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Self-attracting self-avoiding walk

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Abstract
This article is concerned with self-avoiding walks (SAW) on $\mathbb{Z}^d$ that are subject to a self-attraction. The attraction, which rewards instances of adjacent parallel edges, introduces difficulties that are not present in ordinary SAW. Ueltschi has shown how to overcome these difficulties for sufficiently regular infinite-range step distributions and weak self-attractions (Ueltschi in Probab Theory Relat Fields 124(2):189–203, 2002). This article considers the case of bounded step distributions. For weak self-attractions we show that the connective constant exists, and, in $d \geq 5$, carry out a lace expansion analysis to prove the mean-field behaviour of the critical two-point function, hereby addressing a problem posed by den Hollander (Random Polymers, vol. 1974. Springer-Verlag, Berlin, 2009).

Keywords Self-interacting random walk · Self-attracting walk · Self-avoiding walk · Linear polymers · Lace expansion · Critical phenomena · Hammersley-Welsh argument

Mathematics Subject Classification Primary 60K35; Secondary 60D05 · 82B27

1 Introduction

1.1 Model definition

Let $\mathbb{Z}^d$ denote the d-dimensional integer lattice with nearest-neighbour edges, and assume $d \geq 2$. Let $\mathbb{P}$ be the law of a random walk on the vertices of $\mathbb{Z}^d$ with i.i.d.

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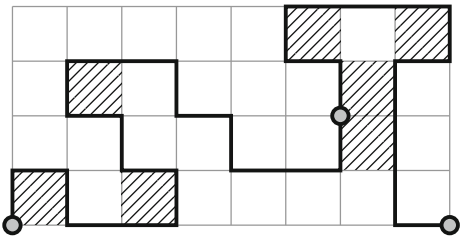


Fig. 1 A self-avoiding walk ω. Shaded plaquettes indicate the seven pairs of adjacent edges of ω

increments distributed according to a step distribution D. Letting $\{\pm e_i\}_{i=1}^d$ denote the standard generators of $\mathbb{Z}^d$, a *plaquette* is a collection of vertices of the form $\{x, x+u, x+v, x+u+v\}$ where $u \notin \{\pm v\}$ and u and v are in $\{\pm e_i\}_{i=1}^d$. Two edges $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of $\mathbb{Z}^d$ are *adjacent* if $\{x_1, y_1, x_2, y_2\}$ is a plaquette.

A walk is a sequence of vertices in $\mathbb{Z}^d$, and the *edges* of a walk ω are the pairs $\{\omega_i, \omega_{i+1}\}$ of consecutive vertices. Note that, for general increment distributions D, the edges of a walk in the support of $\mathbb{P}$ are not necessarily edges of $\mathbb{Z}^d$. Define $\mathsf{adj}(\omega)$ to be the collection of pairs of edges of ω that are adjacent edges of $\mathbb{Z}^d$, and let $|\mathsf{adj}(\omega)|$ be the cardinality of this set. See Fig. 1.

Let $\mathbb{P}_n$ denote the law induced by $\mathbb{P}$ on n-step walks that begin at the origin $o \in \mathbb{Z}^d$, and recall that a walk is *self-avoiding* if it does not visit any vertex more than once (see Sect. 3.1 for a more precise definition). The models we are interested in are perturbations $\mathsf{P}_{n,\kappa}$ of $\mathbb{P}_n$ defined by

$$\mathsf{P}_{n,\kappa}(\omega) \propto \mathbb{1}_{\{\omega \in \Gamma_n\}} \mathcal{W}_\kappa(\omega), \qquad \kappa \geq 0, \tag{1.1}$$

where Γ_n is the set of n-step self-avoiding walks with initial vertex $\omega_0 = o$ and

$$\mathcal{W}_\kappa(\omega) := e^{-H_\kappa(\omega)} \mathbb{P}_n(\omega), \qquad e^{-H_\kappa(\omega)} := (1+\kappa)^{|\mathsf{adj}(\omega)|}. \tag{1.2}$$

The symbol $:=$ indicates equality by definition. The law $\mathsf{P}_{n,\kappa}$ on n-step walks is called (*n-step*) *attracting self-avoiding walk* with *attraction strength* κ, *or* (*n-step*) *κ-ASAW*. When the length of the walk is irrelevant the adjective n-step will be dropped. We think of the right-hand side of (1.1) as defining the κ-ASAW *weight* of a walk ω. The probability of a walk is proportional to its weight.

The law $\mathsf{P}_{n,0}$ defined by (1.1) is the law of n-step *self-avoiding walk* (SAW) [3]. Physically, self-avoiding walk is a model of a linear polymer in a good solvent. The self-avoidance constraint represents the inability of two molecules in the polymer to occupy the same space. If $\kappa > 0$, walks under the κ-ASAW law are attracted to themselves. Physically, this is a model of a linear polymer in a poor solvent, see [4, Section 6.3] and [2, Chapter 6]. The molecules huddle together to escape exposure to the surrounding solvent.

1.2 Lack of submultiplicativity

Before stating our results, we briefly discuss the central difficulty of the model. Let $c_n(\kappa)$ denote the normalization constant that makes $\mathsf{P}_{n,\kappa}$ a probability measure, i.e.,

$$c_n(\kappa) := \sum_{\omega \in \Gamma_n} \mathcal{W}_\kappa(\omega). \tag{1.3}$$

Note that $c_n(\kappa)$ is implicitly also a function of the step distribution D.

The first mathematical fact one learns about self-avoiding walk is that when $\kappa = 0$ the sequence $(c_n(\kappa))_{n\geq 1}$ is submultiplicative, i.e.,

$$c_{n+m}(0) \leq c_n(0)c_m(0).$$

This bound arises because any $(n + m)$-step self-avoiding walk can be split into an n-step self-avoiding walk and an m-step self-avoiding walk. Simple estimates and Fekete's lemma [5, Lemma 1.2.1] on submultiplicative sequences imply that $(c_n(0))^{1/n}$ converges as $n \to \infty$; see [6, Section 1.2].[1]

The basic difficulty in the study of κ-ASAW is that the sequence $(c_n(\kappa))_{n\geq 1}$ is generally not submultiplicative for $\kappa > 0$. To see that submultiplicativity cannot hold in general, consider the nearest-neighbour step distribution $D(x) = (2d)^{-1}\mathbb{1}_{\{\|x\|_1=1\}}$. Submultiplicativity of the sequence $c_n(\kappa)$ would imply

$$c_n(\kappa) \leq c_1(\kappa)^n = 1,$$

which cannot hold for fixed $n \geq 3$ when κ is sufficiently large, as the left-hand side is a polynomial in κ of degree at least 1.

2 Results

Henceforth it will be assumed that $D(x)$ is invariant under the symmetries of $\mathbb{Z}^d$ (namely, reflections in hyperplanes and rotations by $\pi/2$ about coordinate axes), and that $D(e_1) := p_1 > 0$. Thus, $D(\pm e_i) = p_1$ for both choices of sign and all choices of $i = 1, 2, \ldots, d$.

The next two subsections present our main results, Theorem 2.1 and Theorem 2.3, and in Sects. 2.3 and 2.4 we discuss the main ideas of the proofs and briefly describe how our results fit into the literature.

2.1 Connective constants

The limiting value $\mu(\kappa)$ of $(c_n(\kappa))^{1/n}$, if the limit exists, is called the *connective constant with self-attraction* κ. We will prove that the connective constant of κ-ASAW exists for κ sufficiently small despite not knowing if submultiplicativity holds.

Theorem 2.1 *Let $d \geq 2$. There exists a $\kappa_0 = \kappa_0(D) > 0$ such that for $0 < \kappa < \kappa_0$ the limit $\mu(\kappa) = \lim_{n\to\infty}(c_n(\kappa))^{1/n}$ exists.*

Note that the dependence of κ_0 on D implicitly means that κ_0 may depend on the dimension d. The remainder of this section briefly describes the proof of Theorem 2.1,

[1] Note that our definition of c_n involves D, i.e., we are enumerating weighted self-avoiding walks.

although we delay a discussion of how the lack of submultiplicativity is overcome to Sect. 2.3. The proof appears in Sect. 4.

An n-step self-avoiding walk ω is a *bridge* if $\pi_1(\omega_0) < \pi_1(\omega_j) \leq \pi_1(\omega_n)$ for all $j = 1, \ldots, n$, where π_1 denotes projection onto the first coordinate. A key observation for the proof of Theorem 2.1 is that $\mathcal{W}_\kappa$ is supermultiplicative on bridges when $\kappa \geq 0$. This implies the connective constant for bridges, $\mu_{\mathsf{B}}(\kappa)$, exists.

A classical argument due to Hammersley and Welsh shows that the number of n-step self-avoiding bridges is the same, up to sub-exponential corrections, as the number of n-step self-avoiding walks [7]; see also [6, Section 3.1]. An immediate consequence is that $\mu_{\mathsf{B}}(0) = \mu(0)$. To prove the existence of $\mu(\kappa)$, we adapt the Hammersley-Welsh argument to $\kappa > 0$; i.e., we prove that the difference in the κ-ASAW weight of n-step bridges and n-step walks is sub-exponential in n.

The Hammersley-Welsh argument involves "unfolding" self-avoiding walks by reflecting segments of the walk through well-chosen hyperplanes. It is during unfolding that the lack of submultiplicativity must be overcome.

2.2 Mean-field behaviour

To formulate Theorem 2.3, our main lace expansion result, we require further assumptions on the step distribution D.

Definition 1 Let $L > 0$. A step distribution D is *spread-out with parameter* L if it has the form

$$D(x) = \begin{cases} \frac{h(x/L)}{\sum_{x \in \mathbb{Z}^d \setminus \{o\}} h(x/L)} & x \neq o \\ 0 & x = o, \end{cases} \tag{2.1}$$

where $h\colon [-1, 1]^d \to [0, \infty)$ is a piecewise continuous function such that $h(0) > 0$, 0 is a point of continuity, and

(i) h is invariant under the symmetries of $\mathbb{Z}^d$, and
(ii) $\int h(x)\, dx = 1$.

In what follows when we consider a "spread-out step distribution" we mean the one-parameter family of step distributions obtained by choosing a single function h. Note that $h(0) > 0$ and 0 being a point of continuity for h implies that the denominator in (2.1) is positive and $D(e_1) = p_1 > 0$ if L is taken sufficiently large. We will implicitly assume L is at least this large in what follows. The variance of D will be denoted by $\sigma^2 := \sum_{x \in \mathbb{Z}^d} \|x\|_2^2 D(x)$.

Example 2.2 Consider $h(x) = 2^{-d}$ on $[-1, 1]^d$. This leads to $D(x)$ being uniformly distributed on vertices $x \neq o$ with $\|x\|_\infty \leq L$.

The proof of Theorem 2.1 relies in part on establishing that κ-ASAW is repulsive in an averaged sense; roughly speaking, this means that a walk under the κ-ASAW law is not typically attracted to its earlier trajectory. This idea, which is explained more precisely in Sect. 2.3, also turns out to enable a lace expansion analysis of κ-ASAW at *criticality*, a notion which we introduce in the next two definitions.

Definition 2 The *susceptibility* of κ-ASAW is the power series

$$\chi_\kappa(z) := \sum_{n=0}^{\infty} c_n(\kappa) z^n. \tag{2.2}$$

Definition 3 The *critical point* $z_c = z_c(D, \kappa)$ of κ-ASAW is defined to be

$$z_c := \sup\{z \geq 0 \mid \chi_\kappa(z) < \infty\}.$$

For SAW, submultiplicativity implies that $z_c(0) = \mu(0)^{-1}$. For κ small enough that Theorem 2.1 applies, it remains true that $z_c(\kappa) = \mu(\kappa)^{-1}$ by the Cauchy–Hadamard characterization of the radius of convergence.

Before stating our main result on the behaviour of κ-ASAW at the critical point $z_c(\kappa)$, we require a few more definitions. Precise formulations of the classes of walks involved in these definitions can be found in Sect. 3.1.

The *two-point function* of κ-ASAW is defined, for $z \geq 0$ and $x \in \mathbb{Z}^d$, by

$$G_{z,\kappa}(x) := \sum_{n \geq 0} \sum_{\omega \in \Gamma_n(x)} z^n \mathcal{W}_\kappa(\omega). \tag{2.3}$$

Note that the inner sum is restricted to self-avoiding walks that end at x; only n-step walks with positive probability under $\mathsf{P}_{n,\kappa}$ contribute.

The κ-ASAW two-point function should be compared with the *simple random walk* two-point function

$$S_z(x) := \sum_{n \geq 0} \sum_{\omega \in \mathsf{W}_n(x)} z^n \mathbb{P}_n[\omega], \tag{2.4}$$

in which the inner sum is over $\mathsf{W}_n(x)$, the set of all n-step walks with initial vertex $o \in \mathbb{Z}^d$ and terminal vertex $x \in \mathbb{Z}^d$. The term "simple" is used to indicate that the associated law on n-step walks is $\mathbb{P}_n$, although this may not be a nearest-neighbour walk. Let $|||x|||$ denote $\max\{\|x\|_2, 1\}$, where $\|\cdot\|_2$ is the Euclidean norm. Despite the notation, $|||\cdot|||$ is not a norm.

Theorem 2.3 *Let $d \geq 5$. For sufficiently spread-out step distributions, with parameter $L \geq L_0(D)$, there is a $\kappa_0 > 0$ such that if $0 \leq \kappa \leq \kappa_0$ and $\alpha > 0$ then*

$$G_{z_c,\kappa}(x) = \frac{a_d}{\sigma^2 |||x|||^{d-2}} \left(1 + O(L^{\alpha-2}) + O\left(\frac{L^2}{|||x|||^{2-\alpha}}\right)\right), \tag{2.5}$$

where σ^2 is the variance of the step distribution D, the constants implicit in the $O(\cdot)$ notation may depend on κ and α, and $a_d = 2^{-1}\pi^{-d/2} d\Gamma(\frac{d}{2} - 1)$. Here $\Gamma(\frac{d}{2} - 1)$ is the evaluation of Euler's Gamma function at $\frac{d}{2} - 1$.

Theorem 2.3 shows that the critical two-point function of κ-ASAW has the same asymptotics as the critical ($z = 1$) two-point function of simple random walk in

$d \geq 5$. In the language of critical exponents, see [4, p.12], this says that $\eta = 0$, i.e., this is a verification that κ-ASAW has mean-field behaviour. We have not attempted to optimize the relation between κ_0 and L in our proof, as our primary interest is in the existence of $\kappa_0 > 0$ for finite L.

It is typically difficult to apply the lace expansion to models containing attracting interactions, as these attractions make it difficult to obtain what are known as diagrammatic bounds. We are able to overcome this difficulty as the *on average repulsion* that κ-ASAW satisfies is compatible with calculating such bounds. This is discussed in more detail in Sect. 2.3. Once the diagrammatic bounds are obtained the remaining part of the lace expansion analysis is well understood and can be adapted from existing arguments [8]. We recall how this can be done in Appendix A. The proof of Theorem 2.3 is carried out in Sect. 6.

2.3 Main idea

The proofs of Theorems 2.1 and 2.3 are essentially independent, but they share a common idea which we explain here.

Let Γ denote the set of all self-avoiding walks, not necessarily starting at the origin o. Writing a walk ω as a concatenation $\omega = \omega^1 \circ \omega^2$ of two subwalks determines an interaction conditional on ω^1, i.e.,

$$\mathbb{1}_{\{\omega\in\Gamma\}} e^{-H_\kappa(\omega)} = \left(\mathbb{1}_{\{\omega^1\in\Gamma\}} e^{-H_\kappa(\omega^1)}\right)\left(\mathbb{1}_{\{\omega\in\Gamma\}} e^{-H_\kappa(\omega^2;\,\omega^1)}\right), \tag{2.6}$$

where this formula defines $H_\kappa(\cdot\,;\omega^1)$. Explicitly,

$$\exp(-H_\kappa(\omega^2;\omega^1)) := (1+\kappa)^{|\mathsf{adj}(\omega^2)|}(1+\kappa)^{|\mathsf{adj}(\omega^1,\,\omega^2)|}, \tag{2.7}$$

where $\mathsf{adj}(\omega^1,\,\omega^2)$ is the set of pairs of adjacent edges $\{f_1,\,f_2\}$ with $f_i \in \omega^i, i = 1, 2$.

For SAW, i.e., $\kappa = 0$, the interaction is trivial: $e^{-H_0(\omega)} = e^{-H_0(\omega^2;\omega^1)} = 1$. Submultiplicativity therefore follows from (2.6) and the observation that

$$\mathbb{1}_{\{\omega\in\Gamma\}} = \mathbb{1}_{\{\omega^1\circ\omega^2\in\Gamma\}} \leq \mathbb{1}_{\{\omega^2\in\Gamma\}}.$$

For $\kappa > 0$ it is not generally true that $e^{-H_\kappa(\eta;\,\omega^1)} \leq e^{-H_\kappa(\eta)}$. See Fig. 1 and consider splitting the walk into the indicated subwalks.

Equation (2.6) highlights a tension between self-avoidance and self-attraction. Energetic rewards of $(1+\kappa)$ due to the conditional interaction only occur if the walk ω^2 has edges adjacent to edges in ω^1. Such an edge in ω^2 carries an entropic penalty, as the potential configurations of ω^2 are reduced. Thus there is both an entropic benefit and an energetic penalty to dropping the conditional interaction due to ω^1 when (2.6) is summed over a suitable class of walks.

More explicitly, if ω^2 contains an edge $\{x_1, y_1\}$ adjacent to an edge $\{x_2, y_2\}$ of ω^1, then typically there is a self-avoiding modification of ω^2 that traverses $\{x_2, y_2\}$ instead of $\{x_1, y_1\}$. The modified walk will be longer than the original, and will be

assigned zero weight by the conditional interaction. However, it has positive κ-ASAW weight. The entropic gain of ignoring ω^1 can therefore be estimated by considering the possible modifications to ω^2 and estimating the energetic cost of the modifications. The energetic cost decreases as κ decreases, and for κ sufficiently small we will show that the entropic benefit outweighs the energetic penalty. This idea, which involves a weighted version of the multivalued map principle [9, Section 2.0.1], has been fruitful in obtaining upper bounds on the number of self-avoiding polygons of given length [9].

