

Synchronization and random long time dynamics for mean-field plane rotators

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Abstract We consider the natural Langevin dynamics which is reversible with respect to the mean-field plane rotator (or classical spin XY) measure. It is well known that this model exhibits a phase transition at a critical value of the interaction strength parameter K , in the limit of the number N of rotators going to infinity. A Fokker–Planck PDE captures the evolution of the empirical measure of the system as $N \rightarrow \infty$, at least for finite times and when the empirical measure of the system at time zero satisfies a law of large numbers. The phase transition is reflected in the fact that the PDE for K above the critical value has several stationary solutions, notably a stable manifold—in fact, a circle—of stationary solutions that are equivalent up to rotations. These stationary solutions are actually unimodal densities parametrized by the position of their maximum (the synchronization phase or center). We characterize the dynamics on times of order N and we show substantial deviations from the behavior of the solutions of the PDE. In fact, if the empirical measure at time zero converges as $N \rightarrow \infty$ to a probability measure (which is away from a thin set that we characterize) and if time is speeded up by N , the empirical measure reaches almost instantaneously a small neighborhood of the stable manifold, to which it then sticks and on which a non-trivial random dynamics takes place. In fact the synchronization center performs a Brownian motion with a diffusion coefficient that we compute. Our approach therefore provides, for one of the basic statistical mechanics systems with continuum symmetry, a detailed characterization of the macroscopic deviations from

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the large scale limit—or law of large numbers—due to finite size effects. But the interest for this model goes beyond statistical mechanics, since it plays a central role in a variety of scientific domains in which one aims at understanding synchronization phenomena.

Keywords Coupled rotators · Fokker–Planck PDE · Kuramoto synchronization model · Finite size corrections to scaling limits · Long time dynamics · Diffusion on stable invariant manifold

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1 Introduction

1.1 Overview

In a variety of instances partial differential equations are a faithful approximation—in fact, a law of large numbers—for particle systems in suitable limits. This is notably the case for stochastic interacting particle systems, for which the mathematical theory has gone very far [24]. The closeness between the particle system and PDE is typically proven in the limit of systems with a large number N of particles or for infinite systems under a space rescaling involving a large parameter N —for example a spin or particle system on \mathbb{Z}^d and the lattice spacing scaled down to $\frac{1}{N}$ —and up to a time horizon which may depend on N . Of course the question of capturing the finite N corrections has been taken up too, and the related central limit theorems as well as large deviations principles have been established (see [24] and references therein). *Sizable* deviations from the law of large numbers, not just small fluctuations or rare events, can be observed beyond the time horizon for which the PDE behavior has been established and these phenomena can be very relevant.

The first examples that come to mind are the ones in which the PDE has multiple isolated stable stationary points: metastability phenomena happens on exponentially long time scales [29]. Deviations on substantially shorter time scales can also take place and this is the case for example of the noise induced escape from stationary unstable solutions, which is particularly relevant in plenty of situations: for example for the model in [30, Ch. 5] phase segregation originates from homogeneous initial data via this mechanism, on times proportional to the logarithm of the size of the system. The logarithmic factor is directly tied to the exponential instability of the stationary solution (see [30] for more literature on this phenomenon). Of course, the type of phenomena happen also in finite dimensional random dynamical systems, in the limit of small noise, but we restrict this quick discussion to infinite dimensional models and PDEs.

In the case on which we focus the deviations also happen on time scales substantially shorter than the exponential ones, but the mechanism of the phenomenon does not involve exponential instabilities. In the system we consider there are multiple

stationary solutions, but they are not (or, at least, not all) isolated, and hence they are not stable in the standard sense. Deviations from the PDE behavior happen as a direct result of the cumulative effect of the fluctuations. More precisely, this phenomenon is due to the presence of whole stable manifold of stationary solutions: the deterministic limit dynamics has no dumping effect along the tangential direction to the manifold so, for the finite size system, the weak noise does have a macroscopic effect on a suitable time scale that depends on how large the system is. We review the mathematical literature on this type of phenomena in Sect. 1.6, after stating our results.

Apart for the general interest on deviations from the PDE behavior, the model we consider—mean-field plane rotators—is a fundamental one in mathematical physics and, more generally, it is the basic model for synchronization phenomena. Our results provide a sharp description of the long time dynamics of this model for general initial data.

1.2 The model

Consider the set of ordinary stochastic differential equations

$$d\varphi_t^{j,N} = \frac{1}{N} \sum_{i=1}^N J(\varphi_t^{j,N} - \varphi_t^{i,N}) dt + dW_t^j \tag{1.1}$$

with $j = 1, 2, \dots, N$, $\{W_j\}_{j=1,2,\dots}$ is an IID collection of standard Brownian motions and $J(\cdot) = -K \sin(\cdot)$. With abuse of notation, when writing $\varphi_t^{j,N}$ we will actually mean $\varphi_t^{j,N} \bmod(2\pi)$ and for us (1.1), supplemented with an (arbitrary) initial condition, will give origin to a diffusion process on \mathbb{S}^N , where \mathbb{S} is the circle $\mathbb{R}/(2\pi\mathbb{Z})$.

The choice of the interaction potential $J(\cdot)$ is such that the (unique) invariant probability of the system is

$$\pi_{N,K}(d\varphi) \propto \exp\left(\frac{K}{N} \sum_{i,j=1}^N \cos(\varphi_i - \varphi_j)\right) \lambda_N(d\varphi), \tag{1.2}$$

where λ_N is the uniform probability measure on \mathbb{S}^N . Moreover, the evolution is reversible with respect to $\pi_{N,K}$, which is the well known Gibbs measure associated to mean-field plane rotators (or classical XY model).

We are therefore considering the simplest Langevin dynamics of mean-field plane rotators and it is well known that such a model exhibits a phase transition, for $K > K_c := 1$, that breaks the continuum symmetry of the model (for a detailed mathematical physics literature we refer to [6]). The continuum symmetry of the model is evident both in the dynamics (1.1) and in the equilibrium measure (1.2): if $\{\varphi_t^{j,N}\}_{t \geq 0, j=1, \dots, N}$ solves (1.1), so does $\{\varphi_t^{j,N} + c\}_{t \geq 0, j=1, \dots, N}$, c an arbitrary constant, and $\pi_{N,K} \Theta_c^{-1} = \pi_{N,K}$, where Θ_c is the rotation by an angle c , that is $(\Theta_c \varphi)_j = \varphi_j + c$ for every j .

1.3 The $N \rightarrow \infty$ dynamics and the stationary states

The phase transition can be understood also taking a dynamical standpoint. Given the mean-field set up it turns out to be particularly convenient to consider the empirical measure

$$\mu_{N,t}(d\theta) := \frac{1}{N} \sum_{j=1}^N \delta_{\varphi_t^{j,N}}(d\theta), \tag{1.3}$$

which is a probability on (the Borel subsets of) \mathbb{S} . It is well known, see [6] (for detailed treatment and original references), that if $\mu_{N,0}$ converges weakly for $N \rightarrow \infty$, then so does $\mu_{N,t}$ for every $t > 0$. Actually, the process itself $t \mapsto \{\mu_{N,t}\}$, seen as an element of $C^0([0, T], \mathcal{M}_1)$, where $T > 0$ and \mathcal{M}_1 is the space of probability measures on \mathbb{S} equipped with the weak topology, converges to a non-random limit which is the process that concentrates on the unique solution of the non-local PDE (* denotes the convolution)

$$\partial_t p_t(\theta) = \frac{1}{2} \partial_\theta^2 p_t(\theta) - \partial_\theta ((J * p_t)(\theta) p_t(\theta)), \tag{1.4}$$

with initial condition prescribed by the limit of $\{\mu_{N,0}\}_{N=1,2,\dots}$ (by [21] this solution is smooth for $t > 0$). We insist on the fact that $p_t(\cdot)$ is a probability density: $\int_{\mathbb{S}} p_t(\theta) d\theta = 1$. We will often commit the abuse of notation of writing $p(\theta)$ when $p \in \mathcal{M}_1$ and p has a density. Much in the same way, if $p(\cdot)$ is a probability density, p , or $p(d\theta)$, is the probability measure.

It is worthwhile to point out that $(J * p)(\theta) = -\Re(\hat{p}_1)K \sin(\theta) + \Im(\hat{p}_1)K \cos(\theta)$ with $\hat{p}_1 := \int_{\mathbb{S}} p(\theta) \exp(i\theta) d\theta$. This is to say that the nonlinearity enters only through the first Fourier coefficient of the solution, a peculiarity that allows to go rather far in the analysis of the model. Notably, starting from this observation one can easily (once again details and references are given in [6]) see that all the stationary solutions to (1.4), in the class of probability densities, can be written, up to a rotation, as

$$q(\theta) := \frac{\exp(2Kr \cos(\theta))}{2\pi I_0(2Kr)}, \tag{1.5}$$

where $2\pi I_0(2Kr)$ is the normalization constant written in terms of the modified Bessel function of order zero ($I_j(x) = (2\pi)^{-1} \int_{\mathbb{S}} (\cos \theta)^j \exp(x \cos(\theta)) d\theta$, for $j = 0, 1$) and r is a non-negative solution of the fixed point equation $r = \Psi(2Kr)$, with $\Psi(x) = I_1(x)/I_0(x)$. Since $\Psi(\cdot) : [0, \infty) \rightarrow [0, 1)$ is increasing, concave, $\Psi(0) = 0$ and $\Psi'(0) = 1/2$ we readily see that if (and only if) $K > 1$ there exists a non-trivial (i.e. non-constant) solution to (1.4). Let us not forget however that $\Psi(0) = 0$ implies that $r = 0$ is a solution and therefore the constant density $\frac{1}{2\pi}$ is a solution no matter what the value of K is. From now on we set $K > 1$ and choose $r = r(K)$, the unique positive solution of the fixed point equation, so that the probability density $q(\cdot)$ in (1.5) is non trivial and it achieves the unique maximum at 0 and the minimum at π .

