



Static large deviations for a reaction–diffusion model

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Abstract

We consider the superposition of a symmetric simple exclusion dynamics, speeded-up in time, with a spin-flip dynamics in a one-dimensional interval with periodic boundary conditions. We prove the large deviations principle for the empirical measure under the stationary state. We deduce from this result that the stationary state is concentrated on the stationary solutions of the hydrodynamic equation which are stable.

Keywords Reaction–diffusion equations · Hydrostatics · Large deviations · Nonequilibrium stationary states

Mathematics Subject Classification 82C22 · 60F10 · 82C35

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1 Introduction

Nonequilibrium thermodynamics has aroused a lot of interest in the last decades. Since the beginning of the 2000's, much attention has been devoted to the investigation of nonequilibrium stationary states which describe a steady flow through a system, [6,18] and references therein.

Over the last years, a general approach to examine nonequilibrium stationary states, called the Macroscopic Fluctuation Theory, has been developed based on a dynamical large deviations principle for the empirical current [2,4,8]. Among the major achievements of the MFT was the deduction of a time-independent variational formula for the quasi-potential, the functional obtained by minimizing the dynamical large deviations rate functional over all trajectories which start from the stationary density profile and produces a fixed fluctuation [3,20], and the proof that the quasi-potential is Gâteaux differentiable at some density profile if and only if the time-dependent variational formula which defines the quasi-potential has a unique minimizer [5].

At the same time, adapting to the infinite-dimensional setting the strategy proposed by Freidlin and Wentzell [24] for stochastic perturbations of finite-dimensional dynamical systems, Bodineau and Giacomin [10] and Farfán [21] proved a large deviations principle for the empirical measure under the nonequilibrium stationary state for conservative dynamics in contact with reservoirs in which the large deviations rate functional is given by the quasi-potential.

We consider in this article the stochastic evolution obtained by superposing a speeded-up symmetric simple exclusion process with a spin-flip dynamics on a one-dimensional interval with periodic boundary conditions. The hydrodynamic equation induced by the microscopic dynamics, the partial differential equation which describes the macroscopic evolution of the density, is given by a reaction–diffusion equation of type

$$\partial_t \rho = (1/2) \Delta \rho + B(\rho) - D(\rho), \quad (1.1)$$

where Δ represents the Laplacian and where B and D are non-negative polynomials.

We investigate the static large deviations of the empirical measure under the stationary state. In contrast with the previous dynamics [10,21], in which the hydrodynamic equation has a unique stationary solution which is a global attractor of the dynamical system generated by the PDE, in reaction–diffusion models, for appropriate choices of the functions B , D , the hydrodynamic equation possesses more than one stationary solution.

The existence of multiple stationary solutions to the PDE (1.1) raises new problems and new questions. For instance, the conjecture that the stationary state does not put mass on the unstable solutions of the hydrodynamic equation (1.1). For reaction–diffusion models it has only been proved in [31] that the stationary state is concentrated on the set of classical solutions to the semilinear elliptic equation

$$(1/2) \Delta \rho + B(\rho) - D(\rho) = 0. \quad (1.2)$$

The main result of this article, Theorem 4, establishes a large deviations principle for the empirical measure under the stationary state. The quasi-potential, the rate

functional of this large deviations principle, is represented through a time-dependent variational problem involving the dynamical large deviations rate functional. The value of the quasi-potential at a measure ϱ is given in terms of the infimum of the dynamical large deviations rate functional over all trajectories which start from a stationary solution of the hydrodynamic equation and end at ϱ .

A consequence of this result is that the stationary measure is concentrated at the stable, stationary solutions of the hydrodynamic equation. This is the content of Theorem 5, the second main result of the article.

The proof of Theorem 5 is based on two properties of the reaction–diffusion equation (1.1). First, it is known from [14] that all solutions of (1.1) converge to solutions of the semilinear elliptic equation (1.2). In particular, there are no time-periodic solutions. Second, we assume that Eq. (1.2) has only a finite number of solutions, modulo translations. This property holds when the polynomial $F(\rho) = B(\rho) - D(\rho)$ satisfies the hypotheses of Lemma 2. Theorem 5 further requires a characterization of the unstable stationary solutions of (1.1). Under the conditions of Lemma 3 on F , this set consists of all non-constant solutions and all constant solutions associated to local maxima of the potential V , where $V'(\rho) = -F(\rho)$.

We conclude this introduction with some comments. This stochastic dynamics has been introduced by De Masi, Ferrari and Lebowitz in [16]. The authors proved the hydrodynamic limit of the system by duality arguments and the fluctuations of the density field. The dynamical large deviations principle for the empirical density starting from a product measure appeared in [27], following the ideas presented in [29]. Bodineau and Lagouge in [11, 12] proved the dynamical large deviations principle for the empirical current, while two of the authors of this article extended in [31] the dynamical large deviations principle to the case in which the process starts from a deterministic configuration.

The stationary states of the symmetric simple exclusion process are the Bernoulli product measures. The introduction of the spin-flip dynamics creates long range correlations. Although the local distribution of particles remains very close to a Bernoulli product measure due to the speeding-up of the exclusion dynamics, the long range correlations affect substantially the macroscopic behavior of the system. The purpose of this article is to study this effect at the level of the large deviations.

There is a huge literature on large deviations for reaction–diffusion equations perturbed by Gaussian or Lévy noise in finite and infinite dimensions after the seminal paper by Faris and Jona-Lasinio [22]. We refer to the recent books [15, 19] for references on the subject. The noise created by the microscopic spin-flip dynamics considered in this article is of a different nature. This is reflected in the dynamical large deviations rate functions in which singular exponential terms appear. This is one of the sources of technical problems faced in order to prove the regularity conditions of the dynamical rate functional needed to derive the static large deviations principle.

We leave to the end of the next section technical comments and remarks on the proofs and on the assumptions, and we mention here some open problems for future research. It would be interesting to extend to this model the results described at the beginning of this introduction which were obtained from the MFT for one-dimensional conservative interacting particle systems in contact with reservoirs: an alternative time-independent variational formula for the quasi-potential, and a description of the optimal trajectory

which solves the time-dependent variational formula defining the quasi-potential. This has been done in [26] in the case where the reaction–diffusion model is reversible, but it remains an open problem in the non-reversible setting. In this general situation the only available information is an expansion of the quasi-potential around a constant stable stationary point obtained by Basile and Jona-Lasinio [1]. A description of the metastable behavior of the reaction–diffusion model when the difference $B(\rho) - D(\rho)$ forms a double-well potential is also a challenging open problem.

2 Notation and results

Throughout this article, we use the following notation. \mathbb{N}_0 stands for the set $\{0, 1, \dots\}$. For a function $f : X \rightarrow \mathbb{R}$, defined on some set X , let $\|f\|_\infty = \sup_{x \in X} |f(x)|$. We sometimes denote the interval $[0, \infty)$ by \mathbb{R}_+ .

2.1 Reaction–diffusion model

Let $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$, $N \geq 1$, be the one-dimensional discrete torus with N points. Denote by X_N the set $\{0, 1\}^{\mathbb{T}_N}$ and by η the elements of X_N , called configurations. For each $x \in \mathbb{T}_N$, $\eta(x)$ represents the occupation variable at site x so that $\eta(x) = 1$ if the site x is occupied for the configuration η , and $\eta(x) = 0$ if the site is vacant. For each $x \neq y \in \mathbb{T}_N$, denote by $\eta^{x,y}$, resp. by η^x , the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$, resp. by flipping the occupation variable $\eta(x)$:

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise,} \end{cases} \quad \eta^x(z) = \begin{cases} \eta(z) & \text{if } z \neq x, \\ 1 - \eta(z) & \text{if } z = x. \end{cases}$$

Consider the superposition of the speeded-up symmetric simple exclusion process with a spin-flip dynamics. The generator of this X_N -valued, continuous-time Markov chain acts on functions $f : X_N \rightarrow \mathbb{R}$ as

$$\mathcal{L}_N f = N^2 \mathcal{L}_K f + \mathcal{L}_G f,$$

where \mathcal{L}_K is the generator of a symmetric simple exclusion process (Kawasaki dynamics),

$$(\mathcal{L}_K f)(\eta) = (1/2) \sum_{x \in \mathbb{T}_N} [f(\eta^{x,x+1}) - f(\eta)],$$

and \mathcal{L}_G is the generator of a spin-flip dynamics (Glauber dynamics),

$$(\mathcal{L}_G f)(\eta) = \sum_{x \in \mathbb{T}_N} c(\tau_x \eta) [f(\eta^x) - f(\eta)].$$

In the last formula, $c(\eta)$ represents a strictly positive, cylinder function, that is, a function $c : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}_+$ which depends only on a finite number of coordinates $\eta(y)$. For a sufficiently large N , c can be regarded as a function on X_N . $\{\tau_x : x \in \mathbb{Z}\}$ represents the group of translations defined by $(\tau_x \eta)(y) = \eta(x + y)$, $y \in \mathbb{T}_N$, where the sum is carried modulo N .

Note that the Kawasaki dynamics has been speeded-up by a factor N^2 , which corresponds to the diffusive scaling. Setting the jump rates of the Glauber part to be 0, we retrieve the symmetric simple exclusion dynamics speeded up by N^2 , whose static large deviation principle has been derived with several different boundary conditions in [3,10–12,21].

Fix a topological space X . Let $D(I, X)$, $I = [0, T]$, $T > 0$, or $I = \mathbb{R}_+$, be the space of right-continuous trajectories from I to X with left-limits, endowed with the Skorohod topology. Let $\{\eta_t^N : N \geq 1\}$ be the continuous-time Markov process on X_N whose generator is given by \mathcal{L}_N . For a probability measure ν on X_N , denote by \mathbb{P}_ν the probability measure on $D(\mathbb{R}_+, X_N)$ induced by the process η_t^N starting from ν . The expectation with respect to \mathbb{P}_ν is represented by \mathbb{E}_ν . Denote by \mathbb{P}_η the measure \mathbb{P}_ν when the probability measure ν is the Dirac measure concentrated on the configuration η . Analogously, \mathbb{E}_η stands for the expectation with respect to \mathbb{P}_η .

2.2 Hydrodynamics

Let \mathbb{T} be the one-dimensional continuous torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$. Denote by $L^p(\mathbb{T})$, $p \geq 1$, the space of all real p -th integrable functions $G : \mathbb{T} \rightarrow \mathbb{R}$ with respect to the Lebesgue measure $d\theta$: $\int_{\mathbb{T}} |G(\theta)|^p d\theta < \infty$. The corresponding norm is denoted by $\|\cdot\|_p$:

$$\|G\|_p^p := \int_{\mathbb{T}} |G(\theta)|^p d\theta.$$

In particular, $L^2(\mathbb{T})$ is a Hilbert space equipped with the inner product

$$\langle G, H \rangle = \int_{\mathbb{T}} G(\theta)H(\theta)d\theta.$$

For a function G in $L^2(\mathbb{T})$, we also denote by $\langle G \rangle$ the integral of G with respect to the Lebesgue measure: $\langle G \rangle := \int_{\mathbb{T}} G(\theta)d\theta$.

Let $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{T})$ be the space of all nonnegative measures on \mathbb{T} with total mass bounded by 1, endowed with the weak topology. For a measure ϱ in \mathcal{M}_+ and a continuous function $G : \mathbb{T} \rightarrow \mathbb{R}$, denote by $\langle \varrho, G \rangle$ the integral of G with respect to ϱ :

$$\langle \varrho, G \rangle = \int_{\mathbb{T}} G(\theta)\varrho(d\theta).$$

The space \mathcal{M}_+ is metrizable. Indeed, if $e_0(\theta) = 1$, $e_k(\theta) = \sqrt{2} \cos(2\pi k\theta)$ and $e_{-k}(\theta) = \sqrt{2} \sin(2\pi k\theta)$, $k \in \mathbb{N}$, one can define a distance d on \mathcal{M}_+ by

$$d(\varrho_1, \varrho_2) := \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} |\langle \varrho_1, e_k \rangle - \langle \varrho_2, e_k \rangle|, \tag{2.1}$$

and one can check that the topology induced by this distance corresponds to the weak topology.

Note that \mathcal{M}_+ is compact under the weak topology, and that, by Schwarz inequality, for all density profiles $\gamma, \gamma' : \mathbb{T} \rightarrow [0, 1]$,

$$d(\gamma, \gamma') \leq 3\|\gamma - \gamma'\|_2. \tag{2.2}$$

In the previous formula we abuse of notation by writing $d(\gamma, \gamma')$ for $d(\gamma(\theta)d\theta, \gamma'(\theta)d\theta)$.

Denote by $C^m(\mathbb{T})$, m in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions on \mathbb{T} which are m times differentiable and whose m -th derivative is continuous. Given a function G in $C^2(\mathbb{T})$, we shall denote by ∇G and ΔG the first and second derivatives of G , respectively.

Let $\nu_\rho = \nu_\rho^N$, $0 \leq \rho \leq 1$, be the Bernoulli product measure on X_N with the density ρ . Define the continuous functions $B, D : [0, 1] \rightarrow \mathbb{R}$ by

$$B(\rho) = \int [1 - \eta(0)] c(\eta) d\nu_\rho, \quad D(\rho) = \int \eta(0) c(\eta) d\nu_\rho.$$

Let $\pi^N : X_N \rightarrow \mathcal{M}_+$ be the function which associates to a configuration η the positive measure obtained by assigning mass N^{-1} to each particle of η ,

$$\pi^N(\eta) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x) \delta_{x/N},$$

where δ_θ stands for the Dirac measure which has a point mass at $\theta \in \mathbb{T}$. Let $\pi_t^N = \pi^N(\eta_t^N)$, $t \geq 0$. The next result was proved by De Masi, Ferrari and Lebowitz in [16] for the first time. We refer to [16,27,28] for its proof.

Theorem 1 Fix a measurable function $\gamma : \mathbb{T} \rightarrow [0, 1]$. Let ν_N be a sequence of probability measures on X_N associated to γ , in the sense that

$$\lim_{N \rightarrow \infty} \nu_N \left(\left| \langle \pi^N, G \rangle - \int_{\mathbb{T}} G(\theta) \gamma(\theta) d\theta \right| > \delta \right) = 0, \tag{2.3}$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T} \rightarrow \mathbb{R}$. Then, for every $t \geq 0$, every $\delta > 0$ and every continuous function $G : \mathbb{T} \rightarrow \mathbb{R}$,

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\nu_N} \left(\left| \langle \pi_t^N, G \rangle - \int_{\mathbb{T}} G(\theta) \rho(t, \theta) d\theta \right| > \delta \right) = 0,$$

where $\rho : [0, \infty) \times \mathbb{T} \rightarrow [0, 1]$ is the unique weak solution of the Cauchy problem

$$\begin{cases} \partial_t \rho = (1/2)\Delta \rho + F(\rho) \text{ on } \mathbb{T}, \\ \rho(0, \cdot) = \gamma(\cdot), \end{cases} \tag{2.4}$$

where $F(\rho) = B(\rho) - D(\rho)$.

The definition, the existence and the uniqueness of weak solutions of the Cauchy problem (2.4) are discussed in Sect. 3.

2.3 The reaction–diffusion equation

We present in this subsection the results on the reaction–diffusion equation (2.4) needed in this section. Let S be the set of all classical solutions of the semilinear elliptic equation:

$$(1/2)\Delta\rho + F(\rho) = 0 \text{ on } \mathbb{T}. \tag{2.5}$$

Classical solution means a $[0, 1]$ -valued function ρ in $C^2(\mathbb{T})$ which satisfies the Eq. (2.5) for any $\theta \in \mathbb{T}$. We also denote by \mathcal{M}_{sol} the set of all absolutely continuous measures whose density is a classical solution of (2.5):

$$\mathcal{M}_{\text{sol}} := \{\bar{\rho} \in \mathcal{M}_+ : \bar{\rho}(d\theta) = \bar{\rho}(\theta)d\theta, \bar{\rho} \in S\}.$$

Next lemma is Theorem D of [14].

Lemma 1 *Let $\rho : [0, \infty) \times \mathbb{T} \rightarrow [0, 1]$ be the unique weak solution of the Cauchy problem (2.4). Then, there exists a density profile ρ_∞ in S such that ρ_t converges to ρ_∞ as $t \rightarrow \infty$ in $C^2(\mathbb{T})$.*

This result excludes the existence of time-periodic solutions of Eq. (2.4), a phenomenon which occurs if the function F is allowed to depend on $\nabla\rho$ as well (cf. [23] and references therein).

We turn to the description of the set S . Denote by \mathfrak{R} the set of roots of F in $[0, 1]$. It is clear that for all $r \in \mathfrak{R}$, the constant function $\rho : \mathbb{T} \rightarrow \mathbb{R}$ given by $\rho(\theta) = r, \theta \in \mathbb{T}$, is an element of S . There might be also non-constant periodic solutions.

Let $V : [0, 1] \rightarrow \mathbb{R}$ be a potential such that $F(\rho) = -V'(\rho)$. If the polynomial F has degree 1, as $V'(0) < 0 < V'(1)$, Eq. (2.5) has a unique solution, which is a global attractor for the dynamical system induced by the reaction–diffusion equation (2.4), and given by $\rho(\theta) = r$, where r is the unique root of F .

Assume that the degree of F is larger than or equal to 2. Denote by m_1, \dots, m_n the local minima of V in $[0, 1]$ and by M_1, \dots, M_m the local maxima in this interval. Since $V'(0) < 0 < V'(1)$, $n = m + 1 \geq 1$ and $m_1 < M_1 < \dots < M_{n-1} < m_n$.

Denote by \sim the equivalence relation in $C^2(\mathbb{T})$ defined by $\rho \sim \rho'$ if there exists $\theta' \in \mathbb{T}$ such that $\rho'(\theta) = \rho(\theta + \theta')$ for all $\theta \in \mathbb{T}$. Of course, if ρ is a periodic solution and $\rho' \sim \rho$, then ρ' is also a solution.

Lemma 2 *Suppose that all zeros of F are real and that all critical points of V are local minima or local maxima. Then, the elliptic equation (2.5) with periodic boundary conditions has at most a finite number of solutions, modulo the equivalence relation introduced above.*

Since we could not find the previous result explicitly stated in the literature, we sketch the proof of this result. The terminology employed can be found in [32]. By Proposition 1.5.2 in [32], F is an $A - B$ function on all intervals (m_1, M_1) ,

$\dots, (M_{n-1}, m_n)$. The diagram of the Hamiltonian system $\dot{p} = q, \dot{q} = V'(p)$ shows that the periodic solutions of (2.5) are bounded below and above by two consecutive minima of the potential V .

Solutions of (2.5) with periodic boundary conditions can be mapped to solutions of (2.5) with Dirichlet boundary conditions. Indeed, fix two consecutive minima m_j, m_{j+1} of V , and a solution ρ of (2.5) taking values in $[m_j, m_{j+1}]$. Let $F_j(r) = F(M_j + r)$, so that $F_j(0) = 0$ because M_j is a local maximum of V . Note that F_j is an $A - B$ function on $(m_j - M_j, 0) \cup (0, m_{j+1} - M_j)$. Let $\theta_0 = \min\{\theta \geq 0 : \rho(\theta) = M_j\}$. Define $\phi : [0, 1] \rightarrow [m_j - M_j, m_{j+1} - M_j]$ by $\phi(\theta) = \rho(\theta + \theta_0) - M_j$. It is clear that ϕ is a solution of $(1/2)\Delta v + F_j(v) = 0$ with Dirichlet boundary conditions.

Since F_j is an $A - B$ function on the intervals $(m_j - M_j, 0)$ and $(0, m_{j+1} - M_j)$, by Propositions 3.1.3, 3.1.4 and Theorem 3.1.9 in [32], the time-map of the Eq. (2.5) with Dirichlet boundary conditions is strictly convex and converges to $+\infty$ at the boundary. In particular, for each branch there exist at most two distinct solutions if $V''(M_j) = 0$ and at most one solution if $V''(M_j) < 0$. Since there is a finite number of branches whose time-map takes value less than or equal to 1, there is a finite number of distinct solutions of (2.5) with Dirichlet boundary conditions. As all solutions with periodic boundary conditions can be mapped to solutions with Dirichlet boundary conditions, the lemma is proved.

We turn to the heteroclinic orbits of (2.4). A complete description has been obtained in [23]. We state here a partial result which fulfills our needs. It asserts that all non-constant stationary solutions are unstable, as well as all constant solutions associated to local maxima of V .

Fix two stationary solutions $\phi \neq \psi$ of (2.4). A trajectory $\rho(t, \cdot), t \in \mathbb{R}$, is called a heteroclinic orbit from ϕ to ψ if $\lim_{t \rightarrow -\infty} \rho(t, \cdot) = \phi, \lim_{t \rightarrow +\infty} \rho(t, \cdot) = \psi$ and if ρ solves (2.4) for every $t \in \mathbb{R}$. Convergences are meant in $C^1(\mathbb{T})$. A solution ϕ of (2.5) is said to be unstable if there exist $\psi \approx \phi$ and a heteroclinic orbit from ϕ to ψ .

For a solution ϕ of (2.5), denote by \mathcal{L}_ϕ the linear operator on $C^2(\mathbb{T})$ given by

$$\mathcal{L}_\phi h = (1/2)\Delta h - V''(\phi)h. \quad (2.6)$$

If ϕ is not constant, $\nabla\phi$ is an eigenfunction associated to the eigenvalue 0. A non-constant solution ϕ of (2.5) is said to be hyperbolic if all eigenvalues of \mathcal{L}_ϕ have non-zero real parts, except the eigenvalue $\lambda = 0$, whose associated eigenspace has dimension 1.

The eigenvalue 0 of the operator \mathcal{L}_ϕ is associated to the orbit $\rho(t, \theta) = \phi(\theta + t)$. Actually, we prove in Lemma 16 that the cost of this orbit along a stationary set vanishes. Moreover, the existence of a positive eigenvalue of \mathcal{L}_ϕ is related to the existence of a heteroclinic orbit starting from ϕ and, therefore, to the instability of ϕ .

Lemma 3 *Assume the conditions of Lemma 2 and that all local maxima of V are non-degenerate: $V''(M_j) \neq 0, 1 \leq j < n$. Then, for each non-constant solution ϕ of (2.5), there exist heteroclinic orbits from ϕ to ϕ_j and from ϕ to ϕ_{j+1} , where $\phi_k(\theta) = m_k, \theta \in \mathbb{T}$, and $j = \max\{k < n : m_k < \phi(\theta) \forall \theta \in \mathbb{T}\}$. There exist also heteroclinic orbits from ψ_j to ϕ_j and from ψ_j to $\phi_{j+1}, 1 \leq j < n$, where $\psi_j(\theta) = M_j, \theta \in \mathbb{T}$.*

This result follows from Theorems 1.3 and 1.4 in [23]. We just have to show that the hypotheses of these theorems are in force. As $V'(0) < 0 < V'(1)$, the solutions are bounded below by 0 and above by 1, so that F is dissipative.

We claim that all non-constant solution ϕ of (2.5) are hyperbolic. Indeed, fix such a function. As we have seen in the sketch of the proof of Lemma 2, there exists $1 \leq j < n$ such that $m_j \leq \phi(\theta) \leq m_{j+1}$ for all $\theta \in \mathbb{T}$. Denote by Π the orbit map associated to the polynomial F (cf. [32, page 51]). Since $V''(M_j) < 0$, by [32, Proposition 1.5.2], F is an $A - B$ function in (m_j, m_{j+1}) . Therefore, by [32, Theorem 2.1.3], $\Pi'(r) \neq 0$ for $r \neq M_j$. Hence, by the proof of [23, Lemma 4.4], ϕ is hyperbolic.

We turn to the proof of the lemma. Fix $1 \leq j < n$ and a non-constant solution ϕ of (2.5) taking values in the interval $[m_j, m_{j+1}]$. We show that there exist heteroclinic orbits from ϕ to ϕ_{j+1} and from ψ_j to ϕ_{j+1} . Similar arguments permit to replace ϕ_{j+1} by ϕ_j .