2.4 Discussion

Ueltschi [1] considered a model of SAWs with an attracting reward for pairs of nearest-neighbour vertices under the assumption that the step distribution $D(x)$ and attraction strength κ satisfy

$$\inf_{|x-y|=1,\, y\neq 0} \frac{D(y)}{D(x)} = \Delta > 0, \qquad (1+\kappa)^{2d} \leq 1 + \frac{\Delta^2}{2d(1+\kappa)^{2d-1}}. \tag{2.8}$$

Note that the condition on $D(x)$ in (2.8) implies the step distribution has infinite range. Given (2.8) it can be shown that the entropic reward of ignoring ω^1 outweighs the energetic cost. The fact that $D(x)$ has infinite range and is "smooth" allows the use of a length preserving transformation to prove the model is submultiplicative. Using this idea Ueltschi also carries out a lace expansion analysis via the inductive approach of [10]; the length-preserving nature of the transformation is important for the application of the inductive method. Ueltschi's result was significant for being the first application of the lace expansion to a self-attracting random walk. Self-attracting interactions, which are also called non-repulsive, are typically difficult to handle with lace expansion methods [4, Section 6.3].

The problem of analysing models of self-attracting self-avoiding walks under weaker hypotheses on the step distributions was raised by den Hollander [2, Chapter 4.8(5)], and our work addresses this question when $d \geq 5$. As described in Sect. 2.3 the main idea is to combine energy-entropy methods with classical techniques for self-avoiding walk.

Beyond our main theorems, an important aspect of this work is that it suggests that energy-entropy methods may be more generally useful in the context of the lace expansion. In particular there is no need to restrict to length-preserving transformations as in [1] (although length-preserving transformations do simplify technical aspects due to [10]). This is significant as energy-entropy methods should be a fairly robust way to overcome a lack of repulsion caused by weak attractions. Roughly speaking, the key step in such an argument is to first subdivide an object, and then to prove the gain in conformational freedom that arises when forgetting one part outweighs the loss of energetic attractions. Our proof implements this strategy for κ-ASAW, and it is plausible it could be implemented for other models, e.g., weakly self-attracting lattice trees in high dimensions via an adaptation of [8,11].

It is worth noting that energy-entropy arguments are carried out by finding a transformation that estimates the number of new configurations that are

available. Finding a transformation is a combinatorial and analytic problem, in contrast to other approaches to overcoming a lack of repulsion via correlation inequalities [4,12], resummation identities [13], or asymmetry assumptions [14].

We end this section by mentioning two recent related works on self-avoiding random walks subject to self-attraction. Firstly, there has been interesting progress [15] on den Hollander's problem for *weakly* self-avoiding walk (WSAW) with a contact self-attraction when $d = 4$. The authors prove Gaussian decay of the critical two-point function when the self-attraction and self-repulsion strengths are sufficiently small by making use of a rigorous renormalization group analysis. The techniques of [15] are wholly different than those of the present paper, and an analysis of self-attracting WSAW when $d \geq 5$ via lace expansion techniques would be a very interesting complement to the results of [15]. Secondly, in [16] it has been shown that a related model known as *prudent self-avoiding walk* undergoes a collapse transition in $d = 2$ when the self-attraction is strong enough.

3 Initial definitions, path transformations

3.1 Conventions

By a common abuse of notation $\mathbb{Z}^d$ will denote the d-dimensional hypercubic lattice, i.e., the graph with vertex set $\mathbb{Z}^d$ and edge set $E(\mathbb{Z}^d) := \{\{x, y\} \mid \|x - y\|_1 = 1\}$. Recall that the standard generators of $\mathbb{Z}^d$ will be denoted $e_1, \ldots, e_d$. $|A|$ will denote the cardinality of a finite set A, and $A \sqcup B$ will denote the union of disjoint sets A and B.

For $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$, an *n-step walk* is a sequence $(\omega_i)_{i=0}^n$, where $\omega_i \in \mathbb{Z}^d$ for $0 \leq i \leq n$ and $\omega_i \neq \omega_{i-1}$, $1 \leq i \leq n$. For such a walk let $|\omega| := n$, and for $0 \leq i < j \leq |\omega|$ we write $\omega_{[i,j]}$ to denote the walk $(\omega_i, \omega_{i+1}, \ldots, \omega_j)$. Let $\mathsf{W}_n(x, y)$ be the set of n-step walks with $\omega_0 = x$ and $\omega_n = y$. We omit the first argument if $x = o$, the origin of $\mathbb{Z}^d$, and let $\mathsf{W}(x, y) := \bigsqcup_{n \geq 0} \mathsf{W}_n(x, y)$. We also let W_n denote the set of all n-step walks, with no constraints on the initial or final vertices.

A walk is *self-avoiding* if $\omega_i \neq \omega_j$ for $i \neq j$, and is a *self-avoiding polygon* if $|\omega| > 2$, $\omega_i \neq \omega_j$ for $0 \leq i < j < |\omega|$, and $\omega_0 = \omega_{|\omega|}$. Note that polygons are rooted and oriented, which is a somewhat non-standard definition. Let $\Gamma_n(x, y)$ denote the set of n-step self-avoiding walks from x to y and $\tilde{\Gamma}_n(x)$ denote the set of n-step self-avoiding polygons with initial vertex x. For self-avoiding walks let $\Gamma_n(x) := \Gamma_n(o, x)$, and again we will omit the subscript n to indicate a union over n. Γ and $\tilde{\Gamma}$ denote the sets of all self-avoiding walks and polygons, respectively, with no restrictions on the initial vertex.

Let $\mathsf{adj}(A, B)$ denote the set of plaquettes spanned by pairs of adjacent edges $e \in A$, $f \in B$ for subsets $A, B \subset E(\mathbb{Z}^d)$, and let $\mathsf{adj}(A) := \mathsf{adj}(A, A)$. Let $E(\omega) := \{\{\omega_i, \omega_{i+1}\}\}_{i=0}^{|\omega|-1}$ be the set of edges traversed by a walk ω. By a slight abuse of notation we will write $\mathsf{adj}(\omega, \eta)$ in place of $\mathsf{adj}(E(\omega), E(\eta))$, and $\mathsf{adj}(\omega) := \mathsf{adj}(\omega, \omega)$.

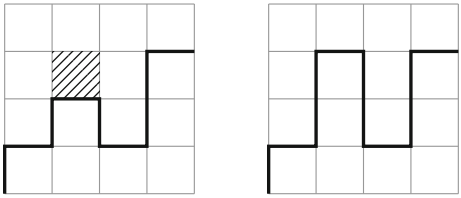


Fig. 2 Illustration of a flip applied to a self-avoiding walk ω at the shaded plaquette P, which is flippable

3.2 Transformations by symmetries of $\mathbb{Z}^d$; basic path operations

For $x \in \mathbb{Z}^d$, let $\mathcal{T}_x$ denote the operator of translation by x, i.e., $\mathcal{T}_x f(y) = f(y - x)$ for f a function on $\mathbb{Z}^d$. Translations will also act on subsets or collections of subsets of $\mathbb{Z}^d$ by identifying sets with indicator functions. For example, if $\omega \in \mathsf{W}_n$, $\mathcal{T}_x\, \omega$ is the n-step walk $(\omega_0 + x, \omega_1 + x, \ldots, \omega_n + x)$.

The projection operator $\pi_i : \mathbb{Z}^d \to \mathbb{Z}$ maps $x = (x_1, \ldots, x_d)$ to x_i. To lighten notation, let $\pi_i^{-1}(x) = \pi_i^{-1}(\pi_i(x))$ denote the hyperplane passing through x with normal e_i. The reflection operator $\mathcal{R}_i : \mathbb{Z}^d \to \mathbb{Z}^d$ reflects any vertex in the coordinate hyperplane $\pi_i^{-1}(o)$.

If $\omega^1 \in \mathsf{W}_m$ and $\omega^2 \in \mathsf{W}_n$ their *concatenation* $\eta = \omega^1 \circ \omega^2$ is the $(n+m)$-step walk with $\eta_i = \omega_i^1$ for $0 \le i \le m$, and $\eta_{m+i} = \mathcal{T}_{(\omega_m^1 - \omega_0^2)}\, \omega_i^2$ for $0 \le i \le n$. This translation moves the initial vertex of ω^2 to the terminal vertex of ω^1, so the concatenated walk continues from where ω^1 ends.

3.3 Flips

Let ω be a walk and P be a plaquette such that there is a unique i such that $\omega_j \in P$ iff $j \in \{i, i+1\}$. We will call such a plaquette P *flippable*. This condition implies that ω has exactly one edge in the plaquette P, and no other vertices in P. Otherwise P is *not flippable*. Since flippability is defined in terms of ω we will write, for example, *flippable for* ω to indicate this dependence.

Suppose P is a flippable plaquette for ω, and that (ω_i, ω_{i+1}) is the unique edge of ω in P. The *flip of* ω *at* P, denoted $\mathcal{F}_P(\omega)$, is the walk ω' that replaces (ω_i, ω_{i+1}) with the traversal of P along the three edges distinct from $\{\omega_i, \omega_{i+1}\}$. See Fig. 2. If P is not flippable, define $\mathcal{F}_P(\omega) = \omega$. Two plaquettes P_1 and P_2 are said to be *disjoint* if they have no vertices in common. Sets of disjoint flippable plaquettes are what will be entropically important in what follows. The next three lemmas establish useful properties of $\mathcal{F}_P$.

Lemma 3.1 *For any plaquette P and vertices $x, y \in \mathbb{Z}^d$, $\mathcal{F}_P : \mathsf{W}(x, y) \to \mathsf{W}(x, y)$, $\mathcal{F}_P : \Gamma(x, y) \to \Gamma(x, y)$, and $\mathcal{F}_P$ is invertible. If P' is disjoint from P, then $\mathcal{F}_P \circ \mathcal{F}_{P'} = \mathcal{F}_{P'} \circ \mathcal{F}_P$.*

Proof To prove $\mathcal{F}_P : \mathsf{W}(x, y) \to \mathsf{W}(x, y)$ it suffices to prove that $\mathcal{F}_P$ does not change the endpoints of a walk. This is immediate as the first and last vertices of $\mathcal{F}_P(\omega)$ in P are the same as the first and last vertices of ω in P.

If $\omega \in \Gamma(x, y)$ and P is not flippable for ω, then $\mathcal{F}_P(\omega) = \omega$, so the image is in $\Gamma(x, y)$. If P is flippable, then ω contains one edge of P and no other vertices; since $\mathcal{F}_P$ only modifies ω on P the result is self-avoiding.

Invertibility of $\mathcal{F}_P$ is clear, as if P is not flippable for ω then $\mathcal{F}_P$ is the identity, while if P is flippable then ω can be recovered from $\mathcal{F}_P(\omega)$ by replacing the traversal of three consecutive edges of P in $\mathcal{F}_P(\omega)$ by the traversal of the single unoccupied edge of P. Lastly, commutativity holds as flips at disjoint plaquettes modify the walk on disjoint sets of edges. □

Given a disjoint set of plaquettes $B = \{P_1, \ldots, P_k\}$, define $\mathcal{F}_B(\omega) := \mathcal{F}_{P_1} \circ \ldots \circ \mathcal{F}_{P_k}(\omega)$; the commutativity of flips at disjoint plaquettes implies the definition of $\mathcal{F}_B$ is unambiguous.

Lemma 3.2 *Let $\eta \circ \omega \in \Gamma$ and let $B \subset \mathsf{adj}(\eta, \omega)$ be a disjoint set of plaquettes that are flippable for ω. Then ω is uniquely determined by $\mathcal{F}_B(\omega)$ and η.*

Proof Since the plaquettes in B are disjoint, we can consider them separately. For each $P \in B$, there is at least one edge of P in $\tilde{\omega} = \mathcal{F}_P(\omega)$ that is also in η. Suppose there is one edge $(\tilde{\omega}_i, \tilde{\omega}_{i+1})$. Then P is the plaquette $\{\tilde{\omega}_{i-1}, \tilde{\omega}_i, \tilde{\omega}_{i+1}, \tilde{\omega}_{i+2}\}$. If there are two edges of $\tilde{\omega}$ in η then P is the plaquette spanned by the two edges, and three or four edges being in η contradicts $\eta \circ \omega \in \Gamma$.

Thus, given η and $\tilde{\omega}$, we can determine B. By Lemma 3.1 $\mathcal{F}_B$ is invertible, so ω is uniquely determined. □

Suppose a self-avoiding walk η is composed of two subwalks ω^1 and ω^2, i.e., $\eta = \omega^1 \circ \omega^2$. It will be convenient to abuse notation and write $\mathsf{adj}(\omega^1, \omega^2)$ in place of $\mathsf{adj}(\omega^1, \mathcal{T}_x\omega^2)$, where $\mathcal{T}_x$ is the translation that takes the initial vertex of ω^2 to the final vertex of ω^1. As it will be contextually clear we are discussing pairs of adjacent edges between ω^1 and ω^2, this should not cause any confusion.

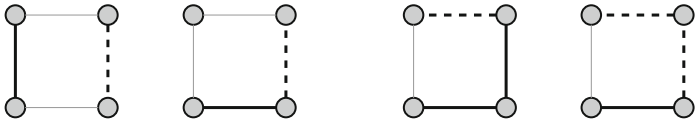
The next lemma says most plaquettes in $\mathsf{adj}(\omega^1, \omega^2)$ are flippable for ω^2; to quantify this we define

$$k_0 := 2d(d-1). \tag{3.1}$$

Lemma 3.3 *If $\omega^1 \circ \omega^2 \in \Gamma$, there are at most k_0 plaquettes in $\mathsf{adj}(\omega^1, \omega^2)$ that are not flippable for ω^2. If $\omega^1 \circ \omega^2 \in \tilde{\Gamma}$, there are at most $2k_0$ plaquettes in $\mathsf{adj}(\omega^1, \omega^2)$ that are not flippable for ω^2.*

Proof Without loss of generality, assume that ω_0^2 is the endpoint of ω^1, and call the vertices in common to ω^2 and ω^1 *points of concatenation*. The proof characterises when $P \in \mathsf{adj}(\omega^1, \omega^2)$ is not flippable for ω^2 case by case, depending on how many edges of P are contained in $\omega^1 \circ \omega^2$. By the definition of $P \in \mathsf{adj}(\omega^1, \omega^2)$ there are at least two such edges.

First, note that a self-avoiding walk or polygon containing four edges in a single plaquette is a four step self-avoiding polygon. The claim is true in this case, as there are exactly two adjacent pairs of edges. Henceforth we may assume there are no plaquettes containing four edges.



(a) P contains 2 edges of $\omega^1 \circ \omega^2$. **(b)** P contains 3 edges of $\omega^1 \circ \omega^2$.

Fig. 3 Illustration of the cases arising in the proof of Lemma 3.3. Solid black lines represent edges of ω^1, while dashed black lines represent edges of ω^2

Suppose that ω^1 and ω^2 each contain exactly one edge of P. If P does not contain a point of concatenation, then P is flippable for ω^2 since the two vertices of P in ω^1 are not in ω^2. See Fig. 3a. If P contains a point of concatenation it may or may not be flippable for ω^2.

Suppose that $\omega^1 \circ \omega^2$ contains three edges of P. Note that the three edges must occur sequentially in $\omega^1 \circ \omega^2$ since this walk is self-avoiding or a self-avoiding polygon, and hence P must contain a point of concatenation if it is to be in $\mathsf{adj}(\omega^1, \omega^2)$. If two edges belong to ω^1, then P is flippable for ω^2. Otherwise, P is not flippable for ω^2. See Fig. 3b.

Thus $P \in \mathsf{adj}(\omega^1, \omega^2)$ and P not being flippable implies there is a point of concatenation in P. As there are at most two points of concatenation, this verifies the claim, as each vertex of $\mathbb{Z}^d$ is contained in k_0 plaquettes. □

The cost $p_1^2 = \frac{\mathbb{P}_{n+2}(\omega')}{\mathbb{P}_n(\omega)}$ is the additional cost of the modified walk ω' according to the *a priori* measure $\mathbb{P}$. Recall the definition of $\mathcal{W}_\kappa$ in (1.2). The next lemma is our basic estimate for the energetic penalty of a flip. For future reference, define

$$\mathrm{flip}_\kappa := p_1^{-2}(1+\kappa)^{2d-4}. \tag{3.2}$$

Lemma 3.4 *Let ω be a self-avoiding walk and P a flippable plaquette. Let $\omega' = \mathcal{F}_P(\omega)$. Then*

$$\frac{\mathcal{W}_\kappa(\omega')}{\mathcal{W}_\kappa(\omega)} \geq \mathrm{flip}_\kappa^{-1}. \tag{3.3}$$

Proof The factor of p_1^2 comes from comparing the *a priori* measures in the definition of $\mathcal{W}_\kappa$. What remains is to bound the difference $|\mathsf{adj}(\omega)| - \left|\mathsf{adj}(\omega')\right|$.

The flip creates at least one pair of adjacent edges in ω' that was not present in ω, namely the pair of adjacent edges of ω' in P. There are $2d-2$ edges that are potentially adjacent to the unique edge of ω in P, and the hypothesis of P being flippable for ω implies at least one of these edges is not in ω. Thus at most $2d-3$ adjacent pairs of edges in ω are not present in ω'. This proves (3.3). □

In what follows it will be necessary to flip many plaquettes. Lemma 3.1 guarantees the result will be self-avoiding if the flipped plaquettes are disjoint. The next lemma guarantees that every collection of plaquettes has a positive density subset of disjoint plaquettes. Define $\alpha = \alpha(d)$ by

$$\alpha(d) := \frac{1}{1 + 8(d-1)^2}. \tag{3.4}$$

Lemma 3.5 *Given a finite set A of plaquettes in $\mathbb{Z}^d$, there exists a subset of pairwise disjoint plaquettes of A of size $\lceil \alpha\, |A| \rceil$.*

Proof The constant α is $(1+R)^{-1}$, where R is the number of plaquettes $P' \neq P$ sharing a vertex with P. The remainder of the proof verifies the value of R claimed in (3.4) by performing inclusion-exclusion on the number of vertices $P' \neq P$ shares with P; note that this number is at most two.