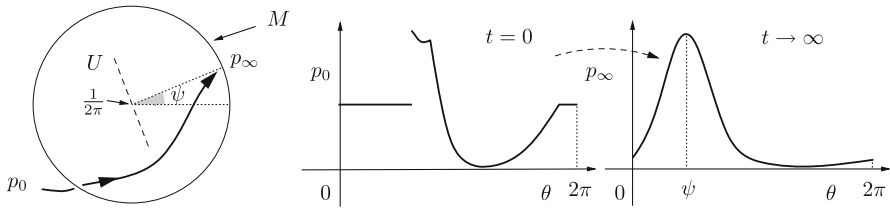


Fig. 1 The limit evolution (1.4) instantaneously smoothen an arbitrary initial probability and, unless the Fourier decomposition such an initial condition has zero coefficients corresponding to the first harmonics (the hyperplane U), it drives it to a point p_∞ —a synchronized profile—on the invariant manifold M and of course it stays there for all times. This has been proven in [21], here we are interested in what happens for the finite size— N —system and we show that the PDE approximation is faithful up to times much shorter than N : on times proportional to N synchronization is kept and the center of synchronization ψ performs a Brownian motion on \mathbb{S}

Note that the rotation invariance of the system immediately yields that there is a whole family of stationary solution:

$$M = \{q_\psi(\cdot) : q_\psi(\cdot) := q(\cdot - \psi) \text{ and } \psi \in \mathbb{S}\}, \tag{1.6}$$

and, when $x \in \mathbb{R}$, $q_x(\cdot)$ of course means $q_{x \bmod(2\pi)}(\cdot)$. M , which is more practically viewed as a manifold (in a suitable function space, see Sect. 2.2 below), is invariant and stable for the evolution. The proper notion of stability is given in the context of *normally hyperbolic manifolds* (see [32] and references therein), but the full power of such a concept is not needed for the remainder. Nevertheless let us stress that in [21] one can find a complete analysis of the global dynamic phase diagram, notably the fact that unless $p_0(\cdot)$ belongs to the stable manifold U of the unstable solution $\frac{1}{2\pi}$ —the solution corresponding to $r = 0$ in (1.5)— $p_t(\cdot)$ converges (also in strong norms, controlling all the derivatives) to one of the points in M , see Fig. 1. There is actually an explicit characterization of U :

$$U = \left\{ p \in \mathcal{M}_1 : \int_{\mathbb{S}} \exp(i\theta) p(d\theta) = 0 \right\}. \tag{1.7}$$

As a matter of fact, it is easy to realize that if $p_0(\cdot) \in U$ then (1.4) reduces to the heat equation $\partial_t p_t(\theta) = \frac{1}{2} \partial_\theta^2 p_t(\theta)$ which of course relaxes to $\frac{1}{2\pi}$.

1.4 Random dynamics on M : the main result

In spite of the stability of M , $q_\psi(\cdot)$ itself is not stable, simply because if we start nearby, say from $q_{\psi'}$, the solution of (1.4) does not converge to $q_\psi(\cdot)$. The important point here is that the linearized evolution operator around $q(\cdot) \in M$ (q is an arbitrary element of M , not necessarily the one in (1.5): the phase ψ of q_ψ is explicit only when its absence may be misleading)

$$L_q u(\theta) := \frac{1}{2} u'' - [uJ * q + qJ * u]', \tag{1.8}$$

with domain $\{u \in C^2(\mathbb{S}, \mathbb{R}) : \int_{\mathbb{S}} u = 0\}$ is symmetric in $H_{-1,1/q}$ —a weighted H_{-1} Hilbert space that we introduce in detail in Sect. 2.1—and it has compact resolvent. Moreover the spectrum of L_q , which is of course discrete, lies in $(-\infty, 0]$ and the eigenvalue 0 has a one dimensional eigenspace, generated by q' . So q' is the only *neutral direction* and it corresponds precisely to the tangent space of M at $q(\cdot)$: all other directions, in function space, are contracted by the linear evolution and the nonlinear part of the evolution does not alter substantially this fact [21,25].

Let us now step back and recall that our main concern is with the behavior of (1.1), with N large but finite, and not (1.4). In a sense the finite size, i.e. finite N , system is close to a suitable stochastic perturbation of (1.4): the type of stochastic PDE, with noise vanishing as $N \rightarrow \infty$, needs to be carefully *guessed* [19], keeping in particular in mind that we are dealing with a system with one conservation law. We will tackle directly (1.1), but the heuristic picture that one obtains by thinking of an SPDE with vanishing noise is of help. In fact the considerations we have just made on L_q suggest that if one starts the SPDE on M , the solution keeps very close to M , since the deterministic part of the dynamics is contractive in the orthogonal directions to M , but a (slow, since the noise is small) random motion on M arises because in the tangential direction the deterministic part of the dynamics is *neutral*. This is indeed what happens for the model we consider for N large. The difficulty that arises in dealing with the interacting diffusion system (1.1) is that one has to work with (1.3), which is not a function. Of course one can mollify it, but the evolution is naturally written and, to a certain extent, *closed* in terms of the empirical measure, and we do not believe that any significative simplification arises in proving our main statement for a mollified version. Working with the empirical measure imposes a clarification from now: as we explain in Sect. 2.1 and Appendix A, if μ and $\nu \in \mathcal{M}_1$, then $\mu - \nu$ can be seen as an element of H_{-1} (or, as a matter of fact, also as an element of a weighted H_{-1} space).

Here is the main result that we prove (recall that $K > 1$):

Theorem 1.1 *Choose a positive constant τ_f and a probability $p_0 \in \mathcal{M}_1 \setminus U$. If for every $\varepsilon > 0$*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\|\mu_{N,0} - p_0\|_{-1} \leq \varepsilon \right) = 1, \tag{1.9}$$

then there exist a constant ψ_0 that depends only on $p_0(\cdot)$ and, for every N , a continuous process $\{W_{N,\tau}\}_{\tau \geq 0}$, adapted to the natural filtration of $\{W_{N,j}^i\}_{j=1,2,\dots,N}$, such that $W_{N,\cdot} \in C^0([0, \tau_f]; \mathbb{R})$ converges weakly to a standard Brownian motion and for every $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{\tau \in [\varepsilon N, \tau_f]} \|\mu_{N,\tau N} - q_{\psi_0 + D_K W_{N,\tau}}\|_{-1} \leq \varepsilon \right) = 1, \tag{1.10}$$

where $\varepsilon_N := C/N$, $C = C(K, p_0, \varepsilon) > 0$, and

$$D_K := \frac{1}{\sqrt{1 - (I_0(2Kr))^{-2}}}. \tag{1.11}$$

The result is saying that, unless one starts on the stable manifold of the unstable solution (see Remark 2.5 for what one expects if $p_0 \in U$), the empirical measure reaches very quickly a small neighborhood of the manifold M : this happens on a time scale of order one, as a consequence of the properties of the deterministic evolution law (1.4) (Fig. 1), and, since we are looking at times of order N , this happens almost instantaneously. Actually, in spite of the fact that the result just addresses the limit of the empirical measure, the drift along M is due to fluctuations: the noise pushes the empirical measure away from M but the deterministic part of the dynamics projects back the trajectory to M and the net effect of the noise is a random shift—in fact, a rotation—along the manifold (this is taken up in more detail in the next section, where we give a complete heuristic version of the proof of Theorem 1.1).

Remark 1.2 Without much effort, one can upgrade this result to much longer times: if we set $\tau_f(N) = N^a$ with an arbitrary $a > 1$, there exists an adapted process $W_{N,\tau}^a$ converging to a standard Brownian motion such that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{\tau \in [\varepsilon_N, \tau_f(N)]} \left\| \mu_{N,\tau N^a} - q_{\psi_0 + D_K N^{a-1} W_{N,\tau}^a} \right\|_{-1} \leq \varepsilon \right) = 1. \tag{1.12}$$

This is due to the fact that our estimates ultimately rely on moment estimates, cf. Sect. 3. These estimates are obtained for arbitrary moments and we choose the moment sufficiently large to get uniformity for times $O(N)$, but working for times $O(N^a)$ would just require choosing larger moments. We have preferred to focus on the case $a = 1$; this is the natural scale, that is the scale in which the center of the probability density converges to a Brownian motion and not to an accelerated Brownian motion (this is really due to the fact that we work on \mathbb{S} and marks a difference with [4,9] where one can rescale the space variable).

1.5 The synchronization phenomena viewpoint

The model (1.1) we consider is actually a particular case of the Kuramoto synchronization model (the full Kuramoto model includes quenched disorder in terms of random constant speeds for the rotators, see [1,6] and references therein). The mathematical physics literature and the more bio-physically oriented literature use somewhat different notations reflecting a slightly different viewpoint. In the synchronization literature one introduces the synchronization degree $r_{N,t}$ and the synchronization center $\Psi_{N,t}$ via

$$r_{N,t} \exp(i\Psi_{N,t}) := \frac{1}{N} \sum_{j=1}^N \exp(i\varphi_t^{j,N}) \left(= \int_{\mathbb{S}} \exp(i\theta) \mu_{N,t}(\mathrm{d}\theta) \right), \tag{1.13}$$

which clearly correspond to the parameters r and ψ that appear in the definition of M , but $r_{N,t}$ and $\Psi_{N,t}$ are defined for N finite and also far from M . Note that if (1.9) holds, then both $r_{N,t}$ and $\Psi_{N,t}$ converge in probability as $N \rightarrow \infty$ to the limits r and ψ , with $r \exp(i\psi) = \int_{\mathbb{S}} \exp(i\theta) p_0(d\theta)$ and the assumption that $p_0 \notin U$ just means $r \neq 0$. Here is a straightforward consequence of Theorem 1.1:

Corollary 1.3 *Under the same hypotheses and definitions as in Theorem 1.1 we have that the stochastic process $\Psi_{N,N} \in C^0([\varepsilon, \tau_f]; \mathbb{S})$ converges weakly, for every $\varepsilon \in (0, \tau_f]$, to $(\psi_0 + D_K W) \bmod(2\pi)$.*

It is tempting to prove such a result by looking directly at the evolution of $\Psi_{N,t}$:

$$\begin{aligned} d\Psi_{N,t} = & \left(-K + \frac{1}{2Nr_{N,t}^2} \right) \frac{1}{N} \sum_{j=1}^N \sin \left(2 \left(\varphi_t^{j,N} - \Psi_{N,t} \right) \right) dt \\ & + \frac{1}{r_{N,t}N} \sum_{j=1}^N \cos \left(\varphi_t^{j,N} - \Psi_{N,t} \right) dW_j(t). \end{aligned} \quad (1.14)$$

But this clearly requires a control of the evolution of the empirical measure, so it does not seem that (1.14) could provide an alternative way to many of the estimates that we develop, namely convergence to a neighborhood of M and persistence of the proximity to M (see Sects. 3 and 5). On the other hand, it seems plausible that one could use (1.14) to develop an alternative approach to the dynamics on M , that is an alternative to Sect. 4. While this can be interesting in its own right, since the notion of synchronization center that we use in the proof and $\Psi_{N,t}$ are almost identical (where they are both defined, that is close to M) we do not expect substantial simplifications.

1.6 A look at the literature and perspectives

Results related to our work have been obtained in the context of SPDE models with vanishing noise. In [8, 15] one dimensional stochastic reaction diffusion equations with bistable potential (also called *stochastic Cahn–Allen* or *model A*) are analyzed for initial data that are close to profiles that connect the two phases. It is shown that the location of the phase boundary performs a Brownian motion. These results have been improved in a number of ways, notably to include *small asymmetries* that result in a drift for the arising diffusion process [7] and to deal with macroscopically finite volumes [4] (which introduce a repulsive effect approaching the boundary). Also the case of stochastic phase field equations has been considered [5].

For interacting particle systems results have been obtained for the zero temperature limit of d -dimensional Brownian particles interacting via local pair potentials in [16]: in this case the *frozen clusters* perform a Brownian motion and, in one dimension, also the merging of clusters is analyzed [17]. In this case the very small temperature is the small noise from which cluster diffusion originates. With respect to [16, 17], our results hold for any super-critical interaction, but of course our system is of mean

field type. It is also interesting to observe that for the model in [16, 17] establishing the stability of the frozen clusters is the crucial issue, because the motion of the center of mass is a martingale, i.e. there is no drift. A substantial part of our work is in controlling that the drift of the center of synchronization vanishes (and controlling the drift is a substantial part also of [4, 5, 7, 8, 15]). This is directly related to the content of Sect. 1.5.