The diagram of the Hamiltonian system $\dot{p} = q, \dot{q} = V'(p)$ shows that the periodic solutions of (2.5) which takes value in the interval $[m_j, m_{j+1}]$ are either (i) $\phi_j, \phi_{j+1}, \psi_j$ or (ii) a non-constant periodic solution whose maximal value belongs to (M_j, m_{j+1}) and minimal value to (m_j, M_j) . Moreover, if ϕ, ψ are such non-constant periodic solutions, either $\min_x \psi(x) < \min_x \phi(x) < M_j < \max_x \phi(x) < \max_x \psi(x)$ or the opposite.

We start with a heteroclinic orbit from ψ_j to ϕ_{j+1} . We may use the heteroclinic orbit from M_j to m_{j+1} for the ODE $\dot{x}(t) = V'(x(t))$ to obtain a heteroclinic orbit from ψ_j to ϕ_{j+1} which remains constant in space.

Consider now a non-constant solution ϕ of (2.5) such that $m_j \leq \phi(\theta) \leq m_{j+1}$. We repeat here the arguments of the proof of Theorem 1.3 in [23] presented at the end of page 111. Let $z(h)$ be the number of strict sign changes of a function $h : \mathbb{T} \rightarrow \mathbb{R}$. Since $z(\nabla\phi) \geq 2$, by [23, Proposition 3.1(b)], the unstable dimension of ϕ , denoted by $i(\phi)$ in [23], is larger than or equal to 1. By the positivity of the first eigenfunction of the operator \mathcal{L}_ϕ , one obtains a trajectory $\rho(t, \theta), t \in \mathbb{R}$, which solves (2.4) and such that $\rho(t, \theta) > \phi(\theta), \lim_{t \rightarrow -\infty} \rho(t) = \phi$. Let $\psi = \lim_{t \rightarrow +\infty} \rho(t)$, which exists in view of Lemma 1. As $m_j \leq \phi(\theta) \leq m_{j+1}$, we have that $m_j \leq \psi(\theta) \leq m_{j+1}$. By the Sturm property, $z(\rho(t) - \phi)$ decreases in time. Since it is equal to 0 for t close to $-\infty, z(\psi - \phi) = 0$. Hence, ψ can not be ψ_j or one of the non-constant solutions taking values in the interval $[m_j, m_{j+1}]$. Thus, ψ must be ϕ_j or ϕ_{j+1} . Since $\psi \geq \phi, \psi = \phi_{j+1}$, which proves the lemma.

We conclude this subsection with an example which fulfills the assumptions of Lemma 3. Fix $0 < a < b$ and consider the reaction–diffusion equation

$$\partial_t \rho = (1/2)\Delta \rho - V'(\rho), \quad \text{where} \quad V(\rho) = \frac{b}{4} (2\rho - 1)^4 - \frac{a}{2} (2\rho - 1)^2. \tag{2.7}$$

This is the so-called Chafee–Infante equation [13]. It is clear that the potential V satisfies the assumptions of Lemma 3. Actually, in this case all stationary solutions and all heteroclinic orbits are known. We examine this example in Sect. 8, where we present microscopic jump rates which fulfill the hypotheses of Theorem 4 below and whose hydrodynamic equation is given by (2.7) with $0 < a < b$.

2.4 Hydrostatics

Since the jump rate $c(\eta)$ is strictly positive, the Markov process η_t^N is irreducible in X_N . We denote by μ^N the unique stationary probability measure under the dynamics. We review in this subsection the asymptotic behavior of the empirical measure under the stationary state μ^N .

Denote by \mathcal{P}^N the probability measure on \mathcal{M}_+ defined by $\mathcal{P}^N := \mu^N \circ (\pi^N)^{-1}$. The following theorem has been established in [31]. It is a consequence of the law of large numbers for the empirical measure, stated in Theorem 1, and of the asymptotic behavior of the solutions of the reaction–diffusion equation, stated in Lemma 1.

Theorem 2 *The sequence of measures $\{\mathcal{P}^N : N \geq 1\}$ is asymptotically concentrated on the set \mathcal{M}_{sol} . Namely, for any $\delta > 0$, we have*

$$\lim_{N \rightarrow \infty} \mathcal{P}^N \left(\varrho \in \mathcal{M}_+ : \inf_{\bar{\varrho} \in \mathcal{M}_{\text{sol}}} d(\varrho, \bar{\varrho}) \geq \delta \right) = 0.$$

Note that this result does not exclude the possibility that the stationary measure gives a positive weight to a neighborhood of an unstable stationary solution of Eq. (2.4).

2.5 Dynamical large deviations

Let $\mathcal{M}_{+,1}$ be the closed subset of \mathcal{M}_+ consisting of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_{+,1} = \{ \varrho \in \mathcal{M}_+ : \varrho(d\theta) = \rho(\theta)d\theta, 0 \leq \rho(\theta) \leq 1 \text{ a.e. } \theta \in \mathbb{T} \}.$$

Fix $T > 0$, and denote by $C^{m,n}([0, T] \times \mathbb{T})$, m, n in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions defined on $[0, T] \times \mathbb{T}$ which are m times differentiable in the first variable and n times in the second one, and whose derivatives are continuous. Let $\mathcal{Q}_{T,\eta} = \mathcal{Q}_{T,\eta}^N$, $\eta \in X_N$, be the probability measure on $D([0, T], \mathcal{M}_+)$ induced by the measure-valued process π_t^N starting from $\pi^N(\eta)$.

For each path $\pi(t, d\theta) = \rho(t, \theta)d\theta$ in $D([0, T], \mathcal{M}_{+,1})$, define the energy \mathcal{Q}_T as

$$\mathcal{Q}_T(\pi) = \sup_{G \in C^{0,1}([0,T] \times \mathbb{T})} \left\{ 2 \int_0^T dt \langle \rho_t, \nabla G_t \rangle - \int_0^T dt \int_{\mathbb{T}} d\theta G(t, \theta)^2 \right\}. \quad (2.8)$$

It is known (cf. [7, Subsection 4.1]) that the energy $\mathcal{Q}_T(\pi)$ is finite if and only if ρ has a generalized derivative, denoted by $\nabla \rho$, and this generalized derivative is square integrable on $[0, T] \times \mathbb{T}$:

$$\int_0^T dt \int_{\mathbb{T}} d\theta |\nabla \rho(t, \theta)|^2 < \infty.$$

Moreover, it is easy to see that the energy \mathcal{Q}_T is convex and lower semicontinuous.

For each function G in $C^{1,2}([0, T] \times \mathbb{T})$, define the functional $\bar{J}_{T,G} : D([0, T], \mathcal{M}_{+,1}) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \bar{J}_{T,G}(\pi) &= \langle \pi_T, G_T \rangle - \langle \pi_0, G_0 \rangle - \int_0^T dt \langle \pi_t, \partial_t G_t + \frac{1}{2} \Delta G_t \rangle \\ &\quad - \frac{1}{2} \int_0^T dt \langle \chi(\rho_t), (\nabla G_t)^2 \rangle \\ &\quad - \int_0^T dt \{ \langle B(\rho_t), e^{G_t} - 1 \rangle + \langle D(\rho_t), e^{-G_t} - 1 \rangle \}, \end{aligned}$$

where $\chi(r) = r(1 - r)$ is the mobility. Let $J_{T,G} : D([0, T], \mathcal{M}_+) \rightarrow [0, \infty]$ be the functional defined by

$$J_{T,G}(\pi) = \begin{cases} \bar{J}_{T,G}(\pi) & \text{if } \pi \in D([0, T], \mathcal{M}_{+,1}), \\ \infty & \text{otherwise,} \end{cases} \tag{2.9}$$

and let $I_T : D([0, T], \mathcal{M}_+) \rightarrow [0, \infty]$ be the functional given by

$$I_T(\pi) = \begin{cases} \sup J_{T,G}(\pi) & \text{if } \mathcal{Q}_T(\pi) < \infty, \\ \infty & \text{otherwise,} \end{cases} \tag{2.10}$$

where the supremum is carried over all functions G in $C^{1,2}([0, T] \times \mathbb{T})$. We sometimes abuse of notation by writing $I_T(\rho)$ for $I_T(\pi)$ and we write J_G for $J_{T,G}$ to keep notation simple.

An explicit formula for the functional I_T at smooth trajectories was obtained in Lemma 2.1 of [27]. Let ρ be a function in $C^{2,3}([0, T] \times \mathbb{T})$ with $c \leq \rho \leq 1 - c$, for some $0 < c < 1/2$. Then, there exists a unique solution $H \in C^{1,2}([0, T] \times \mathbb{T})$ of the partial differential equation

$$\partial_t \rho = (1/2) \Delta \rho - \nabla(\chi(\rho) \nabla H) + B(\rho) e^H - D(\rho) e^{-H},$$

and the rate functional $I_T(\rho)$ can be expressed as

$$\begin{aligned} I_T(\rho) &= \frac{1}{2} \int_0^T dt \langle \chi(\rho_t), (\nabla H_t)^2 \rangle \\ &\quad + \int_0^T dt \langle B(\rho_t), 1 - e^{H_t} + H_t e^{H_t} \rangle + \int_0^T dt \langle D(\rho_t), 1 - e^{-H_t} - H_t e^{-H_t} \rangle. \end{aligned}$$

For a measurable function $\gamma : \mathbb{T} \rightarrow [0, 1]$, define the dynamical large deviations rate function $I_T(\cdot|\gamma) : D([0, T], \mathcal{M}_+) \rightarrow [0, \infty]$ as

$$I_T(\pi|\gamma) = \begin{cases} I_T(\pi) & \text{if } \pi(0, d\theta) = \gamma(\theta) d\theta, \\ \infty & \text{otherwise.} \end{cases}$$

The next result, which establishes a dynamical large deviations principle for the measure-valued process π^N with rate functional $I_T(\cdot|\gamma)$ has been presented in [31] under the assumption that the functions B and D are concave on $[0, 1]$. We refer to [11, 12, 27] for different versions.

Theorem 3 *Assume that the functions B and D are concave in $[0, 1]$. Fix $T > 0$ and a measurable function $\gamma : \mathbb{T} \rightarrow [0, 1]$. Consider a sequence η^N of initial configurations in X_N associated to γ in the sense that $\langle \pi^N(\eta^N), G \rangle$ converges to $\int_{\mathbb{T}} G(\theta)\gamma(\theta)d\theta$, as $N \uparrow \infty$, for all continuous function $G : \mathbb{T} \rightarrow \mathbb{R}$. Then, the measure \mathcal{Q}_{T, η^N} on $D([0, T], \mathcal{M}_+)$ satisfies a large deviation principle with the rate function $I_T(\cdot|\gamma)$. That is, for each closed subset $\mathcal{C} \subset D([0, T], \mathcal{M}_+)$,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{Q}_{T, \eta^N}(\mathcal{C}) \leq - \inf_{\pi \in \mathcal{C}} I_T(\pi|\gamma),$$

and for each open subset $\mathcal{O} \subset D([0, T], \mathcal{M}_+)$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{Q}_{T, \eta^N}(\mathcal{O}) \geq - \inf_{\pi \in \mathcal{O}} I_T(\pi|\gamma).$$

Moreover, the rate function $I_T(\cdot|\gamma)$ is lower semicontinuous and it has compact level sets.

2.6 Static large deviations

We state in Theorem 4 below the main result of this paper, a large deviations principle for the empirical measure under the stationary measure.

Assume that the semilinear elliptic equation (2.5) admits at most a finite number of solutions, modulo translations. More precisely, assume that there exists $l \geq 1$ and density profiles $\bar{\rho}_1, \dots, \bar{\rho}_l$ in $C^2(\mathbb{T})$, such that

$$\mathcal{M}_{\text{sol}} = \{\bar{q}_i(d\theta - \omega) = \bar{\rho}_i(\theta - \omega)d\theta : 1 \leq i \leq l, \omega \in \mathbb{T}\}.$$

Lemma 2 provides conditions on the potential V which guarantee that this condition is in force. Let \mathcal{M}_i , $1 \leq i \leq l$, be the subset of \mathcal{M}_{sol} given by $\mathcal{M}_i = \{\bar{q}_i(d\theta - \omega) : \omega \in \mathbb{T}\}$.

Define the functionals $V_i : \mathcal{M}_+ \rightarrow [0, \infty]$, $1 \leq i \leq l$, by

$$V_i(\varrho) = \inf \left\{ I_T(\pi|\gamma) : T > 0, \gamma(\theta)d\theta \in \mathcal{M}_i, \pi \in D([0, T], \mathcal{M}_+) \text{ and } \pi_T = \varrho \right\}, \quad (2.11)$$

which is the minimal cost to create the measure ϱ from the set \mathcal{M}_i . We prove in Lemma 16 that in the previous variational formula we may replace the condition $\gamma(\theta)d\theta \in \mathcal{M}_i$ by the more restrictive condition $\gamma(\theta)d\theta = \bar{q}_i$ for some fixed $\bar{q}_i \in \mathcal{M}_i$: for all $\bar{q}_i \in \mathcal{M}_i$,

$$V_i(\varrho) = \inf \left\{ I_T(\pi | \gamma) : T > 0, \gamma(\theta)d\theta = \bar{\varrho}_i, \pi \in D([0, T], \mathcal{M}_+) \text{ and } \pi_T = \varrho \right\}. \tag{2.12}$$

By an abuse of notation, we sometimes write $V_i(\gamma)$ instead of $V_i(\gamma(\theta)d\theta)$, where $\gamma : \mathbb{T} \rightarrow [0, 1]$ is a density profile.

By translation invariance, $V_i(\gamma) = V_i(\gamma')$ if $\gamma'(\cdot) = \gamma(\cdot - \omega)$ for some $\omega \in \mathbb{T}$. In particular, V_i is constant on the set $\mathcal{M}_j, j \neq i$, and $v_{ij} = V_i(\bar{\varrho}_j)$ is well defined, where $\bar{\varrho}_j$ is any element of \mathcal{M}_j . Moreover, by choosing $T = 1$ and $\pi_t = (1 - t)\bar{\varrho}_i + t\bar{\varrho}_j, t \in [0, 1]$, in the infimum of (2.12) yields that v_{ij} is finite for any $i \neq j$. Finally, by Lemmata 12 and 16, $V_i(\varrho) = 0$ for any $\varrho \in \mathcal{M}_i$.

Following [24, Chapter 6], denote by $\mathcal{T}(i), i \in \mathcal{V} := \{1, \dots, l\}$, the set of all oriented, weighted, rooted trees whose vertices are all the elements of \mathcal{V} and whose root is i . The edges are oriented from the child to the parent, and the weight v_{mn} is assigned to the oriented edge (m, n) . Denote by $\kappa(g)$ the sum of the weights of the tree $g \in \mathcal{T}(i)$ and by w_i the minimal weight of all trees in $\mathcal{T}(i)$:

$$w_i = \min_{g \in \mathcal{T}(i)} \kappa(g), \quad \kappa(g) = \sum_{(m,n) \in g} v_{mn}.$$

Since v_{ij} is finite for $i \neq j$, so are w_i and $w = \min_{1 \leq i \leq l} w_i$. We will see below in (2.15) that the non-negative parameter w_i corresponds to the exponential weight of a neighborhood of the set \mathcal{M}_i under the stationary state.

Note that for all $i \neq j$,

$$w_i \leq w_j + v_{ji}. \tag{2.13}$$

Indeed, let g be a graph in $\mathcal{T}(j)$ such that $w_j = \kappa(g)$. Denote by $(a, b), a \neq b \in \mathcal{V}$, the oriented edge where a is the child and b the parent. Let i' be the parent of i in g . Of course, i' might be j . Denote by g' the tree in $\mathcal{T}(i)$ obtained from g by adding the oriented edge (j, i) and removing the the edge (i, i') , and note that $\kappa(g) + v_{ji} = \kappa(g') + v_{ii'}$. Since w_i is the minimal value of $\kappa(\tilde{g}), \tilde{g} \in \mathcal{T}(i), w_i \leq \kappa(g') \leq \kappa(g') + v_{ii'} = \kappa(g) + v_{ji} = w_j + v_{ji}$.

For each $1 \leq i \leq l$, define the functions $W_i, W : \mathcal{M}_+ \rightarrow [0, \infty]$ by

$$W_i(\varrho) = w_i - w + V_i(\varrho), \quad W(\varrho) = \min_{1 \leq i \leq l} W_i(\varrho). \tag{2.14}$$

Note that for all $\varrho \in \mathcal{M}_i$,

$$W(\varrho) = \bar{w}_i := w_i - w. \tag{2.15}$$

Indeed, fix $\varrho \in \mathcal{M}_i$. In view of the definition of W , we have to show that $\min_{1 \leq j \leq l} \{\bar{w}_j + V_j(\varrho)\} = \bar{w}_i$. The minimum is less than or equal to \bar{w}_i because $V_i(\varrho) = 0$. On the other hand, since $V_j(\varrho) = v_{ji}$ and since, by (2.13), $w_i \leq w_j + v_{ji}, \bar{w}_i \leq \bar{w}_j + V_j(\varrho)$ for $j \neq i$.

The following theorem is the main result of this paper.

Theorem 4 Assume that the jump rates are strictly positive and that the functions B and D are concave on $[0, 1]$. Assume, furthermore, that the semilinear elliptic equation (2.5) admits at most a finite number of solutions, modulo translations. Then, the sequence of probability measures $\{\mathcal{P}^N; N \geq 1\}$ satisfies a large deviation principle on \mathcal{M}_+ with speed N and rate function W . Namely, for each closed set $\mathcal{C} \subset \mathcal{M}_+$ and each open set $\mathcal{O} \subset \mathcal{M}_+$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{C}) \leq - \inf_{\varrho \in \mathcal{C}} W(\varrho),$$

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{O}) \geq - \inf_{\varrho \in \mathcal{O}} W(\varrho).$$

Moreover, the rate functional W is bounded on $\mathcal{M}_{+,1}$, it is lower semicontinuous, and it has compact level sets.

2.7 The support of the stationary measure μ^N

The next result improves on Theorem 2 and asserts that the stationary measure μ^N is concentrated on neighborhoods of stable equilibria of the reaction–diffusion equation (2.4). This result has been conjectured in [11, 12].

As in the previous subsection, assume that the semilinear elliptic equation (2.5) admits at most a finite number of solutions, modulo translations. Denote by \mathfrak{M} the set of local minima of V , and by $\mathcal{S} \subset \mathcal{M}_{\text{sol}}$ the set of associated density profiles:

$$\mathcal{S} = \{\varrho(d\theta) = m d\theta : m \in \mathfrak{M}\}.$$

The elements of \mathcal{S} are called stable solutions. Lemma 30 justifies this terminology. It states that the quasi-potential associated to each $\varrho \in \mathcal{S}$ is strictly positive outside any neighborhood of ϱ . More precisely, for every $\bar{\varrho}_i(d\theta) = \bar{\rho}_i(\theta)d\theta \in \mathcal{S}$ and $\varepsilon > 0$, there exists $c > 0$ such that $\inf_{\gamma \notin \mathcal{B}_\varepsilon(\bar{\varrho}_i)} V_i(\gamma) \geq c$, where $\mathcal{B}_\varepsilon(\bar{\varrho}_i)$ represents a ball in \mathcal{M}_+ of radius ε centered at $\bar{\varrho}_i$.

Denote by $I_s, I_u \subset \{1, \dots, l\}$ the set of indices associated to stable, unstable density profiles, respectively:

$$I_s = \{j : \bar{\rho}_j(\theta) d\theta \in \mathcal{S}\}, \quad I_u = \{1, \dots, l\} \setminus I_s.$$

Theorem 5 Assume that the hypotheses of Theorem 4 are in force, and that for all $i \in I_u$ there exists $j \in I_s$ such that

$$v_{ij} = 0. \tag{2.16}$$

Then, for all $\varepsilon > 0$ there exist $c > 0$ and $N_0 \geq 1$ such that for all $N \geq N_0$,

$$\mathcal{P}^N \left(\mathcal{M}_+ \setminus \left[\bigcup_{j \in I_s} \mathcal{B}_\varepsilon(\bar{\varrho}_j) \right] \right) \leq e^{-cN}.$$

Of course, one expects the stationary measure μ^N to be concentrated on neighborhoods of the density profiles associated to the global minima of the potential V . This problem remains an open question. A finer estimate than the one provided by the large deviations might be needed to answer this open question. We refer to [25] for a similar problem in the context of the pinned Wiener measure, and to [9] and references therein for the study of the concentration of measures in the situation where the rate functional has more than one minimizer.

Lemma 3 provides a set of sufficient conditions, expressed in terms of the potential V , for assumption (2.16) to hold. Indeed, by Lemma 29, if there exists a heteroclinic orbit from \mathcal{M}_i to \mathcal{M}_j , then $v_{ij} = 0$. Hence, by Lemma 3, if all zeros of F are real, all critical points of V are local minima or local maxima and all local maxima of V are non-degenerate, hypothesis (2.16) is in force.

Fix a stationary solution $\phi = \bar{\rho}_i$ of (2.5). There are at least three different possible definitions of instability: (i) the operator \mathcal{L}_ϕ , defined in (2.6), has an eigenvalue with positive real part. (ii) there exists $\psi \approx \phi$ and a heteroclinic orbit from ϕ to ψ . (iii) $v_{ij} = 0$ for some $j \neq i$. We presented in the proof of Lemma 3 a sketch of the proof that (i) \Rightarrow (ii) under some additional hypotheses. Lemma 29 asserts that (ii) \Rightarrow (iii). We believe that the other implications hold, at least with some extra assumptions, but we were not able to prove them.

2.8 Comments and remarks

The characterization of the global attractor of the solutions of reaction–diffusion equations [14,23] has only been achieved in dimension 1, not to mention the description of the heteroclinic orbits. This is the main obstacle to extend the previous result to higher dimensions. Although it is true that the dynamical large deviations principle has been derived only in one dimension [31], it should not be very difficult to extend it to higher dimensions.

The results presented in this article can be proved for one-dimensional reaction–diffusion models with Dirichlet or Neumann boundary conditions. The description of the heteroclinic orbits in these contexts is simpler than the one with periodic boundary conditions (cf. [13] for the case of Dirichlet boundary conditions).

As mentioned above, it is an open, and very appealing, problem to show that the stationary measure μ^N is concentrated on neighborhoods of the density profiles associated to global minima of the potential V . To apply the method presented in the article to solve this question would require a sharp estimate of the cost of the instanton, the trajectory which drives the system from a stable equilibrium to another. In view of Theorem 9 below, it is clear that the instanton in the case of a double well potential with non-constant stationary profiles is the trajectory which crosses the non-constant stationary solution with one period. To estimate the cost of this trajectory seems to be out of reach.

The previous questions lead us to the problem of the metastability of the dynamics. It is challenging to describe the metastable behavior of these reaction–diffusion models.

The hypothesis that the functions B and D are concave is only needed in the proof of the dynamical large deviations principle [31], and we never use it in this paper. If one

is able to prove this dynamical result without the concavity assumption, the arguments presented in this article provide a proof of the static large deviations principle without the concavity assumption.

As mentioned in the introduction, the strategy of the proof consists in adapting to our infinite-dimensional setting the Freidlin and Wentzell approach [24] to prove a large deviation principle for the stationary state of a small perturbation of a dynamical system. This has been done before in [10,21] for conservative evolutions in contact with reservoirs. However, in the context of reaction–diffusion models the existence of several stationary solutions to the hydrodynamic equation introduces additional difficulties.

The proof relies on a representation of the stationary state of the reaction–diffusion model in terms of the invariant measure of a discrete-time Markov chain induced by the successive visits to the neighborhoods of the stationary solutions of the hydrodynamic equation.

The proof of the static large deviations principle can be decomposed in essentially three steps. We first need to derive some regularity properties of the dynamical large deviations rate functional. For instance, that any trajectory which remains in a long time interval far apart (in the L^2 -topology) from the stationary solutions of the hydrodynamic equation pays a strictly positive cost. Or that the quasi-potential is lower semicontinuous in the weak topology.

