Probab. Theory Relat. Fields (2018) 171:685–708
https://doi.org/10.1007/s00440-017-0790-0

The box-crossing property for critical two-dimensional oriented percolation

H. Duminil-Copin[1,2] · V. Tassion[2] · A. Teixeira[3]

Received: 1 November 2016 / Revised: 19 June 2017 / Published online: 29 June 2017

Abstract We consider critical oriented Bernoulli percolation on the square lattice $\mathbb{Z}^2$. We prove a Russo–Seymour–Welsh type result which allows us to derive several new results concerning the critical behavior:

- We establish that the probability that the origin is connected to distance n decays polynomially fast in n.
- We prove that the critical cluster of 0 conditioned to survive to distance n has a typical width w_n satisfying $\varepsilon n^{2/5} \le w_n \le n^{1-\varepsilon}$ for some $\varepsilon > 0$.

The sub-linear polynomial fluctuations contrast with the supercritical regime where w_n is known to behave linearly in n. It is also different from the critical picture obtained for non-oriented Bernoulli percolation, in which the scaling limit is non-degenerate in both directions. All our results extend to the graphical representation of the one-dimensional contact process.

✉ A. Teixeira
augusto@impa.br

H. Duminil-Copin
duminil@ihes.fr

V. Tassion
vincent.tassion@unige.ch

[1] Institut des Hautes Études Scientifiques, Le Bois-Marie 35, route de Chartres, 91440 Bures-sur-Yvette, France

[2] Département de mathématiques, Université de Genéve, 2-4 rue du lièvre, 64, 1211 Geneva 4, Switzerland

[3] Instituto Nacional de Matemática Pura e Aplicada - IMPA, Estrada Dona Castorina 110, Rio de Janeiro, RJ 22460-320, Brazil

Keywords Percolation · Oriented percolation · Critical behaviour · Contact process · Renormalization

Mathematics Subject Classification 60K35 · 82B43 · 82C43

1 Introduction

1.1 Motivation

Oriented percolation, which is a directed version of classical Bernoulli percolation (introduced by Broadbent and Hammersley [3] to understand percolation of a liquid in a porous medium), provides a model for a variety of physical systems in chemistry, solid state physics, and astrophysics. At a theoretical level, it is one of the simplest system exhibiting a phase transition, and has been as such an object of intensive study in the last 50 years. It is also related to the geometric representation of the one-dimensional contact process introduced by Harris [15,16] and is therefore interesting from the point of view of particles systems as well. We refer to [11] for a review on the subject and for further references.

The model is defined as follows. Consider the rotated (and rescaled) square lattice $\mathbb{L} := \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 \text{ even}\}$. Each vertex $x \in \mathbb{L}$ is connected to the vertices $x + (-1, 1)$ and $x + (1, 1)$ by two oriented edges, see Fig. 1. Let $p \in [0, 1]$. Each oriented edge is said to be *open* with probability p, and *closed* with probability $1 - p$, independently of the state of the other edges. The law of the set of open edges is denoted by $\mathbb{P}_p$.

In oriented percolation, we study the connectivity properties of the random graph with vertex set $\mathbb{L}$, and edge set given by the open oriented edges. These open oriented edges should be understood as the set of edges allowing us to go upwards in the system. An *open path* is a collection of vertices $x_0, x_1, \ldots, x_k$ such that the oriented edge (x_i, x_{i+1}) is open for every $0 \le i < k$. Two vertices x and y are said to be *connected* (denoted $x \to y$) if there exists an open path starting at x and ending at y. Let C_0 be the connected component of the origin, i.e. the set of vertices x such that $0 \to x$. In what follows, $0 \to \infty$ denotes the event that C_0 is infinite.

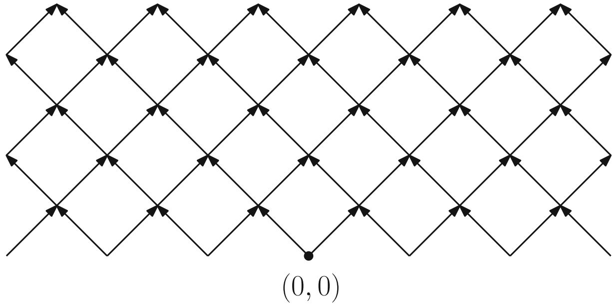

Fig. 1 The lattice $\mathbb{L}$ with the oriented edges

One of the main interest of the model lies in the existence of a phase transition at a value $p_c \in (0, 1)$ such that $\mathbb{P}_p(0 \to \infty) = 0$ if $p < p_c$, and above which $\mathbb{P}_p(0 \to \infty) > 0$ if $p > p_c$ (see [1,4] for non-trivial lower and upper bounds on p_c). For $p < p_c$, connectivity properties are known to decay exponentially fast (see [13] for the original proof, and [11] for more details), while for $p > p_c$, the global shape of C_0 converges to a cone of opening $\alpha(p) > 0$ and Gaussian fluctuations on the boundary of C_0, as proved in [12,17]. An alternative proof of exponential decay for $p < p_c$ together with a proof of the mean-field bound $\mathbb{P}_p(0 \to \infty) \geq c(p - p_c)$ was provided recently in [5]. These results are just a few examples illustrating the more general motto that the subcritical and supercritical phases $p < p_c$ and $p > p_c$ are now well understood.

In [7] and in [2] respectively, the authors proved that $\alpha(p_c) = 0$ and $\mathbb{P}_{p_c}[0 \to \infty] = 0$. These results naturally raise the question of quantitative bounds on the probability of being connected to distance n and the typical width of large connected components at criticality. In this paper, we provide polynomial upper bounds on these quantities (some lower bounds were proved previously in [8]).

1.2 Main results

The main results of this paper deal with the critical phase $p = p_c$. The first theorem states that the probability that 0 is connected to distance n decays polynomially fast. For $n \geq 0$, define $\ell_n := \mathbb{Z} \times \{n\}$.

Theorem 1.1 *There exists $\varepsilon > 0$ such that for every $n \geq 1$,*

$$\frac{\varepsilon}{n^{1/2-\varepsilon}} \leq \mathbb{P}_{p_c}(0 \to \ell_n) \leq \frac{1}{n^{\varepsilon}}.$$

It was already known that $\lim_{n\to\infty} \sqrt{n}\mathbb{P}_{p_c}(0 \to \ell_n) = +\infty$ (see e.g. Eq. (1.9) in [8]) and $\lim_{n\to\infty} \mathbb{P}_{p_c}(0 \to \ell_n) = 0$ (see [2]). The novelty of this paper is the polynomial n^{ε} improvement in these estimates.

The second theorem deals with the typical width of the set of vertices connected to the origin. More precisely, let

$$R_n := \max\{x \in \mathbb{Z} : \exists y \leq 0 \text{ even such that } (y, 0) \to (x, n)\}.$$

Note that when $0 \to \ell_n$, then R_n is the first coordinate of the right-most point of C_0. In some sense, the quantity R_n can be understood as the width of a typical cluster that reaches distance n. The next theorem provides non-trivial polynomial bounds on R_n.

Theorem 1.2 *There exists $\varepsilon > 0$ such that for every $n \geq 1$,*

$$\varepsilon n^{2/5} \leq \mathbb{E}_{p_c}(R_n \,|\, 0 \to \ell_n) \leq n^{1-\varepsilon}. \tag{1.1}$$

The lower bound $\mathbb{E}_{p_c}(R_n \,|\, 0 \to \ell_n) \geq \varepsilon n^{1/4}$ for all n (and $\varepsilon n^{2/5}$ for infinitely many scales) was proved in [8]. The novelty of the paper lies in the upper bound. We wish to highlight the fact that the existence of $\varepsilon > 0$ in the $n^{1-\varepsilon}$ upper bound is maybe the most important feature of the previous theorem. It implies that large connected components are rather thin. We should mention that it was shown that $\alpha(p) \searrow 0$ as $p \searrow p_c$, thus suggesting that the scaling limit indeed needs to be rescaled differently in the x and y coordinates (contrarily to the non-oriented cases where both directions play symmetric roles). The quantitative polynomial bound seems to be new.

We believe that the techniques developed to prove the two previous theorems should be very useful to study more delicate properties of the critical phase. In order to emphasize the technique, we isolate one important technical statement, called the *box-crossing property*, which we consider as one of the main new inputs of the paper.

The statement of the box-crossing property involves crossing probabilities. A *vertical crossing* of a box $B = [a, b] \times [c, d]$ is an open path of vertices *in* B from the bottom $[a, b] \times \{c\}$ to the top $[a, b] \times \{d\}$ of B. A *left–right crossing* is an open path of vertices from the left $\{a\} \times [c, d]$ to the right $\{b\} \times [c, d]$ of B. Similarly, one define a *right-left crossing* of B.

