

For the first term in (4.54), we recall the definition of the vector flow $z_s(u)$ as in (2.25). Since $\|V'(z)\|_{C^1} = \mathrm{O}(1)$, we have

$$\overline{-\tilde{m}_s(z_s)} + \tilde{m}_s(z_s') = \partial_s(\bar{z}_s - z_s') + \mathrm{O}(|\bar{z}_s - z_s'|). \tag{4.56}$$

Therefore,

$$\begin{aligned}
&\frac{2}{N^2}\int_0^t \frac{(\overline{-\tilde{m}_s(z_s)} + \tilde{m}_s(z_s'))}{(\bar{z}_s - z_s')^3}\mathrm{d}s \\
&= \frac{2}{N^2}\int_0^t \frac{\partial_s(\bar{z}_s - z_s')}{(\bar{z}_s - z_s')^3}\mathrm{d}s + \mathrm{O}\left(\frac{1}{N^2}\int_0^t \frac{\mathrm{d}s}{\mathrm{Im}[z_s]^2}\right) \\
&= -\frac{1}{N^2(\bar{z}_t - z_t')^2} + \frac{1}{N^2(\bar{u} - u')^2} + \mathrm{O}\left(\frac{1}{N^2\,\mathrm{Im}[z_t]}\right) \\
&= -\frac{1}{N^2(\bar{z}_t - z_t')^2} + \mathrm{O}\left(\frac{1}{N^2t^2} + \frac{1}{N^2\,\mathrm{Im}[z_t]}\right),
\end{aligned} \tag{4.57}$$

where we used $|\bar{u} - u'| \geq \mathrm{Im}[u] + \mathrm{Im}[u'] \geq ct$. This finishes the proof of Proposition 4.12. □

Proof of Theorem 4.2 Let the event Ω be as above. Thanks to the estimates Theorem 3.1 and Lemma 4.11 which hold on Ω, we can bound the first term on the RHS of (4.42) by

$$\begin{aligned}
&\left|\int_0^t (\tilde{m}_s(z_s) - m_s(z_s))\,\partial_z\left(\tilde{m}_s(z_s) + \frac{V'(z_s)}{2}\right)\mathrm{d}s\right| \\
&\leq C\int_0^t \frac{M}{N\,\mathrm{Im}[z_s]}\left(\frac{M}{N\,\mathrm{Im}[z_s]^2} + \frac{1}{t}\right)\mathrm{d}s \\
&= \mathrm{O}\left(\frac{M^2}{(N\,\mathrm{Im}[z_t])^2} + \frac{M\log N}{Nt}\right),
\end{aligned} \tag{4.58}$$

where we used (4.15).

For the second term on the righthand side of (4.42), by Proposition 3.9 we have on the event Ω

$$\left|\frac{1}{\pi}\int_0^t\int_{\mathbb{C}} \partial_{\bar{w}}\tilde{g}(z_s, w)(\tilde{m}_s(w) - m_s(w))\mathrm{d}^2w\mathrm{d}t\right| \leq \frac{CtM(\log N)^2}{N}. \tag{4.59}$$

We can rewrite the third term on the righthand side of (4.42) as

$$\frac{2-\beta}{\beta N^2}\int_0^t\sum_{i=1}^N \frac{\mathrm{d}s}{(\lambda_i(s) - z_s)^3} = \frac{2-\beta}{2\beta N}\int_0^t \partial_z^2\tilde{m}_s(z_s)\mathrm{d}s. \tag{4.60}$$

Thanks to Lemma 4.11, and (4.15) we have

$$\left|\int_0^t \partial_z^2 \tilde{m}_s(z_s)\mathrm{d}s\right| \le C\int_0^t \left(\frac{1}{N\,\mathrm{Im}[z_s]^3} + \frac{1}{t\,\mathrm{Im}[z_s]}\right)\mathrm{d}s = \mathrm{O}\left(\frac{1}{N(\mathrm{Im}[z_t])^2} + \frac{\log N}{t}\right). \tag{4.61}$$