The second step consists in obtaining sharp large deviations bounds for the invariant measure of the discrete-time Markov chain. The final step, whose proofs are similar to the ones presented in [10,21], consists in estimating the minimal cost to create a measure starting from a stationary solution of the hydrodynamic equation.

The topology is one of the main technical difficulties in the argument. The weak topology is imposed by the dynamical large deviations principle which has been derived in this set-up, and one is forced to prove all regularity properties of the rate functionals in this topology. To overcome this obstacle, we systematically use the smoothening properties of the hydrodynamic equation. Lemma 19 is a good illustration of this strategy.

The article is organized as follows. In Sect. 3, we present the main properties of the weak solutions of the Cauchy problem (2.4), and in Sects. 4 and 5, we examine the dynamical and the static large deviations rate functionals. These sections are purely analytical, and no probabilistic argument is used. In Sect. 6, we prove the static large deviations principle, and, in Sect. 7, the concentration of the stationary measure μ^N . In Sect. 8 we present a reaction–diffusion model which fulfills the hypotheses of Theorem 5.

3 The reaction–diffusion equation

We present in this section several properties of the weak solutions of the Cauchy problem (2.4). When we did not find a reference, we present a proof of the result. Throughout this section and in the next ones, C_0 represents a finite, positive constant which depends only on F and which may change from line to line. As mentioned above,

this section and the following two ones are purely analytical, and no probabilistic arguments appear.

We first define two concepts of solutions.

Definition 1 A measurable function $\rho : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ is said to be a weak solution of the Cauchy problem (2.4) in the layer $[0, T] \times \mathbb{T}$ if for every function G in $C^{1,2}([0, T] \times \mathbb{T})$,

$$\begin{aligned} & \langle \rho_T, G_T \rangle - \langle \gamma, G_0 \rangle - \int_0^T dt \langle \rho_t, \partial_t G_t \rangle \\ &= \frac{1}{2} \int_0^T dt \langle \rho_t, \Delta G_t \rangle + \int_0^T dt \langle F(\rho_t), G_t \rangle. \end{aligned} \tag{3.1}$$

Definition 2 A measurable function $\rho : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ is said to be a mild solution of the Cauchy problem (2.4) in the layer $[0, T] \times \mathbb{T}$ if for any t in $[0, T]$

$$\rho_t = P_t \gamma + \int_0^t P_{t-s} F(\rho_s) ds, \tag{3.2}$$

where $\{P_t : t \geq 0\}$ stands for the semigroup on $L^2(\mathbb{T})$ generated by $(1/2)\Delta$.

Next proposition asserts that the two notions of solutions are equivalent. We refer to Proposition 6.3 of [31] for the proof.

Proposition 1 *Definitions 1 and 2 are equivalent. Moreover, there exists a unique weak solution of the Cauchy problem (2.4).*

The next result is contained in Proposition 2.1 of [17].

Proposition 2 *Let ρ be the unique weak solution of the Cauchy problem (2.4). Then ρ is infinitely differentiable over $(0, \infty) \times \mathbb{T}$.*

Let $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$, and let $c_0 : \{0, 1\}^{\mathbb{Z}_*} \rightarrow \mathbb{R}_+$ be the cylinder function defined by $c_0(\xi) = c(\xi^{(0)})$, where $\xi^{(0)}$ is the configuration of $\{0, 1\}^{\mathbb{Z}}$ defined by $\xi^{(0)}(x) = \xi(x)$, $x \neq 0$, $\xi^{(0)}(0) = 0$. The cylinder function $c_1 : \{0, 1\}^{\mathbb{Z}_*} \rightarrow \mathbb{R}_+$ is defined analogously with $\xi^{(0)}$ replaced by $\xi^{(1)}$, where $\xi^{(1)}(0) = 1$.

Note that c_0 and c_1 are strictly positive cylinder functions because so is $c(\eta)$. Hence, if ν_ρ^* represents the Bernoulli product measure on $\{0, 1\}^{\mathbb{Z}_*}$ with density ρ , the polynomial $\widehat{B}(\rho)$ defined by $\widehat{B}(\rho) = E_{\nu_\rho^*}[c_0(\eta)]$ is strictly positive. Similarly, the polynomial $\widehat{D}(\rho)$ defined by $\widehat{D}(\rho) = E_{\nu_\rho^*}[c_1(\eta)]$ is strictly positive.

By definition, $B(\rho) = E_{\nu_\rho}[\{1-\eta(0)\}c(\eta)] = (1-\rho) E_{\nu_\rho^*}[c_0(\eta)] = (1-\rho)\widehat{B}(\rho)$, and $D(\rho) = \rho\widehat{D}(\rho)$. Hence,

$$B(\rho) = (1 - \rho) \widehat{B}(\rho), \quad D(\rho) = \rho \widehat{D}(\rho), \tag{3.3}$$

where $\widehat{B}(\rho)$ and $\widehat{D}(\rho)$ are strictly positive polynomials. In particular, $F(0) = B(0) - D(0) = \widehat{B}(0) > 0$ and $F(1) = B(1) - D(1) = -\widehat{D}(1) < 0$.

Denote by $x_a(t)$, $0 \leq a \leq 1$, the solution of the ODE

$$\dot{x}(t) = F(x(t)) \quad (3.4)$$

with initial condition $x(0) = a$. Since $F(1) < 0 < F(0)$, $x_0(t)$ (resp. $x_1(t)$) is strictly increasing (resp. decreasing) and $x_0(t) \rightarrow x_0$ (resp. $x_1(t) \rightarrow x_1$), where x_0 (resp. x_1) is the smallest (resp. largest) solution of $F(x) = 0$. The next result is a simple application of the maximum principle.

Lemma 4 *Let $\gamma : \mathbb{T} \rightarrow [0, 1]$ be a density profile such that $a \leq \gamma(\theta) \leq b$ for a.e. $\theta \in \mathbb{T}$. Denote by $\rho^\gamma(t, \theta)$ the unique weak solution of (2.4) with initial condition γ . Then, $x_a(t) \leq \rho(t, \theta) \leq x_b(t)$ for all $t \geq 0$. In particular, for any $t > 0$, there exists $\varepsilon = \varepsilon(t) > 0$ such that $\varepsilon \leq \rho^\gamma(t, \theta) \leq 1 - \varepsilon$ for all $\theta \in \mathbb{T}$ and all initial density profiles $\gamma : \mathbb{T} \rightarrow [0, 1]$. Moreover, there exists $\delta > 0$, depending only on F , such that*

$$\delta \leq \bar{\rho}_i \leq 1 - \delta \quad (3.5)$$

for all $1 \leq i \leq l$, $\bar{\rho}_i(\theta) d\theta \in \mathcal{M}_i$.

Lemma 5 *There exists a finite constant C_0 , depending only on F , such that for any density profile $\gamma : \mathbb{T} \rightarrow [0, 1]$ and any $t > 0$,*

$$\|\rho_t\|_2^2 + \int_0^t \|\nabla \rho_s\|_2^2 ds \leq C_0(1+t),$$

where $\rho(t, \theta)$ stands for the unique weak solution of (2.4) with initial condition γ .

Proof Fix a density profile $\gamma : \mathbb{T} \rightarrow [0, 1]$. By Proposition 2, and since ρ is the weak solution of (2.4), for any $0 < s < t$, by an integration by parts,

$$\|\rho_t\|_2^2 = \|\rho_s\|_2^2 - \int_s^t \|\nabla \rho_r\|_2^2 dr + 2 \int_s^t dr \int_{\mathbb{T}} \rho_r(\theta) F(\rho_r(\theta)) d\theta.$$

Since ρ_r is absolutely bounded by 1, we complete the proof of the lemma by letting $s \downarrow 0$. \square

A similar argument provides a bound on the distance between a solution of the hydrodynamic equation and a constant stationary solution. Recall that $\alpha \in (0, 1)$ is an attractor of the ODE (3.4) if there exists $\varepsilon > 0$ such that the solution $x(t)$ of the ODE with initial condition x_0 converges to α as $t \uparrow \infty$ if $|x_0 - \alpha| < \varepsilon$. Note in particular that $F(\alpha) = 0$ if α is an attractor.

Lemma 6 *Let $\varepsilon > 0$, let α be an attractor of the ODE (3.4), and let $\bar{\rho}_\alpha$ be the density profile given by $\bar{\rho}_\alpha(\theta) = \alpha$, $\theta \in \mathbb{T}$. There exists $\delta_{10} = \delta_{10}(\varepsilon, \alpha) > 0$ such that for any density profile $\gamma : \mathbb{T} \rightarrow [0, 1]$ such that $\|\gamma - \bar{\rho}_\alpha\|_2 \leq \delta_{10}$, ρ_t converges in the sup norm to $\bar{\rho}_\alpha$ as $t \uparrow \infty$, where $\rho_t(\theta) = \rho(t, \theta)$ is the unique weak solution of (2.4) with initial condition γ . Moreover, for all $t \geq 1$, $\|\rho_t - \bar{\rho}_\alpha\|_\infty \leq \varepsilon$.*

Proof Fix $\varepsilon > 0$, $\alpha \in (0, 1)$ such that $F(\alpha) = 0$, a density profile $\gamma : \mathbb{T} \rightarrow [0, 1]$ and recall the notation introduced in the statement of the lemma. Let ρ be the weak solution of (2.4) with initial condition γ . Repeating the computation presented in the proof of Lemma 5, we obtain that for every $0 < s < t$,

$$\|\rho_t - \bar{\rho}_\alpha\|_2^2 = \|\rho_s - \bar{\rho}_\alpha\|_2^2 - \int_s^t \|\nabla \rho_r\|_2^2 dr + 2 \int_s^t dr \int_{\mathbb{T}} [\rho_r(\theta) - \bar{\rho}_\alpha(\theta)] F(\rho_r(\theta)) d\theta.$$

Since $F(\alpha) = 0$, we may subtract $F(\bar{\rho}_\alpha(\theta))$ from $F(\rho_r(\theta))$ in the last integral and bound the product by $C_0[\rho_r(\theta) - \bar{\rho}_\alpha(\theta)]^2$, where C_0 is the Lipschitz constant of F . By letting $s \downarrow 0$ and then applying Gronwall inequality, we obtain that

$$\|\rho_t - \bar{\rho}_\alpha\|_2^2 + \int_0^t \|\nabla \rho_r\|_2^2 dr \leq \|\rho_0 - \bar{\rho}_\alpha\|_2^2 e^{2C_0 t} \tag{3.6}$$

for all $t \geq 0$.

Choose $\varepsilon_0 > 0$ so that $(\alpha - 3\varepsilon_0, \alpha + 3\varepsilon_0)$ is contained in the basin of attraction of α for the ODE (3.4) and set $\varepsilon_1 = \min\{\varepsilon/2, \varepsilon_0\}$. Choose $\delta_{10} = \varepsilon_1 e^{-C_0}$ and let γ be an initial profile such that $\|\gamma - \bar{\rho}_\alpha\|_2 \leq \delta_{10}$. By (3.6), for all $0 \leq t \leq 1$,

$$\|\rho_t - \bar{\rho}_\alpha\|_2 \leq \varepsilon_1 \quad \text{and} \quad \int_0^1 \|\nabla \rho_r\|_2^2 dr \leq \varepsilon_1^2. \tag{3.7}$$

In particular, there exists $0 \leq s \leq 1$ such that $\|\nabla \rho_s\|_2 \leq \varepsilon_1$, so that

$$\sup_{\theta \neq \omega \in \mathbb{T}} |\rho(s, \theta) - \rho(s, \omega)| \leq \|\nabla \rho_s\|_1 \leq \|\nabla \rho_s\|_2 \leq \varepsilon_1.$$

Therefore, by (3.7), for all $\theta \in \mathbb{T}$,

$$|\rho(s, \theta) - \alpha| \leq \sup_{\theta \neq \omega \in \mathbb{T}} |\rho(s, \theta) - \rho(s, \omega)| + \|\rho_s - \bar{\rho}_\alpha\|_1 \leq \varepsilon_1 + \|\rho_s - \bar{\rho}_\alpha\|_2 \leq \varepsilon$$

because $2\varepsilon_1 \leq \varepsilon$.

Let $x_\pm(t)$, $t \geq 0$, be the solution of the ODE (3.4) with initial condition $x_\pm(0) = \alpha \pm 2\varepsilon_1$. By the previous estimate, $x_-(0) \leq \rho(s, \theta) \leq x_+(0)$ for all $\theta \in \mathbb{T}$. Hence, by Lemma 4, $x_-(t) \leq \rho(s+t, \theta) \leq x_+(t)$ for all $\theta \in \mathbb{T}$, $t \geq 0$. Since the basin of attraction of the ODE is contained in $(\alpha - 3\varepsilon_1, \alpha + 3\varepsilon_1)$, $x_\pm(t) \rightarrow \alpha$, as $t \rightarrow \infty$, and $x_-(0) \leq x_-(t) \leq x_+(t) \leq x_+(0)$ for all $t \geq 0$. In particular, ρ_t converges in the sup norm to $\bar{\rho}_\alpha$, as $t \rightarrow \infty$, and $\|\rho_{s+t} - \bar{\rho}_\alpha\|_\infty \leq \|\rho_s - \bar{\rho}_\alpha\|_\infty \leq \varepsilon$ for all $t \geq 0$. This completes the proof of the lemma because $s \leq 1$. \square

Similar arguments permit to estimate the distance between two solutions of the reaction–diffusion equation (2.4).

Lemma 7 *There exists a constant $C_0 > 0$ such that for any weak solutions $\rho^j, j = 1, 2$, of the Cauchy problem (2.4) with initial profile ρ_0^j and for any $t > 0$,*

$$\|\rho_t^1 - \rho_t^2\|_2 \leq e^{C_0 t} \|\rho_0^1 - \rho_0^2\|_2.$$

Proof From (3.2), for any $t \geq 0$ and $j = 1, 2$,

$$\rho_t^j = P_t \rho_0^j + \int_0^t P_{t-s} F(\rho_s^j) ds.$$

Therefore

$$\begin{aligned} \|\rho_t^1 - \rho_t^2\|_2 &\leq \|P_t(\rho_0^1 - \rho_0^2)\|_2 + \int_0^t \|P_{t-s}(F(\rho_s^1) - F(\rho_s^2))\|_2 ds \\ &\leq \|\rho_0^1 - \rho_0^2\|_2 + \|F'\|_\infty \int_0^t \|\rho_s^1 - \rho_s^2\|_2 ds. \end{aligned}$$

In the last inequality, we used the fact that the operator norm of P_t is equal to 1. To conclude the proof of the lemma, it remains to apply Gronwall inequality. \square

For each function ρ in $L^2(\mathbb{T})$, let $\mathbb{B}_\delta(\rho), \delta > 0$, be the δ -open neighborhood of ρ in $L^2(\mathbb{T})$. Recall also that we denote by $\mathcal{B}_\delta(\varrho)$ be the δ -open neighborhood of ϱ in \mathcal{M}_+ . We sometimes represent the neighborhood $\mathcal{B}_\delta(\varrho)$ by $\mathcal{B}_\delta(\gamma)$ when $\varrho(d\theta) = \gamma(\theta)d\theta$.

Lemma 8 *Let $\bar{\rho} : \mathbb{T} \rightarrow [0, 1]$ be a classical solution to the Eq. (2.5), and set $\bar{\varrho}(d\theta) = \bar{\rho}(\theta)d\theta$. For any $\varepsilon > 0$ and $0 < T < T'$, there exists $\delta_{11} = \delta_{11}(\varepsilon, T, T') \in (0, \varepsilon)$ such that for any density profile $\rho_0 : \mathbb{T} \rightarrow [0, 1], \rho_0(\theta)d\theta \in \mathcal{B}_{\delta_{11}}(\bar{\varrho})$, it holds that $\rho(t, \theta)d\theta \in \mathcal{B}_\varepsilon(\bar{\varrho})$ for all $0 \leq t \leq T'$ and that $\rho_t \in \mathbb{B}_\varepsilon(\bar{\rho})$ for all $T \leq t \leq T'$, where $\rho_t(\theta) = \rho(t, \theta)$ is the unique weak solution of the Cauchy problem (2.4) with initial condition ρ_0 .*

Proof Fix $\varepsilon > 0$ and $0 < T < T'$. Let $\zeta_1 = (1/3)\varepsilon e^{-C_0 T'}$, where C_0 is the constant appearing in Lemma 7. Fix a density profile $\rho_0 : \mathbb{T} \rightarrow [0, 1]$, and let $\pi_t(d\theta) = \rho(t, \theta)d\theta$, where ρ_t is the unique weak solution of the Cauchy problem (2.4) with initial condition ρ_0 . Recall the definition of the complete orthogonal normal basis $\{e_k; k \in \mathbb{Z}\}$ introduced just before (2.1). By (2.1) and (3.1), for any $t \geq 0$,

$$d(\pi_t, \bar{\varrho}) \leq d(\pi_0, \bar{\varrho}) + \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \left| \frac{1}{2} \int_0^t ds \langle \rho_s, \Delta e_k \rangle + \int_0^t ds \langle F(\rho_s), e_k \rangle \right|.$$

The first term on the right hand side is bounded by $\varepsilon/2$ if $\rho_0 \in \mathcal{B}_{\zeta_2}(\bar{\varrho})$, where $\zeta_2 = \varepsilon/2$, while the second one is less than or equal to

$$t \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \{C_0 + (2\pi k)^2\} = C_0 t,$$

because ρ_s is bounded by 1, F by a constant C_0 and $\|e_k\|_2 = 1$. Hence, if $T_1 = \varepsilon/2C_0$ and $\pi_0 \in \mathcal{B}_{\zeta_2}(\bar{\varrho})$,

$$\pi_t \in \mathcal{B}_\varepsilon(\bar{\varrho}) \text{ for all } 0 \leq t \leq T_1. \tag{3.8}$$

We turn to the L^2 -estimate. From the Eq. (3.2), we have

$$\begin{aligned} \|\rho_t - \bar{\rho}\|_2 &\leq \|P_t(\rho_0 - \bar{\rho})\|_2 + \int_0^t \|P_{t-s}[F(\rho_s) - F(\bar{\rho})]\|_2 ds \\ &\leq \|P_t(\rho_0 - \bar{\rho})\|_2 + t\|F'\|_\infty, \end{aligned} \tag{3.9}$$

since the operator norm of P_t is equal to 1. Let $\tilde{\rho} = \rho_0 - \bar{\rho}$ and, for each $t > 0$, $\tilde{\rho}_t = P_t\tilde{\rho}$. It is easy to see that, for any $k \in \mathbb{Z}$,

$$\langle \tilde{\rho}_t, e_k \rangle = \langle \tilde{\rho}, P_t e_k \rangle = e^{-2\pi^2 k^2 t} \langle \tilde{\rho}, e_k \rangle.$$

Therefore, from Parseval’s relation,

$$\|\tilde{\rho}_t\|_2^2 = \sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 t} \langle \tilde{\rho}, e_k \rangle^2. \tag{3.10}$$

Set $T_2 := \min\{\zeta_1/2\|F'\|_\infty, T_1, T\}$ and choose a large enough positive integer k_1 so that

$$\sum_{|k| > k_1} e^{-4\pi^2 k^2 T_2} \leq \zeta_1^2/8.$$

To estimate the first terms of the series, observe that

$$\sum_{|k| \leq k_1} e^{-4\pi^2 k^2 t} \langle \tilde{\rho}, e_k \rangle^2 \leq 4^{k_1} \left(\sum_{|k| \leq k_1} 2^{-|k|} |\langle \tilde{\rho}, e_k \rangle| \right)^2 \leq 4^{k_1} d(\rho_0, \bar{\rho})^2.$$

Hence, if we set $\zeta_3 = \zeta_1/2^{k_1+2}$, this last expression is bounded by $\zeta_1^2/16 \leq \zeta_1^2/8$ provided $\rho_0(\theta)d\theta$ belongs to $\mathcal{B}_{\zeta_3}(\bar{\varrho})$. Therefore, by (3.9), (3.10) and the choice of ζ_3 ,

$$\|\rho_{T_2} - \bar{\rho}\|_2 \leq \zeta_1 \tag{3.11}$$

if $\pi_0 \in \mathcal{B}_{\zeta_3}(\bar{\varrho})$.

Let $\delta_{11} = \min\{\zeta_2, \zeta_3\}$, and note that δ_{11} depends only on ε, T, T' . By (3.8) and (3.11), and since $T_2 \leq T_1$, for all $\pi_0 \in \mathcal{B}_{\delta_{11}}(\bar{\varrho})$,

$$\pi_t \in \mathcal{B}_\varepsilon(\bar{\varrho}) \text{ for all } 0 \leq t \leq T_2 \text{ and } \|\rho_{T_2} - \bar{\rho}\|_2 \leq \zeta_1.$$

By Lemma 7, by the previous estimate and by definition of ζ_1 , for all $T_2 \leq t \leq T'$, $\pi_0 \in \mathcal{B}_{\delta_{11}}(\bar{\varrho})$,

$$\|\rho_t - \bar{\rho}\|_2 \leq e^{C_0(t-T_2)} \|\rho_{T_2} - \bar{\rho}\|_2 \leq e^{C_0 T'} \zeta_1 \leq \varepsilon/3.$$

Moreover, by this bound and by (2.2), for all $T_2 \leq t \leq T'$, $\pi_0 \in \mathcal{B}_{\delta_{11}}(\bar{\varrho})$,

$$d(\pi_t, \bar{\varrho}) \leq 3 \|\rho_t - \bar{\rho}\|_2 \leq \varepsilon.$$

This completes the proof of the lemma since $T_2 \leq T$. □

The previous results permit to strengthen Lemma 6.

Lemma 9 *Let $\varepsilon > 0$, let α be an attractor of the ODE (3.4), and let $\bar{\varrho}_\alpha(d\theta) = \bar{\rho}_\alpha(\theta)d\theta$, $\bar{\rho}_\alpha(\theta) = \alpha$, $\theta \in \mathbb{T}$. There exists $\delta_{12} = \delta_{12}(\varepsilon, \alpha) > 0$ such that for any density profile $\gamma : \mathbb{T} \rightarrow [0, 1]$ such that $\gamma(\theta)d\theta \in \mathcal{B}_{\delta_{12}}(\bar{\varrho}_\alpha)$, ρ_t converges in the sup norm to $\bar{\rho}_\alpha$, as $t \uparrow \infty$, where $\rho_t(\theta) = \rho(t, \theta)$ is the unique weak solution of (2.4) with initial condition γ . Moreover, $\pi_t(d\theta) = \rho(t, \theta)d\theta$ belongs to $\mathcal{B}_\varepsilon(\bar{\varrho}_\alpha)$ for all $t \geq 0$.*

Proof Fix $\varepsilon > 0$. Denote by ζ_1 the constant $\delta_{10} = \delta_{10}(\varepsilon, \alpha)$ provided by Lemma 6. Let $\zeta_2 = \min\{\zeta_1, \varepsilon\}$, and let δ_{12} be the constant $\delta_{11} = \delta_{11}(\zeta_2, 1/2, 2)$ provided by Lemma 8 with $\bar{\rho} = \bar{\rho}_\alpha$.

Fix $\gamma : \mathbb{T} \rightarrow [0, 1]$ such that $\gamma(\theta)d\theta \in \mathcal{B}_{\delta_{12}}(\bar{\varrho}_\alpha)$. Denote by $\rho_t(\theta) = \rho(t, \theta)$ the weak solution of the hydrodynamic equation with initial condition γ . By Lemma 8, $\|\rho_t - \bar{\rho}_\alpha\|_2 \leq \zeta_1$ and $\pi_t(d\theta) = \rho(t, \theta)d\theta$ belongs to $\mathcal{B}_\varepsilon(\bar{\varrho}_\alpha)$ for all $t \leq 2$.