Any vertex x is contained in exactly $4\binom{d}{2}$ plaquettes. To see this, note these plaquettes are in bijection with sets $\{(u_i, \sigma_i)\}_{i=1,2}$, where $u_1 \neq u_2$ are distinct generators of $\mathbb{Z}^d$ and $\sigma_i \in \{\pm 1\}$: these sets identify the unique plaquette containing $x, x+\sigma_1 u_1, x+\sigma_2 u_2$. Since every edge belongs to exactly $2(d-1)$ plaquettes, the total number of plaquettes sharing a vertex with a plaquette P is therefore

$$R = 4\left((4\binom{d}{2} - 1) - (2(d-1) - 1) \right) = 8(d-1)^2,$$

where the factors of -1 correct for the presence of P in our counts. □

4 Existence of the connective constant: proof of Theorem 2.1

4.1 Half-space walks and bridges

We begin by recalling the basic definitions used in the Hammersley-Welsh argument.

Definition 4 The set B_n of *n-step bridges* is the subset of $\omega \in \Gamma_n$ such that

$$0 = \pi_1(\omega_0) < \pi_1(\omega_i) \leq \pi_1(\omega_n), \qquad 1 \leq i \leq n. \tag{4.1}$$

The set H_n of *n-step half-space walks* is the set of $\omega \in \Gamma_n$ such that

$$0 = \pi_1(\omega_0) < \pi_1(\omega_i), \qquad 1 \leq i \leq n. \tag{4.2}$$

Let $\mathsf{H} := \sqcup_{n \geq 0} \mathsf{H}_n$ and $\mathsf{B} := \sqcup_{n \geq 0} \mathsf{B}_n$. The *masses* $h_n(\kappa)$ of n-step half-space walks and $b_n(\kappa)$ of half-space bridges are defined by

$$h_n(\kappa) := \sum_{\omega \in \mathsf{H}_n} \mathcal{W}_\kappa(\omega), \qquad b_n(\kappa) := \sum_{\omega \in \mathsf{B}_n} \mathcal{W}_\kappa(\omega). \tag{4.3}$$

Note that the inclusions $\mathsf{B}_n \subset \mathsf{H}_n \subset \Gamma_n$ imply that $b_n(\kappa) \leq h_n(\kappa) \leq c_n(\kappa)$.

While self-attraction on adjacent edges ruins the submultiplicativity of self-avoiding walks, it enhances the supermultiplicativity of bridges.

Proposition 4.1 *Let $\kappa \geq 0$. The limit $\mu_B(\kappa) = \lim_{n\to\infty} (b_n(\kappa))^{1/n}$ exists and is equal to $\sup_{n \geq 1} \left(b_n(\kappa)\right)^{1/n}$.*

Proof The definition of a bridge implies that the concatenation $\omega^1 \circ \omega^2$ of two bridges ω^1 and ω^2 is a bridge. Each pair of adjacent edges in ω^i remains adjacent in $\omega^1 \circ \omega^2$. Any other pair of adjacent edges in the concatenation receives weight $1 + \kappa \geq 1$, so

$$\sum_{\omega^1 \in \mathsf{B}_{n_1}} \sum_{\omega^2 \in \mathsf{B}_{n_2}} \mathcal{W}(\omega^1)\mathcal{W}(\omega^2) \leq \sum_{\eta \in \mathsf{B}_{n_1+n_2}} \mathcal{W}(\eta) \mathbb{1}_{\{\eta = \omega^1 \circ \omega^2, \omega^i \in \mathsf{B}_{n_i}\}}$$

for all choices of $n_1, n_2 \in \mathbb{N}$. The left-hand side is $b_{n_1}(\kappa) b_{n_2}(\kappa)$. Ignoring the indicator on the right-hand side gives an upper bound $b_{n_1+n_2}(\kappa)$. Thus $b_n(\kappa)$ is supermultiplicative, and the proposition follows by Fekete's lemma. □

4.2 Unfolding I. Classical unfolding

This section recalls how half-space walks can be unfolded into a concatenation of bridges. We do this because a multivalued extension of this procedure will be introduced in the next section. We omit the proofs of the various facts that we recall; for details see [6, Section 3.1].

Definition 5 Let $\omega \in \mathsf{H}$. The *(first) bridge point* $\tau(\omega)$ of ω is the maximal index i satisfying $\pi_1(\omega_i) = \max_j \pi_1(\omega_j)$.

Definition 6 The *span* of a self-avoiding walk is

$$\mathrm{span}(\omega) := \max_j \pi_1(\omega_j) - \min_j \pi_1(\omega_j). \tag{4.4}$$

Note that if ω is an n-step bridge then $\mathrm{span}(\omega) = \pi_1(\omega_n)$.

Given a half-space walk $\omega \in \mathsf{H}_n$, let $x := -\omega_{\tau(\omega)}$ and define the *initial bridge* $\omega^b := \omega_{[0,\tau(\omega)]}$ and the *remainder* $\omega^h := \mathcal{T}_x\, \omega_{[\tau(\omega),n]}$. We will write $\omega = (\omega^b, \omega^h)$ in what follows to indicate this decomposition into an initial bridge and a remainder. The following properties of the decomposition are important.

(i) $\omega = \omega^b \circ \omega^h$,
(ii) $\mathcal{R}_1(\omega^h)$ is a half-space walk,
(iii) $\mathrm{span}(\omega^b) = \mathrm{span}(\omega)$, and
(iv) $\mathrm{span}(\mathcal{R}_1(\omega^h)) < \mathrm{span}(\omega^b)$, as ω never revisits the coordinate hyperplane $\pi_1^{-1}(o)$ after ω_0.

Definition 7 The *classical unfolding map* $\Psi : \mathsf{H} \to \mathsf{B}$ is recursively defined as follows. Ψ is the identity map on B. Otherwise, if $\omega \in \mathsf{H} \backslash \mathsf{B}$, let $\omega = (\omega^b, \omega^h)$ and define $\Psi(\omega) = \omega^b \circ \Psi(\mathcal{R}_1(\omega^h))$.

In words, Ψ reflects the remainder ω^h of the walk ω through $\pi_1^{-1}(\omega_{\tau(\omega)})$, the affine hyperplane with normal e_1 that contains the endpoint $\omega_{\tau(\omega)}$ of ω^b. Since the reflection

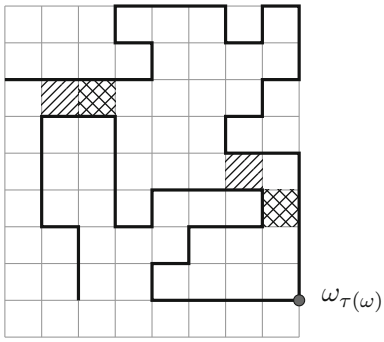


Fig. 4 The figure depicts a half-space walk, along with its decomposition into ω^b and ω^h. Shaded plaquettes are in $\mathsf{adj}(\omega^b, \omega^h)$; crosshatched plaquettes indicate a choice of subset of $\mathsf{adj}^\star(\omega^b, \omega^h)$

of ω^h is itself a half-space walk, this procedure can be iterated until the first bridge point of the newest half-space walk is also the endpoint of the walk. The recursion terminates at some depth $r = r(\omega)$ as the spans of the half-space walks produced are strictly decreasing. See Figs. 4 and 5 for one step of this procedure (the meaning of the shaded plaquettes will be explained in the next section).

Thus, Ψ produces a sequence of bridges ω^{b_i}, $i = 1, \ldots, r$, and $\Psi(\omega) = \omega^{b_1} \circ \cdots \circ \omega^{b_r}$. This sequence of bridges is called the *classical bridge decomposition* of ω. In what follows, the bridges in the classical bridge decomposition will always be denoted by $\{\omega^{b_i}\}_{i=1}^r$. Let us record some properties of this decomposition.

Proposition 4.2 (i) $\mathrm{span}(\Psi(\omega)) = \sum_{i=1}^r \mathrm{span}(\omega^{b_i})$,
(ii) *the sequence* $(\mathrm{span}(\omega^{b_i}))_{i=1}^r$ *of spans is strictly decreasing in* i, *and*
(iii) *given* $\Psi(\omega)$ *and* $\{\mathrm{span}(\omega^{b_i})\}_{i=1}^r$, ω *is uniquely determined.*

Again we will not prove these claims, but let us remark that the third property holds as knowing the lengths of the spans indicates the locations at which to fold the bridge $\Psi(\omega)$ in order to undo the unfolding procedure.

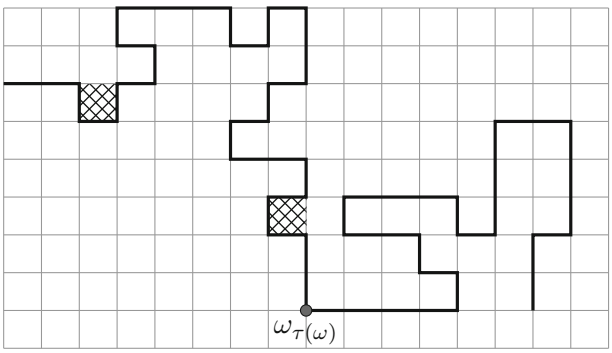


Fig. 5 The figure depicts the image in $\Phi(\omega)$ of the half-space walk ω depicted in Fig. 5, when the subset B of plaquettes at which flips occur is the set of crosshatched plaquettes

4.3 Unfolding II. Multivalued unfolding

This section describes a multivalued extension of the classical unfolding map. Roughly speaking, this multivalued extension quantifies an entropic gain in unfolding a half-space walk ω into bridges. The gain arises because the unfolded walk may have fewer adjacent edges than the original walk ω.

Definition 8 The *marked unfolding map* $\bar{\Phi}$ is recursively defined on H by

$$\bar{\Phi}(\omega) := \Big((\omega^b, \mathsf{adj}_1(\omega^b, \omega^h)), \bar{\Phi}(\mathcal{R}_1(\omega^h))\Big), \tag{4.5}$$

$$\mathsf{adj}_1(\omega^b, \omega^h) := \{P \in \mathsf{adj}(\omega^b, \omega^h) \mid P \text{ flippable for } \omega^b\}. \tag{4.6}$$

Let $r := r(\omega)$ denote the number of bridges generated by this recursion. The image of $\bar{\Phi}(\omega)$ will be denoted $((\omega^{b_i}, \mathsf{adj}_i))_{i=1}^r$, where $\mathsf{adj}_r := \emptyset$ and for $1 \le i < r$ we have used adj_i as shorthand for the set of plaquettes defined by (4.6) in the ith step of the recursion.

The marked unfolding map is an extension of Ψ. It records the plaquettes at which ω^{b_i} is flippable with respect to the remainder of the half-space walk with initial bridge ω^{b_i}. Denote the set of discovered flippable plaquettes by $\mathsf{adj}_{\bar{\Phi}}(\omega)$, i.e.,

$$\mathsf{adj}_{\bar{\Phi}}(\omega) := \bigsqcup_i \mathcal{T}_i \, \mathsf{adj}_i,$$

where $\mathcal{T}_i$ is the translation that translates the bridge ω^{b_i} to its location in the bridge $\Psi(\omega)$.

For $k \in \mathbb{N}$, let $\mathsf{H}_n^k \subset \mathsf{H}_n$ be the set of n-step half-space walks with $\left|\mathsf{adj}_{\bar{\Phi}}(\omega)\right| = k$. To define the multivalued extension of Ψ on H_n^k, we will make use of the following facts. First, Lemma 3.5 implies there exists a subset $\mathsf{adj}^\star_{\bar{\Phi}}(\omega) \subset \mathsf{adj}_{\bar{\Phi}}(\omega)$ of size $\lceil \alpha k \rceil$ such that the plaquettes in $\mathsf{adj}^\star_{\bar{\Phi}}$ are pairwise disjoint. In what follows we will assume the set $\mathsf{adj}^\star_{\bar{\Phi}}$ has been chosen according to some arbitrary (but definite) procedure. Second, Lemma 3.1 implies that if B is a finite collection of pairwise vertex-disjoint plaquettes and $x \in \mathbb{Z}^d$, then $\mathcal{F}_B = \prod_{P \in B} \mathcal{F}_P$ is an unambiguously defined map from $\Gamma(x)$ to $\Gamma(x)$.

Definition 9 Let $0 < \delta < \frac{1}{2}$, $n \in \mathbb{N}$, and $0 \le k \le n$. The *multivalued unfolding map* $\Phi\colon \mathsf{H}_n^k \to 2^{\mathsf{B}_{n+2\lceil \alpha\delta k \rceil}}$ is defined by

$$\Phi(\omega) := \left\{ \omega' \,\middle|\, \omega' = \mathcal{F}_B(\Psi(\omega)),\, B \in \binom{\mathsf{adj}^\star_{\bar{\Phi}}(\omega)}{\lceil \delta\alpha k \rceil} \right\}, \tag{4.7}$$

where $\binom{A}{k}$ denotes the k-element subsets of a set A.

The next lemma gives the basic properties of Φ; in particular it verifies the stated codomain in the previous definition. To lighten the notation, define

$$\alpha_k := \lceil \alpha k \rceil, \qquad \delta_k := \lceil \delta\alpha k \rceil. \tag{4.8}$$

Lemma 4.3 *Let $\omega \in H_n^k$. Then*

(i) *Every $P \in \mathsf{adj}_{\bar{\Phi}}(\omega)$ is flippable for $\Psi(\omega)$.*
(ii) *Each walk $\omega' \in \Phi(\omega)$ is a bridge in $\mathsf{B}_{n+2\delta_k}$.*
(iii) *The half-space walk ω can be reconstructed from any $\omega' \in \Phi(\omega)$ given the spans* $\mathrm{span}(\omega^{b_i})$.

Proof Recall $\omega = \omega^{b_1} \circ \omega^h$. We begin by noting some properties of the plaquettes in $\mathsf{adj}(\omega^{b_1}, \omega^h)$. Since ω is a half-space walk, no plaquette in $\mathsf{adj}(\omega^{b_1}, \omega^h)$ contains vertices in the half-space $\pi_1^{-1}((-\infty, 0])$. Similarly, no plaquette contains vertices in the half-space $\pi_1^{-1}(\big[\mathrm{span}(\omega^{b_1}) + 1, \infty\big))$, as ω^h is contained in $\pi_1^{-1}(\big[1, \mathrm{span}(\omega^{b_1})\big])$.

We first prove (i). A moment of thought shows that each plaquette in $B \subset \mathsf{adj}_1(\omega^{b_1}, \omega^h) \subset \mathsf{adj}^\star_{\bar{\Phi}}(\omega)$ is flippable for $\omega^{b_1} \circ \mathcal{R}_1(\omega^h)$. Iterating this argument for each bridge in the bridge decomposition of ω implies each plaquette in $\mathsf{adj}_{\bar{\Phi}}(\omega)$ is flippable for $\Psi(\omega)$.

We next prove (ii). The preceding shows each $\omega' \in \Phi(\omega)$ is given by $\omega' = \mathcal{F}_B(\omega) = \mathcal{F}_B(\omega^{b_1}) \circ \cdots \circ \mathcal{F}_B(\omega^{b_r})$ for $B \subset \mathsf{adj}^\star_{\bar{\Phi}}(\omega)$. This is because each plaquette in B contains exactly one edge of $\Psi(\omega)$, and this edge is located in exactly one subwalk. By Lemma 3.1 and the first paragraph of the proof, each $\omega^j = \mathcal{F}_B(\omega^{b_j})$ is a bridge with span $\mathrm{span}(\omega^{b_j})$ for $j = 1, \ldots, r$. As a concatenation of bridges is a bridge and each flip adds exactly two edges, this completes the proof, as each ω' results from applying $\mathcal{F}_B$ with $|B| = \delta_k$.

Lastly we prove (iii). By construction, the last bridge ω^{b_r} in the classical bridge decomposition has no flips applied to it in the formation of ω^r. Given $\mathrm{span}(\omega^j) = \mathrm{span}(\omega^{b_j})$ for each j, $\omega^r = \omega^{b_r}$ is determined. Hence, by Lemma 3.2, $\omega^{b_{r-1}}$ can be reconstructed. Iterating this procedure reconstructs ω from ω', establishing (iii). □

Having established the basic properties of the multivalued unfolding map, we turn to estimating the weight of n-step half-space walks in terms of the weights of bridges. The next lemma gives the basic relation between these objects.

For $n \in \mathbb{N}$, let $P(n)$ denote the number of partitions of n into distinct natural numbers. Let $\mathsf{B}_n(\ell)$ denote the set of n-step bridges ω with $\mathrm{span}(\omega) \le \ell$, and recall the definition of flip_κ from (3.2).

Proposition 4.4 *Consider SAW on $\mathbb{Z}^d$. For $n, k \in \mathbb{N}$ and $0 \le k \le n$,*

$$\sum_{\omega \in H_n^k} \mathcal{W}_\kappa(\omega) \le P(n) \binom{\alpha_k}{\delta_k}^{-1} \mathrm{flip}_\kappa^{\delta_k} \sum_{\omega' \in \mathsf{B}_{n+2\delta_k}(n)} (1+\kappa)^{k+3d(d-1)\sqrt{n}} \mathcal{W}_\kappa(\omega'). \tag{4.9}$$

Proof We begin by comparing the weight of $\omega \in \mathsf{H}_n^k$ to the weight of a generic $\omega' \in \Phi(\omega)$. Recall that $k_0 = 2d(d-1)$, and that $r = r(\omega)$ is the number of bridges created by the unfolding map.

(i) At each unfolding, there are at most k_0 plaquettes that are not flippable for ω^b in $\mathsf{adj}(\omega^b, \omega^h)$ by Lemma 3.3. Hence,

$$\mathcal{W}_\kappa(\omega) \le (1+\kappa)^{k+rk_0} \mathcal{W}_\kappa(\omega^{b_1} \circ \cdots \circ \omega^{b_r}),$$

as there are r unfolding steps.

