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Reversibility of whole-plane SLE

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Abstract The main result of this paper is that, for $\kappa \in (0, 4]$, whole-plane SLE_κ satisfies reversibility, which means that the time-reversal of a whole-plane SLE_κ trace is still a whole-plane SLE_κ trace. In addition, we find that the time-reversal of a radial SLE_κ trace for $\kappa \in (0, 4]$ is a disc SLE_κ trace with a marked boundary point. The main tool used in this paper is a stochastic coupling technique, which is used to couple two whole-plane SLE_κ traces so that they overlap. Another tool used is the Feynman–Kac formula, which is used to solve a PDE. The solution of this PDE is then used to construct the above coupling.

Mathematics Subject Classification 60G · 30C

1 Introduction

The stochastic Loewner evolution (SLE) introduced by Oded Schramm [1] describes some random fractal curves in plane domains that satisfy conformal invariance and Domain Markov Property. These two properties make SLEs the most suitable candidates for the scaling limits of many two-dimensional lattice models at criticality. These models are proved or conjectured to converge to SLE with different parameters (e.g., [2–7]). For basics of SLE, the reader may refer to [8] and [9].

There are several different versions of SLEs, among which chordal SLE and radial SLE are the most well-known. A chordal or radial SLE trace is a random fractal curve that grows in a simply connected plane domain from a boundary point. A chordal SLE

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trace ends at another boundary point, and a radial SLE trace ends an interior point. Their behaviors both depend on a positive parameter κ. When $\kappa \in (0, 4]$, both traces are simple curves, and all points on the trace other than the initial and final points lie inside the domain. When $\kappa > 4$, the traces have self-intersections.

A stochastic coupling technique was introduced in [10] to prove that, for $\kappa \in (0, 4]$, chordal SLE_κ satisfies reversibility, which means that if β is a chordal SLE_κ trace in a domain D from a to b, then after a time-change, the time-reversal of β becomes a chordal SLE_κ trace in D from b to a. The technique was later used [11,12] to prove Duplantier's duality conjecture, which says that, for $\kappa > 4$, the boundary of the hull generated by a chordal SLE_κ trace looks locally like an $\mathrm{SLE}_{16/\kappa}$ trace. The technique was also used to prove that the radial or chordal SLE_2 can be obtained by erasing loops on a planar Brownian motion [13], and the chordal $\mathrm{SLE}(\kappa, \rho)$ introduced in [2] also satisfies reversibility for $\kappa \in (0, 4]$ and $\rho \geq \kappa/2 - 2$ [14].

Since the initial point and final point of a radial SLE are topologically different, the time-reversal of a radial SLE trace can not be a radial SLE trace. However, we may consider whole-plane SLE instead, which describes a random fractal curve in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that grows from one interior point to another interior point. Whole-plane SLE is related to radial SLE as follows: conditioned on the initial part of a whole-plane SLE_κ trace, the rest part of such trace has the distribution of a radial SLE_κ trace that grows in the complementary domain of the initial part of this trace. The main result of this paper is the following theorem.

Theorem 1.1 *Whole-plane SLE_κ satisfies reversibility for $\kappa \in (0, 4]$.*

The theorem in the case $\kappa = 2$ has been proved in [15]. The proof used the reversibility of loop-erased random walk (LERW for short, see [16]) and the convergence of LERW to whole-plane SLE_2. In this paper we will obtain a slightly more general result: the reversibility of whole-plane $\mathrm{SLE}(\kappa, s)$ process, which is defined by adding a constant drift to the driving function for the whole-plane SLE_κ process. This is the statement of Theorem 7.1.

To get some idea of the proof, let's first review the proof of the reversibility of chordal SLE_κ in [10]. We constructed a pair of chordal SLE_κ traces γ_1 and γ_2 in a simply connected domain D, where γ_1 grows from a boundary point a_1 to another boundary point a_2, γ_2 grows from a_2 to a_1, and these two traces commute in the following sense. Fix $j \neq k \in \{1, 2\}$, if T_k is a stopping time for γ_k, then conditioned on $\gamma_k(t)$, $t \leq T_k$, the part of γ_j before hitting $\gamma_k(t)((0, T_k])$ has the distribution of a chordal SLE_κ trace that grows from a_j to $\gamma_k(T_k)$ in $D_k(T_k)$, which is a component of $D\backslash\gamma_k(t)((0, T_k])$. In the case $\kappa \leq 4$, a.s. γ_j hits $\gamma_k(t)((0, T_k])$ exactly at $\gamma_k(T_k)$, so γ_j visits $\gamma_k(T_k)$ before any $\gamma_k(t)$, $t < T_k$. Since this holds for any stopping time T_k for γ_k, the two traces a.s. overlap, which implies the reversibility.

To prove the reversibility of whole-plane SLE_κ, we want to construct two whole-plane SLE_κ traces in $D = \widehat{\mathbb{C}}$, one is γ_1 from a_1 to a_2, the other is γ_2 from a_2 to a_1, so that γ_1 and γ_2 commute. Here we can not expect that they commute in exactly the same sense as in the above paragraph. Note that conditioned on $\gamma_k(t)$, $t \leq T_k$, the part of γ_j before hitting $\gamma_k(t)$, $t \leq T_k$, can not have the distribution of a whole-plane SLE_κ trace in $D_k(T_k)$ from a_1 to $\gamma_k(T_k)$ because now the complementary domain $D_k(T_k)$ is topologically different from $\widehat{\mathbb{C}}$, while whole-plane SLEs are only defined in $\widehat{\mathbb{C}}$. Since the conditional curve grows from an interior point to a boundary point, it is neither a radial SLE trace nor a chordal SLE trace.

Thus, we need to define SLE traces in simply connected domains that grow from an interior point to a boundary point. We use the idea of defining whole-plane SLE using radial SLE. The situation here is a little different: after a positive initial part, the rest part of the curve grows in a doubly connected domain. Another difference is that there is a marked point on the boundary of the initial domain. In this paper, we use the annulus Loewner equation introduced in [17] together with an annulus drift function $\Lambda = \Lambda(t, x)$ to define the so-called annulus SLE(κ, Λ) process in a doubly connected domain D, which starts from a point $a \in \partial D$, and whose growth is affected by a marked point $b \in \partial D$. In the case when a and b lie on different boundary components, by shrinking the boundary component containing a to a singlet, we get the so called disc SLE(κ, Λ), which describes a random curve that grows in a simply connected domain and starts from an interior point.

We find that if $\Lambda = \kappa \frac{\Gamma'}{\Gamma}$, where Γ is a positive differentiable function defined on $(0, \infty) \times \mathbb{R}$ that solves a linear PDE and satisfies some periodic condition [see (4.1) and (4.2)], then using the coupling technique we could construct a coupling of two whole-plane SLE$_\kappa$ traces: γ_1 and γ_2, which commute in the sense that, conditioned on one curve up to a finite stopping time T, the other curve is a disc SLE(κ, Λ) trace in the remaining domain, and its marked point is the tip point of the first curve at T.

The main new idea in the current paper is an application of a Feynman–Kac representation, which is used to get a formal solution of the PDE for Γ in the case $\kappa \in (0, 4]$. Using Fubini's Theorem, Itô's formula, and some estimations, we prove that the formal solution Γ_κ is smooth and solves the PDE. We then find that $\Lambda_\kappa := \kappa \frac{\Gamma_\kappa'}{\Gamma_\kappa}$ satisfies the property that the marked point for an annulus or disc SLE(κ, Λ_κ) process is a subsequential limit point of the trace. This property implies that, if two whole-plane SLE$_\kappa$ traces commute as in the previous paragraph, then they must overlap. So the main theorem is proved. Moreover, from the relation between whole-plane SLE$_\kappa$ and radial SLE$_\kappa$, we conclude that, for $\kappa \in (0, 4]$, the time-reversal of a radial SLE$_\kappa$ trace is a disc SLE(κ, Λ_κ) trace.

The marked point and the initial point of an annulus SLE(κ, Λ) process could also lie on the same boundary component. In this case, if $\Lambda = \kappa \frac{\Gamma'}{\Gamma}$, and Γ satisfies a similar linear PDE [see (4.48)], then for a doubly connected domain D with two boundary points a_1 and a_2 on the same boundary component, we can construct a pair of annulus SLE(κ, Λ) traces γ_1 and γ_2 in D, which commute with each other. If an SLE process in a doubly connected domain is the scaling limit of some random path in a lattice, which satisfies reversibility at the discrete level, then such SLE should satisfy reversibility. We hope that the work in this paper will shed some light on the study of these processes.

The study on the commutation relations of SLE in doubly connected domains continues the work in [18] by Dubédat, who used some tools from Lie Algebra to obtain commutation conditions of SLE in simply connected domains. The annulus SLE(κ, Λ_κ) process used to prove the reversibility of whole-plane SLE$_\kappa$ was later studied in [19]. When $\kappa = 8/3$, the process satisfies the restriction property, which is similar to the restriction property for chordal $\mathrm{SLE}_{8/3}$ (see [2]). For $\kappa \in (0,4]\backslash\{8/3\}$, it satisfies some "weak" restriction property.

Lawler [20] used a different method to define annulus SLE_κ processes for $\kappa \in (0,4]$, which agree with our annulus $\mathrm{SLE}(\kappa,\Lambda_\kappa)$ processes. His construction uses the Brownian loop measures. The "strong" ($\kappa = 8/3$) and "weak" ($\kappa \neq 8/3$) restriction properties of Lawler's annulus SLE processes are immediate from the definition; and the reversibility of these processes follows from the chordal reversibility. However, the reversibility of whole-plane SLE is not proved in [20]. To get the whole-plane reversibility, some additional work is required based on Lawler's work. In this paper, the reversibility of annulus $\mathrm{SLE}(\kappa,\Lambda_\kappa)$ and the reversibility of whole-plane SLE_κ are proved separately, and the coupling technique is applied in both proofs, which are similar though.

Miller and Sheffield [21] recently proved the reversibility of whole-plane SLE for all $\kappa \in [0,8]$. Their proof uses the imaginary geometry of Gaussian free field developed in their earlier papers (c.f. [22]).

This paper is organized as follows. In Sect. 2, we introduce some symbols and notations. In Sect. 3, we review several versions of Loewner equations. In Sect. 3.4, we define annulus $\mathrm{SLE}(\kappa,\Lambda)$ and disc $\mathrm{SLE}(\kappa,\Lambda)$ processes, whose growth is affected by one marked boundary point. In Sect. 4 we prove that when Γ solves (4.1) or (4.48), there is a commutation coupling of two annulus $\mathrm{SLE}(\kappa,\Lambda)$ processes, where $\Lambda = \kappa\frac{\Gamma'}{\Gamma}$. In Sect. 5, we construct a coupling of two whole-plane SLE processes, which is similar to the coupling in the previous section. In Sect. 6, we solve PDE (4.1) using a Feynman–Kac expression, and the solution is then used in Sect. 7 to prove the reversibility of whole-plane SLE_κ process. In fact, we obtain a slightly more general result: the reversibility of skew whole-plane SLE_κ processes for $\kappa \in (0,4]$. In the last section, we find some solutions to the PDE for Γ and Λ when $\kappa \in \{0,2,3,4,16/3\}$, which can be expressed in terms of well-known special functions.

2 Preliminary

2.1 Symbols

Throughout this paper, we will use the following symbols. Let $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For $p > 0$, let $\mathbb{A}_p = \{z \in \mathbb{C} : 1 > |z| > e^{-p}\}$ and $\mathbb{S}_p = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < p\}$. For $p \in \mathbb{R}$, let $\mathbb{T}_p = \{z \in \mathbb{C} : |z| = e^{-p}\}$ and $\mathbb{R}_p = \{z \in \mathbb{C} : \operatorname{Im} z = p\}$. Then $\partial\mathbb{D} = \mathbb{T}$, $\partial\mathbb{H} = \mathbb{R}$, $\partial\mathbb{A}_p = \mathbb{T}\cup\mathbb{T}_p$, and $\partial\mathbb{S}_p = \mathbb{R}\cup\mathbb{R}_p$. Let e^i denote the map $z \mapsto e^{iz}$. Then e^i is a covering map from $\mathbb{H}$ onto $\mathbb{D}$, and from $\mathbb{S}_p$ onto $\mathbb{A}_p$; and it maps $\mathbb{R}$ onto $\mathbb{T}$ and maps $\mathbb{R}_p$ onto $\mathbb{T}_p$. For a doubly connected domain D, we use $\operatorname{mod}(D)$ to denote its modulus. For example, $\operatorname{mod}(\mathbb{A}_p) = p$.

A conformal map in this paper is a univalent analytic function. A conjugate conformal map is defined to be the complex conjugate of a conformal map. Let $I_0(z) = 1/\overline{z}$ be the reflection w.r.t. $\mathbb{T}$. Then I_0 is a conjugate conformal map from $\widehat{\mathbb{C}}$ onto itself, fixes $\mathbb{T}$, and interchanges 0 and ∞. Let $\widetilde{I}_0(z) = \overline{z}$ be the reflection w.r.t. $\mathbb{R}$. Then $\widetilde{I}_0$ is a conjugate conformal map from $\mathbb{C}$ onto itself and satisfies $e^i \circ \widetilde{I}_0 = I_0 \circ e^i$. For

$p > 0$, let $I_p(z) := e^{-p}/\overline{z}$ and $\widetilde{I}_p(z) = ip + \overline{z}$. Then I_p and $\widetilde{I}_p$ are conjugate conformal automorphisms of $\mathbb{A}_p$ and $\mathbb{S}_p$, respectively. Moreover, I_p interchanges $\mathbb{T}_p$ and $\mathbb{T}$, $\widetilde{I}_p$ interchanges $\mathbb{R}_p$ and $\mathbb{R}$, and $I_p \circ e^i = e^i \circ \widetilde{I}_p$.

We will frequently use functions $\cot(z/2), \tan(z/2), \coth(z/2), \tanh(z/2), \sin(z/2)$, $\cos(z/2)$, $\sinh(z/2)$, and $\cosh(z/2)$. For simplicity, we write 2 as a subscript. For example, $\cot_2(z)$ means $\cot(z/2)$, and we have $\cot_2'(z) = -\frac{1}{2}\sin_2^{-2}(z)$.

An increasing function in this paper will always be strictly increasing. For a real interval J, we use $C(J)$ to denote the space of real continuous functions on J. The maximal solution to an ODE or SDE with initial value is the solution with the biggest definition domain.

Many functions in this paper depend on two variables. In some of these functions, the first variable represents time or modulus, and the second variable does not. In this case, we use ∂_t and ∂_t^n to denote the partial derivatives w.r.t. the first variable, and use $'$, $''$, and the superscripts (h) to denote the partial derivatives w.r.t. the second variable. For these functions, we say that it has period r (resp. is even or odd) if it has period r (resp. is even or odd) in the second variable when the first variable is fixed. Some functions in Sects. 4 and 5 depend on two variables: t_1 and t_2, which both represent time. In this case we use ∂_j to denote the partial derivative w.r.t. the j-th variable, $j = 1, 2$.

In this paper, a domain is a connected open subset of $\widehat{\mathbb{C}}$, and a continuum is a connected compact subset of $\widehat{\mathbb{C}}$ that contains more than one point. A continuum K is called a hull in $\mathbb{C}$ if $K \subset \mathbb{C}$ and $\widehat{\mathbb{C}}\backslash K$ is connected. In this case, there is a unique conformal map f_K from $\widehat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ onto $\widehat{\mathbb{C}}\backslash K$ and satisfies $\lim_{z\to\infty} f_K(z)/z = a_K$ for some positive number a_K. Then a_K is called the capacity of K, and is denoted by $\mathrm{cap}(K)$.

A doubly connected domain in this paper is a domain whose complement is a disjoint union of two continuums. Let D be a doubly connected domain. If K is a relatively closed subset of D, has positive distance from one boundary component of D, and if $D\backslash K$ is also doubly connected, then we call K a hull in D, and call the number $\mathrm{mod}(D) - \mathrm{mod}(D\backslash K)$ the capacity of K in D, and let it be denoted by $\mathrm{cap}_D(K)$.

2.2 Brownian motions

Throughout this paper, a Brownian motion means a standard one-dimensional Brownian motion, and $B(t)$, $0 \le t < \infty$, will always be used to denote a Brownian motion. This means that $B(t)$ is continuous, $B(0) = 0$, and $B(t)$ has independent increment with $B(t) - B(s) \sim N(0, t-s)$ for $t \ge s \ge 0$. For $\kappa \ge 0$, the rescaled Brownian motion $\sqrt{\kappa}B(t)$ will be used to define annulus SLE$_\kappa$. The symbols $B_*(t)$, $\widehat{B}_*(t)$, or $\widetilde{B}_*(t)$ will also be used to denote a Brownian motion, where the $*$ stands for subscript. Let $(\mathcal{F}_t)_{t\ge 0}$ be a filtration. By saying that $B(t)$ is an $(\mathcal{F}_t)$-Brownian motion, we mean that $(B(t))$ is $(\mathcal{F}_t)$-adapted, and for any fixed $t_0 \ge 0$, $B(t_0 + t) - B(t_0)$, $t \ge 0$, is a Brownian motion independent of $\mathcal{F}_{t_0}$.

Definition 2.1 Let $\kappa > 0$ and $(\mathcal{F}_t)_{t\in\mathbb{R}}$ be a right-continuous filtration. A process $B^{(\kappa)}(t)$, $t \in \mathbb{R}$, is called a pre-$(\mathcal{F}_t)$-$(\mathbb{T};\kappa)$-Brownian motion if $(e^i(B^{(\kappa)}(t)))$ is $(\mathcal{F}_t)$-adapted, and for any $t_0 \in \mathbb{R}$,

$$B_{t_0}(t) := \frac{1}{\sqrt{\kappa}}\Big(B^{(\kappa)}(t_0+t) - B^{(\kappa)}(t_0)\Big), \quad 0 \le t < \infty, \tag{2.1}$$

is an $(\mathcal{F}_{t_0+t})$-Brownian motion. If $(\mathcal{F}_t)$ is generated by $(e^i(B^{(\kappa)}(t)))$, then we simply call $(B^{(\kappa)}(t))$ a pre-$(\mathbb{T};\kappa)$-Brownian motion.

Remark The name of the pre-$(\mathbb{T};\kappa)$-Brownian motion comes from the fact that $B_{\mathbb{T}}(t) := e^i(B^{(\kappa)}(t))$, $t \in \mathbb{R}$, is a Brownian motion on $\mathbb{T}$ with speed κ: for every $t_0 \in \mathbb{R}$, $B_{\mathbb{T}}(t_0)$ is uniformly distributed on $\mathbb{T}$; and $B_{\mathbb{T}}(t_0+t)/B_{\mathbb{T}}(t_0)$, $t \ge 0$, has the distribution of $e^i(\sqrt{\kappa}B(t))$, $t \ge 0$, and is independent of $B_{\mathbb{T}}(t)$, $t \le t_0$. One may construct $B^{(\kappa)}(t)$ as follows. Let $B_+(t)$ and $B_-(t)$, $t \ge 0$, be two independent Brownian motions. Let $\mathbf{x}$ be a random variable uniformly distributed on $[0, 2\pi)$, which is independent of $(B_\pm(t))$. Let $B^{(\kappa)}(t) = \mathbf{x} + \sqrt{\kappa}B_{\text{sign}(t)}(|t|)$ for $t \in \mathbb{R}$. Then $B^{(\kappa)}(t)$, $t \in \mathbb{R}$, is a pre-$(\mathbb{T};\kappa)$-Brownian motion.

Definition 2.2 Let $B^{(\kappa)}(t)$, $t \in \mathbb{R}$, be a pre-$(\mathcal{F}_t)$-$(\mathbb{T};\kappa)$-Brownian motion, where $(\mathcal{F}_t)$ is right-continuous, and every $\mathcal{F}_t$ contains all eligible events w.r.t. the process $(e^i(B^{(\kappa)}(t)))$. Suppose T is an $(\mathcal{F}_t)$-stopping time, and $T > t_0$ for a deterministic number $t_0 \in \mathbb{R}$. We say that $X(t)$ satisfies the $(\mathcal{F}_t)$-adapted SDE

$$dX(t) = a(t)dB^{(\kappa)}(t) + b(t)dt, \quad -\infty < t < T,$$

if $e^i(X(t))$, $a(t)$, and $b(t)$ are continuous and $(\mathcal{F}_t)$-adapted, and if for any deterministic number t_0 with $t_0 < T$, $X_{t_0}(t) := X(t_0+t) - X(t_0)$ satisfies the following $(\mathcal{F}_{t_0+t})_{t\ge 0}$-adapted SDE with the traditional meaning (c.f. Chapter IV, Section 3 of [23]):

$$dX_{t_0}(t) = a_{t_0}(t)\sqrt{\kappa}dB_{t_0}(t) + b_{t_0}(t)dt, \quad 0 \le t < T - t_0,$$

where $B_{t_0}(t)$ is given by (2.1), $a_{t_0}(t) := a(t_0+t)$, and $b_{t_0}(t) := b(t_0+t)$. Note that $B_{t_0}(t)$ is an $(\mathcal{F}_{t_0+t})_{t\ge 0}$-Brownian motion, $X_{t_0}(t)$, $a_{t_0}(t)$ and $b_{t_0}(t)$ are all $(\mathcal{F}_{t_0+t})_{t\ge 0}$-adapted.

2.3 Special functions

We now introduce some functions that will be used to define annulus Loewner equations. For $t > 0$, define

$$\mathbf{S}(t,z) = \lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{2kt}+z}{e^{2kt}-z} = \text{P.V.} \sum_{2|n} \frac{e^{nt}+z}{e^{nt}-z},$$

$$\mathbf{H}(t,z) = -i\mathbf{S}(t, e^i(z)) = -i\,\text{P.V.} \sum_{2|n} \frac{e^{nt}+e^{iz}}{e^{nt}-e^{iz}} = \text{P.V.} \sum_{2|n} \cot_2(z - int).$$

Then $\mathbf{H}(t,\cdot)$ is a meromorphic function in $\mathbb{C}$, whose poles are $\{2m\pi + i2kt : m, k \in \mathbb{Z}\}$, which are all simple poles with residue 2. Moreover, $\mathbf{H}(t,\cdot)$ is an odd function and takes real values on $\mathbb{R}\backslash\{\text{poles}\}$; $\text{Im}\,\mathbf{H}(t,\cdot) \equiv -1$ on $\mathbb{R}_t$; $\mathbf{H}(t, z+2\pi) = \mathbf{H}(t,z)$ and $\mathbf{H}(t, z+i2t) = \mathbf{H}(t,z) - 2i$ for any $z \in \mathbb{C}\backslash\{\text{poles}\}$.

The power series expansion of $\mathbf{H}(t,\cdot)$ near 0 is

$$\mathbf{H}(t,z)=\frac{2}{z}+\mathbf{r}(t)z+O(z^3), \tag{2.2}$$

where $\mathbf{r}(t)=\sum_{k=1}^{\infty}\sinh^{-2}(kt)-\frac{1}{6}$. As $t\to\infty$, $\mathbf{S}(t,z)\to\frac{1+z}{1-z}$, $\mathbf{H}(t,z)\to\cot_2(z)$, and $\mathbf{r}(t)\to-\frac{1}{6}$. So we define $\mathbf{S}(\infty,z)=\frac{1+z}{1-z}$, $\mathbf{H}(\infty,z)=\cot_2(z)$, and $\mathbf{r}(\infty)=-\frac{1}{6}$. Then $\mathbf{r}$ is continuous on $(0,\infty]$, and (2.2) still holds when $t=\infty$. In fact, we have $\mathbf{r}(t)-\mathbf{r}(\infty)=O(e^{-t})$ as $t\to\infty$, so we may define $\mathbf{R}$ on $(0,\infty]$ by $\mathbf{R}(t)=-\int_t^{\infty}(\mathbf{r}(s)-\mathbf{r}(\infty))ds$. Then $\mathbf{R}$ is continuous on $(0,\infty]$, $\mathbf{R}(t)=O(e^{-t})$ as $t\to\infty$, and for $0<t<\infty$,

$$\mathbf{R}'(t)=\mathbf{r}(t)-\mathbf{r}(\infty). \tag{2.3}$$

Let $\mathbf{S}_I(t,z)=\mathbf{S}(t,e^{-t}z)-1$ and $\mathbf{H}_I(t,z)=-i\mathbf{S}_I(t,e^{iz})=\mathbf{H}(t,z+it)+i$. It is easy to check:

$$\mathbf{S}_I(t,z)=\mathrm{P.V.}\sum_{2\nmid n}\frac{e^{nt}+z}{e^{nt}-z},\quad \mathbf{H}_I(t,z)=\mathrm{P.V.}\sum_{2\nmid n}\cot_2(z-int). \tag{2.4}$$

So $\mathbf{H}_I(t,\cdot)$ is a meromorphic function in $\mathbb{C}$ with poles $\{2m\pi+i(2k+1)t:m,k\in\mathbb{Z}\}$, which are all simple poles with residue 2; $\mathbf{H}_I(t,\cdot)$ is an odd function and takes real values on $\mathbb{R}$; and $\mathbf{H}_I(t,z+2\pi)=\mathbf{H}_I(t,z)$, $\mathbf{H}_I(t,z+i2t)=\mathbf{H}_I(t,z)-2i$ for any $z\in\mathbb{C}\backslash\{\text{poles}\}$.

It is possible to express $\mathbf{H}$ and $\mathbf{H}_I$ using classical functions. Let $\theta(\nu,\tau)$ and $\theta_k(\nu,\tau)$, $k=1,2,3$, be the Jacobi theta functions defined in Chapter V, Section 3 of [24]. Define $\Theta(t,z)=\theta\left(\frac{z}{2\pi},\frac{it}{\pi}\right)$ and $\Theta_I(t,z)=\theta_2\left(\frac{z}{2\pi},\frac{it}{\pi}\right)$. Then Θ_I has period 2π, Θ has antiperiod 2π, and

$$\mathbf{H}=2\,\frac{\Theta'}{\Theta},\quad \mathbf{H}_I=2\,\frac{\Theta_I'}{\Theta_I}. \tag{2.5}$$

These follow from the product representations of Θ and Θ_I. For example,

$$\Theta_I(t,z)=\prod_{m=1}^{\infty}(1-e^{-2mt})\left(1-e^{-(2m-1)t}e^{iz}\right)\left(1-e^{-(2m-1)t}e^{-iz}\right). \tag{2.6}$$

Both Θ and Θ_I solve the heat equation

$$\partial_t\Theta=\Theta'',\quad \partial_t\Theta_I=\Theta_I''. \tag{2.7}$$

So $\mathbf{H}$ and $\mathbf{H}_I$ solve the PDE:

$$\partial_t\mathbf{H}=\mathbf{H}''+\mathbf{H}'\mathbf{H},\quad \partial_t\mathbf{H}_I=\mathbf{H}_I''+\mathbf{H}_I'\mathbf{H}_I. \tag{2.8}$$

We rescale the functions $\mathbf{H}$ and $\mathbf{H}_I$ as follows. For $t>0$ and $z\in\mathbb{C}$, let

$$\widehat{\mathbf{H}}(t,z)=\frac{\pi}{t}\mathbf{H}\left(\frac{\pi^2}{t},\frac{\pi}{t}z\right)+\frac{z}{t},\quad \widehat{\mathbf{H}}_I(t,z)=\frac{\pi}{t}\mathbf{H}_I\left(\frac{\pi^2}{t},\frac{\pi}{t}z\right)+\frac{z}{t}. \tag{2.9}$$

Since $\widehat{\mathbf{H}}$ and $\widehat{\mathbf{H}}_I$ have period 2π,

$$\widehat{\mathbf{H}}(t,z+2kt)=\widehat{\mathbf{H}}(t,z)+2k,\quad \widehat{\mathbf{H}}_I(t,z+2kt)=\widehat{\mathbf{H}}_I(t,z)+2k,\quad k\in\mathbb{Z}. \tag{2.10}$$

From the identities for θ in [24] or formula (3) in [25], we see $\mathbf{H}(t,z)=i\frac{\pi}{t}\mathbf{H}\left(\frac{\pi^2}{t},i\frac{\pi}{t}z\right)-\frac{z}{t}$. So

$$\widehat{\mathbf{H}}(t,z)=-i\mathbf{H}(t,-iz)=\text{P. V.}\sum_{2|n}\coth_2(z-nt). \tag{2.11}$$

Since $\mathbf{H}_I(t,z)=\mathbf{H}(t,z+it)+i$,

$$\widehat{\mathbf{H}}_I(t,z)=\widehat{\mathbf{H}}(t,z+\pi i)=\text{P. V.}\sum_{2|n}\tanh_2(z-nt). \tag{2.12}$$

From (2.8) and (2.9) we may check that

$$-\partial_t\widehat{\mathbf{H}}=\widehat{\mathbf{H}}''+\widehat{\mathbf{H}}'\widehat{\mathbf{H}},\quad -\partial_t\widehat{\mathbf{H}}_I=\widehat{\mathbf{H}}_I''+\widehat{\mathbf{H}}_I'\mathbf{H}_I. \tag{2.13}$$

From (2.11) and (2.12) we see that $\widehat{\mathbf{H}}(t,\cdot)\to\coth_2$ and $\widehat{\mathbf{H}}_I(t,\cdot)\to\tanh_2$ as $t\to\infty$.
From (2.4) we see that as $t\to\infty$, $\mathbf{H}_I(t,z)\to 0$, so its derivatives about z also tend to 0. The following lemma gives some estimations of these limits.

Lemma 2.1 *If* $|\operatorname{Im} z|<t$, *then*

$$|\mathbf{H}_I(t,z)|\le\frac{4e^{|\operatorname{Im} z|-t}}{(1-e^{|\operatorname{Im} z|-t})^2(1-e^{2(|\operatorname{Im} z|-t)})}. \tag{2.14}$$

If $t\ge|\operatorname{Im} z|+2$, *then* $|\mathbf{H}_I(t,z)|<5.5e^{|\operatorname{Im} z|-t}$. *For any* $h\in\mathbb{N}$, *if* $t\ge|\operatorname{Im} z|+h+2$, *then* $|\mathbf{H}_I^{(h)}(t,z)|<15\sqrt{h}e^{|\operatorname{Im} z|-t}$.

Proof From (2.4), if $|\operatorname{Im} z|<t$, then

$$\begin{aligned}|\mathbf{H}_I(t,z)|&=\left|\sum_{k=0}^{\infty}\left(\frac{e^{(2k+1)t}+e^{iz}}{e^{(2k+1)t}-e^{iz}}+\frac{e^{-(2k+1)t}+e^{iz}}{e^{-(2k+1)t}-e^{iz}}\right)\right|\\&=\left|\sum_{k=0}^{\infty}\frac{2\sin(z)}{\cosh((2k+1)t)-\cos(z)}\right|\\&\le\sum_{k=0}^{\infty}\frac{2e^{|\operatorname{Im} z|}}{\cosh((2k+1)t)-\cosh(|\operatorname{Im} z|)}.\end{aligned} \tag{2.15}$$

Here we use the facts that $|\sin(z)|\le e^{|\operatorname{Im} z|}$ and $|\cos(z)|\le\cosh(|\operatorname{Im} z|)<\cosh(t)$. Let $h_0=t-|\operatorname{Im} z|>0$. Then for $k\ge 0$,

$$\begin{aligned}&\cosh((2k+1)t)-\cosh(|\operatorname{Im} z|)\\&\quad=2\sinh_2((2k+1)t+|\operatorname{Im} z|)\sinh_2((2k+1)t-|\operatorname{Im} z|)\end{aligned}$$

$$= \frac{1}{2}e^{((2k+1)t+|\operatorname{Im} z|)/2}(1-e^{-(2k+1)t-|\operatorname{Im} z|})e^{((2k+1)t-|\operatorname{Im} z|)/2}(1-e^{-(2k+1)t+|\operatorname{Im} z|})$$
$$\geq \frac{1}{2}e^{((2k+1)t+|\operatorname{Im} z|)/2}e^{((2k+1)t-|\operatorname{Im} z|)/2}\left(1-e^{-h_0}\right)^2 = \frac{1}{2}e^{(2k+1)t}\left(1-e^{-h_0}\right)^2.$$

So the RHS of (2.15) is not bigger than

$$\sum_{k=0}^{\infty}\frac{4e^{|\operatorname{Im} z|}e^{-(2k+1)t}}{(1-e^{-h_0})^2} = \frac{4e^{|\operatorname{Im} z|-t}}{(1-e^{-h_0})^2(1-e^{-2t})} \leq \frac{4e^{-h_0}}{(1-e^{-h_0})^2(1-e^{-2h_0})}.$$

So we proved (2.14).