Note that a left–right crossing does not imply a right-left crossing (nor the other way around), as it is the case for non-oriented percolation. If such a vertical (resp. left–right, right-left) crossing exists, we say that B is *crossed vertically* (resp. from left to right, from right to left). Define

$$V_p(m, n) := \mathbb{P}_p([0, m] \times [0, n] \text{ is crossed vertically}),$$
$$H_p(m, n) := \mathbb{P}_p([0, m] \times [0, n] \text{ is crossed from left to right}).$$

By symmetry, $H_p(m, n)$ is also the probability that $[0, m] \times [0, n]$ is crossed from right to left. We are now ready to state our main technical statement.

Theorem 1.3 (The box-crossing property) *There exist a sequence of integers* $(w_n)_{n\geq 1}$ *and a constant* $c_1 > 0$ *such that*

$$c_1 \leq H_{p_c}(3w_n, n) \leq H_{p_c}(w_n, 3n) \leq 1 - c_1. \tag{1.2}$$
$$c_1 \leq V_{p_c}(w_n, 3n) \leq V_{p_c}(3w_n, n) \leq 1 - c_1. \tag{1.3}$$

We wish to highlight that similar statements are also available in the context of critical non-oriented percolation, with $w_n = n$ in this case. We will see that w_n is of the same order as $\mathbb{E}_{p_c}(R_n \mid 0 \to \ell_n)$ and can therefore be intuitively understood as the typical width of a connected component of height n. As opposed to the non-oriented case, we will show that w_n is not growing linearly but is in fact smaller than $n^{1-\varepsilon}$.

The different rectangles involved in the previous statement will be the "elementary bricks" for all the constructions made in this article. The quantities on the right correspond to "crossings in the easy direction", while those on the left corresponds to "crossings in the hard direction", meaning that compared to a "square box" of size w_n times n, the events on the left involve rectangles which are three times longer in the direction of crossing, while the events on the right involve rectangles which are three times larger orthogonally to the direction of crossing. The proof of the box-crossing property is based on an analog in the oriented case of the Russo–Seymour–Welsh (RSW) result for two-dimensional non-oriented Bernoulli percolation (see [6] for a

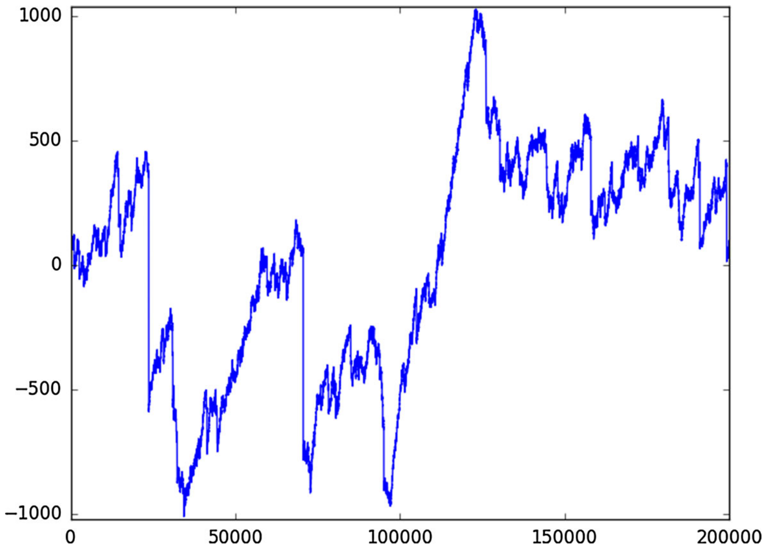

Fig. 2 On the *left*, C_0 conditioned on $0 \to \ell_{8000}$ and $0 \nrightarrow \ell_{10000}$. Above, the process obtained by taking the right-most particle of C_0 conditioned on $0 \to \ell_{200\,000}$

recent survey on this subject). This RSW result is stated as Theorem 2.1 in Sect. 2. The reader should be careful that the specificities of the oriented case make the proof of the RSW result very different from the non-oriented case, and that the denomination simply refer to the fact that crossings of rectangles in the hard direction are expressed in terms of crossing of rectangles in the easy direction.

Generalization to other two-dimensional models We work with a specific choice of model but we believe that the proof extends *mutatis mutandis* to oriented percolation on $\mathbb{Z}^2$ where edges are oriented from x to $x + (0, 1)$, $x + (-1, 0)$ and $x + (1, 0)$, and to the geometric representation of the one-dimensional contact process.

Applications and open problems For non-oriented percolation, non-trivial bounds on crossing probabilities is the key step towards the understanding of the critical and near-critical phases. We believe that the box-crossing property established in this paper should lead to similar applications in the oriented case. For instance, scaling relations can be studied using [9,10].

Let us mention that studying the limit of C_0 conditioned on $0 \to \ell_n$ and computing the exact value of critical exponents is a major open question. In particular, two objects of special interest in the oriented case are the set of "renewal points" (i.e. heights that intersect C_0 only once), and the process of the "right-most particle" $n \mapsto R_n$, see Fig. 2.

Note that the process $n \mapsto R_n$ presents macroscopic jumps, as it becomes clear from Fig. 2 and the box-crossing properties of Theorem 1.3. In particular, one should consider the Skorokhod topology when attempting to understand the scaling limit of this process.

1.3 Preliminaries

Further notation We will always work with intersections of sets with $\mathbb{L}$. For instance, $[a,b]\times[c,d]$ will mean the intersection of $\mathbb{L}$ with the corresponding part of the plane. We write $A\to B$ for the event that there exist $x\in A$ and $y\in B$ with $x\to y$. Below, we will drop the subscript p_c in the notation and write for instance $\mathbb{P}$, $H(m,n)$ and $V(m,n)$ for $\mathbb{P}_{p_c}$, $H_{p_c}(m,n)$ and $V_{p_c}(m,n)$. Importantly, we will keep the subscript p when p is not a priori equal to p_c.

One input from percolation theory: the square root trick We will use repeatedly (see [14]) the classical Harris–Fortuin–Kasteleyn–Ginibre (FKG) inequality: for two increasing events[1] E and F,

$$\mathbb{P}_p(E\cap F)\geq \mathbb{P}_p(E)\mathbb{P}_p(F). \tag{FKG}$$

Let us also mention the following trivial application of the FKG inequality, called the *square-root trick*: for any increasing events $A_1,\ldots,A_N$,

$$\max\{\mathbb{P}_p(A_n):1\leq n\leq N\}\geq 1-\big(1-\mathbb{P}_p(A_1\cup\cdots\cup A_N)\big)^{1/N}. \tag{SRT}$$

Organization of the paper Section 2 is devoted to the proof of the Russo–Seymour–Welsh type result. This result is then used in Sect. 3 to derive the box-crossing property. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2.

2 Russo–Seymour–Welsh type result

This section is dedicated to the proof of a Russo–Seymour–Welsh theorem for oriented percolation. It enables us to express crossing probabilities of rectangles with different aspect ratios. We include also a technical (and easy) result at the end of this section. In this section it will be convenient to use crossing probabilities for rectangle which may have non integer dimensions. If r, s are two real numbers, we set

$$H_p(r,s)=H_p(\lceil r\rceil,\lceil s\rceil)\quad\text{and}\quad V_p(r,s)=V_p(\lceil r\rceil,\lceil s\rceil), \tag{2.1}$$

where $\lceil r\rceil$ denotes the upper integer part of r.

Theorem 2.1 (RSW type result) *For any $\alpha\in(\frac{3}{4},1)$, there exist $\varepsilon\in(0,1)$ and an increasing homeomorphism $g_0:[0,1]\to[0,1]$ such that for any $m,n\geq 1$,*

$$\min\big\{V_p(m,3n),H_p(3m,n)\big\}\geq g_0\big(\min\big\{V_p(m,\alpha\varepsilon n),H_p(\alpha m,\varepsilon n)\big\}\big).$$

On the left, if a rectangle of size m times n is our reference, the crossing probabilities involve rectangles which are three times longer in the direction of crossing. On the right, if a rectangle of size m times εn is our reference, the crossing probabilities involve

[1] An event E is increasing if it is stable with respect to opening edges.

rectangles which are slightly shorter in the direction of crossing. This is reminiscent of the classical RSW theory for non-oriented percolation: crossing probabilities in the hard direction can be bounded from below by expressions involving crossing probabilities in the easy direction.

The heights of the rectangles are very different on the left and the right (there is a factor roughly ε between the two), which is a major difference between the oriented and the non-oriented cases. Said differently, in order to obtain estimations on probabilities of crossings of rectangles in the hard direction, one needs to pay a cost on the height of the rectangle. The following example illustrates perfectly why changing the height is necessary in the oriented case: think of the extremal case of the horizontal crossing from left to right of a rectangle of size n times n. In this case, it is simply impossible to cross horizontally a rectangle of size $2n$ times n due to the direction of the edges. Note moreover that for non-oriented percolation, there is no need to increase the hight of the box in Russo–Seymour–Welsh's theorem.