It follows that

$$\left|\frac{2-\beta}{\beta N^2}\int_0^t \sum_{i=1}^N \frac{\mathrm{d}s}{(\lambda_i(s)-z_s)^3}\right| = \mathrm{O}\left(\frac{1}{(N\,\mathrm{Im}[z_t])^2} + \frac{\log N}{Nt}\right). \tag{4.62}$$

By combining the above estimates we see that on the event Ω, we have

$$\tilde{m}_t(z_t) - m_t(z_t) = \mathrm{O}\left(\frac{M^2}{(N\,\mathrm{Im}[z_t])^2} + \frac{M\log N}{Nt}\right) + \sqrt{\frac{2}{\beta N^3}}\int_0^t \sum_{i=1}^N \frac{\mathrm{d}B_i(s)}{(\lambda_i(s)-z_s)^2}. \tag{4.63}$$

In the following we show that the Brownian integrals are asymptotically jointly Gaussian. We fix some $u_j \in \{E + \mathrm{i}\eta : E_0 - r \le E \le E_0 + r, \eta_* \le \eta \le 1\}$, $j = 1, 2, \dots, k$ such that

$$z_t(u_j) \in \{E + \mathrm{i}\eta : E \in [E_0 - r/2, E_0 + r/2], M^2/N \ll \eta \ll t\}, \quad j = 1, 2, \dots, k. \tag{4.64}$$

For $1 \le j \le k$. Let

$$X_j(t) = \mathrm{Im}[z_t(u_j)]\sqrt{\frac{2}{\beta N}}\int_0^t \sum_{i=1}^N \frac{\mathrm{d}B_i(s)}{(\lambda_i(s)-z_s(u_j))^2}, \quad j = 1, 2, \dots, k. \tag{4.65}$$

We compute their joint characteristic function,

$$\mathbb{E}\left[\exp\left\{\mathrm{i}\sum_{j=1}^k a_j\,\mathrm{Re}[X_j(t)] + b_j\,\mathrm{Im}[X_j(t)]\right\}\right]. \tag{4.66}$$

Since $\sum_{j=1}^k a_j\,\mathrm{Re}[X_j(t)] + b_j\,\mathrm{Im}[X_j(t)]$ is a martingale, the following is also a martingale

$$\exp\left\{\mathrm{i}\sum_{j=1}^k a_j\,\mathrm{Re}[X_j(t)] + b_j\,\mathrm{Im}[X_j(t)]\} + \frac{1}{2}\left\langle\sum_{j=1}^k a_j\,\mathrm{Re}[X_j(t)] + b_j\,\mathrm{Im}[X_j(t)]\right\rangle\right\}. \tag{4.67}$$

In particular, its expectation is one. By Proposition 4.12, on the event Ω (as defined in the proof of Proposition 3.8), the quadratic variation is given by

$$\begin{aligned} &\frac{1}{2}\left\langle \sum_{j=1}^{k} a_j \operatorname{Re}[X_j(t)] + b_j \operatorname{Im}[X_j(t)] \right\rangle \\ &\quad = -\sum_{1\leq j,\ell\leq k} \operatorname{Re}\left[\frac{(a_j-\mathrm{i}b_j)(a_\ell+\mathrm{i}b_\ell)\operatorname{Im}[z_t(u_j)]\operatorname{Im}[z_t(u_\ell)]}{2\beta(z_t(u_j)-\overline{z_t(u_\ell)})^2}\right] \\ &\qquad + \mathrm{O}\left(\frac{M}{N\min_j\{z_t(u_j)\}} + \frac{\max_j\{\operatorname{Im}[z_t(u_j)]\}}{t}\right). \end{aligned} \tag{4.68}$$

Therefore,

$$\begin{aligned} (4.66) = \exp&\left\{\sum_{1\leq j,\ell\leq k} \operatorname{Re}\left[\frac{(a_j-\mathrm{i}b_j)(a_\ell+\mathrm{i}b_\ell)\operatorname{Im}[z_t(u_j)]\operatorname{Im}[z_t(u_\ell)]}{2\beta(z_t(u_j)-\overline{z_t(u_\ell)})^2}\right]\right\} \\ &+ \mathrm{O}\left(\frac{M}{N\min_j\{\operatorname{Im}[z_t(u_j)]\}} + \frac{\max_j\{\operatorname{Im}[z_t(u_j)]\}}{t}\right). \end{aligned} \tag{4.69}$$