If $t \geq |\operatorname{Im} z|+2$, then $4/((1-e^{|\operatorname{Im} z|-t})^2(1-e^{2(|\operatorname{Im} z|-t)})) \leq 4/((1-e^{-2})^2(1-e^{-4})) < 5.5$. From (2.14) we have $|\mathbf{H}_I(t,z)| < 5.5e^{|\operatorname{Im} z|-t}$. Now we assume $h \in \mathbb{N}$ and $t \geq |\operatorname{Im} z|+h+2$. Then for any $w \in \mathbb{C}$ with $|w-z| = h$, we have $t \geq |\operatorname{Im} w|+2$, so $|\mathbf{H}_I(t,w)| < 5.5e^{|\operatorname{Im} w|-t} \leq 5.5e^h e^{|\operatorname{Im} z|-t}$. From Cauchy's integral formula and Stirling's formula, we have

$$|\mathbf{H}_I^{(h)}(t,z)| \leq 5.5\frac{h!e^h}{h^h}e^{|\operatorname{Im} z|-t} \leq 5.5\sqrt{2\pi h}e^{1/(12h)}e^{|\operatorname{Im} z|-t} < 15\sqrt{h}e^{|\operatorname{Im} z|-t}.$$

□

3 Loewner equations

3.1 Whole-plane Loewner equation

To motivate the definition of the whole-plane Loewner equation, let's start with the well-known radial Leowner equation with reflection about the unit circle $\mathbb{T}$. Let $T \in (0,\infty]$. Let $\beta_I : [0,T) \to \mathbb{C}$ be a simple curve with $\beta_I(0) \in \mathbb{T}$ and $\beta_I(t) \in \mathbb{C}\backslash\overline{\mathbb{D}}$ for $t \in (0,T)$. Let $K_I(t) = \overline{\mathbb{D}} \cup \beta_I((0,t])$, $0 \leq t < T$. Then each $K_I(t)$ is a hull in $\mathbb{C}$, and the capacity increases continuously in t. After a time-change, we may assume that $\operatorname{cap}(K_I(t)) = e^t$, $0 \leq t < T$. Let $g_I(t,\cdot)$ be the unique conformal map from $\mathbb{C}\backslash K_I(t)$ conformally onto $\mathbb{C}\backslash\overline{\mathbb{D}}$ with normalization $\lim_{z\to\infty} z/g_I(t,z) = e^t$. It turns out that there is $\xi \in C([0,T))$ such that $g_I(t,\cdot)$ satisfies the radial Loewner equation

$$\partial_t g_I(t,z) = g_I(t,z)\frac{e^{i\xi(t)}+g_I(t,z)}{e^{i\xi(t)}-g_I(t,z)} \tag{3.1}$$

with initial value $g_I(0,z) = z$. In fact, each $g_I(t,\cdot)^{-1}$ extends continuously to $\mathbb{T}$, and maps $e^{i\xi(t)}$ to $\beta_I(t)$, and the function ξ is determined by β_I up to an integer multiple of 2π.

Let $a \in \mathbb{R}$ and $T \in (a,\infty]$. Now suppose a simple curve $\beta_I : [a,T) \to \mathbb{C}$ satisfies $\beta_I(0) \in e^a\mathbb{T}$ and $\beta_I(t) \in \mathbb{C}\backslash e^a\overline{\mathbb{D}}$ for $t \in (a,T)$. Let $K_I(t) = e^a\overline{\mathbb{D}} \cup \beta_I((a,t])$, $a \leq t < T$. Assume that $\operatorname{cap}(K_I(t)) = e^t$, $a \leq t < T$. Then the mappings $g_I(t,\cdot)$ determined by $K_I(t)$ also satisfy (3.1) for some $\xi \in C([a,T))$, and the initial value now is $g_I(0,z) = e^{-a}z$. Let a tend to $-\infty$, then the initial point of β_I approaches 0.

So let's consider a simple curve $\beta_I : [-\infty, T) \to \mathbb{C}$ with $\beta_I(-\infty) = 0$. Let $K_I(t) = \beta_I([-\infty, t])$, $-\infty < t < T$. Assume that $\text{cap}(K_I(t)) = e^t$, $-\infty < t < T$. Then the mappings $g_I(t, \cdot)$ determined by $K_I(t)$ still satisfy (3.1) for some $\xi \in C((-\infty, T))$, and they have an asymptotic initial value at $t = -\infty$:

$$\lim_{t\to-\infty} e^t g_I(t, z) = z, \quad z \in \mathbb{C}\backslash\{0\}. \tag{3.2}$$

For this reason, we also call (3.1) the whole-plane Loewner equation.

We now reverse the above process. Let $T \in (-\infty, \infty]$ and $\xi \in C((-\infty, T))$. Let $g_I(t, \cdot)$, $-\infty < t < T$, be the solution of the whole-plane Loewner equation (3.1) with the asymptotic initial value (3.2). Note that for each fixed z, (3.1) is an ODE in t. For each $t \in (-\infty, T)$, let $K_I(t)$ be the set of $z \in \mathbb{C}$ at which $g_I(t, \cdot)$ is not defined. Then $K_I(t)$ and $g_I(t, \cdot)$, $-\infty < t < T$, are called the whole-plane Loewner hulls and maps driven by ξ.

Remark Since the asymptotic initial value is used, the existence and uniqueness of the solution is not trivial. From Proposition 4.21 in [8] we know that $K_I(t)$ and $g_I(t, \cdot)$ exist and are determined by $e^{i\xi(s)}$, $-\infty < s \le t$. Moreover, each $g_I(t, \cdot)$ maps $\widehat{\mathbb{C}}\backslash K_I(t)$ conformally onto $\widehat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ and fixes ∞, and $g_I(t, z) = e^{-t}z + O(1)$ near ∞. So each $K_I(t)$ is a hull in $\mathbb{C}$ with $\text{cap}(K_I(t)) = e^t$. The whole-plane Loewner equation can be viewed as a mapping which takes the driving function ξ to a family of hulls $(K_I(t))$ or conformal maps $(g_I(t, \cdot))$. The family $(K_I(t))$ increases in t, but may not be simple curves.

We say that ξ generates a whole-plane Loewner trace β_I if

$$\beta_I(t) := \lim_{|z|>1, z\to e^{i\xi(t)}} g_I(t, \cdot)^{-1}(z)$$

exists for $t \in (-\infty, T)$, and if $\beta_I(t)$, $-\infty \le t < T$, is a continuous curve in $\mathbb{C}$. Such a trace, if it exists, starts from 0, i.e., $\beta_I(-\infty) := \lim_{t\to-\infty} \beta_I(t) = 0$. The trace is called simple if $\beta_I(t)$, $-\infty \le t < T$, has no self-intersection. If ξ generates a whole-plane Loewner trace β_I, then for each t, $\mathbb{C}\backslash K_I(t)$ is the unbounded component of $\mathbb{C}\backslash\beta_I([-\infty, t])$. In particular, if β_I is simple, then $K_I(t) = \beta_I([-\infty, t])$ for each t, and we recover an earlier picture.

Let $\kappa > 0$. A pre-$(\mathbb{T}; \kappa)$-Brownian motion a.s. generates a whole-plane Loewner trace, which is called a standard whole-plane SLE_κ trace. The trace goes from 0 to ∞, i.e., $\lim_{t\to\infty} \beta_I(t) = \infty$. If $\kappa \in (0, 4]$, the trace is simple. If the driving function is the sum of a pre-$(\mathbb{T}; \kappa)$-Brownian motion and $s_0 t$ for some constant $s_0 \in \mathbb{R}$, then we also get a whole-plane Loewner trace, which is called a standard whole-plane $\text{SLE}(\kappa, s_0)$ trace. The trace still goes from 0 to ∞ as $t \to \infty$, and is simple when $\kappa \le 4$. For any $z_1 \ne z_2 \in \widehat{\mathbb{C}}$, we may define whole-plane SLE_κ and $\text{SLE}(\kappa, s_0)$ trace from z_1 to z_2 via Möbius transform.

Remark Whole-plane SLE_κ is related to radial SLE in the way that, if $T \in \mathbb{R}$ is fixed, then conditioned on $K_I(t)$, $-\infty < t \le T$, the curve $\beta_I(T + t)$, $t \ge 0$, is the radial SLE_κ trace in $\widehat{\mathbb{C}}\backslash K_I(T)$ from $\beta_I(T)$ to ∞. Whole-plane $\text{SLE}(\kappa, s_0)$ is related to radial

SLE$(\kappa, -s_0)$ (the radial Loewner process driven by $\sqrt{\kappa}B(t) - s_0 t$) in a similar way. The additional negative sign is due to the reflection about $\mathbb{T}$.

We will need the following inverted whole-plane Loewner process, which grows from ∞. For $-\infty < t < T$, let $K(t) = I_0(K_I(t))$ and $g(t,\cdot) = I_0 \circ g_I(t,\cdot) \circ I_0$. Then for each t, $g(t,\cdot)$ maps $\widehat{\mathbb{C}} \backslash K(t)$ conformally onto $\mathbb{D}$ and fixes 0. Moreover, $g(t,\cdot)$ satisfies (3.1) with some initial value at $-\infty$. We call $K(t)$ and $g(t,\cdot)$ the inverted whole-plane Loewner hulls and maps driven by ξ. If ξ generates a whole-plane Loewner trace β_I, then $\beta(t) := I_0 \circ \beta_I(t)$ is a continuous curve in $\widehat{\mathbb{C}}$ that satisfies $\beta(-\infty) = \infty$ and $\beta(t) = \lim_{\mathbb{D} \ni z \to e^{i\xi(t)}} g(t,\cdot)^{-1}(z)$, $-\infty < t < T$. We call β the inverted whole-plane Loewner trace driven by ξ.

Let $K_I(t)$ and $g_I(t,\cdot)$, $-\infty < t < T$, be as before. Let $\widetilde{K}_I(t) = (e^i)^{-1}(K_I(t))$, $-\infty < t < T$. It is easy to see that there exists a unique family $\widetilde{g}_I(t,\cdot)$, $-\infty < t < T$, such that, $\widetilde{g}_I(t,\cdot)$ maps $\mathbb{C} \backslash \widetilde{K}_I(t)$ conformally onto $-\mathbb{H}$, $e^i \circ \widetilde{g}_I(t,\cdot) = g_I(t,\cdot) \circ e^i$, and $\widetilde{g}_I$ satisfies

$$\partial_t \widetilde{g}_I(t,z) = \cot_2(\widetilde{g}_I(t,z) - \xi(t)), \quad -\infty < t < T, \tag{3.3}$$

and the initial value at $-\infty$:

$$\lim_{t \to -\infty} (\widetilde{g}_I(t,z) - it) = z.$$

Then we call $\widetilde{K}_I(t)$ and $\widetilde{g}_I(t,\cdot)$ the covering whole-plane Loewner hulls and maps driven by ξ.

For $-\infty < t < T$, let $\widetilde{K}(t) = \widetilde{I}_0(\widetilde{K}_I(t))$ and $\widetilde{g}(t,\cdot) = \widetilde{I}_0 \circ \widetilde{g}_I(t,\cdot) \circ \widetilde{I}_0$. Then $\widetilde{K}(t) = (e^i)^{-1}(K(t))$ and $e^i \circ \widetilde{g}(t,\cdot) = g(t,\cdot) \circ e^i$. We call $\widetilde{K}(t)$ and $\widetilde{g}(t,\cdot)$ the inverted covering whole-plane Loewner hulls and maps driven by ξ. Then for each $t \in (-\infty, T)$, $\widetilde{g}(t,\cdot)$ maps $\mathbb{C} \backslash \widetilde{K}(t)$ conformally onto $\mathbb{H}$, and satisfies (3.3) for $t \in (-\infty, T)$ and the initial value at $-\infty$:

$$\lim_{t \to -\infty} (\widetilde{g}(t,z) + it) = z. \tag{3.4}$$

3.2 Annulus Loewner equation

The annulus Loewner equation was introduced in [17] to describe curves in a doubly connected domain. Let $p \in (0,\infty)$. To motivate the definition, we consider a simple curve $\beta(t)$, $0 \le t < T$, with $\beta(0) \in \mathbb{T}$ and $\beta(t) \in \mathbb{A}_p$, $0 < t < T$. Let $K(t) = \beta((0,t])$, $0 \le t < T$. Then each $K(t)$ is a hull in $\mathbb{A}_p$, and $\operatorname{cap}_{\mathbb{A}_p}(K(t))$ increases continuously. After a time-change, we may assume that $\operatorname{cap}_{\mathbb{A}_p}(K(t)) = t$ for all t. For each t, there exists $g(t,\cdot)$, which maps $\mathbb{A}_p \backslash K(t)$ conformally onto $\mathbb{A}_{p-t}$, and maps $\mathbb{T}_p$ onto $\mathbb{T}_{p-t}$. Such $g(t,\cdot)$ is unique only up to a rotation. There are different ways to make $g(t,\cdot)$ unique. For example, we may fix a point on $w_0 \in \mathbb{T}_p$ and require that $e^{-t} g(t,\cdot)$ fixes w_0. The normalization used here does not have a clear geometric meaning. The work in [17] shows that the $g(t,\cdot)$ can be chosen to satisfy annulus Loewner equation of modulus p for some $\xi \in C([0,T))$:

$$\partial_t g(t,z) = g(t,z)\mathbf{S}(p-t, g(t,z)/e^{i\xi(t)}), \quad 0 \le t < T, \quad g(0,z) = z, \tag{3.5}$$

We now reverse the above process. Let $\xi \in C([0,T))$ where $0 < T \le p$. Let $g(t,\cdot)$ be the solution of the ODE (3.5). For $0 \le t < T$, let $K(t)$ denote the set of $z \in \mathbb{A}_p$ such that the solution $g(s,z)$ blows up before or at time t. We call $K(t)$ and $g(t,\cdot)$, $0 \le t < T$, the annulus Loewner hulls and maps of modulus p driven by ξ. It turns out that, fo each t, $K(t)$ is a hull in $\mathbb{A}_p$ with $\mathrm{cap}_{\mathbb{A}_p}(K(t)) = t$, and $g(t,\cdot)$ maps $\mathbb{A}_p \backslash K(t)$ conformally onto $\mathbb{A}_{p-t}$ and maps $\mathbb{T}_p$ onto $\mathbb{T}_{p-t}$. To see that $g(t,\cdot)$ maps $\mathbb{T}_p$ onto $\mathbb{T}_{p-t}$, one may note that (3.5) implies that $\partial_t \ln|g(t,z)| = \mathrm{Re}\,\mathbf{S}(p-t, g(t,z)/e^{i\xi(t)})$, and $\mathrm{Re}\,\mathbf{S}(r,\cdot) \equiv 1$ on $\mathbb{T}_r$ because $\mathrm{Im}\,\mathbf{H}(r,\cdot) \equiv -1$ on $\mathbb{R}_r$ and $\mathbf{H}(r,z) = -i\mathbf{S}(t, e^i(z))$.

We say that ξ generates an annulus Loewner trace β of modulus p if

$$\beta(t) := \lim_{\mathbb{A}_{p-t} \ni z \to e^{i\xi(t)}} g(t,\cdot)^{-1}(z) \tag{3.6}$$

exists for all $t \in [0,T)$, and if $\beta(t)$, $0 \le t < T$, is a continuous curve. The curve lies in $\mathbb{A}_p \cup \mathbb{T}$ and starts from $e^{i\xi(0)} \in \mathbb{T}$. The trace is called simple if β has no self-intersection and stays inside $\mathbb{A}_p$ for $t > 0$. In that case, we have $K(t) = \beta((0,t])$ for each t, and recover the picture at the beginning of this subsection.

Remark 1. If ξ generates an annulus Loewner trace β, then for each t, $\mathbb{A}_p \backslash K(t)$ is the component of $\mathbb{A}_p \backslash \beta((0,t])$ whose boundary contains $\mathbb{T}_p$. If the trace is simple, then $K(t) = \beta((0,t])$ for each t.
2. Let $\beta(t)$, $0 \le t < T$, be a simple curve with $\beta(0) \in \mathbb{T}$ and $\beta(t) \in \mathbb{A}_p$ for $0 < t < T$. If it is parameterized by capacity in $\mathbb{A}_p$, i.e., $\mathrm{cap}_{\mathbb{A}_p}(\beta((0,t])) = t$ for each t, then it is an annulus Loewner trace of modulus p. In the general case, let $u(t) = \mathrm{cap}_{\mathbb{A}_p}(\beta((0,t]))$. Then $\beta(u^{-1}(t))$ is an annulus Loewner trace of modulus p.
3. If $\xi(t) = \sqrt{\kappa}B(t)$, $0 \le t < p$, then a.s. ξ generates an annulus Loewner trace. If $\kappa \in (0,4]$, the trace is simple. From Girsanov theorem, the above still hold if ξ is a semimartingale, whose stochastic part is $\sqrt{\kappa}B(t)$, and whose drift part is a continuously differentiable function.

The covering annulus Loewner equation of modulus p driven by the above ξ is

$$\partial_t \widetilde{g}(t,z) = \mathbf{H}(p-t, \widetilde{g}(t,z) - \xi(t)), \quad \widetilde{g}(0,z) = z. \tag{3.7}$$

For $0 \le t < T$, let $\widetilde{K}(t)$ denote the set of $z \in \mathbb{S}_p$ such that the solution $\widetilde{g}(s,z)$ blows up before or at time t. Then for $0 \le t < T$, $\widetilde{g}(t,\cdot)$ maps $\mathbb{S}_p \backslash \widetilde{K}(t)$ conformally onto $\mathbb{S}_{p-t}$ and maps $\mathbb{R}_p$ onto $\mathbb{R}_{p-t}$. We call $\widetilde{K}(t)$ and $\widetilde{g}(t,\cdot)$, $0 \le t < T$, the covering annulus Loewner hulls and maps of modulus p driven by ξ. Let $K(t)$ and $g(t,\cdot)$ be the notations appeared above. Then we have $\widetilde{K}(t) = (e^i)^{-1}(K(t))$ and $e^i \circ \widetilde{g}(t,\cdot) = g(t,\cdot) \circ e^i$ for $0 \le t < T$.

Since $\widetilde{g}(t,\cdot)$ maps $\mathbb{R}_p$ onto $\mathbb{R}_{p-t}$ and $\mathbf{H}_I(t,z) = \mathbf{H}(t, z+it) + i$, we have

$$\partial_t\, \mathrm{Re}\,\widetilde{g}(t,z) = \mathbf{H}_I(p-t, \mathrm{Re}\,\widetilde{g}(t,z) - \xi(t)), \quad z \in \mathbb{R}_p.$$

Differentiating the above formula w.r.t. z, we obtain

$$\partial_t \widetilde{g}'(t,z) = \widetilde{g}'(t,z)\mathbf{H}_I'(p-t, \operatorname{Re}\widetilde{g}(t,z) - \xi(t)), \quad z \in \mathbb{R}_p. \tag{3.8}$$

If ξ generates an annulus Loewner trace β of modulus p, then a.s.

$$\widetilde{\beta}(t) := \lim_{\mathbb{S}_{p-t} \ni z \to \xi(t)} \widetilde{g}(t,\cdot)^{-1}(z)$$

exists for $0 \le t < T$, and $\widetilde{\beta}(t)$, $0 \le t < T$, is a continuous curve in $\mathbb{S}_p \cup \mathbb{R}$ started from $\widetilde{\beta}(0) = \xi(0) \in \mathbb{R}$. Such $\widetilde{\beta}$ is called the covering annulus Loewner trace of modulus p driven by ξ. And we have $\beta = e^i \circ \widetilde{\beta}$. If β is a simple trace, then $\widetilde{\beta}$ has no self-intersection, stays inside $\mathbb{S}_p$ for $t > 0$, and $\widetilde{K}(t) = \widetilde{\beta}((0,t]) + 2\pi\mathbb{Z}$ for each t.

Let $K_I(t) = I_p(K(t))$, $g_I(t,\cdot) = I_{p-t} \circ g(t,\cdot) \circ I_p$, $\widetilde{K}_I(t) = \widetilde{I}_p(\widetilde{K}(t))$, and $\widetilde{g}_I(t,\cdot) = \widetilde{I}_{p-t} \circ \widetilde{g}(t,\cdot) \circ \widetilde{I}_p$. Then $K_I(t)$ is a hull in $\mathbb{A}_p$ with $\operatorname{cap}_{\mathbb{A}_p}(K_I(t)) = t$, and $g_I(t,\cdot)$ maps $\mathbb{A}_p \backslash K_I(t)$ conformally onto $\mathbb{A}_{p-t}$ and maps $\mathbb{T}$ onto $\mathbb{T}$. Moreover, $\widetilde{K}_I(t) = (e^i)^{-1}(K_I(t))$, $\widetilde{g}_I(t,\cdot)$ maps $\mathbb{S}_p \backslash \widetilde{K}_I(t)$ conformally onto $\mathbb{S}_{p-t}$, maps $\mathbb{R}$ onto $\mathbb{R}$, satisfies $e^i \circ \widetilde{g}_I(t,\cdot) = g_I(t,\cdot) \circ e^i$, and the equation

$$\partial_t \widetilde{g}_I(t,z) = \mathbf{H}_I(p-t, \widetilde{g}_I(t,z) - \xi(t)), \quad \widetilde{g}(0,z) = z. \tag{3.9}$$

We call $K_I(t)$ and $g_I(t,\cdot)$ (resp. $\widetilde{K}_I(t)$ and $\widetilde{g}_I(t,\cdot)$) the inverted annulus (resp. inverted covering annulus) Loewner hulls and maps of modulus p driven by ξ. The inverted hulls grow from the smaller circle $\mathbb{T}_p$ instead of the unit circle $\mathbb{T}$.

3.3 Disc Loewner equation

We now review the definition of the disc Loewner equation, which is used to describe a simple curve in a simply connected domain started from an interior point. The relation between the disc Loewner equation and the annulus Loewner equation is similar to that between the whole-plane Loewner equation and the radial Loewner equation. Intuitively, one considers the inverted annulus Loewner equations of modulus p so that the hulls start from $\mathbb{T}_p$, and then lets $p \to \infty$.

Let $T \in (-\infty, 0]$ and $\xi \in C((-\infty, T))$. Let $g_I(t,\cdot)$, $-\infty < t < T$, be the solution of

$$\partial_t g_I(t,z) = g_I(t,z)\mathbf{S}_I\left(-t, g_I(t,z)/e^{i\xi(t)}\right), \quad -\infty < t < T;$$
$$\lim_{t\to-\infty} g_I(t,z) = z, \quad \forall z \in \overline{\mathbb{D}}\backslash\{0\}. \tag{3.10}$$

For each $t \in (-\infty, T)$, let $K_I(t)$ be the set of $z \in \mathbb{D}$ at which $g_I(t,\cdot)$ is not defined. Then $K_I(t)$ and $g_I(t,\cdot)$, $-\infty < t < T$, are called the disc Loewner hulls and maps driven by ξ.

Remark From Proposition 4.1 and 4.2 in [17] we know that $K_I(t)$ and $g_I(t,\cdot)$ exist and are determined by $e^{i\xi(s)}$, $-\infty < s \le t$. Moreover, each $g_I(t,\cdot)$ maps $\mathbb{D}\backslash K_I(t)$ conformally onto $\mathbb{A}_{-t}$ and maps $\mathbb{T}$ onto $\mathbb{T}$.

We say that ξ generates a disc Loewner trace β if

$$\beta_I(t) := \lim_{\mathbb{A}_{-t}\ni z\to e^{t+i\xi(t)}} g_I(t,\cdot)^{-1}(z)$$

exists for every $t \in (-\infty, T)$, and if $\beta_I(t)$, $-\infty \le t < T$, is a continuous curve in $\mathbb{D}$ with $\beta_I(-\infty) = 0$. The trace is called simple if it has no self-intersection. If ξ generates a disc Loewner trace β_I, then for each t, $\mathbb{C}\setminus K_I(t)$ is the unbounded component of $\mathbb{C}\setminus\beta_I([-\infty, t])$. In particular, if β_I is simple, then $K_I(t) = \beta_I([-\infty, t])$ for each t.

Let $\beta_I(t)$, $-\infty \le t < T$, be a simple curve in $\mathbb{D}$ with $\beta_I(-\infty) = 0$. If it is parameterized by capacity in $\mathbb{D}$, i.e., $\operatorname{mod}(\mathbb{D}\setminus\beta_I([-\infty, t])) = -t$ for each t, then β_I is a disc Loewner trace. In the general case, let $u(t) = -\operatorname{mod}(\mathbb{D}\setminus\beta_I([-\infty, t]))$, then $\beta_I(u^{-1}(t))$ is a disc Loewner trace.

We will need the following inverted disc Loewner process, which grows from ∞. For $-\infty < t < T$, let $K(t) = I_0(K_I(t))$ and $g(t,\cdot) = I_{-t} \circ g(t,\cdot) \circ I_0$. Then each $g(t,\cdot)$ maps $\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}\setminus K(t)$ conformally onto $\mathbb{A}_{-t}$ and maps $\mathbb{T}$ onto $\mathbb{T}_{-t}$. Moreover, $g(t,\cdot)$ satisfies (3.10) with $\mathbf{S}_I$ replaced by $\mathbf{S}$. We call $K(t)$ and $g(t,\cdot)$, $-\infty < t < T$, the inverted disc Loewner hulls and maps driven by ξ. If ξ generates a disc Loewner trace β_I, then $\beta := I_0 \circ \beta_I$ is called the inverted disc Loewner trace driven by ξ.

The covering disc Loewner hulls and maps are defined as follows. Let $\widetilde{K}_I(t) = (e^i)^{-1}(K_I(t))$, $-\infty < t < T$. There is a unique family $\widetilde{g}_I(t,\cdot)$, $-\infty < t < T$, which satisfy that, for each t, $\widetilde{g}_I(t,\cdot)$ maps $\mathbb{H}\setminus\widetilde{K}_I(t)$ conformally onto $\mathbb{S}_{-t}$ and maps $\mathbb{R}$ onto $\mathbb{R}$, $e^i \circ \widetilde{g}_I(t,\cdot) = g_I(t,\cdot) \circ e^i$, and the following hold:

$$\partial_t \widetilde{g}_I(t,z) = \mathbf{H}_I(-t, \widetilde{g}_I(t,z) - \xi(t)); \tag{3.11}$$

$$\lim_{t\to-\infty} \widetilde{g}_I(t,z) = z. \tag{3.12}$$

We call $\widetilde{K}_I(t)$ and $\widetilde{g}_I(t,\cdot)$ the covering disc Loewner hulls and maps driven by ξ. Let $\widetilde{K}(t) = \widetilde{I}_0(\widetilde{K}_I(t))$ and $\widetilde{g}(t,\cdot) = \widetilde{I}_{-t} \circ \widetilde{g}_I(t,\cdot) \circ \widetilde{I}_0$. Then $\widetilde{g}(t,\cdot)$ maps $-\mathbb{H}\setminus\widetilde{K}(t)$ conformally onto $\mathbb{S}_{-t}$, maps $\mathbb{R}$ onto $\mathbb{R}_{-t}$, $e^i \circ \widetilde{g}(t,\cdot) = g(t,\cdot) \circ e^i$, and satisfies $\partial_t \widetilde{g}(t,z) = \mathbf{H}(-t, \widetilde{g}(t,z) - \xi(t))$. We call $\widetilde{K}(t)$ and $\widetilde{g}(t,\cdot)$ the inverted covering disc Loewner hulls and maps driven by ξ.

Remark Now we summarize the conformal maps that appear in the this section so far. The relations between a (inverted) whole-plane, annulus, or disc Loewner map $g(t,\cdot)$ or $g_I(t,\cdot)$ and its corresponding covering map $\widetilde{g}(t,\cdot)$ or $\widetilde{g}_I(t,\cdot)$ are $g(t,\cdot) \circ e^i = e^i \circ \widetilde{g}(t,\cdot)$ and $g_I(t,\cdot) \circ e^i = e^i \circ \widetilde{g}_I(t,\cdot)$. The relation between the inverted pair $\widetilde{g}(t,\cdot)$ and $\widetilde{g}_I(t,\cdot)$ depends on the three cases. For the whole-plane Loewner maps,

$$\widetilde{g}_I(t,\cdot) : \mathbb{C}\setminus\widetilde{K}_I(t) \overset{\text{Conf}}{\twoheadrightarrow} -\mathbb{H}, \quad \widetilde{g}(t,\cdot) : \mathbb{C}\setminus\widetilde{K}(t) \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}, \quad \widetilde{I}_0 \circ \widetilde{g}(t,\cdot) = \widetilde{g}_I(t,\cdot) \circ \widetilde{I}_0.$$

For the annulus Loewner maps of modulus p,

$$\widetilde{g}(t,\cdot) : (\mathbb{S}_p\setminus\widetilde{K}(t); \mathbb{R}_p) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{S}_{p-t}; \mathbb{R}_{p-t}), \quad \widetilde{g}_I(t,\cdot) : (\mathbb{S}_p\setminus\widetilde{K}_I(t); \mathbb{R}) \overset{\text{Conf}}{\twoheadrightarrow} (\mathbb{S}_{p-t}; \mathbb{R}),$$
$$\widetilde{I}_{p-t} \circ \widetilde{g}_I(t,\cdot) = \widetilde{g}(t,\cdot) \circ \widetilde{I}_p, \quad t \in [0, p).$$

For the disc Loewner maps,

$$\widetilde{g}_I(t,\cdot):(\mathbb{H}\backslash\widetilde{K}_I(t);\mathbb{R})\stackrel{\text{Conf}}{\twoheadrightarrow}(\mathbb{S}_{-t};\mathbb{R}),\quad \widetilde{g}(t,\cdot):(-\mathbb{H}\backslash\widetilde{K}(t);\mathbb{R})\stackrel{\text{Conf}}{\twoheadrightarrow}(\mathbb{S}_{-t};\mathbb{R}_{-t}),$$
$$\widetilde{I}_{-t}\circ\widetilde{g}(t,\cdot)=\widetilde{g}_I(t,\cdot)\circ\widetilde{I}_0,\quad t\in(-\infty,0).$$

The relation between $g(t,\cdot)$ and $g_I(t,\cdot)$ depends on the three cases in a similar way.

3.4 SLE with marked points

Definition 3.1 A covering crossing annulus drift function is a real valued $C^{0,1}$ differentiable function defined on $(0,\infty)\times\mathbb{R}$. A covering crossing annulus drift function with period 2π is called a crossing annulus drift function.

Definition 3.2 Suppose Λ is a covering crossing annulus drift function. Let $\kappa>0$, $p>0$, and $x_0, y_0\in\mathbb{R}$. Let $\xi(t)$, $0\le t<p$, be the maximal solution to the SDE

$$d\xi(t)=\sqrt{\kappa}dB(t)+\Lambda(p-t,\xi(t)-\operatorname{Re}\widetilde{g}(t,y_0+pi))dt,\quad \xi(0)=x_0,\tag{3.13}$$

where $\widetilde{g}(t,\cdot)$, $0\le t<p$, are the covering annulus Loewner maps of modulus p driven by ξ. Then the covering annulus Loewner trace of modulus p driven by ξ is called the covering (crossing) annulus SLE(κ,Λ) trace in $\mathbb{S}_p$ started from x_0 with marked point y_0+pi.

Definition 3.3 Suppose Λ is a crossing annulus drift function. Let $\kappa>0$, $p>0$, $a\in\mathbb{T}$ and $b\in\mathbb{T}_p$. Choose $x_0,y_0\in\mathbb{R}$ such that $a=e^{ix_0}$ and $b=e^{-p+iy_0}$. Let $\xi(t)$, $0\le t<p$, be the maximal solution to (3.13). The annulus Loewner trace β driven by ξ is called the (crossing) annulus SLE(κ,Λ) trace in $\mathbb{A}_p$ started from a with marked point b.

The above definition does not depend on the choices of x_0 and y_0 because Λ has period 2π, and for any $n\in\mathbb{Z}$, the annulus Loewner objects driven by $\xi(t)+2n\pi$ agree with those driven by $\xi(t)$.

A covering chordal-type annulus drift function is a real valued $C^{0,1}$ differentiable function defined on $(0,\infty)\times(\mathbb{R}\backslash 2\pi\mathbb{Z})$. The word "covering" is omitted if the function has period 2π. If Λ is a chordal-type annulus drift function, using the same idea, we may define the annulus SLE(κ,Λ) processes, where the initial point $a=e^{ix_0}$ and marked point $b=e^{iy_0}$ both lie on $\mathbb{T}$ and are distinct. The driving function ξ is the solution to (3.13) with $\operatorname{Re}\widetilde{g}(t,y_0+pi)$ replaced by $\widetilde{g}(t,y_0)$.

Via conformal maps, we can then define annulus SLE(κ,Λ) process and trace in any doubly connected domain started from one boundary point with another boundary point being marked. Here Λ is a chordal-type or crossing annulus drift function depending on whether or not the initial point and the marked point lie on the same boundary component. Let $\Lambda_I(p,x)=-\Lambda(p,-x)$, then Λ_I is called the dual of Λ. If W is a conjugate conformal map of $\mathbb{A}_p$, and Λ_I is the dual of Λ, then $(W(K(t)))$ is an annulus SLE(κ,Λ_I) process in $W(\mathbb{A}_p)$ started from $W(a)$ with marked point $W(b)$.

