Probab. Theory Relat. Fields (2015) 162:155–198
DOI 10.1007/s00440-014-0568-6

Barriers, exit time and survival probability for unimodal Lévy processes

Krzysztof Bogdan · Tomasz Grzywny · Michał Ryznar

Received: 31 October 2013 / Revised: 14 May 2014 / Published online: 7 June 2014

Abstract We give superharmonic functions and derive sharp bounds for the expected exit time and probability of survival for isotropic unimodal Lévy processes in smooth domains.

Keywords Lévy–Khintchine exponent · Unimodal isotropic Lévy process · Lévy measure · First exit time · Survival probability · Superharmonic function

Mathematics Subject Classification (2010) Primary 31B25 · 60J50; Secondary 60J75 · 60J35

1 Introduction

A function is called barrier for an open set if it is superharmonic inside and vanishes outside, near a part of the boundary of the set. Barriers are important for studying boundary behavior of solutions to the Dirichlet problem [2,4]. From a general

The research was supported in part by NCN grants 2011/03/B/ST1/00423 and 2012/07/B/ST1/03356 and the Alexander von Humboldt Foundation.

K. Bogdan (✉)
Institute of Mathematics of Polish Academy of Sciences, Warszawa, Poland
e-mail: krzysztof.bogdan@pwr.edu.pl

M. Ryznar
e-mail: michal.ryznar@pwr.edu.pl

K. Bogdan · T. Grzywny · M. Ryznar
Institute of Mathematics and Computer Science, Wrocław University of Technology,
ul. Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland
e-mail: tomasz.grzywny@pwr.edu.pl

perspective, understanding boundary asymptotics of superharmonic functions gives detailed information on the behavior of the underlying Markov process at the boundary. The information is obtained by using maximum principle, super-mean value property and Doob's conditioning. Calculation of barriers is extremely delicate for open sets with Lipschitz regularity, even for the Laplacian and cones in $\mathbb{R}^d$, see, e.g., [3], [14, Section 3]. The situation is somewhat easier for smooth open sets. For instance, the Laplacian in a half-space has barriers which are linear functions, correspondingly for smooth sets approximately linear barriers exist. Similar results, with non-linear boundary decay, are known for the fractional Laplacian and generators of convolution semigroups corresponding to *complete subordinate Brownian motions* with *weak scaling* (see [34,38] and Sect. 7 for discussion and references). Recall that for a sub-Markovian semigroup $(P_t, t \geqslant 0)$ we have $\mathcal{A} f(x) = \lim_{t\to 0^+}[P_t f(x) - f(x)]/t \leqslant 0$ if f is bounded, the limit exists and $f(x) = \max f \geqslant 0$. Accordingly, we say that operator $\mathcal{A}$ on $C_c^\infty(\mathbb{R}^d)$ satisfies the *positive maximum principle* if for every $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi(x) = \sup_{y\in\mathbb{R}^d}\varphi(y) \geqslant 0$ implies $\mathcal{A}\varphi(x) \leqslant 0$. The most general operators which have this property are of the form

$$\begin{aligned}\mathcal{A}\varphi(x) = &\sum_{i,j=1}^d a_{ij}(x) D_{x_i} D_{x_j}\varphi(x) + b(x)\nabla\varphi(x) + q(x)\varphi(x)\\ &+ \int\limits_{\mathbb{R}^d} \big(\varphi(x+y) - \varphi(x) - y\nabla\varphi(x)\, \mathbf{1}_{|y|<1}\big)\ \nu(x, dy).\end{aligned}$$

Here for every $x \in \mathbb{R}^d$, $a(x) = (a_{ij}(x))_{i,j=1}^n$ is a real nonnegative definite symmetric matrix, vector $b(x) = (b_i(x))_{i=1}^d$ has real coordinates, $q(x) \leqslant 0$, and $\nu(x,\cdot)$ is a Lévy measure. The description is due to Courrège, see, e.g., [28, Proposition 2.10]. For translation invariant (convolution) operators of this type, a, b, q, and ν are independent of x. If we further assume rotation invariance and conservativeness ($\mathcal{A}1 = 0$), then

$$\mathcal{A}\varphi(x) = \sigma\Delta\varphi(x) + \lim_{\varepsilon\to 0^+}\int\limits_{|y|>\varepsilon} [\varphi(x+y) - \varphi(x)]\,\nu(dy), \tag{1.1}$$

where $\sigma \geqslant 0$ and ν is *isotropic*. (1.1) gives the general setting of our paper; we shall also consider the corresponding isotropic Lévy processes X.

It is in general difficult to determine barriers for non-local Markov generators, even in the setting of (1.1) and for smooth open sets. In fact the wide range of Lévy measures ν results in a comparable variety of boundary asymptotics of superharmonic functions, not fully codified by the existing calculus. The situation might even seem hopeless but it is not. For instance, the expected exit time $x \mapsto \mathbb{E}_x\tau_D$ of X from open bounded set $D \subset \mathbb{R}^d$ is a barrier for D. We shall effectively estimate this function for smooth open sets D and *unimodal* Lévy processes X by giving barriers for the ball of arbitrary radius. To this end we use the renewal function V of the ladder-height process of one-dimensional projections of X: the barriers are defined as compositions of V with the distance to the complement of the ball. This and a similar definition of functions subharmonic in the complement of the ball yield sharp estimates for the expected exit time for open sets $D \subset \mathbb{R}^d$ which are of class $C^{1,1}$. We also obtain sharp estimates for the probability of X surviving in D longer than given time $t > 0$, even for some unbounded D and rather general *unimodal* Lévy processes.

Thus, V allows for calculations accurate enough to exhibit specific super- and subharmonic functions for the considered processes. The idea of using V in this context comes from Kim et al. [34] (see Introduction and p. 931 ibid.) and has already proved very fruitful for *complete subordinate Brownian motions*.

When verifying superharmonicity, we calculate a version of the infinitesimal generator on the composition of V with the distance to the complement of the ball. In view of the curvature of the sphere, the calculation requires good control of V'. We carry out calculations assuming that V' satisfies a Harnack-type condition (**H**), described in (3.7) below. When using (**H**) we only need to estimate certain weighted integrals of V' (given, e.g., by Lemma 3.7), rather than individual values of V'. The condition (**H**) holds, e.g., for *special subordinate Brownian motions*, a class of processes wider than the complete subordinate Brownian motions. We should note that V is defined implicitly but in the considered isotropic setting it enjoys simple sharp estimates in terms of more elementary functions: the Lévy–Khintchine exponent ψ of X and the following Pruitt's function h [41] (see (2.5) below for details),

$$h(r) = \frac{\sigma^2 d}{r^2} + \int\limits_{\mathbb{R}^d} \left(\frac{|z|^2}{r^2} \wedge 1 \right) \nu(dz), \quad r > 0. \tag{1.2}$$

Namely, it follows from Proposition 2.4 and (3.1) that for unimodal Lévy processes with unbounded ψ we have

$$h(r) \approx \psi(1/r) \approx 1/V(r)^2, \quad r > 0. \tag{1.3}$$

On the other hand, the control of V' is hard. For instance continuity and monotonicity of V', although common, are open to conjectures (we actually know that V' may fail to be monotone for some unimodal Lévy processes, see Remark 8). For complete subordinate Brownian motions good control results from the fact that V' is completely monotone. This sheds light on the results obtained by Chen et al. (cf. [17,36] and Sect. 7.2 below). Our approach allows to lift this structure requirement that X is a subordinate Brownian motion, thanks to new ideas employing unimodality, scaling and (approximating) Dynkin's operator.

The basic object of interest in our study is $\mathbb{E}^x \tau_{B_r}$, the expected exit time from the ball B_r centered at the origin and with radius $r > 0$, for arbitrary starting point $x \in \mathbb{R}^d$ of X (for detailed definitions see Sect. 2). When $x = 0$, the classical result of Pruitt [41] (see p. 954, Theorem 1 and (3.2) ibid.) provides in our setting constants $c = c(d)$ and $C = C(d)$ such that

$$\frac{c}{h(r)} \leqslant \mathbb{E}^0 \tau_{B_r} \leqslant \frac{C}{h(r)}, \qquad r > 0. \tag{1.4}$$

Pruitt's estimate may be called *sharp*, meaning that the ratio of its extreme sides is bounded. One of our main contributions is the following inequality,

$$\frac{c^*}{\sqrt{h(r)h(r-|x|)}} \leqslant \mathbb{E}^x \tau_{B_r} \leqslant \frac{C}{\sqrt{h(r)h(r-|x|)}}, \quad x \in B(0,r), \tag{1.5}$$

where $c^* = c^*(r, d, X) > 0$ is *non-increasing* in r and $C = C(d)$. The estimate holds for unimodal Lévy processes under condition (**H**) on V'. The estimate is *sharp* up to the boundary of the ball. As we note in Lemma 2.3, the upper bound in (1.5) easily follows from the one-dimensional case (2.18), cf. [26]. The lower bound is much more delicate. To the best of our knowledge the lower bound was only known for complete subordinate Brownian motions satisfying certain scaling conditions (see Theorem 1.2 and Proposition 2.7 in [30]). Our results cover in a uniform way isotropic stable process, relativistic stable process, sums of two independent isotropic stable processes (also with Gaussian component), geometric stable processes, variance gamma processes, conjugate to geometric stable processes [44] and much more which could not be treated by previous methods. The fact that c in (1.5) depends on r is a drawback if one needs to consider large r. In many situations, however, we may actually choose c independent of r. For example if X is a special subordinate Brownian motion, then we have $c = c(d)$, which follows by combining Theorem 4.1 with Lemma 7.5 below. We conjecture that in the case of isotropic Lévy processes, one can always choose c depending only on d. This is certainly true in the one-dimensional case, see (2.18). For $d \geqslant 2$ the conjecture is strongly supported by comparison of (1.4) and (1.5).

We test super- and subharmonicity by means of Dynkin's generator of X in a way suggested by [13]. We also rely on our recent bounds for the semigroups of weakly scaling unimodal Lévy processes on the whole of $\mathbb{R}^d$ [10], and results of Grzywny [24]. As we indicated above, delicate properties of V, indeed of V', are used to prove (1.5) by way of calculating Dynkin's operator on functions defined with the help of V. Fortunately, the resulting asymptotics is directly expressed by V, rather than by V', and may also be described by means of the Lévy–Khintchine exponent ψ or h, which we indeed do in (1.5) (estimates expressed in terms of h may be considered the most explicit, because h is given by a direct integration without cancellations).

On a general level our development rests on estimates for Dynkin-type generators acting on smooth test functions (Sect. 2) and compositions of V (Sect. 3). This explains our restriction to $C^{1,1}$ open sets: we approximate them by translations and rotations of the half-space $\mathbb{H} = \{x \in \mathbb{R}^d : x_1 > 0\}$, and $V(x_1)$ is harmonic for X on $\mathbb{H}$. Noteworthy, the so-called boundary Harnack principle (BHP) for harmonic functions of X is negligible in our development; it is superseded in estimates by the ubiquitous function V. Barriers resulting from V provide access to asymptotics of the expected exit time, survival probability, Green function, harmonic measure, distribution of the exit time and the heat kernel. In fact, our estimates imply explicit decay rate for nonnegative harmonic functions near the boundary of $C^{1,1}$ open sets, see Proposition 7.6. Furthermore, in [8] we give applications to heat kernels for the corresponding Dirichlet problem in $C^{1,1}$ open sets. We also expect applications to Hardy-type inequalities, cf. [2].

It would be of considerable interest to further extend our estimates to Markov processes with isotropic Lévy kernels $dy \mapsto \nu(x, dy)$ or to isotropic Lévy processes with the Lévy measure approximately unimodal in the sense of (4.3). Partial results in this direction are given in Corollary 4.3. We like to note that the case of Lipschitz open sets apparently requires approach based on BHP and is bound to produce less explicit estimates. We refer the reader to [3,9] for more information and bibliography on this subject. In this connection we like to note that BHP fails for non-convex open sets for the so-called truncated stable Lévy processes [32].

The rest of the paper is composed as follows. In Sect. 2 we estimate tails of X_t and X_{τ_D} by means of $\mathbb{E}^x\tau_D$, V or h. In Lemma 3.8 and 3.9 of Sect. 3 we give mildly super- and subharmonic functions for the ball and the complement of the ball, respectively. In Sect. 4 we estimate the expected exit time: Theorem 4.1 provides (1.5) and Theorem 4.6 states (with more detail) the following estimates of the expected exit time of unimodal Lévy processes with unbounded Lévy–Khintchine exponent from $C^{1,1}$ open bounded sets D, under mild conditions including (**H**),

$$\mathbb{E}^x\tau_D \approx V(\delta_D(x)), \quad x \in \mathbb{R}^d.$$

In Sect. 5 we consider the case of transient X, and estimate the probability of ever hitting the ball from outside in, say, dimension $d \geqslant 3$, by using the estimates of Grzywny [24] for potential kernel: $U(x) \leqslant cV^2(|x|)/|x|^d$ for $x \in \mathbb{R}^d$, and for the capacity of the ball: $Cap(\overline{B_r}) \approx r^d/V^2(r)$ for $r > 0$. In Sect. 6 under weak scaling conditions we estimate the survival probability:

$$\mathbb{P}^x(\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1, \qquad x \in \mathbb{R}^d, \quad 0 < t \leqslant CV(r_0)^2.$$

Here r_0 is the $C^{1,1}$-localization radius of D. The result is new even for complete subordinate Brownian motions. Further estimates and information are given as we proceed.

In Sect. 7 we discuss the role and validity of (**H**) and give specific examples of Lévy processes manageable by our methods. Since $V(\delta_D(x)) \approx [\psi(1/\delta_D(x))]^{-1/2}$, our estimates are often entirely explicit.

As we advance, the reader should observe the assumptions specified at the beginning of each section: as a rule they bind the statements of the results in that section. Notably, a large part of our estimates, especially of the upper bounds, are valid under minimal assumptions including isotropy and, usually but not always (cf. Sect. 2), unimodality of X. Scaling, unimodality, pure-jump character of X and the Harnack-type condition (**H**) on V' are commonly assumed to prove matching lower bounds. We strive to make explicit the dependence of constants in our estimates on characteristics of D and X. Some of the constants depend only on d for all isotropic Lévy processes, others depend on the assumption of unimodality, the parameters in the weak scaling and other analytic properties of X expressed through the Lévy measure. Good control of constants in estimates at scale $r > 0$ necessitates the use of rather intrinsic quantities $\mathcal{I}(r)$ and $\mathcal{J}(r)$ introduced in Sect. 4. Such control is especially important for the study of unbounded sets.

2 Preliminaries

We write $f(x) \approx g(x)$ and say that f and g are *comparable* if $f, g \geqslant 0$ and there is a positive number C, called comparability constant, such that $C^{-1} f(x) \leqslant g(x) \leqslant Cf(x)$ for all considered x. We write $C = C(a, \ldots, z)$ to indicate that (constant) C may be so chosen to depend only on $a, \ldots, z$. Constant may change values from place to place except for capitalized numbered constants (C_1, C_2 etc.), which are the same at each occurrence.

We consider the Euclidean space $\mathbb{R}^d$ of arbitrary dimension $d \in \mathbb{N}$. All sets, functions and measures considered below are assumed Borel. Let $B(x, r) = \{y \in \mathbb{R}^d : |x-y| < r\}$, the open ball with center at $x \in \mathbb{R}^d$ and radius $r > 0$, and let $B_r = B(0, r)$. We denote by $\omega_d = 2\pi^{d/2}/\,\Gamma(d/2)$ the surface measure of the unit sphere in $\mathbb{R}^d$. We also consider *exterior* sets $B^c(x, r) = (B(x, r))^c = \{y \in \mathbb{R}^d : |x - y| \geqslant r\}$, $B_r^c = (B(0, r))^c$ and $\overline{B}_r^c = (\overline{B(0, r)})^c$. For $D \subset \mathbb{R}^d$ we consider the distance to the complement of D:

$$\delta_D(x) = \text{dist}(x, D^c), \qquad x \in \mathbb{R}^d.$$

We say that D is of class $C^{1,1}$ at scale r if $r > 0$, D is open nonempty set in $\mathbb{R}^d$ and for every $Q \in \partial D$ there are balls $B(x', r) \subset D$ and $B(x'', r) \subset D^c$ tangent at Q. Thus, $B(x', r)$ and $B(x'', r)$ are the *inner* and *outer* balls at Q, respectively. Estimates for $C^{1,1}$ open sets often rely on the inclusion $B(x', r) \subset D \subset B(x'', r)^c$, domain monotonicity of the considered quantities and on explicit calculations for the extreme sides of the inclusion. If D is $C^{1,1}$ at some unspecified scale (hence also at all smaller scales), then we simply say D is $C^{1,1}$. The $C^{1,1}$-*localization radius*,

$$r_0 = r_0(D) = \sup\{r : D \text{ is } C^{1,1} \text{ at scale } r\},$$

describes the local geometry of such D, while the *diameter*,

$$\text{diam}(D) = \sup\{|x - y| : \ x, y \in D\},$$

depends on the global geometry of D. The ratio $\text{diam}(D)/r_0(D) \geqslant 2$ is called the *distortion* of D. We remark that $C^{1,1}$ open sets may be defined by using local coordinates and Lipschitz condition on the gradient of the function defining their boundary (see, e.g., [1, Section 2]), hence the notation $C^{1,1}$. They can also be *localized* near the boundary without much changing the distortion [11, Lemma 1]. Some of the comparability constants in our estimates depend on D only through d and the distortion of D.

We denote by $C_c(D)$ the class of the continuous functions on $\mathbb{R}^d$ with compact support in (arbitrary) open $D \subset \mathbb{R}^d$, and we let $C_0(D)$ denote the closure of $C_c(D)$ in the supremum norm.