We start the proof of the theorem by a key lemma allowing us to increase the width of rectangles which are crossed horizontally.

Lemma 2.2 *For any* $\alpha \in (\frac{3}{4}, 1)$*, there exists an increasing homeomorphism* $g_1 : [0, 1] \to [0, 1]$ *such that for any* $k, \ell \geq 1$*,*

$$H_p(k, \ell) \geq g_1\big(\min\big\{V_p(k, \ell), H_p(\alpha k, \ell/2)\big\}\big).$$

Note that in the statement above, we allow the variables k and ℓ to take non-integer values.

Proof Let us first assume that $k/2$, $\ell/2$ and αk are integers (this is purely for convenience as can be seen at the end of the proof). Introduce the boxes

$$B = [-k/2, k/2] \times [0, \ell] \quad \text{and} \quad B_r = [0, \alpha k] \times [\ell/2, \ell].$$

illustrated on Fig. 3. Let E be the event that there exists an open path in $B \cup B_r$ starting from the bottom of B and ending on the right side of B_r. Let us prove that

$$\mathbb{P}_p[E] \geq H_p(\alpha k, \ell/2)\Big(1 - \sqrt{1 - V_p(k, \ell)}\Big). \tag{2.2}$$

In order to get this inequality, we use a "conditioning on the top-most left–right crossing of B_r" argument, as illustrated on Fig. 3. This type of reasoning is now classical in percolation.

For a configuration ω containing a left–right crossing of B_r, define Γ to be the top-most left–right crossing[2] of B_r. When there is no such crossing, set $\Gamma = \emptyset$. Since the box B_r is crossed from left to right with probability $H(\alpha k, \ell/2)$, we have

$$H(\alpha k, \ell/2) = \sum_{\gamma \neq \emptyset} \mathbb{P}_p(\Gamma = \gamma), \tag{2.3}$$

[2] Formally, this can be seen as the largest left–right crossing of B_r for the natural lexicographical order on path induced by the lexicographical order on vertices and the order that the edge going left from a vertex is smaller than the edge going right.

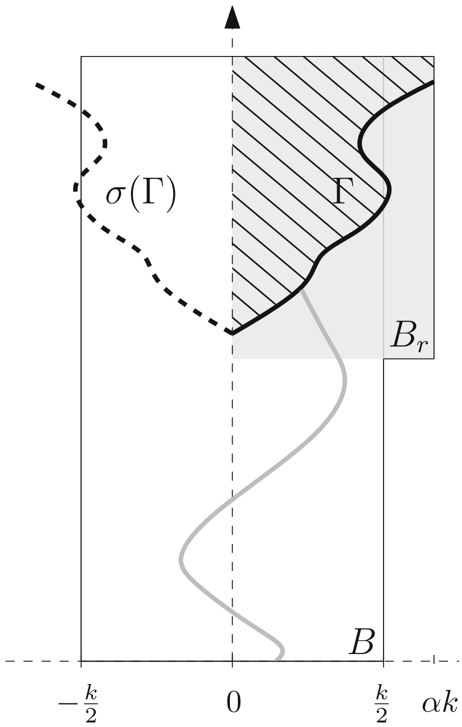

Fig. 3 Construction of the event E. First, we require that the *box* B_r be crossed from *left* to *right*, and we explore the top-most *left–right* crossing Γ in B_r. After this exploration, the edges in the hatched region have been discovered. Then we ask that in the unexplored region there exists an open path (in *grey*) connecting the *bottom* side of B to Γ

where the sum is over all the possible left to right paths in B_r. Fix for a moment such a path γ. Introduce the orthogonal symmetry σ with respect to the axis $x = 0$. Define S_γ to be the set of vertices of B which are reachable from a vertex of the bottom of B by an oriented path of edges not crossing $\gamma \cup \sigma(\gamma)$. Let J_γ be the event that there exists a path in S_γ connecting the bottom side of B to γ inside S_γ. Using symmetry and the square root trick, together with the fact that any path crossing B vertically must contain a path reaching γ or $\sigma(\gamma)$ in S_γ, we find

$$\mathbb{P}_p(J_\gamma) \geq 1 - \sqrt{1 - V_p(k, \ell)}. \tag{2.4}$$

Now, if $\Gamma = \gamma$ and J_γ occurs then the event E occurs. Therefore, summing over all the possible paths γ, we obtain

$$\begin{aligned}\mathbb{P}_p(E) &\geq \sum_{\gamma \neq \emptyset} \mathbb{P}_p(\{\Gamma = \gamma\} \cap J_\gamma) \\ &= \sum_{\gamma \neq \emptyset} \mathbb{P}_p(\Gamma = \gamma)\mathbb{P}_p(J_\gamma).\end{aligned} \tag{2.5}$$

In the second line, we used that the event $\Gamma = \gamma$ is measurable with respect to the edges with both ends in $B \setminus S_\gamma$ while J_γ is measurable with respect to the edges in S_γ, therefore these two events are independent. We finally obtain Eq. (2.2) by combining the equation above together with (2.3) and (2.4).

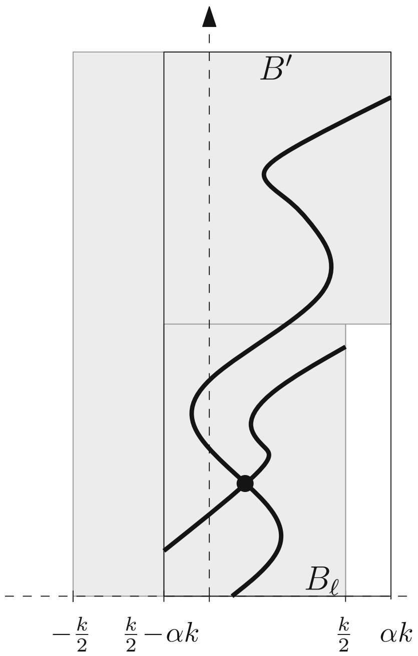

Fig. 4 Two possible cases when the event $E \cap C_\ell$ occurs: on the *left* picture, the left–right crossing of B_ℓ intersect the path realizing E, and on the the *right* picture the two paths do not intersect. In both cases, the *box* B' is crossed from *left* to *right*

We now conclude the proof. Consider the boxes

$$B_\ell = [k/2 - \alpha k, k/2] \times [0, \ell/2],$$
$$B' = [k/2 - \alpha k, \alpha k] \times [0, \ell].$$

Let C_ℓ be the event that B_ℓ is crossed from left to right. On the event $E \cap C_\ell$, there must exist a path from left to right in the box B'. Indeed, we are in one of the two following cases (illustrated on Fig. 4):

- A crossing from left to right in B_ℓ intersects a crossing from the bottom of B to the right of B_r, thus creating a left–right crossing in B'.
- No crossing from left to right in B_ℓ intersects a crossing from the bottom of B to the right of B_r, in such case any of the latter paths contains a left–right crossing B'.

Since $2\alpha - \frac{1}{2} > 1$, we deduce that

$$\begin{aligned} H_p(k, \ell) &\geq \mathbb{P}_p(B' \text{ is crossed from left to right}) \\ &\geq \mathbb{P}_p(E \cap C_\ell) \\ &\overset{\text{(FKG)}}{\geq} \mathbb{P}_p(C_\ell)\mathbb{P}_p(E) \\ &\overset{(2.2)}{\geq} H_p(\alpha k, \ell/2)^2\Big(1 - \sqrt{1 - V_p(k, \ell)}\Big). \end{aligned}$$

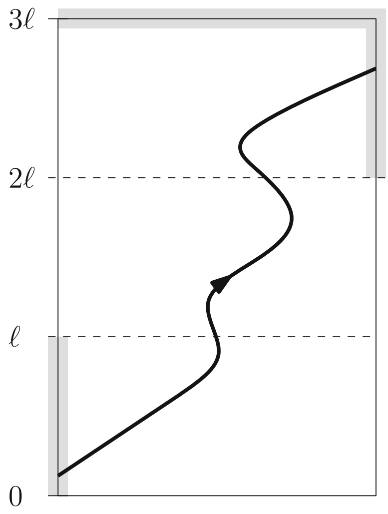

Fig. 5 Diagrammatic representation of the event $G(k, \ell)$

This finishes the proof of the case where k and ℓ are two even integers. For general real values k large enough and $\ell \geq 1$, one may do the same proof with $B = [-\lceil k/2 \rceil, \lceil k/2 \rceil] \times [0, \lceil \ell \rceil]$ and $B_r = [0, \lceil k/2 \rceil] \times [\lceil \ell \rceil - \lceil \ell/2 \rceil, \lceil \ell \rceil]$ provided that $2\lceil \alpha k \rceil - \lceil k/2 \rceil \geq k$. Finally, note that by choosing g_1 properly, we may cover the case of small values of k. □

The next trivial lemma will be useful in the proof. For $k, \ell \geq 1$ integers, let $G(k, \ell)$ be the event that $\{0\} \times [0, \ell]$ is connected to $\{k\} \times [2\ell, 3\ell]$ or $[0, k] \times \{3\ell\}$ (see Fig. 5).