(ii) As $\omega' = \mathcal{F}_B(\omega^{b_1} \circ \cdots \circ \omega^{b_r})$ and $|B| = \delta_k$, applying Lemma 3.4 δ_k times implies

$$\mathcal{W}_\kappa(\omega^{b_1} \circ \cdots \circ \omega^{b_r}) \leq \mathrm{flip}_\kappa^{\delta_k} \mathcal{W}_\kappa(\omega').$$

This implies

$$\mathcal{W}_\kappa(\omega) \leq (1+\kappa)^{k+r(\omega)k_0} \mathrm{flip}_\kappa^{\delta_k} \mathcal{W}_\kappa(\omega'). \tag{4.10}$$

Next we apply the multivalued map principle. Let $\Phi(H_n^k) = \cup_{\omega \in H_n^k} \Phi(\omega)$.

$$\sum_{\omega \in \mathsf{H}_n^k} \mathcal{W}_\kappa(\omega)\, |\Phi(\omega)| = \sum_{\omega \in \mathsf{H}_n^k} \sum_{\omega' \in \Phi(\omega)} \mathcal{W}_\kappa(\omega) \tag{4.11}$$

$$= \sum_{\omega' \in \Phi(\mathsf{H}_n^k)} \sum_{\omega \in \Phi^{-1}(\omega')} \mathcal{W}_\kappa(\omega) \tag{4.12}$$

$$\leq \sum_{\omega' \in \Phi(\mathsf{H}_n^k)} \sum_{\omega \in \Phi^{-1}(\omega')} (1+\kappa)^{k+r(\omega)k_0} \mathrm{flip}_\kappa^{\delta_k} \mathcal{W}_\kappa(\omega'). \tag{4.13}$$

where the inequality is by (4.10). To make the inner sum uniform in ω, we use Proposition 4.2. The spans of the bridges in the bridge decomposition of ω are distinct positive integers summing to $\mathrm{span}(\Psi(\omega)) \leq n$. This implies that $r(\omega)$ is at most $\frac{3}{2}\sqrt{n}$, as the sum of the first $\frac{3}{2}\sqrt{n}$ positive integers exceeds n. Hence

$$\sum_{\omega \in \mathsf{H}_n^k} |\Phi(\omega)|\, \mathcal{W}_\kappa(\omega) \leq \sum_{\omega' \in \Phi(\mathsf{H}_n^k)} \left|\Phi^{-1}(\omega')\right| (1+\kappa)^{k+\frac{3}{2}\sqrt{n}k_0} \mathrm{flip}_\kappa^{\delta_k} \mathcal{W}_\kappa(\omega'). \tag{4.14}$$

Next we estimate $|\Phi(\omega)|$ and $\left|\Phi^{-1}(\omega')\right|$.

(i) Lemma 4.3 implies that

$$|\Phi(\omega)| = \binom{\alpha_k}{\delta_k}. \tag{4.15}$$

Note that this is the number of subsets B of $\mathsf{adj}^\star_{\bar{\Phi}}(\omega)$ of size δ_k. The claim is true as (i) all plaquettes in B are flippable for $\Psi(\omega)$, and (ii) every distinct choice of a subset B in the definition of Φ results in a distinct image.

(ii) By Lemma 4.3 (iii) we can reconstruct ω from an image $\omega' \in \Phi(\omega)$ given the sequence of spans in the classical bridge decomposition of ω. The maximal possible span of $\Psi(\omega)$, the bridge produced by the classical bridge decomposition, is n, and by Proposition 4.2 the spans form a partition of $\mathrm{span}(\Psi(\omega))$. Thus the number of preimages $\left|\Phi^{-1}(\omega')\right|$ is at most $P(n)$, the number of partitions of n.

This establishes (4.9) if the index set $\mathsf{B}_{n+2\delta_k}(n)$ of the sum on the right-hand side is replaced with $\Phi(\mathsf{H}_n^k)$. By Lemma 4.3 (ii), $\Phi(\mathsf{H}_n^k) \subset \mathsf{B}_{n+2\delta_k}(n)$, as $\mathrm{span}(\omega') = \mathrm{span}(\Psi(\omega)) \leq n$. The proposition thus follows as each summand is non-negative. □

Lemma 4.5 *For κ sufficiently small, there are $\delta > 0$, $K > 0$, and $a > 0$ such that*

$$\binom{\alpha_k}{\delta_k}^{-1} (\mathrm{flip}_\kappa \mu_{\mathsf{B}}^2)^{\delta_k} (1+\kappa)^k \le K e^{-ak} \tag{4.16}$$

for all $k \in \mathbb{N}$.

Proof We will show that for $0 < \delta < \frac{1}{2}$ small enough there is a κ' such that for $\kappa \in [0, \kappa']$, there are $a, K > 0$ such that (4.16) holds. This implies the statement of the lemma. Since it suffices to prove the validity of (4.16) for $k > k' := (\delta\alpha)^{-1}$, we will restrict attention to such k.

We begin by estimating the combinatorial prefactor in (4.16). Recall the definition of α_k and δ_k in (4.8). By using (i) $\binom{n}{k} \ge (n/k)^k$ for $1 \le k \le n$, (ii) $\max\{1, x\} \le \lceil x \rceil \le x + 1$ for $x > 0$, and (iii) $k > k'$ and $\delta < \frac{1}{2}$, we obtain

$$\binom{\alpha_k}{\delta_k}^{-1} \le \left(\frac{\delta_k}{\lceil \alpha k \rceil}\right)^{\delta_k} \le (\delta + (\alpha k)^{-1})^{\delta_k} \le (2\delta)^{\delta\alpha k}. \tag{4.17}$$

Thus, when $k > k'$, the left-hand side of (4.16) is bounded above by

$$\left[2\delta (\mathrm{flip}_\kappa \mu_{\mathsf{B}}^2)^{\frac{\lceil \delta\alpha k \rceil}{\delta\alpha k}} (1+\kappa)^{(\delta\alpha)^{-1}}\right]^{\delta\alpha k}, \tag{4.18}$$

and the claim will follow by showing that the quantity in square brackets is strictly less than one.

By Proposition 4.1, $\mu_{\mathsf{B}}(\kappa) = \sup_n \left(b_n(\kappa)\right)^{1/n}$, and this latter quantity is at most $\sup_n \left(c_n(\kappa)\right)^{1/n}$. This, in turn, is bounded above by $(1+\kappa)^{2d-2}$, as each edge in a self-avoiding walk can be adjacent to at most $2d - 2$ others. Since $\delta\alpha k > 1$ when $k > k'$, we can bound above the bracketed quantity of (4.18) by

$$2\delta (\mathrm{flip}_\kappa (1+\kappa)^{4(d-1)})^2 (1+\kappa)^{(\delta\alpha)^{-1}}. \tag{4.19}$$

Recalling the definition (3.2) of flip_κ, it follows that when $\kappa = 0$ the quantity in (4.19) is strictly less than one, for $\delta = \delta(p_1)$ sufficiently small. Since the expression in (4.19) is continuous in κ for δ fixed, it is strictly less than one for small positive κ. This proves the claim. □

The next theorem is a consequence of a stronger result due to Hardy and Ramanujan; it will be needed to estimate the mass of n-step half-space walks.

Theorem 4.6 (Hardy–Ramanujan [17]) *Let $P(n)$ denote the number of partitions of n into distinct parts. Then*

$$\log P(n) \sim \pi \sqrt{\frac{n}{3}}, \qquad n \to \infty, \tag{4.20}$$

meaning that the ratio of the two sides tends to 1 as $n \to \infty$.

Proposition 4.7 *Consider SAW on $\mathbb{Z}^d$. For κ sufficiently small, there are $a_1, K_1 > 0$ such that*

$$h_n(\kappa) = \sum_{\omega \in \mathsf{H}_n} \mathcal{W}_\kappa(\omega) \le K_1 e^{a_1\sqrt{n}} \mu_{\mathsf{B}}^n. \tag{4.21}$$

Proof By Lemma 4.4 and Lemma 4.5,

$$\sum_{\omega \in \mathsf{H}_n^k} \mathcal{W}_\kappa(\omega) \le K e^{-ak} \mu_{\mathsf{B}}^{-2\delta_k} (1+\kappa)^{3d(d-1)\sqrt{n}} P(n) \sum_{\omega' \in \mathsf{B}_{n+2\delta_k}(n)} \mathcal{W}_\kappa(\omega').$$

By Proposition 4.1, $b_\ell(\kappa) \le \mu_{\mathsf{B}}(\kappa)^\ell$. Hence, dropping the constraint that the spans of bridges on the right-hand side of (4.3) are at most n yields

$$\sum_{\omega \in \mathsf{H}_n^k} \mathcal{W}_\kappa(\omega) \le K e^{-ak} (1+\kappa)^{3d(d-1)\sqrt{n}} P(n) \mu_{\mathsf{B}}^n. \tag{4.22}$$

The right-hand side of (4.22) is summable in k, and the left-hand side sums to $h_n(\kappa)$. By Theorem 4.6, $(1+\kappa)^{3d(d-1)\sqrt{n}} P(n)$ is at most $K_1 e^{a_1\sqrt{n}}$ for constants $K_1, a_1 > 0$; this completes the proof. □

Proof of Theorem 2.1 To prove $\mu(\kappa) = \mu_{\mathsf{B}}(\kappa)$, it suffices to prove that there exist K' and a' such that

$$b_n(\kappa) \le c_n(\kappa) \le K' e^{a'\sqrt{n}} \mu_{\mathsf{B}}(\kappa)^n. \tag{4.23}$$

For $\omega \in \Gamma_n$, let m be the maximal i such that $\pi_1(\omega_i)$ is minimized. Let $\omega^1 = \omega_{[0,m]}$ and $\omega^2 = \mathcal{T}_{(-\omega_m)}\, \omega_{[m,n]}$. The first part of the proof is to define a multivalued map Ψ that assigns to (ω^1, ω^2) a set of pairs of half-space walks $\{(\eta^1, \eta^2)\}$ in an injective way. Note that ω^2 is a half-space walk.

The *reversal* of a walk $\eta = (\eta_i)_{i=1}^m$ is the walk $(\eta_{m-i})_{i=0}^{m-1}$; this walk begins at the endpoint of η and goes back to the initial vertex. Translate the reversal of ω^1 so that the walk begins at the origin, i.e., consider the reversal of $\mathcal{T}_{(-\omega_m)}\omega^1$. This is almost a half-space walk; the only problem is that it may visit the coordinate hyperplane $\pi_1^{-1}(o)$ more than just at the initial vertex.

We now define Ψ. Set $\eta^2 := \omega^2$. Define $\tilde{\omega}^1$ to be the concatenation of the one-step self-avoiding walk from o to e_1 with the reversal of ω^1. Let $x := e_1 - \omega_m$ denote the vector along which ω^1 is translated when forming this concatenation. Note that $\tilde{\omega}^1$ is a half-space walk.

Let $\mathsf{adj} = \mathsf{adj}(\omega^1, \omega^2)$, and suppose this set contains k flippable plaquettes for ω^1. Let $\mathsf{adj}^\star \subset \mathsf{adj}$ be a subset of disjoint flippable plaquettes of size α_k; such a subset exists by Lemma 3.5. Then

$$\Psi(\omega) := \left\{ (\eta^1, \eta^2) \mid \eta^1 = \mathcal{F}_{\mathcal{T}_x B}(\tilde{\omega}^1),\, B \in \binom{\mathsf{adj}^\star}{\delta_k} \right\}.$$

By an argument as in the proof of Lemma 4.3, η^1 is a half-space walk, as any flips do not change the minimal value of the first coordinate of $\tilde{\omega}^1$. Note that if $\eta^2 \in \mathsf{B}_m$, then $\eta^1 \in \mathsf{B}_{n-m+2\delta_k+1}$.

We now verify that it is possible to reconstruct (ω^1, ω^2), and hence ω itself, from any image (η^1, η^2). Given η^1, the translation applied to ω^1 is determined, as flips do not change the endpoints of walks. The translation applied to ω^1 determines the translation applied to η^2, and hence determines ω^2. Hence, by Lemma 3.2, ω^1 is determined since we know ω^2 and η^1.

Recall that $\sqcup$ indicates a union of disjoint sets, and note $\Gamma_n = \sqcup_{k=0}^n \Gamma_n^k$, where Γ_n^k is the set of self-avoiding walks such that $\left|\mathsf{adj}(\omega^1, \omega^2)\right| = k$. Proceeding as in the proof of Lemma 4.4 yields

$$c_n(\kappa) \leq \sum_{m=0}^{n} \sum_{k \leq m} \binom{\alpha_k}{\delta_k}^{-1} (\mathrm{flip}_\kappa)^{\delta_k} (1+\kappa)^{k+k_0} h_m(\kappa) h_{n-m+2\delta_k+1}(\kappa).$$

The exponent of $(1+\kappa)$ is $k + k_0$ as we have used Lemma 3.3 in the unfolding step that created the pair of half-space walks. Using Proposition 4.7 to estimate the factors $h_\ell(\kappa)$ and Lemma 4.5 to estimate the remaining terms yields

$$c_n(\kappa) \leq K' \sum_{m=0}^{n} \sum_{k \leq m} e^{-ak} e^{a_1\sqrt{m}} e^{a_1\sqrt{n-m+2\delta_k+1}} \mu_{\mathsf{B}}^n,$$

where K' is the product of the constant prefactors. The inequality $\sqrt{x} + \sqrt{y} \leq \sqrt{2x+2y}$ combined with the fact that k is at most n implies a further upper bound of the form

$$c_n(\kappa) \leq K''(n+1)e^{a'\sqrt{n}} \mu_{\mathsf{B}}^n$$

for some $K'', a' > 0$. As this establishes (4.23), the proof is complete. □

5 Averaged submultiplicativity and initial consequences

The transformation used to estimate entropic gains in Sect. 4 flipped a fixed fraction $\delta\alpha$ of flippable plaquettes, and this led to p_1-dependent constants in estimates, where we recall p_1 depends on the step distribution D. This was convenient as it gave us precise control over the length added to a walk. For a lace expansion analysis the lack of uniformity in p_1 complicates matters, and this section defines a greedier transformation that enables estimates uniform in p_1 when $\kappa = \kappa(p_1)$ is chosen correctly. The price to pay is less control over the increase in length of a walk.

5.1 κ-ASAW with a memory

We first define memories, which will play the role of boundary conditions for self-avoiding walks. Recall that $\tilde{\Gamma}(o)$ is the set of self-avoiding polygons that begin at the origin.

Definition 10 A *memory* is a walk $\eta \in \bigcup_x \Gamma(x, o) \cup \tilde{\Gamma}(o)$.

By a slight abuse of notation, let $\omega \cap \eta$ denote the set of vertices in common to two walks ω and η. We will now define the law of κ-ASAW conditional on having memory η. Let $\Gamma_n^o := \cup_x \Gamma_n(x)$ denote the set of n-step SAW with initial vertex o. For $n \in \mathbb{N}$, $\kappa \geq 0$, and a memory η, define *n-step κ-ASAW with memory η* to be the law $\mathsf{P}_{n,\kappa}^{\eta}$ on Γ_n^o given by

$$\mathsf{P}_{n,\kappa}^{\eta}(\omega) \propto \mathcal{W}_\kappa(\omega; \eta) \mathbb{1}_{\{\omega \in \Gamma_n^o,\, \omega_{[1,n]} \cap \eta = \emptyset\}}, \tag{5.1}$$

where

$$\mathcal{W}_\kappa(\omega; \eta) := e^{-H_\kappa(\omega;\eta)} \mathbb{P}_n(\omega), \quad e^{-H_\kappa(\omega;\eta)} = (1+\kappa)^{|\mathsf{adj}(\omega)|}(1+\kappa)^{|\mathsf{adj}(\eta,\omega)|}, \tag{5.2}$$

where we have recalled the definition of $H_\kappa(\omega; \eta)$ from (2.7) for the convenience of the reader. Thus $\mathsf{P}_{n,\kappa}^{\eta}$ is supported on self-avoiding walks that intersect η only at their initial vertex $\omega_0 = o$, and $\mathsf{P}_{n,\kappa}^{\eta}$ gives a reward of $(1+\kappa)$ for each pair of adjacent edges in ω and for each time an edge of ω is adjacent to an edge of η.

5.2 Averaged submultiplicativity

Let η be a memory. Define

$$\Gamma_{n,k}^{\eta}(x) := \left\{\omega \in \Gamma_n(x) \mid \mathsf{P}_{n,\kappa}^{\eta}(\omega) > 0,\, \left|\mathsf{adj}_1(\omega, \eta)\right| = k\right\},$$

where (by a slight abuse of the notation in (4.6)) $\mathsf{adj}_1(\omega, \eta)$ is the set of plaquettes that are flippable for ω in $\mathsf{adj}(\omega, \eta)$. The (n, k) *two-point function with memory η* is

$$c_{n,k}^{\eta}(x) := \sum_{\omega \in \Gamma_{n,k}^{\eta}(x)} \mathcal{W}_\kappa(\omega; \eta). \tag{5.3}$$

The dependence of $c_{n,k}^{\eta}(x)$ on κ is left implicit. Let $c_n^{\eta}(x) := \sum_{k \geq 0} c_{n,k}^{\eta}(x)$.

Definition 11 Let $z \geq 0$, $\kappa \geq 0$, and $x \in \mathbb{Z}^d$. The *two-point function $G_{z,\kappa}^{\eta}(x)$ with memory η* is defined to be

$$G_{z,\kappa}^{\eta}(x) := \sum_{n \geq 0} z^n c_n^{\eta}(x) \tag{5.4}$$

If $\eta = \emptyset$ this is identically the 2-point function of κ-ASAW as defined in (2.3).

Recall that the critical point $z_c(\kappa)$ was specified in Definition 3. The next proposition will imply that $G_{z,\kappa}^{\eta}$ is well-defined when $z < z_c(\kappa)$.

Definition 12 Define $z_0(\kappa) := (1+\kappa)^{-2(d-1)}$.

This choice of z_0 will be useful in Sect. 6.5 for making comparisons with simple random walk. Recall that k_0 was defined in (3.1).

Proposition 5.1 *Suppose that $\kappa_0(d, p_1)$ is sufficiently small and that $z \geq z_0(\kappa)$. Then, for all $\kappa < \kappa_0$,*

$$G^{\eta}_{z,\kappa}(x) \leq (1+\kappa)^{k_0} G_{z,\kappa}(x). \tag{5.5}$$

In particular, $G^{\eta}_{z,\kappa}(x)$ is an absolutely convergent power series when $|z| < z_c(\kappa)$.

Proof We begin by defining a multivalued map Φ that will quantify the entropic gain of forgetting a memory. Let $\omega \in \Gamma^{\eta}_{m,k}$. By Lemma 3.5, there exists a set $\mathsf{adj}^\star \subset \mathsf{adj}(\eta, \omega)$ of $\alpha_k = \lceil \alpha k \rceil$ disjoint flippable plaquettes for ω. Define Φ by

$$\Phi(\omega) = \{\omega' \mid \omega' = \mathcal{F}_B(\omega),\, B \subset \mathsf{adj}^\star\}. \tag{5.6}$$

The definition of Φ implicitly depends on η through $\mathsf{adj}^\star$, but we suppress this dependence from the notation as η is fixed.