A Lévy process is a stochastic process $X = (X_t,\ t \geqslant 0)$ with values in $\mathbb{R}^d$, stochastically independent increments, cádlág paths and such that $\mathbb{P}(X(0) = 0) = 1$ [42]. We use $\mathbb{P}$ and $\mathbb{E}$ to denote the distribution and the expectation of X on the space of cádlág paths $\omega : [0, \infty) \to \mathbb{R}^d$, in fact X may be considered as the canonical map:

$X_t(\omega) = \omega(t)$ for $t \geqslant 0$. In what follows, we shall use the Markovian setting for X, that is we define the distribution $\mathbb{P}^x$ and the expectation $\mathbb{E}^x$ for the Lévy process starting from arbitrary point $x \in \mathbb{R}^d$: $\mathbb{E}^x F(X) = \mathbb{E}F(x + X)$ for Borel functions $F \geqslant 0$ on paths. For $t \geqslant 0$, $x \in \mathbb{R}^d$, $f \in C_0(\mathbb{R}^d)$ we let $P_t f(x) = \mathbb{E}^x f(X_t)$, the semigroup of X. The distribution of X_t under $\mathbb{E} = \mathbb{E}^0$ is denoted $p_t(dx)$ ($t \geqslant 0$) and forms a convolution semigroup of probability measures on $\mathbb{R}^d$.

We define the time of the first exit of X from (Borel) $D \subset \mathbb{R}^d$:

$$\tau_D = \inf\{t > 0 : \ X_t \notin D\}.$$

This random variable gives rise to a number of important objects in the potential theory of X. We shall focus on the expected exit time,

$$s_D(x) = \mathbb{E}^x \tau_D, \qquad x \in \mathbb{R}^d, \tag{2.1}$$

and the survival probability

$$\mathbb{P}^x(\tau_D > t), \qquad x \in \mathbb{R}^d,\ t > 0.$$

We shall also use the harmonic measure of D for X defined as

$$\Omega_D^x(A) = \mathbb{P}^x(X_{\tau_D} \in A), \qquad x \in \mathbb{R}^d, \quad A \subset \mathbb{R}^d.$$

A real-valued function f on $\mathbb{R}^d$ is called *harmonic* (for X) on open $D \subset \mathbb{R}^d$ if for every open U such that $\overline{U}$ is a compact subset of D, we have

$$f(x) = \mathbb{E}^x f(X_{\tau_U}) = \int_{U^c} f(y)\Omega_U^x(dy), \quad x \in U, \tag{2.2}$$

and the integral is absolutely convergent. In particular, if g is defined on D^c, and $f(x) = \mathbb{E}^x g(X_{\tau_D})$ is absolutely convergent for $x \in D$, then f is harmonic on D. This follows from the strong Markov property of X [6]. A function f is called *regular* harmonic in D if (2.2) holds for $U = D$.

2.1 Isotropic Lévy processes

Lévy measure is a (nonnegative Borel) measure concentrated on $\mathbb{R}^d \backslash \{0\}$ such that

$$\int_{\mathbb{R}^d} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty. \tag{2.3}$$

We call measure on $\mathbb{R}^d$ *isotropic* if it is invariant upon linear isometries of $\mathbb{R}^d$ (i.e. symmetric if $d = 1$). A Lévy process X_t [42] is called isotropic if all the measures

$p_t(dx)$ are isotropic. Isotropic Lévy processes are characterized by Lévy–Khintchine (characteristic) exponents of the form

$$\psi(\xi) = \sigma^2|\xi|^2 + \int\limits_{\mathbb{R}^d} (1 - \cos\langle \xi, x\rangle)\, \nu(dx), \tag{2.4}$$

with isotropic Lévy measure ν and $\sigma \geqslant 0$. To be specific, by the Lévy–Khintchine formula,

$$\mathbb{E}\, e^{i\langle \xi, X_t\rangle} = \int\limits_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p_t(dx) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d.$$

Unless explicitly stated otherwise, in what follows we assume that X_t is an isotropic Lévy process in $\mathbb{R}^d$ with Lévy measure ν and characteristic exponent $\psi \not\equiv 0$ (we shall make additional assumptions in Sects. 2.2, 2.3 and 3). Since ψ is a radial function, we shall often write $\psi(u) = \psi(x)$, where $x \in \mathbb{R}^d$ and $u = |x| \geqslant 0$. For the first coordinate X^1_t of X_t we obtain the same function $\psi(u)$. Clearly, $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$.

For $r > 0$ we define, after [41],

$$\begin{aligned} K(r) &= \int\limits_{B_r} \frac{|z|^2}{r^2}\nu(dz), \quad L(r) = \nu\left(B_r^c\right), \\ h(r) &= \frac{\sigma^2 d}{r^2} + K(r) + L(r) = \frac{\sigma^2 d}{r^2} + \int\limits_{\mathbb{R}^d} \left(\frac{|z|^2}{r^2} \wedge 1\right) \nu(dz). \end{aligned} \tag{2.5}$$

We note that $0 \leqslant K(r) \leqslant h(r) < \infty$, $L(r) \geqslant 0$, h is (strictly) positive and decreasing, and L is non-increasing. The corresponding quantities for X^1_t, say $K_1(r), L_1(r), h_1(r)$, are given by the Lévy measure $\nu_1 = \nu \circ x_1^{-1}$ on $\mathbb{R}$ [42, Proposition 11.10], in particular

$$h_1(r) = \frac{\sigma^2}{r^2} + \int\limits_{\mathbb{R}} \left(\frac{u^2}{r^2} \wedge 1\right) \nu_1(du) = \frac{\sigma^2}{r^2} + \int\limits_{\mathbb{R}^d} \left(\frac{|z_1|^2}{r^2} \wedge 1\right) \nu(dz), \quad r > 0.$$

We see that

$$h_1(r) \leqslant h(r) \leqslant h_1(r)d, \qquad r > 0. \tag{2.6}$$

We shall make connections to the expected exit time of X for general open sets $D \subset \mathbb{R}^d$. By domain-monotonicity of exit times and Pruitt's estimate (1.4), we have

$$s_D(x) \leqslant s_{B(x,\mathrm{diam}(D))}(x) \leqslant \frac{C}{h(\mathrm{diam}(D))} < \infty. \tag{2.7}$$

Our first lemma is a slight improvement of [24, Lemma 3].

Lemma 2.1 *If $r > 0$ and $x \in B_{r/2}$, then $\mathbb{P}^x(|X_{\tau_D}| \geqslant r) \leqslant 24\, h(r)\, \mathbb{E}^x \tau_D$.*

Proof We assume that $\mathbb{E}^x \tau_D < \infty$, otherwise the result is trivial. Let $r > 0$. Let $\mathcal{A}$ be the generator of the semigroup of X acting on $C_0(\mathbb{R}^d)$. If $\phi \in C_c^2(\mathbb{R}^d)$, then ϕ is in the domain of $\mathcal{A}$, by Dynkin's formula,

$$\mathbb{E}^x \int_0^{\tau_D} \mathcal{A}\phi(X_s)ds = \mathbb{E}^x \phi(X_{\tau_D}) - \phi(x), \quad x \in \mathbb{R}^d, \tag{2.8}$$

and the generator may be calculated pointwise as

$$\mathcal{A}\phi(x) = \sigma^2 \Delta\phi(x) + \int \left[\phi(x+z) - \phi(x) - \mathbf{1}_{|z|<1} \langle z, \nabla\phi(x)\rangle\right] \nu(dz).$$

Since ν is symmetric, we can replace $\mathbf{1}_{|z|<1}$ in the above equation by $\mathbf{1}_{|z|<r}$. We shall use a function $g : [0, \infty) \mapsto [-1, 0]$ such that $g(t) = -1$ for $0 \leqslant t \leqslant 1/2$, $g(t) = 0$ for $t \geqslant 1$, and $\operatorname{ess\,sup}_{t \geqslant 0} |g'(t)|$ and $\operatorname{ess\,sup}_{t \geqslant 0} |g''(t)|$ are finite. In fact, we initially let $g'' = 16$ on $(1/2, 3/4)$ and $g'' = -16$ on $(3/4, 1)$, which gives $\|g''\|_\infty = 16$ and $\|g'\|_\infty = 4$. We then have

$$4 \sup_{t \geqslant 0} |g'(t)| + \frac{1}{2} \sup_{t \geqslant 0} |g''(t)| = 24, \tag{2.9}$$

$$2(d-1) \sup_t |g'(t)| + \sup_t |g''(t)| = 8(d+1). \tag{2.10}$$

These will only slightly increase as we modify g'' to be continuous (such modified $g \in C^2$ is used below). Denote

$$\phi_r(y) = g(|y|/r), \quad y \in \mathbb{R}^d.$$

We first consider ϕ_1. Let $v, z \in \mathbb{R}^d$ be such that $|v+z|+|v| \geqslant 1/2$. There is a number θ between $|v|$ and $|v+z|$, such that

$$\begin{aligned}
\phi_1(v+z) - \phi_1(v) &= g'(|v|)(|v+z| - |v|) + (1/2)g''(\theta)(|v+z| - |v|)^2 \\
&= g'(|v|)\frac{(|v+z|^2 - |v|^2)}{|v+z| + |v|} + (1/2)g''(\theta)(|v+z| - |v|)^2 \\
&= g'(|v|)\frac{|z|^2 + 2\langle v, z\rangle}{|v+z| + |v|} + (1/2)g''(\theta)(|v+z| - |v|)^2 \\
&= g'(|v|)\frac{\langle v, z\rangle}{|v|} + g'(|v|)\frac{\langle v, z\rangle}{|v|}\frac{|v| - |v+z|}{|v+z| + |v|} \\
&\quad + g'(|v|)\frac{|z|^2}{|v+z| + |v|} + (1/2)g''(\theta)(|v+z| - |v|)^2.
\end{aligned}$$

Since $|v+z|+|v| \geqslant 1/2$, we have

$$\left| g'(|v|) \frac{\langle v, z\rangle}{|v|} \frac{|v|-|v+z|}{|v+z|+|v|} \right| \leqslant |g'(|v|)| \frac{|z|^2}{|v+z|+|v|} \leqslant 2|g'(|v|)||z|^2.$$

Also,

$$\frac{1}{2} g''(\theta)(|v+z|-|v|)^2 \leqslant \frac{1}{2}|g''(\theta)||z|^2.$$

Since

$$\langle z, \nabla \phi_1(v)\rangle = g'(|v|) \frac{\langle v, z\rangle}{|v|},$$

we obtain

$$\left|\phi_1(v+z)-\phi_1(v)-\mathbf{1}_{|z|<1} \langle z, \nabla \phi_1(v)\rangle\right| \leqslant \left(4 \sup_t |g'(t)| + \frac{1}{2} \sup_t |g''(t)|\right) |z|^2.$$

If $|v+z|+|v| < 1/2$, then the latter inequality is also true because the left-hand side equals 0. By changing variables we have

$$|\phi_r(v+z)-\phi_r(v)-\mathbf{1}_{|z|<r} \langle z, \nabla \phi_r(v)\rangle| \leqslant \left(4 \sup_t |g'(t)| + \frac{1}{2} \sup_t |g''(t)|\right) |z/r|^2.$$

We also note that

$$|\Delta \phi_1(z)| = |(d-1)g'(|z|)/|z| + g''(|z|)| \leqslant 2(d-1) \sup_t |g'(t)| + \sup_t |g''(t)|.$$

Applying (2.8) to $\phi_r(y) = g(|y|/r)$ we get

$$\mathbb{P}^x\left(|X_{\tau_D}| \geqslant r\right) \leqslant \mathbb{E}^x[\phi_r(X_{\tau_D})+1] = \mathbb{E}^x \int_0^{\tau_D} \mathcal{A}\phi_r(X_s)ds, \qquad |x| \leqslant r/2. \tag{2.11}$$

By the preceding estimates,

$$\begin{aligned} \mathcal{A}\phi_r(v) &= \sigma^2 \Delta \phi_r(v) + \int \left(\phi_r(v+z)-\phi_r(v)-\mathbf{1}_{|z|<r} \langle z, \nabla \phi_r(x)\rangle\right) \nu(dz) \\ &\leqslant \sigma^2 \frac{2(d-1)\sup_t |g'(t)| + \sup_t |g''(t)|}{r^2} \\ &\quad + \frac{4\sup_t |g'(t)| + \frac{1}{2}\sup_t |g''(t)|}{r^2} \int_{|z|<r} |z|^2 \nu(dz) + \nu(B_r^c). \end{aligned} \tag{2.12}$$

By (2.11), (2.12), (2.9), (2.10) and (2.5), we get the result. □

Remark 1 The approach generalizes to other stopping times, e.g. deterministic times $t > 0$:

$$\mathbb{P}^x\left(|X_t| \geqslant r\right) \leqslant 24\, h(r)\, t, \quad r > 0, \quad |x| \leqslant r/2. \tag{2.13}$$

Recall that $p_t(dx)$ has no atoms if and only if ψ is unbounded (if and only if $\nu(\mathbb{R}^d) = \infty$ or $\sigma > 0$) [42, Theorem 30.10]. In fact, if $\sigma > 0$ or if $d \geqslant 2$ and $\nu(\mathbb{R}^d) = \infty$, then $(p_t, t > 0)$ have lower semicontinuous density functions [49, (4.6)]. We further note that the *resolvent measures*

$$A \mapsto \int_0^\infty p_t(A) e^{-qt} dt, \qquad q > 0,$$

are absolutely continuous if and only if p_t, $t > 0$, are absolutely continuous. This consequence of symmetry of p_t is proved in [23, Theorem 6], see also [42, Remark 41.13].

2.2 Isotropic Lévy processes with unbounded characteristic exponent

Unless explicitly stated otherwise, in what follows X is an isotropic Lévy process with unbounded Lévy–Khintchine exponent ψ.

Let $M_t = \sup_{s \leqslant t} X_s^1$ and let L_t be the local time at 0 for $M_t - X_t^1$, the first coordinate of X reflected at the supremum [6,22]. We consider its right-continuous inverse, L_s^{-1}, called the ascending ladder time process for X_t^1. We also define the ascending ladder-height process, $H_s = X^1_{L_s^{-1}} = M_{L_s^{-1}}$. The pair (L_t^{-1}, H_t) is a two-dimensional subordinator [6,22]. In fact, since X_t^1 is symmetric and has infinite Lévy measure or nonzero Gaussian part, by [22, Corollary 9.7], the Laplace exponent of (L_t^{-1}, H_t) is

$$-\frac{1}{t}\log\left(\mathbb{E}\exp[-\tau L_t^{-1} - \xi H_t]\right) = c_+ \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log\left[\tau + \psi(\theta\xi)\right]}{1+\theta^2} d\theta\right\}, \quad \tau, \xi \geqslant 0,$$

In what follows we let $c_+ = 1$, thus normalizing the local time L [22]. In particular, L_s^{-1} is then the standard 1/2-stable subordinator (see also [21, (4.4.1)]), and the Laplace exponent of H_t is

$$\kappa(\xi) = -\frac{1}{t}\log\left(\mathbb{E}\exp[-\xi H_t]\right) = \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{\log\psi(\xi\zeta)}{1+\zeta^2}\, d\zeta\right\}, \quad \xi \geqslant 0. \tag{2.14}$$

The renewal function V of the ascending ladder-height process H is defined as

$$V(x) = \int_0^\infty \mathbb{P}(H_s \leqslant x) ds, \quad x \in \mathbb{R}. \tag{2.15}$$

Thus, $V(x) = 0$ if $x < 0$ and V is non-decreasing. It is also well known that V is subadditive,

$$V(x+y) \leqslant V(x) + V(y), \quad x, y \in \mathbb{R}, \tag{2.16}$$

and $V(\infty) = \infty$. Furthermore, from (2.14) we infer that the Laplace transform $\mathcal{L}V$ of V is

$$\mathcal{L}V(\xi) = \frac{1}{\xi\kappa(\xi)}. \tag{2.17}$$

Both V and its derivative V' play a crucial role in our development. They were studied by Silverstein as g and ψ in [45], see (1.8) and Theorem 2 ibid., respectively. If resolvent measures of X_t^1 are absolutely continuous, then it follows from [45, Theorem 2] that $V(x)$ is absolutely continuous and harmonic on $(0,\infty)$ for the process X_t^1, in fact, V is invariant for the process X_t^1 killed on exiting $(0,\infty)$. Also, V' is a positive harmonic function for X_t^1 on $(0,\infty)$, hence V is actually (strictly) increasing. Notably, the definition of V is rather implicit and the study of V poses problems. In fact, we shall shortly present sharp estimates of V by means of (simpler) functions ψ and h, but decay properties of V' are more delicate and they are not yet fully understood. Under structure assumptions satisfied for complete subordinate Brownian motions, V' is monotone, in fact completely monotone (cf. Lemma 7.5). This circumstance stimulated much of the progress made in [17,36]. The methods presented below in this paper address more general situations, e.g. when the Lévy–Khintchine exponent ψ has *weak scaling* or when X has a nonzero Gaussian part (see Sect. 7.1).

By [21, Corollary 4 and Theorem 3] and [44, Remark 3.3 (iv)] the following result holds.

Lemma 2.2 *We have* $\lim_{\xi\to\infty} \kappa(\xi)/\xi = \sigma$. *Furthermore, if* $\sigma > 0$, *then* V' *is continuous, positive and bounded by* $\lim_{t\to 0^+} V'(t) = \sigma^{-1}$. *In fact* V' *is bounded if and only if* $\sigma > 0$.

As we indicated in Sect. 1, estimates of $\mathbb{E}^x\tau_{B_r}$, the expected exit time from the ball play an important role in this paper. The upper bound (1.4), sharp at the center of the ball, was given by Pruitt in [41, p. 954]. It was later generalized to more general Markov processes by Schilling in [43, Remark 4.8]. For every symmetric Lévy process X on $\mathbb{R}^1$ with unbounded Lévy–Khintchine exponent ψ, the following bound with absolute constant $C_0 > 0$ follows from [26, Proposition 3.5] by Grzywny and Ryznar and subadditivity of V,

$$C_0 V(r)V(r-|x|) \leqslant \mathbb{E}^x\tau_{(-r,r)} \leqslant 2V(r)V(r-|x|), \qquad x \in \mathbb{R}, \quad r > 0. \tag{2.18}$$

In Sect. 4 we establish a similar *comparability* result in arbitrary dimension under appropriate conditions on X. The upper bound is, however, simpler, and we can give it immediately.