Since $\|\rho_t - \bar{\rho}_\alpha\|_2 \leq \zeta_1$, by Lemma 6, ρ_t converges in the sup norm to $\bar{\rho}_\alpha$ as $t \uparrow \infty$, and $\|\rho_t - \alpha\|_\infty \leq \varepsilon$ for all $t \geq 2$. In particular, $\pi_t(d\theta) = \rho(t, \theta)d\theta$ belongs to $\mathcal{B}_\varepsilon(\bar{\varrho}_\alpha)$ for all $t \geq 2$. □

4 The dynamical rate function

We present in this section some features of the dynamical rate function needed to prove the properties of the static rate function stated in the next section. The main result of the section asserts that a trajectory can not remain too long far in the L^2 -topology from all stationary solutions of the hydrodynamic equation without paying a fixed positive cost.

The first four lemmata have been proved in [31, Section 4] for the rate functional $I_T(\cdot|\gamma)$. The same arguments apply the functional I_T . The first three extract information on the trajectory $\pi(t, d\theta)$ from the finiteness of the large deviations rate functional. Lemma 10 states that a trajectory with finite rate function is a continuous path in $D([0, T], \mathcal{M}_+)$. This lemma is repeatedly used in the rest of this paper, and therefore, is used without any further mention.

Lemma 10 *Fix $T > 0$. Let π be a path in $D([0, T], \mathcal{M}_+)$ such that $I_T(\pi)$ is finite. Then π belongs to $C([0, T], \mathcal{M}_{+,1})$.*

Lemma 11 states that the density $\rho(t, \theta)$ of a trajectory $\pi(t, d\theta)$ with finite rate function belongs to \mathcal{H}_1 for almost all t , where \mathcal{H}_1 represents the space of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ which have a general derivative in $L^2(\mathbb{T})$.

Lemma 11 *There exists a finite constant $C_0 > 0$ such that for any $T > 0$ and for any path $\pi(t, d\theta) = \rho(t, \theta)d\theta$ in $D([0, T], \mathcal{M}_{+,1})$ with finite energy,*

$$\mathcal{E}_T(\rho) := \int_0^T dt \int_{\mathbb{T}} d\theta \frac{|\nabla \rho(t, \theta)|^2}{\chi(\rho(t, \theta))} \leq C_0 \{I_T(\pi) + T + 1\}.$$

The next result characterizes the weak solutions of the hydrodynamic equation as the trajectories at which the dynamical large deviations rate functional vanishes.

Lemma 12 *Fix $T > 0$. The density ρ of a path $\pi(t, d\theta) = \rho(t, \theta)d\theta$ in $D([0, T], \mathcal{M}_{+,1})$ is the weak solution of the Cauchy problem (2.4) with initial profile γ if and only if $I_T(\pi|\gamma) = 0$. Moreover, in that case*

$$\mathcal{E}_T(\rho) < \infty.$$

The next result is extremely useful. In the expression of the functionals $J_{T,G}$ and I_T , terms appearing such as $\int_0^T \langle B(\rho_t), G_t \rangle dt$, where G is a smooth function, are not continuous for the weak topology, but only for the L^1 -topology. The next lemma establishes that if the cost of a sequence π^n of trajectories is uniformly bounded and if this sequence converges weakly to some trajectory π , then the sequence of density profiles converges in L^2 -topology. The proof of this result follows from the computations presented in the proof of Theorem 4.7 in [31].

Lemma 13 *Fix $T > 0$. Let $\{\pi^n(t, d\theta) = \rho^n(t, \theta)d\theta : n \geq 1\}$ be a sequence of trajectories in $D([0, T], \mathcal{M}_{+,1})$. Assume that there exists a finite constant C such that*

$$\sup_{n \geq 1} I_T(\pi^n) \leq C.$$

If ρ^n converges to ρ weakly in $L^2(\mathbb{T} \times [0, T])$, then ρ^n converges to ρ strongly in $L^2(\mathbb{T} \times [0, T])$.

Recall the definition of the neighborhoods $\mathbb{B}_\delta(\rho)$ and $\mathcal{B}_\delta(\varrho)$ introduced just before the statement of Lemma 8. For each $\delta > 0$ and $T > 0$, denote by $\mathbb{D}_{T,\delta}$ the set of trajectories $\pi(t, d\theta) = \rho(t, \theta)d\theta$ in $D([0, T], \mathcal{M}_{+,1})$ such that $\rho_t \notin \mathbb{B}_\delta(\bar{\rho})$ for all $0 \leq t \leq T$ and $\bar{\rho} \in S$.

Next lemma states that a trajectory can not stay a long time interval far, in the L^2 -topology, from all stationary solutions of the hydrodynamic equation without paying an appreciable cost. This result plays a fundamental role in the proof of the lower semi-continuity of the functional W . To enhance its interest, note that L^2 -neighborhoods are much thinner than the neighborhoods of the weak topology.

Lemma 14 *For every $\delta > 0$ there exists $T = T(\delta) > 0$ such that*

$$\inf_{\pi \in \mathbb{D}_{T,\delta}} I_T(\pi) > 0.$$

Proof Assume that the assertion of the lemma is false. Then, there exists some $\delta > 0$ such that, for any $n \in \mathbb{N}$,

$$\inf_{\pi \in \mathbb{D}_{n,\delta}} I_n(\pi) = 0.$$

In this case there exists a sequence of trajectories $\{\pi^n(t, d\theta) = \rho^n(t, \theta)d\theta : n \geq 1\}$, $\pi^n \in \mathbb{D}_{n,\delta}$, such that $I_n(\pi^n) \leq 1/n$. Since I_T has compact level sets, by using a Cantor’s diagonal argument and passing to a subsequence if necessary, we obtain a path $\pi(t, d\theta) = \rho(t, \theta)d\theta$ in $D(\mathbb{R}_+, \mathcal{M}_{+,1})$ such that π^n converges to π in $D([0, T], \mathcal{M}_+)$ for any $T > 0$. Moreover, by Lemma 13, ρ^n converges to ρ strongly in $L^2([0, T] \times \mathbb{T})$ for all $T > 0$.

Since I_T is lower semicontinuous, $I_T(\pi) = 0$ for all $T > 0$. By Lemma 12, the density of π , denoted by ρ so that $\pi(t, d\theta) = \rho(t, \theta)d\theta$, is the unique weak solution of the Eq. (2.4) with initial condition $\rho(0, \cdot)$. Hence, by Lemma 1, ρ_t converges in $C^2(\mathbb{T})$ to some density profile $\rho_\infty \in S$. Therefore, there exists some $T_0 > 0$ such that

$$\|\rho_t - \rho_\infty\|_2 \leq \delta/2,$$

for any $t \geq T_0$. Hence, since π^n belongs to $\mathbb{D}_{n,\delta}$, for $n \geq T_0 + 1$

$$\begin{aligned} \int_0^{T_0+1} \|\rho_t^n - \rho_t\|_2 dt &\geq \int_{T_0}^{T_0+1} \|\rho_t^n - \rho_t\|_2 dt \\ &\geq \int_{T_0}^{T_0+1} (\|\rho_t^n - \rho_\infty\|_2 - \|\rho_t - \rho_\infty\|_2) dt \\ &\geq \delta - \delta/2 = \delta/2, \end{aligned}$$

which contradicts the strong convergence of ρ^n to ρ in $L^2([0, T_0 + 1] \times \mathbb{T})$ and we are done. □

Analogously, for each $\delta > 0$ and $T > 0$, denote by $\mathcal{D}_{T,\delta}$ the set of trajectories $\pi(t, d\theta) = \rho(t, \theta)d\theta$ in $D([0, T], \mathcal{M}_{+,1})$ such that $\pi_t \notin \mathcal{B}_\delta(\bar{\varrho})$ for all $0 \leq t \leq T$ and $\bar{\varrho} \in \mathcal{M}_{\text{sol}}$. A similar result also holds for the set $\mathcal{D}_{T,\delta}$.

Corollary 1 *For every $\delta > 0$, there exists $T > 0$ such that*

$$\inf_{\pi \in \mathcal{D}_{T,\delta}} I_T(\pi) > 0.$$

Proof The assertion follows from Lemma 14 and the fact that

$$\{\varrho(d\theta) = \rho(\theta)d\theta : \rho \in \mathbb{B}_\delta(\bar{\rho})\} \subset \mathcal{B}_{3\delta}(\bar{\rho}),$$

for every $\bar{\rho} \in S$ and every $\delta > 0$ in view of (2.2). □

In the proof of the static large deviations principle, it will be useful to estimate $V_i(\varrho)$ for some measure $\varrho(d\theta) = \gamma(\theta)d\theta$. If γ is a smooth density profile, this can be achieved by joining $\bar{\varrho}_i$ to ϱ through a linear interpolation $\pi_t = (1 - t/T)\bar{\varrho}_i + (t/T)\varrho$,

$T > 0$, and by estimating the cost of the path π . This is the content of Lemma 17. For a general measure $\varrho(d\theta) = \gamma(\theta)d\theta$, we need first to smooth the density profile γ . We use the hydrodynamic equation to do that. Fix $\varepsilon > 0$ small, and denote by $\rho(t, \theta)$ the solution of the hydrodynamic equation starting from γ , $0 \leq t \leq \varepsilon$. By Proposition 2, $\rho(\varepsilon, \cdot)$ is smooth. We may use the first part of this argument to joint $\bar{\varrho}_i$ to $\rho(\varepsilon, \theta)d\theta$. To connect $\rho(\varepsilon, \theta)d\theta$ to $\gamma(\theta)d\theta$ we use the backward path $\hat{\pi}_t(d\theta) = \rho(\varepsilon - t, \theta)d\theta$, $0 \leq t \leq \varepsilon$. The cost of this path is estimated in the next lemma.

Lemma 15 *There exists a constant $C_0 > 0$ such that for any $T > 0$, any weak solution ρ of (2.4), and any classical solution $\bar{\rho}$ to the Eq. (2.5),*

$$I_T(\pi) \leq C_0\{T + \|\rho_T - \bar{\rho}\|_1 + \|\rho_0 - \bar{\rho}\|_1\},$$

where π is the trajectory defined by $\pi(t, d\theta) = \rho(T - t, \theta)d\theta$.

Proof For any test function $G \in C^{1,2}([0, T] \times \mathbb{T})$, $J_{T,G}(\pi)$ can be rewritten as

$$\begin{aligned} J_{T,G}(\pi) &= \int_0^T dt \langle \nabla \rho_t, \nabla \widehat{G}_t \rangle - \frac{1}{2} \int_0^T dt \langle \chi(\rho_t), (\nabla \widehat{G}_t)^2 \rangle \\ &\quad - \int_0^T dt \langle B(\rho_t), e^{\widehat{G}_t} + \widehat{G}_t - 1 \rangle - \int_0^T dt \langle D(\rho_t), e^{-\widehat{G}_t} - \widehat{G}_t - 1 \rangle, \end{aligned}$$

where $\widehat{G}(t, \theta) = G(T - t, \theta)$. The first line on the right hand side is bounded above by

$$\frac{1}{2} \int_0^T dt \int_{\mathbb{T}} d\theta \frac{|\nabla \rho(t, \theta)|^2}{\chi(\rho(t, \theta))} = \frac{1}{2} \mathcal{E}_T(\rho). \tag{4.1}$$

Since for any $0 < \rho < 1$ and any $a \in \mathbb{R}$

$$\begin{aligned} -B(\rho)e^a + D(\rho)a + D(\rho) &\leq D(\rho) \log(D(\rho)/B(\rho)), \\ -D(\rho)e^{-a} - B(\rho)a + B(\rho) &\leq B(\rho) \log(B(\rho)/D(\rho)), \end{aligned}$$

and since $B(\rho) = (1 - \rho)\widehat{B}(\rho)$, $D(\rho) = \rho\widehat{D}(\rho)$, where $\widehat{B}(\rho)$, $\widehat{D}(\rho)$ are the strictly positive functions introduced in (3.3), the second line on the right hand side is bounded above by

$$\begin{aligned} &\int_0^T dt \langle D(\rho_t) \log(D(\rho_t)/B(\rho_t)) + B(\rho_t) \log(B(\rho_t)/D(\rho_t)) \rangle \\ &\leq C_0 T - \int_0^T dt \langle D(\rho_t) \log(1 - \rho_t) + B(\rho_t) \log(\rho_t) \rangle \end{aligned} \tag{4.2}$$

To estimate this last term, let $h(x) = x \log x + (1 - x) \log(1 - x)$ and note that

$$\begin{aligned} \partial_t h(\rho_t) &= [\log(\rho_t) - \log(1 - \rho_t)] \partial_t \rho_t \\ &= [\log(\rho_t) - \log(1 - \rho_t)] \left[\frac{1}{2} \Delta \rho_t + F(\rho_t) \right]. \end{aligned}$$

This equation is justified since, by Proposition 2, any weak solution of the Eq. (2.4) is smooth for $t > 0$. Therefore,

$$\begin{aligned}
 - \int_0^T dt \langle D(\rho_t) \log(1 - \rho_t) + B(\rho_t) \log(\rho_t) \rangle &= - \langle h(\rho_T) \rangle + \langle h(\rho_0) \rangle \\
 - \frac{1}{2} \mathcal{E}_T(\rho) - \int_0^T dt \langle D(\rho_t) \log(\rho_t) + B(\rho_t) \log(1 - \rho_t) \rangle.
 \end{aligned}$$

Adding and subtracting $\langle h(\bar{\rho}) \rangle$, since $\bar{\rho}$ takes value in a compact interval of $(0, 1)$, the first two terms on the right hand side can be bounded above by

$$C_0 \{ \|\rho_T - \bar{\rho}\|_1 + \|\rho_0 - \bar{\rho}\|_1 \},$$

for some $C_0 > 0$. Since the last term in the penultimate displayed formula is bounded by $C_0 T$, we have shown that (4.2) is less than or equal to

$$-\frac{1}{2} \mathcal{E}_T(\rho) + C_0 \{ T + \|\rho_T - \bar{\rho}\|_1 + \|\rho_0 - \bar{\rho}\|_1 \}.$$

This estimate together with (4.1) completes the proof of the lemma. □

The last result of this section states that the cost to move inside a set of static solutions is zero.

Lemma 16 Fix $1 \leq i \leq l$. For all $\bar{q}_1, \bar{q}_2 \in \mathcal{M}_i$,

$$\inf \{ I_T(\pi | \bar{q}_1) : T > 0, \pi \in D([0, T], \mathcal{M}_+), \pi_T = \bar{q}_2 \} = 0.$$

Proof If \mathcal{M}_i is a singleton, then the conclusion is clear. Assume that \mathcal{M}_i is not a singleton and fix $\bar{q}_1, \bar{q}_2 \in \mathcal{M}_i$ so that $\bar{\rho}_k(d\theta) = \bar{\rho}_k(\theta)d\theta$, $k = 1, 2$, and $\bar{\rho}_2(\theta) = \bar{\rho}_1(\theta + \theta_0)$ for some $0 < \theta_0 < 1$.

Fix $a > 0$ small, and let $\rho(t, \theta) = \bar{\rho}_1(\theta + at)$ so that $\rho(0, \cdot) = \bar{\rho}_1(\cdot)$, $\rho(\theta_0/a, \cdot) = \bar{\rho}_2(\cdot)$. Let $T = \theta_0/a$, $\pi_t(d\theta) = \rho(t, \theta)d\theta$. Since $\partial_t \rho = a \nabla \rho$ and $(1/2)\Delta \rho + F(\rho) = 0$, an integration by parts gives that for any smooth function $G : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$,

$$\begin{aligned}
 J_{T,G}(\pi) &= - \int_0^T dt \langle a \rho_t, \nabla G_t \rangle - \frac{1}{2} \int_0^T dt \langle \chi(\rho_t), (\nabla G_t)^2 \rangle \\
 &\quad - \int_0^T dt \langle B(\rho_t), e^{G_t} - 1 - G_t \rangle - \int_0^T dt \langle D(\rho_t), e^{-G_t} - 1 + G_t \rangle.
 \end{aligned}$$

The second line is negative, while the first one, by Young’s inequality and by Lemma 4, is less than or equal to

$$\frac{a^2 T}{2} \int_{\mathbb{T}} \frac{\bar{\rho}_1(\theta)}{1 - \bar{\rho}_1(\theta)} d\theta \leq C_0 a \theta_0.$$

To complete the proof it remains to let $a \rightarrow 0$. □

5 The static rate functional W

In this section, we present some properties of the quasi-potential W . The main result asserts that the functional W , introduced in (2.14), is lower semicontinuous for the weak topology.

The first main result states that W is continuous at each measure $\bar{q}_i \in \mathcal{M}_i$ in the L^2 -topology. The second one states that W is lower semicontinuous in the weak topology.

We start with an estimate of $V_i(\varrho)$ for measures $\varrho(d\theta) = \gamma(\theta)d\theta$ whose density is close to \mathcal{M}_i in the L^2 -topology. This estimate together with Lemma 15 will allow us to prove that V_i is continuous for the L^2 -topology.

Let \mathbb{D} be the space of measurable functions on \mathbb{T} bounded below by 0 and bounded above by 1, endowed with the L^2 -topology:

$$\mathbb{D} = \{ \rho : \mathbb{T} \rightarrow [0, 1] : 0 \leq \rho(\theta) \leq 1 \text{ a.e. } \theta \in \mathbb{T} \}.$$

For each $1 \leq i \leq l$, let $\mathbb{V}_i : \mathbb{D} \rightarrow [0, +\infty]$ be the functional given by $\mathbb{V}_i(\rho) = V_i(\rho(\theta)d\theta)$. Note that the topology of $\mathcal{M}_{+,1}$ is the weak topology, while the one of \mathbb{D} is the L^2 -topology. Recall that we denote by \mathcal{H}_1 the Sobolev space of functions G with generalized derivatives ∇G in $L^2(\mathbb{T})$. For each $h > 0$ and each $\delta > 0$, let \mathbb{D}_δ^h be the subset of \mathbb{D} consisting of those profiles ρ satisfying the following conditions:

- (A) $\rho \in \mathcal{H}_1$ and $\int_{\mathbb{T}} (\nabla \rho(\theta))^2 d\theta \leq h$.
- (B) $\delta \leq \rho(\theta) \leq 1 - \delta$ a.e. in \mathbb{T} .

Lemma 17 *For each $1 \leq i \leq l$, $h > 0$, $\delta > 0$ and an increasing C^1 -diffeomorphism $\alpha : [0, 1] \rightarrow [0, 1]$, there exist constants $C_1 = C_1(\delta, h) > 0$ and $C_2 = C_2(\delta, \alpha) > 0$ such that for any ρ in \mathbb{D}_δ^h and $\bar{\rho}_i(\theta)d\theta$ in \mathcal{M}_i*

$$\mathbb{V}_i(\rho) \leq C_1 \int_0^1 \alpha(t)^2 dt + C_2 \|\rho - \bar{\rho}_i\|_1.$$

Proof Fix $1 \leq i \leq l$, $h > 0$, $\delta > 0$ and let $\alpha : [0, 1] \rightarrow [0, 1]$ be an increasing C^1 -diffeomorphism. Let $\rho \in \mathbb{D}_\delta^h$ and $\bar{\rho}_i(\theta)d\theta \in \mathcal{M}_i$. Consider the path $\pi_t^\alpha(d\theta) = \rho^\alpha(t, \theta)d\theta$ in $C([0, 1], \mathcal{M}_+)$ with density given by $\rho_t^\alpha = (1 - \alpha(t))\bar{\rho}_i + \alpha(t)\rho$. It is clear that π^α belongs to $D([0, 1], \mathcal{M}_{+,1})$, and it follows from condition (A) that $\mathcal{Q}_1(\pi^\alpha)$ is finite. From the definition of ρ^α it follows that $\nabla \rho_t^\alpha = \alpha(t)(\nabla \rho - \nabla \bar{\rho}_i) + \nabla \bar{\rho}_i$ and that $\partial_t \rho_t^\alpha = \alpha'(t)(\rho - \bar{\rho}_i)$. Since $(1/2)\Delta \bar{\rho}_i = -F(\bar{\rho}_i)$, $J_{1,G}(\pi^\alpha)$ can be rewritten as

$$\begin{aligned} J_{1,G}(\pi^\alpha) &= \frac{1}{2} \int_0^1 dt \left\{ \alpha(t) \langle (\nabla \rho - \nabla \bar{\rho}_i), \nabla G_t \rangle - \langle \chi(\rho_t^\alpha), (\nabla G_t)^2 \rangle \right\} \\ &+ \int_0^1 dt \left\{ \alpha'(t)(\rho - \bar{\rho}_i) + F(\bar{\rho}_i) \right\} G_t - B(\rho_t^\alpha)(e^{G_t} - 1) - D(\rho_t^\alpha)(e^{-G_t} - 1). \end{aligned} \tag{5.1}$$

By Young’s inequality, the first term on the right hand side of (5.1) is bounded by

$$\frac{1}{8} \int_0^1 \alpha(t)^2 \left\langle \frac{(\nabla \rho - \nabla \bar{\rho}_i)^2}{\chi(\rho_t^\alpha)} \right\rangle dt \leq C_1 \int_0^1 \alpha(t)^2 dt$$

for some finite constant $C_1 = C_1(\delta, h)$. To derive the last inequality we used the fact that $\bar{\rho}_i$ is bounded away from 0 and 1 and conditions (A) and (B) on ρ .

To conclude the proof it is enough to show that the second term on the right hand side of (5.1) is bounded by

$$C_2 \|\rho - \bar{\rho}_i\|_1$$

for some constant $C_2 = C_2(\delta, \alpha)$.

Consider the function $\Phi : \mathbb{R} \times (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\Phi(H, \rho, G) = HG - B(\rho)(e^G - 1) - D(\rho)(e^{-G} - 1).$$

If we set $H_t = \alpha'(t)(\rho - \bar{\rho}_i) + F(\bar{\rho}_i)$, it is clear that the second term on the right hand side of (5.1) can be expressed as

$$\int_0^1 \langle \Phi(H_t, \rho_t^\alpha, G_t) \rangle dt.$$

It follows from a straightforward computation that for any fixed $H \in \mathbb{R}$ and $\rho \in (0, 1)$, the function $\Phi(H, \rho, \cdot)$ reaches a maximum at

$$G(H, \rho) = \log \left(\frac{H + \sqrt{H^2 + 4B(\rho)D(\rho)}}{2B(\rho)} \right).$$

From condition (B) and Lemma 4, there exists a constant $c_\delta > 0$ such that $c_\delta \leq \rho^\alpha \leq 1 - c_\delta$. On the other hand, since $\Phi(H, \rho, 0) = 0$ and since $G(F(\rho), \rho) = 0, \Phi(F(\rho), \rho, G(F(\rho), \rho)) = 0$ for any $\rho \in \mathbb{R}$. Therefore, as $(H, \rho) \mapsto \Phi(H, \rho, G(H, \rho))$ is a Lipschitz-continuous function on the interval $[-\|\alpha'\|_\infty - \|F\|_\infty, \|\alpha'\|_\infty + \|F\|_\infty] \times [c_\delta, 1 - c_\delta]$,

$$\begin{aligned} \Phi(H_t, \rho_t^\alpha, G_t) &\leq \Phi(H_t, \rho_t^\alpha, G(H_t, \rho_t^\alpha)) \\ &= \Phi(H_t, \rho_t^\alpha, G(H_t, \rho_t^\alpha)) - \Phi(F(\rho_t^\alpha), \rho_t^\alpha, G(F(\rho_t^\alpha), \rho_t^\alpha)) \\ &\leq C_2 |H_t - F(\rho_t^\alpha)| \\ &\leq C_2 \{ \alpha'(t) + \|F'\|_\infty \alpha(t) \} |\rho - \bar{\rho}_i| \end{aligned}$$

for some finite constant $C_2 = C_2(\delta, \alpha)$. These bounds give the desired conclusion. \square

We are now in a position to prove that the functional V_i is continuous in the L^2 -topology.

Theorem 6 *For each $1 \leq i \leq l$, the function \mathbb{V}_i is continuous at $\bar{\rho}_i$ in \mathbb{D} .*

Proof Fix $1 \leq i \leq l$, and let $\{\rho^n : n \geq 1\}$ be a sequence in \mathbb{D} converging to $\bar{\rho}_i$. Denote by $\lambda^n, n \geq 1$, the weak solution to the Eq. (2.4) with initial condition ρ^n .