Lemma 2.3 *For any integer $C > 0$ and any integers $k, \ell \geq 1$,*

$$V_p(k, C\ell) \geq \mathbb{P}_p\big(G(k, \ell)\big)^{2C}.$$

Proof For an integer $i \geq 0$, let F_i be the event that $\{0\} \times [i\ell, (i+1)\ell]$ is connected to $\{0\} \times [(i+2)\ell, (i+3)\ell]$ inside the strip $[0, k] \times \mathbb{Z}$. First, by translation invariance, the probability of F_i is equal to the probability of F_0.

Observe that the model features a symmetry with respect to vertical reflections (with respect to the axis $y = 0$). Moreover, the event F_0 occurs as soon $G(k, \ell)$ occurs together with a symmetric version of it (see Fig. 6). Therefore, by the FKG inequality, we have for every $i \geq 0$

$$\mathbb{P}_p(F_i) = \mathbb{P}_p(F_0) \geq \mathbb{P}_p(G(k, \ell))^2.$$

Finally, if all the events F_i occur for $0 \leq i < C$, the box $[0, k] \times [\ell, \ell + C\ell]$ is crossed vertically. The lemma thus follows from the FKG inequality. □

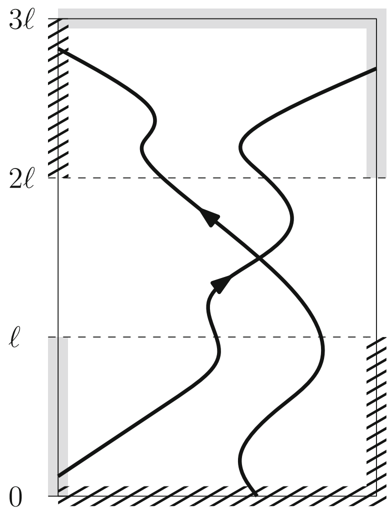

Fig. 6 The event F_0 obtained by intersecting $G(k, \ell)$ and a symmetric version of it

Proof of Theorem 2.1 Without loss of generality, we may assume that αm is an integer. We start by proving the bound on $H_p(3m, n)$ assuming the bound on $V_p(m, 3n)$. For $k \geq m$ and $\ell \leq 3n$, Lemma 2.2 implies

$$H_p(k, \ell) \geq g_1\big(\min\big\{V_p(m, 3n), H_p(\alpha k, \ell/2)\big\}\big).$$

By iterating the statement above s times, we get for every $s \geq 1$

$$H_p(\alpha^{1-s} m, n) \geq g_1^{(s)}\big(\min\big\{V_p(m, 3n), H_p(\alpha m, n/2^s)\big\}\big).$$

Fix $s = s(\alpha)$ such $\alpha^{1-s} \geq 3$ and set $\varepsilon = \varepsilon(\alpha) = 2^{-s}$. Then the equation above implies the desired inequality. Note that this is the only place where the constant ε is used: it guarantees that the height of the rectangles obtained via the iteration of Lemma 2.2 is always smaller than n (and hence a fortiori $3n$).

Let us now focus on the lower bound on $V_p(m, 3n)$. Let $\ell = \alpha\varepsilon n/12$ and let g_2 be an homeomorphism defined through:

$$g_*(x) = 1 - (1-x)^{1/12},\ g_\#(x) = 1 - (1-x)^{1/2} \text{ and } g_2(x) = g_\# \circ g_*(x). \quad (2.6)$$

We may assume without loss of generality that ℓ is an integer. We divide the proof in two cases.

Case 1 $H_p(\alpha m, 2\ell) < g_2(H_p(\alpha m, \varepsilon n))$.

For $i = 0, \ldots, 11$, let A_i be the event that there exists an open path from $\{0\} \times [i\ell, (i+1)\ell]$ to $\{\alpha m\} \times [0, 12\ell]$ in the strip $[0, \alpha m] \times \mathbb{Z}$. Since for every i, $\mathbb{P}_p(A_0) \geq \mathbb{P}_p(A_i)$, the square-root trick implies that there exists some i with

$$\mathbb{P}_p(A_0) \geq 1 - (1 - H_p(\alpha m, \varepsilon n))^{1/12} = g_*\big(H_p(\alpha m, \varepsilon n)\big).$$

Now, if A_0 occurs, then either $[0, \alpha m] \times [0, 2\ell]$ is crossed horizontally, or the event $G(\alpha m, \ell)$ occurs. As a consequence, the square-root trick used one more time implies that

$$\max\{H_p(\alpha m, 2\ell), \mathbb{P}_p(G(\alpha m, \ell))\} \geq g_2(H_p(\alpha m, \varepsilon n)).$$

(This is the definition of g_2 used above). The assumption on $H_p(\alpha m, \varepsilon n)$ implies that

$$\mathbb{P}_p(G(\alpha m, \ell)) \geq g_2(H_p(\alpha m, \varepsilon n)),$$

so that Lemma 2.3 applied to $k = \alpha m$, ℓ and $C > 16/\alpha\varepsilon$ gives

$$V_p(m, 3n) \geq g_3(H_p(\alpha m, \varepsilon n)).$$

Case 2 $H_p(\alpha m, 2\ell) \geq g_2(H_p(\alpha m, \varepsilon n))$.
In such case, Lemma 2.2 implies that

$$\begin{aligned} H_p(m, 4\ell) &\geq g_1(\min\left\{V_p(m, 4\ell), g_2(H_p(\alpha m, 2\ell))\right\}) \\ &\geq g_1(\min\left\{V_p(m, \alpha\varepsilon n), g_2(H_p(\alpha m, \varepsilon n))\right\}). \end{aligned} \tag{2.7}$$

Since $G(m, 4\ell)$ occurs as soon as there exists a left–right crossing of $[0, m] \times [0, 4\ell]$ and a vertical crossing of $[0, m] \times [0, 12\ell]$, the FKG inequality implies immediately that

$$\mathbb{P}_p(G(m, 4\ell)) \geq H_p(m, 4\ell) V_p(m, 12\ell). \tag{2.8}$$

Since $12\ell \leq \alpha\varepsilon n$, (2.7) and (2.8) can be combined to obtain

$$\mathbb{P}_p(G(m, 4\ell)) \geq g_4(\min\{V_p(m, \alpha\varepsilon n), H_p(\alpha m, \varepsilon n)\}).$$

Lemma 2.3 applied with $k = m$, ℓ and $C > 8/\alpha\varepsilon$ gives

$$V_p(m, 3n) \geq g_5(\min\{V_p(m, \alpha\varepsilon n), H_p(\alpha m, \varepsilon n)\}), \tag{2.9}$$

thus concluding the proof in this case as well. □

Let us mention the following technical statement, which will be useful in the next sections.

Lemma 2.4 *For any* $\Delta > \delta > 1$*, there exists* $C > 0$ *such that for any* $n, m \geq 1$*,*

$$\max\{V_p(\Delta m, n), H_p(m, \Delta n)\} \leq g_6\big(\max\left\{V_p(\delta m, n), H_p(m, \delta n)\right\}\big),$$

where $g_6(x) = 1 - (1-x)^C$ *for any* $x \in [0, 1]$.

Proof Let us present the proof for $V_p(\Delta m, n)$ (the proof for $H_p(m, \Delta n)$ can be adapted easily). Set $\varepsilon < (\delta - 1)$ and an integer $K > \Delta/\varepsilon$. We may assume without loss of generality that εm and δm are two integers.

Define the two collections of boxes

$$\mathscr{F} = \Big\{[k\varepsilon m, (k\varepsilon + \delta)m] \times [0, n], 0 \le k < K\Big\},$$
$$\mathscr{E} = \Big\{[k\varepsilon m, (k\varepsilon + 1)m] \times [0, n], 0 \le k < K\Big\}.$$

For $[0, \Delta m] \times [0, n]$ to be crossed vertically, then one of the boxes in $\mathscr{F}$ must be crossed vertically, or one of the boxes in $\mathscr{E}$ must be crossed from left to right, or one of the boxes in $\mathscr{E}$ must be crossed from right to left. In other words, the event that $[0, \Delta m] \times [0, n]$ is crossed vertically is contained in the union of $3K$ events of probability smaller or equal to $V_p(\delta m, n)$ and $H_p(m, n)(\le H_p(m, \delta n))$. The square-root trick implies that

$$V_p(\Delta m, n) \le 1 - (1 - x)^{3K},$$

where $x := \max\big\{V_p(\delta m, n), H_p(m, \delta n)\big\}$. The proof follows by setting $C = 3K$. □

Remark 2.5 Combined with Theorem 1.3 below, Lemma 2.4 shows that for any $\Delta > 1$, $H(w_n, \Delta n)$ and $V(\Delta w_n, n)$ are bounded by $1 - c(\Delta) < 1$ uniformly in $n \ge 1$.