Since by (4.63),

$$\Gamma_t(z_t(u_j)) = X_j(t) + \mathrm{O}\left(\frac{M^2}{N\operatorname{Im}[z_t]} + \frac{M\log N\operatorname{Im}[z_t(u_j)]}{t}\right), \tag{4.70}$$

and so (4.2) follows. This finishes the proof of Theorem 4.2. □

Proof of Corollary 4.4 The corollary follows from Theorem 4.2 and the rigidity estimate in Theorem 3.1 by the same argument as in [49].

We can approximate the test function $\psi_{\eta,E}(x)$ by its convolution with a Cauchy distribution on the scale η, as ε goes to 0,

$$\psi_{\eta,E}^{(\varepsilon)} := \psi_{\eta,E} * \frac{1}{\pi}\frac{(\varepsilon\eta)}{x^2+(\varepsilon\eta)^2} \to \psi_{\eta,E}, \quad \text{in } H^s. \tag{4.71}$$

We let

$$\mathcal{L}(\psi_{\eta,E}^{(\varepsilon)}) := \sum_{i=1}^{N} \psi_{\eta,E}^{(\varepsilon)}(\lambda_i(t)) - N\int_{\mathbb{R}} \psi_{\eta,E}^{\varepsilon}(x)\mathrm{d}\mu_t(x). \tag{4.72}$$

Then thanks to Theorem 4.2, $\mathcal{L}(\psi_{\eta,E}^{(\varepsilon)})$ is asymptotically Gaussian,

$$\mathbb{E}\left[e^{\mathrm{i}\xi\mathcal{L}(\psi_{\eta,E}^{(\varepsilon)})}\right] = \exp\left\{-\frac{\xi^2}{4\beta\pi^2}\operatorname{Re}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{\psi(x)-\psi(y)}{x-y+2\mathrm{i}\varepsilon}\right)^2\mathrm{d}x\mathrm{d}y\right\} + \mathrm{o}(1). \tag{4.73}$$

We can approximate the characteristic function of $\mathcal{L}(\psi_{\eta,E})$ by that of $\mathcal{L}(\psi^{(\varepsilon)}_{\eta,E})$,

$$\left|\mathbb{E}\left[e^{\mathrm{i}\xi\mathcal{L}(\psi_{\eta,E})}\right]-\mathbb{E}\left[e^{\mathrm{i}\xi\mathcal{L}(\psi^{(\varepsilon)}_{\eta,E})}\right]\right|\leq|\xi|\mathbb{E}\left[\left(\mathcal{L}(\psi_{\eta,E})-\mathcal{L}(\psi^{(\varepsilon)}_{\eta,E})\right)^2\right]^{1/2}. \quad (4.74)$$

By the Littlewood–Paley type decomposition argument developed in [49, Section 3], the righthand side of (4.74) can be bounded by the variance of the Stieltjes transform. It follows from the rigidity estimate in Theorem 3.1 by the same argument as in [49, Proposition 4.1], we have the following bound on the variance of the Stieltjes transform

$$\mathbb{E}\left[|\tilde{m}_t(z)-m_t(z)|^2\right]\leq\frac{C}{N^2\eta^{2+\delta}}, \quad (4.75)$$

uniformly for any $z\in\{E+\mathrm{i}\eta: E\in[E_0-r/2,E_0+r/2], 0\leq\eta\leq 1$, where $\delta>0$ can be arbitrarily small. Therefore, [49, Theorem 5] implies that

$$\mathbb{E}\left[\left(\mathcal{L}(\psi_{\eta,E})-\mathcal{L}(\psi^{(\varepsilon)}_{\eta,E})\right)^2\right]^{1/2}\leq C(\eta)\|\psi_{\eta,E}-\psi^{(\varepsilon)}_{\eta,E}\|_{H^s}, \quad (4.76)$$

provided that $2+\delta\leq 2s$.