To begin, we compare the weight of $\omega \in \Gamma^{\eta}_{m,k}(x)$ under $\mathcal{W}_\kappa(\cdot;\eta)$ to the weight $\mathcal{W}_\kappa(\cdot)$ of its image $\Phi(\omega)$. We claim that

$$z^m \mathcal{W}_\kappa(\omega;\eta) \leq (1+\kappa)^{k_0} \left(\frac{(1+\kappa)^{\frac{k}{\alpha_k}}}{1+z^2p_1^2(1+\kappa)^{-(2d-4)}} \right)^{\alpha_k} \sum_{\omega' \in \Phi(\omega)} z^{|\omega'|} \mathcal{W}_\kappa(\omega') \tag{5.7}$$

To see this, note that at each $P \in \mathsf{adj}^\star$ a flip may or may not occur. Hence the definition of Φ implies

$$\sum_{\omega' \in \Phi(\omega)} z^{|\omega'|} \mathcal{W}_\kappa(\omega') \geq \prod_{j=1}^{\lceil \alpha k \rceil} (1+z^2p_1^2(1+\kappa)^{-(2d-4)}) z^m \mathcal{W}_\kappa(\omega). \tag{5.8}$$

Formally, this bound arises by applying Lemma 3.4 $|B|$ times to $\omega' = \mathcal{F}_B(\omega)$, and then summing over all $B \subset \mathsf{adj}^\star$. As $\omega \in \Gamma^{\eta}_{m,k}(x)$, there is a $\sigma \in \{0, 1, \ldots, k_0\}$ such that $\mathcal{W}_\kappa(\omega) = (1+\kappa)^{-k-\sigma}\mathcal{W}_\kappa(\omega;\eta)$. The possible values for σ follows from the proof of Lemma 3.3, which shows there are at most k_0 plaquettes that are not flippable in $\mathsf{adj}(\omega, \eta)$, because such plaquettes only occur at points of concatenation. Inserting this formula for $\mathcal{W}_\kappa(\omega)$ into (5.8) and rearranging gives (5.7).

Since $\kappa \geq 0$,

$$\frac{(1+\kappa)^{\frac{k}{\alpha_k}}}{1+z^2p_1^2(1+\kappa)^{-(2d-4)}} \leq \frac{(1+\kappa)^{\frac{1}{\alpha}}}{1+z^2p_1^2(1+\kappa)^{-(2d-4)}}. \tag{5.9}$$

The right-hand side of (5.9) is at most $(1+\kappa)^{\frac{1}{\alpha}}(1+z_0^2p_1^2(1+\kappa)^{-(2d-4)})^{-1}$, which is strictly less than one when $\kappa = 0$. The right-hand side of (5.9) is continuous in κ

and hence is strictly less than one for small positive κ. Thus, for κ sufficiently small and $z \geq z_0$, (5.7) implies

$$z^m \mathcal{W}_\kappa(\omega;\eta) \leq (1+\kappa)^{k_0} \sum_{\omega' \in \Phi(\omega)} z^{|\omega'|} \mathcal{W}_\kappa(\omega'). \tag{5.10}$$

Sum (5.10) over $\omega \in \Gamma_m^\eta(x) = \sqcup_{k \geq 0} \Gamma_{m,k}^\eta(x)$. To conclude the proof what must be shown is that $\Phi(\omega) \subset \Gamma(x)$, and that if $\omega^i \in \Gamma_m^\eta(x)$, $i = 1, 2$, then

$$\omega^1 \neq \omega^2 \implies \Phi(\omega^1) \cap \Phi(\omega^2) = \emptyset. \tag{5.11}$$

The first claim follows from Lemma 3.1. To prove the second claim, suppose $\gamma^i \in \Phi(\omega^i)$, $i = 1, 2$. Then as γ^i is the result of flipping ω^i at a disjoint set B of flippable plaquettes, Lemma 3.2 implies ω^i is determined by γ^i and η. Hence if $\gamma^1 = \gamma^2$, then $\omega^1 = \omega^2$. □

For our lace expansion analysis it will be necessary to have a slight generalization of Proposition 5.1. Let η be a memory and define

$$\bar{\mathsf{P}}_{n,\kappa}^\eta(\omega) \propto \mathcal{W}_\kappa(\omega;\eta) \mathbb{1}_{\{\omega \in \Gamma_n^o, \omega_{[1,n-1]} \cap \eta = \emptyset\}}. \tag{5.12}$$

Thus $\bar{\mathsf{P}}_{n,\kappa}^\eta$ is a law on n-step self-avoiding walks that do not intersect η except for (i) at $\omega_0 = o$ and (ii) possibly at ω_n. Let $\bar{G}_{z,\kappa}^\eta(x)$ denote the two-point function for walks with law $\bar{\mathsf{P}}_{n,\kappa}^\eta$.

Corollary 5.2 *Proposition 5.1 holds for $\bar{G}_{z,\kappa}^\eta$ in place of $G_{z,\kappa}^\eta$ after changing $(1+\kappa)^{k_0}$ to $(1+\kappa)^{2k_0}$.*

Proof The proof for $\bar{G}_{z,\kappa}^\eta$ is, *mutatis mutandis*, the proof of Proposition 5.1. The extra factor of $(1+\kappa)^{k_0}$ arises as the proof of Lemma 3.3 allows for the existence of $2k_0$ plaquettes in $\mathsf{adj}(\eta, \omega)$ that are not flippable. This is because there may be k_0 such plaquettes for each point of concatenation, of which there are at most 2. □

Corollary 5.3 *Both Proposition 5.1 and Corollary 5.2 hold when the two-point functions are restricted to sums over walks of length at least m for $m \in \mathbb{N}$.*

Proof The map Φ used in the proof of Proposition 5.1 only increases the length of a walk. □

5.3 First applications of averaged submultiplicativity

It will be necessary to temporarily consider κ-ASAW in a finite volume. Precisely, let $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ be a torus of side length L. Define

$$\mathsf{P}_{n,\kappa,\Lambda}(\omega) \propto \left(\prod_{i=0}^{n} \mathbb{1}_{\{\omega_i \in \Lambda\}} \right) \mathsf{P}_{n,\kappa}(\omega).$$

Let $\chi_\Lambda(z)$ be the susceptibility associated to κ-ASAW on Λ, i.e.,

$$\chi_{\Lambda,\kappa}(z) := \sum_{n\geq 0} z^n \sum_{\omega\in\Gamma_n^o} \mathcal{W}_\kappa(\omega) \prod_{i=0}^{n} \mathbb{1}_{\{\omega_i\in\Lambda\}}. \tag{5.13}$$

Note that $\chi_{\Lambda,\kappa}(z)$ is a polynomial in z whenever Λ is finite. We will attach the subscript Λ to other finite volume quantities in an analogous way.

Lemma 5.4 *Let $\kappa_0(d, p_1)$ be as in Proposition 5.1. Fix $\kappa < \kappa_0(d, p_1)$ and $z \geq z_0$. On a finite torus Λ,*

$$-\frac{d}{dz}\left(\chi_{\Lambda,\kappa}(z)\right)^{-1} \leq (1+\kappa)^{k_0} z_0^{-1}. \tag{5.14}$$

Proof The proof we will give is the standard one for self-avoiding walk, with averaged submultiplicativity (Proposition 5.1) replacing submultiplicativity. While Proposition 5.1 is for κ-ASAW on $\mathbb{Z}^d$, the proof applies immediately to Λ as well: the only property of the graph that was used in the proof is that flips of flippable plaquettes are well-defined.

As $\chi_{\Lambda,\kappa}(z)$ is a polynomial in z, we can compute

$$\frac{d}{dz}\left[z\chi_{\Lambda,\kappa}(z)\right] = \sum_{y\in\Lambda}\sum_{\omega\in\Gamma(y)} (|\omega|+1) z^{|\omega|} \mathcal{W}_\kappa(\omega) \tag{5.15}$$

$$= \sum_{x,y\in\Lambda}\sum_{\eta\in\Gamma(x)}\sum_{\omega'\in\Gamma(x,y)} z^{|\eta|} z^{|\omega'|} \mathcal{W}_\kappa(\eta\circ\omega') \mathbb{1}_{\{\eta\circ\omega'\in\Gamma\}} \tag{5.16}$$

$$= \sum_{x,y\in\Lambda}\sum_{\eta\in\Gamma(x)} z^{|\eta|} \mathcal{W}_\kappa(\eta) G^\eta_{z,\kappa,\Lambda}(y-x). \tag{5.17}$$

Equation (5.16) follows by using the fact that for $\omega \in \Gamma$,

$$|\omega| + 1 = \sum_x \mathbb{1}\{\omega_i = x \text{ for some } i\},$$

and then splitting ω into $\eta = \omega_{[0,i]}$ and ω'. The third equality follows from the definition of the conditional weight $\mathcal{W}_\kappa(\cdot\,;\eta)$ and the translation invariance of $G^\eta_{z,\kappa,\Lambda}$ on a torus.

By (the finite-volume version of) Proposition 5.1, the memory η can be ignored to obtain an upper bound. Summing over y results in a factor $\chi_{\Lambda,\kappa}(z)$, as does the sum over x. The result is

$$\frac{d}{dz}\left[z\chi_{\Lambda,\kappa}(z)\right] \leq (1+\kappa)^{k_0} (\chi_{\Lambda,\kappa}(z))^2. \tag{5.18}$$

Computing the derivative on the left-hand side, rearranging, and using $\chi_{\Lambda,\kappa}(z) \geq 0$ proves the lemma. □

The validity of Lemma 5.4 for all $z \geq z_0$ and all finite Λ implies the continuity of the phase transition for κ-ASAW and a mean-field lower bound on the critical exponent γ. Proposition 5.6 below is a formal statement of these facts. We include a proof for completeness, although the implication given Lemma 5.4 is well-known [18]. We need one preparatory lemma.

Lemma 5.5 *For any $z < z_c$ there are constants $c_1(z), c_2(z)$ such that $G_{z,\kappa}(x) \leq c_1(z)\exp(-c_2(z)\|x\|_\infty)$.*

Proof This follows from the fact that the summation defining $G_{z,\kappa}(x)$ contains only walks of length at least $C\|x\|_\infty$ for some $C > 0$, and hence is contained in the tail of the convergent sum $\chi_\kappa(z)$. For more details, see [4, p.11–12]. □

Proposition 5.6 *Let $\kappa_0(d, p_1)$ be as in Proposition 5.1. For $0 < \kappa < \kappa_0(d, p_1)$ and $z_0 \leq z < z_c$,*

$$\chi_\kappa(z) \geq \frac{(1+\kappa)^{-k_0} z_0}{z_c - z}, \tag{5.19}$$

where $\chi_\kappa(z)$ is defined by (2.2).

Proof Note that $\chi_{\Lambda,\kappa}(z) \geq 1$ due to the contribution of the zero-step walk. Fixing $\epsilon > 0$ and integrating (5.14) from $z \geq z_0$ to $z_c + \epsilon$ implies

$$\big(\chi_{\Lambda,\kappa}(z)\big)^{-1} - \big(\chi_{\Lambda,\kappa}(z_c+\epsilon)\big)^{-1} \leq (1+\kappa)^{k_0} z_0^{-1}(z_c - z) + \epsilon(1+\kappa)^{k_0} z_0^{-1}. \tag{5.20}$$

Let $\bar{\chi}_{\Lambda,\kappa}(z)$ be the contribution to the infinite volume susceptibility $\chi_\kappa(z)$ due to walks restricted to the finite box Λ without periodic boundary conditions. $\bar{\chi}_{\Lambda,\kappa}(z)$ is monotone in Λ and converges to $\chi_\kappa(z)$. Moreover, $\chi_{\Lambda,\kappa}(z)$ dominates $\bar{\chi}_{\Lambda,\kappa}(z)$, and hence $\big(\chi_{\Lambda,\kappa}(z_c+\epsilon)\big)^{-1}$ tends to zero as $\Lambda \uparrow \mathbb{Z}^d$ by the definition of z_c.

To conclude it is enough to prove that $\big(\chi_{\Lambda,\kappa}(z)\big)^{-1}$ converges to $(\chi_\kappa(z))^{-1}$ as $\Lambda \uparrow \mathbb{Z}^d$, as we can then take $\Lambda \uparrow \mathbb{Z}^d$ in (5.20), followed by $\epsilon \to 0$. To see this we note that Proposition 5.1 implies

$$\big|\bar{\chi}_{\Lambda,\kappa}(z) - \chi_{\Lambda,\kappa}(z)\big| \leq \sum_{x \in \partial\Lambda} G_{z,\kappa}(x)\chi_{\Lambda,\kappa}(z)$$

by reasoning as from (5.15) to (5.17); here $\partial\Lambda$ denotes the boundary vertices of Λ. Dividing through by $\chi_{\Lambda,\kappa}(z)$ and using that $G_{z,\kappa}(x)$ decays exponentially in $\|x\|_\infty$ by Lemma 5.5 proves that $\bar{\chi}_{\Lambda,\kappa}(z)$ has the same limit as $\chi_{\Lambda,\kappa}(z)$. This proves the claim as we have already noted that $\bar{\chi}_{\Lambda,\kappa}(z)$ converges to $\chi_\kappa(z)$. □

6 A lace expansion for κ-ASAW

This section derives and analyzes a lace expansion for κ-ASAW to prove Theorem 2.3. As self-avoiding walk is the special case $\kappa = 0$ of κ-ASAW, many aspects of our

analysis mirror the SAW case. In what follows we therefore focus on the details specific to $\kappa \neq 0$, and give precise statements and references for the details that we omit.

For the reader unfamiliar with the lace expansion, the following pointers to the literature may be helpful. Our derivation of a lace expansion for κ-ASAW in Sect. 6.2 is an adaptation of the Brydges-Spencer expansion [19]; [4, Ch. 3.2–3.3] and [6, Ch. 5.2] contain pedagogical accounts of this method. For the derivation of diagrammatic bounds our approach in Sect. 6.4 is similar to [8], but the unfamiliar reader may wish to consult [4, Ch. 4] or [6, Ch. 5.4] as a first introduction to the notion of diagrammatic bounds. It is at this step that we make use of averaged submultiplicativity. Lastly, for the convergence of the expansion and proof of Theorem 2.3 we make use of the method of [8]. We recall the method of [8] in Appendix A, and our proof of Theorem 2.3 is by a verification of the hypotheses of this method.

6.1 Explicit κ-ASAW interaction

In this section we give an algebraic formulation of the κ-ASAW weight. Let

$$U_{ij}(\omega) := \mathbb{1}_{\{\omega_i=\omega_j\}} - \kappa \mathbb{1}_{\{\{\omega_i,\omega_{i+1},\omega_{j-1},\omega_j\} \text{ is a plaquette}\}}. \tag{6.1}$$

Lemma 6.1 *If $\omega \in W_n$ is an n-step walk with $\omega_0 = o$, then $P_{n,\kappa}(\omega)$ is proportional to*

$$\mathcal{W}_\kappa(\omega)\mathbb{1}_{\{\omega\in\Gamma\}} = \mathbb{P}_n(\omega) \prod_{\substack{0\le i<j\le n\\ j>i+1}} (1 - U_{ij}(\omega)). \tag{6.2}$$

Proof Recall that $\omega_{i+1} \neq \omega_i$ by the definition of a walk. As $\omega_i = \omega_j$ precludes $\{\omega_i, \omega_{i+1}, \omega_{j-1}, \omega_j\}$ being a plaquette, the first indicator in (6.1) encodes the self-avoidance constraint between ω_i and ω_j for $j > i + 1$. The second indicator encodes the self-attraction between edges; each attraction is counted with weight $1 + \kappa$ when ω_i is the first visit to a plaquette and ω_j is the last visit to the plaquette, which requires $j \geq i + 3$. □

6.2 Derivation of the lace expansion

In this section we derive a lace expansion for κ-ASAW using the so-called algebraic method [4]. A similar expansion for a vertex attraction was derived in [1].

For $a \leq b$ integers let $[a, b] := \{a, a+1, \ldots, b\}$, and $[a, a-1] = \emptyset$. An *edge* $\{i, j\}$ is an element of $\binom{[a,b]}{2}$ with $|i - j| > 1$; $\{i, j\}$ will be abbreviated ij. A *graph* G on $[a, b]$ is a (possibly empty) set of edges, and G is *connected* if for all $j \in [a + 1, b - 1]$ there are $i < j < k$ such that ik is an edge in G. Let $\mathsf{G}_{[a,b]}$ denote the set of graphs on $[a, b]$, and $\mathsf{G}^{\mathsf{c}}_{[a,b]}$ the set of connected graphs. Let ω be a walk and define

$$K_{[a,b]}(\omega) := \sum_{G\in\mathsf{G}_{[a,b]}} \prod_{ij\in G} -U_{ij}(\omega), \quad \text{and} \tag{6.3}$$

$$J_{[a,b]}(\omega) := \sum_{G \in \mathsf{G}^{\mathsf{c}}_{[a,b]}} \prod_{ij \in G} -U_{ij}(\omega). \tag{6.4}$$

These definitions imply $K_{[a,a]} = K_{[a,a+1]} = J_{[a,a]} = J_{[a,a+1]} = 1$ due to the contribution of the empty graph.

Expanding the product of $(1 - U_{ij})$ over ij in (6.2) and using (2.3) implies the two-point function can be written in terms of graphs:

$$G_{z,\kappa}(x) = \sum_{n=0}^{\infty} \sum_{\omega \in \mathsf{W}_n(x)} z^n \mathbb{P}_n(\omega) K_{[0,n]}(\omega). \tag{6.5}$$

In what follows, we manipulate (6.5) by rewriting the term $K_{[0,n]}(\omega)$ to obtain a convolution equation for $G_{z,\kappa}$. The first step is a well-known lemma. For notational convenience we will use the convention that $\sum_{j=b+k}^{b} f(j) = 0$ if $k \geq 1$.