3 The box-crossing property

This section is devoted to the proof of Theorem 1.3. With the help of the RSW result from the previous section, the proof of the theorem is not more than a proper definition for w_n. Theorem 2.1 does the work for us, since it enables us to invoke two classical results on crossing probabilities (see the lemma below), which are somehow not specific to oriented percolation.

Lemma 3.1 (Finite size criteria for $p < p_c$ and $p > p_c$) *There exists* $\eta > 0$ *such that for* $p \in (0, 1)$ *and* $m, n \ge 1$,

- *If* $\max\{V_p(2m, n), H_p(m, 2n)\} < \eta$, *then* $p < p_c$ *and there exists* $c > 0$ *such that for any* $N \ge 1$,
$$\mathbb{P}_p\big(0 \to \ell_N\big) \le \exp(-cN).$$
- *If* $\min\{V_p(m, 2n), H_p(2m, n)\} > 1 - \eta$, *then* $p > p_c$ *and there exists* $c > 0$ *such that for any* $N \ge 1$,
$$\mathbb{P}_p\big(0 \to \ell_N, 0 \nrightarrow \infty\big) \le \exp(-cN). \tag{3.1}$$

Before proving this lemma, let us show the theorem. Recall that we omit the subscript p_c.

Proof of Theorem 1.3 Let $n \ge 1$ large enough. Set η to be the constant in the previous lemma. Fix any α and α' with $3/4 < \alpha' < \alpha < 1$. Then use Theorem 2.1 to obtain $\varepsilon = \varepsilon(\alpha') > 0$ such that the conclusion of the theorem holds with α' (and consequently also for α). Introduce now

$$w_n := \inf\big\{m \ge 0 : H(\alpha m, \varepsilon n) \le V(m, \alpha\varepsilon n)\big\}. \tag{3.2}$$

Note that w_n diverges with n. Introduce the following notation:

$$H_- := H(\alpha(w_n - 1), \varepsilon n) \quad H_+ := H(\alpha w_n, \varepsilon n)$$
$$V_- := V(w_n - 1, \alpha\varepsilon n) \qquad V_+ := V(w_n, \alpha\varepsilon n).$$

The definition of w_n implies that $H_+ \leq V_+$ and $H_- > V_-$.

Proof of the lower bound Using monotonicity, and then Lemma 2.4 applied to $\Delta = 2/\alpha$ and some $\delta \in (1, 1/\alpha)$, we obtain

$$\begin{aligned}\max\{V(2w_n, \varepsilon n), H(w_n, 2\varepsilon n)\} &\leq \max\{V(2w_n, \alpha\varepsilon n), H(\alpha w_n, 2\varepsilon n)\}\\ &\leq g_6\big(\max\{V(\delta\alpha w_n, \alpha\varepsilon n), H(\alpha w_n, \delta\alpha\varepsilon n)\}\big)\\ &\leq g_6(\max\{V_-, H_+\}).\end{aligned}$$

The first item of Lemma 3.1 thus implies that $\max\{V_-, H_+\} \geq g_6^{-1}(\eta)$. This gives that either $H_- > V_- \geq g_6^{-1}(\eta)$ or $V_+ \geq H_+ \geq g_6^{-1}(\eta)$. In either case, Theorem 2.1 may be applied to get

$$\min\{V(w_n, 3n), H(3w_n, n)\} \geq g_0(g_6^{-1}(\eta)).$$

Proof of the upper bound We use Theorem 2.1 with our chosen $\alpha' \in (3/4, \alpha)$ to conclude that

$$\begin{aligned}\min\{V(w_n, 3n), H(3w_n, n)\} &\geq g_0\big(V(w_n, \alpha'\varepsilon n) \wedge H(\alpha' w_n, \varepsilon n)\big)\\ &\geq g_0(\min\{V_+, H_-\}).\end{aligned}$$

The second item of Lemma 3.1 thus implies that $\min\{V_+, H_-\} \leq g_0^{-1}(1-\eta)$. This gives that either $H_+ \leq V_+ \leq g_0^{-1}(1-\eta)$, or $V_- < H_- \leq g_0^{-1}(1-\eta)$. In either case, Lemma 2.4 may be applied with $\delta = 1/\alpha$ and $\Delta = 3/(\alpha\varepsilon)$ to get

$$\max\{V(3w_n, n), H(w_n, 3n)\} \leq g_6(g_0^{-1}(1-\eta)), \tag{3.3}$$

concluding the proof of the theorem. □

Proof of Lemma 3.1 *Proof of the first item* Introduce the sequence of scales $m_k = 2^k m$ and $n_k = 2^k n$ for $k \geq 0$ and set

$$u_k = \max\big\{H_p(m_k, 2n_k), V_p(2m_k, n_k)\big\}.$$

A vertical crossing of the box $[0, 4m_k] \times [0, 2n_k]$ must contain vertical crossings of the boxes $[0, 4m_k] \times [0, n_k]$ and $[0, 4m_k] \times [n_k, 2n_k]$. Lemma 2.4 (and the trivial bound $g_6(x) := 1 - (1-x)^C \leq Cx$) thus implies that $V_p(2m_{k+1}, n_{k+1}) \leq (Cu_k)^2$. Doing the same with $H_p(m_{k+1}, 2n_{k+1})$, we deduce that

$$u_{k+1} \leq (Cu_k)^2.$$

By choosing $\eta < \frac{1}{eC^2}$ small enough, $u_0 < \eta$ implies that $u_k \leq \exp(-2^k)$ for any $k \geq 0$. To conclude, fix $N \geq 1$ and let K be the unique integer such that $n_K \leq N < 2n_K$. The event $0 \to \ell_N$ implies that one of the three rectangles $[-m_K, m_K] \times [0, n_K]$, $[0, m_K] \times [2n_K]$ or $[-m_K, 0] \times [0, 2n_K]$ must be crossed "in the easy direction", we deduce that

$$\mathbb{P}_p\big(0 \to \ell_N\big) \leq 3u_K \leq \exp\{-cN\},$$

for a constant $c > 0$ small enough. This finishes the proof of exponential decay.

The fact that $p < p_c$ follows from the observation that the condition $\max\{V_p(2m, n), H_p(m, 2n)\} < \eta$ is satisfied for some $p' > p$, and that therefore $p < p' \leq p_c$.

Proof of the second item For this proof, we consider a dependent percolation defined on a renormalized lattice. More precisely, given integers $m, n \geq 1$, associate to every $x = (i, j) \in \mathbb{L}$ the boxes

$$\begin{aligned} B_x &:= [0, m] \times [0, 2n] \ + \ (im, jn), \\ B_x^+ &:= [0, 2m] \times [0, n] \ + \ \big(im, (j+1)n\big), \\ B_x^- &:= [0, 2m] \times [0, n] \ + \ \big((i-1)m, (j+1)n\big). \end{aligned}$$

Say that the edge $(x, x + (1, 1))$ is *open* if B_x is crossed vertically and B_x^+ is crossed from left to right. Analogously, say that the edge $(x, x + (-1, 1))$ is *open* if B_x is crossed vertically and B_x^- is crossed from right to left. Denote the induced percolation measure $\mathbb{P}_p^{m,n}$.

On the event that there is an infinite path of open edges starting from the origin (using the above definition), then there is also an infinite open path on the original lattice, starting from $[0, m] \times [0, n]$.

Note that the above percolation measure is 3-dependent, as defined below (7.60) of [14], p. 178. Therefore, using a result by Liggett, Schonmann and Stacey (see Theorem (7.65) of [14]), we conclude that there exists an $\varepsilon > 0$ such that if

$$\mathbb{P}_p^{n,m}\Big(\big((0, 0), (1, 1)\big) \text{ is open}\Big) > 1 - \varepsilon, \tag{3.4}$$

then there exists $c > 0$ such that for every $N \geq 1$,

$$\min\{V_p(N, 2N), H_p(2N, N)\} \geq 1 - \exp(-cN),$$

(see for instance the contour counting argument presented in Section 10 of [11] for additional details). The claim follows since (3.4) is directly implied by the assumption in the statement.

The above implies that $p > p_c$ since $\min\{V_p(m, 3n), H_p(3m, n)\} > 1 - \eta$ is satisfied for some $p' < p$.

Note that the above argument is very particular of two dimensional percolation. First, in higher dimensions there is no guarantee that vertical and horizontal crossings intersect. Moreover, for large enough dimensions it is expected that crossings between macroscopic objects occur with high probability even at criticality.