By Lemma 5, there exists a constant $C_0 > 0$, independent of n , such that

$$\int_0^1 dt \int_{\mathbb{T}} |(\nabla \lambda_t^n)(\theta)|^2 d\theta \leq C_0$$

for all $n \geq 1$. Fix $0 < \zeta < 1$. For each $n \geq 1$, there exists $\zeta \leq T_n \leq 2\zeta$ such that

$$\int_{\mathbb{T}} |(\nabla \lambda_{T_n}^n)(\theta)|^2 d\theta \leq C_0/\zeta.$$

Moreover, by Lemma 4, there exists a constant $0 < c_\zeta < 1/2$, independent of n , such that $c_\zeta \leq \lambda_{T_n}^n(\theta) \leq 1 - c_\zeta$ for all $n \geq 1$ and θ in \mathbb{T} . Therefore, the density profiles $\lambda_{T_n}^n, n \geq 1$, belong to the set $\mathbb{D}_{c_\zeta}^{C_0/\zeta}$ introduced just above Lemma 17.

By definition (2.11) of the functional V_i ,

$$\mathbb{V}_i(\rho^n) \leq \mathbb{V}_i(\lambda_{T_n}^n) + I_{T_n}(\pi^n),$$

where $\pi^n(t, d\theta) = \lambda^n(T_n - t, \theta)d\theta, 0 \leq t \leq T_n$. Therefore, it is enough to prove that

$$\limsup_{n \rightarrow \infty} \mathbb{V}_i(\lambda_{T_n}^n) = \limsup_{\zeta \downarrow 0} \limsup_{n \rightarrow \infty} I_{T_n}(\pi^n) = 0.$$

Since $\lambda_{T_n}^n$ belongs to the set $\mathbb{D}_{c_\zeta}^{C_0/\zeta}$, by Lemma 17, there exist constants $C_1 = C_1(\zeta) > 0$ and $C_2 = C_2(\zeta, \alpha) > 0$ such that

$$\mathbb{V}_i(\lambda_{T_n}^n) \leq C_1 \int_0^1 \alpha(t)^2 dt + C_2 \|\lambda_{T_n}^n - \bar{\rho}_i\|_2,$$

for any increasing C^1 -diffeomorphism $\alpha : [0, 1] \rightarrow [0, 1]$. By Lemma 7 and since $T_n \leq 2\zeta \leq 2$,

$$\|\lambda_{T_n}^n - \bar{\rho}_i\|_2 \leq C_0 \|\rho^n - \bar{\rho}_i\|_2 \tag{5.2}$$

for some finite constant $C_0 > 0$, independent of n , and whose value may change from line to line. Since ρ^n converges to $\bar{\rho}_i$ in L^2 -topology,

$$\limsup_{n \rightarrow \infty} \mathbb{V}_i(\lambda_{T_n}^n) \leq C_1 \inf_{\alpha} \left\{ \int_0^1 \alpha(t)^2 dt \right\} = 0.$$

It remains to prove that

$$\limsup_{\zeta \downarrow 0} \limsup_{n \rightarrow \infty} I_{T_n}(\pi^n) = 0. \tag{5.3}$$

By Lemma 15, by (5.2), and since $T_n \leq 2\zeta$,

$$I_{T_n}(\pi^n) \leq C_0 \left\{ T_n + \|\lambda_{T_n}^n - \bar{\rho}_i\|_2 + \|\rho^n - \bar{\rho}_i\|_2 \right\} \leq C_0 \left\{ \zeta + \|\rho^n - \bar{\rho}_i\|_2 \right\}.$$

To complete the proof of (5.3) and the one of the lemma, it remains to recall that $\rho^n \rightarrow \bar{\rho}_i$ in $L^2(\mathbb{T})$. □

The proof of the next result is similar to the one of Proposition 4.9 in [21]. The lemma asserts that the functional V_i is uniformly bounded in $\mathcal{M}_{+,1}$.

Lemma 18 *The function W is finite if and only if ϱ belongs to $\mathcal{M}_{+,1}$. Moreover, for all $1 \leq i \leq l$,*

$$\sup_{\varrho \in \mathcal{M}_{+,1}} V_i(\varrho) < \infty,$$

so that $\sup_{\varrho \in \mathcal{M}_{+,1}} W(\varrho) < \infty$.

Proof Fix $\varrho \in \mathcal{M}_+$, and suppose that $W(\varrho) < \infty$. By the definition (2.14) of the functional W , there exists $1 \leq i \leq l$ such that $V_i(\varrho) < \infty$. Hence, by (2.11), there exist $\bar{\rho} \in \mathcal{M}_i$, $T < \infty$ and a trajectory $\pi_t(d\theta)$, $0 \leq t \leq T$, such that $\pi_T = \varrho$, $I_T(\pi|\bar{\rho}) < \infty$. By (2.10) and (2.9), $\pi \in D([0, T], \mathcal{M}_{+,1})$, proving that $\varrho = \pi_T$ belongs to $\mathcal{M}_{+,1}$, as claimed.

To prove the reciprocal assertion and the uniform bound, fix $1 \leq i \leq l$, $\varrho \in \mathcal{M}_{+,1}$, $\varrho(d\theta) = \rho(\theta)d\theta$, and denote by λ the weak solution of the Eq. (2.4) with initial condition ρ . By Lemmata 4 and 5, there exist constants $0 < a < 1/2$ and $C_0 > 0$ such that $a \leq \lambda_t(\theta) \leq 1 - a$ for all $t \geq 1$, $\theta \in \mathbb{T}$, and

$$\int_0^2 dt \int_{\mathbb{T}} |(\nabla \lambda_t)(\theta)|^2 d\theta \leq C_0.$$

In particular, there exists $1 \leq T \leq 2$ such that λ_T belongs to the set $\mathbb{D}_a^{C_0}$.

By the definition (2.11) of the functional V_i ,

$$V_i(\rho) \leq V_i(\lambda_T) + I_T(\pi),$$

where $\pi(t, d\theta) = \lambda(T - t, \theta)d\theta$, $0 \leq t \leq T$.

As λ_T belongs to the set $\mathbb{D}_a^{C_0}$, by Lemma 17, $V_i(\lambda_T) \leq C_1$ for some finite constant C_1 which depends only on F . On the other hand, by Lemma 15 and since $\|\gamma\|_1 \leq 1$ for all density profile γ , $I_T(\pi) \leq C_2$, which completes the proof of the lemma in view of the definition of the functional W . □

We now turn to the proof that the functionals V_i are lower semicontinuous for the weak topology. The idea of the proof is very simple. Let ϱ^n be a sequence converging to ϱ . Since, by Lemma 18, V_i is finite, there exists a trajectory π_t^n , $0 \leq t \leq T_n$, such that $\pi_0^n \in \mathcal{M}_i$, $\pi_{T_n}^n = \varrho^n$, $V_i(\varrho^n) \leq I_{T_n}(\pi^n) + 1/n \leq C$. We will now use the lower semicontinuity of I_T and the fact that the level sets are compact to conclude. If the sequence T_n is uniformly bounded, say by T , we may add a piece of length $T - T_n$ to

the trajectory π^n letting it to stay at $\pi_0^n \in \mathcal{M}_i$ in the time interval $[0, T - T_n]$. In this way, we obtain a new sequence, denoted by $\bar{\pi}_i^n$, of trajectories such that $\bar{\pi}_0^n \in \mathcal{M}_i$, $\bar{\pi}_T^n = \varrho^n$, $V_i(\varrho^n) \leq I_{T_n}(\pi^n) + 1/n = I_T(\bar{\pi}_i^n) + 1/n \leq C$. Since the level sets are compact, we may extract a converging subsequence. Denote by π the limit and observe that $\pi_0 \in \mathcal{M}_i$, $\pi_T = \varrho$. By the lower semicontinuity and by definition of V_i , $V_i(\varrho) \leq I_T(\pi) \leq \liminf_n V_i(\varrho^n)$, and we are done.

Of course, it might happen that the sequence T_n is not bounded, and this is the main difficulty. In this case, we will use Lemma 14 to claim that the trajectory π^n may not spend too much time outside an L^2 -neighborhood of a stationary solution. Hence, all the proof consists in replacing the long intervals of time at which the trajectory stays close to a stationary profile $\bar{\rho}_i$ by one which remains only a time interval of length 2. This is done by showing in Lemma 19 below that it is possible to go from a neighborhood of $\bar{\rho}_i$ to $\bar{\rho}_i$ in time 1 by paying a small cost and by using Theorem 6 to obtain a trajectory which goes from $\bar{\rho}_i$ to a neighborhood of $\bar{\rho}_i$ in time 1 by paying a small cost.

Lemma 19 Fix $1 \leq i \leq l$ and $\bar{\varrho}_i(d\theta) = \bar{\rho}_i(\theta)d\theta \in \mathcal{M}_i$. For any $\varepsilon > 0$ there exists $\delta_{13} = \delta_{13}(\varepsilon) > 0$ such that for any $\varrho(d\theta) = \gamma(\theta)d\theta$ in $\mathcal{B}_{\delta_{13}}(\bar{\varrho}_i)$ there exists a path $\pi(t, d\theta) = \rho(t, \theta)d\theta$ in $D([0, 1], \mathcal{M}_+)$ such that $\pi_0 = \varrho$, $\pi_1 = \bar{\varrho}_i$, $I_1(\pi) \leq \varepsilon$, and $\pi_t \in \mathcal{B}_\varepsilon(\bar{\varrho}_i)$ for all $0 \leq t \leq 1$.

Proof Fix $1 \leq i \leq l$, $\varepsilon > 0$, and a density profile $\gamma : \mathbb{T} \rightarrow \mathbb{R}$. Let $\rho(t, \theta)$ be the unique weak solution of (2.4) with initial condition γ , and let $\pi_t(d\theta) = \rho(t, \theta)d\theta$.

Fix $0 < \zeta_1 < \varepsilon$, to be chosen later, and set $T = 1/4$, $T' = 1/2$. Let $\zeta_2 > 0$ be the constant $\delta_{11}(\zeta_1, 1/4, 1/2)$ given by Lemma 8 for $\bar{\rho} = \bar{\rho}_i$. Assume that $\gamma \in \mathcal{B}_{\zeta_2}(\bar{\varrho}_i)$. According to Lemma 8, $\rho_s(\theta) d\theta$ belongs to $\mathcal{B}_{\zeta_1}(\bar{\varrho}_i)$ for all $0 \leq s \leq 1/2$ and $\|\rho_s - \bar{\rho}_i\|_2 \leq \zeta_1$ for all $1/4 \leq s \leq 1/2$.

By Proposition 2, ρ_t belongs to $C^\infty(\mathbb{T})$ for all $t > 0$. By Lemma 4, there exists $a > 0$, depending only on F , such that $a \leq \rho(s, \theta) \leq 1 - a$ for all $\theta \in \mathbb{T}$ and $1/4 \leq s \leq 1/2$. On the other hand, by Lemma 5, there exists a finite constant C_0 , depending only on F , such that

$$\int_0^{1/2} dt \int_{\mathbb{T}} |(\nabla \rho)(t, \theta)|^2 d\theta \leq C_0.$$

In particular, there exists $T_1 \in [1/4, 1/2]$ such that

$$\int_{\mathbb{T}} |(\nabla \rho)(T_1, \theta)|^2 d\theta \leq 4C_0, \tag{5.4}$$

so that ρ_{T_1} belongs to $\mathbb{D}_a^{4C_0}$

Recall that $T_1 \leq 1/2$. Let $\alpha : [0, 1/2] \rightarrow [0, 1]$ be an increasing C^1 -diffeomorphism, and define the trajectory ρ_t^α , $T_1 \leq t \leq T_1 + 1/2$, by $\rho^\alpha(T_1 + s, \theta) = \alpha(s)\bar{\rho}_i + [1 - \alpha(s)]\rho_{T_1}$, $0 \leq s \leq 1/2$. By a similar computation to the one presented in the proof of Lemma 17, and since $\|\rho_{T_1} - \bar{\rho}_i\|_1 \leq \|\rho_{T_1} - \bar{\rho}_i\|_2 \leq \zeta_1$

$$I_{[T_1, T_1+1/2]}(\rho^\alpha) \leq C_1 \int_0^{1/2} [1 - \alpha(t)]^2 dt \|\nabla \rho_{T_1} - \nabla \bar{\rho}_i\|_2^2 + C_2(\alpha) \zeta_1,$$

where C_1 is a finite constant depending only on F , and $C_2(\alpha)$ one which also depends on α . In view of (5.4),

$$I_{[T_1, T_1+1/2]}(\rho^\alpha) \leq C_3 \int_0^{1/2} [1 - \alpha(t)]^2 dt + C_2(\alpha) \zeta_1,$$

for some finite constant C_3 independent of γ .

Choose an increasing C^1 -diffeomorphism $\alpha : [0, 1/2] \rightarrow [0, 1]$ which turns the first term on the right hand side bounded by $\varepsilon/2$. Note that this diffeomorphism does not depend on γ . For this fixed α , choose ζ_1 small enough for the second term to be less than or equal to $\varepsilon/2$. To complete the proof of the first assertion of the lemma, juxtapose the trajectories $\rho_t, 0 \leq t \leq T_1, \rho_t^\alpha, T_1 \leq t \leq T_1 + 1/2$, and the constant one $\bar{\rho}_i, T_1 + 1/2 \leq t \leq 1$.

We turn to the assertion that $\pi_t \in \mathcal{B}_\varepsilon(\bar{q}_i)$ for all $0 \leq t \leq 1$. By the second paragraph of the proof, and by definition of $T_1, \pi_t \in \mathcal{B}_{\zeta_1}(\bar{q}_i) \subset \mathcal{B}_\varepsilon(\bar{q}_i)$ for all $0 \leq t \leq T_1$. By definition of the trajectory $\rho^\alpha, d(\pi_t, \bar{q}_i) \leq d(\pi_{T_1}, \bar{q}_i) < \varepsilon$ for $T_1 \leq t \leq T_1 + 1/2$. This completes the proof of the lemma since $\pi_t = \bar{q}_i$ for $T_1 + 1/2 \leq t \leq 1$. \square

We have now all the elements to prove the main result of the section.

Theorem 7 *For each $1 \leq i \leq l, V_i$ is lower semicontinuous. In particular, the rate function W is also lower semicontinuous.*

Proof To keep notation simple, we prove the theorem in the case where all solutions of (2.5) are constant in space, or equivalently, assume that

$$\mathcal{M}_{\text{sol}} = \{\bar{q}_i(d\theta) = \bar{\rho}_i d\theta : i = 1, \dots, l\}.$$

It is not difficult to extend the argument to the general case by invoking Lemma 16.

Fix $1 \leq i \leq l, q \in \mathbb{R}_+$, and let

$$\mathcal{V}_i^{(q)} = \{q \in \mathcal{M}_+ : V_i(q) \leq q\}.$$

By the proof of Lemma 18, $\mathcal{V}_i^{(q)} \subset \mathcal{M}_{+,1}$. We claim that $\mathcal{V}_i^{(q)}$ is a closed subset of \mathcal{M}_+ . To see this, let $\{q^n(d\theta) = \rho^n(\theta)d\theta : n \geq 1\}$ be a sequence in $\mathcal{V}_i^{(q)}$ converging to some $q(d\theta) = \rho(\theta)d\theta$ in \mathcal{M}_+ .

From (2.11), for each $n \geq 1$, there exist $T_n > 0$ and a path π^n in $C([0, T_n], \mathcal{M}_{+,1})$ such that $\pi_0^n = \bar{q}_i, \pi_{T_n}^n = q^n$ and

$$I_{T_n}(\pi^n | \bar{q}_i) \leq V_i(q^n) + 1/n \leq q + 1. \tag{5.5}$$

Assume first that the sequence $\{T_n : n \geq 1\}$ is bounded above by some $T < \infty$. In this case, let $\hat{\pi}^n$ be the trajectory which remains at \bar{q}_i in the time interval $[0, T - T_n]$ and then follows the trajectory π_n :

$$\widehat{\pi}_t^n = \begin{cases} \bar{q}_i & 0 \leq t \leq T - T_n, \\ \pi_n(t - T + T_n) & T - T_n \leq t \leq T. \end{cases}$$

Note that $\widehat{\pi}_T^n = \varrho^n$ for all $n \geq 1$, and that $I_T(\widehat{\pi}^n | \bar{q}_i) = I_{T_n}(\pi^n | \bar{q}_i) \leq q + 1/n$ for all $n \geq 1$.

By Theorem 3, the functional $I_T(\cdot | \bar{q}_i)$ has compact level sets and is lower semi-continuous. There exists, in particular, a subsequence n_j and a trajectory $\pi \in D([0, T], \mathcal{M}_+)$ such that $\widehat{\pi}^{n_j} \rightarrow \pi$, $I_T(\pi | \bar{q}_i) \leq q$. Since $\pi_T = \lim_j \widehat{\pi}_T^{n_j} = \lim_j \pi_{T_n_j}^{n_j} = \lim_j \varrho^{n_j} = \varrho$, by definition of V_i , $V_i(\varrho) \leq q$, which completes the proof of the theorem in the case where the sequence T_n is bounded.

Suppose now that the sequence $\{T_n : n \geq 1\}$ is unbounded. In view of (5.5), by Lemma 14, the path π^n may not remain too long outside an L^2 -neighborhood of one of the stationary profiles. We use this observation, Lemma 19 and Theorem 6, to construct from π^n a new path on a bounded time interval by replacing the long intervals of time in which π^n remained close to a stationary profile by a path defined in a time interval of length 2 which connects the entrance time in a neighborhood of a stationary profile $\bar{\rho}_i$ to $\bar{\rho}_i$ and from this profile to the exit time of the neighborhood. The details are given below.

Fix $\varepsilon > 0$. By Theorem 6, there exists ζ_1 , such that $V_j(\pi) \leq \varepsilon$ if $\pi(d\theta) = \rho(\theta)d\theta$ and $\|\rho - \bar{\rho}_j\|_2 \leq \zeta_1$ for any $1 \leq j \leq l$. Let ζ_2 be the constant $\delta_{13}(\varepsilon)$ given by Lemma 19 and set $\zeta = \min\{\zeta_1, \zeta_2\}$.

Let \mathcal{L}_j , $1 \leq j \leq l$, be the closed L^2 -neighborhood of $\bar{\rho}_j$: $\mathcal{L}_j = \{\varrho(d\theta) = \rho(\theta)d\theta : \rho \in \mathbb{B}_\zeta[\bar{\rho}_j]\}$, where $\mathbb{B}_\zeta[\bar{\rho}_j]$ represents the closure of $\mathbb{B}_\zeta(\bar{\rho}_j)$, and let $\mathcal{L} = \cup_{1 \leq j \leq l} \mathcal{L}_j$. Assume that ζ is sufficiently small so that $\{\mathcal{L}_j\}_{j=1}^l$ are mutually disjoint. Note that \mathcal{L}_j is a closed subset of $\mathcal{M}_{+,1}$. We define a sequence of entrances and exit times associated to the sets \mathcal{L}_j . Recall that $\pi_0^n = \bar{q}_i$, and set $\tau_1^n = 0$. Let σ_1^n be the last exit time from \mathcal{L}_i :

$$\sigma_1^n = \sup \{t \leq T_n : \pi_t^n \in \mathcal{L}_i\}.$$

Note that the path π_t^n may visit several neighborhoods \mathcal{L}_j , $j \neq i$, in the time interval $[0, \sigma_1^n]$, and that it does not return to \mathcal{L}_i after σ_1^n . Suppose that τ_k^n, σ_k^n , $1 \leq k < p$, have already been introduced. Define

$$\tau_p^n := \inf \{\sigma_{p-1}^n \leq t \leq T_n : \pi_t^n \in \mathcal{L}\}, \quad \sigma_p^n := \sup \{t \leq T_n : \pi_t^n \in \mathcal{L}_{j(p)}\},$$

where $j(p)$ is the index of the neighborhood visited at time τ_p^n : $j(p) = a$ if $\pi_{\tau_p^n}^n \in \mathcal{L}_a$. By convention, if $\pi_t^n \notin \mathcal{L}$ for all $\sigma_{p-1}^n \leq t \leq T_n$, we set τ_p^n and σ_p^n to be ∞ for all $p' \geq p$ and we do not define $j(p)$. Note that $j(1) = i$ and that $j(p) \neq j(q)$ if $q \neq p$.

Denote by S_n the set of neighborhoods visited by π^n , $S_n = \{j(p) : \tau_p(\pi^n) < \infty\} = \{j(1), \dots, j(b)\}$. By the choice of ζ and by Lemma 19, there exist paths $\pi^{1,+}$, $\pi^{m,-}$, $\pi^{m,+}$, $2 \leq m < b$, $\pi^{b,-}$ such that

$$\pi_0^{k,-} = \pi^n(\tau_{j(k)}), \quad \pi_1^{k,-} = \bar{q}_{j(k)}, \quad \pi_0^{k,+} = \bar{q}_{j(k)}, \quad \pi_1^{k,+} = \pi^n(\sigma_{j(k)}),$$

and such that $I_1(\pi_1^{1,+}) \leq 2\varepsilon$, $I_1(\pi_1^{m,-}) + I_1(\pi_1^{m,+}) \leq 3\varepsilon$, $2 \leq m < b$, $I_1(\pi_1^{b,-}) \leq \varepsilon$. Here and below, to keep notation simple, we denote sometimes π_t^n by $\pi^n(t)$. If $\sigma_b^n \leq T_n$ there exists also a path $\pi^{b,+}$, such that

$$\pi_0^{b,+} = \bar{q}_{j(b)}, \quad \pi_1^{b,+} = \pi^n(\sigma_{j(b)}),$$

and such that $I_1(\pi_1^{b,+}) \leq 2\varepsilon$.

We construct below a path $\tilde{\pi}^n$ from the previous paths and from π^n under the assumption that $\sigma_b^n \leq T_n$. The construction can be easily adapted to the cases $\sigma_b^n = +\infty$. For $1 \leq c < b$, let

$$R_c = 2c + \sum_{a=1}^c (\tau_{j(a+1)} - \sigma_{j(a)}).$$

This sequence represents the times at which the path $\tilde{\pi}^n$ visits the measures $\bar{q}_{j(c+1)}$. Set

$$\tilde{T}_n = R_{b-1} + 1 + T_n - \sigma_{j(b)},$$

and define the path $\tilde{\pi}^n$ in $C([0, \tilde{T}_n], \mathcal{M}_{+,1})$ as follows:

$$\tilde{\pi}^n(t) = \begin{cases} \pi^{1,+}(t) & 0 \leq t \leq 1, \\ \pi^n(\sigma_{j(1)} + t - 1) & 1 \leq t \leq 1 + \tau_{j(2)} - \sigma_{j(1)}, \\ \pi^{2,-}(t - 1 - \tau_{j(2)} + \sigma_{j(1)}) & 1 + \tau_{j(2)} - \sigma_{j(1)} \leq t \leq R_1, \\ \pi^{2,+}(t - R_1) & R_1 \leq t \leq R_1 + 1, \\ \dots & \dots \\ \pi^{b,+}(t - R_{b-1}) & R_{b-1} \leq t \leq R_{b-1} + 1, \\ \pi^n(\sigma_{j(b)} + t - R_{b-1} - 1) & R_{b-1} + 1 \leq t \leq R_{b-1} + 1 + T_n - \sigma_{j(b)}. \end{cases}$$

From the definition of $\tilde{\pi}^n$ it is clear that $\tilde{\pi}_0^n = \bar{q}_i$, $\tilde{\pi}^n(\tilde{T}_n) = \pi^n(T_n) = \varrho$ and that

$$I_{\tilde{T}_n}(\tilde{\pi}^n) \leq I_{T_n}(\pi^n) + 3b\varepsilon \leq I_{T_n}(\pi^n) + 3l\varepsilon. \tag{5.6}$$

In particular, by (5.5) and (5.6), $I_{\tilde{T}_n}(\tilde{\pi}^n)$ is uniformly bounded. The time spent by $\tilde{\pi}^n$ in \mathcal{L}^c is at least $\tilde{T}_n - 2l$. Therefore, by Lemma 14 and by the previous uniform bound on $I_{\tilde{T}_n}(\tilde{\pi}^n)$, the sequence \tilde{T}_n is uniformly bounded. At this point, we may repeat the arguments presented in the first part of the proof, which are solely based on a uniform bound for the sequence $I_{\tilde{T}_n}(\tilde{\pi}^n)$, provided by (5.5) and (5.6), and on a uniform bound of the sequence \tilde{T}_n , to conclude. \square

We conclude this section with a result needed in the next one. In Lemma 19 we constructed paths from $\mathcal{B}_\delta(\mathcal{M}_i)$ to \mathcal{M}_i whose costs are small. In the next result, we prove a partial converse statement by showing that there are measures ϱ at distance δ from \mathcal{M}_i for which there exist paths from \mathcal{M}_i to ϱ whose cost is small.