Definition 3.4 Let $\kappa \ge 0$, $b \in \mathbb{T}$, and Λ be a crossing annulus drift function. Choose $y_0 \in \mathbb{R}$ such that $e^{iy_0} = b$. Let $B_*^{(\kappa)}(t)$, $t \in \mathbb{R}$, be a pre-$(\mathbb{T};\kappa)$-Brownian motion. Suppose $\xi(t)$, $-\infty < t < 0$, satisfies the following SDE with the meaning in Definition 2.2:

$$d\xi(t) = dB_*^{(\kappa)}(t) + \Lambda(-t, \xi(t) - \widetilde{g}_I(t, y_0))dt, \quad -\infty < t < 0,$$

where $\widetilde{g}_I(t,\cdot)$ are the disc Loewner maps driven by ξ. Then we call the disc Loewner trace driven by ξ the disc SLE(κ,Λ) trace in $\mathbb{D}$ started from 0 with marked point b.

Via conformal maps, we can define disc SLE(κ,Λ) trace in any simply connected domain started from an interior point with a marked boundary point.

4 Coupling of two annulus SLE traces

The goal of this section is to prove Theorem 4.1 below, which says that when certain PDE is satisfied, we may couple two annulus SLE$(\kappa;\Lambda)$ processes such that they commute with each other. Although this result will not be used directly in the proof of the whole-plane reversibility, we prove this theorem because on the one hand, the result may be used in the future, and on the other hand, the proof will serve as a reference for a more complicated proof of the theorem about coupling two whole-plane SLE processes.

After some preparation in Sect. 4.1, the construction formally starts from Sect. 4.2, which resembles Section 3 of [10]. The extra complexity comes from the appearance of covering maps and inverted maps. Then we construct a two-dimensional local martingale M in Sect. 4.3, which resembles Section 4 of [10]. In the same subsection, we derive the boundedness of M when the two processes are stopped at some exiting time. In Sect. 4.4, we first construct local commutation couplings using M, then construct the global commutation coupling using the coupling technique, and finishes the proof.

Theorem 4.1 *Let $\kappa > 0$ and $s_0 \in \mathbb{R}$. Suppose Γ is a positive $C^{1,2}$ differentiable function on $(0,\infty)\times\mathbb{R}$ that satisfies*

$$\partial_t \Gamma = \frac{\kappa}{2}\Gamma'' + \mathbf{H}_I \Gamma' + \left(\frac{3}{\kappa} - \frac{1}{2}\right)\mathbf{H}_I'\Gamma; \tag{4.1}$$

$$\Gamma(t, x+2\pi) = e^{\frac{2\pi s_0}{\kappa}}\Gamma(t,x), \quad t > 0,\ x \in \mathbb{R}. \tag{4.2}$$

We call Γ a partition function following Gregory Lawler's terminology in [20]. Let $\Lambda = \kappa\frac{\Gamma'}{\Gamma}$. Then Λ is a crossing annulus drift function. Let $\Lambda_1 = \Lambda$ and Λ_2 be the dual of Λ. Then for any $p > 0$, $a_1, a_2 \in \mathbb{T}$, there is a coupling of two curves: $\beta_1(t)$, $0 \le t < p$, and $\beta_2(t)$, $0 \le t < p$, such that for $j \ne k \in \{1,2\}$, the following hold.

(i) β_j is an annulus SLE(κ,Λ_j) trace in $\mathbb{A}_p$ started from a_j with marked point $a_{I,k}{:=}I_p(a_k)$.

(ii) If $t_k < p$ is a stopping time w.r.t. $(\beta_k(t))$, then conditioned on $\beta_k(t)$, $0 \le t \le t_k$, after a time-change, $\beta_j(t)$, $0 \le t < T_j(t_k)$ is the annulus SLE(κ,Λ_j) process in

a connected component of $\mathbb{A}_p \backslash I_p(\beta_k((0, t_k]))$ *started from* a_j *with marked point* $I_p(\beta_k(t_k))$*, where* $T_j(t_k)$ *is the first time that* β_j *visits* $I_p \circ \beta_k((0, t_k])$*, which is set to be* p *if such time does not exist.*

Remark 1. The Λ satisfies the PDE:

$$\partial_t \Lambda = \frac{\kappa}{2}\Lambda'' + \Big(3 - \frac{\kappa}{2}\Big)\mathbf{H}_I'' + \Lambda \mathbf{H}_I' + \mathbf{H}_I \Lambda' + \Lambda\Lambda'. \tag{4.3}$$

On the other hand, if Λ satisfies (4.3), then there is Γ such that $\Lambda = \kappa\frac{\Gamma'}{\Gamma}$ and (4.1) holds.

2. The theorem also holds for $\kappa = 0$ if Λ satisfies (4.3) with $\kappa = 0$.

4.1 Transformations of PDE

Lemma 4.1 *Let* $\sigma, s_0 \in \mathbb{R}$*. Suppose* Γ*,* Ψ*, and* Ψ_{s_0} *are functions defined on* $(0, \infty) \times \mathbb{R}$*, which satisfy* $\Psi = \Gamma \Theta_I^{\frac{2}{\kappa}}$*,* $\Psi_s = \Gamma_s \Theta_I^{\frac{2}{\kappa}}$*, and* $\Psi_{s_0}(t, x) = e^{-\frac{s_0 x}{\kappa} - \frac{s_0^2 t}{2\kappa}}\Psi(t, x)$*. Then the following PDEs are equivalent:*

$$\partial_t \Gamma = \frac{\kappa}{2}\Gamma'' + \mathbf{H}_I \Gamma' + \Big(\sigma - \frac{1}{\kappa} + \frac{1}{2}\Big)\mathbf{H}_I' \Gamma; \tag{4.4}$$

$$\partial_t \Psi = \frac{\kappa}{2}\Psi'' + \sigma \mathbf{H}_I' \Psi; \tag{4.5}$$

$$\partial_t \Psi_{s_0} = \frac{\kappa}{2}\Psi_{s_0}'' + s_0 \Psi_{s_0}' + \sigma \mathbf{H}_I' \Psi_{s_0}. \tag{4.6}$$

Proof This follows from (2.5), (2.7), and some straightforward computations. □

Remark When $\sigma = \frac{4}{\kappa} - 1$, (4.4) agrees with (4.1).

Lemma 4.2 *Let* $\sigma, s_0 \in \mathbb{R}$*. Suppose* Ψ_{s_0} *is positive, has period* 2π*, and solves (4.6) in* $(0, \infty) \times \mathbb{R}$*. Then* $\Psi_{s_0}(t, x) \to C$ *as* $t \to \infty$ *for some constant* $C > 0$*, uniformly in* $x \in \mathbb{R}$*.*

Proof Fix $t_0 > 0$ and $x_0 \in \mathbb{R}$. For $0 \le t < t_0$, let $X_{x_0}(t) = x_0 + \sqrt{\kappa}B(t) + st$ and

$$M(t) = \Psi_{s_0}(t_0 - t, X_{x_0}(t)) \exp\left(\sigma \int_0^t \mathbf{H}_I'(t_0 - r, X_{x_0}(r))dr\right).$$

From (4.6) and Itô's formula, $M(t)$, $0 \le t < t_0$, is a local martingale. Since Ψ_{s_0} and $\mathbf{H}_I'$ are continuous on $(0, \infty) \times \mathbb{R}$ and have period 2π, we see that, for any $t_1 \in (0, t_0]$, $M(t)$, $0 \le t \le t_0 - t_1$, is uniformly bounded, so it is a bounded martingale. Thus,

$$\Psi_{s_0}(t_0, x_0) = M(0) = \mathbf{E}\left[\Psi_{s_0}(t_1, X_{x_0}(t_0 - t_1)) \exp\left(\sigma \int_0^{t_0 - t_1} \mathbf{H}_I'(t_0 - r, X_{x_0}(r))dr\right)\right]. \tag{4.7}$$

Now suppose $t_0 > t_1 \geq 3$. From Lemma 2.1, we see that,

$$\int\limits_0^{t_0-t_1} |\mathbf{H}_I'(t_0 - r, X_{x_0}(r))|dr \leq \int\limits_0^{t_0-t_1} 15e^{r-t_0}dr \leq 15e^{-t_1}. \tag{4.8}$$

Let $\varepsilon > 0$. Choose $t_1 \geq 3$ such that $15\sigma e^{-t_1} < \varepsilon/3$. For $t \in [t_1, \infty)$ and $x \in \mathbb{R}$, define

$$\Psi_{s_0,t_1}(t, x) = \mathbf{E}\,[\Psi_{s_0}(t_1, X_x(t - t_1))].$$

As $t \to \infty$, the distribution of $e^i(X_x(t - t_1))$ tends to the uniform distribution on $\mathbb{T}$. Since Ψ_{s_0} is positive, continuous, and has period 2π, we see that $\Psi_{s_0,t_1}(t, x) \to \frac{1}{2\pi}\int_0^{2\pi} \Psi_{s_0}(t_1, x)dx > 0$ as $t \to \infty$, uniformly in $x \in \mathbb{R}$. Thus, $\lim_{t\to\infty} \ln(\Psi_{s_0,t_1})$ converges uniformly in $x \in \mathbb{R}$. So there is $t_2 > t_1$ such that if $t_a, t_b \geq t_2$ and $x_a, x_b \in \mathbb{R}$, then $|\ln(\Psi_{t_1}(t_a, x_a)) - \ln(\Psi_{t_1}(t_b, x_b))| < \varepsilon/3$. From (4.7) and (4.8) we see that

$$|\ln(\Psi_{s_0}(t, x)) - \ln(\Psi_{s_0,t_1}(t, x))| \leq 15\sigma e^{-t_1} < \varepsilon/3, \quad t \geq t_1,\ x \in \mathbb{R}.$$

Thus, $|\ln(\Psi_{s_0}(t_a, x_a)) - \ln(\Psi_{s_0}(t_b, x_b))| < \varepsilon$ if $t_a, t_b \geq t_2$ and $x_a, x_b \in \mathbb{R}$. So $\lim_{t\to\infty} \ln(\Psi_{s_0})$ converges uniformly in $x \in \mathbb{R}$, which implies the conclusion of the lemma. □

Lemma 4.3 *Let $s_0 \in \mathbb{R}$. Suppose Γ is positive, satisfies (4.2), and solves (4.4). Then there is $C > 0$ such that $\Gamma_{s_0}(t, x) := C^{-1}e^{-\frac{s_0 x}{\kappa} - \frac{s_0^2 t}{2\kappa}}\Gamma(t, x)$ has period 2π and satisfies $\lim_{t\to\infty} \Gamma_{s_0}(t, x) = 1$, uniformly in $x \in \mathbb{R}$.*

Proof Let Ψ_{s_0} be given by Lemma 4.1. Since $\Theta_I > 0$, Ψ_{s_0} is positive and solves (4.6). Since Γ satisfies (4.2) and Θ_I has period 2π, Ψ_{s_0} also has period 2π. From Lemma 4.2, there is $C > 0$ such that $\Psi_{s_0} \to C$ as $t \to \infty$, uniformly in $x \in \mathbb{R}$. Let $\Gamma_{s_0}(t, x) := C^{-1}e^{-\frac{s_0 x}{\kappa} - \frac{s_0^2 t}{2\kappa}}\Gamma(t, x)$. Then $\Gamma_{s_0} = C^{-1}\Psi_{s_0}\Theta_I(t, x)^{-\frac{2}{\kappa}}$. From (2.6), $\Theta_I \to 1$ as $t \to \infty$, uniformly in $x \in \mathbb{R}$. Since Θ_I has period 2π, we get the desired conclusion. □

4.2 Ensemble

Let $p > 0$ and $\xi_1, \xi_2 \in C([0, p))$. For $j = 1, 2$, let $g_j(t, \cdot)$ (resp. $g_{I,j}(t, \cdot)$), $0 \leq t < p$, be the annulus (resp. inverted annulus) Loewner maps of modulus p driven by ξ_j. Let $\widetilde{g}_j(t, \cdot)$ and $\widetilde{g}_{I,j}(t, \cdot)$, $0 \leq t < p$, $j = 1, 2$, be the corresponding covering Loewner maps. Suppose ξ_j generates a simple annulus Loewner trace of modulus p: β_j, $j = 1, 2$. Let $\beta_{I,j} = I_p \circ \beta_j$, $j = 1, 2$, be the inverted trace. Define

$$\begin{aligned}\mathcal{D} &= \{(t_1, t_2) : \beta_1((0, t_1]) \cap \beta_{I,2}((0, t_2]) = \emptyset\} \\ &= \{(t_1, t_2) : \beta_{I,1}((0, t_1]) \cap \beta_2((0, t_2]) = \emptyset\}.\end{aligned} \tag{4.9}$$

For $(t_1, t_2) \in \mathcal{D}$, we define

$$\begin{aligned} \mathrm{m}(t_1, t_2) &= \mathrm{mod}(\mathbb{A}_p \setminus \beta_1([0, t_1]) \setminus \beta_{I,2}([0, t_2])) \\ &= \mathrm{mod}(\mathbb{A}_p \setminus \beta_{I,1}([0, t_1]) \setminus \beta_2([0, t_2])). \end{aligned} \tag{4.10}$$

Fix any $j \neq k \in \{1, 2\}$ and $t_k \in [0, p)$. Let $T_j(t_k)$ be the maximal number such that for any $t_j < T_j(t_k)$, we have $(t_1, t_2) \in \mathcal{D}$. As $t_j \to T_j(t_k)^-$, the spherical distance between $\beta_j((0, t_j])$ and $\beta_{I,k}((0, t_k])$ tends to 0, so $\mathrm{m}(t_1, t_2) \to 0$. For $0 \le t_j < T_j(t_k)$, let $\beta_{j,t_k}(t_j) = g_{I,k}(t_k, \beta_j(t_j))$. Then $\beta_{j,t_k}(t_j)$, $0 \le t_j < T_j(t_k)$, is a simple curve that starts from $g_{I,k}(t_k, e^{i\xi_j(t_j)}) \in \mathbb{T}$, and stays inside $\mathbb{A}_p$ for $t_j > 0$. Let

$$v_{j,t_k}(t_j) = \mathrm{cap}_{\mathbb{A}_{p-t_k}}(\beta_{j,t_k}((0, t_j])) = p - t_k - \mathrm{m}(t_1, t_2). \tag{4.11}$$

Then v_{j,t_k} is continuous and increasing and maps $[0, T_j(t_k))$ onto $[0, S_{j,t_k})$ for some $S_{j,t_k} \in (0, p - t_k]$. Since $\mathrm{m} \to 0$ as $t_j \to T_j(t_k)$, $S_{j,t_k} = p - t_k$. Then $\gamma_{j,t_k}(t) := \beta_{j,t_k}(v_{j,t_k}^{-1}(t))$, $0 \le t < p - t_k$, are the annulus Loewner trace of modulus $p - t_k$ driven by some $\zeta_{j,t_k} \in C([0, p - t_k))$. Let $\gamma_{I,j,t_k}(t)$ be the corresponding inverted annulus Loewner trace. Let $h_{j,t_k}(t, \cdot)$ and $h_{I,j,t_k}(t, \cdot)$ be the corresponding annulus and inverted annulus Loewner maps. Let $\widetilde{h}_{j,t_k}(t, \cdot)$, and $\widetilde{h}_{I,j,t_k}(t, \cdot)$ be the corresponding covering maps.

For $0 \le t_j < T_j(t_k)$, let $\xi_{j,t_k}(t_j)$, $\beta_{I,j,t_k}(t_j)$, $g_{j,t_k}(t_j, \cdot)$, $g_{I,j,t_k}(t_j, \cdot)$, $\widetilde{g}_{j,t_k}(t_j, \cdot)$, and $\widetilde{g}_{I,j,t_k}(t_j, \cdot)$ be the time-changes of $\zeta_{j,t_k}(t)$, $\gamma_{I,j,t_k}(t)$, $h_{j,t_k}(t, \cdot)$, $h_{I,j,t_k}(t, \cdot)$, $\widetilde{h}_{j,t_k}(t, \cdot)$, and $\widetilde{h}_{I,j,t_k}(t, \cdot)$, respectively, via the map v_{j,t_k}. For example, this means that $\xi_{j,t_k}(t_j) = \zeta_{j,t_k}(v_{j,t_k}(t_j))$ and $g_{j,t_k}(t_j, \cdot) = h_{j,t_k}(v_{j,t_k}(t_j), \cdot)$.

For $0 \le t_j < T_j(t_k)$, let

$$G_{I,k,t_k}(t_j, \cdot) = g_{j,t_k}(t_j, \cdot) \circ g_{I,k}(t_k, \cdot) \circ g_j(t_j, \cdot)^{-1}, \tag{4.12}$$

$$\widetilde{G}_{I,k,t_k}(t_j, \cdot) = \widetilde{g}_{j,t_k}(t_j, \cdot) \circ \widetilde{g}_{I,k}(t_k, \cdot) \circ \widetilde{g}_j(t_j, \cdot)^{-1}. \tag{4.13}$$

Then $G_{I,k,t_k}(t_j, \cdot)$ maps $\mathbb{A}_{p-t_j} \setminus g_j(t_j, \beta_{I,k}((0, t_k]))$ conformally onto $\mathbb{A}_{\mathrm{m}(t_1,t_2)}$ and maps $\mathbb{T}$ onto $\mathbb{T}$; $e^i \circ \widetilde{G}_{I,k,t_k}(t_j, \cdot) = G_{I,k,t_k}(t_j, \cdot) \circ e^i$; and $\widetilde{G}_{I,k,t_k}(t_j, \cdot)$ maps $\mathbb{R}$ onto $\mathbb{R}$. Since $\gamma_{j,t_k}(t) = \beta_{j,t_k}(v_{j,t_k}^{-1}(t))$, from (3.6) and a similar formula for γ, we find that $e^{i\xi_{j,t_k}(t_j)} = G_{I,k,t_k}(t_j, e^{i\xi_j(t_j)})$ for $0 \le t_j < T_j(t_k)$. So there is $n \in \mathbb{Z}$ such that $\widetilde{G}_{I,k,t_k}(t_j, \xi_j(t_j)) = \xi_{j,t_k}(t_j) + 2n\pi$ for $0 \le t_j < T_j(t_k)$. Since $\zeta_{j,t_k} + 2n\pi$ generates the same annulus Loewner hulls as ζ_{j,t_k}, we may choose ζ_{j,t_k} such that for $0 \le t_j < T_j(t_k)$,

$$\xi_{j,t_k}(t_j) = \widetilde{G}_{I,k,t_k}(t_j, \xi_j(t_j)). \tag{4.14}$$

For $0 \le t_j < T_j(t_k)$, let

$$A_{j,h}(t_1, t_2) = \widetilde{G}^{(h)}_{I,k,t_j}(t_k, \xi_j(t_j)), \quad h = 1, 2, 3, \tag{4.15}$$

$$A_{j,S}(t_1, t_2) = \frac{A_{j,3}(t_1, t_2)}{A_{j,1}(t_1, t_2)} - \frac{3}{2}\left(\frac{A_{j,2}(t_1, t_2)}{A_{j,1}(t_1, t_2)}\right)^2. \tag{4.16}$$

Then $A_{j,S}(t_1,t_2)$ is the Schwarzian derivative of $\widetilde{G}_{I,k,t_j}(t_k,\cdot)$ at $\xi_j(t_j)$. A standard argument using Lemma 2.1 in [17] shows that, for $0\le t_j<T_j(t_k)$,

$$v'_{j,t_k}(t_j)=|G'_{I,k,t_k}(t_j,\xi_j(t_j))|^2=\widetilde{G}'_{I,k,t_k}(t_j,\xi_j(t_j))^2=A_{j,1}(t_1,t_2)^2, \tag{4.17}$$

so from (4.11) we have

$$\partial_j\,\mathrm{m}=-A_{j,1}^2. \tag{4.18}$$

Moreover, for $0\le t_j<T_j(t_k)$,

$$\partial_t\widetilde{g}_{j,t_k}(t_j,z)=A_{j,1}(t_1,t_2)^2\mathbf{H}(\mathrm{m}(t_1,t_2),\widetilde{g}_{j,t_k}(t_j,z)-\xi_{j,t_k}(t_j)); \tag{4.19}$$
$$\partial_t\widetilde{g}_{I,j,t_k}(t_j,z)=A_{j,1}(t_1,t_2)^2\mathbf{H}_I(\mathrm{m}(t_1,t_2),\widetilde{g}_{I,j,t_k}(t_j,z)-\xi_{j,t_k}(t_j)). \tag{4.20}$$

From (4.13) we have

$$\widetilde{G}_{I,k,t_k}(t_j,\cdot)\circ\widetilde{g}_j(t_j,z)=\widetilde{g}_{j,t_k}(t_j,\cdot)\circ\widetilde{g}_{I,k}(t_k,z). \tag{4.21}$$

Differentiate (4.21) w.r.t. t_j. Let $w=\widetilde{g}_j(t_j,z)\to\xi_j(t_j)$. From (3.7), (4.14), (4.19), and (2.2) we get

$$\partial_t\widetilde{G}_{I,k,t_k}(t_j,\xi_j(t_j))=-3\widetilde{G}''_{I,k,t_k}(t_j,\xi_j(t_j))=-3A_{j,2}(t_1,t_2). \tag{4.22}$$

Differentiate (4.21) w.r.t. t_j and z, and let $w=\widetilde{g}_j(t_j,z)\to\xi_j(t_j)$. Then we get

$$\frac{\partial_t\widetilde{G}'_{I,k,t_k}(t_j,\xi_j(t_j))}{\widetilde{G}'_{I,k,t_k}(t_j,\xi_j(t_j))}=\frac{1}{2}\cdot\left(\frac{A_{j,2}}{A_{j,1}}\right)^2-\frac{4}{3}\cdot\frac{A_{j,3}}{A_{j,1}}+A_{j,1}^2\mathbf{r}(\mathrm{m})-\mathbf{r}(p-t_j). \tag{4.23}$$

Note that both $G_{I,k,t_k}(t_j,\cdot)$ and $g_{I,k,t_j}(t_k,\cdot)$ map $\mathbb{A}_{p-t_j}\setminus\beta_{I,k,t_j}((0,t_k])$ conformally onto $\mathbb{A}_{\mathrm{m}(t_1,t_2)}$ and maps $\mathbb{T}$ onto $\mathbb{T}$. So they differ by a multiplicative constant of modulus 1. Thus, there is $C_k(t_1,t_2)\in\mathbb{R}$ such that

$$\widetilde{G}_{I,k,t_k}(t_j,\cdot)=\widetilde{g}_{I,k,t_j}(t_k,\cdot)+C_k(t_1,t_2). \tag{4.24}$$

Interchanging j and k in (4.24), we see that there is $C_j(t_1,t_2)\in\mathbb{R}$ such that

$$\widetilde{G}_{I,j,t_j}(t_k,\cdot)=\widetilde{g}_{I,j,t_k}(t_j,\cdot)+C_j(t_1,t_2). \tag{4.25}$$

From (4.13) we have

$$\widetilde{g}_{I,j,t_k}(t_j,\cdot)\circ\widetilde{g}_k(t_k,\cdot)+C_j=\widetilde{g}_{k,t_j}(t_k,\cdot)\circ\widetilde{g}_{I,j}(t_j,\cdot), \tag{4.26}$$
$$\widetilde{g}_{I,k,t_j}(t_k,\cdot)\circ\widetilde{g}_j(t_j,\cdot)+C_k=\widetilde{g}_{j,t_k}(t_j,\cdot)\circ\widetilde{g}_{I,k}(t_k,\cdot). \tag{4.27}$$

From the definition of inverted annulus Loewner maps, we have

$$\widetilde{g}_{j,t_k}(t_j,\cdot)=\widetilde{I}_{\mathrm{m}(t_1,t_2)}\circ\widetilde{g}_{I,j,t_k}(t_j,\cdot)\circ\widetilde{I}_{p-t_k},\quad \widetilde{g}_j(t_j,\cdot)=\widetilde{I}_{p-t_j}\circ\widetilde{g}_{I,j}(t_j,\cdot)\circ\widetilde{I}_p;$$
$$\widetilde{g}_{I,k,t_j}(t_k,\cdot)=\widetilde{I}_{\mathrm{m}(t_1,t_2)}\circ\widetilde{g}_{k,t_j}(t_k,\cdot)\circ\widetilde{I}_{p-t_j},\quad \widetilde{g}_{I,k}(t_k,\cdot)=\widetilde{I}_{p-t_k}\circ\widetilde{g}_k(t_k,\cdot)\circ\widetilde{I}_p.$$

From (4.27) and the above formulas, we get $\widetilde{g}_{k,t_j}(t_k,\cdot)\circ\widetilde{g}_{I,j}(t_j,\cdot)+C_k = \widetilde{g}_{I,j,t_k}(t_j,\cdot)\circ\widetilde{g}_k(t_k,\cdot)$. Comparing this formula with (4.26), we see that $C_1+C_2\equiv 0$. Now we define X_1 and X_2 on $\mathcal{D}$ by

$$\begin{aligned}X_j(t_1,t_2)&=\xi_{j,t_k}(t_j)-\widetilde{g}_{I,j,t_k}(t_j,\xi_k(t_k))\\&=\widetilde{G}_{I,k,t_k}(t_j,\xi_j(t_j))-\widetilde{g}_{I,j,t_k}(t_j,\xi_k(t_k)).\end{aligned}\tag{4.28}$$

From (4.24), (4.25), and $C_1+C_2\equiv 0$, we have

$$X_1+X_2\equiv 0.\tag{4.29}$$

Since $\mathbf{H}_I'''$ is even, we may define Q on $\mathcal{D}$ by

$$Q=\mathbf{H}_I'''(\mathrm{m},X_1)=\mathbf{H}_I'''(\mathrm{m},X_2).\tag{4.30}$$

Differentiate (4.20) w.r.t. z twice. We get

$$\frac{\partial_t\widetilde{g}'_{I,j,t_k}(t_j,z)}{\widetilde{g}'_{I,j,t_k}(t_j,z)}=A_{j,1}^2\mathbf{H}_I'(\mathrm{m},\widetilde{g}_{I,j,t_k}(t_j,z)-\xi_{j,t_k}(t_j)).\tag{4.31}$$

$$\partial_t\left(\frac{\widetilde{g}''_{I,j,t_k}(t_j,z)}{\widetilde{g}'_{I,j,t_k}(t_j,z)}\right)=A_{j,1}^2\mathbf{H}_I''(\mathrm{m},\widetilde{g}_{I,j,t_k}(t_j,z)-\xi_{j,t_k}(t_j))\widetilde{g}'_{I,j,t_k}(t_j,z).\tag{4.32}$$

Let $z=\xi_k(t_k)$ in (4.20), (4.31), and (4.32). Since $\mathbf{H}_I$ and $\mathbf{H}_I''$ are odd and $\mathbf{H}_I'$ is even, from (4.25) and (4.28) we have

$$\partial_j\widetilde{g}_{I,j,t_k}(t_j,\xi_k(t_k))=-A_{j,1}^2\mathbf{H}_I(\mathrm{m},X_j).\tag{4.33}$$

$$\frac{\partial_j A_{k,1}}{A_{k,1}}=A_{j,1}^2\mathbf{H}_I'(\mathrm{m},X_j).\tag{4.34}$$

$$\partial_j\left(\frac{A_{k,2}}{A_{k,1}}\right)=-A_{j,1}^2\mathbf{H}_I''(\mathrm{m},X_j)A_{k,1}.\tag{4.35}$$

Differentiate (4.32) w.r.t. z again, and let $z=\xi_k(t_k)$. Since $\mathbf{H}_I'''$ is even, we get

$$\partial_j\left(\frac{A_{k,3}}{A_{k,1}}-\left(\frac{A_{k,2}}{A_{k,1}}\right)^2\right)=A_{j,1}^2[\mathbf{H}_I'''(\mathrm{m},X_j)A_{k,1}^2-\mathbf{H}_I''(\mathrm{m},X_j)A_{k,2}],$$

which together with (4.30) and (4.35) implies that

$$\partial_j A_{k,S}=A_{j,1}^2A_{k,1}^2Q.\tag{4.36}$$

Define F on $\mathcal{D}$ by

$$F(t_1,t_2)=\exp\left(\int_0^{t_2}\int_0^{t_1} A_{1,1}(s_1,s_2)^2 A_{2,1}(s_1,s_2)^2 Q(s_1,s_2)ds_1 ds_2\right), \quad (4.37)$$

Since $\widetilde{g}_{I,j,t_k}(0,\cdot)=\widetilde{h}_{I,j,t_k}(0,\cdot)=\text{id}$, when $t_j=0$, we have $A_{k,1}=1$, $A_{k,2}=A_{k,3}=0$, hence $A_{k,S}=0$. From (4.36), we see that

$$\frac{\partial_j F}{F}=A_{j,S}. \quad (4.38)$$

Remark There is an explanation of F in terms of Brownian loop measure. If R is a function on $(0,\infty)$ that satisfies $R'(t)=\mathbf{r}(t)+\frac{1}{t}$, then

$$-\frac{1}{3}\ln F(t_1,t_2)-R(t_1,t_2)+R(t_1,0)+R(0,t_2)-R(0,0)$$

is the Brownian loop measure of the loops in $\mathbb{A}_p$ that intersect both $\beta_1([0,t_1])$ and $\beta_{I,2}([0,t_2])$.

4.3 Martingales in two time variables

Let $a_1,a_2\in\mathbb{T}$ be as in Theorem 4.1. Let $a_{I,j}=I_p(a_j)\in\mathbb{T}_p$, $j=1,2$. Choose $x_1,x_2\in\mathbb{R}$ such that $a_j=e^{ix_j}$, $j=1,2$. Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions. For $j=1,2$, let $(\mathcal{F}_t^j)$ be the complete filtration generated by $(B_j(t))$. Let Γ, Λ, Λ_1, and Λ_2 be as in Theorem 4.1. Since Γ satisfies (4.2), Λ_j, $j=1,2$, has period 2π, which implies that they are annulus drift functions. For $j=1,2$, let $\xi_j(t_j)$, $0\le t_j<p$, be the solution to the SDE:

$$d\xi_j(t_j)=\sqrt{\kappa}dB_j(t_j)+\Lambda_j(p-t_j,\xi_j(t_j)-\widetilde{g}_{I,j}(t_j,x_{3-j}))dt_j, \quad \xi_j(0)=x_j. \quad (4.39)$$

Then (ξ_1) and (ξ_2) are independent. For simplicity, suppose $\kappa\in(0,4]$ (for the case $\kappa>4$, we may work on Loewner chains and apply Proposition 2.1 in [17]). Then for $j=1,2$, a.s. (ξ_j) generates a simple annulus Loewner trace β_j, which is an annulus SLE(κ,Λ_j) trace β_j in $\mathbb{A}_p$ started from a_j with marked point $a_{I,3-j}$. We may apply the results in the prior subsection.

As the annulus Loewner objects driven by ξ_j, β_j, $\beta_{I,j}=I_p\circ\beta_j$, $(g_{I,j}(t_j,\cdot))$, $(\widetilde{g}_j(t_j,\cdot))$, and $(\widetilde{g}_{I,j}(t_j,\cdot))$ are all $(\mathcal{F}^j_{t_j})$-adapted. Fix $j\ne k\in\{1,2\}$. Since β_j is $(\mathcal{F}^j_{t_j})$-adapted and $(g_{I,k}(t_k,\cdot))$ is $(\mathcal{F}^k_{t_k})$-adapted, we see that $(t_1,t_2)\mapsto\beta_{j,t_k}(t_j)=g_{I,k}(t_k,\beta_j(t_j))$ defined on $\mathcal{D}$ is $(\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{t_2})$-adapted. Since $\widetilde{g}_{j,t_k}(t_j,\cdot)$ and $\widetilde{g}_{I,j,t_k}(t_j,\cdot)$ are determined by $\beta_{j,t_k}(s_j)$, $0\le s_j\le t_j$, they are $(\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{t_2})$-adapted. From (4.13),

$(\widetilde{G}_{I,k,t_k}(t_j,\cdot))$ is $(\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{t_2})$-adapted. From (4.14), $(\xi_{j,t_k}(t_j))$ is also $(\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{t_2})$-adapted. From (4.10), (4.28), (4.15), and (4.16), we see that (m), (X_j), $(A_{j,h})$, $h=1,2,3$, and $(A_{j,S})$ are all $(\mathcal{F}^1_{t_1}\times\mathcal{F}^2_{t_2})$-adapted.