Lemma 6.2 (Lemma 5.2.2 of [6]) *For any walk ω and $a < b$,*

$$K_{[a,b]}(\omega) = K_{[a+1,b]}(\omega) + \sum_{j=a+2}^{b} J_{[a,j]}(\omega) K_{[j,b]}(\omega). \tag{6.6}$$

For $x \in \mathbb{Z}^d$, define $\Pi_{z,\kappa}(x)$ by

$$\Pi_{z,\kappa}(x) := \sum_{n=2}^{\infty} \sum_{\omega \in \mathsf{W}_n(x)} z^n \mathbb{P}_n(\omega) J_{[0,n]}(\omega). \tag{6.7}$$

It is not *a priori* clear that the series defining $\Pi_{z,\kappa}$ is convergent. When convergence is unknown we will interpret the series and the formulas in which it occurs as formulas relating formal power series in z. We recall that for $f, g \colon \mathbb{Z}^d \to \mathbb{R}$ the convolution of f and g is $(f * g)(x) := \sum_{y \in \mathbb{Z}^d} f(y) g(x - y)$, and we also recall that the step distribution D was introduced in (2.1).

Proposition 6.3 (Theorem 5.2.3 of [6]) *As formal power series in z,*

$$G_{z,\kappa}(x) = \mathbb{1}_{\{x=o\}} + (zD * G_{z,\kappa})(x) + (\Pi_{z,\kappa} * G_{z,\kappa})(x). \tag{6.8}$$

This is an equality between functions when all terms of (6.8) are absolutely convergent in z.

Proof Consider the contribution of n-step walks to $G_{z,\kappa}(\omega)$, i.e.,

$$\sum_{\omega \in \Gamma_n(x)} z^n \mathcal{W}_\kappa(\omega) = \sum_{\omega \in \mathsf{W}_n(x)} z^n \mathbb{P}_n(\omega) K_{[0,n]}(\omega). \tag{6.9}$$

Since $U_{ij}(\omega)$ only depends on those ω_ℓ with $i \leq \ell \leq j$, $K_{[0,j]}(\omega) J_{[j,n]}(\omega)$ is equal to $K_{[0,j]}(\omega_{[0,j]}) J_{[j,n]}(\omega_{[j,n]})$. Hence, by using (6.6) to rewrite the factor $K_{[0,n]}$ in (6.9), the right-hand side of (6.9) can be rewritten as

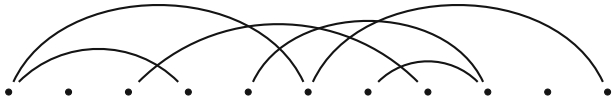


Fig. 6 An illustration of the graph G with edge set $\{\{0,3\},\{0,5\},\{2,7\},\{4,8\},\{5,10\},\{6,8\}\}$. The vertices are labelled $0,1,\ldots,10$ from left to right

$$\sum_{\omega\in\mathsf{W}_n(x)} zD(\omega_1-\omega_0)z^{n-1}\mathbb{P}_{n-1}(\omega_{[1,n]})K_{[1,n]}(\omega)$$
$$+\sum_{\omega\in\mathsf{W}_n(x)}\sum_{j=2}^{n} z^j\mathbb{P}_j\left(\omega_{[0,j]}\right)J_{[0,j]}(\omega_{[0,j]})z^{n-j}\mathbb{P}_{n-j}\left(\omega_{[j,n]}\right)K_{[j,n]}(\omega_{[j,n]}). \quad (6.10)$$

To conclude, sum (6.9) over all n. The left-hand side is $G_{z,\kappa}(x)$. For $n=0$, the right-hand side is $\mathbb{1}_{\{x=o\}}$, the contribution of the zero-step walk. For $n\geq 1$ we use (6.10). The factors of $G_{z,\kappa}$ arise by (6.5) and the translation invariance of $\mathcal{W}_\kappa$. The term $\Pi_{z,\kappa}$ arises from the sum over j by (6.7). This proves (6.8) in the sense of formal power series, as the coefficient of z^n consists of only finitely many terms for all n. □

6.3 Representation of $\pi^{(m)}_{z,\kappa}(x)$

To make use of Proposition 6.3 requires control on $\Pi_{z,\kappa}$. Obtaining this control is at the heart of the lace expansion, and requires some further definitions.

Definition 13 A connected graph G is a *lace* if for any edge $ij\in G$ the graph $G\backslash ij$ is not connected. Let $\mathsf{L}^{(m)}_{[a,b]}$ denote the set of laces on $[a,b]$ with m edges, and $\mathsf{L}_{[a,b]}=\bigsqcup_{m\geq 1}\mathsf{L}^{(m)}_{[a,b]}$.

Lemma 6.4 (pp.126–128 of [6]) *For $G\in\mathsf{G}^{\mathsf{c}}_{[a,b]}$, let $\mathcal{L}(G)$ be the graph with edges $\{s_it_i\}$ defined by $s_1=a$, $t_1=\max\{t\mid s_1t\in G\}$, and*

$$t_{i+1}=\max\{t\mid \exists s<t_i \text{ such that } st\in G\},\quad s_{i+1}=\min\{s\mid st_{i+1}\in G\}.$$

Then (i) $\mathcal{L}(G)$ is a lace and (ii) if $\mathcal{L}(G)=\mathcal{L}(H)$ then $\mathcal{L}(G\cup H)=\mathcal{L}(G)$.

Figures 6 and 7 depict a graph G and its associated lace $\mathcal{L}(G)$.

$\mathsf{G}_{[a,b]}$ is partially ordered by inclusion, and Lemma 6.4 implies that for every lace L there is a maximal graph G such that $\mathcal{L}(G)=L$. The edges of the maximal graph G can be partitioned into $L\sqcup\mathsf{C}(L)$, where $\mathsf{C}(L)$ are called the *compatible edges*. Explicitly, $ij\in\mathsf{C}(L)$ if and only if $ij\notin L$ and $\mathcal{L}(L\cup\{ij\})=L$. It follows that

$$J_{[a,b]}(\omega)=\sum_{L\in\mathsf{L}_{[a,b]}}\prod_{ij\in L}-U_{ij}(\omega)\prod_{i'j'\in\mathsf{C}(L)}(1-U_{i'j'}(\omega)). \quad (6.11)$$

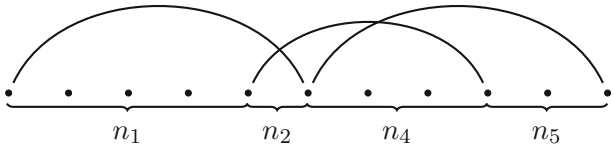


Fig. 7 The lace $\mathcal{L}(G)$ associated to the graph G from Fig. 6 has the edge set $\{\{0,5\},\{4,8\},\{5,10\}\}$. $\mathcal{L}(G)$ subdivides the vertex set into intervals of length $n_1 = 4$, $n_2 = 1$, $n_3 = 0$, $n_4 = 3$, and $n_5 = 2$

Define $J^{(m)}_{[a,b]}$ to be the contribution to (6.11) given by $L \in \mathsf{L}^{(m)}_{[a,b]}$, and let

$$\pi^{(m)}_{z,\kappa}(x) := \sum_{n=2}^{\infty} \sum_{\omega \in \mathsf{W}_n(x)} z^n \mathbb{P}_n(\omega) J^{(m)}_{[0,n]}(\omega). \tag{6.12}$$

These definitions imply that as formal power series

$$\Pi_{z,\kappa}(x) = \sum_{m \geq 1} \pi^{(m)}_{z,\kappa}(x), \tag{6.13}$$

and our approach to controlling $\Pi_{z,\kappa}$ will be to estimate the terms $\pi^{(m)}_{z,\kappa}$. This will be done by re-expressing $\pi^{(m)}_{z,\kappa}$ in terms of sums of collections of walks subject to conditional interactions $\mathcal{W}_\kappa(\cdot\,;\eta)$ and estimating these sums. The remainder of this section introduces the definitions needed to make the reformulation of $\pi^{(m)}_{z,\kappa}$ precise. As noted previously, the arguments are similar to standard ones [4,6]; see also [1].

A lace $L \in \mathsf{L}^{(m)}_{[0,n]}$ partitions $[0,n]$ into $2m-1$ intervals with disjoint interiors, and this induces a composition of n into $2m-1$ parts. See Fig. 7. This composition yields a vector $\boldsymbol{n} := (n_1, n_2, \ldots, n_{2m-1})$ of interval lengths with the properties

$$\begin{aligned} n_j &\geq 0, \qquad j \in \{3,5,7,\ldots,2m-3\} \\ n_j &\geq 1, \qquad j \in [0,2m-1] \setminus \{3,5,7,\ldots,2m-3\}, \end{aligned} \tag{6.14}$$

and $n = \sum_{j=1}^{2m-1} n_j$. Conversely, to each such $\boldsymbol{n}$ there is associated a unique lace graph [4, Exercise 3.6].

Let $M_i := \sum_{j=1}^{i} n_j$. By our convention for sums, $M_k := 0$ for $k \leq 0$. Define $I_j := \big(M_{j-1}, M_j\big]$ for $j = 1,2,3,\ldots,2m-1$ and $I_k := \emptyset$ for $k \leq 0$. Let $\mathsf{C}_i(L) := \{i'j' \mid j' \in I_i\}$ be the set of compatible edges $i'j'$ whose right endpoint j' is in I_i. Observing that $\mathsf{C}(L) = \bigsqcup_{i=1}^{2m-1} \mathsf{C}_i(L)$ since the intervals I_i are a partition of $(0,n]$, these definitions imply that, for any lace $L \in \mathsf{L}^{(m)}_{[0,n]}$ and walk $\omega \in \mathsf{W}_n$,

$$\prod_{ij \in \mathsf{C}(L)} (1 - U_{ij}(\omega)) = \prod_{i=1}^{2m-1} \prod_{i'j' \in \mathsf{C}_i(L)} (1 - U_{i'j'}(\omega)). \tag{6.15}$$

Because $\mathsf{C}_i(L)$ consists of edges whose right endpoint is in I_i, the second product on the right-hand side of (6.15) describes an interaction between the vertices $\{\omega_j\}_{j \in I_i}$

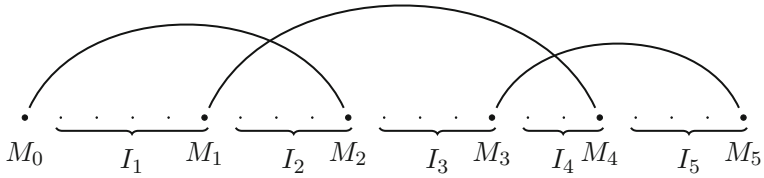


Fig. 8 A lace $\{s_1t_1, s_2t_2, s_3t_3\}$ with $s_1 = M_0$, $s_2 = M_1$, and $s_3 = M_3$. Its corresponding intervals I_i, $i = 1, \ldots, 5$ are also illustrated

and the vertices $\{\omega_{j'}\}_{j' \le M_i}$. Proposition 6.5 below controls this interaction; achieving this control requires characterising the edges $ij \in \mathsf{C}(L)$.

Recalling that $M_k = 0$ and $I_k = \emptyset$ for $k \le 0$, the compatible edges can be characterized as follows. Let $M^k_{2k-2} = M_{2k-2}$ if $k \ne 1$, and $M^1_0 = \emptyset$. For $k \ge 0$, $i < j$, and $m \ge 1$:

(i) if $j \in I_{2k+1}$, then $i \in \bigcup_{\ell=2k-1}^{2k+1} I_\ell \cup \{M_{2k-2}\}$, and $j = M_{2m-1} = n$ implies $i > M_{2m-3}$.
(ii) if $j \in I_{2k+2}$, then $i \in \bigcup_{\ell=2k-1}^{2k+2} I_\ell \cup \{M_{2k-2}\}$, and $j = M_{2k+2}$ implies $i > M_{2k-1}$.

This classification follows by considering the procedure defined in Lemma 6.4. For $m = 1$ the only incompatible edge is $0n$. For $m \ge 2$ the constraints fall into three types: for I_{2k+2}, $k \ge 1$; for I_{2k+1}, $k \le m-2$; and for I_2, I_3, I_4 and I_{2m-1}. The case I_2 is distinguished because the only endpoint of a lace between s_1 and t_1 is s_2, while for $i \ge 2$ we have $s_i < t_{i-1} \le s_{i+1} < t_i$. This distinction is manifested above by the triviality of the intervals I_k for $k \le 0$. I_3 and I_4 are distinguished as the initial vertex of a compatible edge cannot be 0, and I_{2m-1} is distinguished for a similar reason. See Fig. 8.

Definition 14 For $\boldsymbol{n}$ a composition of n into $2m-1$ parts and $\omega \in \mathsf{W}_n$, let $\boldsymbol{\omega} := (\omega^{(i)})_{i=1}^{2m-1}$ denote the vector of walks determined by $\omega^{(i)} := \omega_{[M_{i-1}, M_i]}$. Note $\omega^{(i)}$ has length n_i.

Using the characterisation of compatible edges above we will now rewrite the right-hand side of (6.15) in terms of the subwalks $\omega^{(i)}$; this requires some further definitions. To keep the notation to a minimum we will in fact give an upper bound.

When $m = 1$ define

$$A^{(1)}_1 := \{\omega^{(1)} \in \Gamma \cup \tilde{\Gamma}\}, \quad \eta^{(1)} := \emptyset.$$

For $m \ge 2$ we introduce: $\eta^{(1)} := \emptyset$; $\eta^{(2)} := \omega^{(1)}$; for $k = 1, \ldots, m-1$, $\eta^{(2k+1)} := \omega^{(2k-1)} \circ \omega^{(2k)}$; and for $k = 2, \ldots, m-1$, $\eta^{(2k)} := \omega^{(2k-3)} \circ \omega^{(2k-2)} \circ \omega^{(2k-1)}$. For $m \ge 2$, $1 \le k \le 2m-1$, and $L \in \mathsf{L}^{(m)}$ define $A^{(m)}_k = A^{(m)}_k(L)$ by

$$A^{(m)}_k := \{\omega^{(k)} \in \Gamma, \omega^{(k)}_i \ne \eta^{(k)}_j \text{ if } i'j' \in \mathsf{C}_k(L)\},$$

where i' is the index such that $\omega^{(k)}_i$ is the i'th vertex in $\omega = \omega^{(1)} \circ \cdots \circ \omega^{(2m-1)}$, and similarly for j'.

Proposition 6.5 *Let $m \geq 1$ and $1 \leq k \leq 2m-1$ be integers, let $L \in \mathsf{L}^{(m)}$ have an associated length vector $\boldsymbol{n}$, let $\omega \in \mathsf{W}_n$, and let $\omega^{(k)}$ be the kth subwalk of ω determined by L as in Definition 14. Then*

$$\mathbb{P}_{n_k}(\omega^{(k)}) \prod_{i'j' \in \mathsf{C}_k(L)} (1 - U_{i'j'}(\omega)) \leq \mathbb{1}_{A_k^{(m)}} \mathcal{W}_\kappa(\omega^{(k)}; \eta^{(k)}). \tag{6.16}$$

Proof The proposition is largely a translation of the definition of a compatible edge $i'j' \in \mathsf{C}_k(L)$ and the definitions of the walks $\eta^{(k)}$.

Let us first consider $m = 1$, so $k = 1$. In this case $\eta^{(1)} = \emptyset$. Each factor $1 - U_{i'j'}(\omega)$ for $i'j' \in \mathsf{C}_1(L)$ enforces $\omega_{i'}^{(1)} \neq \omega_{j'}^{(1)}$ (in the case $k = 1$ there is no distinction between primed and unprimed indices). Since the only edge that is *not* compatible is $0n$, we see that $\omega_{[0,n-1]}$ is self-avoiding, but $\omega_n = \omega_0$ is not forbidden. Hence $\omega^{(1)} \in \Gamma \cup \tilde{\Gamma}$, which is precisely the constraint that $A_1^{(1)}$ occurs.

Similarly, the factors of $1 - U_{i'j'}(\omega)$ encode the reward $(1+\kappa)$ if $\{i', i'+1, j'-1, j'\}$ is a plaquette. Since $0n$ is *not* compatible, if the first and last edges of $\omega^{(1)}$ are parallel this reward is not present. Since including this reward gives an upper bound we do so, as it allows the right-hand side to be written as $\mathcal{W}_\kappa(\omega^{(1)}; \emptyset)$.

For $m \geq 2$ and $1 \leq k \leq 2m-1$, the considerations are almost exactly the same. The fact that $\omega^{(k)} \in \Gamma$ follows as $i'j' \in \mathsf{C}_k(L)$ if $0 \leq i < j \leq n_k$ and $j > i+1$ by the classification of compatible edges for a lace, and this implies $\omega_i^{(k)} \neq \omega_j^{(k)}$. The definitions of $A_k^{(m)}$ arise from observing that the set of compatible edges have endpoints i' such that $\omega_{i'}$ falls into one of the walks comprising $\eta^{(k)}$. We obtain an upper bound by including additional rewards $(1+\kappa)$ corresponding to incompatible edges, and this recreates the weight $\mathcal{W}_\kappa(\omega^{(k)}; \eta^{(k)})$. □

6.4 Diagrammatic bounds

Proposition 6.5 leads to an upper bound for $\pi_{z,\kappa}^{(m)}(x)$ in terms of a sum over collections $\boldsymbol{\omega}$ of interacting walks. This will be used to estimate the size of $\left|\pi_{z,\kappa}^{(m)}(x)\right|$ in terms of convolutions of $G_{z,\kappa}(x)$; the resulting bounds are what are known as *diagrammatic bounds*. The next definition will be used in upper-bounding the factors $-U_{ij}(\omega)$ in (6.11).

Definition 15 Define $r_\kappa(x) := \mathbb{1}_{\{x=o\}} + \kappa \mathbb{1}_{\{\|x\|_\infty = 1\}}$.

In what follows $\kappa = \kappa(p_1)$ will be a function of p_1, chosen such that $\kappa \leq \kappa_0(d, p_1)$; in particular Proposition 5.1 applies when $z \geq z_0 = (1+\kappa)^{-2(d-1)}$. Let $H_{z,\kappa}(x) := G_{z,\kappa}(x) - \delta_{x,o}$ be the sum of contributions to $G_{z,\kappa}(x)$ due to non-trivial walks.