Lemma 20 *For every $1 \leq i \leq l$ and $\varepsilon > 0$, there exists $\delta_{14} = \delta_{14}(\varepsilon) > 0$ such that for all $0 \leq \delta < \delta_{14}$, there exists a measure $\varrho \in \mathcal{M}_{+,1}$ such that $d(\varrho, \mathcal{M}_i) = \delta$ and $V_i(\varrho) \leq \varepsilon$.*

Proof Fix $1 \leq i \leq l$, $\bar{\varrho}_i \in \mathcal{M}_i$, and $\varepsilon > 0$. Recall the definition of the constant c_1 introduced in (3.5), and let $0 < a \leq c_1/2$. Let $\pi(t, d\theta)$, $0 \leq t \leq 1$, be the trajectory $\pi(t, d\theta) = \rho(t, \theta)d\theta$, $\rho(t, \theta) = \bar{\rho}_i(\theta) + at$. On the one hand, by definition (2.1) of the distance, $d(\mathcal{M}_i, \pi_1) = d(\bar{\varrho}_i, \pi_1) = a$. On the other hand, for every smooth function $G : [0, 1] \times \mathbb{T} \rightarrow \mathbb{R}$, since $(1/2)\Delta\rho_t = (1/2)\Delta\bar{\rho}_i = -F(\bar{\rho}_i)$,

$$J_{1,G}(\pi) \leq \int_0^1 dt \left\{ \left\{ a + [F(\bar{\rho}_i) - F(\rho_t)] \right\} G_t \right\} - \langle B(\rho_t), e^{G_t} - 1 - G_t \rangle.$$

Note that we omitted in the previous expression the terms $\chi(\rho_t)[\nabla G_t]^2$ and $D(\rho_t)[e^{-G_t} - 1 + G_t]$ which are positive. By definition of ρ_t , $F(\bar{\rho}_i) - F(\rho_t)$ is absolutely bounded by $C_F a$, where C_F stands for the Lipschitz constant of the function F . By definition, $a \leq c_1/2$ and by (3.5), $c_1 \leq \bar{\rho}_i \leq 1 - c_1$. Thus, $c_1/2 \leq \rho_t \leq 1 - c_1/2$. Let $b = \inf\{B(x) : c_1/2 \leq x \leq 1 - c_1/2\} > 0$, so that

$$J_{1,G}(\pi) \leq \int_0^1 dt \left\{ (1 + h_t) a G_t - b [e^{G_t} - 1 - G_t] \right\},$$

where h_t is absolutely bounded by C_F . Assume that $0 \leq a < a_1$ where a_1 is chosen so that $(C_F - 1)a_1 < b$. The right hand side of the previous expression is bounded by $\psi_b((1 \pm C_F)a)$, uniformly in G , where $\psi_b(x) := (x + b) \log[1 + (x/b)] - x$. Therefore, $V_i(\varrho) \leq I_1(\pi) \leq \psi_b((1 \pm C_F)a)$. Since $\psi_b(0) = 0$, there exists a_0 such that $\psi_b((1 \pm C_F)a) \leq \varepsilon$, for any $0 \leq a < a_0$. The assertion of the lemma holds provided we choose $\delta_{14} = a_0 \wedge a_1$ and $\varrho = \bar{\varrho}_i + ad\theta$ for $0 \leq a < \delta_{14}$. \square

Remark 1 Actually, we proved the existence of trajectory π_t , $0 \leq t \leq 1$, such that $I_1(\pi) \leq \varepsilon$, $\pi_0 = \bar{\varrho}_i \in \mathcal{M}_i$, $d(\pi_1, \mathcal{M}_i) = \delta$.

6 The static large deviations principle

We prove in this section Theorem 4. As we said before, the proof is based on a representation of the stationary state of the reaction–diffusion model in terms of the invariant probability measure of a discrete-time Markov chain. In the first part of this section, we introduce the discrete-time Markov chain and we prove in Proposition 3 sharp upper and lower bounds for its invariant probability measure.

For any $0 < \beta_0 < \beta_1$, let B_i be the open neighborhoods, and Γ_i be the closed neighborhoods given by

$$B = \bigcup_{i=1}^l B_i, \quad \text{where } B_i := \mathcal{B}_{\beta_0}(\mathcal{M}_i) := \{\varrho \in \mathcal{M}_+ : \inf_{\bar{\varrho} \in \mathcal{M}_i} d(\varrho, \bar{\varrho}) < \beta_0\}.$$

$$\Gamma = \bigcup_{i=1}^l \Gamma_i, \quad \text{where } \Gamma_i = \{\varrho \in \mathcal{M}_+ : \beta_1 \leq \inf_{\bar{\varrho} \in \mathcal{M}_i} d(\varrho, \bar{\varrho}) \leq 2\beta_1\}.$$

To stress the dependence of B and Γ on β_0, β_1 we sometimes denote B, Γ by $B(\beta_0), \Gamma(\beta_1)$, respectively.

For $N \geq 1$ and a subset A of \mathcal{M}_+ , let $A^N = (\pi^N)^{-1}(A)$ and let $H_A^N : D(\mathbb{R}_+, X_N) \rightarrow [0, +\infty]$ be the hitting time of A^N :

$$H_A^N = \inf \{t \geq 0 : \eta_t \in A^N\}.$$

The first result of the section states that the reaction–diffusion model reaches the set B in finite time with high probability.

Lemma 21 *For every $\delta > 0$, there exist $T_0, C_0, N_0 > 0$, depending on δ , such that*

$$\sup_{\eta \in X_N} \mathbb{P}_\eta \left[H_{B(\delta)}^N \geq kT_0 \right] \leq \exp \{-kC_0N\}$$

for all $N \geq N_0$ and all $k \geq 1$.

Proof Fix $\delta > 0$. By Corollary 1, there exist $T_0 > 0$ and $C_0 > 0$, which depend on δ , such that

$$\inf_{\pi \in \mathcal{D}_{T_0, \delta}} I_{T_0}(\pi) > C_0,$$

where $\mathcal{D}_{T_0, \delta} = D([0, T_0], \mathcal{M}_+ \setminus B(\delta))$. For each integer $N \geq 1$, denote by η^N a configuration in X_N such that

$$\mathbb{P}_{\eta^N} \left[H_B^N \geq T_0 \right] = \sup_{\eta \in X_N} \mathbb{P}_\eta \left[H_B^N \geq T_0 \right].$$

By the compactness of \mathcal{M}_+ , every subsequence of $\pi^N(\eta^N)$ contains a subsequence converging to some ϱ in \mathcal{M}_+ . Moreover, since each configuration in X_N has at most one particle per site, any limit point ϱ belongs to $\mathcal{M}_{+,1}$. From this observation and since $\mathcal{D}_{T_0, \delta}$ is a closed subset of $D([0, T_0], \mathcal{M}_+)$, by the dynamical large deviations upper bound, there exists a measure $\varrho(d\theta) = \gamma(\theta)d\theta$ in $\mathcal{M}_{+,1}$ such that

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N} \left[H_B^N \geq T_0 \right] &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \log Q_{T_0, \eta^N}(\mathcal{D}_{T_0, \delta}) \\ &\leq - \inf_{\pi \in \mathcal{D}_{T_0, \delta}} I_{T_0}(\pi | \gamma) < -C_0. \end{aligned}$$

In particular, there exists $N_0 \geq 1$ such that for every integer $N \geq N_0$,

$$\sup_{\eta \in X_N} \mathbb{P}_\eta \left[H_B^N \geq T_0 \right] \leq \exp\{-C_0N\}.$$

To complete the proof, we proceed by induction, applying the strong Markov property. Suppose that the statement of the lemma is true for all integers $j < k$. Let $N \geq N_0$

and let $\hat{\eta}$ be a configuration in X_N . By the strong Markov property,

$$\begin{aligned} \mathbb{P}_{\hat{\eta}} \left[H_B^N \geq kT_0 \right] &= \mathbb{E}_{\hat{\eta}} \left[\mathbf{1} \left\{ H_B^N \geq T_0 \right\} \mathbb{P}_{\eta_{T_0}} \left[H_B^N \geq (k-1)T_0 \right] \right] \\ &\leq \mathbb{P}_{\hat{\eta}} \left[H_B^N \geq T_0 \right] \sup_{\eta \in X_N} \mathbb{P}_{\eta} \left[H_B^N \geq (k-1)T_0 \right] \\ &\leq \exp \{-k C_0 N\}, \end{aligned}$$

which completes the proof. □

Corollary 2 For every $\delta > 0$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in X_N} \mathbb{E}_{\eta} \left(H_{B(\delta)}^N \right) \leq 0.$$

Proof Fix $\delta > 0$ and denote by $T_0, C_0, N_0 \geq 1$ the constants provided by Lemma 21. For every integer $N \geq N_0$ and for every configuration η in X_N ,

$$\mathbb{E}_{\eta} \left(H_B^N \right) \leq T_0 \sum_{k=0}^{\infty} \mathbb{P}_{\eta} \left(H_B^N \geq kT_0 \right) \leq T_0 \sum_{k=0}^{\infty} \exp \{-kC_0N\} \leq \frac{T_0}{1 - e^{-C_0N}},$$

which proves the corollary. □

We have now all elements to introduce the discrete-time Markov chain. Let ∂B^N (which depends on β_0)¹ be the set of configurations η in X_N for which there exists a finite sequence of configurations $\{\eta^i : 0 \leq i \leq k\}$ in X_N with η^0 in Γ^N , $\eta^k = \eta$, and such that

- (a) For every $1 \leq i \leq k$, the configuration η^i can be obtained from η^{i-1} by a jump of the dynamics (either from the stirring mechanism or from the non-conservative spin flip dynamics).
- (b) The unique configuration of the sequence which can enter into B^N after a jump is η^k .

We similarly define the set $\partial B_i^N, 1 \leq i \leq l$. It is clear that for N large enough and β_1 small enough,

$$\partial B^N = \bigcup_{i=1}^l \partial B_i^N.$$

Let $\tau = \tau^N : D(\mathbb{R}_+, X_N) \rightarrow [0, \infty]$ be the stopping time given by

$$\tau = \inf \left\{ t > 0 : \text{there exist } s < t \text{ such that } \eta_s \in \Gamma^N \text{ and } \eta_t \in \partial B^N \right\}. \tag{6.1}$$

Set $\tau_1 := \tau$. We recursively define the sequence of stopping times $\{\tau_k : k \geq 1\}$ by

$$\tau_k = \inf \left\{ \tau_k > 0 : \text{there exist } s < t \text{ such that } \tau_{k-1} < s, \eta_s \in \Gamma^N \text{ and } \eta_t \in \partial B^N \right\}.$$

¹ Note that for N large the set ∂B^N does not depend on β_1 .

This sequence generates a discrete-time Markov chain ξ_k on ∂B^N by setting $\xi_k = \eta_{\tau_k}$. The arguments presented before Lemma 5.1 in [21] show that this chain is irreducible. Denote by ν^N its unique invariant probability measure.

Define \tilde{v}_{ij} by

$$\tilde{v}_{ij} = \inf\{I_T(\pi|\bar{\rho}) : T > 0, \bar{\rho}(\theta)d\theta \in \mathcal{M}_i, \pi \in \mathcal{A}_T, \pi_T \in \mathcal{M}_j\},$$

where $\mathcal{A}_T = \{\pi \in C([0, T], \mathcal{M}_+) : \pi_t \notin \mathcal{M}_{\text{sol}} \text{ for all } 0 < t < T\}$.

The jumps of the Markov chain $\{\xi_k : k \geq 1\}$ correspond to paths from ∂B_i^N to ∂B_j^N which do not visit other boundaries. They are thus related to the dynamical large deviations principle. In Lemmata 22 and 24 we estimate the probability of a jump from ∂B_i^N to ∂B_j^N for $j \neq i$. These estimates provide sharp bounds for the invariant probability measure of the chain ξ_k , alluded to at the beginning of this section.

Lemma 22 *For every $1 \leq i \neq j \leq l, \varepsilon > 0$ and $0 < \beta_1 < (1/4) \min_{a \neq b} d(\mathcal{M}_a, \mathcal{M}_b)$, there exists $0 < \delta_{15} < \beta_1$ such that for all $\beta_0 < \delta_{15}$*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \inf_{\eta \in \partial B_i^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) \geq -\tilde{v}_{ij} - \varepsilon.$$

Proof Fix $\varepsilon > 0, 1 \leq i \neq j \leq l$, and $0 < \beta_1 < (1/4) \min_{a \neq b} d(\mathcal{M}_a, \mathcal{M}_b)$. By definition of \tilde{v}_{ij} , there exist $T > 0, \bar{\rho}_i(d\theta) = \bar{\rho}_i(\theta)d\theta \in \mathcal{M}_i, \bar{\rho}_j \in \mathcal{M}_j$, and $\pi \in \mathcal{A}_T$ such that $\pi_0 = \bar{\rho}_i, \pi_T = \bar{\rho}_j, I_T(\pi|\bar{\rho}_i) \leq \tilde{v}_{ij} + \varepsilon$. Note that $\pi_t \notin \mathcal{M}_i$ for all $0 < t \leq T$ because $\pi \in \mathcal{A}_T$.

Let t_0 be the first time the path π_t is at distance β_1 from \mathcal{M}_i : $t_0 = \min\{t \geq 0 : d(\pi_t, \mathcal{M}_i) \geq \beta_1\}$. Since π belongs to \mathcal{A}_T and $\pi_T \in \mathcal{M}_j, \zeta_2 = \inf_{t_0 \leq t \leq T} d(\pi_t, \cup_{k \neq j} \mathcal{M}_k) > 0$. Let ζ_3 be the constant $\delta_{13}(\min\{\beta_1, \varepsilon\})$ given by Lemma 19. Set $\delta_{15} = (1/2) \min\{\beta_1, \zeta_2, \zeta_3\}$ and fix $\beta_0 < \delta_{15}$.

For each integer $N > 0$, let η^N be a configuration in ∂B_i^N such that

$$\mathbb{P}_{\eta^N}(\eta_\tau \in \partial B_j^N) = \inf_{\eta \in \partial B_i^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N). \tag{6.2}$$

Recall that every subsequence of $\pi^N(\eta^N)$ contains a sub-subsequence converging in \mathcal{M}_+ to some measure ϱ which belongs to $\mathcal{M}_{+,1}$. We may therefore assume that $\pi^N(\eta^N)$ converges to $\varrho(d\theta) = \gamma(\theta)d\theta$ which belongs to the closure of B_i : $\varrho \in \mathcal{B}_{\beta_0}(\bar{\rho}'_i)$, for some $\bar{\rho}'_i \in \mathcal{M}_i$.

Since $\beta_0 < \zeta_3$, we may apply Lemma 19 to connect ϱ to $\bar{\rho}'_i \in \mathcal{M}_i$. Denote by $\pi'_t, 0 \leq t \leq 1$, the path given by Lemma 19 and such that $\pi'_0 = \varrho, \pi'_1 = \bar{\rho}'_i \in \mathcal{M}_i, I_1(\pi'|\gamma) \leq \varepsilon$ and $\pi'_t \in \mathcal{B}_{\beta_1}(\bar{\rho}'_i), 0 \leq t \leq 1$. We may apply Lemma 16 to connect $\bar{\rho}'_i$ to $\bar{\rho}_i$, the initial point of the path introduced in the first paragraph of the proof. By Lemma 16, there exists T' and a path $\pi'' \in C([0, T'], \mathcal{M}_+)$ such that $\pi''_0 = \bar{\rho}'_i, \pi''_{T'} = \bar{\rho}_i$ and $I_{T'}(\pi''|\bar{\rho}'_i) \leq \varepsilon$.

Concatenate the paths π', π'' and π to obtain a path $\tilde{\pi}$ in $C([0, T + T' + 1], \mathcal{M}_+)$ such that $\tilde{\pi}_0 = \varrho, \tilde{\pi}_{T+T'+1} \in \mathcal{M}_j, I_{T+T'+1}(\tilde{\pi}''|\varrho) \leq \tilde{v}_{ij} + 3\varepsilon$. Moreover,

$d(\pi_t, \mathcal{M}_i) < \beta_1$ for $0 \leq t \leq 1 + T'$. In particular, $\tilde{\pi}_t$ reaches a distance β_1 from \mathcal{M}_i for the first time at $t = 1 + T' + t_0$, and $\inf_{t_0 \leq t \leq T} d(\tilde{\pi}_{1+T'+t}, \cup_{k \neq j} \mathcal{M}_k) = \zeta_2 > \beta_0$.

Denote by \mathcal{N} a $\beta_0/2$ -neighborhood in $D([0, T + T' + 1], \mathcal{M}_+)$ of the path $\tilde{\pi}$. It follows from the last observation of the previous paragraph and from the fact that $\tilde{\pi}_{T+T'+1} \in \mathcal{M}_j$ that $\mathcal{N} \subset \{\eta_\tau \in \partial B_j^N\}$. Since \mathcal{N} is an open set, by the lower bound of the dynamical large deviations principle, and by (6.2)

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \inf_{\eta \in \partial B_i^N} \log \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) &= \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(\eta_\tau \in \partial B_j^N) \\ &\geq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(\mathcal{N}) \geq - \inf_{\tilde{\pi} \in \mathcal{N}} I_{T+T'+1}(\tilde{\pi}|\gamma) \geq - I_{T+T'+1}(\tilde{\pi}|\gamma). \end{aligned}$$

This completes the proof of the lemma because $I_{T+T'+1}(\tilde{\pi}|\varrho) \leq \tilde{v}_{ij} + 3\varepsilon$. □

The proof of the upper bound for $\mathbb{P}_\eta(\eta_\tau \in \partial B_j^N)$, $\eta \in \partial B_i^N$, requires a lower bound for the dynamical large deviations rate functional. For $T > 0$ and $\zeta > 0$, let $\mathcal{C}_j = \mathcal{C}_j(T, \zeta)$ be the closed subset of $D([0, T], \mathcal{M}_+)$ consisting of all paths π for which there exists some time $t \in [0, T]$ such that $\pi(t)$ belongs to $\Gamma_j(\zeta)$ or $\pi(t-)$ belongs to $\Gamma_j(\zeta)$.

Lemma 23 *For every $1 \leq i \neq j \leq l$, $\varepsilon > 0$, there exist $\delta_{16} = \delta_{16}(\varepsilon) > 0$ and $T = T(\varepsilon) > 0$ such that for all $\delta' < \delta_{16}$, $T' \geq T$, $\gamma(\theta)d\theta \in \Gamma_i(\delta')$,*

$$\inf_{\pi \in \mathcal{C}_j(T', \delta')} I_{T'}(\pi|\gamma) \geq v_{ij} - \varepsilon.$$

Proof Fix $1 \leq i \neq j \leq l$ and assume that the assertion of the lemma is false. In that case, there exists $\varepsilon > 0$ such that for every $\zeta > 0$ and $T > 0$, there exist $\zeta' < \zeta$, $T' \geq T$, $\gamma(\theta)d\theta \in \Gamma_i(\zeta')$ and $\pi \in \mathcal{C}_j(T', \zeta')$ with

$$I_{T'}(\pi|\gamma) < v_{ij} - \varepsilon/2.$$

In particular, taking the sequences $\zeta_n = 1/n$, $T_n = 1$, $n \geq 1$, there exist $\zeta'_n < 1/n$, $T'_n \geq 1$, $\gamma_n \in \Gamma_i(\zeta'_n)$ and $\pi_n \in \mathcal{C}_j(T'_n, \zeta'_n) \cap C([0, T'_n], \mathcal{M}_{+,1})$ with

$$I_{T'_n}(\pi^n|\gamma_n) < v_{ij} - \varepsilon/2. \tag{6.3}$$

Since π^n belongs to $\mathcal{C}_j(T'_n, \zeta'_n) \cap C([0, T'_n], \mathcal{M}_{+,1})$, there exists $0 < \tilde{T}_n \leq T'_n$ such that $\pi^n(\tilde{T}_n) \in \{\varrho \in \mathcal{M}_{+,1} : \zeta'_n \leq \inf_{\bar{\varrho} \in \mathcal{M}_j} d(\varrho, \bar{\varrho}) \leq 2\zeta'_n\}$.

Assume first that the sequence of times $\{\tilde{T}_n : n \geq 1\}$ is bounded above by some $T > 0$. For each integer $n > 0$, let $\hat{\pi}^n$ be the path in $C([0, T - \tilde{T}_n], \mathcal{M}_{+,1})$ given by $\hat{\pi}_t^n(d\theta) = \hat{\rho}^n(t, \theta)d\theta$, where $\hat{\rho}^n$ is the solution of the hydrodynamic equation (2.4) with initial condition $\rho^n(\tilde{T}_n)$, where $\pi^n(\tilde{T}_n, d\theta) = \rho^n(\tilde{T}_n, \theta)d\theta$. Since $d(\pi^n(\tilde{T}_n), \mathcal{M}_j) \leq 2\zeta'_n$, by Lemma 8, $\hat{\pi}^n(T - \tilde{T}_n)$ converges to some element of \mathcal{M}_j .

Let $\tilde{\pi}^n$ be the path in $C([0, T], \mathcal{M}_{+,1})$ given by

$$\tilde{\pi}_t^n = \begin{cases} \pi_t^n & \text{if } 0 \leq t \leq \tilde{T}_n, \\ \hat{\pi}^n(t - \tilde{T}_n) & \text{if } \tilde{T}_n \leq t \leq T. \end{cases}$$

By definition of $\tilde{\pi}^n$, $I_T(\hat{\pi}^n) = I_T(\hat{\pi}^n|\gamma_n) = I_{T_n}(\pi^n|\gamma_n) < v_{ij} - \varepsilon/2$. Since I_T has compact level sets and since $\pi_0^n(d\theta) = \gamma_n(\theta)d\theta$ belongs to $\Gamma_i(\zeta'_n) \cap \mathcal{M}_{+,1}$, there exists a subsequence of $\tilde{\pi}^n$ converging to some π in $C([0, T], \mathcal{M}_{+,1})$ such that $\pi_0 \in \mathcal{M}_i$, $\pi_T \in \mathcal{M}_j$, and $I_T(\pi) \leq v_{ij} - \varepsilon/2$, which contradicts the definition of v_{ij} .

If the sequence $\{\tilde{T}_n : n \geq 1\}$ is not bounded, we may repeat the reasoning presented in the proof of Theorem 7 to replace the path π^n by a path $\bar{\pi}^n$ which satisfies an inequality analogous to (6.3) (with extra factors of ε) and whose entry time to the set $\{\varrho \in \mathcal{M}_+ : \zeta'_n \leq \inf_{\bar{\varrho} \in \mathcal{M}_j} d(\varrho, \bar{\varrho}) \leq 2\zeta'_n\}$ is uniformly bounded in n . This completes the proof of the lemma, since the bounded case has been treated above. \square

We now prove the upper bound.