As a matter of fact, in spite of the fact that our work deals directly with an interacting system, and not with an SPDE model, our approach is closer to the one in the SPDE literature. However, as we have already pointed out, a non negligible point is that we are forced to perform an analysis in distribution spaces, in fact Sobolev spaces with negative exponent, in contrast to the approach in the space of continuous functions in [4, 5, 7, 8, 15]. We point out that approaches to dynamical mean field type systems via Hilbert spaces of distribution has been already taken up in [14] but in our case the specific use of weighted Sobolev spaces is not only a technical tool, but it is intimately related to the geometry of the contractive invariant manifold M . In this sense and because of the iterative procedure we apply—originally introduced in [8]—our work is a natural development of [4, 8].

An important issue about our model that we have not stressed at all is that propagation of chaos holds (see e.g. [18]), in the sense that if the initial condition is given by a product measure, then this property is approximately preserved, at least for finite times. Recently much work has been done toward establishing quantitative estimates of chaos propagation (see for example the references in [11]). On the other hand, like for the model in [11], we know that, for our model, chaos propagation eventually breaks down: this is just because one can show by Large Deviations arguments that the empirical measure at equilibrium converges in law as $N \rightarrow \infty$ to the random probability density $q_X(\cdot)$, with X a uniform random variable on \mathbb{S} . But using Theorem 1.1 one can go much farther and show that chaos propagation breaks down at times proportional to N . From Theorem 1.1 one can actually extract also an accurate description of how the correlations build up due to the random motion on M .

It is natural to ask whether the type of results we have proven extend to the case in which random natural frequencies are present, that is to the disordered version of the model we consider that goes under the name of Kuramoto model. The question is natural because for the limit PDE [13, 26] there is a contractive manifold similar to M [20]. However the results in [27] suggest that a nontrivial dynamics on the contractive manifold is observed rather on times proportional to \sqrt{N} and one expects a dynamics with a nontrivial random drift. The role of disorder in this type of models is not fully elucidated (see however [12] on the critical case) and the global long time dynamics represents a challenging issue.

The paper is organized as follows: we start off (Sect. 2) by introducing the precise mathematical set-up and a number of technical results. This will allow us to present quantitative heuristic arguments and sketch of proofs. In Sect. 3 we prove that if the system is close to M , it stays so for a long time. We then move on to analyzing the dynamics on M (Sect. 4) and it is here that we show that the drift is negligible. Section 5 provides the estimates that guarantee that we do approach M and in Sect. 6 we collect all these estimates and complete the proof our main result (Theorem 1.1).

2 More on the mathematical set-up and sketch of proofs

2.1 On the linearized evolution

We introduce the Hilbert space $H_{-1,1/q}$ or, more generally, the space $H_{-1,w}$ for a general weight $w \in C^1(\mathbb{S}; (0, \infty))$ by using the rigged Hilbert space structure [10] with pivot space $\mathbb{L}_0^2 := \{u \in \mathbb{L}^2 : \int_{\mathbb{S}} u = 0\}$. In this way given an Hilbert space $V \subset \mathbb{L}_0^2$, V dense in \mathbb{L}_0^2 , for which the canonical injection of V into \mathbb{L}_0^2 is continuous, one automatically obtains a representation of V' —the dual space—in terms of a third Hilbert space into which \mathbb{L}_0^2 is canonically and densely injected. If V is the closure of $\{u \in C^1(\mathbb{S}; \mathbb{R}) : \int u = 0\}$ under the squared norm $\int_{\mathbb{S}} (u')^2/w$, that is $H_{1,1/w}$, the third Hilbert space is precisely $H_{-1,w}$. The duality between $H_{1,1/w}$ and $H_{-1,w}$ is denoted in principle by $\langle \cdot, \cdot \rangle_{H_{1,1/w}, H_{-1,w}}$, but less cumbersome notations will be introduced when the duality is needed (for example, below we drop the subscripts).

It is not difficult to see that for $u, v \in H_{-1,w}$

$$(u, v)_{-1,w} = \int_{\mathbb{S}} w \mathcal{U} \mathcal{V}, \tag{2.1}$$

where \mathcal{U} , respectively \mathcal{V} , is the primitive of u (resp. v) such that $\int_{\mathbb{S}} w \mathcal{U} = 0$ (resp. $\int_{\mathbb{S}} w \mathcal{V} = 0$), see [6, § 2.2]. More precisely, $u \in H_{-1,w}$ if there exists $\mathcal{U} \in \mathbb{L}^2(\mathbb{S}; \mathbb{R})$ such that $\int_{\mathbb{S}} \mathcal{U} w = 0$ and $\langle u, h \rangle = - \int_{\mathbb{S}} \mathcal{U} h'$ for every $h \in H_{1,1/w}$. One sees directly also that by changing w one produces equivalent $H_{1,w}$ norms [22, §2.1] so, when the geometry of the Hilbert space is not crucial, one can simply replace the weight by 1, and in this case we simply write H_{-1} . Occasionally we will need also H_{-2} which is introduced in an absolutely analogous way.

Remark 2.1 One observation that is of help in estimating weighted H_{-1} norms is that computing the norm of u requires access to \mathcal{U} : in practice if one identifies a primitive $\tilde{\mathcal{U}}$ of u , then $\|u\|_{-1,w}^2 \leq \int_{\mathbb{S}} \tilde{\mathcal{U}}^2 w$. This is just because $\tilde{\mathcal{U}} = \mathcal{U} + c$ for some $c \in \mathbb{R}$ and $\int_{\mathbb{S}} \tilde{\mathcal{U}}^2 w = \int_{\mathbb{S}} \mathcal{U}^2 w + c^2 \int_{\mathbb{S}} w$.

The reason for introducing weighted H_{-1} spaces is because, as one can readily verify, L_q , given in (1.8), is symmetric in $H_{-1,1/q}$. A deeper analysis (cf. [6]) shows that L_q is essentially self-adjoint, with compact resolvent. The spectrum of $-L_q$ lies in $[0, \infty)$, there is an eigenvalue $\lambda_0 = 0$ with one dimensional eigenspace generated by q' . We therefore denote the set of eigenvalues of $-L_q$ as $\{\lambda_0, \lambda_1, \dots\}$, with $\lambda_1 > 0$ and $\lambda_{j+1} \geq \lambda_j$ for $j = 1, 2, \dots$. The set of eigenfunctions is denoted by $\{e_j\}_{j=0,1,\dots}$ and let us point out that it is straightforward to see that $e_j \in C^\infty(\mathbb{S}; \mathbb{R})$. Moreover, if $u \in C^2(\mathbb{S}; \mathbb{R})$ is even (respectively, odd), then $L_q u$ is even (respectively, odd): the notion of parity is of course the one obtained by observing that $u \in C^2(\mathbb{S}; \mathbb{R})$ can be extended to a periodic function in $C^2(\mathbb{R}; \mathbb{R})$. This implies that one can choose $\{e_j\}_{j=0,1,\dots}$ with e_j that is either even or odd, and we will do so.

Remark 2.2 By rotation symmetry the eigenvalues do not depend on the choice of $q(\cdot) \in M$, but the eigenfunctions do depend on it, even if in a rather trivial way: the

eigenfunction of $L_{q\psi}$ and $L_{q\psi'}$ just differ by a rotation of $\psi' - \psi$. We will often need to be precise about the choice of $q(\cdot)$ and for this it is worthwhile to introduce the notations

$$L_\psi := L_{q\psi} \text{ and } -L_\psi e_{\psi,j} = \lambda_j e_{\psi,j}. \tag{2.2}$$

The eigenfunctions are normalized in $H_{-1,1/q\psi}$.

Remark 2.3 Some expressions involving weighted H_{-1} norms can be worked out explicitly. For example a recurrent expression in what follows is $(u, q')_{1,1/q}$, for $u \in H_{-1}$ and $q \in M$. If \mathcal{U} is the primitive of u such that $\int_{\mathbb{S}} \mathcal{U}/q = 0$, then we have $(u, q')_{1,1/q} = \int_{\mathbb{S}} \mathcal{U}(q-c)/q = \int_{\mathbb{S}} \mathcal{U}$, where c is uniquely defined by $\int_{\mathbb{S}} (q-c)/q = 0$, but of course the explicit value of c is not used in the final expression. In practice however it may be more straightforward to use an arbitrary primitive $\tilde{\mathcal{U}}$ of u (i.e. $\int_{\mathbb{S}} \tilde{\mathcal{U}}/q$ is not necessarily zero) for which we have

$$(u, q')_{1,1/q} = \int_{\mathbb{S}} \tilde{\mathcal{U}} \left(1 - \frac{c}{q}\right). \tag{2.3}$$

Since now c appears, let us make it explicit:

$$c = \frac{2\pi}{\int_{\mathbb{S}} 1/q} = \frac{1}{2\pi I_0^2(2Kr)}. \tag{2.4}$$

2.2 About the manifold M

As we have anticipated, we look at the set of stationary solutions M , defined in (2.2), as a manifold. For this we introduce

$$\tilde{H}_{-1} := \left\{ \mu : \mu - \frac{1}{2\pi} \in H_{-1} \right\}, \tag{2.5}$$

which is a metric space equipped with the distance inherited from H_{-1} , that is $\text{dist}(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{-1}$. We have $M \subset \tilde{H}_{-1}$ and M can be viewed as a smooth one dimensional manifold in \tilde{H}_{-1} . The tangent space at $q \in M$ is $q'\mathbb{R}$ and for every $u \in H_{-1}$ we define the projection P_q^o on this tangent space as $P_q^o u = (u, q')_{-1,1/q} q' / (q', q')_{-1,1/q}$. The following result is proven in [32, p. 501] (see also [22, Lemma 5.1]):

Lemma 2.4 *There exists $\sigma > 0$ such that for all $p \in N_\sigma$ with*

$$N_\sigma := \cup_{q \in M} \{ \mu \in \tilde{H}_{-1} : \|\mu - q\|_{-1} < \sigma \}, \tag{2.6}$$

there is one and only one $q =: v(\mu) \in M$ such that $(\mu - q, q')_{-1,1/q} = 0$. Furthermore, the mapping $\mu \mapsto v(\mu)$ is in $C^\infty(\tilde{H}_{-1}, \tilde{H}_{-1})$, and (with D the Fréchet derivative)

$$Dv(\mu) = P_{v(\mu)}^o. \quad (2.7)$$

Note that the empirical (probability) measure $\mu_{N,t}$ that describes our system at time t is in \tilde{H}_{-1} (see Appendix A) and Lemma 2.4 guarantees in particular that as soon as it is sufficiently close to M there is a well defined projection $v(\mu_{N,t})$ on the manifold. Since the manifold is isomorphic to \mathbb{S} it is practical to introduce, for $\mu \in \tilde{H}_{-1}$, also $\mathfrak{p}(\mu) \in \mathbb{S}$, uniquely defined by $v(\mu) = q_{\mathfrak{p}(\mu)}$. It is immediate to see that the projection \mathfrak{p} is $C^\infty(\tilde{H}_{-1}, \mathbb{S})$.