Fix $j\ne k\in\{1,2\}$ and any $(\mathcal{F}^k_t)$-stopping time $t_k\in[0,p)$. Let $\mathcal{F}^{j,t_k}_{t_j}=\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k}$, $0\le t_j<p$. Then $(\mathcal{F}^{j,t_k}_{t_j})_{0\le t_j<p}$ is a filtration. Since $(B_j(t_j))$ is independent of $\mathcal{F}^k_{t_k}$, it is also an $(\mathcal{F}^{j,t_k}_{t_j})$-Brownian motion. Thus, (4.39) is an $(\mathcal{F}^{j,t_k}_{t_j})$-adapted SDE. From now on, we will apply Itô's formula repeatedly, all SDE will be $(\mathcal{F}^{j,t_k}_{t_j})$-adapted, and t_j ranges in $[0,T_j(t_k))$.

From (4.22), (4.28), (4.15), and (4.33), we see that X_j satisfies

$$\partial_j X_j = A_{j,1}\partial\xi_j(t_j)+\left(\frac{\kappa}{2}-3\right)A_{j,2}\partial t_j + A^2_{j,1}\mathbf{H}_I(\mathrm{m},X_j)\partial t_j. \tag{4.40}$$

Let $\Gamma_1=\Gamma$ and $\Gamma_2(t,x)=\Gamma(t,-x)$. Then for $j=1,2$, $\Lambda_j=\frac{\Gamma_j'}{\Gamma_j}$ and Γ_j satisfies (4.1). From (4.29), we may define Y on $\mathcal{D}$ by

$$Y=\Gamma_1(\mathrm{m},X_1)=\Gamma_2(\mathrm{m},X_2). \tag{4.41}$$

From (4.1), (4.18), (4.40), and (4.41), we have

$$\begin{aligned}\frac{\partial_j Y}{Y} &= \frac{1}{\kappa}\Lambda_j(\mathrm{m},X_j)A_{j,1}\partial\xi_j(t_j)\\ &\quad-\left(\frac{3}{\kappa}-\frac{1}{2}\right)\left(A^2_{j,1}\mathbf{H}_I'(\mathrm{m},X_j)+\Lambda_j(\mathrm{m},X_j)A_{j,2}\right)\partial t_j.\end{aligned} \tag{4.42}$$

From (4.23) we have

$$\begin{aligned}\frac{\partial_j A_{j,1}}{A_{j,1}} &= \frac{A_{j,2}}{A_{j,1}}\cdot\partial\xi_j(t_j)+\left(\frac{1}{2}\cdot\left(\frac{A_{j,2}}{A_{j,1}}\right)^2+\left(\frac{\kappa}{2}-\frac{4}{3}\right)\cdot\frac{A_{j,3}}{A_{j,1}}\right)\partial t_j\\ &\quad+A^2_{j,1}\mathbf{r}(\mathrm{m})\partial t_j-\mathbf{r}(p-t_j)\partial t_j.\end{aligned}$$

Let

$$\alpha=\frac{6-\kappa}{2\kappa},\qquad c=\frac{(8-3\kappa)(\kappa-6)}{2\kappa}.$$

Actually, c is the central charge for SLE$_\kappa$. Then we compute

$$\frac{\partial_j A^\alpha_{j,1}}{A^\alpha_{j,1}}=\alpha\cdot\frac{A_{j,2}}{A_{j,1}}\cdot\partial\xi_j(t_j)+\frac{c}{6}A_{j,S}\partial t_j+\alpha A^2_{j,1}\mathbf{r}(\mathrm{m})\partial t_j-\alpha\mathbf{r}(p-t_j)\partial t_j. \tag{4.43}$$

Recall the $\mathbf{R}$ defined in Sect. 2.3. Define $\widehat{M}$ on $\mathcal{D}$ by

$$\widehat{M}=A^\alpha_{1,1}A^\alpha_{2,1}F^{-\frac{c}{6}}Y\exp(\alpha\mathbf{R}(\mathrm{m})). \tag{4.44}$$

Then $\widehat{M}$ is positive. From (2.3), (4.18), (4.34), (4.38), (4.42), and (4.43), we have

$$\frac{\partial_j \widehat{M}}{\widehat{M}} = \alpha \frac{A_{j,2}}{A_{j,1}} \partial \xi_j(t_j) + \frac{A_{j,1}}{\kappa} \Lambda_j(\mathrm{m}, X_j) \partial \xi_j(t_j) \\ - \alpha \mathbf{r}(p - t_j) \partial t_j + \alpha A_{j,1}^2 \mathbf{r}(\infty) \partial t_j. \tag{4.45}$$

When $t_k = 0$, we have $A_{j,1} = 1$, $A_{j,2} = 0$, $\mathrm{m} = p - t_j$, and $X_j = \xi_j(t_j) - \widetilde{g}_{I,j}(t_j, x_k)$, so the RHS of (4.45) becomes $\frac{1}{\kappa} \Lambda_j(p - t_j, \xi_j(t_j) - \widetilde{g}_{I,j}(t_j, x_k)) \partial \xi_j(t_j)$. Define M on $\mathcal{D}$ by

$$M(t_1, t_2) = \frac{\widehat{M}(t_1, t_2) \widehat{M}(0, 0)}{\widehat{M}(t_1, 0) \widehat{M}(0, t_2)}. \tag{4.46}$$

Then M is also positive, and $M(\cdot, 0) \equiv M(0, \cdot) \equiv 1$. From (4.39) and (4.45) we have

$$\frac{\partial_j M}{M} = \left[\left(3 - \frac{\kappa}{2}\right) \frac{A_{j,2}}{A_{j,1}} + \Lambda_j(\mathrm{m}, X_j) A_{j,1} - \Lambda_j(p - t_j, \xi_j(t_j) - \widetilde{g}_{I,j}(t_j, x_k)) \right] \frac{\partial B_j(t_j)}{\sqrt{\kappa}}. \tag{4.47}$$

So when $t_k \in [0, p)$ is a fixed $(\mathcal{F}_t^k)$-stopping time, M is a local martingale in t_j.

Let $\mathcal{J}$ denote the set of Jordan curves in $\mathbb{A}_p$ that separate $\mathbb{T}$ and $\mathbb{T}_p$. For $J \in \mathcal{J}$ and $j = 1, 2$, let $T_j(J)$ be the first time that β_j visits J. It is also the first time that $\beta_{I,j}$ visits $I_p(J)$. Let JP denote the set of pairs $(J_1, J_2) \in \mathcal{J}^2$ such that $I_p(J_1) \cap J_2 = \emptyset$ and $I_p(J_1)$ is surrounded by J_2. This is equivalent to that $I_p(J_2) \cap J_1 = \emptyset$ and $I_p(J_2)$ is surrounded by J_1. Then for every $(J_1, J_2) \in \mathrm{JP}$, $\beta_{I,1}((0, t_1]) \cap \beta_2((0, t_2]) = \emptyset$ when $t_1 \le T_1(J_1)$ and $t_2 \le T_2(J_2)$, so $[0, T_1(J_1)] \times [0, T_2(J_2)] \subset \mathcal{D}$.

Lemma 4.4 *There are positive continuous functions $N_L(p)$ and $N_S(p)$ defined on $(0, \infty)$ that satisfies $N_L(p), N_S(p) = O(pe^{-p})$ as $p \to \infty$ and the following properties. Suppose K is an interior hull in $\mathbb{D}$ containing 0, g maps $\mathbb{D} \setminus K$ conformally onto $\mathbb{A}_p$ for some $p \in (0, \infty)$ and maps $\mathbb{T}$ onto $\mathbb{T}$, and $\widetilde{g}$ is an analytic function that satisfies $e^i \circ \widetilde{g} = g \circ e^i$. Then for any $x \in \mathbb{R}$, $|\ln(\widetilde{g}'(x))| \le N_L(p)$ and $|S\widetilde{g}(x)| \le N_S(p)$, where $S\widetilde{g}(x)$ is the Schwarzian derivative of $\widetilde{g}$ at x, i.e., $S\widetilde{g}(x) = \widetilde{g}'''(x)/\widetilde{g}'(x) - \frac{3}{2}(\widetilde{g}''(x)/\widetilde{g}'(x))^2$.*

Proof Let $f = g^{-1}$ and $\widetilde{f} = \widetilde{g}^{-1}$. Then $e^i \circ \widetilde{f} = f \circ e^i$. Since $\widetilde{f}'(\widetilde{g}(x)) = 1/\widetilde{g}'(x)$ and $S\widetilde{f}(\widetilde{g}(x)) = -S\widetilde{g}(x)/\widetilde{g}'(x)^2$, we suffice to prove the lemma for f. Let $P(p, z) = -\mathrm{Re}\,\mathbf{S}_I(p, z) - \ln|z|/p$ and $\widetilde{P}(p, z) = P(p, e^{iz}) = \mathrm{Im}\,\mathbf{H}_I(p, z) + \mathrm{Im}\, z/p$. Then $P(p, \cdot)$ vanishes on $\mathbb{T}$ and $\mathbb{T}_p \setminus \{e^{-p}\}$ and is harmonic inside $\mathbb{A}_p$. Moreover, when $z \in \mathbb{A}_p$ is near e^{-p}, $P(p, z)$ behaves like $-\mathrm{Re}(\frac{e^{-p}+z}{e^{-p}-z}) + O(1)$. Thus, $-P(p, \cdot)$ is a renormalized Poisson kernel in $\mathbb{A}_p$ with the pole at e^{-p}. Since $\ln|f|$ is negative and harmonic in $\mathbb{A}_p$ and vanishes on $\mathbb{T}$, there is a positive measure μ_K on $[0, 2\pi)$ such that

$$\ln|f(z)| = -\int P(p, z/e^{i\xi}) d\mu_K(\xi), \quad z \in \mathbb{A}_p,$$

which implies that

$$\operatorname{Im} \widetilde{f}(z) = \int P(p, e^{iz}/e^{i\xi}) d\mu_K(\xi) = \int \widetilde{P}(p, z-\xi) d\mu_K(\xi), \quad z \in \mathbb{S}_p$$

So for any $x \in \mathbb{R}$ and $h = 1, 2, 3$, $\widetilde{f}^{(h)}(x) = \int \frac{\partial^h}{\partial x^{h-1}\partial y} \widetilde{P}(p, x-\xi) d\mu_K(\xi)$. Let

$$m_p = \inf_{x\in\mathbb{R}} \frac{\partial}{\partial y} \widetilde{P}(p, x), \quad M_p = \sup_{x\in\mathbb{R}} \frac{\partial}{\partial y} \widetilde{P}(p, x),$$
$$M_p^{(h)} = \sup_{x\in\mathbb{R}} \left| \frac{\partial^h}{\partial_x^{h-1}\partial y} \widetilde{P}(p, x) \right|, \quad h = 2, 3.$$

We have $0 < m_p < M_p < \infty$ and $m_p|\mu_K| \le \widetilde{f}' \le M_p|\mu_K|$ on $\mathbb{R}$. Since $\widetilde{f}(2\pi) = \widetilde{f}(0) + 2\pi$, we get $1/M_p \le |\mu_K| \le 1/m_p$. Thus, $m_p/M_p \le \widetilde{f}' \le M_p/m_p$ and $|\widetilde{f}^{(h)}| \le M_p^{(h)}/m_p, h = 2, 3$, from which follows that $|S\widetilde{f}| \le \frac{M_p^{(3)}M_p}{m_p^2} + \frac{3}{2}\left(\frac{M_p^{(2)}M_p}{m_p^2}\right)^2$ on $\mathbb{R}$. Since $\widetilde{P}(p, z) = \operatorname{Im} \mathbf{H}_I(p, z) + \operatorname{Im} z/p$, we see that $\frac{\partial}{\partial y}\widetilde{P}(p, x) = \mathbf{H}_I'(p, x) + \frac{1}{p}$ and $\frac{\partial^h}{\partial x^{h-1}\partial y}\widetilde{P}(p, x) = \mathbf{H}_I^{(h)}(p, x)$, $h = 2, 3$. From Lemma 2.1, $M_p, m_p = \frac{1}{p} + O(e^{-p})$ and $M_p^{(h)} = O(e^{-p})$, $h = 2, 3$, as $p \to \infty$. So we have $\ln(M_p/m_p) = O(pe^{-p})$ and $\frac{M_p^{(3)}M_p}{m_p^2} + \frac{3}{2}\left(\frac{M_p^{(2)}M_p}{m_p^2}\right)^2 = O(pe^{-p})$. □

Proposition 4.1 *(Boundedness) Fix* $(J_1, J_2) \in \mathrm{JP}$. *Then* $|\ln(M)|$ *is bounded on* $[0, T_1(J_1)] \times [0, T_2(J_2)]$ *by a constant depending only on* J_1 *and* J_2.

Proof In this proof, we say a function is uniformly bounded if its values on $[0, T_1(J_1)] \times [0, T_2(J_2)]$ are bounded in absolute value by a constant depending only on p, J_1, and J_2. If there is no ambiguity, let $\Omega(A, B)$ denote the domain bounded by sets A and B, and let $\mathrm{mod}(A, B)$ denote the modulus of this domain if it is doubly connected. Let $J_{I,2} = I_0(J_2)$. Let $p_0 = \mathrm{mod}(J_1, J_{I,2}) > 0$. If $t_1 \le T_1(J_1)$ and $t_2 \le T_2(J_2)$, since $\Omega(J_1, J_{I,2})$ disconnects $K_1(t_1)$ and $K_{I,2}(t_2)$ in $\mathbb{A}_p$, $\mathrm{m}(t_1, t_2) \ge p_0$. Since $\mathrm{m} \le p$ always holds, $\mathrm{m} \in [p_0, p]$ on $[0, T_1(J_1)] \times [0, T_2(J_2)]$. Since $\mathbf{R}$ is continuous on $(0, \infty)$, $\mathbf{R}(\mathrm{m})$ is uniformly bounded. Since $Q = \mathbf{H}_I'''(\mathrm{m}, X_1)$ and $\mathbf{H}_I'''$ is continuous and has period 2π, Q is uniformly bounded. From Lemma 4.4, for $j = 1, 2$, $|\ln(A_{j,1})| \le N_L(\mathrm{m})$, so it is uniformly bounded. From (4.38), $\ln(F)$ is uniformly bounded. Let $s_0 \in \mathbb{R}$ be as in Theorem 4.1. Let $\Gamma_{s_0} > 0$ be defined by Lemma 4.3, and $Y_{s_0} = \Gamma_{s_0}(X_1)$. Then Γ_{s_0} has period 2π. So $\ln(Y_{s_0})$ is uniformly bounded. Define $\widehat{M}_{s_0}$ and M_{s_0} using (4.44) and (4.46) with Y and $\widehat{M}$ replaced by Y_{s_0} and $\widehat{M}_{s_0}$, respectively. Then $\ln(\widehat{M}_{s_0})$ and $\ln(M_{s_0})$ are uniformly bounded because their factors are. Now it suffices to show that $\ln(M) - \ln(M_{s_0})$ is uniformly bounded. We have

$$\ln(M(t_1,t_2))-\ln(M_{s_0}(t_1,t_2))=\frac{s_0}{\kappa}(X_1(t_1,t_2)-X_1(t_1,0)-X_1(0,t_2)+X_1(0,0))$$
$$+\frac{s_0^2}{2\kappa}(\mathrm{m}(t_1,t_2)-\mathrm{m}(t_1,0)-\mathrm{m}(0,t_2)+\mathrm{m}(0,0)).$$

The second term on the RHS of the above formula is uniformly bounded because $\mathrm{m}\in[p_0,p]$. So it suffices to show that $X_1(t_1,t_2)-X_1(t_1,0)-X_1(0,t_2)+X_1(0,0)$ is uniformly bounded. Let

$$\widetilde{G}(t_1,t_2)=\widetilde{G}_{I,2,t_2}(t_1,\xi_1(t_1)),\quad \widetilde{g}(t_1,t_2)=\widetilde{g}_{I,1,t_2}(t_1,\xi_2(t_2)).$$

From (4.28) we have $X_1=\widetilde{G}-\widetilde{g}$. So it suffices to show that $\widetilde{G}(t_1,t_2)-\widetilde{G}(t_1,0)-\widetilde{G}(0,t_2)+\widetilde{G}(0,0)$ and $\widetilde{g}(t_1,t_2)-\widetilde{g}(t_1,0)-\widetilde{g}(0,t_2)+\widetilde{g}(0,0)$ are both uniformly bounded. From (4.20) we have $\partial_1\widetilde{g}(t_1,t_2)=A_{1,1}^2\mathbf{H}_I(\mathrm{m}(t_1,t_2),\widetilde{g}(t_1,t_2)-\xi_{1,t_2}(t_1))$. Since $A_{1,1}^2$ is uniformly bounded, $\mathrm{m}\in[p_0,p]$, and $\mathbf{H}_I$ is continuous and has period 2π, $\widetilde{g}(t_1,t_2)-\widetilde{g}(0,t_2)$ is uniformly bounded. Thus, $\widetilde{g}(t_1,t_2)-\widetilde{g}(t_1,0)-\widetilde{g}(0,t_2)+\widetilde{g}(0,0)$ is uniformly bounded. Let $\widetilde{G}_d(t_1,t_2)=\widetilde{G}(t_1,t_2)-\xi_1(t_1)$. Then $\widetilde{G}(t_1,t_2)-\widetilde{G}(t_1,0)-\widetilde{G}(0,t_2)+\widetilde{G}(0,0)=\widetilde{G}_d(t_1,t_2)-\widetilde{G}_d(t_1,0)-\widetilde{G}_d(0,t_2)+\widetilde{G}_d(0,0)$. To finish the proof it suffices to show that $\widetilde{G}_d$ is uniformly bounded.

Let J be a Jordan curve which is disjoint from J_1 and $I_p(J_2)$, and separates these two curves. Let $\widetilde{J}=(e^i)^{-1}(J)$. Since $\widetilde{G}_d(t_1,t_2)=\widetilde{G}_{I,2,t_2}(t_1,\xi_1(t_1))-\xi_1(t_1)$, from the Maximum principle, we suffice to show that $\sup_{z\in\widetilde{g}_1(t_1,\widetilde{J})}(\widetilde{G}_{I,2,t_2}(t_1,z)-z)$ is uniformly bounded. Recall from (4.13) that $\widetilde{G}_{I,2,t_1}(t_1,\cdot)=\widetilde{g}_{1,t_2}(t_1,\cdot)\circ\widetilde{g}_{I,2}(t_2,\cdot)\circ\widetilde{g}_1(t_1,\cdot)^{-1}$. So we suffice to show that the following three quantities are uniformly bounded:

$$\sup_{z\in\widetilde{J}}|\widetilde{g}_1(t_1,z)-z|,\quad \sup_{z\in\widetilde{J}}|\widetilde{g}_{I,2}(t_2,z)-z|,\quad \sup_{z\in\widetilde{g}_{I,2}(t_2,\widetilde{J})}|\widetilde{g}_{1,t_2}(t_1,z)-z|.$$

The uniformly boundedness of these quantities follow from similar arguments. We only work on the last one since it is the hardest. From (4.19) we have

$$\widetilde{g}_{1,t_2}(t_1,z)-z=\int_0^{t_1}A_{1,1}(s,t_2)^2\mathbf{H}(\mathrm{m}(s,t_2),\widetilde{g}_{1,t_2}(s,z)-\xi_{1,t_2}(s))ds.$$

Since $\int_0^{t_1}A_{1,1}(s,t_2)^2ds=\mathrm{m}(0,t_2)-\mathrm{m}(t_1,t_2)$ is uniformly bounded, we suffice to show that

$$\sup_{z\in\widetilde{g}_{I,2}(t_2,\widetilde{J})}|\mathbf{H}(\mathrm{m}(t_1,t_2),\widetilde{g}_{1,t_2}(t_1,z)-\xi_{1,t_2}(t_1))|$$

is uniformly bounded. From the properties of $\mathbf{H}$, we suffice to show that there is a constant $h>0$ such that $\operatorname{Im}\widetilde{g}_{1,t_2}(t_1,\cdot)\circ\widetilde{g}_{I,2}(t_2,z)\ge h$ for any $z\in\widetilde{J}$. This is equivalent to that $|g_{1,t_2}(t_1,\cdot)\circ g_{I,2}(t_2,z)|\le e^{-h}$ for any $z\in J$. We suffice to show that the extremal distance (c.f. [26]) between $\mathbb{T}$ and $g_{1,t_2}(t_1,\cdot)\circ g_{I,2}(t_2,J)$ is bounded below by a positive constant depending only on p, J, J_1 and J_2. From conformal invariance, that is equal to the extremal distance between J and $\mathbb{T}_p \cup \beta_I((0, t_2])$, which is not smaller than the extremal distance between J and $I_p(J_2)$ since $I_p(J_2)$ separates J from $\mathbb{T}_p \cup \beta_I((0, t_2])$. So we are done. □

4.4 Local couplings and global coupling

Let μ_j denote the distribution of (ξ_j), $j = 1, 2$. Let $\mu = \mu_1 \times \mu_2$. Then μ is the joint distribution of (ξ_1) and (ξ_2), since ξ_1 and ξ_2 are independent. Fix $(J_1, J_2) \in \text{JP}$. From the local martingale property of M and Proposition 4.1, we have $\mathbf{E}_\mu[M(T_1(J_1), T_2(J_2))] = M(0, 0) = 1$. Define ν_{J_1,J_2} by $d\nu_{J_1,J_2}/d\mu = M(T_1(J_1), T_2(J_2))$. Then ν_{J_1,J_2} is a probability measure. Let ν_1 and ν_2 be the two marginal measures of ν_{J_1,J_2}. Then $d\nu_1/d\mu_1 = M(T_1(J_1), 0) = 1$ and $d\nu_2/d\mu_2 = M(0, T_2(J_2)) = 1$, so $\nu_j = \mu_j$, $j = 1, 2$. Suppose temporarily that the joint distribution of (ξ_1) and (ξ_2) is ν_{J_1,J_2} instead of μ. Then the distribution of each (ξ_j) is still μ_j.

Fix an $(\mathcal{F}_t^2)$-stopping time $t_2 \le T_2(J_2)$. From (4.39), (4.47), and Girsanov theorem (c.f. Chapter VIII, Section 1 of [23]), under the probability measure ν_{J_1,J_2}, there is an $(\mathcal{F}_{t_1}^1 \times \mathcal{F}_{t_2}^2)_{t_1 \ge 0}$-Brownian motion $\widetilde{B}_{1,t_2}(t_1)$ such that $\xi_1(t_1)$, $0 \le t_1 \le T_1(J_1)$, satisfies the $(\mathcal{F}_{t_1}^1 \times \mathcal{F}_{t_2}^2)_{t_1 \ge 0}$-adapted SDE:

$$d\xi_1(t_1) = \sqrt{\kappa} d\widetilde{B}_{1,t_2}(t_1) + \left(3 - \frac{\kappa}{2}\right) \frac{A_{1,2}}{A_{1,1}} dt_1 + \Lambda_1(\mathrm{m}, X_1) A_{1,1} dt_1,$$

which together with (4.14), (4.22), and Itô's formula implies that

$$d\xi_{1,t_2}(t_1) = A_{1,1}\sqrt{\kappa} d\widetilde{B}_{1,t_2}(t_1) + A_{1,1}^2 \Lambda_1(\mathrm{m}, \xi_{1,t_2}(t_1) - \widetilde{g}_{I,1,t_2}(t_1, \xi_2(t_2))) dt_1.$$

Recall that $\zeta_{1,t_2}(s_1) = \xi_{1,t_2}(v_{1,t_2}^{-1}(s_1))$ and $\widetilde{h}_{I,1,t_2}(s_1, \cdot) = \widetilde{g}_{I,1,t_2}(v_{1,t_2}^{-1}(s_1), \cdot)$. So from (4.11) and (4.17), there is another Brownian motion $\widehat{B}_{1,t_2}(s_1)$ such that for $0 \le s_1 \le v_{1,t_2}(T_1(J_1))$,

$$d\zeta_{1,t_2}(s_1) = \sqrt{\kappa} d\widehat{B}_{1,t_2}(s_1) + \Lambda_1(p - t_2 - s_1, \zeta_{1,t_2}(s_1) - \widetilde{h}_{I,1,t_2}(s_1, \xi_2(t_2))) ds_1.$$

Moreover, the initial values is $\zeta_{1,t_2}(0) = \xi_{1,t_2}(0) = \widetilde{G}_{I,2,t_2}(0, x_1) = \widetilde{g}_{I,2}(t_2, x_1)$. Thus, after a time-change, $g_{I,2}(t_2, \beta_1(t_1))$, $0 \le t_1 \le T_1(J_1)$, is a partial annulus SLE(κ, Λ_1) trace in $\mathbb{A}_{p-t_2}$ started from $g_{I,2}(t_2, a_1)$ with marked point $I_{p-t_2} \circ e^i(\xi_2(t_2))$. This means that, conditioning on $\mathcal{F}_{t_2}^2$, after a time-change, $\beta_1(t_1)$, $0 \le t_1 \le T_1(J_1)$, is a partial annulus SLE(κ, Λ_1) trace in $\mathbb{A}_p \backslash \beta_{I,2}((0, t_2])$ started from a_1 with marked point $\beta_{I,2}(t_2)$. Similarly, the above statement holds true if the subscripts "1" and "2" are exchanged.

The joint distribution ν_{J_1,J_2} is a local coupling such that the desired properties in the statement of Theorem 4.1 holds true up to the stopping time $T_1(J_1)$ and $T_2(J_2)$. Then we can apply the coupling technique developed in Section 7 of [10] to construct a global coupling using the local couplings for different pairs (J_1, J_2).

The coupling technique is composed of several steps. First, let $\{(J_1^k, J_2^k) : k \in \mathbb{N}\}$ denote the set of all pairs in JP such that J_j^k, $k \in \mathbb{N}$, $j = 1, 2$, are polygonal curves, whose vertices have rational coordinates. Second, for every $n \in \mathbb{N}$, one may find a coupling of β_1 and β_2 such that, for every $1 \le k \le n$, if β_1 is stopped at $\tau_{J_1^k}$, and β_2 is stopped at $\tau_{J_2^k}$, then the joint distribution is $\nu_{J_1^k, J_2^k}$. To construct such coupling, we work on the two-dimensional random process M. One may prove that there is a process M_n defined on $[0, p]^2$, which satisfies the following properties:

1. M_n is a martingale in one variable, when the other variable is fixed;
2. $M_n = 1$ when either variable is 0;
3. $M_n = M$ on $[0, \tau_{J_1^k}] \times [0, \tau_{J_2^k}]$, $1 \le k \le n$.

To construct M_n, we use vertical lines $\{t_1 = \tau_{J_1^k}\}$ and horizontal lines $\{t_2 = \tau_{J_2^k}\}$, $1 \le k \le n$, to divide the square $[0, p]^2$ into smaller rectangles. First define M_n on

$$\bigcup_{k=1}^{n} [0, \tau_{J_1^k}] \times [0, \tau_{J_2^k}] \cup (\{0\} \times [0, p]) \cup ([0, p] \times \{0\})$$

according to 2 and 3. Then we extend M_n to other smaller rectangles one by one in such a way that in each rectangle R not contained in any $[0, \tau_{J_1^k}] \times [0, \tau_{J_2^k}]$, $M_n(t_1, t_2) = f_R(t_1) g_R(t_2)$ for some suitable functions f_R and g_R. Such extension exists, is unique, and satisfies the desired properties. The reader is referred to Theorem 6.1 in [10] for details. The ν_n is then defined by $d\nu_n/d\mu = M_n(p, p)$. Finally, the global coupling measure ν is any subsequential weal limit of the sequence (ν_n) in some suitable topology.

4.5 Other results

Here we state without proofs some other results which can be proved using the idea in the proof of Theorem 4.1.

Theorem 4.2 *Let* $\kappa > 0$. *Suppose* Γ *is a* $C^{1,2}$ *differentiable function on* $(0, \infty) \times (\mathbb{R} \backslash 2\pi\mathbb{Z})$ *that satisfies*

$$\partial_t \Gamma = \frac{\kappa}{2} \Gamma'' + \mathbf{H}\Gamma' + \left(\frac{3}{\kappa} - \frac{1}{2}\right) \mathbf{H}'\Gamma. \tag{4.48}$$

Let $\Lambda = \kappa \frac{\Gamma'}{\Gamma}$, $\Lambda_1 = \Lambda$, *and* Λ_2 *be the dual of* Λ. *Then for any* $p > 0$ *and* $a_1 \ne a_2 \in \mathbb{T}$, *there is a coupling of two curves:* $\beta_1(t)$, $0 \le t < T_1$, *and* $\beta_2(t)$, $0 \le t < T_2$, *such that for* $j \ne k \in \{1, 2\}$ *the following hold.*

(i) β_j *is an annulus SLE*(κ, Λ_j) *trace in* $\mathbb{A}_p$ *started from* a_j *with marked point* a_k.
(ii) If $t_k \in [0, T_k)$ *is a stopping time w.r.t.* $(K_k(t))$, *then conditioned on* $\beta_k(t)$, $0 \le t \le t_k$, *after a time-change,* $\beta_j(t)$, $0 \le t < T_j(t_k)$, *is a partial annulus SLE*(κ, Λ_j) *process in a component of* $\mathbb{A}_p \backslash \beta_k((0, t_k])$ *started from* a_j *with marked point* $\beta_k(t_k)$, *where* $T_j(t_k)$ *is the first time that* β_j *hits* $\beta_k([0, t_k])$, *and is set to be* T_j *if such time does not exist. If* $\kappa \in (0, 4]$, *the word "partial" could be removed.*

Remark 1. The Λ in the theorem satisfies the following partial differential equation:

$$\partial_t \Lambda = \frac{\kappa}{2}\Lambda'' + \left(3 - \frac{\kappa}{2}\right)\mathbf{H}'' + \Lambda \mathbf{H}' + \mathbf{H}\Lambda' + \Lambda\Lambda'. \tag{4.49}$$

On the other hand, if Λ satisfies (4.49), then there is Γ, which satisfies $\Lambda = \kappa\frac{\Gamma'}{\Gamma}$ and (4.48).

2. Theorem 4.2 also holds for $\kappa = 0$ if Λ solves (4.48).
3. We may also derive similar results for radial SLE(κ, Λ) process and strip SLE(κ, Λ) process. In these two cases, Γ and Λ are functions of a single variable, and $\Lambda = \kappa\frac{\Gamma'}{\Gamma}$. If $\Lambda = \frac{\rho}{2}\cot_2$ or $\Lambda = \frac{\rho}{2}\coth_2$, respectively, in these two cases, then we get the radial SLE(κ, ρ) and strip SLE(κ, ρ) processes, respectively (c.f. [27]). For the radial SLE(κ, Λ) process, to have the commutation relation, we need that Γ solves the ODE

$$0 = \frac{\kappa}{2}\Gamma'' + \cot_2 \Gamma' + \left(\frac{3}{\kappa} - \frac{1}{2}\right)\cot_2' \Gamma + C\Gamma, \tag{4.50}$$

where C is a constant. For the strip SLE(κ, Λ) process, Γ must solves (4.50) with $\cot_2$ replaced by $\coth_2$ to guarantee the existence of the commutation coupling.

5 Coupling of whole-plane SLE

The goal of this section is to prove Theorem 5.1 below, which is about commutation couplings of two whole-plane SLE processes. This result will later be used to prove the whole-plane reversibility. Since the proof is similar to the proof of Theorem 4.1, we will frequently quote the arguments in the previous section.

Theorem 5.1 *Let $\kappa > 0$ and $s_0 \in \mathbb{R}$. Suppose Γ is a positive $C^{1,2}$ differentiable function on $(0, \infty) \times \mathbb{R}$ that satisfies (4.1) and (4.2). We also call Γ a partition function. Let $\Lambda = \kappa\frac{\Gamma'}{\Gamma}$, $\Lambda_1 = \Lambda$, and Λ_2 be the dual of Λ_1. Let $s_1 = s_0$ and $s_2 = -s_0$. Then there is a coupling of two curves $\beta_{I,1}(t)$, $-\infty < t < \infty$, and $\beta_{I,2}(t)$, $-\infty < t < \infty$, such that for $j \ne k \in \{1, 2\}$, the following hold.*

(i) $\beta_{I,j}$ is a whole-plane SLE(κ, s_j) trace in $\widehat{\mathbb{C}}$ from 0 to ∞;

(ii) Let t_k be a finite stopping time w.r.t. $(K_{I,k}(t))$. Then conditioned on $\beta_{I,k}(s)$, $-\infty < s \le t_k$, after a time-change, the curve $\beta_{I,j}(t_j)$, $-\infty < t_j < T_j(t_k)$, is a disc SLE(κ, Λ_j) process in a component of $\widehat{\mathbb{C}} \backslash I_0(\beta_{I,j}([-\infty, t_j]))$ started from 0 with marked point $I_0(\beta_{I,j}(t_j))$, where $T_j(t_k)$ is the first time that β_j hits $\beta_k([-\infty, t_k])$, or ∞ if such time does not exist.