Lemma 2.3 *For all* $r > 0$ *and* $x \in \mathbb{R}^d$ *we have* $\mathbb{E}^x\tau_{B_r} \leqslant 2V(r)V(r-|x|)$.

Proof Since X is isotropic with unbounded Lévy–Khintchine exponent ψ, by Blumenthal's 0-1 law we have $\tau_{B_r} = 0$ $\mathbb{P}^x$-a.s. for all $x \in B_r^c$. Hence, it remains to prove the claim for $x \in B_r$. If $\tau = \inf\{t > 0 : |X_t^1| > r\}$, then domain-monotonicity of the exit times and [26, Proposition 3.5] yield $\mathbb{E}^x \tau_{B_r} \leqslant \mathbb{E}^x \tau \leqslant V(r - |x_1|)V(2r)$. By (2.16) and rotations we obtain the claim. □

We define the maximal characteristic function $\psi^*(u) := \sup_{0 \leqslant s \leqslant u} \psi(s)$, where $u \geqslant 0$.

Proposition 2.4 *The constants in the following comparisons depend only on the dimension,*

$$h(r) \approx h_1(r) \approx \psi^*(1/r) \approx [V(r)]^{-2}, \quad r > 0. \tag{2.19}$$

Proof We shall see that all of the comparisons are absolute, except for the first comparison in (2.19), which depends on the dimension via (2.6). Let $r > 0$. Since X^1 is symmetric,

$$h_1(r) \approx \psi^*(1/r), \tag{2.20}$$

see [24, Corollary 1]. Let $r > 0$ and τ_r be the time of the first exit of X_t^1 from the interval $(-r, r)$. By (2.18) and (1.4), we have $V^2(r) \approx \mathbb{E}^0 \tau_r \approx 1/h_1(r)$. □

Lemma 2.5 *We have* $\lim_{t \to 0^+} t/V(t) = \sigma$.

Proof By Proposition 2.4 and the dominated convergence theorem,

$$\frac{t^2}{V^2(t)} \approx t^2 h_1(t) = \sigma^2 + \int\limits_{\mathbb{R}^d} \left(t^2 \wedge |z_1|^2\right) \nu(dz) \to \sigma^2 \quad \text{as } t \to 0.$$

This ends the proof when X is pure-jump. If $\sigma > 0$, then we use Lemma 2.2. □

The next result on survival probability was known before in the situation when $\psi(r)$ and $r^2/\psi(r)$ are non-decreasing in $r \in (0, \infty)$, see [37, Theorem 4.6].

Proposition 2.6 *For every symmetric Lévy process in* $\mathbb{R}$ *which is not compound Poisson,*

$$\mathbb{P}^x(\tau_{(0,\infty)} \geqslant t) \approx 1 \wedge \frac{1}{\sqrt{t\psi^*(1/x)}}, \quad t, x > 0, \tag{2.21}$$

and the comparability constant is absolute.

Proof Considering that L_s^{-1} is a 1/2-stable subordinator, from [37, Theorem 3.1] we see that

$$\mathbb{P}^x(\tau_{(0,\infty)} \geqslant t) \approx 1 \wedge \frac{V(x)}{\sqrt{t}}, \qquad t,\ x > 0. \tag{2.22}$$

The result now obtains from (2.22) and Proposition 2.4. □

Remark 2 If $\Pi \subset \mathbb{R}^d$ is an open halfspace, $R > 0$ and $B_R \subset \Pi$, then $\mathbb{P}^x(\tau_{B_R} > t) \leqslant \mathbb{P}^x(\tau_\Pi > t)$ and $\mathbb{P}^x(\tau_{\overline{B}_R^c} > t) \geqslant \mathbb{P}^x(\tau_{\overline{\Pi}^c} > t)$. By (2.22), for all $t > 0$ and $x \in \mathbb{R}^d$ we obtain

$$\mathbb{P}^x(\tau_{B_R} > t) \leqslant C\left(1 \wedge \frac{V(\delta_{B_R}(x))}{\sqrt{t}}\right)$$

and

$$\mathbb{P}^x(\tau_{\overline{B}_R^c} > t) \geqslant C^{-1}\left(1 \wedge \frac{V(\delta_{\overline{B}_R^c}(x))}{\sqrt{t}}\right),$$

where C is an absolute constant. Namely, we let $\partial\Pi$ touch ∂B_R at the point closest to x. The inequalities hold for all isotropic Lévy processes which are not compound Poisson.

From (2.19) and definitions of L_1, L and h, we derive the following inequality,

$$L_1(r) \leqslant L(r) \leqslant h(r) \leqslant c/[V(r)]^2, \qquad r > 0. \tag{2.23}$$

Lemma 2.7 *There is $C_1 = C_1(d)$ such that if $r > 0$, $D \subset B_r$ and $x \in D\cap B_{r/2}$, then*

$$\mathbb{P}^x(|X_t| \geqslant r) \leqslant C_1\frac{t}{V^2(r)}, \quad t > 0, \tag{2.24}$$

$$\mathbb{P}^x\left(|X_{\tau_D}| \geqslant r\right) \leqslant C_1\frac{\mathbb{E}^x\tau_D}{V^2(r)}, \quad \textit{and} \tag{2.25}$$

$$\mathbb{E}^x\tau_{B_r} \geqslant V^2(r)/C_1. \tag{2.26}$$

Proof Lemma 2.1, Proposition 2.4 and (2.13) give (2.24) and (2.25), which yield (2.26). □

Corollary 2.8 *There exist $C_2 = C_2(d)$ and $C_3 = C_3(d)$ such that for $t, r > 0$ and $|x| \leqslant r/2$,*

$$\mathbb{P}^x(\tau_{B_r} \leqslant t) \leqslant C_2\frac{t}{V^2(r)},$$

and

$$\mathbb{P}^x(\tau_{B_r} > C_3V^2(r)) \geqslant 1/2.$$

Proof Observe that for $|x| \leqslant r/2$,

$$\mathbb{P}^x(\tau_{B_r} \leqslant t) \leqslant \mathbb{P}^0(\tau_{B_{r/2}} \leqslant t).$$

By Lévy's inequality and (2.24) we obtain the first claim with $C_2 = 8C_1$, because

$$\begin{aligned}\mathbb{P}^0(\tau_{B_{r/2}} \leqslant t) &= \mathbb{P}^0\left(\sup_{s\leqslant t}|X_s| \geqslant r/2\right)\\ &\leqslant 2\mathbb{P}^0(|X_t| \geqslant r/2) \leqslant 2C_1\frac{t}{V^2(r/2)}\\ &\leqslant 8C_1\frac{t}{V^2(r)}.\end{aligned}$$

Taking $t = V^2(r)/(16C_1)$ we prove the second claim with $C_3 = (16C_1)^{-1}$. □

We observe the following regularity of the expected exit time.

Lemma 2.9 *If the resolvent measures of X are absolutely continuous and the open bounded set $D \subset \mathbb{R}^d$ has the outer cone property, then $s_D \in C_0(D)$.*

Proof Recall that s_D is bounded. We also have $s_D(x) = 0$ for $x \in D^c$. Indeed, for $x \in \partial D$, by Blumenthal's 0-1 law we have $\tau_D = 0$ $\mathbb{P}^x$-a.s., because X is isotropic with unbounded Lévy–Khintchine exponent ψ and D has the outer cone property. Due to [23, Theorem 6] and [27, Lemma 2.1], X is strong Feller. Hence, for each $t > 0$, $x \mapsto \mathbb{P}^x(\tau_D > t)$ is upper semicontinuous [19, Proposition 4.4.1, p. 163]. Therefore $s_D(x) = \int_0^\infty \mathbb{P}^x(\tau_D > t)dt$ is also upper semi-continuous. In consequence, $s_D(x) \to 0$ as $\delta_D(x) \to 0$, and so s_D is continuous at ∂D. To prove continuity of s_D on D, we let $D \ni z \to x \in D$, and denote

$$D' = D - (z - x), \quad U = D \cap D', \quad R = D \backslash U.$$

We have $s_D(x) = s_U(x) + \int_R s_D(y)\omega_U^x(dy)$ and $s_D(z) = s_{D'}(x) \geqslant s_U(x)$, thus $s_D(z) \geqslant s_D(x) - \int_R s_D(y)\omega_U^x(dy) \to s_D(x)$, because if $y \in R$, then $\delta_D(y) \leqslant |z - x|$ and $s_D(y)$ is small. We see that s_D is lower semi-continuous on D, hence continuous in D, in fact on $\mathbb{R}^d$. □

Remark 3 The resolvent measures are absolutely continuous in dimensions bigger than one, hence $s_D \in C_0(D)$ if D is an open bounded set with the outer cone property in $\mathbb{R}^d$ and $d \geqslant 2$. This is also the case in dimension $d = 1$ under the assumptions of the next section.

2.3 Isotropic absolutely continuous Lévy measure

In what follows, unless stated otherwise, we assume that X is an isotropic Lévy process in $\mathbb{R}^d$ with the Lévy measure $\nu(dx) = \nu(x)dx$ and unbounded Lévy–Khintchine exponent ψ. In particular, X is symmetric, not compound Poisson, has absolute continuous distribution for all $t > 0$ and absolutely continuous resolvent measures. Indeed, the case of $d \geqslant 2$ was discussed in Sect. 2.1 and Remark 3, and for $d = 1$ we invoke [46, Theorem 1 (i)(ii)]. We may assume that the density functions $x \mapsto p_t(x)$ are lower-semicontinuous for every $t > 0$, see [27, Theorem 2.2].

The transition density of the process X *killed off* open $D \subset \mathbb{R}^d$ is defined by Hunt's formula,

$$p_D(t, x, y) = p(t, x, y) - \mathbb{E}^x\big[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t\big], \qquad t > 0,\ x, y \in \mathbb{R}^d.$$

We call p_D the Dirichlet heat kernel of X on D. The Green function of D for X is defined as

$$G_D(x, y) = \int_0^\infty p_D(t, x, y)dt.$$

Here is a connection between the main objects of our study,

$$s_D(x) = \mathbb{E}^x \tau_D = \int_{\mathbb{R}^d} G_D(x, y) dy = \int_0^\infty \mathbb{P}^x(\tau_D > t) dt. \tag{2.27}$$

If $x \in D$, then the $\mathbb{P}^x$-distribution of $(\tau_D, X_{\tau_D-}, X_{\tau_D})$ restricted to $X_{\tau_D-} \neq X_{\tau_D}$ is given by the following density function [29],

$$(0, \infty) \times D \times \left(\overline{D}\right)^c \ni (s, u, z) \mapsto \nu(z - u) p_D(s, x, u). \tag{2.28}$$

Integrating against ds, du and/or dz gives marginal distributions. For instance, if $x \in D$ then

$$\mathbb{P}^x(X_{\tau_D} \in dz) = \left(\int_D G_D(x, u) \nu(z - u) du\right) dz \quad \text{on } (\overline{D})^c. \tag{2.29}$$

Identities resulting from (2.28) are called Ikeda–Watanabe formulae for X. Noteworthy, they allow for intuitive interpretations in terms of the expected occupation time measures $p_D(s, x, u)du$ and $G_D(x, u)du$, and in terms of the measure of the intensity of jumps, $\nu(z - u)dz$, cf. [7, p. 17].

3 Barriers for unimodal Lévy processes

A measure on $\mathbb{R}^d$ is called isotropic unimodal, in short, unimodal, if it is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with a radial non-increasing density function (such measures may have an atom at the origin). A Lévy process X_t is called (isotropic) unimodal if the distributions $p_t(dx)$ are unimodal. Unimodal Lévy processes are characterized in [48] by unimodal Lévy measures $\nu(dx) = \nu(x)dx = \nu(|x|)dx$. For the unimodal process, $p_t(dx)$ has a radial nonincreasing density $p_t(x)$ on $\mathbb{R}^d \setminus \{0\}$, and an atom at the origin, with mass $\exp[-t\nu(\mathbb{R}^d)]$ (no atom if ψ is unbounded, i.e. if $\sigma > 0$ or $\nu(\mathbb{R}^d) = \infty$). We refer to [10] for additional discussion. Unless explicitly stated otherwise, in what follows we always assume that X is a unimodal Lévy process in $\mathbb{R}^d$ with unbounded Lévy–Khintchine exponent ψ. Recall that by [10, Proposition 2],

$$\psi(u) \leqslant \psi^*(u) \leqslant \pi^2 \psi(u) \quad \text{for } u \geqslant 0. \tag{3.1}$$

For $f : \mathbb{R}^d \to \mathbb{R}$, $t > 0$ and $x \in \mathbb{R}^d$ we consider the (approximating) Dynkin operator,

$$\mathcal{A}_t f(x) = \frac{\mathbb{E}^x f(X_{\tau_{B(x,t)}}) - f(x)}{\mathbb{E}^x \tau_{B(x,t)}},$$

whenever $\mathbb{E}^x f(X_{\tau_{B(x,t)}})$ is well defined. For instance, if $s_D(x) = \mathbb{E}^x \tau_D$ and $0 < t \leqslant \delta_D(x)$, then by the strong Markov property, $s_D(x) = s_{B(x,t)}(x) + \mathbb{E}^x s_D(X_{\tau_{B(x,t)}})$, and so

$$\mathcal{A}_t s_D(x) = -1. \tag{3.2}$$

By a similar argument, if f is harmonic on D, $x \in D$ and $0 < t < \delta_D(x)$, then $\mathcal{A}_t f(x) = 0$, by the (harmonic) mean-value property. In particular, let $\mathbb{H} = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$ and $V_1(x) = V(x_1)$. Since V is harmonic on $(0, \infty) \subset \mathbb{R}$ for X_1, V_1 is harmonic in $\mathbb{H}$ for X and so $\mathcal{A}_t V_1(x) = 0$, if $0 < t < \delta_{\mathbb{H}}(x)$ (this is the main reason why V is relevant for the construction of barriers for $C^{1,1}$ sets in $\mathbb{R}^d$). We also observe the following minimum principle: if x is a point in $\mathbb{R}^d$ and $f(x) = \inf_{y \in \mathbb{R}^d} f(y)$, then $\mathcal{A}_t f(x) \geqslant 0$ for every $t > 0$.

Corollary 3.1 *If $\mathcal{A}_t f(x) < 0$ for some $t > 0$, then $f(x) > \inf_{y \in \mathbb{R}^d} f(y)$.*

Lemma 3.2 *If $f \in C_0(D)$ and for every $x \in D$ there is $t > 0$ such that $\mathcal{A}_t f(x) < 0$, then $f \geqslant 0$ on $\mathbb{R}^d$.*

Proof Since f attains its infimum on $\mathbb{R}^d$, but not on D (cf. Corollary 3.1), we have $f \geqslant 0$. □

We make a simple observation on local regularity of harmonic functions, motivated by [12, proof of Lemma 6] (see [5,47] for more in this direction).

Lemma 3.3 *Let X be an isotropic Lévy process with absolutely continuous Lévy measure. If g is bounded on $\mathbb{R}^d$ and harmonic on open $D \subset \mathbb{R}^d$, then g is continuous on D.*

Proof For $r > 0$, let $\Omega_r(dy) = \mathbb{P}^0(X(\tau_{B_r}) \in dy)$. Note that $g(x) = \int_{\mathbb{R}^d} g(y + x)\Omega_r(dy)$ if $0 < r < \delta_D(x)$. By the isotropy and Ikeda–Watanabe formula, $\Omega_r(dy) = c_r \sigma_r(dy) + \phi_r(y)dy$, where σ_r is the normalized spherical measure on ∂B_r, $0 \leqslant c_r \leqslant 1$, and

$$\phi_r(y) = \begin{cases} \int_{B_r} G_{B_r}(0, v)\nu(y - v)dv, & \text{if } |y| > r, \\ 0 & \text{else.} \end{cases}$$

Let $\rho > 0$. Note that the measure $\int_\rho^{2\rho} c_r \sigma_r(A)dr$ has density function $\omega_d^{-1} \mathbf{1}_{\rho < |x| < 2\rho} \; |x|^{1-d} c_{|x|}$. Therefore $\Phi_\rho(A) = \rho^{-1} \int_\rho^{2\rho} \Omega_r(A)dr$ is absolutely continuous, with density function denoted F_ρ. We have $g(x) = \int_{\mathbb{R}^d} g(y + x)\Phi_\rho(dy) = \int_{\mathbb{R}^d} g(y)F_\rho(y - x)dy$ if $\delta_D(x) > 2\rho$. So, locally on D, g is a convolution of a bounded function with an integrable function, so it is continuous on D. □

Lemma 3.4 *Let $D \subset \mathbb{R}^d$ be an open bounded set with the outer cone property and let function f be continuous on $\mathbb{R}^d$. If $\int_{B_r^c} |f(y)|\nu(y/2)dy < \infty$ for some $r > 0$, then $g(x) = \mathbb{E}^x f(X_{\tau_D})$ is continuous on $\mathbb{R}^d$.*

Proof Since D has the outer cone property, $f(x) = g(x)$ for all $x \in D^c$, and so g is continuous as a function on D^c. To prove the continuity of g on $\overline{D}$, we first notice that

f is locally bounded, hence $\int_{B_r^c} |f(y)|\nu(y/2)dy < \infty$ for *all* $r > 0$. Let $r_1 > 0$ be such that $D \subset B_{r_1/2}$, and let $r \geqslant r_1$. We have

$$g(x) = \mathbb{E}^x f(X_{\tau_D})\mathbf{1}_{B_r}(X_{\tau_D}) + \mathbb{E}^x f(X_{\tau_D})\mathbf{1}_{B_r^c}(X_{\tau_D}), \quad x \in \mathbb{R}^d. \tag{3.3}$$

For clarity, the second term in (3.3) is finite because by the Ikeda–Watanabe formula,

$$\int_{B_r^c} |f(y)| \int_D G_D(x,z)\nu(y-z)dzdy \leqslant \mathbb{E}^x \tau_D \int_{B_r^c} |f(y)|\nu(y/2)dy < \infty. \tag{3.4}$$

It follows that g is well defined and finite everywhere. Of course, g and each term in (3.3) are (regular) harmonic in D. Let $x_0 \in \partial D$, $0 < \rho < r_1/2$ and $|x - x_0| < \rho/2$. We have

$$\begin{aligned} g(x) - g(x_0) = {} & \mathbb{E}^x \left[f(X_{\tau_D}) - f(x_0)\right]\mathbf{1}_{B(x_0,\rho)}(X_{\tau_D}) \\ & + \mathbb{E}^x \left[f(X_{\tau_D}) - f(x_0)\right]\mathbf{1}_{B_r \setminus B(x_0,\rho)}(X_{\tau_D}) \\ & + \mathbb{E}^x \left[f(X_{\tau_D}) - f(x_0)\right]\mathbf{1}_{B_r^c}(X_{\tau_D}). \end{aligned}$$

By Lemma 2.1 and (3.4),

$$\begin{aligned} |g(x) - g(x_0)| \leqslant {} & \sup_{y \in B(x_0,\rho)} |f(y) - f(x_0)| + 48h(\rho)\mathbb{E}^x \tau_D \sup_{y \in B_r} |f(y)| \\ & + \mathbb{E}^x \tau_D \int_{B_r^c} |f(y) - f(x_0)|\nu(y/2)dy. \end{aligned}$$

By this and Remark 3 we see that $g(x) \to g(x_0)$ as $x \to x_0$.