It follows from combining (4.73), (4.74) and (4.76) that

$$\mathbb{E}\left[e^{\mathrm{i}\xi\mathcal{L}(\psi_{\eta,E})}\right]=\exp\left\{-\frac{\xi^2}{4\beta\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{\psi(x)-\psi(y)}{x-y}\right)^2\mathrm{d}x\mathrm{d}y\right\}+\mathrm{o}(1), \quad (4.77)$$

and the claim (4.5) follows. □

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A Universality

In this appendix we roughly sketch the following universality theorem.

Theorem A.1 *Suppose V satisfies Assumption 2.1. Fix $N^{-1}\leq\eta_*\ll r\leq 1$, and assume that the initial data $\lambda(0)$ satisfies Assumption 4.1. For any time t with $\eta_*\ll t\ll(\log N)^{-1}r\wedge(\log N)^{-2}$, and index k such that $E-r/2\leq\lambda_k(0)\leq E+r/2$, there is a constant $\mathfrak{c}>0$ so that the following estimate holds for any smooth test function $O:\mathbb{R}\to\mathbb{R}$ and N large enough,*

$$\left|\mathbb{E}\left[O(\rho N(\lambda_{k+1}(t)-\lambda_k(t)))\right]-\mathbb{E}_\beta\left[O(N(\lambda_{\ell+1}-\lambda_\ell))\right]\right|\leq N^{-\mathfrak{c}}, \quad (A.1)$$

where the second expectation is with respect to the Gaussian β-ensemble, the index ℓ satisfies $\varepsilon N \leq \ell \leq (1-\varepsilon)N$ for any $\varepsilon > 0$ and the scaling factor ρ is defined by

$$\rho := \frac{\mu_t(\gamma_k(t))}{\rho_{\mathrm{sc}}(\gamma_\ell^{\mathrm{sc}})} \tag{A.2}$$

where μ_t is the limiting measure-valued process appearing earlier, $\gamma_i(t)$ are its classical eigenvalue locations, ρ_{sc} is the semicircle density and γ_i^{sc} are its classical eigenvalue locations.

The proof of this theorem is a modification of the proof of universality in [38]. In this work, the above theorem was proven in the case that $\beta = 1, 2$ or 4 and V is quadratic. These assumptions ensured that the process $\lambda_i(t)$ was equal in distribution to the eigenvalues of a random matrix ensemble evolving according to a matrix Brownian motion. The matrix structure was used to prove the rigidity estimates (3.4) in this special case. Therefore the proof given there carries over without change to the case that $\beta \geq 1$ is general and V is quadratic. In order to prove the above theorem we need only show how to deal with general V. The case of general V was carried out also in the work [38], but in the case of equilibrium initial data. This is the case when the DBM with general potential has initial data the equilibrium measure of the β-ensemble with the potential V. Short-time universality was proved, which then implied universality for the initial data as it is stationary. The approach to β-ensembles of [38] is somewhat different than the approach to matrix Brownian motion given in the same work. This is due to the fact that the matrix Brownian motion is handled using a certain family of interpolating ensembles for which the rigidity was not known in the case of general β (the matrix case being handled via matrix estimates). Therefore, the proof of the above theorem comes down to a modification of the methods in [38]. In the interest of brevity, we mention the important changes.

k be as above, and fix a time $\eta_* \ll s \ll (\log N)^{-1} r \wedge (\log N)^{-2}$. Thanks to Theorem 3.1, the eigenvalue rigidity holds at time s. After appropriate re-scalings and translations of the initial data and the potential, let us assume that

$$\gamma_k(s) = \gamma_k^{\mathrm{sc}}, \qquad \mu_t(\gamma_k(s)) = \rho_{\mathrm{sc}}(\gamma_k^{\mathrm{sc}}). \tag{A.3}$$