4 Proofs of the main theorems

4.1 Relation between R_n and w_n

In this section we use the box-crossing property to show that w_n is equal up to constant to several quantities related to R_n. The two first items below will be useful to obtain polynomial bounds on w_n. The last two items are useful to get the main theorems. Below, $x^+ = \max\{x, 0\}$.

Proposition 4.1 *There exist constants $c_3, c_4, c_5, c_6 > 0$ such that for every $n \geq 1$,*

(i) $c_3 w_n \leq \mathbb{E}(R_n^+) \leq \frac{1}{c_3} w_n$,
(ii) $c_4 w_n \leq \sqrt{\mathrm{Var}(R_n)} \leq \frac{1}{c_4} w_n$,
(iii) $c_5 w_n \leq \mathbb{E}(R_n \mid 0 \to \ell_n) \leq \frac{1}{c_5} w_n$,
(iv) $c_6 w_n \leq \sqrt{\mathrm{Var}(R_n \mid 0 \to \ell_n)} \leq \frac{1}{c_6} w_n$.

The proof of this proposition is heavily based on the box-crossing property. In particular, we will use several times the following event E, whose probability is bounded from below using the box-crossing property. Let $B = [-\frac{1}{2}w_n, \frac{5}{2}w_n] \times [0, n]$ and define E (see Fig. 7) to be the event that there exists an open path from $[-\frac{1}{2}w_n, 0] \times \{0\}$ to $[2w_n, \frac{5}{2}w_n] \times \{n\}$ in B. The event E occurs if there exist

- a vertical crossing from $[-\frac{1}{2}w_n, 0] \times \{0\}$ to the top side of B,
- a vertical crossing from the bottom side of B to $[2w_n, \frac{5}{2}w_n] \times \{n\}$,
- a left–right crossing of B.

By the box-crossing property and symmetry, the two first paths exist with probability larger than $c_1/2$, and the third with probability larger than c_1. The FKG inequality implies that

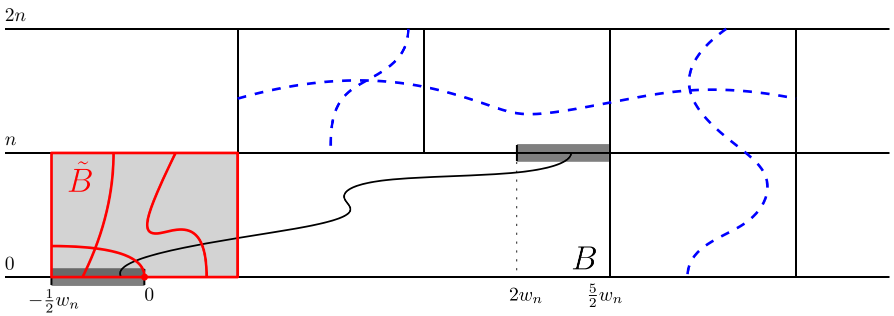

Fig. 7 The crossing in *black* illustrates the occurrence of the event E. The *red* picture illustrates the bound $\mathbb{P}(0 \to \ell_n) \geq \frac{c_1^2}{4}\mathbb{P}(0 \to \partial\tilde{B})$: paths combine to obtain a path from 0 to ℓ_n: if $0 \to \partial\tilde{B}$ implies the existence of a path from 0 to $\partial\tilde{B}\backslash\ell_n$, then the paths from $[-\frac{1}{2}w_n, 0] \times \{0\}$ or $[0, \frac{1}{2}w_n] \times \{0\}$ would cross it to create a path from 0 to ℓ_n in $\tilde{B}$. The *dotted blue lines* denoted dual path (which are not necessarily oriented) preventing the existence of oriented crossings (color figure online)

$$\mathbb{P}(E) \geq \tfrac{1}{4}c_1^3. \tag{4.1}$$

We are now in a position to attack the proof of Proposition 4.1.

Proof We prove each item one after the other.
(i) The lower bound is immediate since

$$\mathbb{E}(R_n^+) \geq 2w_n\mathbb{P}[R_n^+ \geq 2w_n] \geq 2w_n\mathbb{P}(E) \overset{(4.1)}{\geq} 2w_n \cdot \tfrac{1}{4}c_1^3. \tag{4.2}$$

The upper bound follows directly from the following exponential bound on the tail of R_n: for every $k \in \mathbb{N}$,

$$\mathbb{P}(R_n \geq kw_n) \leq (1 - c_1)^k, \tag{4.3}$$

which is obtained as follows. If $R_n \geq kw_n$, then there must exist an open path from left to right inside the rectangle $[0, kw_n] \times [0, n]$. In particular k disjoint rectangles of size w_n by n must be crossed from left to right by an open path. This observation and independence imply that

$$\mathbb{P}(R_n \geq kw_n) \leq H(kw_n, n) \leq H(w_n, n)^k \leq H(w_n, 3n)^k.$$

The box-crossing property implies (4.3). By summing over k, we find that

$$\mathbb{E}(R_n^+) \leq w_n \sum_{k=0}^{\infty} \mathbb{P}(R_n \geq kw_n) \leq \frac{1}{c_1} w_n,$$

which gives the desired upper bound.

(ii) By (4.2), we already know that $\mathbb{P}(R_n \geq 2w_n) \geq \frac{c_1^3}{4}$. Since $R_n \leq w_n$ on the event that $[0, w_n] \times [0, n]$ is not crossed from left to right, we deduce that $\mathbb{P}(R_n \leq w_n) \geq c_1$. This directly implies that the lower bound on the standard deviation.

The upper bound follows once again from the following exponential bound on the tail of $|R_n|$: for every $k \in \mathbb{N}$,

$$\mathbb{P}(|R_n| \geq kw_n) \leq 2(1 - c_1)^k. \tag{4.4}$$

The contribution of $R_n \geq 0$ is controlled by (4.3). For the contribution of $R_n \leq 0$, observe that $R_n \leq -kw_n$ implies that the rectangle $[-kw_n, 0] \times [0, n]$ is not crossed vertically. In particular, k disjoint rectangles of size w_n by n fail to be crossed vertically. Using independence, the box crossing property implies

$$\mathbb{P}(R_n \leq -kw_n) \leq 1 - V(kw_n, n) \leq (1 - V(w_n, n))^k \leq (1 - c_1)^k.$$

(iii) We use a technique similar to the proof of (i). The lower bound is slightly more delicate here because we do not take the positive part of R_n and we therefore have to show that the negative part does not counterbalance the positive part. To achieve this, we use that the law of the cluster of 0 is invariant by the orthogonal reflection σ with respect to the vertical axis $y = 0$.

Let F be the intersection of the event E and its image by σ. The FKG inequality together with (4.1) implies that the event F occurs with probability larger than $c_1^6/16$.

Now, if 0 is connected to ℓ_n and F occurs, then R_n must be larger than $2w_n$. Therefore,

$$\mathbb{E}(R_n \mathbb{1}_{F\cap\{0\to\ell_n\}}) \geq 2w_n \mathbb{P}(F \cap \{0 \to \ell_n\}) \overset{\text{(FKG)}}{\geq} \tfrac{c_1^6}{8} w_n \mathbb{P}(0 \to \ell_n).$$

Furthermore, by invariance of F under symmetry,

$$\mathbb{E}(R_n \mathbb{1}_{F^c\cap\{0\to\ell_n\}}) = \tfrac{1}{2}\mathbb{E}(R_n \mathbb{1}_{F^c\cap\{0\to\ell_n\}}) - \tfrac{1}{2}\mathbb{E}(L_n \mathbb{1}_{F^c\cap\{0\to\ell_n\}}) \geq 0,$$

where L_n is the left-most point of ℓ_n connected to 0.

Summing the two displayed equations above and dividing by $\mathbb{P}(0 \to \ell_n)$ gives

$$\mathbb{E}(R_n \mid 0 \to \ell_n) \geq \tfrac{c_1^6}{8} w_n. \tag{4.5}$$

For the upper bound, we use an exponential domination as in (4.1). The only difference is that here we have to take care of the conditioning. Let $k \geq 1$. If $(R_n \geq kw_n, 0 \to \ell_n)$, then there must exist an open path from 0 to the boundary ∂B of the box $B = [-w_n, w_n] \times [0, n]$, and a left–right crossing of $[w_n, kw_n] \times [0, n]$. Using independence and the box-crossing property, we obtain

$$\begin{aligned}\mathbb{P}(R_n \geq kw_n, 0 \to \ell_n) &\leq H(w_n, n)^{k-1}\mathbb{P}(0 \to \partial B)\\ &\leq (1 - c_1)^{k-1}\mathbb{P}(0 \to \partial B).\end{aligned} \tag{4.6}$$

To conclude the proof, we need to compare the probability of an open path from 0 to ∂B with the probability of an open path from 0 to ℓ_n. We use the following observation. If 0 is connected to ∂B and the two rectangles $[-w_n, 0] \times [0, n]$ and $[0, w_n] \times [0, n]$ are crossed vertically by open paths, then 0 is connected to ℓ_n. The FKG inequality and the box crossing property imply

$$\mathbb{P}(0 \to \ell_n) \overset{\text{(FKG)}}{\geq} V(w_n, n)^2 \mathbb{P}(0 \to \partial B) \geq c_1^2 \mathbb{P}(0 \to \partial B). \tag{4.7}$$

Plugging the inequality in (4.6) and dividing by $\mathbb{P}(0 \to \partial B)$ gives

$$\mathbb{P}(R_n \geq kw_n \mid 0 \to \partial\ell_n) \leq (1 - c_1)^{k-1}/c_1^2, \tag{4.8}$$

which gives the claim after summing over k.