2.3 A quantitative heuristic analysis: the diffusion coefficient

The proof of Theorem 1.1 is naturally split into two parts: the approach to M and the motion on M . The approach to M is based on the properties of the PDE (1.4): in [21] it is shown, using the gradient flow structure of (1.4), that if the initial condition is not on the stable manifold U (see (1.7)) of the unstable stationary solution $\frac{1}{2\pi}$, then the solution converges for time going to infinity to one of the probability densities $q = q_\psi \in M$ (of course ψ is a function of the initial condition), so given a neighborhood of q_ψ after a finite time (how large it depends only on the initial condition), it gets to the chosen neighborhood: due to the regularizing properties of the PDE, such a neighborhood can be even in a topology that controls all the derivatives [21], but here there is no point to use a strong topology, since at the level of interacting diffusions we deal with a measure (that we inject into H_{-1}). And in fact we have to estimate the distance between the empirical measure and the solution to (1.4)—controlling thus the effect of the noise—but this type of estimates on finite time intervals is standard. However here there is a subtle point: the result we are after is a matter of fluctuations and it will not come as a surprise that the empirical measure approaches M but does not reach it (of course: M just contains smooth functions, and $\mu_{N,t}$ is not a function), but it will stay in a $N^{-1/2}$ -neighborhood (measured in the H_{-1} norm). How long will it take to reach such a neighborhood? The approach to M is actually exponential and driven by the spectral gap (λ_1) of the linearized evolution operator (at least close to M). Therefore in order to enter such a $N^{-1/2}$ -neighborhood a time proportional to $\log N$ appears to be needed, as the quick observation that $\exp(-\lambda_1 t) = O(N^{-1/2})$ for $t \geq \log N / (2\lambda_1)$ suggests. The proofs on this stage of the evolution are in Sect. 5: here we just stress that

1. controlling the effect of the noise on the system on times $O(\log N)$ is in any case sensibly easier than controlling it on times of order N , which is our final aim;
2. on times of order N it is no longer a matter of showing that the empirical measure stays close to the solution of the PDE: on such a time scale the noise takes over and the finite N system, which has a non-trivial (random) dynamics, substantially deviates from the behavior of the solution to the PDE, which just converges to one of the stationary profiles.

Let us therefore assume that the empirical measure is in a $N^{-1/2}$ -neighborhood of a given $q = q_\psi$. It is reasonable to assume that the dominating part of the dynamics close to q is captured by the operator L_q and we want to understand the action of the semigroup generated by L_q on the noise that stirs the system, on long times. Note

that we cannot choose arbitrarily long times, in particular not times proportional to N right away, because in view of the result we are after, we expect that on such a time scale the empirical measure of the system is no longer in a neighborhood of q , but close to $q_{\psi'}$ for a $\psi' \neq \psi$. We will actually choose some intermediate time scale $N^{1/10}$ as we will see in Sect. 2.4 and Remark 2.6, that guarantees that working with L_q makes sense, i.e. that the projection of the empirical measure on M is still sufficiently close to q . The point is that the effect of the noise on intermediate times is very different in the tangential direction and the orthogonal directions to M , simply because in the orthogonal direction there is a damping, that is absent in the tangential direction. So on intermediate times the leading term in the evolution of the empirical measure turns out to be the projection of the evolution on the tangential direction, that is $(q', \mu_{N,t} - q)_{-1,1/q} / \|q'\|_{-1,1/q}$. One can now use Remark 2.3 to obtain

$$(q', \mu_{N,t} - q)_{-1,1/q} = - \int_{\mathbb{S}} \mathcal{K}(\theta) (\mu_{N,t}(d\theta) - q(\theta)d\theta), \tag{2.8}$$

with \mathcal{K} a primitive of $1 - c/q$ (c given in Remark 2.3). By applying Itô's formula we see that the term in (2.8) can be written as the sum of a drift term and of a martingale term. It is not difficult to see that to leading order the drift term is zero (a more attentive analysis shows that one has to show that the next order correction does not give a contribution, but we come back to this below). The quadratic variation of the martingale term instead turns out to be equal to t/N times

$$\int_{\mathbb{S}} (\mathcal{K}'(\theta))^2 q(\theta)d\theta = 1 - \frac{(2\pi)^2}{\int_{\mathbb{S}} 1/q} = \|q'\|_{-1,1/q}^2. \tag{2.9}$$

Since $q_{\psi+\varepsilon} = q_{\psi} - \varepsilon q'_{\psi} + \dots$ (note that q'_{ψ} is not normalized), (2.9) suggests that the diffusion coefficient D_K in Theorem 1.1 is $\|q'\|_{-1,1/q}^{-1}$, which coincides with (1.11).

To make this procedure work one has to carefully put together the analysis on the intermediate time scale, by setting up an adequate iterative scheme. Several delicate issues arise and one of the challenging points is precisely to control that the drift can be neglected. In fact the first order expansion of the projection that we have used

$$\mathfrak{P}(q_{\psi} + h) = \psi - \frac{(h, q')_{-1,1/q}}{(q', q')_{-1,1/q}} + O(\|h\|_{-1}^2), \tag{2.10}$$

is not accurate enough and one has to go to the next order, see Lemma 7.5. This is due to the fact that the random contribution, which in principle appears as first order, fluctuates and generates a cancellation, so in the end the term is of second order.

Remark 2.5 It is natural to expect that Theorem 1.1 holds true also when $p_0 \in U$ and this is just because the evolution is attracted to $\frac{1}{2\pi}$ and then the noise will cause an escape from this unstable profile after a time $\propto \log N$, since the exponential instability will make the fluctuations grow exponentially with a rate which is just given by the linearized dynamics (linearized around $\frac{1}{2\pi}$ of course). Arguments in this spirit can

be found for example in [30, Ch. 5], see [3] and references therein for the finite dimensional counterpart. However

1. this is not so straightforward because it requires a good control on the dynamics on and around the heteroclinic orbits linking $\frac{1}{2\pi}$ to M_0 [21, Section 5];
2. the statement would require more details about the initial condition: the simple convergence to a point on U is largely non sufficient (the fluctuations of the initial conditions now matter!);
3. in general the initial phase ψ_0 on M is certainly going to be random: if the initial condition is rotation invariant (at least in law), like if $\{\varphi_0^{j,N}\}_{j=1,\dots,N}$ are IID variables uniformly distributed on \mathbb{S} or if $\varphi_0^{j,N} = 2\pi j/N$, one expects ψ_0 to be uniformly distributed on \mathbb{S} . Note however that uniform distribution of ψ_0 is definitely not expected in the general case and asymmetries in the initial condition should affect the distribution of ψ_0 .

2.4 The iterative scheme

As we have explained in Sect. 2.3, the analysis close to M requires an iterative procedure, which we introduce here. We assume that at $t = 0$ the system is already close to M , while in practice this will happen after some time: in Sect. 6 we explain how to put together the results on the early stage of the evolution and the analysis close to M , that we start here. So, for $\mu_0 = \mu_{N,0} = \frac{1}{N} \sum_{j=1}^N \delta_{\varphi_0^{j,N}}$ such that $\text{dist}(\mu_0, M) \leq \sigma$ (here and below $\text{dist}(\cdot, \cdot)$ is the distance built with the norm of H_{-1}), by Lemma 2.4 we can define $\psi_0 = \mathbb{P}(\mu_0)$. Applying the Itô formula to $v_t = \mu_t - q_{\psi_0}$ ($\mu_t = \mu_{N,t}$), we see that

$$v_t = e^{-tL_{\psi_0}} v_0 - \int_0^t e^{-(t-s)L_{\psi_0}} \partial_{\theta} [v_s J * v_s] ds + Z_t, \tag{2.11}$$

where

$$Z_t = \frac{1}{N} \sum_{j=1}^N \int_0^t \partial_{\theta'} \mathcal{G}_{t-s}^{\psi_0} (\theta, \varphi_s^{j,N}) dW_s^j, \tag{2.12}$$

and $\mathcal{G}_s^{\psi_0}(\theta, \theta')$ is the kernel of $e^{-sL_{\psi_0}}$ in \mathbb{L}^2 . The evolution equation (2.11) and the noise term (2.12) have a meaning in H_{-1} , as well as the recentered empirical measures v_t , and it is in this sense that we will use them: we detail this in Appendix A, where one finds also an explicit expression and some basic facts about the kernel $\mathcal{G}_s^{\psi_0}(\theta, \theta')$. We have started here an abuse of notation that will be persistent through the text: $\partial_{\theta'} \mathcal{G}_{t-s}^{\psi_0}(\theta, \varphi_s^{j,N})$ stands for $\partial_{\theta'} \mathcal{G}_{t-s}^{\psi_0}(\theta, \theta')|_{\theta'=\varphi_s^{j,N}}$.

Equations (2.11)–(2.12) are useful tools as long as we can properly define the phase associated to the empirical measure of the system and that this phase is close to ψ_0 : in view of the result we want to prove, this is expected to be true for a long time, but

it is certainly expected to fail for times of the order of N , since on this timescale the phase does change of an amount that does not vanish as N becomes large.

The idea is therefore to divide the evolution of the particle system up to a final time proportional to N into $n = n_N \xrightarrow{N \rightarrow \infty} \infty$ time intervals $[T_i, T_{i+1}]$, where $T_i = iT$ and $T = T(N)$ is chosen close to a fractional power of N (see Remark 2.6). Moreover i runs from 1 up to $n = n_N$ so that $n_N T(N) = T_{n_N}$ and $\lim_N T_{n_N}/N$ is equal to a positive constant (the τ_f of Theorem 1.1). If the empirical measure μ_t stays close to the manifold M , we can define the projections of μ_{T_k} and successively update the reentering phase at all times T_k . The point then will be essentially to show that the process given by these phases, on the time scale $\propto N$, converges to a Brownian motion.