5.1 Estimations on Loewner maps

Let $\widetilde{g}(t, \cdot)$, $t \in \mathbb{R}$, be the inverted covering *whole-plane* Loewner maps driven by some $\xi \in C(\mathbb{R})$. Let $z \in \mathbb{C}$ and $h(t) = \operatorname{Im} \widetilde{g}(t, z) > 0$ for $t \in (-\infty, \tau_z)$, the interval on which $\widetilde{g}(t, z)$ is defined. From (3.3) we have $-\tanh_2(h(t)) \ge h'(t) \ge -\coth_2(h(t))$, and

$$|\partial_t \widetilde{g}(t,z)+i| \le \frac{2}{e^{\mathrm{Im}\,\widetilde{g}(t,z)}-1} = \frac{2}{e^{h(t)}-1}, \quad t\in(-\infty,\tau_z). \tag{5.1}$$

So $h(t)$ decreases, and $\frac{d}{dt}\ln(\cosh_2(h(t))) \ge -1/2$, which together with (3.4) and integration implies that $\cosh_2(h(t)) \ge \frac{1}{2}e^{\frac{\mathrm{Im}\,z}{2}-\frac{t}{2}}$. Then we have $e^{h(t)} \ge e^{\mathrm{Im}\,z-t}-3$. From (5.1) we see that, if $t<\mathrm{Im}\,z-\ln(8)$, then $|\partial_t\widetilde{g}(t,z)+i| \le \frac{2}{e^{\mathrm{Im}\,z-t}-4} \le 4e^{t-\mathrm{Im}\,z}$. From (3.4) and integration we have

$$|\widetilde{g}(t,z)+it-z| \le 4e^{t-\mathrm{Im}\,z} \le 1/2, \quad \text{if } t\le \mathrm{Im}\,z-\ln(8). \tag{5.2}$$

If $\widetilde{g}(t,\cdot)$ are the covering whole-plane Loewner maps, then from $\widetilde{g}(t,\cdot)=\widetilde{I}_0\circ\widetilde{g}_I(t,\cdot)\circ\widetilde{I}_0$, we have

$$|\widetilde{g}_I(t,z)-it-z| \le 4e^{t+\mathrm{Im}\,z} \le 1/2, \quad \text{if } t\le -\mathrm{Im}\,z-\ln(8). \tag{5.3}$$

Let $\widetilde{g}_I(t,\cdot)$, $-\infty<t<0$, be the covering *disc* Loewner maps driven by some $\xi\in(-\infty,0)$. Let $z\in\mathbb{H}$ and $h(t)=\mathrm{Im}\,\widetilde{g}_I(t,z)>0$ for $t\in(-\infty,\tau_z)$. From Lemma 2.1 and (3.11) we see that if $-t\ge h(t)+2$ then $|h'(t)|\le 5.5e^{h(t)+t}$, so $|\frac{d}{dt}e^{-h(t)}|\le 5.5e^t$. From (3.12) we see that $-t\ge h(t)+2$ when t is close to $-\infty$. Suppose that $-t\ge h(t)+2$ does not hold for all $t\in(-\infty,\tau_z)$, and let t_0 be the first t such that $-t=h(t)+2$. Then $|\frac{d}{dt}e^{-h(t)}|\le 5.5e^t$ on $(-\infty,t_0]$. From (3.12) and integration we have $e^{t_0+2}=e^{-h(t_0)}\ge e^{-\mathrm{Im}\,z}-5.5e^{t_0}$, which implies that $e^{-\mathrm{Im}\,z}\le(e^2+5.5)e^{t_0}<13e^{t_0}$. Thus, if $t\le-\mathrm{Im}\,z-\ln(13)$ then $-t\ge h(t)+2$, so $|h'(t)|\le 5.5e^{h(t)+t}$. From (3.12) and integration, we see that, if $t\le-\mathrm{Im}\,z-\ln(13)$, then $e^{-\mathrm{Im}\,\widetilde{g}_I(t,z)}\ge\frac{7.5}{13}e^{-\mathrm{Im}\,z}$, which implies that $\mathrm{Im}\,\widetilde{g}_I(t,z)\le\mathrm{Im}\,z+\ln(13/7.5)<-t-2$, which, together with Lemma 2.1, implies that $|\mathbf{H}_I(-t,\widetilde{g}_I(t,z)-\xi(t))|\le 5.5\frac{13}{7.5}e^{\mathrm{Im}\,z+t}<10e^{\mathrm{Im}\,z+t}$. From (3.11), (3.12) and integration we have $|\widetilde{g}_I(t,z)-z|\le 10e^{\mathrm{Im}\,z+t}$, if $t\le-\mathrm{Im}\,z-\ln(13)$. If $\widetilde{g}(t,\cdot)$ are the inverted covering disc Loewner maps, then from $\widetilde{g}(t,\cdot)=\widetilde{I}_{-t}\circ\widetilde{g}_I(t,\cdot)\circ\widetilde{I}_0$, we have

$$|\widetilde{g}(t,z)+it-z| \le 10e^{-\mathrm{Im}\,z+t} \le 10/13, \quad \text{if } t\le \mathrm{Im}\,z-\ln(13). \tag{5.4}$$

5.2 Ensemble

The argument in this subsection is parallel to that in Sect. 4.2. Let $\xi_1,\xi_2\in C(\mathbb{R})$. For $j=1,2$, let $g_{I,j}(t,\cdot)$ (resp. $g_j(t,\cdot)$), $t\in\mathbb{R}$, be the whole-plane (resp. inverted whole-plane) Loewner maps driven by ξ_j. Let $\widetilde{g}_{I,j}(t,\cdot)$ and $\widetilde{g}_j(t,\cdot)$, $t\in\mathbb{R}$, $j=1,2$, be the corresponding covering Loewner maps. Suppose ξ_j generates a simple whole-plane Loewner trace: $\beta_{I,j}$, $j=1,2$. Let $\beta_{I,j}=I_0\circ\beta_j$, $j=1,2$, be the inverted trace. Let $K_j(t)$ and $K_{I,j}(t)$ be the corresponding hulls. Define $\mathcal{D}$ and m using (4.9) and (4.10) with 0 replaced by $-\infty$ and $\mathbb{A}_p$ replaced by $\widehat{\mathbb{C}}$. Fix any $j\ne k\in\{1,2\}$ and $t_k\in\mathbb{R}$. Let $T_j(t_k)$ be as defined as before. Then for any $t_j<T_j(t_k)$, we have $(t_1,t_2)\in\mathcal{D}$. Moreover, as $t_j\to T_j(t_k)^-$, $\mathrm{m}(t_1,t_2)\to 0$.

For $-\infty\le t_j<T_j(t_k)$, let $\beta_{I,j,t_k}(t_j)=g_k(t_k,\beta_{I,j}(t_j))$. Then β_{j,t_k} is a simple curve in $\mathbb{D}$ starts from 0. For $-\infty<t_j<T_j(t_k)$, let $v_{j,t_k}(t_j)=-\mathrm{mod}(\mathbb{D}\backslash$

$\beta_{I,j,t_k}([-\infty, t_j])) = -\mathrm{m}(t_1, t_2)$. Then v_{j,t_k} is continuous and increasing and maps $(-\infty, T_j(t_k))$ onto $(-\infty, 0)$. Let $\gamma_{I,j,t_k}(t) = \beta_{I,j,t_k}(v_{j,t_k}^{-1}(t))$, $-\infty \le t < 0$. Then γ_{I,j,t_k} is the disc Loewner trace driven by some $\zeta_{j,t_k} \in C((-\infty, 0))$. Let γ_{j,t_k} be the corresponding inverted disc Loewner trace. Let $h_{I,j,t_k}(t, \cdot)$ and $h_{j,t_k}(t, \cdot)$ be the corresponding disc and inverted disc Loewner maps. Let $\widetilde{h}_{I,j,t_k}(t, \cdot)$ and $\widetilde{h}_{j,t_k}(t, \cdot)$ be the corresponding covering Loewner maps.

For $-\infty < t_j < T_j(t_k)$, let $\xi_{j,t_k}(t_j)$, $\beta_{j,t_k}(t_j)$, $g_{I,j,t_k}(t_j, \cdot)$, $g_{j,t_k}(t_j, \cdot)$, $\widetilde{g}_{I,j,t_k}(t_j, \cdot)$, and $\widetilde{g}_{j,t_k}(t_j, \cdot)$ be the time-changes of $\zeta_{j,t_k}(t)$, $\gamma_{j,t_k}(t)$, $h_{I,j,t_k}(t, \cdot)$, $h_{j,t_k}(t, \cdot)$, $\widetilde{h}_{I,j,t_k}(t, \cdot)$, and $\widetilde{h}_{j,t_k}(t, \cdot)$, respectively, via v_{j,t_k}.

Define $G_{I,k,t_k}(t_j, \cdot)$ and $\widetilde{G}_{I,k,t_k}(t_j, \cdot)$ by (4.12) and (4.13). Then we could choose the driving function ζ_{j,t_k} such that (4.14) holds. Define $A_{j,h}$ and $A_{j,S}$ using (4.15) and (4.16). A standard argument using Lemma 2.1 in [17] shows (4.17) and (4.18) hold here. From the definition of $\widetilde{G}_{I,k,t_k}(t_j, \cdot)$, we get (4.21), which can be differentiated to conclude that (4.22) holds here, and (4.23) holds with $p - t_j$ replaced by ∞. Let X_j be defined by (4.28). Then (4.29) holds. Let Q be defined by (4.30). Then (4.31)–(4.36) still hold.

From Lemma 2.1, we have

$$Q = O(e^{-\mathrm{m}}), \quad \text{as } \mathrm{m} \to \infty. \tag{5.5}$$

From Lemma 4.4, we see that, for $j = 1, 2$,

$$\ln(A_{j,1}), A_{j,S} = O(\mathrm{m}\, e^{-\mathrm{m}}), \quad \text{as } \mathrm{m} \to \infty. \tag{5.6}$$

Since e^{t_j} is the capacity of $K_{I,j}(t_j)$, which contains 0, we have $K_{I,j}(t_j) \subset \{|z| \le 4e^{t_j}\}$. This then implies that $K_j(t_j) \subset \{|z| \ge e^{-t_j}/4\}$, $\widetilde{K}_{I,j}(t_j) \subset \{\mathrm{Im}\, z \ge -t_j - \ln(4)\}$, and $\widetilde{K}_j(t_j) \subset \{\mathrm{Im}\, z \le \ln(4) + t_j\}$. Thus,

$$\{(t_1, t_2) \in \mathbb{R}^2 : t_1 + t_2 < -\ln(16)\} \subset \mathcal{D}, \tag{5.7}$$

$$\mathrm{m}(t_1, t_2) \ge -t_1 - t_2 - \ln(16), \quad \text{if } (t_1, t_2) \in \mathcal{D}. \tag{5.8}$$

From (5.5)–(5.8), we see that $A_{1,1}^2 A_{2,1}^2 Q = O(e^{t_1+t_2})$ as $t_1+t_2 \to -\infty$. Define F on $\mathcal{D}$ using (4.37) with the lower bounds 0 replaced by $-\infty$. From Lemma 4.4, $A_{k,S} \to 0$ as $t_j \to -\infty$. Thus, (4.38) still holds here, and $\ln(F(t_1, t_2)) = \int_{-\infty}^{t_1} \frac{A_{1,S}(s_1,t_2)}{A_{1,1}(s_1,t_2)^2} \cdot A_{1,1}(s_1, t_2)^2 ds_1$. Changing variable with $x(s_1) = \mathrm{m}(s_1, t_2)$, and using (4.18) and (5.6),we conclude that

$$\ln(F) = O(\mathrm{m}\, e^{-\mathrm{m}}), \quad \text{as } \mathrm{m} \to \infty. \tag{5.9}$$

5.3 Martingales in two time variables

The argument in this subsection is parallel to that in Sect. 4.3. Let $(B_1^{(\kappa)}(t))$ and $(B_2^{(\kappa)}(t))$ be two independent pre-$(\mathbb{T}; \kappa)$-Brownian motion. Let $\xi_j(t) = B_j^{(\kappa)}(t) + s_j t$, $t \in \mathbb{R}$, $j = 1, 2$. For simplicity, suppose $\kappa \in (0, 4]$. Then for $j = 1, 2$, a.s. (ξ_j) generates a simple whole-plane Loewner trace $\beta_{I,j}$, which is a whole-plane SLE(κ, s_0)

trace in $\widehat{\mathbb{C}}$ from 0 to ∞. We may apply the results in the prior subsection. For $j = 1, 2$, let $(\mathcal{F}^j_t)_{t\in\mathbb{R}}$ be the complete filtration generated by $e^i(\xi_j)$. The whole-plane Loewner objects driven by ξ_j are all $(\mathcal{F}^j_t)$-adapted, because they are all determined by $(e^i(\xi_j(t)))$. It is easy to check that for $j \ne k \in \{1, 2\}$, the processes (β_{I,j,t_k}), $(\widetilde{g}_{I,j,t_k}(t_j, \cdot))$, $(A_{j,h})$, $h = 1, 2, 3$, $(A_{j,S})$, $(G_{I,j,t_j}(t_k, \cdot))$, $(\widetilde{G}_{I,j,t_j}(t_k, \cdot))$, $(e^i(\xi_{j,t_k}))$, $(e^i(X_j))$, (m), $(\mathbf{H}^{(h)}_I(\mathrm{m}, X_1))$, $(\Gamma_j(\mathrm{m}, X_j))$, $(\Lambda_j(\mathrm{m}, X_j))$, (Q) and (F) defined on $\mathcal{D}$ are all $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{t_2})$-adapted. This is not true for $(\xi_{j,t_k}(t_j))$ and (X_j), but is true for their images under the map e^i. Define Y using (4.41). Then (Y) is also $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{t_2})$-adapted since Γ_j has period 2π.

In this section we work on SDE with the meaning as in Definition 2.2: the stochastic part contains pre-$(\mathbb{T}; \kappa)$-Brownian motions, and the time intervals start from $-\infty$. The traditional Itô's formula works only for time intervals that start from 0 or a finite number. To derive the results in this section, we may truncate the interval "$(-\infty, T)$" by an arbitrary real number c (and we work on the interval $[c, T)$), which is close to $-\infty$. Fix $j \ne k \in \{1, 2\}$ and any $(\mathcal{F}^k_t)$-stopping time $t_k \in \mathbb{R}$. Let $\mathcal{F}^{j,t_k}_{t_j} = \mathcal{F}^j_{t_j} \times \mathcal{F}^k_{t_k}$, $t_j \in \mathbb{R}$. From now on, all SDE will be $(\mathcal{F}^{j,t_k}_{t_j})$-adapted (with the meaning as in Definition 2.2), and t_j ranges in $[0, T_j(t_k))$.

First, we find that (4.40) still holds here, which then implies (4.42). From the modified (4.23), we see that (4.43) holds here with $p - t_j$ replaced by ∞. Let $\widehat{M}$ be defined by (4.44). Then (4.45) holds with $p - t_j$ replaced by ∞.

Define M on $\mathcal{D}$ by

$$M = \widehat{M} \exp\left(\alpha \mathbf{r}(\infty)(\mathrm{m} + t_1 + t_2) + \sum_{j=1,2} -\frac{s_j}{\kappa}\xi_j(t_j) + \frac{s_j^2}{2\kappa} t_j\right). \tag{5.10}$$

Then M is $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{t_2})$-adapted. From the modified (4.45) and that $\xi_j(t_j) = B^{(\kappa)}_j(t_j) + s_j t_j$, we compute

$$\frac{\partial_j M}{M} = \left[\left(3 - \frac{\kappa}{2}\right)\frac{A_{j,2}}{A_{j,1}} + A_{j,1}\Lambda_j(\mathrm{m}, X_j) - s_j\right]\frac{\partial B^{(\kappa)}_j(t_j)}{\kappa}. \tag{5.11}$$

So M is a local martingale in t_j when t_k is a finite stopping time.

Let $\mathcal{J}$ denote the set of Jordan curves in $\mathbb{C}\setminus\{0\}$ that surround 0. For $J \in \mathcal{J}$ and $j = 1, 2$, let $T_j(J)$ denote the first time that β_j hits J. Then $T_j(J)$ is also the first time that $\beta_{I,j}$ hits $I_0(J)$. Let H_J denote the closure of the domain bounded by $I_0(J)$, and let C_J denote the capacity of H_J. If $K_{I,j}(t) \subset H_J$, then $e^t \le C_J$. So we have $T_j(J) \le \ln(C_J)$.

Let JP denote the set of pairs $(J_1, J_2) \in \mathcal{J}^2$ such that $I_0(J_1) \cap J_2 = \emptyset$ and $I_0(J_1)$ is surrounded by J_2. This is equivalent to that $I_0(J_2) \cap J_1 = \emptyset$ and $I_0(J_2)$ is surrounded by J_1. Then for every $(J_1, J_2) \in$ JP, $\beta_{I,1}(t_1) \ne \beta_2(t_2)$ when $t_1 \le T_1(J_1)$ and $t_2 \le T_2(J_2)$, so $(-\infty, T_1(J_1)] \times (-\infty, T_2(J_2)] \subset \mathcal{D}$.

Proposition 5.1 (Boundedness) *Fix* $(J_1, J_2) \in$ JP. *(i)* $|\ln(M)|$ *is bounded on* $(-\infty, T_1(J_1)] \times (-\infty, T_2(J_2)]$ *by a constant depending only on* J_1 *and* J_2. *(ii) Fix any* $j \ne k \in \{1, 2\}$. *Then* $\ln(M) \to 0$ *as* $t_j \to -\infty$ *uniformly in* $t_k \in (-\infty, T_k(J_k)]$.

Proof Let Γ_{s_0} be given by Lemma 4.3. Let $\Gamma_{s_0,1} = \Gamma_{s_0}$, and $\Gamma_{s_0,2}(t,x) = \Gamma_{s_0}(t,-x)$. Define Y_{s_0} on $\mathcal{D}$ by $Y_{s_0} = \Gamma_{s_0,1}(\mathrm{m}, X_1) = \Gamma_{s_0,2}(\mathrm{m}, X_2)$. Then $Y_{s_0} = Y \exp(-\frac{s_0}{\kappa}X_1 - \frac{s_0^2\,\mathrm{m}}{2\kappa})$. From Lemma 4.3,

$$\ln(Y_{s_0}) = o(\mathrm{m}) \quad \text{as } \mathrm{m} \to \infty. \tag{5.12}$$

Define $\widehat{M}_{s_0}$ using (4.44) with Y replaced by Y_{s_0}. From (5.10) we have

$$M = M_{s_0} \exp\left(\left(\alpha \mathbf{r}(\infty) + \frac{s_0^2}{2\kappa}\right)(\mathrm{m} + t_1 + t_2) + \frac{s_0}{\kappa}(X_1 - \xi_1(t_1) + \xi_2(t_2))\right). \tag{5.13}$$

From (5.6), (5.9), (5.12), and that $\mathbf{R}(p) = O(e^{-p})$ as $p \to \infty$, we see that there is a positive continuous function f on $(0,\infty)$ with $\lim_{x\to\infty} f(x) = 0$ such that

$$|\ln(M_{s_0}(t_1,t_2))| \le f(\mathrm{m}(t_1,t_2)). \tag{5.14}$$

Let $\Omega(I_0(J_1), J_2)$ denote the doubly connected domain bounded by $I_0(J_1)$ and J_2. Let $p_0 > 0$ denote its modulus. For $(t_1,t_2) \in (-\infty, T_1(J_1)] \times (-\infty, T_2(J_2)]$, since $\Omega(I_0(J_1), J_2)$ disconnects $K_{I,1}(t_1)$ from $K_2(t_2)$, we have $\mathrm{m}(t_1,t_2) \ge p_0$. On the other hand, $\mathrm{m} \le p$. From (5.14) we see that $\ln(M_{s_0})$ is uniformly bounded. From (5.7), (5.8), (5.14), and that $T_k(J_k) \le C_{J_k} < \infty$, we see that $\ln(M) \to 0$ as $t_j \to -\infty$ uniformly in $t_k \in (-\infty, T_k(J_k)]$. The rest of the proof follows from (5.13) and the following proposition. □

Proposition 5.2 *Fix* $(J_1, J_2) \in \mathrm{JP}$. *(i)* $|X_1 - \xi_1 + \xi_2|$ *and* $|\mathrm{m} + t_1 + t_2|$ *are bounded on* $(-\infty, T_1(J_1)] \times (-\infty, T_2(J_2)]$ *by constants depending only on* J_1 *and* J_2. *(ii) For any* $j \ne k \in \{1,2\}$, $X_1 - \xi_1 + \xi_2 \to 0$ *and* $\mathrm{m} + t_1 + t_2 \to 0$ *as* $t_j \to -\infty$, *uniformly in* $t_k \in (-\infty, T_k(J_k)]$.

Proof Recall that $T_j(J_j) \le C_{J_j} < \infty$ for $j = 1,2$, and $\mathrm{m} \ge p_0 > 0$ on $(-\infty, T_1(J_1)] \times (-\infty, T_2(J_2)]$. If there is no ambiguity, let $\Omega(A,B)$ denote the domain bounded by sets A and B, and let $\mathrm{mod}(A,B)$ denote the modulus of this domain if it is doubly connected.

From (4.28) we have $X_1(t_1,t_2) = \widetilde{G}_{I,2,t_2}(t_1, \xi_1(t_1)) - \widetilde{g}_{I,1,t_2}(t_1, \xi_2(t_2))$. So

$$\begin{aligned}|X_1(t_1,t_2) - \xi_1(t_1) + \xi_2(t_2)| \le\ & |\widetilde{g}_{I,1,t_2}(t_1, \xi_2(t_2)) - \xi_2(t_2)| \\ & + |\widetilde{G}_{I,2,t_2}(t_1, \xi_1(t_1)) - \xi_1(t_1)|.\end{aligned} \tag{5.15}$$

From (3.12) we have $\lim_{t_1\to-\infty} \widetilde{g}_{I,1,t_2}(t_1, \xi_2(t_2)) = \xi_2(t_2)$. From (4.18), (4.20), and Lemma 2.1, we see that there is a deterministic positive decreasing function $f(x)$ with $\lim_{x\to\infty} f(x) = 0$ such that $|\widetilde{g}_{I,1,t_2}(t_1,\xi_2(t_2)) - \xi_2(t_2)| \le f(\mathrm{m}(t_1,t_2))$. Since $\mathrm{m} \ge p_0$ on $(-\infty, T_1(J_1)] \times (-\infty, T_2(J_2)]$, $|\widetilde{g}_{I,1,t_2}(t_1,\xi_2(t_2)) - \xi_2(t_2)|$ is uniformly bounded by $f(p_0)$. From (5.8) and that $T_2(J_2) \le C_{J_2}$, we see that $\widetilde{g}_{I,1,t_2}(t_1, \xi_2(t_2)) - \xi_2(t_2) \to 0$ as $t_1 \to -\infty$, uniformly in $t_2 \in (-\infty, T_2(J_2)]$.

Let J be a Jordan curve separating J_1 and $J_{I,2}$. Let $p_1 = \mathrm{mod}(J, J_1)$ and $p_2 = \mathrm{mod}(J, J_{I,2})$. Let $\widetilde{J} = (e^i)^{-1}(J)$. Let $h_m = \inf\{\mathrm{Im}\, z : z \in \widetilde{J}\}$ and $h_M = \sup\{\mathrm{Im}\, z : z \in \widetilde{J}\}$. Then both h_m and h_M are finite. For $j = 1, 2$, there is $h_j > 0$ depending only on p_j, such that, if $K \subset \mathbb{D}$ is an interior hull with $0 \in K$ and $\mathrm{mod}(\mathbb{D}\backslash K) \ge p_j$, then $K \subset \{|z| \le e^{-h_j}\}$. If $t_1 \le T_1(J_1)$, then J_1 disconnects J from $K_1(t_1)$, so $\mathrm{mod}(J, K_1(t_1)) \ge p_1$. Since $\Omega(J, K_1(t_1))$ is mapped by $g_1(t_1, \cdot)$ conformally onto $\Omega(g_1(t_1, J), \mathbb{T}) \subset \mathbb{D}$, $\mathrm{mod}(g_1(t_1, J), \mathbb{T}) \ge p_1$. Since $g_1(t_1, J)$ surrounds 0, $g_1(t_1, J) \subset \{|z| \le e^{-h_1}\}$. Since $\widetilde{g}_1(t_1, \widetilde{J}) = (e^i)^{-1}(g_1(t_1, J))$, $\widetilde{g}_1(t_1, \widetilde{J}) \subset \{\mathrm{Im}\, z \ge h_1\}$. Similarly, if $t_2 \le T_2(J_2)$, then $\widetilde{g}_{I,2}(t_2, \widetilde{J}) \subset \{\mathrm{Im}\, z \le -h_2\}$. If $t_1 \le T_1(J_1)$ and $t_2 \le T_2(J_2)$, then $g_{1,t_2}(t_1, \cdot) \circ g_{I,2}(t_2, \cdot)$ maps $\mathbb{C}\backslash K_1(t_1)\backslash K_{I,2}(t_2)$ conformally onto $\mathbb{A}_{\mathrm{m}}$. A similar argument shows that the image of J under this map lies in $\{e^{-\mathrm{m}+h_2} \le |z| \le e^{-h_1}\}$. Thus, $\widetilde{g}_{1,t_2}(t_1, \widetilde{g}_{I,2}(t_2, \widetilde{J})) \subset \{h_1 \le \mathrm{Im}\, z \le \mathrm{m} - h_2\}$, if $t_1 \le T_1(J_1)$ and $t_2 \le T_2(J_2)$.

Let $z_0 \in \mathbb{C}\backslash \widetilde{K}_1(t_1)\backslash \widetilde{K}_{I,2}(t_2)$, $w_1 = \widetilde{g}_1(t_1, z_0)$, $w_2 = \widetilde{g}_{I,2}(t_2, z_0)$, and $w_3 = \widetilde{g}_{1,t_2}(t_1, w_2)$. From (5.2), (5.3), and (5.4) we see that

$$|w_1 - (z_0 - it_1)| \le 4e^{t_1 - \mathrm{Im}\, z_0} \le 1/2, \quad \text{if } t_1 \le \mathrm{Im}\, z_0 - \ln(8); \tag{5.16}$$

$$|w_2 - (z_0 + it_2)| \le 4e^{t_2 + \mathrm{Im}\, z_0} \le 1/2, \quad \text{if } t_2 \le -\mathrm{Im}\, z_0 - \ln(8); \tag{5.17}$$

$$|w_3 - (w_2 + i\,\mathrm{m})| \le 10e^{-\mathrm{m} - \mathrm{Im}\, w_2} < 1, \quad \text{if } \mathrm{Im}\, w_2 + \mathrm{m} \ge \ln(13). \tag{5.18}$$

Now let $z_0 \in \widetilde{J}$. From the prior paragraph, $\mathrm{Im}\, \widetilde{g}_1(s, z_0) \ge h_1$ for $s \le t_1$, $\mathrm{Im}\, \widetilde{g}_{I,2}(s, z_0) \le -h_2$ for $s \le t_2$, and $\mathrm{m}(s, t_2) - h_2 \ge \widetilde{g}_{1,t_2}(s, w_2) \ge h_1$ for $s \le t_1$. From (5.1) we have $|\partial_t \widetilde{g}_1(s, z_0) + i| \le \frac{2}{e^{h_1}-1}$, for $s \le t_1$. Similarly, $|\partial_t \widetilde{g}_{I,2}(s, z_0) - i| \le \frac{2}{e^{h_2}-1}$ for $s \le t_2$. If $t_1 \le \mathrm{Im}\, z_0 - \ln(8)$, then from (5.16), $|w_1 - (z_0 - it_1)| \le 1/2$. If $t_1 > \mathrm{Im}\, z_0 - \ln(8)$, we let $t_1' = \mathrm{Im}\, z_0 - \ln(8)$, and $w_1' = \widetilde{g}_1(t_1', z_0)$. Then we have $|w_1' - (z_0 - it_1')| \le 1/2$. From the bound of $|\partial_t \widetilde{g}_1(s, z_0) + i|$, we see that

$$|(w_1 + it_1) - (w_1' + it_1')| \le \frac{2(t_1 - t_1')}{e^{h_1} - 1} \le \frac{2C_{J_1} - 2(\mathrm{Im}\, z_0 - \ln(8))}{e^{h_1} - 1} \le \frac{2C_{J_1} + 2\ln(8) - 2h_m}{e^{h_1} - 1}.$$

Let $A_1 = \frac{1}{2} + \max\left\{0, \frac{2C_{J_1}+2\ln(8)-2h_m}{e^{h_1}-1}\right\}$. Then in all cases we have

$$|w_1 - (z_0 - it_1)| \le A_1. \tag{5.19}$$

Similarly, let $A_2 = \frac{1}{2} + \max\left\{0, \frac{2C_{J_2}+2\ln(8)+2h_M}{e^{h_2}-1}\right\}$. Then we always have

$$|w_2 - (z_0 + it_2)| \le A_2. \tag{5.20}$$

Since $\mathrm{Im}\, z_0 \ge h_m$, we have

$$t_2 - \mathrm{Im}\, w_2 \le A_2 - \mathrm{Im}\, z_0 \le A_2 - h_m. \tag{5.21}$$

If $\operatorname{Im} w_2 + \mathrm{m}(t_1, t_2) \ge \ln(13)$, from (5.18), we have $|w_3 - (w_2 + i\,\mathrm{m}(t_1, t_2))| < 1$. Now suppose that $\operatorname{Im} w_2 + \mathrm{m}(t_1, t_2) < \ln(13)$. We may choose $t_1' < t_1$ such that $\operatorname{Im} w_2 + \mathrm{m}(t_1', t_2) = \ln(13)$. Let $w_3' = \widetilde{g}_{1,t_2}(t_1', w_2)$. Then we have $|w_3' - (w_2 + i\,\mathrm{m}(t_1', t_2))| < 1$. For $s \le t_1$, since $h_1 \le \operatorname{Im} \widetilde{g}_{1,t_2}(s, w_2) \le \mathrm{m}(s, t_2) - h_2$, from Lemma 2.1 we have

$$|\mathbf{H}_I(\mathrm{m}(s, t_2), i\,\mathrm{m}(s, t_2) - \widetilde{g}_{1,t_2}(s, w_2) + \xi_{1,t_2}(s))| \le \frac{4e^{-h_1}}{(1 - e^{-h_1})^3}.$$

Since $\mathbf{H}_I(\mathrm{m}, z) + i = \mathbf{H}_I(\mathrm{m}, z - i\,\mathrm{m}) = -\mathbf{H}_I(\mathrm{m}, i\,\mathrm{m} - z)$, we have

$$|\mathbf{H}(\mathrm{m}(s, t_2), \widetilde{g}_{1,t_2}(s, w_2) - \xi_{1,t_2}(s)) + i| \le \frac{4e^{-h_1}}{(1 - e^{-h_1})^3}, \quad \text{if } s \le t_1.$$

Let $C_0 = \frac{4e^{-h_1}}{(1-e^{-h_1})^3}$. From (4.18), (4.20), (5.8), (5.21), and the above inequality, we have

$$\begin{aligned} |(w_3 - i\,\mathrm{m}(t_1, t_2)) - (w_3' - i\,\mathrm{m}(t_1', t_2))| &\le C_0(\mathrm{m}(t_1', t_2) - \mathrm{m}(t_1, t_2)) \\ &\le C_0(\ln(13) - \operatorname{Im} w_2 + t_1 + t_2 + \ln(16)) \\ &\le C_0(\ln(13) + \ln(16) + C_{J_1} + A_2 - h_m). \end{aligned}$$

Let $A_3 = 1 + \max\{0, C_0(\ln(13) + \ln(16) + C_{J_1} + A_2 - h_m)\}$. Then $|w_3 - (w_2 + i\,\mathrm{m})| \le A_3$ always holds, which together with (5.19) and (5.20) implies that, for any $t_1 \le T_1(J_1)$ and $t_2 \le T_2(J_2)$,

$$|\widetilde{G}_{I,2,t_2}(t_1, w_1) - w_1 - i(\mathrm{m} + t_1 + t_2)| \le A_1 + A_2 + A_3, \quad w_1 \in \widetilde{g}_1(t_1, \widetilde{J}). \tag{5.22}$$

Now $\widetilde{g}_1(t_1, \widetilde{J})$ is a curve with period 2π above $\mathbb{R}$, the function $w \mapsto \widetilde{G}_{I,2,t_2}(t_1, w) - w$ has period 2π, is analytic in $\Omega(\widetilde{g}_1(t_1, \widetilde{J}), \mathbb{R})$, and its imaginary part vanishes on $\mathbb{R}$. Applying the maximum principle to the real part of this function, and using (5.22), we conclude that

$$|\widetilde{G}_{I,2,t_2}(t_1, \xi_1(t_1)) - \xi_1(t_1)| \le A_1 + A_2 + A_3, \quad \text{if } t_1 \le T_1(J_1) \text{ and } t_2 \le T_2(J_2).$$

This together with (5.15) and the estimation of $|\widetilde{g}_{I,1,t_2}(t_1, \xi_2(t_2)) - \xi_2(t_2)|$ implies that $|X_1 - \xi_1 + \xi_2|$ is uniformly bounded on $(-\infty, T_1(J_1)] \times (-\infty, T_2(J_2)]$.