By (3.3), Lemma 3.3 and (3.4) we see that locally on D, g is a uniform limit of continuous functions. Thus, g is continuous on $\mathbb{R}^d$. □

Lemma 3.5 *There is $c = c(d)$ such that for every $r > 0$,*

$$\int_{B_r^c} V(|y|)\nu(y)dy \leqslant \frac{c}{V(r)}.$$

Proof Recall that $L(r) = \nu\left(B_r^c\right)$. Integration by parts and (2.23) yield

$$\begin{aligned} \int_{B_r^c} V(|y|)\nu(y)dy &= \int_r^\infty \omega_d\, \nu(\rho)\rho^{d-1}V(\rho)d\rho = L(r)V(r) + \int_r^\infty V'(\rho)L(\rho)d\rho \\ &\leqslant c/V(r) + c\int_r^\infty V'(\rho)/V^2(\rho)d\rho \leqslant c/V(r), \end{aligned}$$

where $c = c(d)$. □

Our main goal in the remainder of this section is to approximate harmonic functions of X in the ball and the complement of the ball. We start with estimates of auxiliary integrals.

Recall that h is defined in (2.5). In (3.5) below we make an important observation on h'.

Proposition 3.6 *There is $C = C(d)$ such that*

$$\int_0^r V(\rho)\rho^d \nu(\rho)d\rho \leqslant C\frac{r}{V(r)}, \quad r > 0.$$

Proof Recall that $K(u) = \omega_d u^{-2} \int_0^u \rho^{d+1}\nu(\rho)d\rho$, $L(u) = \omega_d \int_u^\infty \rho^{d-1}\nu(\rho)d\rho$, and $h(u) = K(u) + L(u) + u^{-2}\sigma^2 d$. Since ν is non-increasing, hence a.e. continuous, for a.e. $u \in \mathbb{R}$ we have

$$\begin{aligned} h'(u) &= -2u^{-1}K(u) + \omega_d u^{d-1}\nu(u) - \omega_d u^{d-1}\nu(u) - 2u^{-3}\sigma^2 d \\ &= -2u^{-1}\left(K(u) + u^{-2}\sigma^2 d\right). \end{aligned} \tag{3.5}$$

Also,

$$\int_0^r V(\rho)\rho^d \nu(\rho)d\rho \leqslant c_1 \int_0^{r/2} V(u)L(u)du, \tag{3.6}$$

because

$$\rho^d \nu(\rho) = \frac{d2^d}{2^d - 1}\int_{\rho/2}^{\rho} u^{d-1}\nu(\rho)du \leqslant \frac{d2^d}{2^d - 1}\int_{\rho/2}^{\rho} u^{d-1}\nu(u)du \leqslant \frac{d2^d}{\omega_d(2^d - 1)}L(\rho/2).$$

By (2.19), $V(u) \approx h^{-1/2}(u)$, and so (3.5) yields

$$\begin{aligned} V(u)L(u) &\approx h^{-1/2}(u)(h(u) - K(u) - u^{-2}\sigma^2 d) = h^{-1/2}(u)\left(h(u) + \frac{u}{2}h'(u)\right) \\ &= (uh^{1/2}(u))' \quad a.e. \end{aligned}$$

From this and (3.6) we obtain the result

$$\int_0^r V(\rho)\rho^d \nu(\rho)d\rho \leqslant c_2 r h^{1/2}(r) \approx \frac{r}{V(r)}.$$

□

Lemma 3.7 *There exists a constant $C = C(d)$, such that for $0 < x < r$,*

$$\int_0^r V'(y/2) \int_{|y-x|}^r \rho^d \nu(\rho/2) d\rho dy \leqslant C \frac{r}{V(r)},$$

and

$$\int_0^r V'(y/2)|y-x|^{d+1}\nu(|y-x|/2)dy \leqslant C \frac{r}{V(r)}.$$

Proof Since ν is decreasing, we have

$$|y-x|^{d+1}\nu(|y-x|/2) \leqslant 2(d+1) \int_{|y-x|/2}^{|y-x|} \rho^d \nu(\rho/2) d\rho,$$

hence

$$\int_0^r V'(y/2)|y-x|^{d+1}\nu(|y-x|/2)dy \leqslant 2(d+1) \int_0^r V'(y/2) \int_{|y-x|/2}^r \rho^d \nu(\rho/2) d\rho dy.$$

To completely prove the lemma it is enough to estimate the latter integral. It equals

$$\begin{aligned} &2\int_0^r \rho^d \nu(\rho/2) \int_{(x/2-\rho)\vee 0}^{(x/2+\rho)\wedge r/2} V'(z) dz d\rho \\ &\leqslant 2\int_0^r \rho^d \nu(\rho/2)[V(x/2+\rho) - V(x/2-\rho)]d\rho \\ &\leqslant 4\int_0^r \rho^d \nu(\rho/2) V(\rho) d\rho \leqslant cr/V(r), \end{aligned}$$

where we used subadditivity (2.16) of V on $\mathbb{R}$ and Proposition 3.6. □

Recall that $V > 0$ and $V' > 0$ on $(0, \infty)$.

Definition 1 We say that condition (**H**) holds if for every $r > 0$ there is $H_r \geqslant 1$ such that

$$V(z) - V(y) \leqslant H_r\, V'(x)(z-y) \quad \text{whenever} \quad 0 < x \leqslant y \leqslant z \leqslant 5x \leqslant 5r. \tag{3.7}$$

We say that (**H***) holds if $H_\infty = \sup_{r>0} H_r < \infty$.

We consider (**H**) and (**H***) as Harnack type because (**H**) is implied by the following property:

$$\sup_{x\leqslant r,\, y\in[x,5x]} V'(y) \leqslant H_r \inf_{x\leqslant r,\, y\in[x,5x]} V'(y), \qquad r>0. \tag{3.8}$$

Both conditions control relative growth of V. If (**H**) holds, then we may and do chose H_r non-decreasing in r. Each of the following situations imply (**H**):

1. X is a subordinate Brownian motion governed by a special subordinator (see Lemma 7.5).
2. $d \geqslant 3$ and the characteristic exponent of X satisfies WLSC (see (5.1) and Lemma 7.2).
3. $d \geqslant 1$ and the characteristic exponent of X satisfies WLSC and WUSC (see (5.2) and Lemma 7.3).
4. $\sigma > 0$ in (2.4) (see Lemma 7.4).

A more detailed discussion of (**H**) and further examples are given in Sect. 7.

The following Lemma 3.8 and Lemma 3.9 are the main results of this section. They exhibit nonnegative functions which are *superharmonic* (hence barriers) or *subharmonic* near the boundary of the ball, inside or outside of the ball, respectively. The functions are obtained by composing V with the distance to the complement of the ball or to the ball, respectively. Super- and subharmonicity are defined by the left-hand side inequality in (3.9) and (3.16), respectively. The super- and subharmonicity of the considered functions are relatively mild as we have good control of the right-hand sides of these inequalities (see the proof of Theorem 4.1 for an application). In comparison with previous developments, it is the use of Dynkin's operator that allows for calculations which only minimally depend on the differential regularity of V (the dependence on V' is via the mean value type inequality (**H**)).

Lemma 3.8 *Assume that* (**H**) *holds or* $d=1$*. Let* $x_0\in\mathbb{R}^d$*,* $r>0$ *and* $g(x)=V(\delta_{B(x_0,r)}(x))$*. There is a constant* $C_5=C_5(d)$ *such that*

$$0\leqslant \limsup_{t\to 0}\left[-\mathcal{A}_t g(x)\right]\leqslant \frac{C_5\, H_r}{V(r)} \qquad \text{if } 0<\delta_{B(x_0,r)}(x)<r/4. \tag{3.9}$$

Proof In what follows we use the notation $y=(\tilde{y},y_d)$, where $y=(y_1,\ldots,y_d)\in\mathbb{R}^d$ and $\tilde{y}=(y_1,\ldots,y_{d-1})$. Without loosing generality we may consider

$$x_0=(\tilde{0},r) \quad\text{and}\quad x=(\tilde{0},x_d), \quad \text{where } 0<4t<x_d<r/4 \tag{3.10}$$

as shown on Fig. 1 (in dimension $d=1$ we mean $y_d=y$, $x_0=r$ and $x_d=x$). We define

$$R(y)=V(y_d)-g(y), \quad y\in\mathbb{R}^d.$$

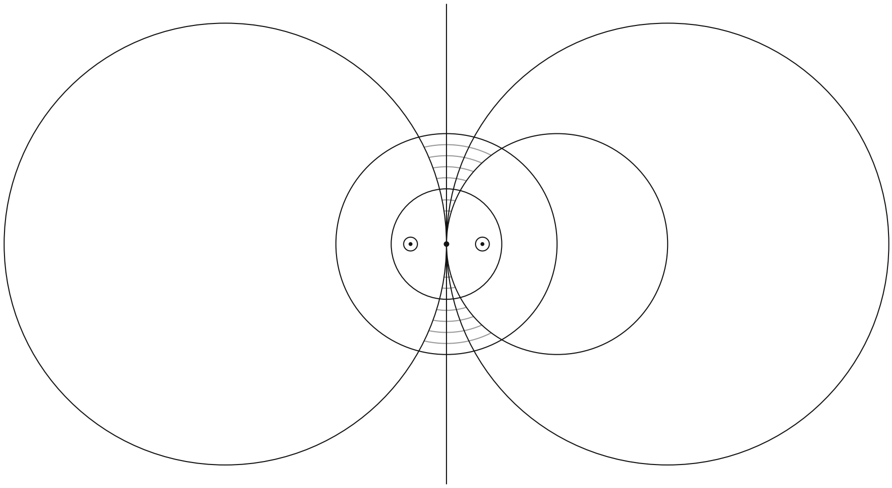

Fig. 1 The settings for the proofs of Lemma 3.8 (*right*) and Lemma 3.9 (*left*)

We note that $R \geqslant 0$ and $R(x) = 0$. Since $V(y_d)$ is harmonic for X in $\{y_d > 0\}$, we have

$$-\mathcal{A}_t g(x) = \mathcal{A}_t R(x) = \frac{1}{\mathbb{E}^x \tau_{B(x,t)}} \mathbb{E}^x[R(X_{\tau_{B(x,t)}})] \geqslant 0.$$

In fact, by (2.29),

$$\mathcal{A}_t R(x) = \frac{1}{\mathbb{E}^x \tau_{B(x,t)}} \mathbb{E}^x[R(X_{\tau_{B(x,t)}}), X_{\tau_{B(x,t)}} \in B(x, 2t)] + \frac{1}{\mathbb{E}^x \tau_{B(x,t)}} \int_{B(x,2t)^c} R(y) \int_{B(x,t)} \nu(y-w) G_{B(x,t)}(x,w)\,dw\,dy. \quad (3.11)$$

We shall split the integral into several parts. First, if $y \in B^c_{r/2} \subset B(x,t)^c$ and $w \in B(x,t)$, then $\nu(y-w) \leqslant \nu(3y/8) \leqslant \nu(y/4)$, and by (2.27),

$$\int_{B(x,t)} \nu(y-w) G_{B(x,t)}(x,w)\,dw \leqslant \mathbb{E}^x \tau_{B(x,t)} \nu(y/4).$$

By this, change of variables, subadditivity (2.16) of V and Lemma 3.5,

$$\frac{1}{\mathbb{E}^x \tau_{B(x,t)}} \int_{B^c_{r/2}} R(y) \int_{B(x,t)} \nu(y-w) G_{B(x,t)}(x,w)\,dw\,dy \leqslant \int_{B^c_{r/2}} R(y)\nu(y/4)\,dy$$
$$\leqslant \int_{B^c_{r/2}} \nu(y/4) V(|y|)\,dy = 4^{d+1} \int_{B^c_{r/8}} \nu(y) V(|y|)\,dy \leqslant c/V(r).$$

If $d = 1$, then $R(y) = 0$ on $B_{r/2} = (-r/2, r/2)$, and the proof is complete.

In what follows we assume that $d \geqslant 2$ and (3.7) holds. We denote (half-ball) $F = B(x_0/2, r/2) \cap \{y_d < r/2\} = \{y : |y|^2/r < y_d < r/2\}$, and we have

$$y_d/2 \leqslant \delta_{B(x_0,r)}(y) \leqslant y_d \quad \text{and} \quad y_d - \delta_{B(x_0,r)}(y) \leqslant |\tilde{y}|^2/r, \quad \text{if } y \in F \tag{3.12}$$

(see the right side of Fig. 1). We leave verification of (3.12) to the reader. By (3.7) and (3.12),

$$R(y) \leqslant H_r V'(y_d/2)\frac{|\tilde{y}|^2}{r}, \quad y \in F. \tag{3.13}$$

If $y \in B(x, 2t) \subset F$, then by (3.10) and (3.12) we further have

$$R(y) \leqslant 4H_r V'(x_d/4)\frac{t^2}{r}. \tag{3.14}$$

By (3.14), (1.4), (2.19) and Lemma 2.5,

$$\begin{aligned}
&\frac{1}{\mathbb{E}^x \tau_{B(x,t)}}\mathbb{E}^x[R(X_{\tau_{B(x,t)}}), X_{\tau_{B(x,t)}} \in B(x, 2t)] \\
&\leqslant \frac{4H_r}{\mathbb{E}^x \tau_{B(x,t)}} V'(x_d/4)\frac{t^2}{r} \leqslant cH_r V'(x_d/4)\frac{t^2}{rV^2(t)} \\
&\to cH_r V'(x_d/4)\frac{\sigma^2}{r}, \quad \text{as } t \to 0.
\end{aligned}$$

If $\sigma > 0$, then by Lemma 2.2 we have $\sup_{x>0} V'(x) \leqslant 1/\sigma$, hence $V(r) \leqslant r/\sigma$ and so

$$V'(x_d/4)\frac{\sigma^2}{r} \leqslant \frac{\sigma}{r} \leqslant \frac{1}{V(r)}.$$

If $y \in B_{r/2} \setminus B(x, 2t)$ and $w \in B(x, t)$, then $|y - w| \geqslant |y - x|/2$. Thus, (3.9) follows if

$$\int_{B_{r/2}} R(y)\nu\left(\frac{y-x}{2}\right)dy \leqslant \frac{C_5 H_r}{V(r)}. \tag{3.15}$$

To prove (3.15), we note the singularity at $y = x \in F$, cover $B_{r/2}$ with sets $\{y \in F : |y_d - x_d| \leqslant |\tilde{y}|\}$, $\{y \in F : |\tilde{y}| < |y_d - x_d|\}$, $\{y \in \mathbb{R}^d : |\tilde{y}| < r/2, -r/2 < y_d \leqslant |y|^2/r\}$, and consider the corresponding integrals. By (3.13), and Lemma 3.7, the first integral does not exceed

$$\frac{H_r}{r}\omega_{d-1}\int_0^r V'(y_d/2)\int_{|y_d-x_d|}^r \rho^d \nu(\rho/2)d\rho\, dy_d \leqslant \frac{CH_r}{V(r)}.$$

Similarly, using (3.13) and Lemma 3.7, we bound the second integral by

$$\frac{H_r}{r}\int_0^r V'(y_d/2)\nu\,(|y_d-x_d|/2)\int_{|\tilde{y}|<|y_d-x_d|}|\tilde{y}|^2 d\tilde{y}dy_d$$
$$=\frac{CH_r}{r}\int_0^r V'(y_d/2)\nu(|y_d-x_d|/2)|y_d-x_d|^{d+1}dy_d\leqslant\frac{CH_r}{V(r)}.$$

By a change of variables, subadditivity (2.16) of V, and Proposition 3.6, we bound the third integral by

$$\int_0^{r/2} V(s)\int_{rs-s^2\leqslant|\tilde{y}|^2<(r/2)^2}\nu(\tilde{y}/2)d\tilde{y}ds\leqslant\omega_{d-1}\int_0^{r/2}\rho^{d-2}\nu(\rho/2)d\rho\int_0^{2\rho^2/r}V(s)ds$$
$$\leqslant\frac{2}{r}\omega_{d-1}\int_0^{r/2}\rho^d\nu(\rho/2)V(2\rho)d\rho\leqslant\frac{C}{V(r)}.$$

This completes the proof of (3.15), and so the proof of the lemma. □

Lemma 3.9 *Assume that* (**H**) *holds or* $d=1$. *Let* $x_0\in\mathbb{R}^d$, $r>0$ *and* $g(x)=V(\delta_{B^c(x_0,r)}(x))$. *There is a constant* $C_6=C_6(d)$ *such that*

$$0\leqslant\limsup_{t\to 0}\mathcal{A}_t g(x)\leqslant\frac{C_6H_r}{V(r)},\qquad \text{if}\quad 0<\delta_{B^c(x_0,r)}(x)<r/4. \tag{3.16}$$

Proof As in the proof of Lemma 3.8, we use the notation $y=(\tilde{y},y_d)$ and without loosing generality we consider $x=(\tilde{0},x_d)$, $0<4t<x_d<r/4$, and $x_0=(\tilde{0},-r)$ (in dimension $d=1$ we mean $y_d=y$, $x_0=-r$ and $x_d=x$). This time we define

$$R(y)=g(y)-V(y_d),\quad y\in\mathbb{R}^d.$$

We have $R\geqslant 0$ and $R(x)=0$. Since $V(y_d)$ is harmonic for X_t at $y_d>0$,

$$\mathcal{A}_t g(x)=\mathcal{A}_t R(x)=\frac{1}{\mathbb{E}^x\tau_{B(x,t)}}\mathbb{E}^x[R(X_{\tau_{B(x,t)}})]\geqslant 0.$$

To prove (3.16) we repeat verbatim the proof of Lemma 3.8, starting from (3.11) there, except for the following minor modification: we replace (3.12) with

$$y_d\leqslant\delta_{B^c(x_0,r)}(y)\leqslant 3y_d/2\quad\text{and}\quad\delta_{B^c(x_0,r)}(y)-y_d\leqslant|\tilde{y}|^2/r,\quad\text{if } y\in F, \tag{3.17}$$

where $F=B(x_0/2,r/2)\cap\{y_d<r/2\}$, as before. □

4 Estimates of the expected exit time

Unless explicitly stated otherwise, we keep assuming that X is a unimodal Lévy process in $\mathbb{R}^d$ with unbounded Lévy–Khintchine exponent ψ. The following theorem gives a sharp estimate for the expected exit time of the ball. Recall that the upper bound in Theorem 4.1 actually holds for arbitrary rotation invariant Lévy process, as proved in Lemma 2.3.