Lemma 24 *For every $1 \leq i \neq j \leq l$, $\varepsilon > 0$, there exists $\delta_{17} = \delta_{17}(\varepsilon)$ such that for all $0 < \beta_0 < \beta_1 < \delta_{17}$,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \partial B_j^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) \leq -v_{ij} + \varepsilon.$$

Proof Fix $1 \leq i \neq j \leq l$, $\varepsilon > 0$. Let $\zeta_1 > 0$, $T > 0$ be chosen according to Lemma 23, and fix $0 < \beta_0 < \beta_1 < \zeta_1$. By the strong Markov property,

$$\sup_{\eta \in \partial B_i^N} \mathbb{P}_\eta(\eta_\tau \in \partial B_j^N) \leq \sup_{\eta \in \Gamma_i^N} \mathbb{P}_\eta(\eta(H_{\partial B^N}) \in \partial B_j^N),$$

where H_D , $D \subset X_N$, represents the hitting time of the set D and $\eta(t) = \eta_t$. For each integer $N > 0$, fix a configuration η^N in Γ_i^N such that

$$\mathbb{P}_{\eta^N}(\eta(H_{\partial B^N}) \in \partial B_j^N) = \sup_{\eta \in \Gamma_i^N} \mathbb{P}_\eta(\eta(H_{\partial B^N}) \in \partial B_j^N).$$

By Lemma 18, $v_{ij} < \infty$. Thus, by Lemma 21, there exists $T_{\beta_0} > 0$ such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in X_N} \mathbb{P}_\eta \left[H_B^N \geq T_{\beta_0} \right] \leq -v_{ij}.$$

We may assume that $T_{\beta_0} > T$, where T is the time introduced at the beginning of the proof. On the other hand, since $\eta^N \in \Gamma_i^N$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N} \left(\eta(H_{\partial B^N}) \in \partial B_j^N, H_B^N \leq T_{\beta_0} \right) \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N} (H_{\Gamma_j}^N \leq T_{\beta_0}).$$

By intersecting the set $\{\eta(H_{\partial B_j^N}) \in \partial B_j^N\}$ with the set $\{H_B^N \leq T_{\beta_0}\}$ and its complement, since

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \{a_N + b_N\} \leq \max \left\{ \limsup_{N \rightarrow \infty} \frac{1}{N} \log a_N, \limsup_{N \rightarrow \infty} \frac{1}{N} \log b_N \right\}, \tag{6.4}$$

it follows from the two previous estimates that

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \Gamma_i^N} \mathbb{P}_\eta(\eta(H_{\partial B_j^N}) \in \partial B_j^N) \\ & \leq \max \left\{ \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(H_{\Gamma_j^N} \leq T_{\beta_0}), -v_{ij} \right\}. \end{aligned} \tag{6.5}$$

Let \mathcal{C}_j be the set introduced in Lemma 23 associated to the pair (β_1, T_{β_0}) . Since \mathcal{C}_j is a closed set, and since $\{H_{\Gamma_j^N} \leq T_{\beta_0}\} \subset \mathcal{C}_j$, by the dynamical large deviations upper bound and by the compactness of \mathcal{M}_+ , there exists $\gamma(\theta)d\theta \in \Gamma_i$ such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(H_{\Gamma_j^N} \leq T_{\beta_0}) \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \log Q_{T_{\beta_0}, \eta^N}(\mathcal{C}_j) \leq - \inf_{\pi \in \mathcal{C}_j} I_{T_{\beta_0}}(\pi | \gamma).$$

By Lemma 23, the last term is bounded above by $-v_{ij} + \varepsilon$. This completes the proof of the lemma in view of (6.5). □

The proof of the next result is similar to the ones of Lemmata 3.1 and 3.2 of chapter 6 in [24]. Recall the notation introduced above Eq. (2.13). Consider a set Ω , which is not assumed to be countable. Denote by $\Omega_i, 1 \leq i \leq l$, a partition of $\Omega: \Omega = \cup_{1 \leq i \leq l} \Omega_i, \Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Let $(Z_n : n \geq 0)$ be a discrete-time Markov chain on Ω and denote by $p(x, dy), x \in \Omega$, the transition probability of the chain Z_n . Assume that any set Ω_j can be reached from any point $x \in \Omega: \sum_{n \geq 0} P_x\{Z_n \in \Omega_j\} > 0$.

Lemma 25 *Suppose that there exist nonnegative numbers $p_{ij}, \tilde{p}_{ij}, 1 \leq i \neq j \leq l$, and a number $a > 1$ such that*

$$\frac{1}{a} p_{ij} \leq P(x, \Omega_j) \leq a \tilde{p}_{ij} \text{ for all } x \in \Omega_i, i \neq j.$$

Then,

$$\frac{1}{a^{2(l-1)}} \frac{Q_i}{\sum_{1 \leq j \leq l} \tilde{Q}_j} \leq \nu(\Omega_i) \leq a^{2(l-1)} \frac{\tilde{Q}_i}{\sum_{1 \leq j \leq l} Q_j}$$

for any invariant probability measure ν , where Q_i, \tilde{Q}_i are given by

$$Q_i = \sum_{g \in \mathcal{T}(i)} \prod_{(m,n) \in g} p_{mn} \quad \text{and} \quad \tilde{Q}_i = \sum_{g \in \mathcal{T}(i)} \prod_{(m,n) \in g} \tilde{p}_{mn}.$$

Let

$$\tilde{w}_i = \min_{g \in \mathcal{T}(i)} \sum_{(m,n) \in g} \tilde{v}_{mn}.$$

By the argument presented in the proof of [24, Lemma 4.1], we have $w_i = \tilde{w}_i$ for all $1 \leq i \leq l$. We are now in a position to state the main result of this subsection.

Proposition 3 *For every $\varepsilon > 0$, there exists $\delta_{18} = \delta_{18}(\varepsilon)$ such that for all $1 \leq i \leq l$, $0 < \beta_0 < \beta_1 < \delta_{18}$.*

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \log v^N(\partial B_i^N) &\leq -\bar{w}_i + \varepsilon, \\ \liminf_{N \rightarrow \infty} \frac{1}{N} \log v^N(\partial B_i^N) &\geq -\bar{w}_i - \varepsilon. \end{aligned}$$

Proof Since $w_i = \tilde{w}_i$, the assertion of this proposition is a straightforward consequence of Lemmata 22, 24 and 25. □

6.1 Lower bound

We prove in this subsection the large deviations lower bound, that is, for any open subset \mathcal{O} of \mathcal{M}_+ ,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{O}) \geq - \inf_{\varrho \in \mathcal{O}} W(\varrho).$$

Recall the definition of the stopping time τ introduced in (6.1). Following [10,21, 24], we represent the stationary measure μ^N of a subset A of X_N as

$$\mu^N(A) = \frac{1}{C_N} \int_{\partial B^N} \mathbb{E}_\eta \left(\int_0^\tau \mathbf{1}\{\eta_s \in A\} ds \right) d\nu^N(\eta), \tag{6.6}$$

where

$$C_N = \int_{\partial B^N} \mathbb{E}_\eta(\tau) d\nu^N(\eta).$$

The first lemma provides an estimate on the normalizing constant C_N .

Lemma 26 *For any $\varepsilon > 0$, there exists $\delta_{19} = \delta_{19}(\varepsilon)$ such that for all $0 < \beta_0 < \beta_1 < \delta_{19}$,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log C_N \leq \varepsilon.$$

Proof Fix $\varepsilon > 0$ and let ζ_1 be a positive number such that $2\zeta_1$ is smaller than the constants $\delta_{13}(\varepsilon)$ introduced in Lemma 19 and smaller than the constant $\delta_{14}(\varepsilon)$ introduced in Lemma 20. Fix $0 < \beta_0 < \beta_1 < \zeta_1$. Since $H_{\Gamma_i}^N < \tau$ when the process starts from ∂B_i^N , by the Strong Markov property,

$$\begin{aligned}
 C_N &= \sum_{i=1}^l \int_{\partial B_i^N} \mathbb{E}_\eta(\tau) d\nu^N(\eta) = \sum_{i=1}^l \int_{\partial B_i^N} \mathbb{E}_\eta(\tau \mathbf{1}\{H_{\Gamma_i}^N < \tau\}) d\nu^N(\eta) \\
 &\leq \sum_{i=1}^l \int_{\partial B_i^N} \mathbb{E}_\eta(H_{\Gamma_i}^N) d\nu^N(\eta) + \sup_{\eta \in X_N} \mathbb{E}_\eta(H_B^N).
 \end{aligned}$$

By Corollary 2 and by (6.4), it remains to show that for every $1 \leq i \leq l$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \partial B_i^N} \mathbb{E}_\eta(H_{\Gamma_i}^N) \leq 3\varepsilon. \tag{6.7}$$

Fix $1 \leq i \leq l$. We claim that there exists N_0 such that for all $N \geq N_0$,

$$\sup_{\eta \in \Gamma_i^N \cup \partial B_i^N} \mathbb{P}_\eta(H_{\Gamma_i}^N \geq 3) \leq 1 - \exp\{-3N\varepsilon\}. \tag{6.8}$$

To prove this assertion, for each integer $N > 0$, consider a configuration η^N such that $\pi^N(\eta^N)$ belongs to $\mathcal{B}_{2\beta_1}[\mathcal{M}_i] := \overline{\mathcal{B}_{2\beta_1}(\mathcal{M}_i)}$ (which contains the set $\Gamma_i^N \cup \partial B_i^N$) and such that

$$\mathbb{P}_{\eta^N}(H_{\Gamma_i}^N < 3) = \inf_{\eta \in \mathcal{B}_{2\beta_1}^N[\mathcal{M}_i]} \mathbb{P}_\eta(H_{\Gamma_i^N} < 3).$$

Recall that each subsequence of $\pi^N(\eta^N)$ contains a sub-subsequence converging in \mathcal{M}_+ to some ϱ which belongs to $\mathcal{M}_{+,1}$. Therefore, we may assume that $\pi^N(\eta^N)$ converges to $\varrho(d\theta) = \gamma(\theta)d\theta$ and that ϱ belongs to the closure of $\mathcal{B}_{2\beta_1}(\bar{\varrho}_i)$: $\varrho \in \mathcal{B}_{2\beta_1}[\bar{\varrho}_i] := \overline{\mathcal{B}_{2\beta_1}(\bar{\varrho}_i)}$ for some $\bar{\varrho}_i \in \mathcal{M}_i$.

Let $\bar{\pi}_0 = \varrho \in \mathcal{B}_{2\beta_1}[\bar{\varrho}_i] \subset \mathcal{B}_{2\varepsilon_1}(\bar{\varrho}_i)$, and let $\bar{\pi}$ be a path in $D([0, 1], \mathcal{M}_{+,1})$ provided by Lemma 19. Let $\tilde{\pi}$ be a path in $D([0, 1], \mathcal{M}_{+,1})$ provided by Remark 1: $\tilde{\pi}_0 = \bar{\varrho}_i, \tilde{\pi}_1 \in \mathcal{M}_+ \setminus \mathcal{B}_{2\beta_1}[\mathcal{M}_i]$ and $I_1(\tilde{\pi}) \leq \varepsilon$. Define the path π in $D([0, 2], \mathcal{M}_{+,1})$ by concatenating the paths $\bar{\pi}$ and $\tilde{\pi}$:

$$\pi_t = \bar{\pi}_t \text{ for } 0 \leq t \leq 1, \quad \pi_t = \tilde{\pi}_{t-1} \text{ for } 1 \leq t \leq 2.$$

The path $\pi_t, 0 \leq t \leq 2$, starts from ϱ , hits \mathcal{M}_i and then $\mathcal{B}_{2\beta_1}[\mathcal{M}_i]^c$. Its cost $I_2(\pi)$ is bounded by 2ε .

Denote by $\Lambda_{\beta_1/2}(\pi)$ the $\beta_1/2$ -open neighborhood of the trajectory π in $D([0, 2], \mathcal{M}_{+,1})$. Since $\Lambda_{\beta_1/2}(\pi) \subset \{H_{\Gamma_i}^N < 3\}$, by the dynamical large deviations lower bound, by definition of the sequence η^N and since $I_2(\pi) \leq 2\varepsilon$, for N large enough,

$$\mathbb{P}_\eta(H_{\Gamma_i}^N < 3) \geq \exp -N \left\{ \inf_{\pi' \in \Lambda_{\beta_1/2}(\pi)} I_2(\pi'|\gamma) + \varepsilon \right\} \geq \exp\{-3N\varepsilon\}$$

for all $\eta \in \mathcal{B}_{2\beta_1}^N[\mathcal{M}_i]$, which proves (6.8).

The estimate (6.8) together with the arguments presented in Lemma 21 and Corollary 2 gives the bound (6.7), which completes the proof. □

Proof of the lower bound of Theorem 4. We first claim that for any open set \mathcal{O} of \mathcal{M}_+ containing some $\bar{q}_i \in \mathcal{M}_i, 1 \leq i \leq l$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{O}) \geq -\bar{w}_i. \tag{6.9}$$

Indeed, fix $1 \leq i \leq l, \varepsilon > 0$ and choose first $\beta_1 > 0$ and then $0 < \beta_0 < \beta_1$ satisfying two conditions: (a) $\beta_0 < \zeta_1$, where ζ_1 is the positive constant $\delta_{11}(\beta_1/2, 1/2, 1)$ provided by Lemma 8 for $\bar{\rho} = \bar{\rho}_i$, and (b) the pair (β_0, β_1) fulfills the lower bound of Proposition 3 and Lemma 26. Assume, moreover, that $\mathcal{B}_{2\beta_1}(\bar{q}_i) \subset \mathcal{O}$. Note that condition (a) entails that $\beta_0 < \beta_1/2$.

By (6.6), and since $\tau \geq H_{\Gamma_i}^N$ if the initial configuration belongs to ∂B_i^N ,

$$\begin{aligned} \mathcal{P}^N(\mathcal{O}) &= \frac{1}{C_N} \int_{\partial B^N} \mathbb{E}_\eta \left(\int_0^\tau \mathbf{1}\{\eta_s \in \mathcal{O}^N\} ds \right) d\nu^N(\eta) \\ &\geq \frac{1}{C_N} \int_{\partial B_i^N} \mathbb{E}_\eta(H_{\Gamma_i}^N) d\nu^N(\eta) \geq \frac{1}{C_N} \nu^N(\partial B_i^N) \inf_{\eta \in \partial B_i^N} \mathbb{P}_\eta(H_{\Gamma_i}^N \geq 1). \end{aligned}$$

By Lemma 26 and Proposition 3, to conclude the proof of claim (6.9), it remains to show that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \inf_{\eta \in \partial B_i^N} \mathbb{P}_\eta(H_{\Gamma_i}^N \geq 1) \geq 0.$$

For each integer $N > 0$, let η^N be a configuration in ∂B_i^N such that

$$\mathbb{P}_{\eta^N}(H_{\Gamma_i}^N \geq 1) = \inf_{\eta \in \partial B_i^N} \mathbb{P}_\eta(H_{\Gamma_i}^N \geq 1).$$

Denote by η^{N_k} a subsequence of η^N which transforms the lim inf in a limit and let $\pi_0(d\theta) = \gamma(\theta)d\theta$ be a limit point of $\pi^{N_k}(\eta^{N_k})$. Observe that $\pi_0 \in \overline{\mathcal{B}_{\beta_0}(\mathcal{M}_i)}$ and denote by $\rho : [0, 1] \times \mathbb{T} \rightarrow [0, 1]$ the unique weak solution of the Cauchy problem (2.4) starting from γ . By Lemma 8 and by the definition of $\beta_0, \beta_1, \pi_t(d\theta) := \rho(t, \theta)d\theta$ belongs to $\mathcal{B}_{\beta_1/2}(\mathcal{M}_i)$ for all $0 \leq t \leq 1$.

Let \mathcal{N} be the subset of $D([0, 1], \mathcal{M}_+)$ given by all trajectories $\pi'_t, 0 \leq t \leq 1$, such that $\sup_{0 \leq t \leq 1} d(\pi'_t, \pi_t) < \beta_1/2$. Note that the set \mathcal{N} is open because π_t is continuous. In particular, since π_t belongs to $\mathcal{B}_{\beta_1/2}(\mathcal{M}_i)$ for any $0 \leq t \leq 1, \{H_{\Gamma_i}^N \geq 1\} \subset (\pi^N)^{-1}(\mathcal{N}) := \{\eta. \in D([0, 1], X_N) : \pi^N(\eta.) \in \mathcal{N}\}$. Therefore,

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(H_{\Gamma_i}^N \geq 1) &\geq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{Q}_{1, \eta^N}(\mathcal{N}) \\ &\geq - \inf_{\pi' \in \mathcal{N}} I_1(\pi' | \gamma) \geq -I_1(\pi | \gamma) = 0, \end{aligned}$$

which completes the proof of the claim.

It follows from (6.9) that there exists a sequence $\varepsilon_N \rightarrow 0$ such that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{B}_{\varepsilon_N}(\mathcal{M}_i)) \geq -\bar{w}_i. \tag{6.10}$$

Fix an open subset \mathcal{O} of \mathcal{M}_+ . In order to prove the lower bound, it is enough to show that for any measure ϱ in \mathcal{O} , $1 \leq i \leq l$, $T > 0$, and any trajectory π in $D([0, T], \mathcal{M}_+)$ with $\pi_0 \in \mathcal{M}_i$, $\pi_T = \varrho$,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{O}) \geq -\bar{w}_i - I_T(\pi). \tag{6.11}$$

To prove this claim, fix an open subset \mathcal{O} of \mathcal{M}_+ , a measure ϱ in \mathcal{O} , $1 \leq i \leq l$, $T > 0$, and a trajectory π in $D([0, T], \mathcal{M}_+)$ with $\pi_0 \in \mathcal{M}_i$, $\pi_T = \varrho$. Since μ^N is the stationary measure,

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{O}) &= \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}_{\mu^N} [\mathbb{P}_\eta (\pi_T^N \in \mathcal{O})] \\ &\geq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \left(\mathcal{P}^N(\mathcal{B}_{\varepsilon_N}(\mathcal{M}_i)) \inf_{\eta \in \mathcal{B}_N} \mathbb{P}_\eta[\pi_T^N \in \mathcal{O}] \right), \end{aligned}$$

where $\mathcal{B}_N = \{\eta : \pi^N(\eta) \in \mathcal{B}_{\varepsilon_N}(\mathcal{M}_i)\}$. Let η^N be a configuration in \mathcal{B}_N such that

$$\mathbb{P}_{\eta^N} [\pi_T^N \in \mathcal{O}] = \inf_{\eta \in \mathcal{B}_N} \mathbb{P}_\eta [\pi_T^N \in \mathcal{O}].$$

Since η^N belongs to \mathcal{B}_N and $\varepsilon_N \rightarrow 0$, we may assume, taking a subsequence if necessary, that $\pi^N(\eta^N)$ converges to some $\bar{\varrho}_i(d\theta) = \bar{\rho}_i(\theta)d\theta \in \mathcal{M}_i$. By (6.10), the expression appearing in the penultimate displayed formula is bounded below by

$$-\bar{w}_i + \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N} [\pi_T^N \in \mathcal{O}] = -\bar{w}_i + \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{Q}_{T, \eta^N}(\mathcal{O}_T),$$

where $\mathcal{O}_T = \{\pi' \in D([0, T], \mathcal{M}_+) : \pi_T \in \mathcal{O}\}$. Since the set \mathcal{O}_T is open, by the lower bound of the dynamical large deviations principle, the previous expression is bounded below by

$$-\bar{w}_i - \inf_{\pi' \in \mathcal{O}_T} I_T(\pi') \geq -\bar{w}_i - I_T(\pi).$$

In view of (2.12), this completes the proof of (6.11) and the one of the lower bound.

6.2 Upper bound

We prove in this subsection the large deviations upper bound. The proof relies on the next two lemmata. The proof of the first one is similar to the proof of Lemma 23 and is left to the reader.

For a closed subset \mathcal{C} of \mathcal{M}_+ and $T > 0$, let C_T be the subset of $D([0, T], \mathcal{M}_+)$ consisting of all paths π for which there exists t in $[0, T]$ such that $\pi(t)$ or $\pi(t-)$ belongs to \mathcal{C} . Note that C_T is a closed subset of $D([0, T], \mathcal{M}_+)$.

Lemma 27 Fix a closed subset \mathcal{C} of \mathcal{M}_+ such that $\inf_{\varrho \in \mathcal{C}} V_i(\varrho) < \infty$. For every $\varepsilon > 0$, there exist $\delta_{20} = \delta_{20}(\mathcal{C}, \varepsilon) > 0$ and $T_{20} > 0$ such that for all $1 \leq i \leq l$, $0 < \beta_1 < \delta_{20}$, $T' \geq T_{20}$, $\gamma(\theta)d\theta \in \Gamma_i$,

$$\inf_{\pi \in C_{T'}} I_{T'}(\pi|\gamma) \geq \inf_{\varrho \in \mathcal{C}} V_i(\varrho) - \varepsilon.$$

Recall the definition of the set B introduced just before Lemma 21 and recall from Corollary 2 that the set B is attained immediately. In particular, if the reaction–diffusion model has to reach a set \mathcal{C} before it hits B , it has to follow straightforwardly the optimal trajectory to \mathcal{C} . The cost of such trajectory has been estimated in the previous lemma, providing the next result.

Lemma 28 Fix $1 \leq i \leq l$ and a closed subset \mathcal{C} of \mathcal{M}_+ . For every $\varepsilon > 0$, there exist $\delta_{21} = \delta_{21}(\mathcal{C}, \varepsilon) > 0$ such that for all $0 < \beta_0 < \beta_1 < \delta_{21}$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \Gamma_i^N} \mathbb{P}_\eta \left[H_C^N < H_B^N \right] \leq - \inf_{\varrho \in \mathcal{C}} V_i(\varrho) + \varepsilon.$$

Proof Fix $\varepsilon > 0$, $1 \leq i \leq l$, and a closed subset \mathcal{C} of \mathcal{M}_+ . We may assume that the left hand side of the inequality appearing in the statement of the lemma is finite. This implies that $\pi_N^{-1}(\mathcal{C}) \cap X_N \neq \emptyset$ for infinitely many N 's. Let $\{\eta^{N_k} : k \geq 1\}$ be a sequence of configurations such that $\pi^{N_k}(\eta^{N_k}) \in \mathcal{C}$. Since \mathcal{M}_+ is compact, taking a subsequence, if necessary, we may assume that $\pi^{N_k}(\eta^{N_k})$ converges to a measure, denoted by ϱ , which belongs to $\mathcal{M}_{+,1}$. Since \mathcal{C} is closed, $\varrho \in \mathcal{C}$ so that $\mathcal{C} \cap \mathcal{M}_{+,1} \neq \emptyset$. In particular, By Lemma 18, $\inf_{\varrho \in \mathcal{C}} V_i(\varrho) < \infty$.

Let ζ_1, R_1 be the constants δ_{20}, T_{20} given by Lemma 27. Fix $0 < \beta_0 < \beta_1 < \zeta_1$. Since $\inf_{\varrho \in \mathcal{C}} V_i(\varrho) < \infty$, by Lemma 21, there exists $R_2 > 0$ such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in X_N} \mathbb{P}_\eta \left[H_B^N \geq R_2 \right] \leq - \inf_{\varrho \in \mathcal{C}} V_i(\varrho). \tag{6.12}$$

Let $T = \max\{R_1, R_2\}$.

Recall the definition of the set C_T introduced just above the statement of Lemma 27 and the fact that C_T is a closed subset of $D([0, T], \mathcal{M}_+)$. Note also that $\{H_C^N \leq T\} \subset C_T$.