Since $G_{I,2,t_2}(t_1, \cdot)$ maps $\mathbb{T}$ onto $\mathbb{T}$, and is conformal in the domain that contains the region between $g_1(t_1, J)$ and $\mathbb{T}$, there must exist $z_1 \in g_1(t_1, J)$ such that $|G_{I,2,t_2}(t_1, z_1)| = |z_1|$. Choose $w_1 \in \widetilde{g}_1(t_1, \widetilde{J})$ such that $e^i(w_1) = z_1$. Then $\operatorname{Im} \widetilde{G}_{I,2,t_2}(t_1, w_1) = \operatorname{Im} w_1$. From (5.22) we get $|\mathrm{m} + t_1 + t_2| \le A_1 + A_2 + A_3$, if $t_1 \le T_1(J_1)$ and $t_2 \le T_2(J_2)$, which finishes the proof of (i).

Now suppose $t_1 + t_2 \le -1 - 2\ln(13) - 2\ln(16)$ and $\operatorname{Im} z_0 = \frac{t_1 - t_2}{2}$. Then

$$\operatorname{Im} z_0 - t_1 = -\operatorname{Im} z_0 - t_2 = -\frac{t_1 + t_2}{2} \ge \frac{1}{2} + \ln(13) + \ln(16) \ge \ln(8).$$

Since $\widetilde{K}_1(t_1) \subset \{\mathrm{Im}\, z \le \ln(4) + \ln(t_1)\}$ and $\widetilde{K}_{I,2}(t_2) \subset \{\mathrm{Im}\, z \ge -\ln(t_2) - \ln(4)\}$, we have $z_0 \in \mathbb{C} \backslash \widetilde{K}_1(t_1) \backslash \widetilde{K}_{I,2}(t_2)$. From (5.16) and (5.17) we have

$$|w_1 - (z_0 - it_1)|, |w_2 - (z_0 + it_2)| \le 4e^{\frac{t_1+t_2}{2}} \le 1/2. \tag{5.23}$$

From (5.8), (5.23), and the upper bound of $t_1 + t_2$, we have

$$\mathrm{Im}\, w_2 + \mathrm{m} \ge \mathrm{Im}\, z_0 + t_2 - \frac{1}{2} - t_1 - t_2 - \ln(16) = -\frac{t_1 + t_2 + 1}{2} - \ln(16) \ge \ln(13).$$

Thus, from (5.18) and the above inequality we have

$$|\widetilde{g}_{1,t_2}(t_1, w_2) - (w_2 + i\,\mathrm{m})| \le 10e^{-\mathrm{m} - \mathrm{Im}\, w_2} \le 264e^{\frac{t_1+t_2}{2}}. \tag{5.24}$$

From (4.13), (5.23), and (5.24) we see that if $t_1 + t_2 \le -1 - 2\ln(13) - 2\ln(16)$, then

$$|\widetilde{G}_{I,2,t_2}(t_1, w_1) - w_1 - i(\mathrm{m} + t_1 + t_2)| \le 272e^{\frac{t_1+t_2}{2}}, \quad w_1 \in \widetilde{g}_1(t_1, \mathbb{R}_{(t_1-t_2)/2}).$$

The argument between (5.22) and the end of part (i) can be used here to show that, if $t_1 + t_2 \le -1 - 2\ln(13) - 2\ln(16)$, then $|\widetilde{G}_{I,2,t_2}(t_1, \xi_1(t_1)) - \xi_1(t_1)| \le 272e^{\frac{t_1+t_2}{2}}$ and $|\,\mathrm{m} + t_1 + t_2| \le 272e^{\frac{t_1+t_2}{2}}$. These inequalities together with the uniform limit of $\widetilde{g}_{I,1,t_2}(t_1, \xi_2(t_2)) - \xi_2(t_2)$ and the fact that $T_2(J_2) \le C_{J_2}$ imply that (ii) hold for $j = 1$ and $k = 2$. Interchanging t_1 and t_2, we find that $\mathrm{m} + t_1 + t_2 \to 0$ and $X_2 - \xi_2 + \xi_1 \to 0$ as $t_2 \to 0$, uniformly in $t_1 \in (-\infty, T_1(J_1)]$. From (4.29) we see that $X_2 - \xi_2 + \xi_1 = -(X_1 - \xi_1 + \xi_2)$, so we have $X_1 - \xi_1 + \xi_2 \to 0$ as $t_2 \to 0$, uniformly in $t_1 \in (-\infty, T_1(J_1)]$. This completes the proof of part (ii). □

Let $\widehat{\mathcal{D}} = \mathcal{D} \cup \{(t_1, -\infty) : t_1 \in [-\infty, \infty)\} \cup \{(-\infty, t_2) : t_2 \in [-\infty, \infty)\}$, and extend M to $\widehat{\mathcal{D}}$ such that $M = 1$ if t_1 or t_2 equals $-\infty$. From Proposition 5.1, we see that M is positive and continuous on $\widehat{\mathcal{D}}$. So for any fixed $j \ne k \in \{1, 2\}$ and any $(\mathcal{F}_t^k)$-stopping time t_k which is uniformly bounded above, M is a local martingale in $t_j \in [-\infty, T_j(t_k))$.

5.4 Local coupling and global coupling

Let μ_j denote the distribution of (ξ_j), $j = 1, 2$. Let $\mu = \mu_1 \times \mu_2$. Then μ is the joint distribution of (ξ_1) and (ξ_2), since ξ_1 and ξ_2 are independent. Fix $(J_1, J_2) \in \mathrm{JP}$. From the local martingale property of M and Proposition 5.1, we have $\mathbf{E}_\mu[M(T_1(J_1), T_2(J_2))] = M(-\infty, -\infty) = 1$. Define ν_{J_1,J_2} by $d\nu_{J_1,J_2} = M(T_1(J_1), T_2(J_2))d\mu$. Then ν_{J_1,J_2} is a probability measure. Let ν_1 and ν_2 be the two marginal measures of ν_{J_1,J_2}. Then $d\nu_1/d\mu_1 = M(T_1(J_1), -\infty) = 1$ and $d\nu_2/d\mu_2 = M(-\infty, T_2(J_2)) = 1$, so $\nu_j = \mu_j$, $j = 1, 2$. Suppose temporarily that the distribution of (ξ_1, ξ_2) is ν_{J_1,J_2} instead of μ. Then the distribution of each (ξ_j) is still μ_j.

We may now use the argument in Sect. 4.4 with a few changes. Here $M(t_1, t_2)$ satisfies (5.11) instead of (4.47); $\xi_j(t_j)$ does not satisfy (4.39), but is a pre-($\mathbb{T}$; κ)-Brownian motion with drift $s_j \cdot t$. The traditional Girsanov theorem needs to be modified to work for the current setting. Eventually, we can conclude that, under the probability measure ν_{J_1,J_2}, for any $j \neq k \in \{1, 2\}$, if t_k is a fixed $(\mathcal{F}^k_t)$-stopping time with $t_k \le T_k(J_k)$, and $g_k(t, \cdot)$, $-\infty < t < \infty$, are the inverted whole-plane Loewner maps driven by ξ_k, then conditioned on $\mathcal{F}^k_{t_k}$, after a time-change, $g_k(t_k, K_{I,j}(t_j))$, $-\infty < t_j \le T_j(J_j)$, is a partial disc SLE($\kappa, \Lambda_j$) process in $\mathbb{D}$ started from 0 with marked point $e^i(\xi_k(t_k))$.

The proof of Theorem 5.1 can be now completed by applying the coupling technique.

6 Partial differential equations

With Theorem 5.1 at hand, to prove the main theorem we need to find particular solutions to (4.1) that satisfy certain properties. This section serves this purpose. From Lemma 4.1 we see that solving (4.1) is equivalent to solving (4.5) with $\sigma = \frac{4}{\kappa} - 1$. Throughout this section, we assume that $\kappa > 0$ and $\sigma \in [0, \frac{4}{\kappa})$, and will find solutions to (4.5) in these cases. In particular, we will obtain solutions to (4.1) when $\kappa \in (0, 4]$.

The solutions to (4.5) is obtained by construction. We will transform (4.5) into a similar PDE (6.26), where $\mathbf{H}_I$ is replaced by $\widehat{\mathbf{H}}_I$. We know that as $t \to \infty$, $\widehat{\mathbf{H}}_I(t, \cdot) \to \coth_2$, and PDE (6.26) tends to another PDE (6.27), which has a simple solution $\widehat{\Psi}_\infty$ given by (6.28). Then we let $\widehat{\Psi}_q = \widehat{\Psi}/\widehat{\Psi}_\infty$, and find that $\widehat{\Psi}$ solves (6.26) if and only if $\widehat{\Psi}_q$ solves PDE (6.29). A formal solution to (6.29) is expressed by a Feynman–Kac formula (6.30), which involves diffusion processes. Such diffusion processes are introduced and studied in Sect. 6.1. In Sect. 6.2 we describe how close is $\widehat{\mathbf{H}}_I(t, \cdot)$ to $\coth_2$ when t is big. In Sect. 6.3 we transform the PDE (4.5) for Ψ into the PDE (6.29) for $\widehat{\Psi}_q$, and give an intuitive reason why the formula (6.30) gives a solution to (6.29). In Sect. 6.4 we prove that the $\widehat{\Psi}_q$ given by (6.30) is smooth, and solves (6.29). So we obtain a solution Γ to (4.1). However, such Γ does not satisfy (4.2). For this purpose, note that (4.1) is a linear PDE, and $\mathbf{H}_I$ has period 2π, so any translation of Γ by an integer multiple of 2π also solves (4.1). The solutions to (4.1) which also satisfy (4.2) will be obtained by summing over all translations of Γ with suitable weights.

The following symbols will be used in this section. For any $n, j \in \mathbb{N}$, we call an j-tuple $\lambda = (\lambda_1, \dots, \lambda_j) \in \mathbb{N}^j$ a partition of n if $\lambda_1 \ge \dots \ge \lambda_j$ and $\sum_{k=1}^j \lambda_k = n$. The length of such partition is denoted by $l(\lambda) = j$. Let $\mathcal{P}_n$ denote the set of all partitions of n. For example, (n) is the only element in $\mathcal{P}_n$ with length 1. Let $\mathcal{P}_\mathbb{N} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ denote the set of all partitions.

6.1 Diffusion processes

Fix $\tau \le 0$. For $x \in \mathbb{R}$, let $u(t, x)$, $t \ge 0$, be the solution to

$$\partial_t u(t, x) = \tau \tanh_2(u(t, x) + \sqrt{\kappa} B(t)); \quad u(0, x) = x. \tag{6.1}$$

Then $X_x(t):=u(t,x)+\sqrt{\kappa}B(t)$ satisfies the SDE

$$dX_x(t)=\sqrt{\kappa}dB(t)+\tau\tanh_2(X_x(t))dt,\quad X_x(0)=x. \tag{6.2}$$

Lemma 6.1 *For any $x\in\mathbb{R}$, we have a.s. $\int_0^\infty \tanh_2'(X_x(t))dt=\infty$ and*

$$\limsup_{t\to\infty} X_x(t)=+\infty,\qquad \liminf_{t\to\infty} X_x(t)=-\infty. \tag{6.3}$$

Proof Fix $x\in\mathbb{R}$. Let $X(t)=X_x(t)$. Define $f(t)=\int_0^t \cosh_2(s)^{-\frac{4}{\kappa}\tau}ds,\quad t\in\mathbb{R}$. Then f is a differentiable increasing odd function and satisfies $\frac{\kappa}{2}f''+\tau\tanh_2 f'=0$. Let $Y(t)=f(X(t))$. From (6.2) and Itô's formula, we have $dY(t)=f'(X(t))\sqrt{\kappa}dB(t)$. Define a time-change function $u(t)=\int_0^t \kappa f'(X(s))^2ds$. Since $\tau\le 0$, $f'(t)\ge 1$, $t\in\mathbb{R}$. Thus, $u(t)\ge t$ for all $t\in\mathbb{R}$. So u maps $[0,\infty)$ onto $[0,\infty)$, and $Y(u^{-1}(t))$, $0\le t<\infty$, has the distribution of a Brownian motion. Thus, (6.3) holds with X replaced by Y, which then implies (6.3). Since X is recurrent, and $\tanh_2'>0$ on $\mathbb{R}$, we immediately have a.s. $\int_0^\infty \tanh_2'(X_x(t))dt=\infty$. □

Lemma 6.2 *For any $b,c>0$ and $x\in\mathbb{R}$,*

$$\mathbb{P}[\exists t\ge 0, |X_x(t)|>ct+b]\le 2e^{\frac{2c}{\kappa}(|x|-b)}. \tag{6.4}$$

Proof First, it is well known that (6.4) holds with $X_x(t)$ replaced by $x+\sqrt{\kappa}B(t)$. So it suffices to show that $(|X_x(t)|)$ is bounded above by a process that has the distribution of $(|x+\sqrt{\kappa}B(t)|)$. This can be proved by using Theorem 4.1 in [28]. Here we give a direct proof.

Let $Y(t)=|X_x(t)|$ From (6.2) and Tanaka–Itô's formula, we have

$$Y(t)=|x|+\sqrt{\kappa}B_0(t)+\frac{\kappa}{2}\tau\int_0^t \tanh_2(Y(s))ds+L(t),\quad t\ge 0, \tag{6.5}$$

where $B_0(t)$ is a Brownian motion, $L(t)$ is a non-decreasing function, which satisfies $L(0)=0$ and is constant on every interval of $\{Y(t)>0\}$.

Fix $t_0\ge 0$. There is $t_0'\in[0,t_0]$ such that $L(t)$ is constant on $[t_0',t_0]$. We may assume t_0' is the smallest such number. There are two cases. Case 1: $t_0'=0$. Then $L(t_0)=L(t_0')=L(0)=0$. Since $\tau\le 0$, from (6.5), $Y(t_0)\le|x|+\sqrt{\kappa}B_0(t)$. Case 2: $t_0'>0$. Then $Y(t_0')=0$. Since $\tau\le 0$, from (6.5),

$$Y(t_0)-|x|-\sqrt{\kappa}B_0(t_0)\le Y(t_0')-|x|-\sqrt{\kappa}B_0(t_0')=-|x|-\sqrt{\kappa}B_0(t_0').$$

Thus, in either case, we have

$$Y(t_0)\le|x|+\sqrt{\kappa}B_0(t_0)+\max\left\{0,\sup_{0\le s\le t_0}\{-|x|-\sqrt{\kappa}B_0(s)\}\right\}.$$

The RHS of the above inequality defines a process that has the distribution of $|x + \sqrt{\kappa}B(t_0)|$, $t_0 \ge 0$ (c.f. Chapter VI, Section 2 of [23]), so the proof is completed. □

Lemma 6.3 *There are $C_n > 0$, $n \in \mathbb{N}$, with $C_1 = 1$, such that*

$$\left|\tanh_2^{(n)}(x)\right| \le C_n \tanh_2'(x) \le \frac{C_n}{2}, \quad x \in \mathbb{R},\ n \in \mathbb{N}.$$

Proof Note that $\tanh_2'(x) = \frac{1}{2}\cosh_2^{-2}(x) \in (0, 1/2]$. So the second "$\le$" holds. By induction, one can prove that for every n, there are $a_j^{(n)} \in \mathbb{R}$, $0 \le j \le n-1$, such that

$$\tanh_2^{(n)}(x) = \sum_{j=0}^{n-1} a_j^{(n)} \cosh_2^{-2-j}(x)\sinh_2^j(x) = \sum_{j=0}^{n-1} a_j^{(n)} \cosh_2^{-2}(x)\tanh_2^j(x).$$

Since $\left|\tanh_2^j(x)\right| \le 1$ and $\cosh_2^{-2} = 2\tanh_2'$, we may choose $C_n = 2\sum_j \left|a_j^{(n)}\right|$. □

Lemma 6.4 *For every $m \in \mathbb{N}$, there is a polynomial $P_m(t)$ of degree $m-1$ such that for any $t > 0$ and $x \in \mathbb{R}$, $|\frac{\partial^m}{\partial x^m} X_x(t)| \le P_m(t)$.*

Proof Since $X_x(t) = u(t, x) + \sqrt{\kappa}B(t)$, $\frac{\partial^n}{\partial x^n} X_x(t) = u^{(n)}(t, x)$. It suffices to show that for every $m \in \mathbb{N}$, there is some polynomial $P_m(t)$ of degree $m-1$, such that

$$|u^{(m)}(t, x)| \le P_m(t), \quad t > 0,\ x \in \mathbb{R}. \tag{6.6}$$

Let $f_x(t) = \tau \tanh_2'(X_x(t))$. Since $\tau \le 0$ and $\tanh_2' > 0$, $f_x(t) \le 0$. Differentiating (6.1) w.r.t. x and using $u'(0, x) = 1$, we get $u'(t, x) = \exp(\int_0^t f_x(s)ds) \in (0, 1]$. Thus, (6.6) holds in the case $n = 1$ with $P_1(t) \equiv 1$.

Let $n \in \mathbb{N}$, $n \ge 2$. Suppose that (6.6) holds for any $m \le n-1$. Differentiating (6.1) n times, by induction we find that there are $b_n(\lambda) \in \mathbb{R}$ for $\lambda \in \mathcal{P}_n$ with $b_n((n)) = \tau$ such that

$$\partial_t u^{(n)}(t, x) = \sum_{\lambda \in \mathcal{P}_n} b_n(\lambda)\tanh_2^{(l(\lambda))}(X_x(t)) \prod_{k=1}^{l(\lambda)} u^{(\lambda_k)}(t, x), \quad u^{(n)}(0, x) = 0. \tag{6.7}$$

Observe that the term $u^{(n)}(t, x)$ appears only once in (6.7), i.e, in the case $\lambda = ((n))$, and the coefficient is $\tau \tanh_2'(X_x(t)) = f_x(t)$. From Lemma 6.3 and induction hypothesis, there is a polynomial $g_x(t)$ of degree $n-2$ such that

$$\partial_t u^{(n)}(t, x) = f_x(t)u^{(n)}(t, x) + g_x(t), \quad u^{(n)}(0, x) = 0.$$

Solving this inequality using the fact that $f_x(t) \le 0$, we can conclude that (6.6) holds in the case $m = n$, which finishes the proof. □

6.2 Some estimations

We will need some estimations about the limits of $\widehat{\mathbf{H}}_I - \tanh_2$ as $t \to \infty$. Let

$$\widehat{\mathbf{H}}_{I,q}(t, z) = \widehat{\mathbf{H}}_I(t, z) - \tanh_2(z). \tag{6.8}$$

From (2.12) we have

$$\widehat{\mathbf{H}}'_{I,q}(t, x) = \sum_{2|n\neq 0} \tanh_2'(x - nt) = \sum_{2|n\neq 0} \frac{1}{2}\cosh_2^{-2}(x - nt) > 0. \tag{6.9}$$

Lemma 6.5 *Let C_n, $n \in \mathbb{N}$, be as in Lemma 6.3. Note that $C_1 = 1$. Then*

$$|\widehat{\mathbf{H}}_{I,q}(t, x)| \le \frac{|x|}{t} + 3 + \frac{2e^{-t}}{1 - e^{-2t}}, \quad t > 0,\ x \in \mathbb{R}. \tag{6.10}$$

$$\left|\widehat{\mathbf{H}}^{(n)}_{I,q}(t, x)\right| \le C_n\left(\frac{1}{2} + \frac{4e^{-t}}{1 - e^{-2t}}\right), \quad t > 0,\ x \in \mathbb{R},\ n \in \mathbb{N}. \tag{6.11}$$

Moreover, for any $c > 0$,

$$|\widehat{\mathbf{H}}_{I,q}(t, x)| \le \frac{2e^{(c-2)t}}{1 - e^{-2t}}, \quad \text{if } t > 0,\ x \in \mathbb{R},\ |x| \le ct. \tag{6.12}$$

$$\left|\widehat{\mathbf{H}}^{(n)}_{I,q}(t, x)\right| \le C_n\frac{4e^{(c-2)t}}{1 - e^{-2t}}, \quad \text{if} \quad t > 0,\ x \in \mathbb{R},\ |x| \le ct,\ n \in \mathbb{N}. \tag{6.13}$$

Proof We first show (6.12). From (2.12) and (6.8) we have

$$\begin{aligned}\widehat{\mathbf{H}}_{I,q}(t, x) &= \sum_{m=1}^{\infty}(\tanh_2(x - 2mt) + \tanh_2(x + 2mt)) \\ &= \sum_{m=1}^{\infty}\left(-\frac{e^{2mt} - e^x}{e^{2mt} + e^x} + \frac{e^{2mt} - e^{-x}}{e^{2mt} + e^{-x}}\right) = \sum_{m=1}^{\infty}\frac{2(e^x - e^{-x})}{e^{2mt} + e^{-2mt} + e^x + e^{-x}}.\end{aligned}$$

Thus,

$$|\widehat{\mathbf{H}}_{I,q}(t, x)| \le \sum_{m=1}^{\infty}\frac{2e^{|x|}}{e^{2mt}} = \frac{2e^{|x|-2t}}{1 - e^{-2t}}. \tag{6.14}$$

Then (6.12) is a direct consequence of this inequality.

Secondly, we show (6.10). Since $|\tanh_2(x)| \le 1$, from (6.8) it suffices to show

$$|\widehat{\mathbf{H}}_I(t, x)| \le \frac{|x|}{t} + 2 + \frac{2e^{-t}}{1 - e^{-2t}}. \tag{6.15}$$

We first consider the case $|x| \le t$. From (6.14) we have

$$|\widehat{\mathbf{H}}_I(t,x)| \le |\tanh_2(x)| + |\widehat{\mathbf{H}}_{I,q}(t,x)| \le 1 + \frac{2e^{|x|-2t}}{1-e^{-2t}} \le 1 + \frac{2e^{-t}}{1-e^{-2t}}. \quad (6.16)$$

Thus, (6.15) holds in this case.

Then we consider the case $|x| \ge t$. There exists $m \in \mathbb{N}$ such that $(2m-1)t \le |x| \le (2m+1)t$. Since $\widehat{\mathbf{H}}_I$ is odd, we only need to consider the case that $(2m-1)t \le x \le (2m+1)t$. Let $x_0 = x - 2mt$. Then $|x_0| \le t$. From (2.10) we have $\widehat{\mathbf{H}}_I(t,x) = 2m + \widehat{\mathbf{H}}_I(t,x_0)$. From (6.16) with $x = x_0$ we have

$$|\widehat{\mathbf{H}}_I(t,x)| \le 2m + |\widehat{\mathbf{H}}_I(t,x_0)| \le 2m + 1 + \frac{2e^{-t}}{1-e^{-2t}} \le \frac{|x|}{t} + 2 + \frac{2e^{-t}}{1-e^{-2t}},$$

where the last inequality uses $\frac{|x|}{t} \ge 2m-1$. So we have (6.15) and (6.10).

Now we prove (6.11) and (6.13). From (6.9) we have

$$\begin{aligned} 0 < \widehat{\mathbf{H}}'_{I,q}(t,x) &= \sum_{2|n\ne 0} \frac{1}{2}\cosh_2^{-2}(|nt-x|) \le \sum_{2|n\ne 0} \frac{1}{2}\cosh_2^{-2}(|n|t-|x|) \\ &= \sum_{m=1}^{\infty} \cosh_2^{-2}(2mt-|x|) \le 4\sum_{m=1}^{\infty} e^{|x|-2mt} = \frac{4e^{|x|-2t}}{1-e^{-2t}}. \end{aligned} \quad (6.17)$$

which implies (6.13) in the case $n=1$. From (6.17) we have $\widehat{\mathbf{H}}'_I(t,x) < \frac{1}{2} + \frac{4e^{-t}}{1-e^{-2t}}$ if $|x| \le t$. Since $\widehat{\mathbf{H}}'_I$ has period $2t$, this inequality holds for all $x \in \mathbb{R}$. Since $\widehat{\mathbf{H}}'_{I,q} < \widehat{\mathbf{H}}'_I$, (6.11) holds in the case $n=1$. From (6.9) and Lemma 6.3 we have $|\widehat{\mathbf{H}}^{(n)}_{I,q}(t,x)| \le C_n \widehat{\mathbf{H}}'_{I,q}(t,x)$. So (6.11) and (6.13) in the case $n \ge 2$ follow from those in the case $n=1$. □

Lemma 6.6 *For every $n \in \mathbb{N} \cup \{0\}$, there is a constant $D_n > 0$ such that for any $j \in \{1,2\}$, $t > 0$, and $x \in \mathbb{R}$,*

$$\left|\partial_t^j \widehat{\mathbf{H}}^{(n)}_{I,q}(t,x)\right| \le D_n \left(\frac{|x|}{t} + 3 + \frac{2e^{-t}}{1-e^{-2t}}\right)^j \left(\frac{1}{2} + \frac{4e^{-t}}{1-e^{-2t}}\right). \quad (6.18)$$

Moreover, for any $n \in \mathbb{N} \cup \{0\}$ and $c > 0$, there is a constant $D_n > 0$, such that

$$\left|\partial_t^j \widehat{\mathbf{H}}^{(n)}_{I,q}(t,x)\right| \le D_n \left(\frac{2e^{(c-2)t}}{1-e^{-2t}}\right)^{j+1}, \quad \text{if } t > 0,\ x \in \mathbb{R},\ |x| \le ct; \quad (6.19)$$

Proof Let $A(t,x) = \frac{|x|}{t} + 3 + \frac{2e^{-t}}{1-e^{-2t}}$, $B(t,x) = \frac{1}{2} + \frac{4e^{-t}}{1-e^{-2t}}$, and $C_c(t,x) = \frac{2e^{(c-2)t}}{1-e^{-2t}}$. In this proof, by $X \lesssim Y$ we mean that there is a constant C such that $X \le CY$. Here C may depend on n if X depends on n. From (6.10), (6.11), (6.12), and (6.13), we see that

$$|\mathbf{H}_{I,q}(t,x)| \lesssim A(t,x), \qquad \left|\mathbf{H}_{I,q}^{(n)}(t,x)\right| \lesssim B(t,x) \lesssim A(t,x), \quad x\in\mathbb{R},\ n\in\mathbb{N}. \tag{6.20}$$

$$\left|\mathbf{H}_{I,q}^{(n)}(t,x)\right| \lesssim C_c(t,x), \quad \text{if } x\in\mathbb{R},\ |x|\le ct,\ n\in\mathbb{N}\cup\{0\}. \tag{6.21}$$

As $t\to\infty$, $\widehat{\mathbf{H}}_I \to \tanh_2$. Then (2.13) becomes $0=\tanh_2''+\tanh_2'\tanh_2$, which can be proved directly. From (2.13), (6.8), and the above equation, we get

$$\partial_t \widehat{\mathbf{H}}_{I,q} = \widehat{\mathbf{H}}_{I,q}'' + \widehat{\mathbf{H}}_{I,q}'\widehat{\mathbf{H}}_{I,q} + \tanh_2' \widehat{\mathbf{H}}_{I,q} + \widehat{\mathbf{H}}_{I,q}'\tanh_2 . \tag{6.22}$$

Then (6.18) and (6.19) in the case $j=1$ and $n=0$ follow from (6.20), (6.21), (6.22), and Lemma 6.3.

Differentiating (2.13) w.r.t. x twice, we get

$$\partial_t \widehat{\mathbf{H}}_I' = \widehat{\mathbf{H}}_I''' + \widehat{\mathbf{H}}_I''\widehat{\mathbf{H}}_I + (\widehat{\mathbf{H}}_I')^2.$$
$$\partial_t \widehat{\mathbf{H}}_I'' = \widehat{\mathbf{H}}_I^{(4)} + \widehat{\mathbf{H}}_I'''\widehat{\mathbf{H}}_I + 3\widehat{\mathbf{H}}_I''\widehat{\mathbf{H}}_I'.$$

Differentiating (2.13) w.r.t. t and using the above two displayed formulas, we obtain

$$\partial_t^2 \widehat{\mathbf{H}}_I = \widehat{\mathbf{H}}_I^{(4)} + 2\widehat{\mathbf{H}}_I'''\widehat{\mathbf{H}}_I + 4\widehat{\mathbf{H}}_I''\widehat{\mathbf{H}}_I' + \widehat{\mathbf{H}}_I''(\widehat{\mathbf{H}}_I)^2 + 2(\widehat{\mathbf{H}}_I')^2\widehat{\mathbf{H}}_I.$$

As $t\to\infty$, this equation tends to the following equation, which can also be checked directly.

$$0=\tanh_2^{(4)} + 2\tanh_2'''\tanh_2 + 4\tanh_2''\tanh_2' + \tanh_2''\tanh_2^2 + 2(\tanh_2')^2\tanh_2 .$$

From (6.8), and the above two equations, we compute

$$\begin{aligned}\partial_t^2 \widehat{\mathbf{H}}_{I,q} =\ & \widehat{\mathbf{H}}_{I,q}^{(4)} + 2\widehat{\mathbf{H}}_{I,q}'''\widehat{\mathbf{H}}_{I,q} + 2\tanh_2'''\widehat{\mathbf{H}}_{I,q}\\ &+2\widehat{\mathbf{H}}_{I,q}'''\widehat{\mathbf{H}}_{I,q} + 4\widehat{\mathbf{H}}_{I,q}''\widehat{\mathbf{H}}_{I,q}'\\ &+4\widehat{\mathbf{H}}_{I,q}''\tanh_2' + \tanh_2''(\widehat{\mathbf{H}}_{I,q})^2 + 2\widehat{\mathbf{H}}_{I,q}''\widehat{\mathbf{H}}_{I,q}\tanh_2 + 2\tanh_2''\widehat{\mathbf{H}}_{I,q}\tanh_2\\ &+\widehat{\mathbf{H}}_{I,q}''(\tanh_2)^2 + \widehat{\mathbf{H}}_{I,q}''(\widehat{\mathbf{H}}_{I,q})^2 + 2(\widehat{\mathbf{H}}_{I,q}')^2\widehat{\mathbf{H}}_{I,q} + 4\widehat{\mathbf{H}}_{I,q}'\tanh_2'\widehat{\mathbf{H}}_{I,q}\\ &+2(\tanh_2')^2\widehat{\mathbf{H}}_{I,q} + 2(\widehat{\mathbf{H}}_{I,q}')^2\tanh_2 + 4\widehat{\mathbf{H}}_{I,q}'\tanh_2'\tanh_2 .\end{aligned} \tag{6.23}$$

Then (6.18) and (6.19) in the case $j=2$ and $n=0$ follow from (6.20), (6.21), (6.23), and Lemma 6.3.

Differentiate (6.22) and (6.23) n times w.r.t. x. We see that $\partial_t \widehat{\mathbf{H}}_{I,q}^{(n)}$ can be expressed as a sum of finitely many terms, whose factors are $\mathbf{H}_{I,q}^{(k)}$ or $\tanh_2^{(k)}$, $k\in\mathbb{N}\cup\{0\}$. In every term, the factors of the kind $\mathbf{H}_{I,q}^{(k)}$ appear at most twice, and the factor $\mathbf{H}_{I,q}$ appears at most once. So we derive (6.18) and (6.19) in the case $j=1$ and $n\in\mathbb{N}$ from (6.20), (6.21), and Lemma 6.3. We see that $\partial_t^2\widehat{\mathbf{H}}_{I,q}^{(n)}$ can be expressed as a sum of finitely many terms, whose factors are constant, $\mathbf{H}_{I,q}^{(k)}$, or $\tanh_2^{(k)}$. In every term, the factors of the kind $\mathbf{H}_{I,q}^{(k)}$ appear at most three times, and the factor $\mathbf{H}_{I,q}$ appears at most twice. So we derive (6.18) and (6.19) in the case $j = 2$ and $n \in \mathbb{N}$ from (6.20), (6.21), and Lemma 6.3. □

6.3 Feynman–Kac expression

We begin with a lemma, which can be proved directly. Recall the definition of $\widehat{\mathbf{H}}_I$ in (2.9).