Theorem 4.1 *If* (**H**) *holds, then there is* $C_7 = C_7(d)$ *such that for* $r > 0$,

$$\frac{C_7}{H_r} V(\delta_{B_r}(x))V(r) \leqslant \mathbb{E}^x \tau_{B_r} \leqslant 2V(\delta_{B_r}(x))V(r), \quad x \in \mathbb{R}^d. \tag{4.1}$$

Proof Due to Lemma 2.3 it suffices to prove the lower bound in (4.1). Of course it holds on $\overline{B}_r^c$. Denote $s(x) = \mathbb{E}^x\tau_{B_r}$ and $g(x) = V(\delta_{B_r}(x))$, $x \in \mathbb{R}^d$. By (2.26), domain-monotonicity of the exit times and subadditivity (2.16) of V, the bound holds on $\overline{B_{3r/4}}$, i.e. there is $C = C(d)$ so large that

$$Cs(x) - V(r)g(x) \geqslant 0 \qquad \text{if} \quad \delta_{B_r}(x) \geqslant r/4.$$

Let $0 < \delta_{B_r}(x) < r/4$. If $t > 0$ is small, then by Lemma 3.8 we have $|\mathcal{A}_t g(x)| \leqslant C_5 H_r / V(r)$, and by (3.2) we obtain

$$\mathcal{A}_t \left[(C_5 H_r + 1)s - V(r)g\right](x) = -(C_5 H_r + 1) - V(r)\mathcal{A}_t g(x) \leqslant -1.$$

Let $c = C \vee (C_5 H_r + 1)$ and $f = cs - V(r)g$, a continuous function. By Corollary 3.1, f cannot attain global minimum on $B_r \setminus \overline{B_{3r/4}}$. Since $f \geqslant 0$ elsewhere, $f \geqslant 0$ everywhere. □

The above argument was inspired by the proof of Green function estimates for the ball and stable Lévy processes given by Bogdan and Sztonyk in [13].

Corollary 4.2 *If D is bounded, convex and $C^{1,1}$ at scale $r > 0$, and if* (**H**) *holds, then*

$$\frac{C_7}{H_r} V(\delta_D(x))V(r) \leqslant \mathbb{E}^x \tau_D \leqslant V(\delta_D(x))V(\mathrm{diam}(D)), \quad x \in \mathbb{R}^d. \tag{4.2}$$

Proof Fix $x \in D$ and consider a strip $\Pi \supset D$ of width not exceeding $\mathrm{diam}(D)$ and ball $B \subset D$ of radius $r \vee \delta_D(x)$ such that $\delta_D(x) = \delta_\Pi(x) = \delta_B(x)$. Since $s_B(x) \leqslant s_D(x) \leqslant s_\Pi(x)$, the result follows from (2.18) and Theorem 4.1. □

Remark 4 All the results in this section also hold if ν is isotropic, infinite and *approximately unimodal* in the sense of (4.3) below. Here is an example and explanation.

Corollary 4.3 *Let X be isotropic with absolutely continuous Lévy measure $\nu(dx) = \nu(|x|)dx$. Let $\nu_0 : (0,\infty) \to (0,\infty)$ be monotone and let C^* be a constant such that*

$$(C^*)^{-1}\nu_0(r) \leqslant \nu(r) \leqslant C^*\nu_0(r), \quad r > 0. \tag{4.3}$$

If (**H**) *holds, then there is* $c = c(d, C^*)$ *such that for* $r > 0$,

$$\mathbb{E}^x \tau_{B_r} \geqslant \frac{c}{H_r} V(\delta_{B_r}(x))V(r), \quad x \in \mathbb{R}^d.$$

Proof Let Y be unimodal with characteristic function $\psi^Y(\xi) = \sigma|\xi|^2 + \int_{\mathbb{R}^d}(1 - \cos\langle\xi, z\rangle)\nu_0(|z|)dz$. By Proposition 2.4 we have $V^Y(r) \approx V(r)$, $r > 0$. By Proposition 3.6,

$$\int_0^r V(\rho)\rho^d \nu(\rho)d\rho \approx \int_0^r V^Y(\rho)\rho^d \nu_0(\rho)d\rho \leqslant C\frac{r}{V(r)}, \quad r > 0. \tag{4.4}$$

The inequality and approximate monotonicity of ν yield extensions of Lemmas 3.7 and 3.8, from which the present corollary follows in a similar manner as Theorem 4.1. □

For $r > 0$ we define functions

$$\mathcal{I}(r) = \inf_{0<\rho\leqslant r/2} \left[\nu(B_r \backslash B_\rho)V^2(\rho)\right], \tag{4.5}$$

and

$$\mathcal{J}(r) = \inf_{0<\rho\leqslant r} \left[L(\rho)V^2(\rho)\right]. \tag{4.6}$$

We note that $\mathcal{J}$ is non increasing. By (2.23),

$$0 \leqslant \mathcal{I}(2r) \leqslant \mathcal{J}(r) \leqslant c(d), \qquad r > 0. \tag{4.7}$$

We shall use $\mathcal{J}$ immediately, but $\mathcal{I}$ shall only be discussed and used later, in Sects. 5 and 6.

Lemma 4.4 *Let* (**H**) *hold. Denote* $D = B_1^c$. *Let* $0 < r < 1$, $x \in D$ *and* $0 < \delta_D(x) \leqslant r/2$. *Let* $x_0 = x/|x|$ *and* $D_1 = B(x_0, r) \cap D$. *There is* $C_8 = C_8(d)$ *such that*

$$\mathbb{E}^x \tau_{D_1} \leqslant C_8 \frac{H_1}{(\mathcal{J}(1))^2} V(\delta_D(x))\, V(r). \tag{4.8}$$

Proof We may and do assume that $x = (\tilde{0}, x_d)$ with $1 < x_d \leqslant 1 + r/2$. If $a, b, c \geqslant 0$, $k \geqslant 2$, $a - b + c \geqslant 0$ and $b \geqslant kc$, then $a \geqslant b - c \geqslant (k-1)c \geqslant kc/2$, so $c \leqslant 2a/k$. We shall use this observation to compare $a(v) = V(\delta_D(v))$, $b(v) = \mathbb{E}^v a(X_{\tau_{D_1}})$ and $s(v) = \mathbb{E}^v \tau_{D_1}$, where $v \in \mathbb{R}^d$. We first let $0 < r \leqslant 1/4$, and consider

$$f(v) = a(v) - b(v) + \frac{C_6 H_1 + 1}{V(1)} s(v), \quad v \in \mathbb{R}^d.$$

If $v \notin D_1$ then $a(v) = b(v)$, $s(v) = 0$, and so $f(v) = 0$. By Lemmas 2.9, 3.4 and 3.5, and by subadditivity of V, $f \in C_0(D_1)$. If $v \in D_1$, and $t > 0$ is small enough, then Lemma 3.9 and (3.2) yield

$$\mathcal{A}_t f(v) \leqslant \frac{C_6 H_1}{V(1)} - \frac{C_6 H_1 + 1}{V(1)} < 0.$$

By Lemma 3.2, $f \geqslant 0$ on D_1. Consider a point $z^* \in \partial B_1 \cap \partial B(x_0, r)$. We have $z_d^* = 1 - r^2/2$. Let $F = \{y : y_d > 1 + r\}$. For all $z \in D_1$, $F - z = \{y : y_d > 1 + r - z_d\} \supset \{y : y_d > r + r^2/2\} = F - z^*$, hence $\inf_{z \in D_1} \nu(F - z) = \nu(F - z^*) = (1/2)L_1(r + r^2/2)$. By Ikeda–Watanabe,

$$\begin{aligned} b(v) &\geqslant V(r)\mathbb{P}^v(X_{\tau_{D_1}} \in F) \geqslant V(r)\mathbb{E}^v \tau_{D_1} \inf_{z \in D_1} \nu(F - z) \\ &= \frac{1}{2} V(r)\mathbb{E}^v \tau_{D_1} L_1(r + r^2/2) \geqslant \frac{1}{2} V(r)\mathbb{E}^v \tau_{D_1} L_1(9r/8). \end{aligned}$$

Since X_t is rotation invariant, there is $c_1 = c_1(d)$ such that $L_1(9r/8) \geqslant 8c_1 L(2r)$. Indeed, B_{2r}^c may be covered by a finite number of rotations of $\{y \in \mathbb{R}^d : |y_d| > 9r/8\}$. Thus, for $r \leqslant 1/4$, we have $L_1(9r/8) \geqslant 8c_1 L(2r) \geqslant \frac{8c_1 \mathcal{J}(1)}{V^2(2r)} \geqslant \frac{2c_1 \mathcal{J}(1)}{V^2(r)}$, hence

$$b(v) \geqslant \mathbb{E}^v \tau_{D_1} \frac{c_1 \mathcal{J}(1)}{V(r)} = \frac{c_1 \mathcal{J}(1)}{C_6 H_1 + 1} \frac{V(1)}{V(r)} \frac{C_6 H_1 + 1}{V(1)} s(v).$$

If $\frac{c_1 \mathcal{J}(1)}{C_6 H_1 + 1} \frac{V(1)}{V(1/4)} \geqslant 2$, then we let $r_0 = 1/4$, else we pick $r_0 > 0$ so that $\frac{c_1 \mathcal{J}(1)}{C_6 H_1 + 1} \frac{V(1)}{V(r_0)} = 2$. By the observation at the beginning of the proof, for $0 < r \leqslant r_0$, we have $s(v) \leqslant 2V(\delta_D(v))V(r)/(c_1 \mathcal{J}(1))$ for all v, in particular for $v = x$.

For $r_0 < r < 1$ we proceed in the following standard way. First assume that $\delta_D(x) \leqslant r_0/2$, and let $D' = B(x_0, r_0) \cap D$. Then by the strong Markov property,

$$s(x) = \mathbb{E}^x \tau_{D_1} = \mathbb{E}^x \tau_{D'} + \mathbb{E}^x s(X_{\tau_{D'}}).$$

As stated in Theorem 4.1, $s(x) \leqslant 2V^2(r)$. By Lemma 2.7, we thus obtain,

$$\begin{aligned} \mathbb{E}^x s(X_{\tau_{D'}}) &\leqslant 2V^2(r)\mathbb{P}^x(|X_{\tau_{D'}} - x_0| \geqslant r_0) \\ &\leqslant 2C_1 V^2(r) \frac{\mathbb{E}^x \tau_{D'}}{V^2(r_0)}. \end{aligned}$$

If this is combined with the estimates already proved, then $c_2 = c_2(d)$ exists such that

$$\begin{aligned} s(x) &\leqslant (2C_1 + 1)\mathbb{E}^x \tau_{D'} \frac{V^2(r)}{V^2(r_0)} \\ &\leqslant c_2 \frac{V(r)}{\mathcal{J}(1)V(r_0)} V(\delta_D(x))V(r) \leqslant c_2 \frac{V(1)}{\mathcal{J}(1)V(r_0)} V(\delta_D(x))V(r). \end{aligned}$$

If $\delta_D(x) \geqslant r_0/2$, then by Lemma 2.3 and subadditivity of V, we trivially have

$$s(x) \leqslant 2V^2(r) \leqslant \frac{2V(r)}{V(r_0/2)} V(\delta_D(x))V(r) \leqslant \frac{4V(1)}{V(r_0)} V(\delta_D(x))V(r).$$

Summarizing, by taking $c_3 = 4 + c_2$, in all the cases we get

$$\mathbb{E}^x \tau_{D_1} \leqslant c_3 \frac{V(1)}{V(r_0)} \left(1 + \frac{1}{\mathcal{J}(1)}\right) V(\delta_D(x))V(r).$$

By the choice of r_0, subadditivity of V and (4.7), $V(1)/V(r_0) \leqslant 4 + 2(C_6 H_1 + 1)/(c_1 \mathcal{J}(1)) \leqslant c_4 H_1/\mathcal{J}(1)$, where $c_4 = c_4(d)$. Therefore,

$$\mathbb{E}^x \tau_{D_1} \leqslant c_4 H_1 \frac{\mathcal{J}(1) + 1}{\mathcal{J}(1)^2} V(\delta_D(x))V(r).$$

This is equivalent to (4.8). □

Corollary 4.5 *Let* (**H**) *hold. Denote $D = B_R^c$. Let $0 < r < R$, $x \in D$, $0 < \delta_D(x) \leqslant r/2$, $x_0 = xR/|x|$. If $D_1 = B(x_0, r) \cap D$, then*

$$\mathbb{E}^x \tau_{D_1} \leqslant C_8 \frac{H_R}{(\mathcal{J}(R))^2} V(\delta_D(x))\, V(r). \tag{4.9}$$

Proof Let $R > 0$, $Y_t = X_t/R$ and denote by V_Y, τ_B^Y, L_Y, $\mathcal{J}_\infty^Y$, $H_\cdot^Y$ the quantities V, τ_B, L, $\mathcal{J}$, $H_\cdot$ corresponding to Y. By (2.14) and (2.17), we infer that $V_Y(s) = V(Rs)$, $s \geqslant 0$. Furthermore, $L_Y(s) = L(Rs)$ for $s > 0$. Hence, we obtain $V_Y^2(s)L_Y(s) = V^2(Rs)L(Rs)$, which shows that $\mathcal{J}^Y(1) = \mathcal{J}(R)$. Also, $H_1^Y = H_R$ and

$$\mathbb{E}^x \tau_{D_1} = \mathbb{E}^{x/R} \tau_{D_1/R}^Y.$$

Here the expectation on the right hand side corresponds to Y. Lemma 4.4 finishes the proof. □

The above argument shall be called *scaling* (a different, *weak* scaling is discussed in Sect. 5).

The following is one of our main results.

Theorem 4.6 *If* (**H**) *holds and $D \subset \mathbb{R}^d$ is open, bounded and $C^{1,1}$ at scale $r > 0$, then $C_9 = C_9(d)$ and $C_{10} = C_{10}(d)$ exist such that*

$$\frac{C_9}{H_r} V(\delta_D(x))V(r) \leqslant \mathbb{E}^x \tau_D \leqslant C_{10} \frac{H_r}{(\mathcal{J}(r))^2} \frac{V^2(\operatorname{diam} D)}{V^2(r)} V(\delta_D(x))V(r), \quad x \in \mathbb{R}^d.$$

Proof Denote $s(x) = \mathbb{E}^x \tau_D$. By Lemma 2.3, $s(x) \leqslant 2V^2(\operatorname{diam}(D))$. Let $Q \in \partial D$ be such that $|x - Q| = \delta_D(x)$. Let $\delta_D(x) \leqslant r/2$. Since D is $C^{1,1}$ at scale r, there exist $x_1 \in \overline{D^c}$ and $x_2 \in \overline{D}$ such that $B(x_1, r) \subset D^c$, $B(x_2, r) \subset D$ and $\{Q\} = \overline{B(x_1, r)} \cap \overline{B(x_2, r)}$. Let $D_1 = B(Q, r) \cap D$. By the strong Markov property and (2.25),

$$s(x) = \mathbb{E}^x \tau_{D_1} + \mathbb{E}^x s(X_{\tau_{D_1}}) \leqslant \mathbb{E}^x \tau_{D_1} + 2V^2(\mathrm{diam}(D))\mathbb{P}^x(|X_{\tau_{D_1}} - Q| > r)$$
$$\leqslant \mathbb{E}^x \tau_{D_1} \left(1 + 2C_1 \frac{V^2(\mathrm{diam}(D))}{V^2(r)}\right).$$

Corollary 4.5 yields the upper bound, since $\mathbb{E}^x \tau_{D_1} \leqslant \mathbb{E}^x \tau_{D_2}$, where $D_2 = B(Q, r) \cap \overline{B(x_1, r)}^c$. The lower bound is a consequence of Theorem 4.1, because $s(x) \geqslant \mathbb{E}^x \tau_{B(x_2, r)}$.