More formally, we construct the following iterative scheme: we choose

$$\sigma = \sigma_N := \left[N^{2\zeta} \sqrt{T/N} \right] \xrightarrow{N \rightarrow \infty} 0, \tag{2.13}$$

for a suitable $\zeta > 0$ (see Remark 2.6), we set $\tau_{\sigma_N}^0 = 0$ and for $k = 1, 2, \dots$ we define

$$\psi_{k-1} := \mathbb{P}(\mu_{T_{k-1}}), \tag{2.14}$$

if $\text{dist}(\mu_{T_{k-1}}, M) \leq \sigma_N$ and

$$\tau_{\sigma_N}^k = \tau_{\sigma_N}^{k-1} \mathbf{1}_{\{\tau_{\sigma_N}^{k-1} < T_{k-1}\}} + \inf\{s \in [T_{k-1}, T_k] : \|\mu_s - q_{\psi_{k-1}}\|_{-1} > \sigma_N\} \mathbf{1}_{\{\tau_{\sigma_N}^{k-1} \geq T_{k-1}\}}. \tag{2.15}$$

Then we set

$$v_t^k := \mu_t - q_{\psi_{k-1}}, \tag{2.16}$$

for $t \in [T_{k-1}, T_k]$ and $t \leq \tau_{\sigma_N}^k$, and otherwise $v_t^k := v_{\tau_{\sigma_N}^k}^k$ for every $t \geq \tau_{\sigma_N}^k$ (of course $\tau_{\sigma_N}^k$ can be smaller than T_{k-1} and, in this case, the definition becomes redundant). Therefore the ν process we have just defined solves for $t \in [T_{k-1}, T_k]$

$$v_t^k = \mathbf{1}_{\{\tau_{\sigma_N}^k < T_{k-1}\}} v_{T_{k-1}}^k + \mathbf{1}_{\{\tau_{\sigma_N}^k \geq T_{k-1}\}} \times \left(e^{-(t \wedge \tau_{\sigma_N}^k - T_{k-1})L_{\psi_{k-1}}} v_{T_{k-1}}^k - \int_{T_{k-1}}^{t \wedge \tau_{\sigma_N}^k} e^{-(t \wedge \tau_{\sigma_N}^k - s)L_{\psi_{k-1}}} \partial_{\theta'} [v_s^k J * v_s^k] ds + Z_{t \wedge \tau_{\sigma_N}^k}^k \right), \tag{2.17}$$

where

$$Z_t^k = \frac{1}{N} \sum_{j=1}^N \int_{T_{k-1}}^t \partial_{\theta'} \mathcal{G}_{t-s}^{\psi_{k-1}} (\theta, \varphi_s^{j,N}) dW_s^j. \tag{2.18}$$

Once again, we refer to Appendix A for the precise meaning of (2.17) and (2.18).

Remark 2.6 For the remainder of the paper we choose $T(N) \sim N^{1/10}$ and $\zeta \leq 1/100$. The two exponents do not have any particular meaning: a look at the argument shows that the exponent for $T(N)$ has in any case to be chosen smaller than $1/2$, but then a number of technical estimates enter the game and we have settled for a value $1/10$ without trying to get the optimal value that comes out of the method we use.

3 A priori estimates: persistence of proximity to M

The aim of this section is to prove that, if we are (say, at time zero) sufficiently close to M , we stay close to M for times $O(N)$. The arguments in this section justify the choice of the proximity parameter σ_N that we have made in the iterative scheme. We first prove some estimates on the size of the noise term and then we will give the estimates on the empirical measure.

3.1 Noise estimates

We define the event

$$B^N = \left\{ \sup_{1 \leq k \leq n-1} \sup_{t \in [T_k, T_{k+1}]} \|Z_t^k\|_{-1} \leq \sqrt{\frac{T}{N}} N^\zeta \right\} \cap \left\{ \sup_{1 \leq k \leq n-1} \sup_{t \in [T_k, T_{k+1}]} \|Z_t^{k,\perp}\|_{-1} \leq \frac{1}{\sqrt{N}} N^\zeta \right\}, \tag{3.1}$$

where $Z_t^{k,\perp}$ is defined precisely like Z_t^k , see (2.18), except for the replacement of $\mathcal{G}_{t-s}^\psi(\cdot, \cdot)$ with $\mathcal{G}_{t-s}^\psi(\theta, \theta') - e_{\psi_{k-1},0}(\theta) f_{\psi_{k-1},0}(\theta')$ [see (3.3)].

Lemma 3.1 $\lim_{N \rightarrow \infty} \mathbb{P}(B^N) = 1$.

Proof In order to perform the estimates we introduce and work with approximated versions of Z_t^k and $Z_t^{k,\perp}$ (see Lemma 7.4). Define for $T_{k-1} < t' < t$

$$Z_{t,t'}^k = \frac{1}{N} \sum_{j=1}^N \int_{T_{k-1}}^{t'} \partial_{\theta'} \mathcal{G}_{t-s}^{\psi_{k-1}}(\theta, \varphi_s^{j,N}) dW_s^j. \tag{3.2}$$

The kernel $\mathcal{G}^{\psi_{k-1}}$ in this case is (cf. Appendix A)

$$\mathcal{G}_s^{\psi_{k-1}}(\theta, \theta') = \sum_{l=0}^{\infty} e^{-s\lambda_l} e_{\psi_{k-1},l}(\theta) f_{\psi_{k-1},l}(\theta'), \tag{3.3}$$

where λ_l are the ordered eigenvalues of $-L_{\psi_{k-1}}$, $e_{\psi_{k-1},l}$ are the associated eigenfunctions of unit norm in $H_{-1,1/q_{\psi_{k-1}}}$, cf. Remark 2.2, and $f_{\psi_{k-1},l}$ are the eigenfunctions of $L_{\psi_{k-1}}^*$, the adjoint in \mathbb{L}^2 (see Appendix A).

Very much in the same way we define

$$Z_{t,t'}^{k,\perp} = \frac{1}{N} \sum_{j=1}^N \int_{T_{k-1}}^{t'} \partial_{\theta'} \mathcal{G}_{t-s}^{\psi_{k-1,\perp}}(\theta, \varphi_s^{j,N}) dW_s^j, \tag{3.4}$$

with

$$\mathcal{G}_s^{\psi_{k-1,\perp}}(\theta, \theta') = \sum_{l=1}^{\infty} e^{-s\lambda_l} e_{\psi_{k-1,l}}(\theta) f_{\psi_{k-1,l}}(\theta'). \tag{3.5}$$

We decompose for $T_{k-1} < s' < s < t$ and $s' < t' < t$

$$\begin{aligned} Z_{t,t'}^k - Z_{s,s'}^k &= \frac{1}{N} \sum_{j=1}^N \int_{T_{k-1}}^{s'} \left(\partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\theta, \varphi_u^{j,N}) - \partial_{\theta'} \mathcal{G}_{s-u}^{\psi_{k-1}}(\theta, \varphi_u^{j,N}) \right) dW_u^j \\ &\quad + \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\theta, \varphi_u^{j,N}) dW_u^j, \end{aligned} \tag{3.6}$$

and an absolutely analogous formula holds for $Z^{k,\perp}$: in fact the bounds for Z^k and $Z^{k,\perp}$ are obtained with the same technique even if the results are slightly different due to the presence of the zero eigenvalue in Z^k . Moreover we apply $\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$, so that we can estimate the two terms in the right-hand side of (3.6) separately.

And we start with the second term of the right-hand side in (3.6): by the orthogonality properties of the eigenvectors we obtain

$$\begin{aligned} &\left\| \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) dW_u^j \right\|_{-1,1/q}^2 \\ &= \frac{1}{N^2} \sum_{l=0}^{\infty} \sum_{j,j'=1}^N \int_{s'}^{t'} \int_{s'}^{t'} e^{-(2t-u-u')\lambda_l} f'_{\psi_{k-1,l}}(\varphi_u^{j,N}) f'_{\psi_{k-1,l}}(\varphi_{u'}^{j',N}) dW_u^j dW_{u'}^{j'}, \end{aligned} \tag{3.7}$$

and by taking the expectation

$$\begin{aligned} &\mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) dW_u^j \right\|_{-1,1/q}^2 \right] \\ &= \frac{1}{N^2} \sum_{l=0}^{\infty} \sum_{j=1}^N \int_{s'}^{t'} e^{-2(t-u)\lambda_l} \mathbb{E} \left[(f'_{\psi_{k-1,l}}(\varphi_u^{j,N}))^2 \right] du. \end{aligned} \tag{3.8}$$

By Corollary 8.6 there exists a constant C_1 such that

$$\mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) dW_u^j \right\|_{-1,1/q}^2 \right] \leq \frac{C_1}{N} \sum_{l=0}^{\infty} \int_{s'}^{t'} e^{-2(t-u)\lambda_l} du. \tag{3.9}$$

Proposition 8.4, Remark 8.3, leads us to

$$\sum_{l=0}^{\infty} \int_{s'}^{t'} e^{-2(t-u)\lambda_l} du \leq \sum_{l=0}^{\infty} \int_{s'}^{t'} e^{-(t-u)\frac{l^2}{C}} du \leq C \sum_{l=0}^{\infty} \frac{1}{l^2} \left(1 - e^{-(t'-s')\frac{l^2}{C}} \right), \tag{3.10}$$

where the addend with $l = 0$ (times C) has to be read as $t' - s'$. The right-most term in (3.10) for $t' - s' \geq 1$ can be bounded by $C(t' - s') + C \sum_{l=1}^{\infty} 1/l^2 \leq 3C(t' - s')$. Instead for $t' - s' < 1$ we decompose the same term and then estimate as follows:

$$\begin{aligned} & C \sum_{l=0}^{\lfloor (t'-s')^{-1/2} \rfloor} \frac{1}{l^2} \left(1 - e^{-(t'-s')\frac{l^2}{C}} \right) + C \sum_{l=\lfloor (t'-s')^{-1/2} \rfloor + 1}^{\infty} \frac{1}{l^2} \left(1 - e^{-(t'-s')\frac{l^2}{C}} \right) \\ & \leq \sum_{l=0}^{\lfloor (t'-s')^{-1/2} \rfloor} (t' - s') + \sum_{l=\lfloor (t'-s')^{-1/2} \rfloor + 1}^{\infty} \frac{C}{l^2} \leq (3 + 2C)\sqrt{t' - s'}, \end{aligned} \tag{3.11}$$

where for the first term we have used $(1 - \exp(-a)) \leq a$, for $a \geq 0$. Therefore we have proven that there exists C such that for every k and every s, s', t, t' such that $T_{k-1} < s' < s < t$ and $s' < t' < t$ we have

$$\mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) dW_u^j \right\|_{-1,1/q}^2 \right] \leq \frac{Ch_1(t' - s')}{N}. \tag{3.12}$$

with

$$h_1(u) := u^{1/2} \mathbf{1}_{(0,1)}(u) + u \mathbf{1}_{[1,\infty)}. \tag{3.13}$$

We can do better in the case of $\mathcal{G}^{\psi_{k-1,\perp}}$, for which a direct inspection of the argument we have just presented shows that the linearly growing term in the estimate can be avoided (since the term $l = 0$ is no longer there) and the net result is

$$\mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1,\perp}}(\cdot, \varphi_u^{j,N}) dW_u^j \right\|_{-1,1/q}^2 \right] \leq \frac{Ch_2(t' - s')}{N}. \tag{3.14}$$

with h_2 defined as

$$h_2(u) = u^{1/2} \mathbf{1}_{[0,1)}(u) + \mathbf{1}_{[1,\infty)}(u). \tag{3.15}$$