(iv) Lower bound - We already know from the previous part that $R_n \geq 2w_n$ with (conditional) probability larger than constant, so that we only need to prove that $\mathbb{P}(R_n \leq \frac{3}{2}w_n | 0 \to \ell_n) \geq c$. In order to see that, let $\tilde{B} = [-\frac{1}{2}w_n, \frac{1}{2}w_n] \times [0, n]$ (see Fig. 7) and the event that

(i) $[0, \frac{1}{2}w_n] \times \{0\}$ and $[-\frac{1}{2}w_n, 0] \times \{0\}$ are both connected to ℓ_n by a path in $\tilde{B}$, and 0 is connected to the boundary of $\tilde{B}$,
(ii) $[\frac{1}{2}w_n, \frac{3}{2}w_n] \times [0, n]$ is not crossed from left to right.

By symmetry and the box-crossing property, each of the two first paths are occurring with probability $\frac{1}{2}c_1$, therefore, the FKG inequality implies that the events in (i) occur with probability larger or equal to

$$\mathbb{P}((i) \text{ occurs}) \geq \tfrac{1}{4}c_1^2\mathbb{P}[0 \to \tilde{B}] \geq \tfrac{1}{4}c_1^2\mathbb{P}[0 \to \ell_n].$$

Since the event in (ii) does not depend on edges in $\tilde{B}$, the box-crossing property implies

$$\mathbb{P}(R_n \leq \tfrac{3}{2}w_n | 0 \to \ell_n) \geq \frac{\mathbb{P}((i) \text{ occurs})\mathbb{P}((ii) \text{ occurs})}{\mathbb{P}[0 \to \ell_n]} \geq \tfrac{1}{4}c_1^3.$$

The upper bound is a consequence of the following exponential domination. Recall that L_n was defined in the proof of (ii) as the left-most point of ℓ_n connected to 0 by an open path. Using $R_n \geq L_n$ and the fact that $-L_n$ has the same law as R_n (conditionally on the existence of an open path from 0 to distance n), we obtain for every $k \geq 1$

$$\begin{aligned}\mathbb{P}(|R_n| \geq kw_n | 0 \to \ell_n) &\leq \mathbb{P}(R_n \geq kw_n | 0 \to \ell_n) + \mathbb{P}(-L_n \leq -kw_n | 0 \to \ell_n) \\ &\leq 2\mathbb{P}(R_n \geq kw_n | 0 \to \ell_n) \overset{(4.8)}{\leq} 2(1-c_1)^{k-1}/c_1^2.\end{aligned}$$

4.2 Polynomial bounds on w_n

We start by proving a polynomial lower bound on w_n using the equivalence with $\sqrt{\mathrm{Var}(R_n)}$.

Proposition 4.2 *Fix $n \geq 1$. There exists a constant $c_7 > 0$ such that for every $n \geq 1$,*

$$w_n \geq c_7 n^{2/5}. \tag{4.9}$$

Proof The starting point of the proof is given by (1.8) in [8], which shows that there exists a constant $c_8 > 0$ such that

$$\mathrm{Var}(R_n) \geq c_8\, n\, \mathbb{P}[0 \to \ell_n]. \tag{4.10}$$

We refer the reader to the original paper for the argument. Let us simply say that it exploits a renewal structure of the right-most open path from $\mathbb{Z}_- \times \{0\}$ to ℓ_n (this path ends at (R_n, n)) by showing that between two consecutive renewal heights, the horizontal increment of the path has variance larger than $\frac{1}{4}$, and then showing that the expected number of renewal heights is at least $c_8\, n\, \mathbb{P}[0 \to \ell_n]$.

For $x \in \{0, \ldots, w_n\}$, let $E(x)$ be the event that there exists a vertical crossing in $B(w_n, 2n)$ that goes through the point (x, n). Note that our choice of the lattice implies that the event $E(x)$ is empty when x has a different parity from n. On the event $E(x)$, there exists an open path starting from ℓ_0 and ending at (x,n), and a path starting from (x,n) and ending on ℓ_{2n}. Hence we have, by independence and symmetry,

$$\mathbb{P}(E(x)) \le \mathbb{P}[0 \to \ell_n]^2.$$

Furthermore, the box-crossing property and the union bound imply

$$c_1 \le V(w_n, 2n) \le \sum_{0\le x\le w_n} \mathbb{P}(E(x)).$$

The combination of the two equations above finally gives

$$\mathbb{P}[0 \to \ell_n] \ge \frac{c_1}{\sqrt{w_n}}. \tag{4.11}$$

Inserting the bound on the variance of R_n obtained via (ii) of Proposition 4.1 in (4.10) gives

$$\left(\frac{w_n}{c_4}\right)^2 \ge \mathrm{Var}(R_n) \ge c_8\, n\, \mathbb{P}[0 \to \ell_n] \ge c_8\, n\, \frac{c_1}{\sqrt{w_n}}, \tag{4.12}$$

which concludes the proof. □

We now show a polynomial upper bound on w_n using the equivalence with $\mathbb{E}(R_n^+)$. The proof is based on sub-additivity properties of $\mathbb{E}(R_n^+)$ (see e.g. [11] for background).

Proposition 4.3 *There exists a constant $\epsilon > 0$ such that for every $n \ge 1$,*

$$\mathbb{E}(R_n^+) \le n^{1-\epsilon}. \tag{4.13}$$

Remark 4.4 This proposition, combined with (i) of Proposition 4.1, immediately implies that

$$c_3 w_n \le n^{1-\varepsilon}. \tag{4.14}$$

Proof The main step in the proof is to show that there exists a constant $c_9 > 0$ such that for every $n \ge 1$,

$$\mathbb{E}(R_{2n}^+) \le (2 - c_9)\mathbb{E}(R_n^+). \tag{4.15}$$

In order to compare $\mathbb{E}(R_n^+)$ with $\mathbb{E}(R_{2n}^+)$, it will be convenient to introduce the following more general variables. For $0 \le m \le n$, define

$$R_{m,n}^+ := \max\left\{0,\ \sup\{x \ge 0 \text{ s.t. } (-\infty, R_m^+] \times \{m\} \to (x + R_m^+, n)\}\right\}. \tag{4.16}$$

Note that $R_{0,n}^+ = R_n^+$ for every $n \ge 0$. Before moving further, let us mention two other useful properties of these variables that follow from the definition. First, translation invariance and independence imply that

$$\mathbb{E}(R_{m,n}^+) = \mathbb{E}(R_{n-m}^+). \tag{4.17}$$

Furthermore, for every percolation configuration ω, we have the following sub-additivity property

$$R^+_{0,n}(\omega) \le R^+_{0,m}(\omega) + R^+_{m,n}(\omega). \tag{4.18}$$

Note that (4.17) and (4.18) already imply that for every n,

$$\mathbb{E}(R^+_{2n}) \le \mathbb{E}(R^+_{0,n}) + \mathbb{E}(R^+_{n,2n}) = 2\mathbb{E}(R^+_n). \tag{4.19}$$

Hence, in order to prove (4.15), we need to show that the inequality above is not sharp. We do this by constructing an event on which R^+_{2n} is significantly smaller than $R^+_{0,n} + R^+_{n,2n}$.

Fix $n \ge 1$. Recall the definition of the event E and let F be the event (see Fig. 7) that

- $[\frac{1}{2}w_n, \frac{3}{2}w_n] \times [n, 2n]$ and $[\frac{5}{2}w_n, \frac{7}{2}w_n] \times [0, 2n]$ are not crossed from left to right,
- $[\frac{1}{2}w_n, \frac{7}{2}w_n] \times [n, 2n]$ is not crossed vertically.