Let η^N be a configuration in Γ_i^N such that

$$\mathbb{P}_{\eta^N} \left[H_C^N \leq T \right] = \sup_{\eta \in \Gamma_i^N} \mathbb{P}_\eta \left[H_C^N \leq T \right].$$

Taking a subsequence if necessary, we may assume that $\pi^N(\eta^N)$ converges to some measure $\varrho(d\theta) = \gamma(\theta)d\theta$ in $\Gamma_i \cap \mathcal{M}_{+,1}$. Since $\{H_C^N \leq T\} \subset C_T$ and since C_T is a closed set, by the dynamical large deviations upper bound and by Lemma 27,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \Gamma_i^N} \mathbb{P}_{\eta^N} (H_C^N < H_B^N \leq T) &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N} (H_C^N \leq T) \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \log Q_{T, \eta^N} (C_T) \leq - \inf_{\pi \in C_T} I_T(\pi | \gamma) \leq - \inf_{\varrho \in \mathcal{C}} V_i(\varrho) + \varepsilon. \end{aligned}$$

By (6.12), by this estimate and by (6.4),

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \Gamma_i^N} \mathbb{P}_{\eta} [H_C^N < H_B^N] \leq - \inf_{\varrho \in \mathcal{C}} V_i(\varrho) + \varepsilon,$$

which completes the proof of the lemma. □

Proof of the upper bound of Theorem 4. Let \mathcal{C} be a closed subset of \mathcal{M}_+ . Assume first that $\mathcal{M}_i \cap \mathcal{C} = \emptyset$ for any $1 \leq i \leq l$. In this case, let $\beta_1 > 0$ be such that $\cup_{1 \leq i \leq l} \mathcal{B}_{2\beta_1}(\mathcal{M}_i) \cap \mathcal{C} = \emptyset$.

By the representation (6.6) of the stationary measure μ^N ,

$$\begin{aligned} \mathcal{P}^N(\mathcal{C}) = \mu^N(\mathcal{C}^N) &= \frac{1}{C_N} \int_{\partial B^N} \mathbb{E}_{\eta} \left(\int_0^{\tau} \mathbf{1}\{\eta_s \in \mathcal{C}^N\} ds \right) d\nu^N(\eta) \\ &\leq \frac{1}{C_N} \sum_{i=1}^l \nu^N(\partial B_i^N) \sup_{\eta \in \partial B_i^N} \mathbb{E}_{\eta} \left(\int_0^{\tau} \mathbf{1}\{\eta_s \in \mathcal{C}^N\} ds \right). \end{aligned}$$

A configuration in X_N can jump to at most $2N$ different configurations and the jump rates are bounded by N^2 . Since any trajectory in $D(\mathbb{R}_+, X_N)$ has to jump at least once before the stopping time τ , the constant C_N appearing in the denominator is bounded below by c_0/N^3 for some positive constant c_0 . Hence, by (6.4) and by Proposition 3, in order to prove the upper bound it is enough to show that for each $1 \leq i \leq l$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \partial B_i^N} \mathbb{E}_{\eta} \left(\int_0^{\tau} \mathbf{1}\{\eta_s \in \mathcal{C}^N\} ds \right) \leq - \inf_{\varrho \in \mathcal{C}} V_i(\varrho) + \varepsilon. \tag{6.13}$$

The time integral appearing in the previous formula vanishes if $\tau \leq H_C^N$. We may therefore introduce the indicator of the set $H_C^N \leq \tau$. After doing this and applying the strong Markov property, we obtain that the left hand side of the previous inequality is less than or equal to

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \partial B_i^N} \mathbb{P}_{\eta} [H_C^N < \tau] \sup_{\eta \in \mathcal{C}^N} \mathbb{E}_{\eta}(\tau).$$

Since the distance between the empirical measure before and after a jump is bounded by C/N , and since $\mathcal{B}_{2\beta_1}(\mathcal{M}_i) \cap \mathcal{C} = \emptyset$, for N large enough, any trajectory in $D(\mathbb{R}_+, X_N)$ starting at some configuration in $\partial B_i^N, \mathcal{C}^N$, satisfies $H_{\Gamma_i}^N \leq H_C^N$,

$\tau \leq H_B^N$, respectively. Hence, by the strong Markov property, the previous expression is bounded above by

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{\eta \in \Gamma_i^N} \mathbb{P}_\eta \left[H_C^N < H_B^N \right] \sup_{\eta \in \mathcal{C}^N} \mathbb{E}_\eta \left(H_B^N \right).$$

By Corollary 2 and Lemma 28, the previous expression is bounded by $-\inf_{\varrho \in \mathcal{C}} V_i(\varrho) + \varepsilon$, which completes the proof of (6.13) and the one of the upper bound in the case $\mathcal{M}_i \cap \mathcal{C} = \emptyset$ for $1 \leq i \leq l$.

We turn to the general case. We first claim that for each $1 \leq i \leq l$, $\varepsilon > 0$, there exists $\zeta_1 > 0$ such that for all $0 < \beta_0 < \zeta_1$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{B}_{\beta_0}(\mathcal{M}_i)) \leq -\bar{w}_i + 2\varepsilon. \tag{6.14}$$

Indeed, fix $\varepsilon > 0$ and set $\zeta_1 = \min\{\delta_{18}, \delta_{19}\}$, where $\delta_{18} > 0$ is the constant provided by Proposition 3 and $\delta_{19} > 0$ is the one given by Lemma 26. Fix $\beta_0 < \zeta_1$. By the representation (6.6) of the stationary measure μ^N ,

$$\begin{aligned} \mathcal{P}^N(\mathcal{B}_{\beta_0}(\mathcal{M}_i)) &= \mu^N(\mathcal{B}_{\beta_0}^N(\mathcal{M}_i)) = \frac{1}{C_N} \int_{\partial B^N} \mathbb{E}_\eta \left(\int_0^\tau \mathbf{1}_{\{\eta_s \in \mathcal{B}_{\beta_0}^N(\mathcal{M}_i)\}} ds \right) dv^N(\eta) \\ &\leq \frac{1}{C_N} \sum_{j=1}^l v^N(\partial B_j^N) \sup_{\eta \in \partial B_j^N} \mathbb{E}_\eta \left(\int_0^\tau \mathbf{1}_{\{\eta_s \in \mathcal{B}_{\beta_0}^N(\mathcal{M}_i)\}} ds \right). \end{aligned}$$

We have seen in the first part of the proof that $\limsup_N N^{-1} \log C_N^{-1} \leq 0$. On the other hand, for $\eta \in \partial B_j^N$, $j \neq i$, $\tau \leq H_{B_i}^N$, so that $\int_0^\tau \mathbf{1}_{\{\eta_s \in \mathcal{B}_{\beta_0}^N(\mathcal{M}_i)\}} ds = 0$. Finally, denote by $\vartheta(t)$, $t > 0$, the time translation of a trajectory by t . For $\eta \in \partial B_i^N$, writing τ as $H_{\Gamma_i}^N + H_B^N \circ \vartheta(H_{\Gamma_i}^N)$, by the strong Markov property, since $\beta_0 < \zeta_1$, and by Proposition 3, the left hand side of (6.14) is bounded by

$$-\bar{w}_i + \varepsilon + \limsup_{N \rightarrow \infty} \frac{1}{N} \log \left\{ \sup_{\eta \in \partial B_i^N} \mathbb{E}_\eta(H_{\Gamma_i}^N) + \sup_{\eta \in \partial \Gamma_i^N} \mathbb{E}_\eta(H_B^N) \right\}.$$

By (6.4), (6.7) and Corollary 2, the limit superior of the previous equation is bounded by ε , which completes the proof of (6.14).

Let \mathcal{C} be a closed subset of \mathcal{M}_+ and fix $\varepsilon > 0$. Let A be the set of indices i such that $\mathcal{C} \cap \mathcal{M}_i \neq \emptyset$. Let ζ_1 be the positive constant introduced in (6.14), and choose $\beta_0 < \zeta_1$ such that $d(\mathcal{C}, \mathcal{M}_j) > \beta_0$ for all $j \in A^c$. Since $\mathcal{C} \subset \cup_{i \in A} \mathcal{B}_{\beta_0}(\mathcal{M}_i) \cup [\mathcal{C} \setminus \{\cup_{i \in A} \mathcal{B}_{\beta_0}(\mathcal{M}_i)\}]$, and since $\mathcal{C} \setminus \{\cup_{i \in A} \mathcal{B}_{\beta_0}(\mathcal{M}_i)\}$ is a closed set which does not intersect the set \mathcal{M}_{sol} , by (6.4), by (6.14) and by the first part of the proof,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{C}) \leq -\min \left\{ \min_{i \in A} \bar{w}_i, \inf_{\pi \in \mathcal{C} \setminus \{\cup_{i \in A} \mathcal{B}_{\beta_0}(\mathcal{M}_i)\}} W(\pi) \right\} + 2\varepsilon.$$

By (2.15), $\bar{w}_i = W(\bar{q}_i)$ for $\bar{q}_i \in \mathcal{M}_i$. On the other hand, since $\mathcal{M}_i \cap \mathcal{C} \neq \emptyset$,

$$\inf_{\pi \in \mathcal{C}} W(\pi) \leq \min_{i \in A} W(\bar{q}_i), \quad \inf_{\pi \in \mathcal{C}} W(\pi) \leq \inf_{\pi \in \mathcal{C} \setminus \{\cup_{i \in A} \mathcal{B}_{\beta_0}(\mathcal{M}_i)\}} W(\pi),$$

which completes the proof of the upper bound.

7 Proof of Theorem 5

We first show that if there exists a heteroclinic orbit from $\phi \in \mathcal{M}_i$ to $\psi \in \mathcal{M}_j$, then the cost of going from \mathcal{M}_i to \mathcal{M}_j vanishes.

Lemma 29 *Suppose that there exists a heteroclinic orbit from $\phi \in \mathcal{M}_i$ to $\psi \in \mathcal{M}_j$. Then, $v_{ij} = 0$.*

Proof Fix $i \neq j$ in $\{1, \dots, l\}$ and assume that there exists a heteroclinic orbit from $\phi \in \mathcal{M}_i$ to $\psi \in \mathcal{M}_j$, denoted by $\rho(t, \theta)$, $t \in \mathbb{R}$. By Proposition 2, ϕ is smooth, and, by Lemma 4, there exists $0 < c < 1/2$ such that $c \leq \phi(\theta) \leq 1 - c$. Since $\rho(t)$, converges in $C^1(\mathbb{T})$ to ϕ , ψ as $t \rightarrow -\infty$, $t \rightarrow +\infty$ respectively, by Lemmata 17, 19 and since the dynamical large deviations rate functional vanishes along the solution of the hydrodynamic equation, $v_{ij} = 0$. \square

We now prove that $\rho(\theta) = r$ is a stable solution of the reaction–diffusion equation (2.5) if r is a local minimum of V .

Lemma 30 *Fix $1 \leq i \leq l$. Let $\bar{q}_i(d\theta) = \bar{\rho}_i(\theta)d\theta$, $\bar{\rho}_i(\theta) = r$, where r is a local minimum of V . Then, for all $\varepsilon > 0$ there exist $c > 0$ such that*

$$\inf \{V_i(\varrho) : \varrho \notin \mathcal{B}_\varepsilon(\bar{q}_i)\} \geq c.$$

Proof Suppose that $\inf\{V_i(\varrho) : \varrho \in \mathcal{B}_\delta(\bar{q}_i)^c\} = 0$ for some $\delta > 0$. In this case there exists a sequence of density profiles γ_n and of trajectories $\pi^n(t, d\theta) = \rho^n(t, \theta)d\theta$, $0 \leq t \leq T_n$, such that $\rho^n(0, \theta) = \bar{\rho}_i$, $\rho^n(T_n, \theta) = \gamma_n(\theta)$, $\gamma_n(\theta)d\theta \in \mathcal{B}_\delta(\bar{q}_i)^c$ and $I_{T_n}(\pi^n) \leq 1/n$.

By Lemma 9, there exists $0 < \varepsilon < \delta$ such that $\pi_t \in \mathcal{B}_\delta(\bar{q}_i)$ for all $t \geq 0$ if $\pi_0 \in \mathcal{B}_{2\varepsilon}(\bar{q}_i)$. Let τ_n be the time the trajectory π^n leaves the set $\mathcal{B}_\varepsilon(\bar{q}_i)$ for ever, and let σ_n be the hitting time of the set $\mathcal{B}_\delta(\bar{q}_i)^c$ after τ_n :

$$\tau_n = \sup\{t \leq T_n : \pi_t^n \in \mathcal{B}_\varepsilon(\bar{q}_i)\}, \quad \sigma_n = \inf\{t \geq \tau_n : \pi_t^n \in \mathcal{B}_\delta(\bar{q}_i)^c\}.$$

Since in the interval $[\tau_n, \sigma_n]$ the trajectory π^n remains in the set $\mathcal{B}_\delta(\bar{q}_i) \setminus \mathcal{B}_\varepsilon(\bar{q}_i)$, if δ is small enough for $\mathcal{B}_\delta(\bar{q}_i) \cap \mathcal{B}_\delta(\mathcal{M}_j) = \emptyset$ for all sets \mathcal{M}_j , $j \neq i$, by Corollary 1, $\sigma_n - \tau_n$ is uniformly bounded by a finite constant, denoted by T .

Extend the definition of π^n from the interval $[0, T_n]$ to \mathbb{R}_+ by following the hydrodynamic trajectory after T_n : $\pi^n(T_n + t, d\theta) = \tilde{\rho}^n(t, \theta)d\theta$, where $\tilde{\rho}^n$ is the solution of the hydrodynamic equation with initial condition $\rho_{T_n}^n$. Let $\tilde{\pi}_t^n$, $0 \leq t \leq T$, be the trajectory defined by $\tilde{\pi}_t^n = \pi^n(\tau_n + t)$. Since π^n belongs to $C([0, T_n], \mathcal{M}_{+,1})$, note

that $\bar{\pi}_0^n \in \partial\mathcal{B}_\varepsilon(\bar{q}_i)$, that $\bar{\pi}^n$ hits the set $\mathcal{B}_\delta(\bar{q}_i)^c$ in the time interval $[0, T]$ and that $I_T(\bar{\pi}^n) \leq 1/n$.

By the compactness of the level sets of I_T , the lower semi-continuity of this functional and the compactness of the space \mathcal{M}_+ , there exists a subsequence $\bar{\pi}^{n_k}$ which converges to some trajectory π such that $\pi_0 \in \partial\mathcal{B}_\varepsilon(\bar{q}_i)$, π hits the set $\mathcal{B}_\delta(\bar{q}_i)^c$ in the time interval $[0, T]$ and $I_T(\pi) = 0$. By Lemma 12, the density of π_t , denoted by ρ_t , is a solution of the hydrodynamic equation. This contradicts the property of ε and concludes the proof of the lemma. \square

Proof of Theorem 5 Recall the definition of the set of indices I_s, I_u . We claim that $\bar{w}_a > 0$ for all $a \in I_u$. To prove this statement, it is enough to show that for each $a \in I_u$, there exists $b \in I_s$ such that $w_a > w_b$.

Fix $a \in I_u$. By assumption, there exists $b \in I_s$ such that $v_{ab} = 0$. We claim that $w_a > w_b$. Indeed, on the one hand, by Lemma 30, $v_{bc} > 0$ for all $c \neq b$. On the other hand, let g be a graph in $\mathcal{T}(a)$ such that $w_a = \kappa(g)$. Recall that we denote by $(d, e), e \neq d \in \mathcal{V}$, the oriented edge where d is the child and e the parent. Let c be the parent of b in g . Of course, c might be a . Denote by g' the tree in $\mathcal{T}(b)$ obtained from g by adding the oriented edge (a, b) and removing the the edge (b, c) , and note that $\kappa(g) + v_{ab} = \kappa(g') + v_{bc}$. Since w_b is the minimal value of $\kappa(\tilde{g}), \tilde{g} \in \mathcal{T}(b)$, $w_b \leq \kappa(g')$ so that $w_b + v_{bc} \leq \kappa(g) + v_{ab} = w_a + v_{ab} = w_a$. The last identity follows from the fact that $v_{ab} = 0$ and the next to last from the fact that $\kappa(g) = w_a$. Since $v_{bc} > 0$, we conclude that $w_b < w_a$, as claimed.

We claim that for every $\delta > 0$,

$$\inf \left\{ W(\pi) : \pi \notin \bigcup_{i \in I_s} \mathcal{B}_\delta(\bar{q}_i) \right\} > 0. \tag{7.1}$$

Fix $\delta > 0$. Since $\bar{w}_a > 0$ for all $a \in I_u$, in view of the definition of W , we only need to check that

$$\inf \left\{ V_j(\pi) : \pi \notin \bigcup_{i \in I_s} \mathcal{B}_\delta(\bar{q}_i) \right\} > 0$$

for each $j \in I_s$. This is the content of Lemma 30, proving (7.1)

To complete the proof of the theorem, it remains to observe that the complement of $\bigcup_{i \in I_s} \mathcal{B}_\delta(\bar{q}_i)$ is a closed set and to apply the upper bound of the static large deviations principle stated in Theorem 4. The theorem is proved. \square

8 The Chafee–Infante equation

We present in this section an example of a reaction–diffusion model which fulfills the hypotheses of Theorems 4 and 5. Actually, in this model a complete description of the stationary solutions and of the heteroclinic orbits is available.

Fix $0 < a < b$ and recall the definition of the potential $V = V_{a,b}$ introduced in (2.7). Denote by $\mathbb{T}_{1/2\pi}$ the one-dimensional torus with length $(2\pi)^{-1}$. Let $p = (1/2)\sqrt{a/b}$, $c = 2(2\pi)^2$ and define $\phi : \mathbb{R}_+ \times \mathbb{T}_{1/2\pi} \rightarrow \mathbb{R}$ as

$$\phi(t, \theta) = \frac{1}{p} \left\{ \rho(ct, 2\pi\theta) - \frac{1}{2} \right\}.$$

A simple computation shows that ρ solves the Eq. (2.7) if and only if ϕ solves

$$\partial_t \phi = \Delta \phi + \lambda(1 - \phi^2)\phi, \tag{8.1}$$

where $\lambda = 4ca = 8(2\pi)^2 a$. Note that ϕ takes values in the interval $[-\sqrt{b/a}, \sqrt{b/a}]$. This is the so-called Chafee–Infante equation [13] with periodic boundary condition.

A complete characterization of the stationary solutions of the Chafee–Infante equation with periodic boundary conditions is presented in [30, Proposition 1.1]. In our context it can be stated as follows. Let ρ_{\pm} be the minima of V : $\rho_{\pm} = (1/2) \pm p = (1/2)[1 \pm \sqrt{a/b}]$.

Theorem 8 *For all $0 < a < b$, the Eq. (8.1) admits three constant stationary solutions: $\psi_{\pm} = \rho_{\pm}$, $\psi_{1/2} = 1/2$. For all nonnegative integers m such that $1 \leq m^2 < \lambda = 32\pi^2 a$, up to translations, there exists a non-constant periodic stationary solution $\phi_m = \phi_{m,\lambda}$ with m periods in $\mathbb{T}_{1/2\pi}$. Moreover, $\lim_{\lambda \downarrow m^2} \phi_{m,\lambda} = 1/2$ in $C^2(\mathbb{T}_{1/2\pi})$. The reaction–diffusion equation (2.4) with potential $V_{a,b}$ has no other stationary solutions.*

The heteroclinic orbits of the Chafee–Infante equation with periodic boundary conditions have been characterized in [23]. Next result follows from Theorems 1.3 and 1.4 of [23],

Theorem 9 *There are heteroclinic orbits from $\psi_{1/2}$ to ψ_{\pm} , and from $\psi_{1/2}$ to ϕ_m for all integers $1 \leq m^2 < \lambda$. Fix $1 \leq n^2 < \lambda$. There are heteroclinic orbits from ϕ_n to ψ_{\pm} , and from ϕ_n to ϕ_m for all integers $1 \leq m^2 < n^2$. There are no other heteroclinic orbits.*

Next proposition follows from the previous results and from Theorem 5.

Proposition 4 *Consider a reaction–diffusion model which satisfies the assumptions of Theorem 4 and which gives rise to the hydrodynamic equation (2.7) with $0 < a < b$. Let $\bar{\rho}_{\pm}(d\theta) = \rho_{\pm}d\theta$. Then, for every $\delta > 0$, there exist $c > 0$ and $N_0 \geq 1$ such that for all $N \geq N_0$,*

$$\mathcal{P}^N(\mathcal{B}_{\delta}(\bar{\rho}_-) \cup \mathcal{B}_{\delta}(\bar{\rho}_+)) \geq 1 - e^{-cN}.$$

If the jump rates are invariant under a global flipping of the configuration: $c(\eta) = c(\mathbf{1} - \eta)$, where $\mathbf{1}$ is the configuration with all sites occupied, there is a symmetry between occupied and vacant sites so that $\mathcal{P}^N(\mathcal{B}_{\delta}(\bar{\rho}_+)) = \mathcal{P}^N(\mathcal{B}_{\delta}(\bar{\rho}_-))$ for all $\delta > 0$. Hence, with this additional assumption, we may refine the previous proposition:

Corollary 3 *Under the assumptions of Proposition 4, if $c(\eta) = c(\mathbf{1} - \eta)$, for every $0 < \delta < (1/2)d(\bar{q}_-, \bar{q}_+)$, there exist $c > 0$ and $N_0 \geq 1$ such that for all $N \geq N_0$,*

$$\left| \mathcal{P}^N(\mathcal{B}_\delta(\bar{q}_-)) - \frac{1}{2} \right| \leq e^{-cN}, \quad \left| \mathcal{P}^N(\mathcal{B}_\delta(\bar{q}_+)) - \frac{1}{2} \right| \leq e^{-cN}.$$

An example. We conclude this section with an example of a reaction–diffusion model satisfying the assumptions of Theorem 4 and whose hydrodynamic equation is given by (2.7).

Consider the reaction–diffusion model whose jump rate $c(\eta)$ is given by

$$c(\eta) = a_2 \mathbf{1}\{\eta_{-1} \neq \eta_1\} + a_1 \mathbf{1}\{\eta_{-1} = \eta_1 = \eta_0\} + a_0 \mathbf{1}\{\eta_{-1} = \eta_1 \neq \eta_0\}.$$

Let ξ be the configuration obtained from η by flipping all occupation variables: $\xi(x) = 1 - \eta(x)$, $x \in \mathbb{T}_N$. Since $[1 - \eta(0)]c(\eta) = \xi(0)c(\xi)$, $B(\rho) = D(1 - \rho)$. Moreover,

$$F(\rho) = \frac{1}{4} \left\{ (a_0 - 3a_1 - 2a_2)(2\rho - 1) - (a_0 + a_1 - 2a_2)(2\rho - 1)^3 \right\}.$$

Fix $0 < \mathfrak{a} < \mathfrak{b}$, and set $a_1 = \mathfrak{a} > 0$. Choose $a_2 \geq \mathfrak{a} + 2\mathfrak{b} > 0$ and set $a_0 = 2a_2 + 4\mathfrak{b} - \mathfrak{a} \geq \mathfrak{a} + 8\mathfrak{b} > 0$. Since the three parameters are positive, the jump rate is strictly positive as required.

As $a_0 - 3a_1 - 2a_2 = 4(\mathfrak{b} - \mathfrak{a})$ and $a_0 + a_1 - 2a_2 = 4\mathfrak{b}$, in the variables \mathfrak{a} , \mathfrak{b} the function F becomes

$$F(\rho) = (\mathfrak{b} - \mathfrak{a})(2\rho - 1) - \mathfrak{b}(2\rho - 1)^3,$$

in conformity with (2.7) for $a = (\mathfrak{b} - \mathfrak{a})/2$, $b = \mathfrak{b}/2$.

Since $B(\rho) = D(1 - \rho)$, D is concave if and only if B is concave. The functions B is concave if $3a_1 + a_0 \leq 4a_2 \leq 4a_0$. We claim that these inequalities are in force. On the one hand, as $\mathfrak{b} > \mathfrak{a} > 0$ and $a_2 > 0$, we have that $a_2 < 2a_2 + 4\mathfrak{b} - \mathfrak{a} = a_0$. On the other hand, $3a_1 + a_0 - 4a_2 = 2a_1 + 4\mathfrak{b} - 2a_2 = 2\mathfrak{a} + 4\mathfrak{b} - 2a_2 \leq 0$ from the definition of a_2 .

This shows that all the assumptions of Theorem 4 are fulfilled.

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