Lemma 6.7 *Let* Ψ *and* $\widehat{\Psi}$ *be functions defined on* $(0, \infty) \times \mathbb{R}$. *The following expressions are equivalent:*

$$\widehat{\Psi}(t, x) = e^{\frac{x^2}{2\kappa t}} \left(\frac{\pi}{t}\right)^{\sigma+\frac{1}{2}} \Psi\left(\frac{\pi^2}{t}, \quad \frac{\pi}{t}x\right). \tag{6.24}$$

$$\Psi(t, x) = e^{-\frac{x^2}{2\kappa t}} \left(\frac{\pi}{t}\right)^{\sigma+\frac{1}{2}} \widehat{\Psi}\left(\frac{\pi^2}{t}, \quad \frac{\pi}{t}x\right). \tag{6.25}$$

If the above two equalities hold, then Ψ *satisfies (4.5) if and only if* $\widehat{\Psi}$ *satisfies*

$$-\partial_t \widehat{\Psi} = \frac{\kappa}{2}\widehat{\Psi}'' + \sigma \widehat{\mathbf{H}}_I' \widehat{\Psi}. \tag{6.26}$$

As $t \to \infty$, $\widehat{\mathbf{H}}_I' \to \tanh_2'$, so (6.26) tends to

$$-\partial_t \widehat{\Psi}_\infty = \frac{\kappa}{2}\widehat{\Psi}_\infty'' + \sigma \tanh_2'(x)\widehat{\Psi}_\infty. \tag{6.27}$$

Let τ be the non-positive root of the equation $\frac{\tau^2}{2\kappa} = \frac{\tau}{4} + \frac{\sigma}{2}$, i.e., $\tau = \kappa/4 - \sqrt{\kappa^2/16 + \kappa\sigma}$. Then $\tau = \frac{\kappa}{2} - 2$ when $\sigma = \frac{4}{\kappa} - 1$. It is easy to check that (6.27) has a simple solution:

$$\widehat{\Psi}_\infty(t, x) = e^{-\frac{\tau^2 t}{2\kappa}} \cosh_2^{\frac{2}{\kappa}\tau}(x). \tag{6.28}$$

Recall the $\widehat{\mathbf{H}}_{I,q}$ defined in (6.8). The proof of the following lemma is straightforward.

Lemma 6.8 *Let* $\widehat{\Psi}$ *and* $\widehat{\Psi}_q$ *be defined on* $(0, \infty) \times \mathbb{R}$, *and satisfy* $\widehat{\Psi} = \widehat{\Psi}_\infty \widehat{\Psi}_q$, *where* Ψ_∞ *is defined by (6.28). Then* $\widehat{\Psi}$ *satisfies (6.26) if and only if* $\widehat{\Psi}_q$ *satisfies*

$$-\partial_t \widehat{\Psi}_q = \frac{\kappa}{2}\widehat{\Psi}_q'' + \tau \tanh_2 \widehat{\Psi}_q' + \sigma \widehat{\mathbf{H}}_{I,q}' \widehat{\Psi}_q. \tag{6.29}$$

Suppose $\widehat{\Psi}_q$ solves (6.29). Let $X_{x_0}(t)$ be as in (6.2). Fix $t_0 > 0$ and $x_0 \in \mathbb{R}$. Let

$$M(t) = \widehat{\Psi}_q(t_0 + t, X_{x_0}(t)) \exp\left(\sigma \int_0^t \widehat{\mathbf{H}}_{I,q}'(t_0 + s, X_{x_0}(s))ds\right).$$

From (6.2), (6.29), and Itô's formula, we see that $M(t)$ is a local martingale. If $M(t)$ is a martingale on $[0, \infty]$, and $\widehat{\Psi}_q \to 1$ as $t \to \infty$, then from $M_0 = \widehat{\Psi}_q(t_0, x_0)$ we have

$$\widehat{\Psi}_q(t_0, x_0) = \mathbf{E}\left[\exp\left(\sigma \int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t_0 + s, X_{x_0}(s))ds\right)\right]. \tag{6.30}$$

This Feynman–Kac formula holds under many additional assumptions. We do not try to prove it. Instead, we now define $\widehat{\Psi}_q$ by (6.30). We will prove that $\widehat{\Psi}_q$ is finite and differentiable, and solves (6.29).

6.4 Regularity

Fix $c_0 \in (1 + \frac{\kappa}{4}\sigma, 2)$. This is possible because $\sigma \in [0, \frac{4}{\kappa})$. Then we have

$$\exp\left(\frac{\sigma}{2(c_0 - 1)} - \frac{2}{\kappa}\right) < 1. \tag{6.31}$$

Throughout this subsection, we use C to denote a positive constant, which depends only on κ, σ, c_0, and could change between lines. The symbol $X \lesssim Y$ means that $X \le CY$ for some C. Let $\alpha(t) = \frac{4}{1-e^{-2t}}$. Then $t^{-1} + 1 \lesssim \alpha(t) \lesssim t^{-1} + 1$. For $m \in \mathbb{N} \cup \{0\}$, let $\mathcal{E}_m$ denote the event that $|X_x(s)| \le s + m$ for all $s \ge 0$. From (6.4) we have

$$\mathbb{P}[\mathcal{E}_m^c] \le 2e^{\frac{2}{\kappa}(|x|-m)}, \quad m \in \mathbb{N} \cup \{0\}. \tag{6.32}$$

Proposition 6.1 *$\widehat{\Psi}_q$ is finite and satisfies*

$$1 \le \widehat{\Psi}_q(t, x) \le \exp\left(C(t^{-1} + 1)e^{(c_0-2)t}\right)(1 + Ce^{\frac{2}{\kappa}|x| - \frac{2}{\kappa}c_0 t}). \tag{6.33}$$

Proof Fix $t > 0$ and $x \in \mathbb{R}$. Assume that $\mathcal{E}_m$ occurs for some $m \in \mathbb{N} \cup \{0\}$. If $s \ge \frac{m - c_0 t}{c_0 - 1}$ then $|X_x(s)| \le s + m \le c_0(s + t)$, so from (6.13) with $C_1 = 1$ we have

$$\widehat{\mathbf{H}}'_{I,q}(t + s, X_x(s)) \le \frac{4e^{(c_0-2)(s+t)}}{1 - e^{-2(s+t)}} \le \alpha(t)e^{(c_0-2)(s+t)}.$$

If $0 \le s \le \frac{m - c_0 t}{c_0 - 1}$, from $-1 \le c_0 - 2$ and (6.11) with $C_1 = 1$, we have

$$\widehat{\mathbf{H}}'_{I,q}(t + s, X_x(s)) < \frac{1}{2} + \frac{4e^{-(s+t)}}{1 - e^{-2(s+t)}} \le \frac{1}{2} + \alpha(t)e^{(c_0-2)(s+t)},$$

Since $c_0 - 2 < 0$, at the event $\mathcal{E}_m$,

$$\int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t + s, X_x(s))ds \le \frac{1}{2} \cdot \frac{(m - c_0 t) \vee 0}{c_0 - 1} + \frac{\alpha(t)e^{(c_0-2)t}}{2 - c_0}; \tag{6.34}$$

Let $H(t) = \exp(\sigma \int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t+s, X_x(s))ds)$. From (6.32) and (6.34) we have

$$\begin{aligned}\widehat{\Psi}_q(t,x) &= \mathbf{E}[1_{\mathcal{E}_{\lfloor c_0 t\rfloor}} H(t)] + \sum_{m=\lfloor c_0 t\rfloor}^{\infty} \mathbf{E}[1_{\mathcal{E}_{m+1}\setminus\mathcal{E}_m} H(t)] \\ &\le \exp\left(\frac{\sigma\alpha(t)e^{(c_0-2)t}}{2-c_0}\right) \\ &\quad + \sum_{m=\lfloor c_0 t\rfloor}^{\infty} 2e^{\frac{2}{\kappa}(|x|-m)} \exp\left(\frac{1}{2}\frac{\sigma(m+1-\lfloor c_0 t\rfloor)}{c_0-1} + \frac{\sigma\alpha(t)e^{(c_0-2)t}}{2-c_0}\right).\end{aligned} \tag{6.35}$$

Change index using $m = l + \lfloor c_0 t\rfloor$. The second term of the RHS of (6.35) equals

$$2\exp\left(\frac{2|x|}{\kappa} - \frac{2\lfloor c_0 t\rfloor}{\kappa} + \frac{\sigma}{2(c_0-1)} + \frac{\sigma\alpha(t)e^{(c_0-2)t}}{2-c_0}\right)\sum_{l=0}^{\infty}\exp\left(\frac{\sigma}{2(c_0-1)} - \frac{2}{\kappa}\right)^l. \tag{6.36}$$

From (6.31), the infinite sum is finite. Thus, from $\widehat{\mathbf{H}}'_{I,q} > 0$, $\sigma \ge 0$, and (6.35), we have

$$1 \le \widehat{\Psi}_q(t,x) \le \exp\left(\frac{\sigma\alpha(t)e^{(c_0-2)t}}{2-c_0}\right)\left(1 + Ce^{\frac{2|x|}{\kappa} - \frac{2c_0 t}{\kappa}}\right).$$

Then (6.33) follows from this formula and that $\alpha(t) \lesssim t^{-1} + 1$. □

Let $n \in \mathbb{N}$. Formally differentiate (6.30) n times w.r.t. x. If the differentiation commutes with the integration and expectation at every time, then we should have

$$\widehat{\Psi}_q^{(n)}(t,x) = \mathbf{E}\left[\exp\left(\sigma\int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t+s, X_x(s))ds\right)\cdot \mathbf{Q}_{0,n}(v_{0,k,\lambda}(t,x))\right], \tag{6.37}$$

where $\mathbf{Q}_{0,n}$ is a polynomial of degree $\le n$ without constant term in the following variables:

$$v_{0,k,\lambda}(t,x) := \int_0^\infty \widehat{\mathbf{H}}_{I,q}^{(k)}(t+s, X_x(s)) \prod_{r=1}^{l(\lambda)} \frac{\partial^{\lambda_r}}{\partial x^{\lambda_r}} X_x(s)ds, \quad k \in \mathbb{N},\ \lambda \in \mathcal{P}_{\mathbb{N}}. \tag{6.38}$$

With $\mathbf{Q}_{0,0} \equiv 1$, (6.37) becomes (6.30). Let $n \in \mathbb{N}\cup\{0\}$. Formally differentiate (6.37) w.r.t. t. If the differentiation commutes with the integration and expectation, then we should have

$$\partial_t \widehat{\Psi}_q^{(n)}(t,x) = \mathbf{E}\left[\exp\left(\sigma\int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t+s,X_x(s))ds\right)\cdot \mathbf{Q}_{1,n}(v_{0,k,\lambda},v_{1,k,\lambda})\right], \quad (6.39)$$

where $\mathbf{Q}_{1,n}$ is a polynomial of degree $\le n+1$ without constant term in the variables $v_{0,k,\lambda}$ defined by (6.38) and

$$v_{1,k,\lambda}(t,x) := \int_0^\infty \partial_t \widehat{\mathbf{H}}_{I,q}^{(k)}(t+s,X_x(s))\prod_{r=1}^{l(\lambda)}\frac{\partial^{\lambda_r}}{\partial x^{\lambda_r}}X_x(s)ds, \quad k\in\mathbb{N}, \lambda\in\mathcal{P}_{\mathbb{N}}\cup\{\mathbb{N}^0\}. \quad (6.40)$$

Here by $\lambda\in\mathbb{N}^0$ we mean that the factor $\prod \frac{\partial^{\lambda_r}}{\partial x^{\lambda_r}}X_x(s)$ disappears. Moreover, in every term of $\mathbf{Q}_{1,n}$, factors $v_{1,k,\lambda}$ appear at most once.

Formally differentiate (6.39) w.r.t. t. If the differentiation commutes with the integration and expectation, then we should have

$$\partial_t^2 \widehat{\Psi}_q^{(n)}(t,x) = \mathbf{E}\left[\exp\left(\sigma\int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t+s,X_x(s))ds\right)\cdot \mathbf{Q}_{2,n}(v_{0,k,\lambda},v_{1,k,\lambda},v_{2,k,\lambda})\right], \quad (6.41)$$

where $\mathbf{Q}_{2,n}$ is a polynomial of degree $\le n+2$ without constant term in the variables $v_{0,k,\lambda}$ defined by (6.38), $v_{1,k,\lambda}$ defined by (6.40), and

$$v_{2,k,\lambda}(t,x) := \int_0^\infty \partial_t^2 \widehat{\mathbf{H}}_{I,q}^{(k)}(t+s,X_x(s))\prod_{j=1}^{l(\lambda)}\frac{\partial^{\lambda_j}}{\partial x^{\lambda_j}}X_x(s)ds, \quad k\in\mathbb{N}, \lambda\in\mathcal{P}_{\mathbb{N}}\cup\{\mathbb{N}^0\}.$$

Moreover, in every term of $\mathbf{Q}_{2,n}$, factors $v_{2,k,\lambda}$ appears at most once; when a factor $v_{2,k,\lambda}$ appears, factors $v_{1,k,\lambda}$ disappear; and when factors $v_{2,k,\lambda}$ disappear, factors $v_{1,k,\lambda}$ appear at most twice.

Now we suppose $\mathcal{E}_m$ occurs for some $m\in\mathbb{N}\cup\{0\}$. Using (6.11), (6.13), (6.38), Lemma 6.4, and the argument in (6.34), we conclude that, for any $k\in\mathbb{N}$ and $\lambda\in\mathcal{P}_{\mathbb{N}}$, there is a polynomial $P_{k,\lambda}$ with no constant term such that

$$|v_{0,k,\lambda}(t,x)| \le P_{k,\lambda}((m-c_0t)\vee 0) + C\alpha(t)e^{(c_0-2)t}. \quad (6.42)$$

Let $j\in\{1,2\}$ and $n\in\mathbb{N}\cup\{0\}$. If $s\ge\frac{m-c_0t}{c_0-1}$ then $|X_x(s)|\le s+m\le c_0(s+t)$, so from (6.19) we have

$$|\partial_t^j \widehat{\mathbf{H}}_{I,q}^{(n)}(t+s,X_x(s))| \le D_n\left(\frac{2e^{(c_0-2)(t+s)}}{1-e^{-2t}}\right)^{j+1} \lesssim \alpha(t)^{j+1}e^{(c_0-2)(t+s)}.$$

If $m\ge c_0t$ and $0\le s\le\frac{m-c_0t}{c_0-1}$, from (6.18) and the definition of $\mathcal{E}_m$, we see that for $j=1,2$,

$$|\partial_t^j \widehat{\mathbf{H}}_{I,q}^{(n)}(t, X_x(s))| \le D_n \left(\frac{|X_x(s)|}{t+s} + 3 + \frac{2e^{-t}}{1-e^{-2t}}\right)^j \left(\frac{1}{2} + \frac{4e^{-t}}{1-e^{-2t}}\right)$$
$$\le D_n \left(\frac{m - c_0 t}{t} + c_0 + 4 + \frac{2e^{-t}}{1-e^{-2t}}\right)^j \left(\frac{1}{2} + \frac{4e^{-t}}{1-e^{-2t}}\right)$$
$$\lesssim ((m - c_0 t)^j + 1)\alpha(t)^{j+1}.$$

Thus, from Lemma 6.4, for $k \in \mathbb{N}$ and $\lambda \in \mathcal{P}_{\mathbb{N}} \cup \{\mathbb{N}^0\}$,

$$|v_{j,k,\lambda}(t, x)| \lesssim \alpha(t)^{j+1}(e^{(c_0-2)t} + P_{j,k,\lambda}((m - c_0 t) \vee 0)), \quad j = 1, 2, \quad (6.43)$$

where $P_{j,k,\lambda}$ is a polynomial with no constant term.

Let $(j, n) \in \{0, 1, 2\} \times (\mathbb{N} \cup \{0\}) \backslash \{(0, 0)\}$. From (6.42), (6.43), and the properties of $\mathbf{Q}_{0,n}$, $n \in \mathbb{N}$, $\mathbf{Q}_{1,n}$ and $\mathbf{Q}_{2,n}$, $n \in \mathbb{N} \cup \{0\}$, we see that, at the event $\mathcal{E}_m$,

$$|\mathbf{Q}_{j,n}| \lesssim \alpha(t)^{2j}[P_{j,n}((m - c_0 t) \vee 0) + Q_{j,n}((m - c_0 t) \vee 0)\alpha(t)^n e^{(c_0-2)t}], \quad (6.44)$$

where $P_{j,n}$ and $Q_{j,n}$ are polynomials, and $P_{j,n}(0) = 0$.

Proposition 6.2 *For* $(j, n) \in \{0, 1, 2\} \times (\mathbb{N} \cup \{0\}) \backslash \{(0, 0)\}$,

$$\mathbf{E}\left[\exp\left(\sigma \int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t+s, X_x(s))ds\right) \cdot |\mathbf{Q}_{j,n}|\right]$$
$$\lesssim \exp\left(C(t^{-1}+1)e^{(c_0-2)t}\right)(t^{-n-2j}+1)(e^{(c_0-2)t} + e^{\frac{2|x|}{\kappa} - \frac{2c_0 t}{\kappa}}), \quad (6.45)$$

Proof Let $H_{j,n}(t) = \exp\left(\sigma \int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t+s, X_x(s))ds\right) \cdot |\mathbf{Q}_{j,n}|$. Recall that (6.34) and (6.44) hold at the event $\mathcal{E}_m$. Using (6.32) and the argument in (6.35) and (6.36), we see that

$$\mathbf{E}[H_{j,n}(t)] = \mathbf{E}[1_{\mathcal{E}_{\lfloor c_0 t\rfloor}} H_{j,n}(t)] + \sum_{m=\lfloor c_0 t\rfloor}^\infty \mathbf{E}[1_{\mathcal{E}_{m+1}\backslash \mathcal{E}_m} H_{j,n}(t)]$$
$$\lesssim \exp\left(C\alpha(t)e^{(c_0-2)t}\right)\alpha(t)^{2j+n}e^{(c_0-2)t} + \exp\left(\frac{2|x|}{\kappa} - \frac{2c_0 t}{\kappa} + C\alpha(t)e^{(c_0-2)t}\right) \cdot$$
$$\cdot \sum_{l=0}^\infty \alpha(t)^{2j}(P_{j,n}(l+1) + Q_{j,n}(l+1)\alpha(t)^n e^{(c_0-2)t}) \exp\left(\frac{\sigma}{2(c_0-1)} - \frac{2}{\kappa}\right)^l.$$

Then (6.45) follows from (6.31) and that $\alpha(t) \lesssim t^{-1} + 1$ and $\alpha(t)^{2j+n} \lesssim t^{-n-2j} + 1$. □

Theorem 6.1 *The function* $\widehat{\Psi}_q$ *is* $C^{\infty,\infty}$ *differentiable and solves (6.29). Moreover, for* $j \in \{0, 1, 2\}$, $n \in \mathbb{N} \cup \{0\}$, *there is a positive continuous function* $c_{j,n}(t)$ *on* $(0, \infty)$ *such that for any* $t \in (0, \infty)$ *and* $x \in \mathbb{R}$, $|\partial_t^j \widehat{\Psi}_q^{(n)}(t, x)| \le c_{j,n}(t)e^{\frac{2}{\kappa}|x|}$.

Proof For $n \in \mathbb{N} \cup \{0\}$, define $\widehat{\Psi}_q^{[0,n]}$, $\widehat{\Psi}_q^{[1,n]}(t,x)$ and $\widehat{\Psi}_q^{[2,n]}(t,x)$ to be equal to the RHS of (6.37), (6.39) and (6.41), respectively. From the above two propositions, these functions are well defined, and there are positive continuous functions $c_{j,n}(t)$ on $(0,\infty)$ such that

$$|\widehat{\Psi}_q^{[j,n]}(t,x)| \le c_{j,n}(t) e^{\frac{2}{\kappa}|x|}, \quad j = 0,1,2, \quad n \in \mathbb{N} \cup \{0\}. \tag{6.46}$$

Let $n \in \mathbb{N} \cup \{0\}$, $j \in \{0,1,2\}$, $t \in (0,\infty)$, and $x_1 < x_2 \in \mathbb{R}$. Since $|\widehat{\Psi}_q^{[j,n+1]}|$ satisfies (6.46), from Fubini's Theorem, we have

$$\int_{x_1}^{x_2} \widehat{\Psi}_q^{[j,n+1]}(t,x)dx = \widehat{\Psi}_q^{[j,n]}(t,x_2) - \widehat{\Psi}_q^{[j,n]}(t,x_1). \tag{6.47}$$

Thus, $\widehat{\Psi}_q^{[j,n]}$ is absolutely continuous in x when t is fixed, and its partial derivative w.r.t. x is a.s. equal to $\widehat{\Psi}_q^{[j,n+1]}$. Since $\widehat{\Psi}_q^{[j,n+1]}$ is continuous in x for fixed t, we see that $\widehat{\Psi}_q^{[j,n]}$ is continuously differentiable in x, and the partial derivative exactly equals $\widehat{\Psi}_q^{[j,n]}$. The above holds for any $n \in \mathbb{N}$, so $\widehat{\Psi}_q^{[j,0]}$ is C^∞ differentiable in x when t is fixed, and $\widehat{\Psi}_q^{[j,n]}$ is its n-th partial derivative w.r.t. x. Especially, since $\widehat{\Psi}_q = \widehat{\Psi}_q^{[0,0]}$, we see that $\widehat{\Psi}_q$ is C^∞ differentiable in x when t is fixed, and $\widehat{\Psi}_q^{[0,n]}$ is its n-th partial derivative w.r.t. x.

A similar argument using Fubini's Theorem shows that, for any $n \in \mathbb{N} \cup \{0\}$, $j \in \{0,1\}$, $\widehat{\Psi}_q^{[j,n]}$ is absolutely continuous in t when x is fixed, and its partial derivative w.r.t. t is a.s. equal to $\widehat{\Psi}_q^{[j+1,n]}$. So $\widehat{\Psi}_q^{[0,n]}$ is continuously differentiable in t when x is fixed, and the partial derivative exactly equals $\widehat{\Psi}_q^{[1,n]}$. From (6.46) and (6.47), we see that $\widehat{\Psi}_q^{[j,n]}$ is locally uniformly Lipschitz continuous in x. We have seen that $\widehat{\Psi}_q^{[j,n]}$ is continuous in t for every fixed x. So $\widehat{\Psi}_q^{[j,n]}$ is continuous in both t and x. Thus, $\widehat{\Psi}_q = \widehat{\Psi}_q^{[0,0]}$ is $C^{1,\infty}$ differentiable.

Fix $t_0 \in (0,\infty)$ and $x_0 \in \mathbb{R}$. Let $M(t) = \mathbf{E}\Big[\exp\Big(\sigma \int_0^\infty \widehat{\mathbf{H}}'_{I,q}(t_0+s, X_{x_0}(s))ds\Big) \Big| \mathcal{F}_t\Big]$, $t \ge 0$. Then $M(t)$, $0 \le t < \infty$, is a uniformly integrable martingale. From (6.30) we have

$$M(t) = \widehat{\Psi}_q(t_0+t, X_{x_0}(t)) \exp\left(\sigma \int_0^t \widehat{\mathbf{H}}'_{I,q}(t_0+s, X_{x_0}(s))ds\right). \tag{6.48}$$

From (6.2), Itô's formula, and the differentiability of $\widehat{\Psi}_q$, we see that $\widehat{\Psi}_q$ solves (6.29) for $t \ge t_0$. Since this is true for any $t_0 \in (0,\infty)$, $\widehat{\Psi}_q$ solves (6.29).

Since $\widehat{\Psi}_q$ is $C^{1,\infty}$ differentiable, the same is true for the RHS of (6.29). Thus, $\partial_t \widehat{\Psi}_q$ is also $C^{1,\infty}$ differentiable. So $\widehat{\Psi}_q$ is $C^{2,\infty}$ differentiable. Iterating this argument, we conclude that $\widehat{\Psi}_q$ is $C^{\infty,\infty}$ differentiable. The previous argument shows that $\partial_t^j \widehat{\Psi}_q^{(n)} = \widehat{\Psi}_q^{[j,n]}$ for any $j \in \{0,1,2\}$ and $n \in \mathbb{N} \cup \{0\}$. The bounds of $|\partial_t^j \widehat{\Psi}_q^{(n)}|$ then follow from (6.46). ◻

Theorem 6.2 *Let* $\widehat{\Psi}_0 = \widehat{\Psi}_q \cdot \widehat{\Psi}_\infty$, *where* $\widehat{\Psi}_\infty$ *is defined by (6.28). Then* $\widehat{\Psi}_0$ *is a positive* $C^{\infty,\infty}$ *differentiable function on* $(0,\infty)\times\mathbb{R}$ *and solves (6.26). Moreover,* $j \in \{0,1,2\}$, *for* $n \in \mathbb{N}\cup\{0\}$, *there is a positive continuous function* $c_{j,n}(t)$ *on* $(0,\infty)$ *such that, for any* $t\in(0,\infty)$ *and* $x\in\mathbb{R}$, $|\partial_t^j \widehat{\Psi}_0^{(n)}(t,x)| \le c_{j,n}(t)e^{\frac{2}{\kappa}|x|}$.

Proof Since $\widehat{\Psi}_q$ and $\widehat{\Psi}_\infty$ are both positive and $C^{\infty,\infty}$ differentiable, the same is true for $\widehat{\Psi}_0 = \widehat{\Psi}_q \cdot \widehat{\Psi}_\infty$. Since $\widehat{\Psi}_q$ solves (6.29), from Lemma 6.8, $\widehat{\Psi}_0$ solves (6.26). From Lemma 6.3, (6.28), and that $\tau \le 0$, we see that for any $j,n\in\mathbb{N}\cup\{0\}$, $|\partial_t^j \widehat{\Psi}_\infty^{(n)}(t,x)|$ is bounded by a positive continuous function in t, which, together with Theorem 6.1, implies the upper bounds of $|\partial_t^j \widehat{\Psi}_0^{(n)}(t,x)|$. □

Theorem 6.3 *Let* Ψ_0 *be the transformation of the above* $\widehat{\Psi}_0$ *via (6.25) (with* $\widehat{\Psi}$ *replaced by* $\widehat{\Psi}_0$*). Then* Ψ_0 *is a* $C^{\infty,\infty}$ *differentiable positive function on* $(0,\infty)\times\mathbb{R}$ *and solves (4.5). Moreover, for* $j\in\{0,1,2\}$, $n\in\mathbb{N}\cup\{0\}$, *there is a function* $h_{j,n}(t,|x|)$, *which is a polynomial in* $|x|$ *for any fixed* t, *and every coefficient is a positive continuous function in* t, *such that for any* $t\in(0,\infty)$ *and* $x\in\mathbb{R}$, $|\partial_t^j \Psi_0^{(n)}(t,x)| \le h_{j,n}(t,|x|)e^{-\frac{x^2}{2\kappa t}+\frac{2\pi|x|}{\kappa t}}$.

Proof Since $\widehat{\Psi}_0>0$, $\Psi_0>0$ also. The differentiability of Ψ_0 is obvious. Since $\widehat{\Psi}_0$ solves (6.29), from Lemma 6.7, Ψ_0 solves (4.5). Let $\Psi_a(t,x) = \widehat{\Psi}_0(\frac{\pi^2}{t},\frac{\pi}{t}x)$. From Theorem 6.2, it is straightforward to check that for every $j\in\{0,1,2\}$, $n\in\mathbb{N}\cup\{0\}$, there is a function $f_{j,n}(t,|x|)$, which is a polynomial in $|x|$ of degree j when t is fixed, and every coefficient is a positive continuous function in t, such that

$$\left|\partial_t^j \Psi_0^{(n)}(t,x)\right| \le f_{j,n}(t,|x|)e^{\frac{2}{\kappa}\frac{\pi}{t}|x|}, \quad t>0, x\in\mathbb{R}. \tag{6.49}$$

It is easy to verify that for every $j\in\{0,1,2\}$, $n\in\mathbb{N}\cup\{0\}$, there is a function $g_{j,n}(t,|x|)$, which is a polynomial in $|x|$, and every coefficient is a positive continuous function in t, such that

$$\left|\partial_t^j\partial_x^n\left(e^{-\frac{x^2}{2\kappa t}}\left(\frac{\pi}{t}\right)^{\sigma+\frac12}\right)\right| \le g_{j,n}(t,|x|)e^{-\frac{x^2}{2\kappa t}}, \quad t>0, x\in\mathbb{R}. \tag{6.50}$$

From (6.25), $\Psi_0(t,x) = e^{-\frac{x^2}{2\kappa t}}(\frac{\pi}{t})^{\sigma+\frac12}\Psi_a(t,x)$. So we get the upper bounds of $|\partial_t^j \Psi_0^{(n)}(t,x)|$ from (6.49) and (6.50). □

Theorem 6.4 *Let* Ψ_0 *be as in the above theorem. Let* $\Gamma_0 = \Psi_0\Theta_I^{-\frac{2}{\kappa}}$ *and* $\Gamma_m(t,x) = \Gamma_0(t,x-2m\pi)$, $m\in\mathbb{Z}$. *For* $s_0\in\mathbb{R}$, *let* $\Gamma_{\langle s_0\rangle} = \sum_{m\in\mathbb{Z}} e^{\frac{2\pi}{\kappa}ms_0}\Gamma_m$. *Then* $\Gamma_{\langle s_0\rangle}$ *is a* $C^{\infty,\infty}$ *differentiable positive function on* $(0,\infty)\times\mathbb{R}$, *satisfies (4.2), and solves (4.4).*

Proof Let $\Psi_m(t,x) = \Psi_0(t,x-2m\pi)$ for $m\in\mathbb{Z}$ and $\Psi_{\langle s_0\rangle} = \sum_{m\in\mathbb{Z}} e^{\frac{2\pi}{\kappa}ms_0}\Psi_m$. Since Θ_I has period 2π, we have $\Gamma_{\langle s_0\rangle} = \Psi_{\langle s_0\rangle}\Theta_I^{-\frac{2}{\kappa}}$. Since Θ_I is a $C^{\infty,\infty}$ differentiable positive function with period 2π, from Lemma 4.1 we suffice to show that $\Psi_{\langle s_0\rangle}$ is a $C^{\infty,\infty}$ differentiable positive function, satisfies (4.2), and solves (4.5). It is clear from the definition that $\Psi_{\langle s_0\rangle}$ satisfies (4.2). Since Ψ_0 is a $C^{\infty,\infty}$ differentiable positive function that solves (4.5), and $\mathbf{H}_I$ has period 2π, every Ψ_m also satisfies these properties. So $\Psi_{\langle s_0\rangle}$ is positive. The upper bounds of $|\partial_t^j \Psi_0^{(n)}(t,x)|$ imply that $\Psi_{\langle s_0\rangle}$ is finite, and the series $\sum_{m\in\mathbb{Z}} e^{\frac{2\pi}{\kappa}ms_0}\partial_t^j\Psi_m^{(n)}$ converges locally uniformly for every $j,n\ge 0$. Fubini's Theorem implies that $\Psi_{\langle s_0\rangle}$ is $C^{\infty,\infty}$ differentiable and $\partial_t^j\Psi_{\langle s_0\rangle}^{(n)}=\sum_{m\in\mathbb{Z}} e^{\frac{2\pi}{\kappa}ms_0}\partial_t^j\Psi_m^{(n)}$ Thus, $\Psi_{\langle s_0\rangle}$ also solves (4.5). □

6.5 Distributions

Proposition 6.3 *Let $p>0$, $s_0\in\mathbb{R}$, and $x_0,y_0\in\mathbb{R}$. Let Γ_m, $m\in\mathbb{Z}$, and $\Gamma_{\langle s_0\rangle}$ be as in Theorem 6.4. Let $\Lambda_*=\kappa\frac{\Gamma_*'}{\Gamma_*}$ for $*\in\{m,\langle s_0\rangle\}$. For $m\in\mathbb{Z}$, let $\widetilde\beta_m$ be the covering annulus SLE(κ,Λ_0) trace in $\mathbb{S}_p$ started from x_0 with marked point $y_0+2m\pi+pi$. Let $\widetilde\beta_{\langle s_0\rangle}$ be the covering annulus SLE$(\kappa,\Lambda_{\langle s_0\rangle})$ trace in $\mathbb{S}_p$ started from x_0 with marked point y_0+pi. Let $\mathbb{P}_{\widetilde\beta,m}$, $m\in\mathbb{Z}$, and $\mathbb{P}_{\widetilde\beta,\langle s_0\rangle}$ denote the distributions of $\widetilde\beta_m$, $m\in\mathbb{Z}$, and $\widetilde\beta_{\langle s_0\rangle}$, respectively. Then*

$$\mathbb{P}_{\widetilde\beta,\langle s_0\rangle}=\sum_{m\in\mathbb{Z}} e^{\frac{2\pi}{\kappa}ms_0}\frac{\Gamma_m(p,x_0-y_0)}{\Gamma_{\langle s_0\rangle}(p,x_0-y_0)}\mathbb{P}_{\widetilde\beta,m}. \tag{6.51}$$

Proof For $m\in\mathbb{Z}$, let $\xi_m(t)$, $0\le t<p$, be the solution to (3.13) with $\Lambda=\Lambda_0$ and y_0 replaced by $y_0+2m\pi$. Let $\xi_{\langle s_0\rangle}(t)$ be the solution to (3.13) with $\Lambda=\Lambda_{\langle s_0\rangle}$. Then the covering annulus Loewner traces of modulus p driven by ξ_m, $m\in\mathbb{Z}$, and $\xi_{\langle s_0\rangle}$ have distributions $\mathbb{P}_{\widetilde\beta,m}$, $m\in\mathbb{Z}$, and $\mathbb{P}_{\widetilde\beta,\langle s_0\rangle}$, respectively. Let $X_m(t)=\xi_m(t)-\mathrm{Re}\,\widetilde g^{\xi_m}(t,y_0+2m\pi+pi)+2m\pi$, $m\in\mathbb{Z}$, and $X_{\langle s_0\rangle}(t)=\xi_{\langle s_0\rangle}(t)-\mathrm{Re}\,\widetilde g^{\xi_{\langle s_0\rangle}}(t,y_0+pi)$. Since $\Gamma_m(t,x)=\Gamma_0(t,x-2m\pi)$, we have $\Lambda_m(t,x)=\Lambda_0(t,x-2m\pi)$. Since $\mathrm{Re}\,g(t,y+pi)=\widetilde g_I(t,y)$ for $y\in\mathbb{R}$, and $\mathbf{H}_I$ is odd and has period 2π, from (3.9), we find that, for $*\in\{m,\langle s_0\rangle\}$, with $\Phi_*:=\Lambda_*+\mathbf{H}_I$, $X_*(t)$ satisfies

$$dX_*(t)=\sqrt{\kappa}dB(t)+\Phi_*(p-t,X_*(t))dt,\quad X_*(0)=x_0-y_0.$$

Let $\mathbb{P}_{X,*}$ denote the distributions of $(X_*(t))$. Since $\xi_*(t)=X_*(t)+y_0-\int_0^t\mathbf{H}_I(p-r,X_*(r))dr$, $0\le t<p$, we suffice to show that (6.51) holds with the subscripts "$\widetilde\beta$" replaced by "X". The rest of the proof is a standard application of Girsanov theorem. One may check that for every $m\in\mathbb{Z}$, $M_m(t):=e^{\frac{2\pi}{\kappa}ms_0}\frac{\Gamma_m(p-t,X_{\langle s_0\rangle}(t))}{\Gamma_{\langle s_0\rangle}(p-t,X_{\langle s_0\rangle}(t))}$ is a nonnegative martingale w.r.t. $\mathbb{P}_{X,\langle s_0\rangle}$, and satisfies that $\frac{dM_m(t)}{M_m(t)}=(\Lambda_m-\Lambda_{\langle s_0\rangle})\frac{dB(t)}{\sqrt{\kappa}}$ and $\sum_{m\in\mathbb{Z}}M_m(t)=1$; and we have $\frac{d\mathbb{P}_{X,m}}{d\mathbb{P}_{X,\langle s_0\rangle}}=\frac{M_m(\infty)}{M_m(0)}$. □

Remark Since $\Gamma_{\langle s_0\rangle}$ satisfies (4.2), $\Lambda_{\langle s_0\rangle}$ has period 2π. So $\Lambda_{\langle s_0\rangle}$ is a crossing annulus drift function, and we could define the annulus SLE$(\kappa,\Lambda_{\langle s_0\rangle})$ process. However, each Λ_m does not have period 2π. It only makes sense to define the covering annulus SLE(κ,Λ_m) processes.