The box-crossing property and the FKG inequality imply that $\mathbb{P}(F) \ge c_1^3$. Since E and F depend on different sets of edges, independence and (4.1) give

$$\mathbb{P}(E \cap F) \ge \tfrac{1}{4}c_1^6.$$

Now, observe that when the event $E \cap F$ occurs, we have $R^+_n \ge 2w_n$, $R^+_{n,2n} = 0$ and $R^+_{2n} \le \frac{3}{2}w_n$. Hence,

$$\mathbb{1}_{E\cap F} R^+_{2n} \le \mathbb{1}_{E\cap F}(R^+_n + R^+_{n,2n} - \tfrac{1}{2}w_n).$$

On the event $(E \cap F)^c$, the trivial bound provided by (4.18) gives

$$\mathbb{1}_{(E\cap F)^c} R^+_{2n} \le \mathbb{1}_{(E\cap F)^c}(R^+_n + R^+_{n,2n}).$$

Summing the two equations above and taking the expectation, we find

$$\mathbb{E}(R^+_{2n}) \le 2\mathbb{E}(R^+_n) - \tfrac{1}{2}w_n\mathbb{P}(E \cap F) \le (2 - \tfrac{1}{8}c_1^6 c_3)\mathbb{E}(R^+_n).$$

In the second inequality, we used the bound $w_n \ge c_3\mathbb{E}(R^+_n)$ provided by Proposition 4.1. This finishes the proof of (4.15), which implies the statement of the proposition along the geometric sequence $n = 2^k$. The general statement of (4.13) follows by sub-additivity. □

4.3 Proof of the main theorems

Proof of Theorem 1.2 By Proposition 4.1, it is sufficient to get the similar bound for w_n. The bounds then follows from Propositions 4.2 and 4.3.

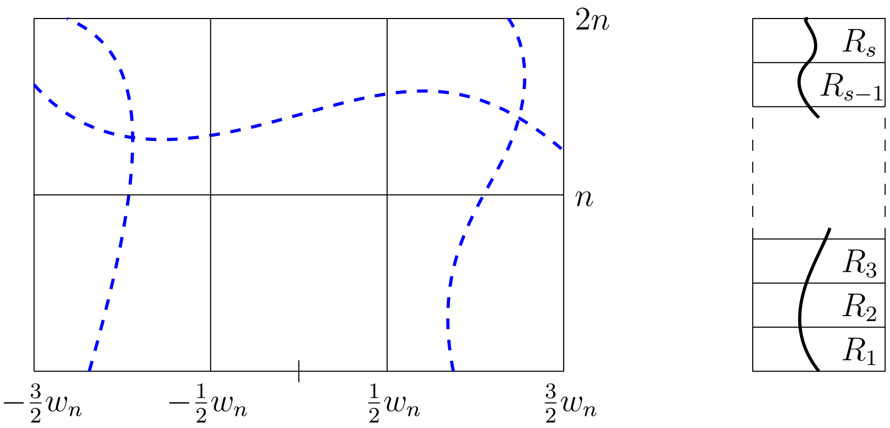

Fig. 8 On the *left*, an illustration of the event E_n. Again, the *blue dotted line* denotes a dual path preventing the existence of an oriented path from $[-\frac{1}{2}w_n, \frac{1}{2}w_n] \times [0, n]$ to the outside of $[-\frac{3}{2}w_n, \frac{3}{2}w_n] \times [0, 2n]$. On the *right*, if a *box* of width $3w_n$ and height sn is crossed vertically, then s *rectangles* of width $3w_n$ and height n are crossed vertically (color figure online)

Proof of Theorem 1.1 The lower bound follows from (4.11) and Proposition 4.2. We now focus on the upper bound.

First, the box-crossing property and the FKG inequality imply that the event E_n defined (see Fig. 8) by

- $[\frac{1}{2}w_n, \frac{3}{2}w_n] \times [0, 2n]$ is not crossed from left to right,
- $[-\frac{3}{2}w_n, -\frac{1}{2}w_n] \times [0, 2n]$ is not crossed from right to left,
- $[-\frac{3}{2}w_n, \frac{3}{2}w_n] \times [0, 2n]$ is not crossed vertically,

satisfies

$$\mathbb{P}(E_n) \overset{\text{(FKG)}}{\geq} H(3w_n, n)V(w_n, 3n)^2 \geq c_1^3. \tag{4.20}$$

Let $r \geq 2$ be a large enough integer that we fix later and set $K := \lfloor \log_r(n/2) \rfloor$. For the event $0 \to \ell_n$ to occur, none of the events E_{r^k}, $1 \leq k \leq K$, should occur. Imagine for a moment that $w_{rn} > 3w_n$ for each n, then the events E_{r^k}, $1 \leq k \leq K$, depend on different sets of edges, so that (4.20) implies

$$\mathbb{P}(0 \to \ell_n) \leq \mathbb{P}\left(\bigcap_{k=1}^{K} E_{r^k}^c\right) = \prod_{k=1}^{K}(1 - \mathbb{P}(E_{r^k})) \leq (1 - c_1^3)^K \leq n^{-c_{10}}.$$

To conclude the proof, we therefore need to show that $w_{rn} \geq 3w_n$, or equivalently, by definition (3.2) of w_{rn}, that

$$H(\alpha 3w_n, \varepsilon rn) > V(3w_n, \alpha\varepsilon rn).$$

On the one hand, provided $r > 1/\varepsilon$, monotonicity and the box-crossing property implies

$$H(\alpha 3w_n, \varepsilon rn) \geq H(3w_n, n) \geq c_1.$$

On the other hand, if the box $[0, 3w_n] \times [0, \alpha\varepsilon rn]$ is crossed vertically, then $s = \lfloor r\alpha\varepsilon \rfloor$ disjoint boxes of width $3w_n$ and height n must be crossed vertically (see Fig. 8). Therefore,

$$V(3w_n, \alpha\varepsilon rn) \leq V(3w_n, n)^s \leq (1 - c_1)^s.$$

Providing r large enough, we may guarantee that $(1 - c_1)^s < c_1$, and therefore $w_{rn} \geq 3w_n$ for every $n \geq 1$.

Remark 4.5 In order to obtain the slightly weaker bound

$$\mathbb{P}(0 \to \ell_n) \geq \frac{1}{n^{(1-\varepsilon)/2}},$$

one may avoid the use of (1.8) in [8] by simply combining the bound $w_n \leq n^{1-\varepsilon}$ with (4.11).

Acknowledgements We are grateful to Daniel Valesin for the careful reading of the first version of this article and for their very helpful comments. The work of the two first authors was supported by a grant from the Swiss NSF and the NCCR SwissMap also funded by the Swiss NSF. The project was initiated during a stay of the third author to the Université de Genève, and the authors are grateful to the institution for making such a stay possible. AT was supported by CNPq grants 306348/2012-8 and 478577/2012-5 and by FAPERJ grant 202.231/2015.

References

1. Balister, P., Bollobás, B., Stacey, A.: Improved upper bounds for the critical probability of oriented percolation in two dimensions. Random Struct. Algorithms **5**(4), 573–589 (1994). (English)
2. Bezuidenhout, C., Grimmett, G.: The critical contact process dies out. Ann. Probab. **18**(4), 1462–1482 (1990)
3. Broadbent, S.R., Hammersley, J.M.: Percolation processes. I. Crystals and mazes. Proc. Camb. Philos. Soc. **53**, 629–641 (1957)
4. Belitsky, V., Ritchie, T.L.: Improved lower bounds for the critical probability of oriented bond percolation in two dimensions. J. Stat. Phys. **122**(2), 279–302 (2006). (English)
5. Duminil-Copin, H., Tassion, V.: A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. Comm. Math. Phys. **343**(2), 725–745 (2016)
6. Duminil-Copin, H., Tassion, V.: Rsw and box-crossing property for planar percolation. In: Proceedings of the International Congress of Mathematical Physics (2016)
7. Durrett, R., Griffeath, D.: Supercritical contact processes on Z. Ann. Probab. **11**(1), 1–15 (1983)
8. Durrett, R., Schonmann, R.H., Tanaka, N.I.: The contact process on a finite set. III. The critical case. Ann. Probab. **17**(4), 1303–1321 (1989)
9. Durrett, R., Schonmann, R.H., Tanaka, N.I.: Correlation lengths for oriented percolation. J. Stat. Phys. **55**(5–6), 965–979 (1989)
10. Durrett, R., Tanaka, N.I.: Scaling inequalities for oriented percolation. J. Stat. Phys. **55**(5–6), 981–995 (1989)
11. Durrett, R.: Oriented percolation in two dimensions. Ann. Probab. **12**(4), 999–1040 (1984)
12. Galves, A., Presutti, E.: Edge fluctuations for the one-dimensional supercritical contact process. Ann. Probab. **15**(3), 1131–1145 (1987)
13. Griffeath, D.: The basic contact processes. Stoch. Process. Appl. **11**(2), 151–185 (1981)

14. Grimmett, G.: Percolation, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 321. Springer, Berlin (1999)
15. Harris, T.E.: Contact interactions on a lattice. Ann. Probab. **2**, 969–988 (1974)
16. Harris, T.E.: Additive set-valued Markov processes and graphical methods. Ann. Probab. **6**(3), 355–378 (1978)
17. Kuczek, T.: The central limit theorem for the right edge of supercritical oriented percolation. Ann. Probab. **17**(4), 1322–1332 (1989)