For the case $\delta_D(x) \geqslant r/2$, we see from (2.26) that $s(x) \geqslant \mathbb{E}^x \tau_{B(x, \delta_D(x))} \geqslant C_1^{-1} V^2(\delta_D(x)) \geqslant (2C_1)^{-1} V(\delta_D(x)) V(r)$. By this, the general upper bound $s(x) \leqslant 2V^2(\mathrm{diam}(D))$ and the observations that $H_r \geqslant 1$ and $\mathcal{J}(r) \leqslant c(d)$, we finish the proof. □

For instance, $V^2(\mathrm{diam}\, D)/V^2(r)$ is bounded by the square of the distortion of D, if r equals the localization radius of D.

In the one-dimensional case in the proof of Theorem 4.6 we may apply (2.18) instead of Theorem 4.1 and Corollary 4.5, to obtain the following improvement.

Corollary 4.7 *If X is a symmetric Lévy process in $\mathbb{R}$ with unbounded Lévy-Khintchine exponent, and $D \subset \mathbb{R}$ is open, bounded and $C^{1,1}$ at scale $r > 0$, then absolute constant $c \geqslant 1$ exists such that* $c^{-1} V(\delta_D(x)) V(r) \leqslant \mathbb{E}^x \tau_D \leqslant c V^2(\mathrm{diam}\, D) V^{-2}(r)\, V(\delta_D(x)) V(r)$ *for* $x \in \mathbb{R}$.

5 Scaling and its consequences

Let X be an isotropic unimodal Lévy process in $\mathbb{R}^d$ with infinite Lévy measure ν. In view of the literature of the subject (cf. [10,34,35,39]), power-like asymptotics of the characteristic exponent of X is a natural condition to consider. Let $I = (\underline{\theta}, \infty)$, where $\underline{\theta} \in [0, \infty)$ and let $\phi \geqslant 0$ be a non-zero function on $(0, \infty)$. We say that ϕ satisfies the *weak lower scaling* condition (at infinity) if there are numbers $\underline{\alpha} > 0$ and $\underline{c} \in (0, 1]$, such that

$$\phi(\lambda\theta) \geqslant \underline{c}\lambda^{\underline{\alpha}}\phi(\theta) \quad \text{for} \quad \lambda \geqslant 1, \quad \theta \in I. \tag{5.1}$$

In short we say that ϕ satisfies WLSC$(\underline{\alpha}, \underline{\theta}, \underline{c})$ and write $\phi \in$ WLSC$(\underline{\alpha}, \underline{\theta}, \underline{c})$. If $\phi \in$ WLSC$(\underline{\alpha}, 0, \underline{c})$, then we say that ϕ satisfies the *global* weak lower scaling condition.

Similarly, we consider $I = (\overline{\theta}, \infty)$, where $\overline{\theta} \in [0, \infty)$ and we say that the weak upper scaling condition holds if there are numbers $\overline{\alpha} < 2$ and $\overline{C} \in [1, \infty)$ such that

$$\phi(\lambda\theta) \leqslant \overline{C}\lambda^{\overline{\alpha}}\phi(\theta) \quad \text{for} \quad \lambda \geqslant 1, \quad \theta \in I. \tag{5.2}$$

In short, $\phi \in$ WUSC$(\overline{\alpha}, \overline{\theta}, \overline{C})$. For *global* weak upper scaling we require $\overline{\theta} = 0$ in (5.2). We write $\phi \in$ WLSC or WUSC if the actual values of the parameters are not important. We shall study consequences of WUSC and WLSC for the characteristic exponent ψ of X_t.

Recall that ψ is a radial function and we use the notation $\psi(u) = \psi(x)$, where $x \in \mathbb{R}^d$ and $u = |x|$. Our estimates below are expressed in terms of V, ψ or ψ^*. In view of Proposition (2.4), these functions yield equivalent descriptions (ψ or ψ^* are even comparable, see (3.1)). Our main goal is to find connections between the scaling conditions on ψ and the magnitude of the quantities $\mathcal{J}$ and $\mathcal{I}$ defined in (4.6) and (4.5). In the preceding section we saw that $\mathcal{J}$ plays a role in estimating the expected exit time from $C^{1,1}$ open sets. The next three results prepare analysis of survival probabilities in Sect. 6. The first one comes from [10, Corollary 15].

Lemma 5.1 *$C = C(d)$ exists such that if $\psi \in$ WUSC$(\overline{\alpha}, \overline{\theta}, \overline{C})$, $a = [(2-\overline{\alpha})C]^{\frac{2}{2-\overline{\alpha}}}\overline{C}^{\frac{\overline{\alpha}-2}{2}}$, then*

$$L(r) \geqslant a\psi(r^{-1}), \qquad 0 < r \leqslant \sqrt{a}/\overline{\theta}.$$

The following result makes use of the complete Bernstein function $\phi(\lambda) \approx \psi(\sqrt{\lambda})$, constructed in the proof of [10, Theorem 26]. For the convenience of the reader we repeat some of the arguments from [10].

Proposition 5.2 (i) *ψ satisfies WUSC if and only if there is $R > 0$, such that $\mathcal{J}(R) > 0$.* (ii) *ψ satisfies WUSC and WLSC (global WUSC and WLSC) if and only if for some $R > 0$ ($R = \infty$, resp.) we have* $\inf_{r<R} \mathcal{I}(r) > 0$.

Proof Assume that ψ satisfies WUSC$(\beta_1, \overline{\theta}, \overline{C})$. By Lemma 5.1 and (2.19), there is a constant c_1 such that $L(r)V^2(r) \geqslant c_1 > 0$ for $r \leqslant \sqrt{a}/\overline{\theta}$, and so $\mathcal{J}(r) \geqslant c_1 > 0$ for such r. On the other hand, if $R \in (0, \infty)$ is given and $\mathcal{J}(R) > 0$, then

$$L(r) \geqslant \mathcal{J}(R)/V^2(r), \qquad r \leqslant R. \tag{5.3}$$

By [44, proof of Theorem 6.2] and (2.3), the following defines a *complete Bernstein function*:

$$\varphi(\lambda) = \int_0^\infty \frac{\lambda}{\lambda+s} s^{-1}\nu(s^{-1/2})s^{-d/2}ds = \int_0^\infty \left(1-e^{-\lambda u}\right)\mu(u)du, \quad \lambda \geqslant 0,$$

where $\mu(u) = \mathcal{L}[\nu(s^{-1/2})s^{-d/2}](u)$. In fact, by changing variables, and (2.5) for $\lambda > 0$ we have

$$\varphi(\lambda) = 2\int_0^\infty \frac{\lambda u^2}{\lambda u^2+1}\nu(u)u^{d-1}du \approx \int_0^\infty \left[1 \wedge (\lambda u^2)\right]\nu(u)u^{d-1}du = \omega_d^{-1}h\left(\lambda^{-1/2}\right).$$

By (2.19), there exists $c_1 = c_1(d)$ such that

$$c_1\varphi(\lambda) \leqslant \psi\left(\sqrt{\lambda}\right) \leqslant c_1^{-1}\varphi(\lambda), \quad \lambda \geqslant 0. \tag{5.4}$$

Since φ is a complete Bernstein function, $\varphi_1(\lambda) = \lambda/\varphi(\lambda)$ is a special Bernstein function (see [44, Definition 11.1 and Proposition 7.1]). Since X_t is pure-jump, $\lim_{|\xi|\to\infty} \psi(\xi)/|\xi|^2 = 0$. Thus, $\lim_{\lambda\to\infty} \varphi_1(\lambda) = \infty$. By [44, (11.9) and Theorem 11.3], the potential measure of the subordinator with the Laplace exponent φ_1 is absolutely continuous with the density function $f(s) = \int_s^\infty \mu(u)du$ since $\varphi(0) = 0$. In particular $\mathcal{L}f = 1/\varphi_1$ (see [44, (5.20)]). We obtain

$$\nu(x) \leqslant \frac{\int_{|x|^{-2}}^{\infty} e^{-s|x|^2}\nu(s^{-1/2})s^{-d/2}ds}{\int_{|x|^{-2}}^{\infty} e^{-s|x|^2}s^{-d/2}ds} \leqslant \frac{\mu(|x|^2)}{\Gamma(1-d/2,1)|x|^{d-2}}, \quad x \neq 0, \tag{5.5}$$

where $\Gamma(1-d/2,1) = \int_1^\infty e^{-u}u^{-d/2}du$ is the upper incomplete gamma integral. Hence,

$$f(r) \geqslant c\int_r^\infty u^{(d-2)/2}\nu(u^{1/2})du = cL(r^{1/2}), \quad r > 0. \tag{5.6}$$

By (5.6), (5.3), (2.19) and (5.4)

$$f(r) \geqslant c\mathcal{J}(R)/V^2(r^{1/2}) \geqslant c_2\varphi(r^{-1}) = \frac{c_2}{r\varphi_1(r^{-1})}, \qquad 0 < r \leqslant R^2. \tag{5.7}$$

Since f is decreasing and $\mathcal{L}f(u) = 1/\varphi_1(u)$, by [10, Lemma 5] we obtain

$$f(s) \leqslant \frac{1}{\gamma(2,1)s^2}\left(\frac{1}{\varphi_1}\right)'(s^{-1}) = \frac{1}{\gamma(2,1)s^2}\frac{\varphi_1'(s^{-1})}{\varphi_1^2(s^{-1})}, \qquad s > 0,$$

where $\gamma(2,1) = \int_0^1 e^{-u}u\,du$ is the lower incomplete gamma integral. By (5.7), $c_3\varphi_1(\lambda) \leqslant \lambda\varphi_1'(\lambda)$, where $\lambda > 1/R^2$ and $c_3 = c_2\gamma(2,1)$. It follows that $\lambda^{-c_3}\phi_1(\lambda)$ is nondecreasing on (R^{-2},∞). By [10, Lemma 11], $\varphi_1 \in \mathrm{WLSC}(c_3, R^{-2}, 1)$. Since φ_1 is concave, $\lambda\varphi_1'(\lambda) \leqslant \varphi_1(\lambda)$, hence $c_3 \leqslant 1$. In fact, φ_1 is not a linear function, because φ is unbounded, and so $c_3 < 1$. Therefore $\varphi \in \mathrm{WUSC}(1-c_3, R^{-2}, 1)$, and so $\psi \in \mathrm{WUSC}(2(1-c_3), R^{-1}, c_1^{-2})$, by (5.4).

To prove the second part of the statement we suppose that ψ satisfies WUSC $(\beta_1, \theta, \overline{C})$ and WLSC$(\beta_2, \theta, \underline{c})$. By [10, Corollary 22] and (2.19),

$$\nu(x) \geqslant \frac{c^*}{V^2(|x|)|x|^d}, \quad |x| \leqslant b/\theta.$$

If $2\rho \leqslant b/\theta$, then by monotonicity of V we have

$$V^2(\rho)\nu(B_{2\rho}\backslash B_\rho) \geqslant V^2(\rho) \int\limits_{B_{2\rho}\backslash B_\rho} \frac{c^* dx}{V^2(|x|)|x|^d} \geqslant \int\limits_{B_\rho\backslash B_{\rho/2}} c^* \frac{dx}{|x|^d}.$$

Therefore $\mathcal{I}(r) \geqslant c^*(1-(1/2)^d)\omega_d/d$ for all $r \leqslant b/\theta$, as needed.

To prove the reverse implication we assume that there exist constants c^* and R, such that for $0 < r < R, \mathcal{I}(r) \geqslant c^*$. By radial monotonicity of ν,

$$\nu(x) \geqslant \frac{c^*}{|B_2| - |B_1|} \frac{1}{V^2(|x|)|x|^d}, \quad |x| < R/2.$$

By (2.19) and [10, Theorem 26], we obtain WLSC and WUSC for ψ. □

Proposition 5.3 *If ψ satisfies WUSC but not WLSC, then* $\liminf_{r\to 0} \mathcal{I}(r) = 0$ *but there is $R > 0$ such that $\mathcal{I}(r) > 0$ for $r < R$.*

Proof Let $R = 2\sup\{r : \nu(r) > 0\}$. We have $R > 0$. If ψ satisfies WUSC, then by Lemma 5.1 and Proposition 2.4, there are $c_1, r_1 > 0$, such that $L(\rho) \geqslant c_1/V^2(\rho)$ for $\rho < r_1$. Since $\lim_{\rho\to 0} V(\rho) = 0$,

$$\liminf_{\rho\to 0} V^2(\rho)\nu(B_r\backslash B_\rho) = \liminf_{\rho\to 0} V^2(\rho)L(\rho) \geqslant c_1,$$

for every $r > 0$. Fix $r \in (0, R)$. There is $r_2 > 0$ such that

$$V^2(\rho)\nu(B_r\backslash B_\rho) \geqslant c_1/2 \quad \text{if } \rho \leqslant r_2.$$

If $r_2 < \rho \leqslant r/2$, then by monotonicity of V,

$$V^2(\rho)\nu(B_r\backslash B_\rho) \geqslant V^2(r_2)\nu(B_r\backslash B_{r/2}) > 0,$$

hence $\mathcal{I}(r) > 0$. If $\liminf_{r\to 0} \mathcal{I}(r) > 0$, then by Proposition 5.2, ψ satisfies also WLSC. □

5.1 Hitting a ball

We shall estimate the probability that X ever hits a fixed ball of radius $R > 0$. If X is transient and its starting point is far from the ball, then the probability of such an event is small; X instead drifts to infinity with probability bounded below by a positive constant. Indeed, define

$$U(x) = \int\limits_0^\infty p_t(x)dt, \qquad x \in \mathbb{R}^d,$$

the potential kernel of X. If the process is transient [40], then U is finite almost everywhere, in fact on $\mathbb{R}^d \setminus \{0\}$. This is the case, e.g. if $d \geqslant 3$. We denote by Cap the capacity with respect to X. Recall that for every non-empty compact set $A \subset \mathbb{R}^d$ there exists a measure μ_A, supported on A (see, e.g., [6, Section II.2]), called the equilibrium measure, such that

$$U\mu_A(x) = \int U(x-y)\mu_A(dy) = \mathbb{P}^x(\tau_{A^c} < \infty), \quad x \in \mathbb{R}^d, \tag{5.8}$$

and $\mu_A(A) = Cap(A)$. The following two lemmas were proved in [24].

Lemma 5.4 (Theorem 3 of [24]) *If $d \geqslant 3$, then there is $C_{15} = C_{15}(d)$ such that*

$$U(x) \leqslant C_{15} \frac{V^2(|x|)}{|x|^d}, \qquad x \in \mathbb{R}^d.$$

We note in passing that lower bounds for U are given in [24] under WLSC.

Lemma 5.5 (Proposition 3 of [24]) *If $d \geqslant 3$, then there is $C_{16} = C_{16}(d)$ such that*

$$C_{16}^{-1} \frac{R^d}{V^2(R)} \leqslant Cap(\overline{B_R}) \leqslant C_{16} \frac{R^d}{V^2(R)}, \qquad R > 0.$$

If $\psi \in \mathrm{WUSC}(\overline{\alpha}, 0, \overline{C})$ and $d > \overline{\alpha} > 0$, then the process X is transient (even if $d < 3$), and we may extend the two previous lemmas by using the weak upper scaling condition.

Lemma 5.6 *If $\psi \in WUSC(\overline{\alpha}, 0, \overline{C})$ and $\overline{\alpha} < d \leqslant 2$, then $c = c(d, \overline{\alpha}, \overline{C})$ exists such that*

$$U(x) \leqslant c \frac{V^2(|x|)}{|x|^d}, \qquad x \in \mathbb{R}^d.$$

Proof Using the global upper weak scaling condition instead of [24, Lemma 1] one can prove as in [24, Lemma 6] that for $d > \overline{\alpha}$,

$$\mathcal{L}\left(\int_{B_{\sqrt{\cdot}}} U(y)dy\right)(\lambda) \approx \frac{1}{\lambda \psi^*(\sqrt{\lambda})}, \tag{5.9}$$

and then we proceed as in the proof of the first part of [24, Theorem 3]. □

Lemma 5.7 *If $\psi \in WUSC(\overline{\alpha}, 0, \overline{C})$ and $\overline{\alpha} < d \leqslant 2$, then $c = c(d, \overline{\alpha}, \overline{C})$ exists such that*

$$c^{-1} \frac{r^d}{V^2(r)} \leqslant Cap(\overline{B_r}) \leqslant c \frac{r^d}{V^2(r)}, \quad r > 0.$$

Proof We follow the proof of [24, Proposition 3], using (5.9) instead of [24, Lemma 6]. □

As a consequence of the above lemmas we obtain the following upper bound of the probability that the process ever hits a ball of arbitrary radius, a close analogue of a well known Brownian result. We note that [39, Lemma 2.5] gives the inequality (5.10), in fact comparability of both sides of (5.10), for $d \geqslant 3$ under the global weak lower scaling condition.

Proposition 5.8 *For $d \geqslant 3$ there exists a constant $C_{17} = C_{17}(d)$ such that for $|x| > R > 0$,*

$$\mathbb{P}^x(\tau_{\overline{B}_R^c} < \infty) \leqslant C_{17} \frac{V^2(|x|)}{|x|^d} \Big/ \frac{V^2(R)}{R^d} . \tag{5.10}$$

If $\overline{\alpha} < d \leqslant 2$ and $\psi \in$WUSC($\overline{\alpha}, 0, \overline{C}$), then (5.10) holds with $C_{17} = C_{17}(d, \overline{\alpha}, \overline{C})$.