Proposition 6.4 *Let $p > 0$ and $x_0, y_0 \in \mathbb{R}$. Let Γ_0 be as in Theorem 6.4, and $\Lambda_0 = \kappa \frac{\Gamma_0'}{\Gamma_0}$. Let $\widetilde{\beta}(t)$, $0 \le t < p$, be the covering annulus SLE(κ, Λ_0) trace in $\mathbb{S}_p$ started from x_0 with marked point $y_0 + pi$. Then a.s.* $\operatorname{dist}(y_0 + pi, \widetilde{\beta}([0, p)) + 2\pi\mathbb{Z}) = 0$.

Proof Let $\xi(t)$ be the driving function, and $\widetilde{g}(t, \cdot)$, $0 \le t < p$, be the covering Loewner maps. Then $\widetilde{g}(t, \cdot)$ maps $\mathbb{S}_p \backslash (\widetilde{\beta}([0, p)) + 2\pi\mathbb{Z})$ conformally onto $\mathbb{S}_{p-t}$, and maps $\mathbb{R}_p$ onto $\mathbb{R}_{p-t}$. From Koebe's 1/4 Theorem, we suffice to show that a.s. $\widetilde{g}'(t, y_0 + pi) \cdot \frac{p}{p-t} \to \infty$ as $t \to p$.

Let $X(t) = \xi(t) - \operatorname{Re} \widetilde{g}(t, y_0 + pi)$ and $\Phi_0 = \Lambda_0 + \mathbf{H}_I$. Then $X(t)$ satisfies the SDE:

$$dX(t) = \sqrt{\kappa} dB(t) + \Phi_0(p - t, X(t))\, dt, \quad 0 \le t < p.$$

From (3.8) we have $\ln(\widetilde{g}'(t, y_0 + pi) \cdot \frac{p}{p-t}) = \int_0^t (\mathbf{H}_I'(p - s, X(s)) + \frac{1}{p-s}) ds$. Let $\widehat{\Phi}_0 = \kappa \frac{\widehat{\Psi}_0'}{\widehat{\Psi}_0}$. Since Ψ_0 and $\widehat{\Psi}_0$ satisfy (6.24), we have $\widehat{\Phi}_0(s, z) = \frac{\pi}{s} \Phi_0(\frac{\pi^2}{s}, \frac{\pi}{s} z) + \frac{z}{s}$. Let $\widehat{p} = \frac{\pi^2}{p}$ and $\widehat{X}(t) = \frac{\widehat{p}+t}{\pi} X(p - \frac{\pi^2}{\widehat{p}+t}), 0 \le t < \infty$. Then $\widehat{X}(0) = \frac{\widehat{p}}{\pi} X(0) = \frac{\pi}{p}(x_0 - y_0)$. Applying Itô's formula and time-change of a semimartingale, we see that $\widehat{X}(t)$ satisfies the SDE:

$$d\widehat{X}(t) = \sqrt{\kappa} \widehat{B}(t) + \widehat{\Phi}_0(\widehat{p} + t, \widehat{X}(t)) dt, \quad 0 \le t < \infty,$$

for some standard Brownian motion $\widehat{B}(t)$. Changing variables using $\widehat{s} = \frac{\pi^2}{p-s} - \widehat{p}$, we get

$$\int_0^t \left(\mathbf{H}_I'(p - s, X(s)) + \frac{1}{p - s} \right) ds$$
$$= \int_0^{\widehat{t}} \left(\mathbf{H}_I' \left(\frac{\pi^2}{\widehat{p} + \widehat{s}}, X \left(p - \frac{\pi^2}{\widehat{p} + \widehat{s}} \right) \right) + \frac{\widehat{p} + s}{\pi^2} \right) \frac{\pi^2}{(\widehat{p} + \widehat{s})^2} d\widehat{s}$$
$$= \int_0^{\widehat{t}} \left(\frac{\pi^2}{(\widehat{p} + \widehat{s})^2} \mathbf{H}_I' \left(\frac{\pi^2}{\widehat{p} + \widehat{s}}, \frac{\pi}{\widehat{p} + \widehat{s}} \widehat{X}(\widehat{s}) \right) + \frac{1}{\widehat{p} + s} \right) d\widehat{s}$$
$$= \int_0^{\widehat{t}} \widehat{\mathbf{H}}_I'(\widehat{p} + \widehat{s}, \widehat{X}(\widehat{s})) d\widehat{s},$$

where $\widehat{t} = \frac{\pi^2}{p-t} - \widehat{p}$, and the last equality follows from (2.9). So we have

$$\lim_{t \to p^-} \ln(\widetilde{g}'(t, y_0 + pi) \cdot \frac{p}{p - t}) = \int_0^\infty \widehat{\mathbf{H}}_I'(\widehat{p} + \widehat{s}, \widehat{X}(\widehat{s})) d\widehat{s} \ge \int_0^\infty \tanh_2'(\widehat{X}(\widehat{s})) d\widehat{s},$$

where the last inequality follows from (2.12).

From Girsanov theorem and the fact that $\kappa\frac{\widehat{\Psi}_0'}{\widehat{\Psi}_0} = \kappa\frac{\widehat{\Psi}_q'}{\widehat{\Psi}_q} + \tau\tanh_2$, we find that the distribution of $(\widehat{X}(t))$ is equivalent to that of $(X_{\frac{\pi}{p}(x_0-y_0)}(t))$ defined by (6.2), and the Radon-Nikodym derivative is $M(\infty)/M(0)$, where $M(t)$ is defined by (6.48). Since $(X_{\frac{\pi}{p}(x_0-y_0)}(t))$ is homogeneous and recurrent, we have a.s. $\int_0^\infty \tanh_2'(X_{\frac{\pi}{p}(x_0-y_0)}(t))dt = \infty$, which implies that a.s. $\int_0^\infty \tanh_2'(\widehat{X}(\widehat{s}))d\widehat{s} = \infty$. Thus, a.s. $\widetilde{g}'(t, y_0+pi)\cdot\frac{p}{p-t} \to \infty$ as $t \to p^-$. □

Corollary 6.1 *Let $p > 0$, $s_0 \in \mathbb{R}$, and $x_0, y_0 \in \mathbb{R}$. Let $\Gamma_{\langle s_0\rangle}$ be as in Theorem 6.4, and $\Lambda_{\langle s_0\rangle} = \kappa\frac{\Gamma_{\langle s_0\rangle}'}{\Gamma_{\langle s_0\rangle}}$. Let $\beta(t)$, $0 \le t < p$, be the annulus SLE$(\kappa, \Lambda_{\langle s_0\rangle})$ trace in $\mathbb{S}_p$ started from e^{ix_0} with marked point e^{-p+iy_0}. Then a.s.* $\text{dist}(e^{-p+iy_0}, \beta([0,p))) = 0$.

Proof This follows immediately from the above two propositions. □

Remark For the reader's convenience, we now make a list of the functions defined in this section. First, $\widehat{\Psi}_q$ is defined by a Feynman–Kac formula (6.30) depending on $\kappa > 0$ and $\sigma \in [0, \frac{4}{\kappa})$. Second, $\widehat{\Psi}_0$ is defined to be the product $\widehat{\Psi}_q\widehat{\Psi}_\infty$, where $\widehat{\Psi}_\infty$ is a simple solution of (6.27) given by (6.28). Third, Ψ_0 is the transformation of the $\widehat{\Psi}_0$ via (6.25). Fourth, the partition functions are defined by $\Gamma_0 = \Psi_0\Theta_I^{-\frac{2}{\kappa}}$, $\Gamma_m(t,x) = \Gamma_0(t, x-2m\pi)$, and $\Gamma_{\langle s_0\rangle} = \sum_{m\in\mathbb{Z}} e^{\frac{2\pi}{\kappa}ms_0}\Gamma_m$. Fifth, the annulus drift functions are defined by $\Lambda_* = \kappa\frac{\Gamma_*'}{\Gamma_*}$.

7 Reversibility

The main result of this section is the theorem below which generalizes Theorem 1.1.

Theorem 7.1 *Let $\kappa \in (0,4]$ and $s_0 \in \mathbb{R}$. If $\beta(t)$, $-\infty \le t < \infty$, is a whole-plane SLE(κ, s_0) trace in $\widehat{\mathbb{C}}$ from a to b, then the reversal of β, up to a time-change, has the distribution of a whole-plane SLE(κ, s_0) trace in $\widehat{\mathbb{C}}$ from b to a.*

Proof From conformal invariance, we only need to consider the case $a = 0$ and $b = \infty$. Let $\Gamma_{\langle s_0\rangle}$ be given by Theorem 6.4 with $\sigma = \frac{4}{\kappa} - 1$. Then $\Gamma_{\langle s_0\rangle}$ solves (4.1) and satisfies (4.2). We now apply Theorem 5.1 to $\Gamma = \Gamma_{\langle s_0\rangle}$. Let Λ_j, s_j and $\beta_{I,j}(t)$, $j = 1, 2$, be given by Theorem 5.1. Then for $j = 1, 2$, $\beta_{I,j}$ is a whole-plane SLE(κ, s_j) trace in $\widehat{\mathbb{C}}$ from 0 to ∞, and satisfies that, for any $t_2 \in \mathbb{Q}$, conditioned on $\beta_{I,2}(s)$, $-\infty \le s \le t_2$, after a time-change, the curve $\beta_{I,1}(t_1)$, $-\infty \le t_1 < T_1(t_2)$, has the distribution of a disc SLE(κ, Λ_1) trace in $\mathbb{C}\backslash I_0(\beta_{I,2}([-\infty, t_2]))$ started from 0 with marked point $\beta_{I,2}(t_2)$, where $T_1(t_2)$ is the maximal number in $(-\infty, +\infty]$ such that $\beta_1(t) \cap \beta_2([-\infty, t_2]) = \emptyset$ for $-\infty < t < T_1(t_2)$.

Let ξ_2 be the driving function for $(\beta_{I,2}(t))$, and $g_2(t,\cdot)$, $-\infty < t < \infty$, be the inverted whole-plane Loewner maps driven by ξ_2. Then $g_2(t,\cdot)$ maps $\mathbb{C}\backslash I_0(\beta_{I,2}([-\infty, t_2]))$ conformally onto $\mathbb{D}$, fixes 0, and takes $\beta_{I,2}(t_2)$ to $e^i(\xi_x(t_2))$. Thus, conditioned on $\beta_{I,2}(s)$, $-\infty \le s \le t_2$, $g_2(t, \beta_{I,1}(t_1))$, $0 \le t_1 < T_1(t_2)$, is a time-change of a disc SLE(κ, Λ_1) trace in $\mathbb{D}$ started from 0 with marked point

$e^i(\xi_x(t_2))$. Since $\Lambda_1 = \Lambda = \kappa \frac{\Gamma'_{\langle s_0 \rangle}}{\Gamma_{\langle s_0 \rangle}}$, from Corollary 6.1 and the relation between the disc SLE(κ, Λ) process and the annulus SLE(κ, Λ) process, we conclude that a.s. $e^i(\xi_2(t_2))$ is a subsequential limit of $g_2(t_2, \beta_{I,1}(t))$ as $t \to T_1(t_2)^-$. Thus, $\beta_2(t_2)$ is a subsequential limit of $\beta_{I,1}(t)$ as $t \to T_1(t_2)^-$. If $T_1(t_2) = \infty$, then $\lim_{t \to T_1(t_2)^-} \beta_{I,1}(t) = \infty = \beta_2(-\infty) \neq \beta_2(t_2)$, which is a.s. a contradiction. So $T_1(t_2) < \infty$ a.s., and we have $\beta_{I,1}(T_1(t_2)) = \lim_{t \to T_1(t_2)^-} \beta_{I,1}(t) = \beta_2(t_2)$ a.s. Since $\mathbb{Q}$ is countable, we conclude that, a.s. $\beta_{I,1}(T_1(t_2)) = \beta_2(t_2)$ for every $t_2 \in \mathbb{Q}$, which implies that a.s. $\beta_2(\mathbb{R}) \subset \beta_{I,1}(\mathbb{R})$. Since both $\beta_{I,1}$ and β_2 are simple, and the initial (resp. final) point of $\beta_{I,1}$ agrees with the final (resp. initial) point of β_2, we see that β_2 is a reversal of $\beta_{I,1}$. Now $\beta_{I,1}$ is a whole-plane SLE(κ, s_0) trace in $\widehat{\mathbb{C}}$ from 0 to ∞, and $\beta_{I,2}$ is a whole-plane SLE$(\kappa, -s_0)$ trace in $\widehat{\mathbb{C}}$ started from 0 to ∞. Since I_0 is conjugate conformal, $\beta_2 = I_0(\beta_{I,2})$ is a whole-plane SLE(κ, s_0) trace in $\widehat{\mathbb{C}}$ from ∞ to 0. So we proved the theorem in the case $a = 0$ and $b = \infty$. □

Theorem 7.2 *If $\beta(t)$, $0 \le t < \infty$, is a radial SLE$(\kappa, -s_0)$ trace in a simply connected domain D from a to b, then a.s. $\lim_{t\to\infty} \beta(t) = b$, and after a time-change, the reversal of β becomes a disc SLE$(\kappa, \Lambda_{\langle s_0 \rangle})$ trace in D started from b with marked point a.*

Proof This follows from the property of the coupling in Theorem 7.1 and the relation between whole-plane SLE(κ, s_0) and radial SLE$(\kappa, -s_0)$. □

Theorem 7.3 *Let D be a doubly connected domain with two boundary points a, b lying on different boundary components. If $\beta(t)$, $0 \le t < p$, is an annulus SLE$(\kappa, \Lambda_{\langle s_0 \rangle})$ trace in D started from a with marked point b, then $\lim_{t\to p} \beta(t) = b$, and after a time-change, the reversal of β becomes an annulus SLE$(\kappa, \Lambda_{\langle s_0 \rangle})$ trace in D started from b with marked point a.*

Proof This follows from the property of the coupling in Theorem 7.1, and the relation between disc SLE$(\kappa, \Lambda_{\langle s_0 \rangle})$ and annulus SLE$(\kappa, \Lambda_{\langle s_0 \rangle})$. □

Remark For $\kappa \in (0, 6)$ and $\sigma = \frac{1}{2} + \frac{1}{\kappa} \in [0, \frac{4}{\kappa})$, the $\Lambda_{\langle 0 \rangle}$ given by Proposition 6.3 can be used to decompose an annulus SLE$_\kappa$ process (without marked point). The statement is similar to Lemma 3.1 in [12].

8 Some particular solutions

In this section, for $\kappa \in \{4, 2, 3, 0, 16/3\}$, we will find solutions to the PDE for Λ ((4.3) and (4.49)) and the PDE for Γ ((4.1) and (4.48)), which can be expressed in terms of $\mathbf{H}$ and $\mathbf{H}_I$. Since $\Lambda = \kappa \frac{\Gamma'}{\Gamma}$, multiplying a function in t to Γ does not change the value of Λ. So we may as well consider the following PDEs for Γ, where $C(t)$ is some real valued continuous depending only on t:

$$\partial_t \Gamma = \frac{\kappa}{2}\Gamma'' + \mathbf{H}_I \Gamma' + \left(\frac{3}{\kappa} - \frac{1}{2}\right) \mathbf{H}_I' \Gamma + C(t)\Gamma. \tag{8.1}$$

$$\partial_t \Gamma = \frac{\kappa}{2}\Gamma'' + \mathbf{H} \Gamma' + \left(\frac{3}{\kappa} - \frac{1}{2}\right) \mathbf{H}' \Gamma + C(t)\Gamma. \tag{8.2}$$

8.1 $\kappa = 4$

Let $\kappa = 4$. From Lemma 4.1 we see that if Ψ solves.

$$\partial_t \Psi = 2\Psi'', \tag{8.3}$$

then $\Gamma = \Psi\Theta_I^{-2/\kappa}$ solves (4.1). Similarly, $\Gamma = \Psi\Theta^{-2/\kappa}$ solves (4.48) if Ψ solves (8.3). The solutions to (8.3) are well-known. For example, we have the following solutions: e^{2c^2t+cx}, $\frac{1}{\sqrt{8\pi t}}e^{-\frac{(x-c)^2}{8t}}$, $e^{-t/2}\sin_2(x-c)$, $\Theta(2t, x-c)$, and $\Theta_I(2t, x-c)$. The function $\Theta_I(2t, x-\pi)$ corresponds to the solution $\Gamma(t, x) = \Theta_I(2t, x-\pi)\Theta_I(t, x)^{-1/2}$ of (8.1), which agrees with the solution given by Sect. 6.3 for $\kappa = 4$ and $\sigma = \frac{4}{\kappa} - 1 = 0$. Some of these solutions are related to the Gaussian free field ([5]) in doubly connected domains.

8.2 $\kappa = 2$

Let $\kappa = 2$. In this case if Ξ on $(0, \infty) \times \mathbb{R}$ solves

$$\partial_t \Xi = \Xi'' + \Xi'\mathbf{H}_I + C(t)\Xi \tag{8.4}$$

then $\Gamma := \Xi'$ solves (8.1). Similarly, if Ξ on $(0, \infty) \times (\mathbb{R}\backslash\{2n\pi : n \in \mathbb{Z}\})$ solves

$$\partial_t \Xi = \Xi'' + \Xi'\mathbf{H} + C(t)\Xi. \tag{8.5}$$

then $\Gamma := \Xi'$ solves (8.2).

From (2.8) we see that $\Xi_1 = \mathbf{H}_I$ solves (8.4) and $\Xi_2 = \mathbf{H}$ solves (8.5) with $C(t) = 0$. It is also easy to check that $\Xi_3(t, x) = t\mathbf{H}_I(t, x) + x$ solves (8.4) and $\Xi_4(t, x) = t\mathbf{H}(t, x) + x$ solves (8.5) with $C(t) = 0$. The Ξ_3 corresponds to the solution $\Gamma(t, x) = t\mathbf{H}_I'(t, x) + 1$, which agrees with the solution given by Sect. 6.3 for $\kappa = 2$ and $\sigma = \frac{4}{\kappa} - 1 = 1$. Such Γ is also the density function of the distribution of the limit point of an annulus SLE_2 trace.

We now derive more solutions. Fix $t > 0$. Let $L_t = \{2n\pi + i2kt : n, k \in \mathbb{Z}\}$. Let $F_{1,t}$ denote the set of odd analytic functions f on $\mathbb{C}\backslash L_t$ such that each $z \in L_t$ is a simple pole of f, 2π is a period of f, and $i2t$ is an antiperiod of f, i.e., $f(z+i2t) = -f(z)$. Let $F_{2,t}$ denote the set of odd analytic functions f on $\mathbb{C}\backslash L_t$ such that each $z \in L_t$ is a simple pole of f, 2π is an antiperiod of f, and $i2t$ is a period of f. Let $F_{3,t}$ denote the set of odd analytic functions f on $\mathbb{C}\backslash L_t$ such that each $z \in L_t$ is a simple pole of f, and both 2π and $i2t$ are antiperiods of f. Define

$$\Xi_1(t, z) = \mathbf{H}(2t, z) - \mathbf{H}_I(2t, z), \quad \Xi_2(t, z) = \frac{1}{2}\mathbf{H}\left(\frac{t}{2}, \frac{z}{2}\right) - \frac{1}{2}\mathbf{H}\left(\frac{t}{2}, \frac{z}{2} + \pi\right),$$
$$\Xi_3(t, z) = \frac{1}{2}\mathbf{H}\left(t, \frac{z}{2}\right) - \frac{1}{2}\mathbf{H}_I\left(t, \frac{z}{2}\right) - \frac{1}{2}\mathbf{H}\left(t, \frac{z}{2} + \pi\right) + \frac{1}{2}\mathbf{H}_I\left(t, \frac{z}{2} + \pi\right).$$

From the properties of $\mathbf{H}$ and $\mathbf{H}_I$, it is easy to check that $F_{j,t}$ is the linear space spanned by $\Xi_j(t,\cdot)$ for $j=1,2,3$. For $j=1,2,3$, Define

$$J_j=\partial_t\Xi_j-\Xi_j''-\Xi_j'\mathbf{H},\quad C_j(t)=\frac{1}{2}\mathrm{Res}_{z=0}\,J_j(t,\cdot).$$

Fix $t>0$. Note that 0 is a simple pole of both $\mathbf{H}(t,\cdot)$ and $\Xi_1(t,\cdot)$ of residue 2. It is easy to conclude that 0 is also a simple pole of $J_1(t,\cdot)$. From that $\Xi_1(t,\cdot)\in F_{1,t}$, that $\mathbf{H}(t,\cdot)$ has period 2π, and that $\mathbf{H}(t,z+2\pi)=\mathbf{H}(t,z)-2i$, it is easy to check that $J_1(t,\cdot)\in F_{1,t}$ as well. So $J_1(t,\cdot)=C_1(t)\Xi_1(t,\cdot)$. Thus, Ξ_1 solves (8.5). Similarly, Ξ_2 and Ξ_3 both solve (8.5).

8.3 $\kappa=3$

Let $\kappa=3$. Let Ξ_j, $j=1,2,3$, be as in the previous subsection. For $j=1,2,3$, let $\Gamma_j=\Xi_j$, and define

$$H_j=\partial_t\Gamma_j-\frac{3}{2}\Gamma_j''-\mathbf{H}\Gamma_j'-\frac{1}{2}\mathbf{H}'\Gamma_j,\quad C_j(t)=\frac{1}{2}\mathrm{Res}_{z=0}\,H_j(t,\cdot).$$

Using the argument in the last subsection, we find that $H_j(t,\cdot)\in F_{j,t}$ for any $t>0$. So $H_j(t,\cdot)=C_j(t)\Gamma_j(t,\cdot)$. Thus, $\Gamma_1,\Gamma_2,\Gamma_3$ solve (8.2). For $j=4,5,6$, let $\Gamma_j(t,z)=\Gamma_{j-3}(t,z+it)$. Since $\mathbf{H}_I(t,z)=\mathbf{H}(t,z+it)+i$, $\Gamma_4,\Gamma_5,\Gamma_6$ solve (8.1).

For $j=2,3$, Γ_j takes positive real values on $(0,2\pi)+4\pi\mathbb{Z}$, takes negative real values on $(-2\pi,0)+4\pi\mathbb{Z}$, and has antiperiod 2π. So $\Lambda_j:=3\frac{\Gamma_j'}{\Gamma_j}$ is a chordal-type annulus drift function that solves (4.49) for $\kappa=3$. It is worth to mention that the annulus SLE$(\kappa;\Lambda_j)$ process preserves the following local martingale, which resembles the $G(\Omega,a,b,z)$ in Proposition 11 of [7]. The proof uses the fact that Γ_j solves (8.2) for $z\in\mathbb{C}\backslash\{\text{poles}\}$.

Proposition 8.1 *Let $j\in\{2,3\}$ and $p>0$. Let $x_0\in\mathbb{R}$ and $z_0\in\mathbb{R}\backslash(x_0+2\pi\mathbb{Z})$. Let $\xi(t)$, $0\le t<T$, be the driving function for the covering annulus SLE$(\kappa;\Lambda_j)$ process in $\mathbb{S}_p$ started from x_0 with marked point z_0. Let $\widetilde{g}_t$, $0\le t<T$, be the covering annulus Loewner maps of modulus p driven by ξ. Then for every $z\in\mathbb{S}_p$,*

$$M_t(z):=\frac{\Gamma_j(p-t,\widetilde{g}_t(z)-\xi(t))}{\Gamma_j(p-t,\widetilde{g}_t(z_0)-\xi(t))}\cdot\frac{\widetilde{g}_t'(z)^{1/2}}{\widetilde{g}_t'(z_0)^{1/2}}$$

is a local martingale for $0\le t<T$.

For $j=1$, $\Gamma_1(t,\cdot)$ takes nonzero pure imaginary values on $\mathbb{R}_t$, the related function Γ_4 agrees with the solution given by Sect. 6.3 for $\kappa=3$ and $\sigma=\frac{4}{\kappa}-1=\frac{1}{3}$ up to a pure imaginary multiplicative constant, and $\Lambda_4:=3\frac{\Gamma_4'}{\Gamma_4}$ is a crossing annulus drift function that solves (4.3) for $\kappa=3$. The annulus SLE$(\kappa;\Lambda_4)$ process also preserves a local martingale. In fact, Proposition 8.1 holds with $z_0\in\mathbb{R}_p$, Λ_j replaced by Λ_4, and Γ_j replaced by Γ_1.

8.4 $\kappa = 0$

Let $\kappa = 0$. Let L_t be as in Sect. 8.2. Let $\mathbf{H}_2(t,z) = \mathbf{H}(t, z/2)$. From (2.8) we have

$$\partial_t \mathbf{H}_2 = 4\mathbf{H}_2'' + 2\mathbf{H}_2'\mathbf{H}_2. \tag{8.6}$$

Let $\Lambda_1 = \mathbf{H} - 2\mathbf{H}_2$. Then for each $t > 0$, $\Lambda_1(t,\cdot)$ is an odd analytic function on $\mathbb{C}\backslash L_t$, and each $z \in L_t$ is a simple pole of Λ_1. From $\mathbf{H}(t, z + 2\pi) = \mathbf{H}(t,z)$ and $\mathbf{H}(t, z + i2t) = \mathbf{H}(t,z) - 2i$ we see that both 4π and $i4t$ are periods of $\Lambda_1(t,\cdot)$. Fix $t > 0$, and define

$$J(z) = \frac{\Lambda_1(t,z)^2}{2} - 2\Lambda_1'(t,z) + 3\mathbf{H}'(t,z).$$

Then J is an even analytic function on $\mathbb{C}\backslash L_t$ and has periods 4π and $i4t$. Fix any $z_0 = 2n_0\pi + i2k_0t \in L_t$ for some $n_0, k_0 \in \mathbb{Z}$. Then $2z_0$ is a period of J, so $J_{z_0}(z) := J(z - z_0)$ is an even function. Thus, $\mathrm{Res}_{z=z_0} J(z) = 0$. The degree of z_0 as a pole of J is at most 2. The principal part of J at z_0 is $\frac{C(z_0)}{(z-z_0)^2}$ for some $C(z_0) \in \mathbb{C}$. Note that $\mathrm{Res}_{z_0} \mathbf{H}(t,z) = 2$ and $\mathrm{Res}_{z_0} \Lambda_1(t,z) = -6$ or 2. In either case, we compute $C(z_0) = 0$. Thus, every $z_0 \in L_t$ is a removable pole of J, which, together with the periods 4π and $i4t$, implies that J is a constant depending only on t. Differentiating J w.r.t. z, we conclude that

$$2\Lambda_1'' = \Lambda_1'\Lambda_1 + 3\mathbf{H}''. \tag{8.7}$$

From $\Lambda_1 = \mathbf{H} - 2\mathbf{H}_2$ we have $2\mathbf{H}_2 = \mathbf{H} - \Lambda_1$. So from (8.6) and (8.7), we have

$$\begin{aligned}\partial_t \mathbf{H} - \partial_t \Lambda_1 &= 2\partial_t \mathbf{H}_2 = 8\mathbf{H}_2'' + 4\mathbf{H}_2'\mathbf{H}_2 = 4\mathbf{H}'' - 4\Lambda_1'' + (\mathbf{H}' - \Lambda_1')(\mathbf{H} - \Lambda_1)\\ &= 4\mathbf{H}'' - 2(\Lambda_1'\Lambda_1 + 3\mathbf{H}'') + (\mathbf{H}' - \Lambda_1')(\mathbf{H} - \Lambda_1)\\ &= -2\mathbf{H}'' - \Lambda_1'\Lambda_1 + \mathbf{H}'\mathbf{H} - \Lambda_1'\mathbf{H} - \mathbf{H}'\Lambda_1.\end{aligned}$$

From the above formula and (2.8), we have

$$\partial_t \Lambda_1 = 3\mathbf{H}'' + \Lambda_1'\Lambda_1 + \mathbf{H}'\Lambda_1 + \Lambda_1'\mathbf{H}. \tag{8.8}$$

Thus, Λ_1 solves (4.49). Note that $\mathbf{H}_I(t, z/2)$ also satisfies (8.6). Let $\Lambda_2(t,z) := \mathbf{H}(t,z) - \mathbf{H}_I(t, \frac{z}{2})$. Then $\Lambda_2(t,\cdot)$ is also an odd analytic function on $\mathbb{C}\backslash L_t$ and has periods 4π and $i4t$. The principal part of $\Lambda_2(t,\cdot)$ at every $z_0 \in L_t$ is also either $\frac{-6}{z-z_0}$ or $\frac{2}{z-z_0}$. Using a similar argument, we conclude that Λ_2 also solves (4.49).

8.5 $\kappa = 16/3$

Let $\kappa = 16/3$. Let Λ_1 and Λ_2 be as in the last subsection. Let $\Lambda_3 = -\Lambda_1/3$. From (8.7) we have

$$0 = \frac{8}{3}\Lambda_3'' + 4\Lambda_3'\Lambda_3 + \frac{4}{3}\mathbf{H}''.$$

From (8.8) we have

$$\partial_t \Lambda_3 = -\mathbf{H}'' - 3\Lambda_3'\Lambda_3 + \mathbf{H}'\Lambda_3 + \Lambda_3'\mathbf{H}.$$

Summing up the above two equalities, we get

$$\partial_t \Lambda_3 = \frac{8}{3}\Lambda_3'' + \frac{1}{3}\mathbf{H}'' + \mathbf{H}'\Lambda_3 + \Lambda_3'\mathbf{H} + \Lambda_3'\Lambda_3.$$

Thus, Λ_3 solves (4.49). Similarly, $\Lambda_4 := -\Lambda_2/3$ also solves (4.49). Here Λ_3 and Λ_4 have period 4π instead of 2π. If we want a solution to (4.3) with period 2π, we may first restrict Λ_3 or Λ_4 to the interval $(0, 2\pi)$ or $(-2\pi, 0)$, and then extend it to $\mathbb{R}\backslash\{2n\pi : n \in \mathbb{Z}\}$ so that the function has period 2π.

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