Note that due to gap universality of the Gaussian β-ensemble [29], it suffices to assume $k = \ell$. We use a coupling idea from [12] which was used again in [38]. For $t \geq 0$, denote now

$$\mathrm{d}x_i(t) = \sqrt{\frac{2}{\beta N}}\mathrm{d}B_i(t) + \frac{1}{N}\sum_{j \neq i} \frac{1}{x_j(t) - x_i(t)}\mathrm{d}t - \frac{1}{2}V(x_i(t))\mathrm{d}t, \qquad x_i(0) = \lambda_i(s) \tag{A.4}$$

and

$$\mathrm{d}y_i(t) = \sqrt{\frac{2}{\beta N}}\mathrm{d}B_i(t) + \frac{1}{N}\sum_{j\neq i}\frac{1}{y_j(t)-y_i(t)}\mathrm{d}t - \frac{y_i}{2}\mathrm{d}t \tag{A.5}$$

where the initial data $y_i(0)$ is an independent Gaussian β-ensemble. Above, the Brownian motion terms are identical. We define the following interpolating processes for $0 \le \alpha \le 1$.

$$\mathrm{d}z_i(t,\alpha) = \sqrt{\frac{2}{\beta N}}\mathrm{d}B_i(t) + \frac{1}{N}\sum_{j:j\neq i}\frac{\mathrm{d}t}{z_i(t,\alpha)-z_j(t,\alpha)} - \frac{1}{2}V'_\alpha(z_i(t,\alpha))\mathrm{d}t, \quad i=1,2,\ldots,N, \tag{A.6}$$

with the potential

$$V_\alpha = \alpha V + (1-\alpha)W, \tag{A.7}$$

and the initial data

$$z_i(0,\alpha) := \alpha x_i(0) + (1-\alpha)y_i(0), \tag{A.8}$$

for $i = 1, 2, \ldots, N$. We can write the difference of the $x_i(t)$ and $y_i(t)$ as an integral in terms of the interpolating process $z_i(t,\alpha)$,

$$x_i(t) - y_i(t) = \int_0^1 (\partial_\alpha z_i(t,\alpha))\mathrm{d}\alpha. \tag{A.9}$$

Thanks to the rigidity and the smoothness of the potential V, the contribution of the potential is to leading order deterministic in the vicinity of $x_k(t)$ and contributes only a drift which is summarized by the movement of the classical eigenvalue location $\gamma_k(t)$. The following "homogenization result" follows from a careful analysis of the differential equation of $\partial_\alpha z_i(t,\alpha)$ essentially the same as in [38]. And the Gap universality Theorem A.1 is a direct consequence.

Lemma A.2 *Let time s, index k, $x_i(t)$ and $y_i(t)$ be as above. There are small constants $\mathfrak{c}, \mathfrak{e} > 0$, For any time $1/N \ll t$, such that $t = (sN)^{\mathfrak{e}}/N$ the following estimate holds with overwhelming probability: for any indices $|i-k| \le (sN)^{\mathfrak{e}}$*

$$(x_i(t) - \gamma_k(t)) - (y_i(t) - \gamma_k^{\mathrm{sc}})) = \zeta(i)_x - \zeta(i)_y + \mathrm{O}\left(\frac{1}{N^{1+\mathfrak{c}}}\right). \tag{A.10}$$

The functions $\zeta(i)_x$ and $\zeta(i)_y$ are mesoscopic linear statistics of the initial data and satisfy the estimate $|\zeta(i)_x - \zeta(i+1)_x| + |\zeta(i)_y - \zeta(i+1)_y| \le C/(N^2 t) \ll 1/N$, which proves gap universality.

The above lemma is the analogue of Theorem 3.1 of [38] in our setting which is proven using the modifications described above. The function $\zeta(i)_x$ is the same as the one appearing in that theorem, and the required estimate follows from Proposition 3.2 of [38]. We remark that in the setting of the current work, we are unable to prove universality of the correlation functions at fixed energy and so the above theorem is stated in terms of the eigenvalue gaps. This is precisely due to the fact that the quantities ζ appearing in the above lemma are mesoscopic linear statistics that are not compactly supported. The result proved in the present work does not apply to these statistics and so we cannot conclude fixed energy universality from the above homogenization result as in [38].

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