Proof We have

$$\mathbb{P}^x(\tau_{\overline{B}_R^c} < \infty) = \int\limits_{\overline{B_R}} U(y-x)\mu_{B_R}(dy).$$

By Lemma 5.4, for $y \in B_R$ and $|x| \geqslant 2R$ we get

$$U(x-y) \leqslant 2^d C_{15}|x|^{-d} V^2(|x|).$$

Hence, by Lemma 5.5,

$$\mathbb{P}^x(\tau_{\overline{B}_R^c} < \infty) \leqslant 2^d C_{15}|x|^{-d} V^2(|x|)\, Cap(\overline{B_R}) \leqslant 2^d C_{15} C_{16} \frac{R^d V^2(|x|)}{|x|^d V^2(R)}.$$

Since $[R^d V^2(|x|)]/[|x|^d V^2(R)] \geqslant 2^{-d}$, for $|x| \leqslant 2R$ we have

$$\mathbb{P}^x(\tau_{\overline{B}_R^c} < \infty) \leqslant 2^d (C_{15}C_{16} + 1) \frac{R^d V^2(|x|)}{|x|^d V^2(R)}, \quad |x| > R.$$

To prove the second claim we use Lemma 5.6 and 5.7 above instead of 5.4 and 5.5. □

The following result is important in Sect. 6.

Corollary 5.9 *If $d \geqslant 3$, then $c = c(d)$ exists such that*

$$\mathbb{P}^x(\tau_{\overline{B}_R^c} = \infty) \geqslant 1/2, \qquad |x| \geqslant cR.$$

If $\overline{\alpha} < d \leqslant 2$ and $\psi \in$WUSC($\overline{\alpha}, 0, \overline{C}$), then the above inequality holds with $c = c(d, \overline{\alpha}, \overline{C})$.

6 Estimates of survival probability

In this section we assume that X is a pure-jump (isotropic) unimodal Lévy process with infinite Lévy measure.

Proposition 6.1 *Let* (**H**) *hold. There are* $C_{11} = C_{11}(d) < 1$ *and* $C_{12} = C_{12}(d)$ *such that if* $R > 0$ *and* $t \leqslant C_{11}V^2(R)$*, then*

$$\mathbb{P}^x(\tau_{B_R} > t) \geqslant C_{12}\,\frac{\mathcal{I}(R)}{H_R}\left(\frac{V(\delta_{B_R}(x))}{\sqrt{t}} \wedge 1\right).$$

Proof Let $R = 1$ and $C_{11} = C_3/64$. Due to Corollary 2.8 and subadditivity of V,

$$\mathbb{P}^0(\tau_{B_{r/8}} > C_{11}V^2(r)) \geqslant 1/2. \tag{6.1}$$

Suppose that $0 < t \leqslant C_{11}V^2(1)$ and pick $r \leqslant 1$ such that $t = C_{11}V^2(r)$. Let $x \in B_1$. If $\delta_{B_1}(x) \geqslant r/8$, then $\mathbb{P}^x(\tau_{B_1} > t) \geqslant 1/2$ by (6.1). To complete the proof for $R = 1$, it is enough to consider the case $\delta_{B_1}(x) < r/8$. Let $\delta_{B_1}(x) < r/8$. Let $r_0 = r/2 \wedge 1/4$ and $D_r = B_1 \backslash B_{1-r_0}$. Notice that $B(z, r/4) \subset B_1$ for $z \in B_{1-r_0}$. By the strong Markov property,

$$\begin{aligned}
\mathbb{P}^x(\tau_{B_1} > t) &\geqslant \mathbb{E}^x\left[\mathbb{P}^{X_{\tau_{D_r}}}(\tau_{B_1} > t);\, X_{\tau_{D_r}} \in B_{1-r_0}\right]\\
&\geqslant \inf_{z\in B_{1-r_0}} \mathbb{P}^z(\tau_{B_1} > t)\mathbb{P}^x\left[X_{\tau_{D_r}} \in B_{1-r_0}\right]\\
&\geqslant \mathbb{P}^0(\tau_{B_{r/4}} > C_{11}V^2(r))\mathbb{P}^x\left[X_{\tau_{D_r}} \in B_{1-r_0}\right]\\
&\geqslant (1/2)\mathbb{P}^x\left[X_{\tau_{D_r}} \in B_{1-r_0}\right].
\end{aligned}$$

If $|z_0| = 1$, then by the Ikeda–Watanabe formula, isotropy and monotonicity of the Lévy density,

$$\mathbb{P}^x\left[X_{\tau_{D_r}} \in B_{1-r_0}\right] \geqslant \mathbb{E}^x\tau_{D_r} \inf_{z\in D_r} \nu(z - B_{1-r_0}) \geqslant \nu(z_0 - B_{1-r_0})\mathbb{E}^x\tau_{D_r}.$$

By Theorem 4.6, subadditivity of V, $\mathbb{E}^x\tau_{D_r} \geqslant \frac{C_9}{H_{r_0/2}}V(r_0/2)V(\delta_{B_1}(x)) \geqslant \frac{C_9}{8H_1}V(r)$ $V(\delta_{B_1}(x))$. Since ν is isotropic, $\nu(z_0 - B_{1-r_0}) \geqslant c_1\nu(B_1 \setminus B_{2r_0}) \geqslant c_1\frac{\mathcal{I}(1)}{4V^2(r)}$, where $c_1 = c_1(d)$. Therefore,

$$\mathbb{P}^x(\tau_{B_1} > t) \geqslant c_1\frac{C_9}{64H_1}\mathcal{I}(1)\frac{V(\delta_{B_1}(x))}{V(r)} = C_{12}\frac{\mathcal{I}(1)}{H_1}\frac{V(\delta_{B_1}(x))}{\sqrt{t}},$$

where $C_{12} = c_1C_9\sqrt{C_3}/512$.

For arbitrary $R > 0$ we use scaling as in the proof of Corollary 4.5. □

Remark 5 The estimate in Proposition 6.1 is sharp if $t \leqslant C_{11}V^2(R)$; a reverse inequality follows immediately from Proposition 2.6. If $t > C_{11}V^2(R)$, then one can use spectral theory to observe exponential decay of the Dirichlet heat kernel and the survival probability in time if, say, $\sup_x p_t(x) < \infty$ for all $t > 0$ (see [10, Corollary 7], [20, Theorem 4.2.5], [25, Theorem 3.1]).

Lemma 6.2 *Let* $R > 0$, $D = \overline{B}_R^c$ *and let* (**H**) *hold. There is* $C_{13} = C_{13}(d)$ *such that,*

$$\mathbb{P}^x(\tau_D > t) \leqslant C_{13} \frac{H_R}{(\mathcal{J}(R))^2} \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)}, \qquad t > 0,\ x \in \mathbb{R}^d.$$

Proof Let $x \in D$ and $x_0 = xR/|x|$. If $0 < t \leqslant V^2(R)$, then we choose r so that $V(r) = \sqrt{t}$, otherwise we set $r = R$. We define

$$D_1 = B(x_0, r) \cap B_R^c.$$

Since $H_R \geqslant 1$ and $\mathcal{J}(R) \leqslant c(d)$, we may assume that $0 < \delta_D(x) \leqslant r/2$. By Corollary 4.5,

$$\mathbb{E}^x \tau_{D_1} \leqslant C_8 \frac{H_R}{(\mathcal{J}(R))^2} V(r) V(\delta_D(x)).$$

By (2.25),

$$\mathbb{P}^x(|X_{\tau_{D_1}} - x_0| \geqslant r) \leqslant C_1 \frac{\mathbb{E}^x \tau_{D_1}}{V^2(r)}.$$

Finally, we get the conclusion:

$$\begin{aligned} \mathbb{P}^x(\tau_D > t) &\leqslant \mathbb{P}^x(\tau_{D_1} > t) + \mathbb{P}^x(|X_{\tau_{D_1}} - x_0| \geqslant r) \leqslant \frac{\mathbb{E}^x \tau_{D_1}}{t} + C_1 \frac{\mathbb{E}^x \tau_{D_1}}{V^2(r)} \\ &\leqslant (C_1 + 1) C_8 H_R (\mathcal{J}(R))^{-2} \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)}. \end{aligned}$$

□

Remark 6 If $d = 1$, then regardless of (**H**), we have for any $t > 0$,

$$\mathbb{P}^x(\tau_D > t) \leqslant C_{13} \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)}.$$

This is easily seen from the above proof and the estimate $\mathbb{E}^x \tau_{D_1} \leqslant 2V(r/2)V(\delta_D(x))$. The estimate is not, however, sharp for large t if D is bounded.

We end this section with bounds for the survival probabilities in the complement of the ball. Noteworthy the constants in the bounds do not depend on the radius.

Theorem 6.3 *Suppose that* $\psi \in WLSC(\underline{\alpha}, 0, \underline{c}) \cap WUSC(\overline{\alpha}, 0, \overline{C})$. *Let* $R > 0$ *and* $D = \overline{B}_R^c$.

(i) *There is a constant* $C^* = C^*(d, \underline{\alpha}, \underline{c}, \overline{\alpha}, \overline{C})$ *such that,*

$$\mathbb{P}^x(\tau_D > t) \leqslant C^* \left(\frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1 \right), \qquad t > 0.$$

(ii) *If* $d > \overline{\alpha}$*, then*

$$\mathbb{P}^x(\tau_D > t) \approx \frac{V(\delta_D(x))}{\sqrt{t} \wedge V(R)} \wedge 1, \qquad t > 0,$$

where the comparability constant depends only on d, $\underline{\alpha}$, $\underline{c}$, $\overline{\alpha}$, $\overline{C}$.

Proof In the proof we make the convention that all the starred constants may only depend on d, $\underline{\alpha}$, $\underline{c}$, $\overline{\alpha}$, $\overline{C}$. By Remark 6 we only need to deal with the first part only for $d \geqslant 2$. By the assumption on ψ and Proposition 5.2, $\inf_{R>0} \mathcal{J}(R) \geqslant c_1^* > 0$. Furthermore, for $d \geqslant 2$, by Lemma 7.2 or Lemma 7.3 we have $H_\infty < \infty$. The first claim now follows from Lemma 6.2.

Let $d > \overline{\alpha}$. By (2.21) we have absolute constant c_2 such that

$$\mathbb{P}^x(\tau_D > t) \geqslant c_2 \frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1, \qquad t > 0, \quad x \in \mathbb{R}^d.$$

Therefore, it is enough to show that there is c_3^* such that

$$\mathbb{P}^x(\tau_D = \infty) \geqslant c_3^* \left(\frac{V(\delta_D(x))}{V(R)} \wedge 1 \right). \tag{6.2}$$

Since $d > \overline{\alpha}$, by Corollary 5.9, there is $c_4^* \geqslant 2$ such that for $|x| \geqslant c_4^* R$, $\mathbb{P}^x(\tau_D = \infty) \geqslant 1/2$. It is now enough to show (6.2) for $R \leqslant |x| \leqslant 3R/2$. Let $F = B(\frac{3Rx}{2|x|}, \frac{R}{2})$. By the strong Markov property,

$$\mathbb{P}^x(\tau_D = \infty) \geqslant \mathbb{E}^x \left(\mathbb{P}^{X_{\tau_F}}(\tau_D = \infty), |X_{\tau_F}| \geqslant c_4^* R \right) \geqslant (1/2)\mathbb{P}^x(|X_{\tau_F}| \geqslant c_4^* R).$$

By the Ikeda–Watanabe formula,

$$\mathbb{P}^x(|X_{\tau_F}| \geqslant c_4^* R) \geqslant \nu(\{y : y_1 \geqslant c_4^* R\})\, \mathbb{E}^x \tau_F.$$

By Theorem 4.1 and subadditivity of V we have $\mathbb{E}^x \tau_F \geqslant c_5^* V(\delta_D(x)) V(R)$. By Lemma 5.1 and Proposition 2.4 for X_t^1 and subadditivity of V we obtain $\nu(\{y : y_1 \geqslant c_4^* R\}) \geqslant c_6^* / V^2(R)$ for some $c_6^* > 0$. This proves (6.2). □

We note that the assumption $d > \overline{\alpha}$ cannot in general be removed from the second part of the theorem. For example, if $d = 1$, then the survival probability of the Cauchy process has asymptotics of logarithmic type, see [9, Remark 10]. Precise estimates of the tails of the hitting time of the ball for the isotropic stable Lévy processes are given in [9]. For the Brownian motion, [15] gives even more-precise estimates of the derivative of the survival probability.

Remark 7 We conclude this section with an obvious but necessary remark: if $B, \overline{B'} \subset \mathbb{R}^d$ are balls (open and closed, correspondingly) and $B \subset D \subset \overline{B'}^c$, then the survival probability of D is bounded as follows

$$\mathbb{P}^x(\tau_B > t) \leqslant \mathbb{P}^x(\tau_D > t) \leqslant \mathbb{P}^x(\tau_{\overline{B'}^c} > t), \qquad x \in \mathbb{R}^d,\ t \geqslant 0.$$

This leads to immediate bounds for the survival probabilities for general $C^{1,1}$ open sets $D \subset \mathbb{R}^d$: If $\psi \in \mathrm{WLSC}(\underline{\alpha}, 0, \underline{c}) \cap \mathrm{WUSC}(\overline{\alpha}, 0, \overline{C})$ and D is $C^{1,1}$ at scale r, then by Proposition 6.1, Remark 5 and Theorem 6.3, there is $C^* = C^*(d, \underline{\alpha}, \overline{\alpha}, \underline{c}, \overline{C})$ such that if $x \in \mathbb{R}^d$ and $t \leqslant C_{11} V^2(r)$, then

$$\frac{1}{C^*}\left(\frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1\right) \leqslant \mathbb{P}^x(\tau_D > t) \leqslant C^*\left(\frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1\right). \tag{6.3}$$

7 Discussion of assumptions and applications

7.1 Condition (H)

Recall that function $v > 0$ is called log-concave if $\log v$ is concave, and if this is the case, then the (right hand side) derivative v' of v exists and v'/v is non-increasing. The next lemma shows that (**H***) is satisfied with $H_\infty = 5$ if V is log-concave.

Lemma 7.1 *If V is log-concave and $0 < x \leqslant y \leqslant z \leqslant 5x$, then $V(z) - V(y) \leqslant 5V'(x)(z-y)$.*

Proof We have $V > 0$ increasing, and $V'/V > 0$ non-increasing. Therefore,

$$\log V(z) - \log V(y) = \int_y^z \frac{V'(s)}{V(s)} ds \leqslant \frac{V'(x)}{V(x)}(z-y),$$

and

$$\log V(z) - \log V(y) = \int_{V(y)}^{V(z)} \frac{1}{u}\, du \geqslant \frac{V(z) - V(y)}{V(z)}.$$

By this and subadditivity of V,

$$V(z) - V(y) \leqslant \frac{V(z)V'(x)}{V(x)}(z-y) \leqslant 5V'(x)(z-y).$$

□

The next lemma shows that for dimension $d \geqslant 3$, the weak lower scaling condition implies (**H**), while the weak global lower scaling implies (**H***). This helps extend many results previously known only for complete subordinate Brownian motions with scaling (see below for definitions).

Remark 8 A sufficient condition for log-concavity of V is that V' be monotone, which is common for subordinate Brownian motions, for instance if the subordinator is special. For complete subordinate Brownian motions, V is even a Bernstein function (see [37, Proposition 4.5]). It is interesting to note that V' is not monotone for the so-called truncated α-stable Lévy processes with $0 < \alpha < 2$ [32]. Indeed, if the Lévy measure has compact support, then by [21, (5.3.4)] the Lévy measure of the ladder-height process (subordinator) has compact support as well. By [44, Proposition 11.16], κ is not a special Bernstein function, therefore by [44, Theorem 11.3], V' is not decreasing. We, however, note that the truncated stable processes have global weak lower scaling with $\underline{\alpha} = \alpha$, and our estimates of the expected exit time for the ball hold for these processes with the comparability constant independent of r. This shows flexibility of our methods.

Lemma 7.2 *If $d \geqslant 3$ and $\psi \in WLSC(\beta, \theta, \underline{c})$, then* (**H**) *holds with $H_R = H_R(\beta, \theta, \underline{c}, R)$ for any $R \in (0, \infty)$. If, furthermore, $\theta = 0$, then* (**H***) *even holds.*

Proof By [24, Corollary 5], the scale invariant Harnack inequality holds for X^1, the one-dimensional projection of X. Namely, for every $R > 0$ there is $C_R < \infty$ such that if $0 < r \leqslant R$, $h \geqslant 0$ on $\mathbb{R}$ and h is harmonic for X^1 on $(-r, r)$, then

$$\sup_{y\in(-r/2,r/2)} h(y) \leqslant C_R \inf_{y\in(-r/2,r/2)} h(y).$$

Since V' is harmonic on $(0, \infty)$ for X^1 and $x_0 \geqslant 2r$, then by spatial homogeneity,

$$\sup_{\theta\in(x_0-r,x_0+r)} V'(\theta) \leqslant C_R \inf_{\theta\in(x_0-r,x_0+r)} V'(\theta).$$

Using the inequality with $(x_0, r) = (x, x/2),\ (9x/4, x),\ (4x, x)$, where $0 < x \leqslant R$ we get

$$\sup_{\theta\in(x/2,5x)} V'(\theta) \leqslant C_R^3 \inf_{\theta\in(x/2,5x)} V'(\theta) \leqslant C_R^3 V'(x).$$

The absolute continuity of V yields the conclusion. □

Lemma 7.3 *Let $d \geqslant 1$ and $\psi \in WLSC(\underline{\alpha}, \theta, \underline{c}) \cap WUSC(\overline{\alpha}, \theta, \overline{C})$. Then* (**H**) *holds with $H_R = H_R(\underline{\alpha}, \overline{\alpha}, \theta, \underline{c}, \overline{C}, R)$ for all $R \in (0, \infty)$. If, furthermore, $\theta = 0$, then* (**H***) *even holds.*

Proof By the same arguments as given in Lemma 7.2 it is enough to show that the scale invariant Harnack inequality holds for X^1. By [10, Corollary 22 and (16)] and Proposition 2.4 applied to X^1, there exists $r_0 > 0$ such that

$$\nu_1(u) \approx \frac{1}{V(|u|)^2|u|^d}, \quad 0 < |u| < r_0/\theta.$$

At first, let $\theta > 0$. By [16, Theorem 5.2] used with auxiliary function

$$\phi(r) = \begin{cases} V^2(r) & \text{if } 0 < r \leqslant r_0/\theta, \\ V^2(r_0/\theta)(r\theta/r_0)^{\overline{\alpha}} & \text{if } r > r_0/\theta, \end{cases}$$

we infer that the scale invariant Harnack inequality holds for X^1.