For what concerns the first term in the right-hand side of (3.6), we have

$$\begin{aligned} & \mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{T_{k-1}}^{s'} \left(\partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) - \partial_{\theta'} \mathcal{G}_{s-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) \right) dW_u^j \right\|_{-1,1/q}^2 \right] \\ &= \frac{1}{N^2} \sum_{l=1}^{\infty} \sum_{j=1}^N \int_{T_{k-1}}^{s'} \left(e^{-(t-u)\lambda_l} - e^{-(s-u)\lambda_l} \right)^2 \mathbb{E} \left[(f'_{\psi_{k-1,l}}(\varphi_u^{j,N}))^2 \right] du, \end{aligned} \tag{3.16}$$

and, by proceeding like for (3.9), we see that the expression in (3.16) is bounded by

$$\frac{C_1}{N} \sum_{l=1}^{\infty} \frac{\left(1 - e^{-\lambda_l(t-s')} \right)^2}{\lambda_l} \leq \frac{C_1 C}{N} \sum_{l=1}^{\infty} \frac{\left(e^{-(t-s')\frac{l^2}{C}} - 1 \right)^2}{l^2}. \tag{3.17}$$

This last term is estimated once again by separating the two cases of $t - s'$ small and large. The net result is that there exists $C > 0$ such that for every k , every s, s' and t such that $T_k < s' < s < t$ we have

$$\begin{aligned} & \mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{T_{k-1}}^{s'} \left(\partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) - \partial_{\theta'} \mathcal{G}_{s-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) \right) dW_u^j \right\|_{-1,1/q}^2 \right] \\ & \leq Ch_2(t - s'). \end{aligned} \tag{3.18}$$

In order to complete the proof of Lemma 3.1 quadratic estimates do not suffice: we need to generalize (3.12), (3.14) and (3.18) to larger exponents. We actually need estimates on moments of order $2m$, with m finite, but sufficiently large, so to apply the standard Kolmogorov Lemma type estimates and get uniform bounds. We are going to use

$$\|a + b\|^m \leq m(\|a\|^m + \|b\|^m), \tag{3.19}$$

but actually we will not track the m dependence of the constants. We aim at showing that the expectation of the moments of order $2m$ of the quantities we are interested in are bounded by the m th power of the estimate we found in the quadratic case, times an m -dependent constant.

For $m = 1, 2, \dots$, the m th-power of the expression in (3.7) gives

$$\begin{aligned} & \left\| \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\theta, \varphi_u^{j,N}) dW_u^j \right\|_{-1,1/q}^{2m} \\ &= \frac{1}{N^{2m}} \sum_{l_1, \dots, l_m=0}^{\infty} \sum_{j_1, j_1', \dots, j_m, j_m'=1}^N F_{l_1}^{j_1}(t, s', t') F_{l_1}^{j_1'}(t, s', t') \dots \\ & \quad \times F_{l_m}^{j_m}(t, s', t') F_{l_m}^{j_m'}(t, s', t'), \end{aligned} \tag{3.20}$$

in which we have introduced the random variables

$$F_l^j(t, s', t') = \int_{s'}^{t'} e^{-\lambda_l(t-u)} f'_{\psi_{k-1,l}}(\varphi_u^{j,N}) dW_u^j. \tag{3.21}$$

We now take the expectation of both terms in (3.20) and all the terms in the sum that do not include an even number of each Brownian motion vanish. The number of non-zero terms in the expectation can thus be bounded by $(2m)!N^m$. Applying the Itô formula to each of these non-zero terms, we get at most $(2m)!/(2^m m!)$ terms (the number of possibilities classifying $2m$ elements in couples) of the type $I_1 \dots I_m$, where

$$I_k = I_k(l_1, l_2) = \int_{s'}^{t'} e^{-(\lambda_{l_1} + \lambda_{l_2})(t-u)} \mathbb{E} \left[f'_{\psi_{k-1,l_1}}(\varphi_u^{j,N}) f'_{\psi_{k-1,l_2}}(\varphi_u^{j,N}) \right] du. \tag{3.22}$$

We now observe that

$$|I_k(l_1, l_2)| \leq \sqrt{I_k(l_1, l_1) I_k(l_2, l_2)}, \tag{3.23}$$

and for each index l_i in the first sum in the right-hand side of (3.20) gives rise either directly to a term $I_k(l_i)$ (for this it is needed that the two terms share the Brownian motion), or in the arising products the terms $I_k(l_i, l_i')$ are associated with a term of the type $I_k(l_i, l_i')$. Therefore the expression obtained after applying Itô formula can be bounded by a sum of terms of the type $\hat{I}_1 \dots \hat{I}_m$, with

$$\hat{I}_k = I_k(l, l) = \int_{s'}^{t'} e^{-2\lambda_l(t-u)} \mathbb{E} \left[(f'_{\psi_{k-1,l}}(\varphi_u^{j,N}))^2 \right] du. \tag{3.24}$$

Therefore we are facing the same estimates that we have encountered in the quadratic case, see (3.8) and (3.9), except of course for combinatorial contribution. In the end

we obtain that there exists $C = C_m$ such that

$$\mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) dW_u^j \right\|_{-1,1/q}^{2m} \right] \leq C \frac{h_1^m(t' - s')}{N^m}, \tag{3.25}$$

$$\mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{s'}^{t'} \partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}, \perp}(\cdot, \varphi_u^{j,N}) dW_u^j \right\|_{-1,1/q}^{2m} \right] \leq C \frac{h_2^m(t' - s')}{N^m}. \tag{3.26}$$

In a similar way

$$\begin{aligned} & \left\| \frac{1}{N} \sum_{j=1}^N \int_{T_{k-1}}^{s'} \left(\partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) - \partial_{\theta'} \mathcal{G}_{s-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) \right) dW_u^j \right\|_{-1,1/q}^{2m} \\ &= \frac{1}{N^{2m}} \sum_{l_1, \dots, l_m=0}^{\infty} \sum_{j_1, j'_1, \dots, j_m, j'_m=1}^N G_{l_1}^{j_1}(s, t, s') G_{l_1}^{j'_1}(s, t, s') \cdots \\ & \quad \times G_{l_m}^{j_m}(s, t, s') G_{l_m}^{j'_m}(s, t, s') \end{aligned} \tag{3.27}$$

with

$$G_l^j(s, t, s') = \int_{T_{k-1}}^{s'} \left(e^{-\lambda_l(t-u)} - e^{-\lambda_l(s-u)} \right) f'_{\psi_{k-1}, l}(\varphi_u^{j,N}) dW_u^j. \tag{3.28}$$

We reduce the problem as above to the study of products of integral terms $J_1 \cdots J_k$ with

$$\begin{aligned} J_k &= \int_{T_{k-1}}^{s'} \left(e^{-\lambda_{l_1}(t-u)} - e^{-\lambda_{l_1}(s-u)} \right) \left(e^{-\lambda_{l_2}(t-u)} - e^{-\lambda_{l_2}(s-u)} \right) \\ & \quad \times \mathbb{E} \left[f'_{\psi_{k-1}, l_1}(\varphi_u^{j,N}) f'_{\psi_{k-1}, l_2}(\varphi_u^{j,N}) \right] du, \end{aligned} \tag{3.29}$$

and then, like before, in terms of products of *diagonal* terms of the type

$$\hat{J}_k = \int_{T_{k-1}}^{s'} \left(e^{-\lambda_l(t-u)} - e^{-\lambda_l(s-u)} \right)^2 \mathbb{E} \left[(f'_{\psi_{k-1}, l}(\varphi_u^{j,N}))^2 \right] du. \tag{3.30}$$

Again we are reduced to the estimating terms that have already appeared in the quadratic case, see (3.16), so we obtain that there exists $C = C_m$ such that

$$\mathbb{E} \left[\left\| \frac{1}{N} \sum_{j=1}^N \int_{T_{k-1}}^{s'} \left(\partial_{\theta'} \mathcal{G}_{t-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) - \partial_{\theta'} \mathcal{G}_{s-u}^{\psi_{k-1}}(\cdot, \varphi_u^{j,N}) \right) dW_u^j \right\|_{-1,1/q}^{2m} \right] \leq C \frac{h_2^m(t-s)}{N^m}. \tag{3.31}$$

We now let $t' \nearrow t$ and $s' \nearrow s$ and by applying Fatou’s Lemma and Lemma 7.4, from(3.6), (3.25), (3.26) and (3.31) we get

$$\mathbb{E} \left[\left\| Z_t^k - Z_s^k \right\|_{-1}^{2m} \right] \leq C \frac{h_1^m(t-s)}{N^m}, \tag{3.32}$$

and

$$\mathbb{E} \left[\left\| Z_t^{k,\perp} - Z_s^{k,\perp} \right\|_{-1}^{2m} \right] \leq C \frac{h_2^m(t-s)}{N^m}. \tag{3.33}$$

The fact that we are allowed to drop the weight in the H_{-1} norm is of course due to the norm equivalence.

We are now in good condition to apply the Garsia–Rodemich–Rumsey Lemma [33]:

Lemma 3.2 *Let p and Ψ be continuous, strictly increasing functions on $(0, \infty)$ such that $p(0) = \Psi(0) = 0$ and $\lim_{t \nearrow \infty} \Psi(t) = \infty$. Given $T > 0$ and ϕ continuous on $(0, T)$ and taking its values in a Banach space $(E, \|\cdot\|)$, if*

$$\int_0^T \int_0^T \Psi \left(\frac{\|\phi(t) - \phi(s)\|}{p(|t-s|)} \right) ds dt \leq B < \infty, \tag{3.34}$$

then for $0 \leq s \leq t \leq T$:

$$\|\phi(t) - \phi(s)\| \leq 8 \int_0^{t-s} \Psi^{-1} \left(\frac{4B}{u^2} \right) p(du). \tag{3.35}$$

We apply Lemma 3.2 with

$$\phi(t) = Z_{t-T_{k-1}}^k, \quad p(u) = u^{\frac{2+\zeta}{2m}} \quad \text{and} \quad \Psi(u) = u^{2m}, \tag{3.36}$$

and $\zeta = 1/100$ (Remark 2.6). With these choices we can find an explicit constant $C = C(m, \zeta)$ such that

$$\|Z_t^k - Z_s^k\|_{-1}^{2m} \leq C(t-s)^\zeta B, \tag{3.37}$$

for every s and t such that $T_{k-1} \leq s < t \leq T_k$ and B is a positive random variable such that

$$\mathbb{E}[B] \leq \frac{C}{N^m} \int_0^T \int_0^T \frac{h_1^m(|t-s|)}{|t-s|^{2+\zeta}} ds dt, \tag{3.38}$$

where C is the constant in (3.32). For $m > 4$ the function $t \mapsto h_1^m(t)/t^{2+\zeta}$, defined for $t > 0$, is increasing (and it tends to zero for $t \searrow 0$). So $\mathbb{E}[B]$ is bounded by $C N^{-m} h_1^m(T)/T^\zeta$ and therefore

$$\mathbb{E} \left[\sup_{T_{k-1} \leq s < t \leq T_k} \frac{\|Z_t^k - Z_s^k\|_{-1}^{2m}}{|t-s|^\zeta} \right] \leq C \frac{T^{m-\zeta}}{N^m}, \tag{3.39}$$

which leads to

$$\mathbb{P} \left[\sup_{T_{k-1} \leq t \leq T_k} \|Z_t^k\|_{-1} \geq \sqrt{\frac{T}{N}} N^\zeta \right] \leq C \frac{1}{N^{m\zeta}}. \tag{3.40}$$