Proposition 6.6 *Fix $\kappa \leq \kappa_0$, $z_0 \leq z < z_c$, and $x \in \mathbb{Z}^d$. The following bounds hold. For $m = 1$,*

$$\left|\pi_{z,\kappa}^{(1)}(x)\right| \leq (1+\kappa)^{2k_0} z r_\kappa(x) (D * H_{z,\kappa})(x). \tag{6.17}$$

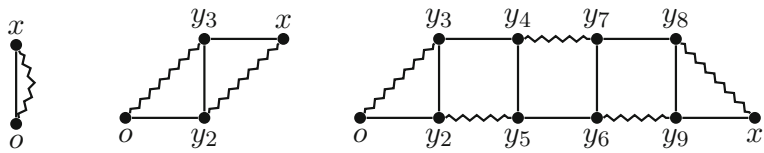


Fig. 9 Diagrammatic representations of Proposition 6.6 in the cases $m = 1, m = 2$ and $m = 5$. Wavy lines represent factors $r(y_{2j+1} - y_{2j-2})$. For $m \geq 2$ the vertical and first and last horizontal straight lines represent factors of $H_{z,\kappa}$. The remaining horizontal straight lines represent factors of $G_{z,\kappa}$. For $m = 1$ the vertical straight line represents $D * H_{z,\kappa}$

For $m \geq 2$, let $\boldsymbol{y} := (y_2, y_3, \ldots, y_{2m-1}) \in \mathbb{Z}^{d(2m-2)}$ denote a $(2m-2)$-tuple of vertices in $\mathbb{Z}^d$. Then

$$\begin{aligned}\left|\pi^{(m)}_{z,\kappa}(x)\right| &\leq (1+\kappa)^{(4m-2)k_0} H_{z,\kappa}(y_2) r_\kappa(y_3) \\ &\quad \times \sum_{\boldsymbol{y}} \Big[\prod_{j=1}^{m-2} \big(H_{z,\kappa}(y_{2j+1} - y_{2j}) G_{z,\kappa}(y_{2j+2} - y_{2j+1}) r_\kappa(y_{2j+3} - y_{2j}) \big) \\ &\quad H_{z,\kappa}(y_{2m-1} - y_{2m-2}) H_{z,\kappa}(x - y_{2m-1}) r_\kappa(x - y_{2m-2}) \Big] \end{aligned} \tag{6.18}$$

where the sum runs over all $\boldsymbol{y} \in \mathbb{Z}^{d(2m-2)}$. If $m = 2$ the product is empty, and hence identically one by definition.

A diagrammatic formulation of these upper bounds can be found in Fig. 9.

Proof of Proposition 6.6 We first outline the strategy of the proof. Recall the definition (6.12) of $\pi^{(m)}_{z,\kappa}(x)$ as a sum over walks $\omega \in \mathsf{W}(x)$ and laces $L \in \mathsf{L}^{(m)}$. The proof will use the decomposition of a walk ω into $2m-1$ subwalks $\omega^{(i)}$ as given by Definition 14. Proposition 6.5 gives a formula for the weight of the ith subwalk in terms of the preceding subwalks, and by using averaged submultiplicativity (Corollary 5.3) we will upper bound the sums over subwalks $\omega^{(i)}$ by factors of $G_{z,\kappa}$ and $H_{z,\kappa}$. The factors of r_κ will arise when we bound the product of $-U_{ij}(\omega)$ over $ij \in L$ in (6.12).

Recall the definition (6.1) of $U_{ij}(\omega)$, and observe that for all walks ω

$$\left|U_{ij}(\omega)\right| \leq r_\kappa(\omega_j - \omega_i), \tag{6.19}$$

which follows by considering separately $\|\omega_j - \omega_i\|_\infty$ being 0, 1, or at least 2.

We first consider the case $m = 1$. By (6.12), (6.11), Proposition 6.5 and the definition (6.1) of $U_{0n}(\omega)$ we have the upper bound

$$\begin{aligned}\left|\pi^{(1)}_{z,\kappa}(x)\right| &\leq \sum_{n\geq 2} \sum_{\omega \in \mathsf{W}_n(x)} z^n \mathcal{W}_\kappa(\omega) \left|U_{0n}(\omega)\right| \mathbb{1}_{\left\{\omega \in \Gamma \cup \tilde{\Gamma}\right\}} \\ &\leq \sum_{y \neq o} z D(y) \sum_{n \geq 1} \sum_{\omega \in \mathsf{W}_n(y,x)} z^{n-1} \mathcal{W}_\kappa(\omega; (o, y)) r_\kappa(x) \mathbb{1}_{\left\{(o,y)\circ\omega \in \Gamma \cup \tilde{\Gamma}\right\}},\end{aligned}$$

where the second inequality follows from (6.19) and explicitly separating out and summing over the location y of the first step of the walk.

Recall $|\omega| = n$ if $\omega \in \mathsf{W}_n$. Let $\mathcal{W}_{z,\kappa}(\omega;\eta) = z^{|\omega|}\mathcal{W}_\kappa(\omega;\eta)$, and recall the definition of $\bar{G}^{\eta}_{z,\kappa}(x)$ from below (5.12). The sum of $\mathcal{W}_{z,\kappa}(\omega;(o,y))\mathbb{1}_{\left\{(o,y)\circ\omega\in\Gamma\cup\tilde{\Gamma}\right\}}$ over $\omega \in \mathsf{W}(y,x)$ of length at least one is at most $\bar{G}^{(o,y)}_{z,\kappa}(y-x) - \delta_{y-x,o}$. The proposition for $m = 1$ now follows from Corollary 5.3 and the translation invariance of $H_{z,\kappa}$.

Next we consider $m \geq 2$; the argument is very similar to $m = 1$. As described in Sect. 6.3, if $L \in \mathsf{L}^{(m)}$ then ω is the concatenation of $2m-1$ walks $\omega^{(i)}$, $\omega^{(i)}$ has length n_i, and the vector $\boldsymbol{n}$ of lengths satisfies (6.14). If $|\omega| = n$, distribute the factor z^n so that there are n_i factors associated to the walk $\omega^{(i)}$. Then by Proposition 6.5 and (6.19),

$$\left|\pi^{(m)}_{z,\kappa}(x)\right| \leq \sum_{\boldsymbol{y}}\sum_{\boldsymbol{\omega}} r_\kappa(y_3)r_\kappa(x-y_{2m-2}) \prod_{j=2}^{m-1} r_\kappa(y_{2j+1}-y_{2j-2}) \prod_{i=1}^{2m-1} \mathbb{1}_{A^{(m)}_i}\mathcal{W}_{z,\kappa}(\omega^{(i)};\eta^{(i)}), \tag{6.20}$$

where the outer sum is over $\boldsymbol{y} = (y_2,\ldots,y_{2m-1}) \in \mathbb{Z}^{d(2m-2)}$, the inner sum is over $\boldsymbol{\omega} = (\omega^{(1)},\ldots,\omega^{(2m-1)})$ whose lengths n_i satisfy (6.14) and such that $\omega^{(1)} \in \mathsf{W}(y_2)$, $\omega^{(i)} \in \mathsf{W}(y_i,y_{i+1})$ for $i = 2,\ldots,2m-2$, and $\omega^{(2m-1)} \in \mathsf{W}(y_{2m-1},x)$. The sum over laces has been replaced with a sum over the possible lengths of the subwalks: see the discussion following (6.14).

We now iteratively sum over the subwalks $\omega^{(i)}$, starting with $i = 2m-1$. Since $\omega^{(j)}$ is fixed for $j = 1\ldots,2m-2$, the walk $\eta^{(2m-1)}$ is determined, and it plays the role of a memory for $\omega^{(2m-1)}$. Temporarily let $\tilde{n} = n_{2m-1}$, $\tilde{\omega} = \omega^{(2m-1)}$, and $\tilde{\eta} = \eta^{(2m-1)}$, and let $\tilde{\mathsf{W}}_{\tilde{n}} = \mathsf{W}_{\tilde{n}}(y_{2m-1},x)$. With these definitions we can upper bound the sum over $\omega^{(2m-1)}$ on the right-hand side of (6.20) by

$$\sum_{\substack{\tilde{\omega}\in\tilde{\mathsf{W}}\\ \tilde{n}\geq 1}} \mathbb{1}_{A^{(m)}_{2m-1}}\mathcal{W}_{z,\kappa}(\tilde{\omega};\tilde{\eta}) \leq (1+\kappa)^{2k_0}\sum_{\substack{\tilde{\omega}\in\tilde{\mathsf{W}}_{\tilde{n}}\\ \tilde{n}\geq 1}} \mathbb{1}_{\{\tilde{\omega}\in\Gamma\}}\mathcal{W}_{z,\kappa}(\tilde{\omega}). \tag{6.21}$$

This bound follows from Corollary 5.3, as the event $A^{(m)}_{2m-1}$ occurs only if $\omega^{(2m-1)}$ does not intersect $\eta^{(2m-1)}$ except for the initial, and possibly terminal, vertices of $\omega^{(2m-1)}$; we have also used the fact that in the sum on the right-hand side of (6.20) the subwalks $\omega^{(i)}$ are all self-avoiding due to the events $A^{(m)}_i$, and that these events imply $\eta^{(2m-1)}$ is self-avoiding or a self-avoiding polygon. The sum on the right-hand side of (6.21) is $H_{z,\kappa}(x-y_{2m-1})$, as there is no contribution from zero-step walks due to the constraint $\tilde{n} \geq 1$.

Repeating this procedure for $2m-2, 2m-3,\ldots,1$ gives the two-point functions in the upper bounds of the proposition. Factors of $H_{z,\kappa}$ arise for edges on which $n_i \geq 1$, and factors of $G_{z,\kappa}$ for those with $n_i \geq 0$; see (6.14). The overall factor of

$(1+\kappa)^{(4m-2)k_0}$ arises as we obtain a factor of $(1+\kappa)^{2k_0}$ for each factor of $H_{z,\kappa}$ and $G_{z,\kappa}$, and there are $2m-1$ of these. □

6.5 Proof of Gaussian decay

We will now prove Theorem 2.3 by making use of Theorem A.6. Recall that $|||x||| := \max\{\|x\|_2, 1\}$.

Proposition 6.7 ([8, Prop. 1.7]) *If $f, g\colon \mathbb{Z}^d \to \mathbb{R}$ satisfy $|f(x)| \le |||x|||^{-a}$ and $|g(x)| \le |||x|||^{-b}$ with $a \ge b > 0$, there exists $C = C(a,b,d)$ such that*

$$|(f * g)(x)| \le \begin{cases} C|||x|||^{-b} & a > d, \\ C|||x|||^{d-(a+b)} & a < d and a + b > d. \end{cases} \tag{6.22}$$

Proposition 6.8 *Let $d > 4$. Suppose $\beta > 0$, $\kappa \le \min\{\kappa_0, \beta\}$, $z_0 \le z \le 2$, and that*

$$G_{z,\kappa}(x) \le \beta |||x|||^{-(d-2)}, \qquad x \ne o. \tag{6.23}$$

If $\beta \le \beta_0(d)$, then there is a $c = c(d) > 0$ such that

$$\left|\Pi_{z,\kappa}(x)\right| \le c\beta \mathbb{1}_{\{x=o\}} + \frac{c\beta^2}{|||x|||^{3(d-2)}}. \tag{6.24}$$

Proof $G_{z,\kappa}(o) = 1$ as only the trivial self-avoiding walk ends at the origin, so (6.23) implies $G_{z,\kappa}(x) \le |||x|||^{-(d-2)}$ and $H_{z,\kappa}(x) \le \beta |||x|||^{-(d-2)}$ for all $x \in \mathbb{Z}^d$. In particular, $\|H_{z,\kappa}\|_\infty \le \beta$.

First consider $\pi^{(1)}_{z,\kappa}(x)$. If $x = o$ then $r_\kappa = 1$, so (6.17), $z \le 2$, and the inequality $\|f * g\|_1 \le \|f\|_\infty \|g\|_1$ imply

$$\left|\pi^{(1)}_{z,\kappa}(o)\right| \le (1+\kappa)^{2k_0} z \sum_{y \in \mathbb{Z}^d} D(y) H_{z,\kappa}(-y) \le 2(1+\kappa)^{2k_0}\beta \tag{6.25}$$

since $\sum_y D(y) = 1$. If $x \ne o$ then $|r_\kappa(x)| \le \kappa \mathbb{1}_{\{\|x\|_\infty = 1\}}$, and an argument as above shows that

$$\begin{aligned} \left|\pi^{(1)}_{z,\kappa}(x)\right| &\le (1+\kappa)^{2k_0} \kappa z \mathbb{1}_{\{\|x\|_\infty=1\}} \sum_{y \in \mathbb{Z}^d} D(y) H_{z,\kappa}(x-y) \\ &\le 2(1+\kappa)^{2k_0}\beta^2 \mathbb{1}_{\{\|x\|_\infty=1\}} \end{aligned} \tag{6.26}$$

where we have used $\kappa \le \beta$ and $z \le 2$.

Next we consider $\pi^{(m)}_{z,\kappa}(x)$ for $m \ge 2$. The factors r_κ in (6.18) imply the collections $\boldsymbol{y}$ of vertices that give a non-zero contribution satisfy

$$\|y_3\|_\infty \le 1, \quad \|x - y_{2m-2}\|_\infty \le 1, \quad \text{and}$$
$$\|y_{2j+3} - y_{2j}\|_\infty \le 1 \text{ for } j = 1, \dots, m-2. \tag{6.27}$$

Given $\boldsymbol{y}$ satisfying (6.27), define $\boldsymbol{\rho}$ to be the collection of vectors

$$\rho_1 := y_3, \quad \rho_{2m+1} := x - y_{2m-2}, \quad \text{and}$$
$$\rho_{2j+1} := y_{2j+1} - y_{2j-2}, \text{ for } j = 1, \dots, m-2. \tag{6.28}$$

Each ρ_j satisfies $\|\rho_j\|_\infty \le 1$. The sum over $\boldsymbol{y}$ in (6.18) can be replaced by a sum over y_i for $i = 2, 4, \dots, 2m-2$ and a sum over the possible $\boldsymbol{\rho}$. Formally, letting $\pi^{(m)} = \pi^{(m)}_{z,\kappa}$, we re-express (6.18) as

$$\pi^{(m)}(x) := (1+\kappa)^{(4m-2)k_0} \sum_{\boldsymbol{y}'} \sum_{\boldsymbol{\rho}} \pi^{(m)}_{\boldsymbol{y}',\boldsymbol{\rho}}(x), \tag{6.29}$$

where the sum over $\boldsymbol{y}'$ is over tuples $(y_2, y_4, \dots, y_{2m-2})$ and this defines the terms $\pi^{(m)}_{\boldsymbol{y}',\boldsymbol{\rho}}(x)$ as the contributions to (6.18) with the vertices y_i determined by $\boldsymbol{y}'$ and $\boldsymbol{\rho}$; we will shortly give an explicit formula for the $\pi^{(m)}_{\boldsymbol{y}',\boldsymbol{\rho}}$. We have abused notation in (6.29), but the bold subscripts will ensure that $\pi^{(m)}_{\boldsymbol{y}',\boldsymbol{\rho}}$ is distinguished from $\pi^{(m)}_{z,\kappa}$ in what follows.

We will now show the proposition follows from the estimate

$$\sum_{\boldsymbol{y}'} \pi^{(m)}_{\boldsymbol{y}',\boldsymbol{\rho}}(x) \le \beta^m C^m |||x|||^{-3(d-2)}, \qquad m \ge 2, \tag{6.30}$$

for a constant $C > 0$ independent of β. Equation (6.30) is uniform in $\boldsymbol{\rho}$, so with (6.29) it implies

$$\left|\pi^{(m)}_{z,\kappa}(x)\right| \le (C'\beta)^m |||x|||^{-3(d-2)}, \qquad m \ge 2, \tag{6.31}$$

where C' can be taken to be $C(1 + (3^d - 1)\kappa)(1+\kappa)^{2k_0}$. The factor of $(1+\kappa)^{2k_0}$ is from the prefactor in (6.29). The factor of $(1 + (3^d - 1)\kappa)$ arises as (i) each ρ_i has $3^d - 1$ non-zero possibilities, (ii) each i with $\|\rho_i\|_\infty = 1$ carries a factor of κ from $r_\kappa(\rho_i)$, and (iii) $r_\kappa(o) = 1$. Summing (6.31) over $m \ge 2$ and combining it with the bounds (6.25) and (6.26) for $m = 1$ implies the proposition. The dependence of β in the proposition is on d alone because $\kappa \le \min\{\kappa_0, \beta\}$ by hypothesis, so C' depends only on the dimension d.

The remainder of the proof establishes (6.30), and for this we need an explicit formula for $\pi^{(m)}_{\boldsymbol{y}',\boldsymbol{\rho}}(x)$. Fix $\boldsymbol{\rho}$, let $H = H_{z,\kappa}$, $G = G_{z,\kappa}$, $r = r_\kappa$, and $y_0 = o$. Recall that $\mathcal{T}_a f(x) = f(x-a)$ for $f : \mathbb{Z}^d \to \mathbb{R}$ and $x, a \in \mathbb{Z}^d$. This yields

$$\pi^{(m)}_{\boldsymbol{y}',\boldsymbol{\rho}}(x) := \sum_{\boldsymbol{y}'} H(y_2) \prod_{j=1}^{m-2} \left(\mathcal{T}_{-\rho_{2j+1}} H(y_{2j-2} - y_{2j}) \mathcal{T}_{\rho_{2j+1}} G(y_{2j+2} - y_{2j-2})\right)$$
$$\mathcal{T}_{\rho_{2m-1}} H(y_{2m-4} - y_{2m-2}) \mathcal{T}_{-\rho_{2m-1}} H(x - y_{2m-4}) \tag{6.32}$$

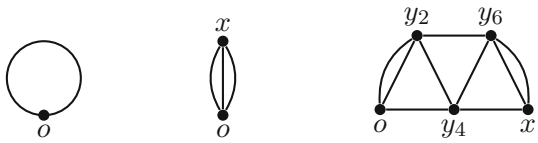


Fig. 10 Diagrammatic representation of Eq. (6.32) when $m = 1, 2, 5$. This is precisely the form of the upper bounds for self-avoiding walk, i.e., $\kappa = 0$. Horizontal lines represent factors of $G_{z,\kappa}$ or its translates, and the remaining lines represent factors of $H_{z,\kappa}$

where the sum is over $y_2, \ldots, y_{2m-2}$.