For $\theta = 0$ we use [18, Theorem 4.12] instead of [16, Theorem 5.2] to get the global scale invariant Harnack inequality for X^1. In consequence we obtain ($\mathbf{H}^*$). □

Lemma 7.4 *If* $\sigma > 0$, *then* ($\mathbf{H}$) *holds.*

Proof If $\sigma > 0$, then V' is positive, continuous and bounded by σ^{-1} (see Lemma 2.2). By Cauchy's mean value theorem, for $R > 0$ we have

$$V(z) - V(y) \leqslant \sigma^{-1}(z - y) \leqslant H_R V'(x)(z - y) \quad 0 < x \leqslant y \leqslant z \leqslant 5R,$$

where $H_R = (\sigma \inf_{z \leqslant 5R} V'(z))^{-1} < \infty$. □

The case when X is a subordinate Brownian motion is of special interest in this theory: we consider a Brownian B motion in $\mathbb{R}^d$ and an independent subordinator η, and we let

$$X(t) = B(2\eta(t)).$$

The process X is then called a subordinate Brownian motion. The monograph [44] is devoted to the study of such processes. Furthermore, X is called a *special* subordinate Brownian motion if the subordinator is *special* (i.e. given by a special Bernstein function [44, Definition 11.1]), and it is called *complete* subordinate Brownian motion if the subordinator is even *complete* [44, Proposition 7.1]. We let φ be the Laplace exponent of the subordinator, i.e.

$$\mathbb{E} \exp[-u\eta(t)] = \exp[-t\varphi(u)], \quad u \geqslant 0.$$

Since

$$\mathbb{E} e^{i\langle \xi, B_t \rangle} = e^{-t|\xi|^2/2}, \quad t \geqslant 0, \ \xi \in \mathbb{R}^d,$$

we have

$$\psi(\xi) = \varphi(|\xi|^2).$$

Then by [37, Theorem 4.4], $V(r) \approx \varphi(r^{-2})^{-1/2}$. For clarity, [37] makes the assumption that φ is unbounded, but it is not necessary for the result. In connection to [37, Remark 4.7] we note that $\varphi(x)$ and $x/\varphi(x)$ are monotone. For instance, by concavity, if $s \geqslant 1$ and $x \geqslant 0$, then $\varphi(sx) \leqslant s\varphi(x)$, hence $sx/\varphi(sx) \geqslant x/\varphi(x)$.

Remark 9 If X is a subordinate Brownian motion, then due to [24, Theorem 7] we may skip the assumption $d \geqslant 3$ in Lemma 7.2. This is related to the fact that Harnack inequality is inherited by orthogonal projections of isotropic unimodal Lévy processes, and every subordinate Brownian motion in dimensions 1 and 2 is a projection of a subordinate Brownian motion in dimension 3 (this observation was used before in [31]).

Lemma 7.5 *If X is a special subordinate Brownian motion, then V is concave.*

Proof By [33, Proposition 2.1], the Laplace exponent κ given by (2.14) is a special Bernstein function. In fact, [33] makes the assumption that the Laplace exponent of a subordinator is a complete Bernstein function, but the same proof works if it is only a special Bernstein function, since it suffices that $|x|^2/\psi(x)$ be negative definite. Then [44, Theorem 11.3] implies that V' is non-increasing, which ends the proof. □

Remark 10 Lemma 7.5 implies ($\mathbf{H}^*$) with $H_\infty = 1$ for special subordinate Brownian motions.

We finish this section with a simple argument leading to boundary Harnack inequality.

Proposition 7.6 *Let ν be continuous in $\mathbb{R}^d \setminus \{0\}$. Assume that ψ satisfies the global weak lower and upper scaling conditions, D is $C^{1,1}$ at scale $\rho > 0$, $z \in \partial D$, $0 < r < \rho$ and $u \geqslant 0$ is regular harmonic in $D \cap B(z, r)$ and vanishes in $B(z, r) \setminus D$. Then positive $c = c(d, \psi)$, $c_1 = c_1(d, \psi)$ exist such that*

$$\frac{u(x)}{u(y)} \leqslant c \frac{V(\delta_D(x))}{V(\delta_D(y))} \leqslant c_1 \sqrt{\frac{\psi(1/\delta_D(y))}{\psi(1/\delta_D(x))}}, \qquad x, y \in D \cap B(z, r/2).$$

Proof [10, Corollary 27] shows that the assumptions of [35] are satisfied. By [35, Lemma 5.5] we get

$$\frac{u(x)}{u(y)} \leqslant c \frac{\mathbb{E}^x \tau_{D \cap B(z,r)}}{\mathbb{E}^x \tau_{D \cap B(z,r)}}, \qquad x, y \in D \cap B(z, r/2).$$

for $c = c(d, \psi)$. In fact [35, Lemma 5.5] is stated only for $r < 1$ but under global weak scaling conditions one can repeat arguments of [35] to obtain the global boundary Harnack inequality (see [36]). Then we estimate the expected exit time of $D \cap B(z, r)$ by using Lemma 7.3, Theorem 4.1, Corollary 4.5 and Proposition 5.2, and we obtain the first inequality. The second inequality follows from Proposition 2.4 and (3.1). □

7.2 Examples

Our results apply to the following unimodal Lévy processes. In each case our sharp bounds for the expected first exit time from the ball apply and the comparability constants depend only on the dimension and the Lévy–Khintchine exponent of the process but not on the radius of the ball. Our estimates of the probability of surviving in B_r and $\overline{B_r}^c$ also hold with constants independent of r if the characteristic exponent of X has global upper and lower scalings (see [10] for a simple discussion of scaling). If the scalings are not global, then the constants may deteriorate as r increases.

Example 1 Chapter 15 of [44] lists more than one hundred cases and classes of complete Bernstein functions. All of those which are unbounded and have killing rate 0 are covered by our results (see Lemma 7.5): we obtain sharp estimates of the expected first exit time from the ball. In fact, the comparability constants depend only on the dimension. This is, e.g., the case for Lévy process with the characteristic exponent

$$\psi(\xi) = \left[|\xi|^{\alpha_2} + (|\xi|^2 + m)^{\alpha_3/2} - m^{\alpha_3/2}\right]^{1-\alpha_1/2} \log^{\alpha_1/2}(1 + |\xi|^{\alpha_4}),$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 2]$, $\alpha_1 + \alpha_2 + \alpha_3 > 0$, $\alpha_2 + \alpha_3 + \alpha_4 > 0$ and $m \geqslant 0$, and also when

$$\psi_2(\xi) = u(|\xi|) - u(0^+),$$

where $u(r) = mr^2 + r^2/\log^{\alpha_1/2}(1 + r^{\alpha_4})$. These include, e.g., isotropic stable process, relativistic stable process, sums of two independent isotropic stable processes (also with Gaussian component) and geometric stable processes, variance gamma processes and conjugate to geometric stable processes [44].

Example 2 Let $0 < \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant 2$, $\alpha^* = \lim_{k\to\infty} \alpha_k$, and define $f(r) = r^{-\alpha_{\lfloor r \rfloor}}$, $r > 0$. Then $f(1/r) \in \mathrm{WLSC}(\alpha_0, 0, 1) \cap \mathrm{WUSC}(\alpha^*, 0, 1)$, if $\alpha^* < 2$. Consider a unimodal Lévy process with Lévy density $\nu(x) = f(|x|)|x|^{-d}$, $x \neq 0$. By [10, Proposition 28], $\psi \in \mathrm{WLSC}(\alpha_0, 0, \underline{c})$. For $d \geqslant 3$ by Lemma 7.2 we get $\mathbb{E}^x \tau_{B_r} \approx 1/\sqrt{f(r) f(r - |x|)}$, where $|x| < r < \infty$, and the comparability constant is independent of r. If $\alpha^* < 2$, then by [24, Proposition 8] $\psi \in \mathrm{WUSC}(\alpha^*, 0, \overline{C})$. The above approximation for $\mathbb{E}^x \tau_{B_r}$ is valid for $d = 2$, too, cf. Lemma 7.3.

Example 3 Let $d \geqslant 3$, $\sigma \geqslant 0$, $\nu(x) = f(|x|)/|x|^d$, $x \in \mathbb{R}^d \setminus \{0\}$. Let $f \geqslant 0$ be non-increasing and let $\beta > 0$ be such that $f(\lambda r) \leqslant c\lambda^{-\beta} f(r)$ for $r > 0$ and $\lambda > 1$ (see [24, Example 2 and 48] and Lemma 7.2). So is the case for the following processes (with $\alpha, \alpha_1 \in (0, 2)$): truncated stable process ($f(r) = r^{-\alpha}\mathbf{1}_{(0,1)}(r)$), tempered stable process ($f(r) = r^{-\alpha}e^{-r}$), isotropic Lamperti stable process ($f(r) = re^{\delta r}(e^r - 1)^{-\alpha-1}$, where $\delta < \alpha + 1$) and layered stable process ($f(r) = r^{-\alpha}\mathbf{1}_{(0,1)}(r) + r^{-\alpha_1}\mathbf{1}_{[1,\infty)}(r)$).

More examples of isotropic processes with scaling may be found in [10, Section 4.1].

Acknowledgments Tomasz Grzywny was supported by the Alexander von Humboldt Foundation and expresses his gratitude for the hospitality of Technische Unversität Dresden and Universität Bielefeld, where the paper was written in main part. We thank the referees for very useful comments and suggestions.

References

1. Aikawa, H., Kilpeläinen, T., Shanmugalingam, N., Zhong, X.: Boundary Harnack principle for p-harmonic functions in smooth Euclidean domains. Potential Anal. **26**(3), 281–301 (2007)
2. Ancona, A.: On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$. J. Lond. Math. Soc. (2) **34**(2), 274–290 (1986)
3. Ancona, A.: On positive harmonic functions in cones and cylinders. Rev. Mat. Iberoam. **28**(1), 201–230 (2012)
4. Armitage, D.H., Gardiner, S.J.: Classical potential theory. Springer Monographs in Mathematics. Springer, London (2001)
5. Bass, R.F., Kassmann, M.: Hölder continuity of harmonic functions with respect to operators of variable order. Commun. Partial Differ. Equ. **30**(7–9), 1249–1259 (2005)
6. Bertoin, J.: Lévy Processes. Cambridge Tracts in Mathematics, vol. 121. Cambridge University Press, Cambridge (1996)
7. Bogdan, K., Byczkowski, T., Kulczycki, T., Ryznar, M., Song, R., Vondraček, Z.: Potential analysis of stable processes and its extensions. In: Graczyk, P., Stos, A. (eds.) Lecture Notes in Mathematics, vol. 1980. Springer, Berlin (2009)
8. Bogdan, K., Grzywny, T., Ryznar M.: Dirichlet heat kernel for unimodal Lévy processes. http://arxiv.org/abs/1402.4589
9. Bogdan, K., Grzywny, T., Ryznar, M.: Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. Ann. Probab. **38**(5), 1901–1923 (2010)
10. Bogdan, K., Grzywny, T., Ryznar, M.: Density and tails of unimodal convolution semigroups. J. Funct. Anal. **266**(6), 3543–3571 (2014)
11. Bogdan, K., Jakubowski, T.: Estimates of the Green function for the fractional Laplacian perturbed by gradient. Potential Anal. **36**(3), 455–481 (2012)
12. Bogdan, K., Kulczycki, T., Kwaśnicki, M.: Estimates and structure of α-harmonic functions. Probab. Theory Relat. Fields **140**(3–4), 345–381 (2008)
13. Bogdan, K., Sztonyk, P.: Estimates of the potential kernel and Harnack's inequality for the anisotropic fractional Laplacian. Studia Math. **181**(2), 101–123 (2007)
14. Burkholder, D.L.: Exit times of Brownian motion, harmonic majorization, and Hardy spaces. Adv. Math. **26**(2), 182–205 (1977)
15. Byczkowski, T., Małecki, J., Ryznar, M.: Hitting times of Bessel processes. Potential Anal. **38**(3), 753–786 (2013)
16. Chen, Z.-Q., Kim, P., Kumagai, T.: On heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces. Acta Math. Sin. **25**(7), 1067–1086 (2009)
17. Chen, Z.-Q., Kim, P., Song, R.: Global heat kernel estimate for relativistic stable processes in exterior open sets. J. Funct. Anal. **263**(2), 448–475 (2012)
18. Chen, Z.-Q., Kumagai, T.: Heat kernel estimates for jump processes of mixed types on metric measure spaces. Probab. Theory Relat. Fields **140**(1–2), 277–317 (2008)
19. Chung, K.L., Walsh, J.B.: Markov processes, Brownian motion, and time symmetry, 2nd edn, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 249. Springer, New York (2005)
20. Davies, E.B.: Heat Kernels and Spectral Theory. Cambridge Tracts in Mathematics, vol. 92. Cambridge University Press, Cambridge (1989)
21. Doney, R.A.: Fluctuation Theory for Lévy Processes, Lecture Notes in Mathematics, vol. 1897. In: Picard, J. (ed.) Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005. Springer, Berlin (2007)
22. Fristedt, B.: Sample functions of stochastic processes with stationary, independent increments. In: Advances in Probability and Related Topics, vol. 3, pp. 241–396. Dekker, New York (1974)
23. Fukushima, M.: On transition probabilities of symmetric strong Markov processes. J. Math. Kyoto Univ. **12**, 431–450 (1972)
24. Grzywny, T.: On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes. Potential Anal. doi:10.1007/s11118-013-9360-y
25. Grzywny, T.: Intrinsic ultracontractivity for Lévy processes. Probab. Math. Stat. **28**(1), 91–106 (2008)
26. Grzywny, T., Ryznar, M.: Potential theory of one-dimensional geometric stable processes. Colloq. Math. **129**(1), 7–40 (2012)
27. Hawkes, J.: Potential theory of Lévy processes. Proc. Lond. Math. Soc. (3) **38**(2), 335–352 (1979)

28. Hoh, W.: Pseudo Differential Operators Generating Markov Processes. Universität Bielefeld, Habilitation (1998)
29. Ikeda, N., Watanabe, S.: On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes. J. Math. Kyoto Univ. **2**, 79–95 (1962)
30. Kim, P., Mimica, A.: Green function estimates for subordinate Brownian motions: Stable and beyond. Trans. Am. Math. Soc. (2014). doi:10.1090/S0002-9947-2014-06017-0
31. Kim, P., Mimica, A.: Harnack inequalities for subordinate Brownian motions. Electron. J. Probab. **17**(37), 23 (2012)
32. Kim, P., Song, R.: Potential theory of truncated stable processes. Math. Z. **256**(1), 139–173 (2007)
33. Kim, P., Song, R., Vondraček, Z.: Boundary Harnack principle for subordinate Brownian motions. Stochastic Process. Appl. **119**(5), 1601–1631 (2009)
34. Kim, P., Song, R., Vondraček, Z.: Two-sided Green function estimates for killed subordinate Brownian motions. Proc. Lond. Math. Soc. (3) **104**(5), 927–958 (2012)
35. Kim, P., Song, R., Vondraček, Z.: Uniform boundary Harnack principle for rotationally symmetric Lévy processes in general open sets. Sci. China Math. **55**(11), 2317–2333 (2012)
36. Kim, P., Song, R., Vondraček, Z.: Global uniform boundary Harnack principle with explicit decay rate and its application. Stochastic Process. Appl. **124**(1), 235–267 (2014)
37. Kwaśnicki, M., Małecki, J., Ryznar, M.: Suprema of Lévy processes. Ann. Probab. **41**(3B), 2047–2065 (2013)
38. Michalik, K.: Sharp estimates of the Green function, the Poisson kernel and the Martin kernel of cones for symmetric stable processes. Hiroshima Math. J. **36**(1), 1–21 (2006)
39. Mimica, A., Vondraček, Z.: Unavoidable collections of balls for isotropic Lévy processes. Stochastic Process. Appl. **124**(3), 1303–1334 (2014)
40. Port, S.C., Stone, C.J.: Infinitely divisible processes and their potential theory. Ann. Inst. Fourier (Grenoble) **21**(2), 157–275 [ibid., **21**(4), 179–265 (1971)]
41. Pruitt, W.E.: The growth of random walks and Lévy processes. Ann. Probab. **9**(6), 948–956 (1981)
42. Sato, K.-I.: Lévy processes and infinitely divisible distributions, volume 68 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1999). Translated from the 1990 Japanese original, Revised by the author
43. Schilling, R.L.: Growth and Hölder conditions for the sample paths of Feller processes. Probab. Theory Relat. Fields **112**(4), 565–611 (1998)
44. Schilling, R.L., Song, R., Vondraček, Z.: Bernstein Functions, Theory and Applications, 2nd edn, de Gruyter Studies in Mathematics, vol. 37. Berlin: Walter de Gruyter & Co. (2012)
45. Silverstein, M.L.: Classification of coharmonic and coinvariant functions for a Lévy process. Ann. Probab. **8**(3), 539–575 (1980)
46. Tucker, H.G.: On a necessary and sufficient condition that an infinitely divisible distribution be absolutely continuous. Trans. Am. Math. Soc. **118**, 316–330 (1965)
47. Wada, M.: Continuity of harmonic functions for non-local Markov generators. Potential Anal. **39**(1), 1–11 (2013)
48. Watanabe, T.: The isoperimetric inequality for isotropic unimodal Lévy processes. Z. Wahrsch. Verw. Gebiete **63**(4), 487–499 (1983)
49. Zabczyk, J.: Sur la théorie semi-classique du potentiel pour les processus à accroissements indépendants. Studia Math. **35**, 227–247 (1970)