Then, (recall $n = n_N = \frac{N}{T}$) we deduce

$$\mathbb{P} \left[\sup_{1 \leq k \leq n} \sup_{T_{k-1} \leq t \leq T_k} \|Z_t^k\|_{-1} \geq \sqrt{\frac{T}{N}} N^\zeta \right] \leq C \frac{1}{T N^{m\zeta-1}}, \tag{3.41}$$

where the right hand side tends to 0 when m is chosen sufficiently large. A similar argument gives for $Z_t^{k,\perp}$

$$\mathbb{P} \left[\sup_{1 \leq k \leq n} \sup_{T_{k-1} \leq t \leq T_k} \|Z_t^{k,\perp}\|_{-1} \geq \frac{1}{\sqrt{N}} N^\zeta \right] \leq C \frac{T^{\zeta-1}}{N^{m\zeta-1}}. \tag{3.42}$$

□

We now give the main result of the section:

Proposition 3.3 *If $\|v_0^1\|_{-1} \leq \frac{N^{2\zeta}}{\sqrt{N}}$ and if the event B^N defined in (3.1) is realized (then, with probability approaching 1 as $N \rightarrow \infty$) we have*

$$\sup_{1 \leq k \leq n} \sup_{t \in [T_{k-1}, T_k]} \|v_t^k\|_{-1} \leq \sqrt{\frac{T}{N}} N^{2\zeta}, \tag{3.43}$$

and

$$\max_{1 \leq k \leq n} \|v_{T_{k-1}}^k\|_{-1} \leq \frac{N^{2\zeta}}{\sqrt{N}}. \tag{3.44}$$

Proof From (2.17) and Lemma 7.2, we get, for all $k = 1 \dots n$ and $t \in [T_{k-1}, T_k]$

$$\|v_t^k\|_{-1} \leq C e^{-\lambda_1(t-T_{k-1})} \|v_{T_{k-1}}^k\|_{-1} + C \int_{T_{k-1}}^t \left(1 + \frac{1}{\sqrt{t-s}}\right) \|v_s^k\|_{-1}^2 ds + \|Z_t^k\|_{-1}. \tag{3.45}$$

The constant C in front of the first term of the right hand side above would be equal to 1 if we were using the $\|\cdot\|_{-1, 1/q\psi_{k-1}}$ norm. Let us assume that $\|v_{T_{k-1}}^k\|_{-1} \leq N^{2\zeta}/\sqrt{N}$, since we are working in B^N we obtain

$$\|v_t^k\|_{-1} \leq C e^{-\lambda_1(t-T_{k-1})} \frac{N^{2\zeta}}{\sqrt{N}} + C(T + \sqrt{T}) \sup_{T_{k-1} \leq s \leq t} \|v_s^k\|_{-1}^2 + \frac{\sqrt{T}}{\sqrt{N}} N^\zeta. \tag{3.46}$$

Therefore we readily see that if we define

$$t^* = \sup \left\{ t \in [T_{k-1}, T_k] : \|v_t^k\|_{-1} > \frac{\sqrt{T}}{\sqrt{N}} N^{2\zeta} \right\}, \tag{3.47}$$

we have that for $t \leq t^*$

$$\|v_t^k\|_{-1} \leq C N^{2\zeta - \frac{1}{2}} + 2CT^2 N^{4\zeta - 1} + \sqrt{T} N^{\zeta - \frac{1}{2}}. \tag{3.48}$$

Therefore since $\lim_N T^3 N^{-1+4\zeta} = 0$ (see Remark 2.6), for N large enough, we have $t^* = T_k$ and (3.43) is reduced to proving $\|v_{T_{k-1}}^k\|_{-1} \leq N^{2\zeta}/\sqrt{N}$ for $k = 1, 2, \dots, n$. This holds for $k = 1$: we are now going to show by induction (3.44) and therefore that the assumption propagates from k to $k + 1$.

To prove the bound on $v_{T_k}^{k+1}$, assuming the bound on $v_{T_{k-1}}^k$, we use the smoothness of the manifold M . Since we are working in B^N , $\tau_\sigma^k = T_k$ and we have

$$\begin{aligned} v_{T_k}^{k+1} &= q_{\psi_{k-1}} + v_{T_k}^k - q_{\psi_k} \\ &= P_{\psi_k}^\perp \left[q_{\psi_{k-1}} + v_{T_k}^k - q_{\psi_k} \right] \\ &= \left(P_{\psi_k}^\perp - P_{\psi_{k-1}}^\perp \right) \left[q_{\psi_{k-1}} + v_{T_k}^k - q_{\psi_k} \right] + P_{\psi_{k-1}}^\perp \left[q_{\psi_{k-1}} - q_{\psi_k} \right] + P_{\psi_{k-1}}^\perp v_{T_k}^k, \end{aligned} \tag{3.49}$$

Since the mapping $\psi \mapsto P_\psi^\perp$ is smooth on the compact M , we have (cf. Sect. 2.2)

$$\|P_{\psi_k}^\perp - P_{\psi_{k-1}}^\perp\|_{\mathcal{L}(H_{-1}, H_{-1})} \leq C |\psi_k - \psi_{k-1}|, \tag{3.50}$$

and the identities

$$\psi_k - \psi_{k-1} = \mathbb{P}(\mu_{T_k}) - \mathbb{P}(\mu_{T_{k-1}}), \tag{3.51}$$

and

$$\mu_{T_k} - \mu_{T_{k-1}} = v_{T_k}^k - v_{T_{k-1}}^k, \tag{3.52}$$

combined with the smoothness of \mathfrak{p} , lead to [using (3.43)]

$$\left\| P_{\psi_k}^\perp - P_{\psi_{k-1}}^\perp \right\|_{\mathcal{L}(H_{-1}, H_{-1})} \leq C \frac{\sqrt{T}}{\sqrt{N}} N^{2\zeta}. \tag{3.53}$$

On the other hand, the smoothness of q_ψ with respect to ψ , (3.51) and (3.52) imply

$$\left\| q_{\psi_{k-1}} + v_{T_k}^k - q_{\psi_k} \right\|_{-1} \leq C \left(\|v_{T_{k-1}}^k\|_{-1} + \|v_{T_k}^k\|_{-1} \right), \tag{3.54}$$

so the first term in the last line of (3.49) is of order $\frac{T}{N} N^{4\zeta}$, which is much smaller than $\frac{N^{2\zeta}}{\sqrt{N}}$ nor $N \rightarrow \infty$, since $\lim_N T N^{2\zeta - \frac{1}{2}} = 0$ (see Remark 2.6). Moreover, Lemma 2.4 implies

$$\left\| P_{\psi_{k-1}}^\perp [q_{\psi_{k-1}} - q_{\psi_k}] \right\|_{-1} = \left\| P_{\psi_{k-1}}^\perp [v(\mu_{T_{k-1}}) - v(\mu_{T_k})] \right\|_{-1} \leq C \|\mu_{T_{k-1}} - \mu_{T_k}\|_{-1}, \tag{3.55}$$

so the second term in the last line of (3.49) is also of order $\frac{T}{N} N^{4\zeta}$. Finally, projecting (2.17) on $\text{Range}(L_{q_{\psi_{k-1}}})$ and by using again Lemma 7.2, we get

$$\begin{aligned} \left\| P_{\psi_{k-1}}^\perp v_t^k \right\|_{-1} &\leq C e^{-\lambda_1(t-T_{k-1})} \|v_{T_{k-1}}^k\|_{-1} \\ &+ C \int_{T_{k-1}}^t \left(1 + \frac{1}{\sqrt{t-s}} \right) \|v_s^k\|_{-1}^2 ds + \|Z_t^{k,\perp}\|_{-1}, \end{aligned} \tag{3.56}$$

which, since $\lim_N T^4 N^{1-5\zeta} = 0$ (see Remark 2.6), leads for N large enough to

$$\left\| P_{\psi_{k-1}}^\perp v_{T_k}^k \right\|_{-1} \leq \frac{N^{3\zeta/2}}{\sqrt{N}}. \tag{3.57}$$

This takes care of the third term in the last line of (3.49) and by collecting the three estimates we obtain (3.44) and the proof is complete. \square

4 The effective dynamics on the tangent space

The following result states that each rotation increment of our discretization scheme is well approximated by the projection the dynamical noise on the tangent space.

Proposition 4.1 *We have the first order approximation in probability: for every $\varepsilon > 0$*

$$\mathbb{P}\left(\left|\sum_{k=1}^n(\psi_k - \psi_{k-1}) - \sum_{k=1}^n \frac{(Z_{T_k}^k, q'_{\psi_{k-1}})_{-1,1/q\psi_{k-1}}}{(q', q')_{-1,1/q}}\right| \leq \varepsilon\right) = 1. \tag{4.1}$$

Proof Lemma 7.5 and Proposition 3.3 give (assuming that B^N is realized: we will do this through all the proof)

$$\begin{aligned} \psi_k - \psi_{k-1} &= -\frac{(v_{T_k}^k, q'_{\psi_{k-1}})_{-1,1/q\psi_{k-1}}}{(q', q')_{-1,1/q}} \\ &\quad - \frac{1}{2\pi I_0^2(2Kr)} \frac{(v_{T_k}^k, (\log q_{\psi_{k-1}})'')_{-1,1/q\psi_{k-1}}}{(q', q')_{-1,1/q}} \frac{(v_{T_k}^k, q'_{\psi_{k-1}})_{-1,1/q\psi_{k-1}}}{(q', q')_{-1,1/q}} \\ &\quad + o\left(\frac{T}{N}\right). \end{aligned} \tag{4.2}$$

Since $\log(q_{\psi_{k-1}})''$ is in $R(L_{q_{\psi_{k-1}}})$, we have

$$(v_{T_k}^k, (\log q_{\psi_{k-1}})'')_{-1,1/q\psi_{k-1}} = ((v_{T_k}^k)^\perp, (\log q_{\psi_{k-1}})'')_{-1,1/q\psi_{k-1}}, \tag{4.3}$$

and thus using again Proposition 3.3 we get for the second term of the right-hand side

$$\begin{aligned} &\left\| \frac{1}{2\pi I_0^2(2Kr)} \frac{(v_{T_k}^k, (\log q_{\psi_{k-1}})'')_{-1,1/q\psi_{k-1}}}{(q', q')_{-1,1/q}} \frac{(v_{T_k}^k, q'_{\psi_{k-1}})_{-1,1/q\psi_{k-1}}}{(q', q')_{-1,1/q}} \right\|_{-1} \\ &\leq C \frac{1}{\sqrt{N}} N^{2\zeta} \frac{\sqrt{T}}{\sqrt{N}} N^{2\zeta}, \end{aligned} \tag{4.4}$$

and hence it is $o(T/N)$, since $\lim_N N^{4\zeta}/\sqrt{T} = 0$ (see Remark 2.6). So only the component on the tangent space of M at the point ψ_{k-1} is of order T/N :

$$\psi_k - \psi_{k-1} = -\frac{(v_{T_k}^k, q'_{\psi_{k-1}})_{-1,1/q\psi_{k-1}}}{(q', q')_{-1,1/q}} + o\left(\frac{T}{N}\right). \tag{4.5}$$