Diagrammatically, see Fig. 10, Equation (6.32) has exactly the form of a self-avoiding walk diagrammatic bound [4], but where the two-point functions G, H have been replaced with their translates. Our estimates for G and H imply there is an $a = a(d)$ such that

$$\left|\mathcal{T}_\rho G_{z,\kappa}(x)\right| \le a(d)|||x|||^{-(d-2)} \tag{6.33}$$

$$\left|\mathcal{T}_\rho H_{z,\kappa}(x)\right| \le a(d)\beta|||x|||^{-(d-2)} \tag{6.34}$$

since $|||x + \rho|||/|||x|||$ is uniformly bounded above when $\|\rho\|_\infty \le 1$. Thus, the two-point functions $\mathcal{T}_\rho G$ and $\mathcal{T}_\rho H$ satisfy, up to a constant depending only on d, the same estimates as do G and H.

The remainder of the proof is standard in lace expansion analyses, and hence we will be somewhat brief. See, e.g., [8, proof of Prop. 1.8(a)] for more details.

Define $\tilde{G}$ and $\tilde{H}$ to be the upper bounds on G and H given by the right-hand sides of Equations (6.33) and (6.34). Let

$$\begin{aligned} A(u, v, x, y) &:= \tilde{H}(v-u)\tilde{G}(y-u)\mathbb{1}_{\{v=x\}}, \\ M^{(2)}(x, y) &:= \tilde{H}(x)^2\tilde{G}(y), \\ M^{(m)}(x, y) &:= \sum_{u,v\in\mathbb{Z}^d} M^{(m-1)}(u, v)A(u, v, x, y), \qquad m \ge 3. \end{aligned}$$

With these definitions, we obtain

$$\sum_{\boldsymbol{y}'} \pi^{(m)}_{\boldsymbol{y}',\boldsymbol{\rho}}(x) \le M^{(m)}(x, x), \qquad m \ge 2, \tag{6.35}$$

where in the case $m = 2$ we have degraded the bound slightly by using the estimate $H \le G$. By (6.33) and (6.34) there is a constant $c' = c'(d) > 0$ such that

$$A(u, v, x, y) \le \frac{c'\beta}{|||v-u|||^{d-2}|||y-u|||^{d-2}}\mathbb{1}_{\{v=x\}}. \tag{6.36}$$

Define

$$\bar{\mathcal{S}} := \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \frac{1}{|||y|||^{d-2}|||x-y|||^{d-2}}.$$

When $d > 4$, $\bar{\mathcal{S}}$ is finite by an elementary convolution estimate [8, Proposition 1.7]. By an induction on m using (6.36) it can be shown [8, p. 381-382] that this implies there is a $C = C(d)$ such that

$$M^{(m)}(x, y) \le (c'\beta)^m (C\bar{\mathcal{S}})^{m-2} \frac{1}{|||x|||^{2(d-2)}|||y|||^{d-2}}, \qquad m \ge 2, \tag{6.37}$$

which proves Equation (6.30). □

Proof of Theorem 2.3 To prove Theorem 2.3, it suffices to verify that there is a κ_0 such that, if $\kappa \le \kappa_0$, the hypotheses of Sect. A.2 on D, $G_{z,\kappa}$, and $\Pi_{z,\kappa}$ are satisfied.

Hypothesis A.1 is trivially satisfied. Hypothesis A.3 is satisfied for $\kappa \le \kappa_0(L)$ for some $\kappa_0(L)$ by Theorem 2.1 which ensures the critical point exists, and Proposition 5.6, which ensures the divergence of the susceptibility.

We now verify the monotonicity hypothesis, (ii), and (iii) of Hypothesis A.4. Since $G_{z,\kappa}(x)$ is an absolutely convergent power series with positive coefficients when $z < z_c$, it is monotone and continuous for $z < z_c$. The exponential decay hypothesis is provided by Lemma 5.5.

To verify (i) of Hypothesis A.4, let $z_0 = (1+\kappa)^{-2(d-1)}$. When $z \le z_0$,

$$\begin{aligned} G_{z,\kappa}(x) &= \sum_n \sum_{\omega\in\Gamma_n(x)} z^n \mathcal{W}_\kappa(\omega) \le \sum_n \sum_{\omega\in\Gamma_n(x)} (1+\kappa)^{-2(d-1)n} \mathcal{W}_\kappa(\omega) \\ &\le \sum_n \sum_{\omega\in\Gamma_n(x)} \mathbb{P}_n(\omega) = G_{1,0}(x). \end{aligned} \tag{6.38}$$

This implies $G_{z_0,\kappa}(x) \le S_1(x)$, as $S_1(x)$ is clearly an upper bound for the SAW two-point function.

For any D, Hypothesis A.5 follows for $G_{z,\kappa}$ by Proposition 6.8 when κ is small enough, with β_0 uniform in κ. Thus, for $\kappa \le \kappa_0(L_0)$, with L_0 the constant of Theorem A.6, we can apply Theorem A.6 by the discussion of Sect. A.4. This proves the theorem. □

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Appendix A: Gaussian asymptotics

This appendix reviews [8, Theorem 1.2], which derives Gaussian asymptotics for critical two-point functions. Our motivation is that the presentation in [8] is, at places, dependent on the particular models being studied. The proofs, however, apply essentially verbatim to other models. Our review axiomatizes sufficient assumptions for models similar to self-avoiding walk. We indicate where these assumptions are used in proofs, but omit the portions of the proofs that purely replicate [8]. We emphasise that the result and techniques are those of [8], and our presentation is primarily for the benefit of the reader who is not familiar with [8].

A.1 Setup

Let $\mathbb{R}_{\geq 0}$ denote the non-negative reals. For $z \in \mathbb{R}_{\geq 0}$, $G_z \colon \mathbb{Z}^d \to \mathbb{R}_{\geq 0}$, $\tilde{\Pi}_z \colon \mathbb{Z}^d \to \mathbb{R}$, and D a probability distribution on $\mathbb{Z}^d$, we consider the convolution equation

$$G_z(x) = \delta_{o,x} + \tilde{\Pi}_z(x) + (zD * (\delta + \tilde{\Pi}_z) * G_z)(x). \tag{A.1}$$

We will further assume that G_z, $\tilde{\Pi}_z$, and D are all $\mathbb{Z}^d$-symmetric, and that $G_z(x)$ is a power series in z with non-negative coefficients. We will see in Section A.4 that the analysis of (A.1) also applies to the convolution equation derived for κ-ASAW in the main body of the text.

The *critical point* z_c is $z_c = \sup\{z \in \mathbb{R}_{\geq 0} \mid \chi(z) < \infty\}$, where the *susceptibility* $\chi(z)$ is defined by

$$\chi(z) := \sum_{x \in \mathbb{Z}^d} G_z(x). \tag{A.2}$$

A.2 Hypotheses and Theorem

Hypothesis A.1 Assume that D is a spread-out step distribution as defined in Definition 1.

Let X_n be a discrete time simple random walk with step distribution D. Let $\sigma^2 = \sum_{x \in \mathbb{Z}^d} D(x)\|x\|_2^2$. Note that σ^2 is comparable to the spread-out parameter L^2. The *non-interacting two-point function* S_μ is defined by

$$S_\mu(x) := \sum_{n=0}^{\infty} \mu^n \mathsf{P}_0\left[X_n = x\right]. \tag{A.3}$$

An important consequence of the form of D is the following proposition. Let $a_d := \frac{d\Gamma(d/2-1)}{2\pi^{d/2}}$, where Γ is Euler's gamma function.

Proposition A.2 ([8, Prop. 1.6]) *Suppose $d > 2$ and Hypothesis A.1 holds. For L sufficiently large, $\alpha > 0$, $\mu \leq 1$, and $x \in \mathbb{Z}^d$,*

$$S_\mu(x) \le \delta_{o,x} + O\left(\frac{1}{L^{2-\alpha}|||x|||^{d-2}}\right) \tag{A.4}$$

$$S_1(x) = \frac{a_d}{\sigma^2}\frac{1}{|||x|||^{d-2}} + O\left(\frac{1}{|||x|||^{d-\alpha}}\right). \tag{A.5}$$

The implicit constants may depend on α, but not on L.

Note that, for fixed d, the leading coefficient in (A.5) is proportional to L^{-2}. The next two hypotheses deal with the critical point and behaviour of G_z for $z_0 \le z < z_c$, where $z_0 > 0$ is a chosen value of the parameter z.

Hypothesis A.3 The critical point z_c satisfies $z_0 < z_c < \infty$. The susceptibility specified by (A.2) diverges as the critical point is approached from below: $\lim_{z\uparrow z_c} \chi(z) = \infty$.

Hypothesis A.4 G_z is well-defined, not identically zero, and monotone increasing in z. For $z_0 \le z < z_c$ and for each $x \in \mathbb{Z}^d$,

(i) $G_{z_0}(x) \le S_1(x)$,
(ii) $G_z(x)$ is continuous for $z \in [z_0, z_c)$, and
(iii) for $t > 0$ and $z \in [z_0, z_c - t)$ there are constants $c(t), C(t) > 0$ such that

$$G_z(x) \le C(t)e^{-c(t)|||x|||}. \tag{A.6}$$

The most substantial hypothesis is the next one.

Hypothesis A.5 Assume

$$G_z(x) \le \beta|||x|||^{-d+2}, \qquad x \ne o. \tag{A.7}$$

Suppose also that $z_0 \le z \le 2$. If $\beta < \beta_0$, there is a constant $c = c(d) > 0$ such that

$$\left|\tilde{\Pi}_z(x)\right| \le c\beta\delta_{o,x} + \frac{c\beta^2}{|||x|||^{3(d-2)}}. \tag{A.8}$$

Theorem A.6 ([8, Theorem 1.2]) *Assume D, G_z, and $\tilde{\Pi}_z$ satisfy the hypotheses of Section A.2. Choose $0 < \alpha < 2$. Let β_0 be the constant of Hypothesis A.5.*

There is an $L_0(d, \alpha, \beta_0)$ such that, for $L \ge L_0$, the function $G_{z_c}\colon \mathbb{Z}^d \to \mathbb{R}$ is well-defined, and there is an $A > 0$ such that

$$G_{z_c}(x) \sim \frac{a_d A}{\sigma^2|||x|||^{2-d}}\left(1 + O\left(\frac{L^2}{|||x|||^{2-\alpha}}\right)\right). \tag{A.9}$$

The implicit constants are uniform in x and L. The values of z_c and A are $1 + O(L^{\alpha-2})$.

A.3 Proof

The next proposition is the heart of the analysis. In what follows we assume the hypotheses of Theorem A.6; in particular, β_0 is given.

Proposition A.7 *Fix $\alpha > 0$. There is an $L_0 = L_0(\beta_0, d, \alpha, z_0)$ such that, for $L \geq L_0$,*

$$G_{z_c}(x) \leq \frac{\text{const}}{L^{2-\alpha} |||x|||^{d-2}}, \qquad x \neq o, \tag{A.10}$$

and $z_c \leq 1 + O(L^{-2+\alpha})$.

Lemma A.8 (Lemma 2.1 [8]) *Let $f : [z_1, z_c) \to \mathbb{R}$, and $a \in (0, 1)$. Suppose*

(i) *f is continuous on $[z_1, z_c)$,*
(ii) *$f(z_1) \leq a$, and*
(iii) *for $z \in [z_1, z_c)$ the inequality $f(z) \leq 1$ implies the inequality $f(z) \leq a$.*

Then $f(z) \leq a$ for all $z \in [z_1, z_c)$.

Proof of Proposition A.7 The proof is essentially that in [8]. We present the steps in which our hypotheses, as opposed to model-specific facts, are used.

Note that it suffices to prove that (A.10) holds for $\alpha < \frac{1}{2}$, as the right-hand side is increasing in α. By Hypothesis A.4 and the monotone convergence theorem, it is enough to prove this for all $z_0 < z < z_c$.

Let K be the optimal constant for the error bound in Proposition A.2:

$$K = \sup_{L \geq 1, x \neq o} L^{2-\alpha} |||x|||^{d-2} S_1(x),$$

and note K is finite by (A.4). Define

$$g_x(z) = (2K)^{-1} L^{2-\alpha} |||x|||^{d-2} G_z(x),$$

and let $g(z) = \sup_{x \neq o} g_x(z)$. To prove (A.10), we will use Lemma A.8 with $f(z) = \max\{g(z), \frac{z}{2z_0}\}$, $z_1 = z_0$, and $a \in \left(\frac{1}{2}, 1\right)$ arbitrary. The claim that $z_c = 1 + O(L^{-2+\alpha})$ will be established in the course of the argument.

Claim *Hypothesis (i) of Lemma A.8 holds.*

Proof For $x \in \mathbb{Z}^d$, $g_x(z)$ is continuous on $[z_0, z_c)$ by Hypothesis A.4. It suffices to show $\sup_{x \neq o} g_x(z)$ is continuous on $[z_0, z_c - t)$ for arbitrarily small $t > 0$.

Fix $t > 0$, and let $z \in [z_0, z_c - t)$. By Hypothesis A.4, $g_x(z)$ decays exponentially in $\|x\|_2$ with decay rate independent of z. Therefore, $\sum_{x \in \mathbb{Z}^d} g_x(z)$ converges exponentially fast with rate independent of z. It follows that the supremum of $g_x(z)$ occurs on $B_R(o)$, the ball of radius R about the origin, for some $R = R(L) > 0$. This proves $\sup_{x \neq o} g_x(z)$ is a continuous function of $z \in [z_0, z_c - t)$ since the supremum of a finite set of continuous functions is continuous. □

Claim *Hypothesis (ii) of Lemma A.8 holds.*

Proof By Hypothesis A.4 and the definition of K, $g_x(z_0) \le \frac{1}{2}$ for all x. Since $a > \frac{1}{2}$, this proves the claim. □

Claim *Hypothesis (iii) of Lemma A.8 holds.*

Proof Fix $z_0 < z < z_c$ and suppose $f(z) \le 1$. Then z is at most $2z_0$, and

$$G_z(x) \le 2z_0 K L^{-2+\alpha} |||x|||^{2-d}, \qquad x \ne o. \tag{A.11}$$

Let $\beta = 2z_0 K L^{-2+\alpha}$. By Hypothesis A.5, when $L^{-2+\alpha}$ is sufficiently small there is a $c > 0$ such that

$$\left|\tilde{\Pi}_z(x)\right| \le c\beta\delta_{o,x} + c\beta^2 |||x|||^{-3(d-2)} \le \frac{c\beta}{|||x|||^{3(d-2)}}. \tag{A.12}$$

By Hypothesis A.4, G_z is not identically zero. Thus $\chi(z) > 0$, and the sum of (A.1) over all $x \in \mathbb{Z}^d$ can be rearranged to give

$$\chi(z) = \frac{1 + \sum_x \tilde{\Pi}_z(x)}{1 - z - z\sum_x \tilde{\Pi}_z(x)} > 0. \tag{A.13}$$

By (A.12), $\|\tilde{\Pi}_z(x)\|_1 < 1$ for L large enough. This implies the numerator, and hence the denominator, of (A.13) is strictly positive. Since $f(z) \le 1$, this implies that

$$z < 1 - z \sum_{x \in \mathbb{Z}^d} \tilde{\Pi}_z(x) \le 1 + O(z_0 L^{-2+\alpha}). \tag{A.14}$$

Thus $\frac{z}{2}$ is bounded above by a for $a \in \left(\frac{1}{2}, 1\right)$, provided that L is large enough.

What remains is to prove $g(z) \le a$ for $a \in \left(\frac{1}{2}, 1\right)$ when L is large enough. This exactly follows the presentation in [8, p. 364], and hence we omit it. □

By Hypothesis A.4 this proves the desired bounds, as we have proven that $f(z) \le a$ for $z_0 \le z < z_c$. The bound on z_c follows from (A.14), which holds as it was derived under the hypothesis that $f(z) \le 1$. □

Proof of Theorem A.6 This follows [8, Theorem 1.2]. The only model specific step in the cited proof is showing that an auxiliary parameter μ_z increases to $\mu_{z_c} = 1$ as $z \uparrow z_c$. We define this parameter below and show that it takes the desired value by Hypothesis A.3.

By (A.12), $\tilde{\Pi}_z(x)$ has a finite second moment when L is large enough. It therefore makes sense to define

$$\lambda_z = \frac{1}{1 + z\sigma^{-2}\sum_x \|x\|_2^2 \tilde{\Pi}_z(x)}, \tag{A.15}$$

$$\mu_z = 1 - \lambda_z \left(1 - z - z \sum_x \tilde{\Pi}_z(x) \right). \tag{A.16}$$

Equation (A.12) implies $\lambda_z \to 1$ as $L \to \infty$ uniformly in $z \in [z, z_c]$. By Equation (A.13) and Hypothesis A.3, as $z \uparrow z_c$, the quantity in brackets in (A.16) tends to zero. Thus, $\mu_{z_c} \uparrow 1$ as $z \uparrow z_c$. □

A.4 Other convolution equations

Consider the equation

$$G_z = \delta + z(D * G_z) + (\Pi_z * G_z). \tag{A.17}$$

If Π satisfies Hypothesis A.5, it is possible to manipulate (A.17) into the form (A.1). To see this, rewrite (A.17) as

$$\begin{aligned} G &= \delta + \Pi + zD * (\delta + \Pi) * G - \Pi * (\delta + zD * G - G) \\ &= \delta + \Pi + zD * (\delta + \Pi) * G + \Pi * \Pi * G, \end{aligned}$$

where, in the second equality, we have used (A.17) to rewrite the term in parentheses, and the subscripts z have been omitted. Rewriting the last factor of G using (A.17) yields

$$G = \delta + \Pi + \Pi^{*2} + zD * (\delta + \Pi + \Pi^{*2}) * G + \Pi^{*3} * G,$$

where A^{*k} is the k-fold autoconvolution of A. Iterating this yields (A.1) with

$$\tilde{\Pi}_z = \sum_{k \geq 1} \Pi^{*k}, \tag{A.18}$$

since $\lim_{n\to\infty} \Pi^{*n} = 0$ under the assumption that Π satisfies Hypothesis A.5. Finally, [8, Proposition 1.7] implies that, if Π_z satisfies Hypothesis A.5, then $\tilde{\Pi}_z$ defined by (A.18) satisfies Hypothesis A.5, for possibly different constants. The change in constants depends only on d. See [8, Section 4.1] for a further discussion of this point. Thus to apply Theorem A.6 to the convolution equation (A.17), it suffices to verify the hypotheses of Section A.2 for G_z, D, and Π.

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