We now decompose this tangent term. Our goal is to show that the projection of the noise Z_{T_k} is the only term that gives a non negligible contribution when N goes to infinity. However, a direct domination of the remainder—the nonlinear part of the evolution equation (2.17)—using the a priori bound $\|v_t^k\|_{-1} \leq \frac{\sqrt{T}}{\sqrt{N}} N^{2\zeta}$ is not sufficient. In fact

$$\left| \left(\int_{T_{k-1}}^{T_k} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta [v_s^k J * v_s^k] ds, q'_{\psi_{k-1}} \right)_{-1,1/q\psi_{k-1}} \right| \leq \frac{T^2}{N} N^{4\zeta}. \tag{4.6}$$

In order to improve this estimate the strategy is to we re-inject (2.17) into the projection $(I_k(T_k), q_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}$, where

$$I_k(t) = \mathbf{1}_{\{\tau_{\sigma_N}^k \geq T_{k-1}\}} \int_{T_{k-1}}^{t \wedge \tau_{\sigma_N}^k} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta [v_s^k J * v_s^k] ds, \tag{4.7}$$

and this leads to a rather long expression

$$\sum_{k=1}^n (\psi_k - \psi_{k-1}) = \sum_{k=1}^n \frac{(Z_{T_k}^k, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q', q')_{-1,1/q}} + \sum_{k=1}^n \sum_{i=1}^9 A_{k,i} + o(1), \tag{4.8}$$

with

$$\begin{aligned} A_{k,1} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} v_{T_{k-1}}^k J * \right. \right. \\ &\quad \left. \left. \times \left(e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} v_{T_{k-1}}^k \right) \right] ds, q'_{\psi_{k-1}} \right)_{\star} \\ A_{k,2} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta [I_k(s)J * I_k(s)] ds, q'_{\psi_{k-1}} \right)_{\star} \\ A_{k,3} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta [Z_s^k J * Z_s^k] ds, q'_{\psi_{k-1}} \right)_{\star} \\ A_{k,4} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} v_{T_{k-1}}^k J * I_k(s) \right] ds, q'_{\psi_{k-1}} \right)_{\star} \\ A_{k,5} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[I_k(s)J * \left(e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} v_{T_{k-1}}^k \right) \right] ds, q'_{\psi_{k-1}} \right)_{\star} \\ A_{k,6} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} v_{T_{k-1}}^k J * Z_s^k \right] ds, q'_{\psi_{k-1}} \right)_{\star} \\ A_{k,7} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q_{\psi_{k-1}}}} \partial_\theta \left[Z_s^k J * \left(e^{-(s-T_{k-1})L_{q_{\psi_{k-1}}}} v_{T_{k-1}}^k \right) \right] ds, q'_{\psi_{k-1}} \right)_{\star} \end{aligned}$$

$$\begin{aligned}
 A_{k,8} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q\psi_{k-1}}} \partial_{\theta} \left[I_k(s)J * Z_s^k \right] ds, q'_{\psi_{k-1}} \right)_{\star} \\
 A_{k,9} &= \mathbf{1}_E \left(\int_{\star} e^{-(T_k-s)L_{q\psi_{k-1}}} \partial_{\theta} \left[Z_s^k J * I_k(s) \right] ds, q'_{\psi_{k-1}} \right)_{\star}, \tag{4.9}
 \end{aligned}$$

where we have used the shortcuts $E = \{\tau_{\sigma_N}^k \geq T_{\psi_{k-1}}\}$, \int_{\star} stands for $\int_{T_{k-1}}^{T \wedge \tau_{\sigma_N}^k}$ and $(\cdot, \cdot)_{\star}$ is $(\cdot, \cdot)_{-1, 1/q\psi_{k-1}}$.

The following bound (a direct consequence of Lemmas 7.2 and 7.3) is now going to be of help:

$$\begin{aligned}
 &\left\| \int_{T_{k-1}}^{T_k} e^{-(T_k-s)L_{q\psi_{k-1}}} \partial_{\theta} [h_1(s)J * h_2(s)] ds \right\|_{-1} \\
 &\leq C \int_{T_{k-1}}^{T_k} \left(1 + \frac{1}{\sqrt{T_k - s}} \right) \|h_1(s)\|_{-1} \|h_2(s)\|_{-1} ds. \tag{4.10}
 \end{aligned}$$

In fact it is not difficult to see that by using Proposition 3.3 and (4.10) we can efficiently bound all the $A_{k,j}$'s, except $A_{k,3}$:

$$\begin{aligned}
 |A_{k,1}| &\leq \frac{1}{N} N^{5\zeta}, \quad |A_{k,2}| \leq \frac{T^5}{N^2} N^{9\zeta}, \quad |A_{k,4}| \leq \frac{T^2}{N^{3/2}} N^{7\zeta}, \quad |A_{k,5}| \\
 &\leq \frac{T^2}{N^{3/2}} N^{7\zeta}, \\
 |A_{k,6}| &\leq \frac{T^{1/2}}{N^{3/2}} N^{4\zeta}, \quad |A_{k,7}| \leq \frac{T^{1/2}}{N^{3/2}} N^{4\zeta}, \quad |A_{k,8}| \leq \frac{T^{7/2}}{N^{3/2}} N^{6\zeta} \quad \text{and} \quad |A_{k,9}| \\
 &\leq \frac{T^{7/2}}{N^{3/2}} N^{6\zeta}. \tag{4.11}
 \end{aligned}$$

Since $T^4 N^{9\zeta-1} \rightarrow 0$ and $N^{5\zeta}/T \rightarrow 0$ (see Remark 2.6), we get (recall that $n = n_N = \frac{N}{T}$)

$$\sum_{k=1}^n (\psi_k - \psi_{k-1}) = \sum_{k=1}^n \frac{(Z_{T_k}^k, q'_{\psi_{k-1}})_{-1, 1/q\psi_{k-1}}}{(q', q')} + \sum_{k=1}^n A_{k,3} + o(1). \tag{4.12}$$

For the $A_{k,3}$ terms we need to use something more sophisticated. To deal with these terms in fact we rely on an averaging phenomena. This method has been used in [4] for the same kind of problem. We write the Doob decomposition

$$\sum_{k=1}^m A_{k,3} = M_m + \sum_{k=1}^m \gamma_k, \tag{4.13}$$

where

$$\gamma_k = \mathbb{E} [A_{k,3} | \mathcal{F}_{T_{k-1}}], \tag{4.14}$$

and M_m is a \mathcal{F}_{T_m} -martingale with brackets

$$\langle M \rangle_m = \sum_{k=1}^m \left(\mathbb{E} [A_{k,3}^2 | \mathcal{F}_{T_{k-1}}] - \gamma_k^2 \right). \tag{4.15}$$

We have

$$\begin{aligned} \gamma_k = \mathbb{E} & \left[\frac{1}{N^2} \sum_{i,j=1}^N \int_{T_{k-1}}^{T_k \wedge \tau_{\sigma_N}^k} dW_s^i \int_{T_{k-1}}^{T_k \wedge \tau_{\sigma_N}^k} dW_{s'}^j \int_{\mathbb{S}} d\theta \left(1 - \frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)} \right) \right. \\ & \left. \times \int_{\mathbb{S}} d\theta'' \partial_{T_k-s} \mathcal{G}_{T_k-s}^{\psi_{k-1}}(\theta, \varphi_s^{i,N}) J(\theta - \theta'') \partial_{T_k-s'} \mathcal{G}_{T_k-s'}^{\psi_{k-1}}(\theta'', \varphi_{s'}^{j,N}) \Big| \mathcal{F}_{T_{k-1}} \right] \mathbf{1}_{\tau_{\sigma_N}^k \geq T_{k-1}}, \end{aligned} \tag{4.16}$$

where $\partial_{\mathcal{G}_t^\psi}(\theta, \theta') := \partial_{\theta'} \mathcal{G}_t^\psi(\theta, \theta')$, and from this we obtain

$$\begin{aligned} \gamma_k = \mathbb{E} & \left[\frac{1}{N} \int_0^{T \wedge \tilde{\tau}_{\sigma_N}} ds \int_{\mathbb{S}} \tilde{\mu}_s(d\theta') \int_{\mathbb{S}} d\theta \left(1 - \frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)} \right) \right. \\ & \left. \int_{\mathbb{S}} d\theta'' \partial_{\theta'} \mathcal{G}_{T-s}^{\psi_{k-1}}(\theta, \theta') J(\theta - \theta'') \partial_{\theta'} \mathcal{G}_{T-s}^{\psi_{k-1}}(\theta'', \theta') \right] \mathbf{1}_{\tau_{\sigma_N}^k \geq T_{k-1}}, \end{aligned} \tag{4.17}$$

with

$$\tilde{\mu}_s := \frac{1}{N} \sum_{j=1}^N \delta_{\varphi_s^{j,N}}, \tag{4.18}$$

where $\{\tilde{\varphi}_s^{j,N}\}_{s \geq 0}$ is a solution of (1.1) depending on $\mathcal{F}_{T_{k-1}}$ only through the initial condition

$$\tilde{\varphi}_0^{j,N} = \varphi_{T_{k-1}}^{j,N}. \tag{4.19}$$

The stopping time $\tilde{\tau}_{\sigma_N}$ is defined as follows:

$$\tilde{\tau}_{\sigma_N} := \inf\{s > 0, \|\tilde{\mu}_s - q_{\psi_{T_{k-1}}}\|_{-1} > \sigma_N\}. \tag{4.20}$$

We now write $\tilde{\mu}_s(d\theta') = q_{\psi_{k-1}}(\theta')d\theta' + \tilde{v}_s^k(d\theta')$ and split the for right hand-side of (4.17) into the corresponding two terms.

The term coming $q_{\psi_{k-1}}(\theta')d\theta'$ is zero as one can see by using the symmetry:

$$\mathcal{G}_s^{\psi_{k-1}}(\psi_{k-1} + \theta, \psi_{k-1} + \theta') = \mathcal{G}_s^{\psi_{k-1}}(\psi_{k-1} - \theta, \psi_{k-1} - \theta') \tag{4.21}$$

which follows from the same statement with $\psi_{k-1} = 0$, which in turn is a consequence of the representation (Appendix A) and of the fact that if e_j is even (respectively, odd) then f_j is even (respectively, odd) too (see Sect. 2.1 and Appendix A).

For the term containing $\tilde{v}_s^k(d\theta')$ instead we get the bound

$$\begin{aligned} & \left| \mathbb{E} \left[\frac{1}{N} \int_{T_{k-1}}^{T_k \wedge \tilde{\tau}_{\sigma_N}} ds \int_{\mathbb{S}} d\tilde{v}_s^k(\theta') \int_{\mathbb{S}} d\theta \int_{\mathbb{S}} d\theta'' \left(1 - \frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)} \right) \right. \right. \\ & \quad \left. \left. \times \partial_{\theta'} \mathcal{G}_{T_k-s}^{\psi_{k-1}}(\theta, \theta') J(\theta - \theta'') \partial_{\theta'} \mathcal{G}_{T_k-s}^{\psi_{k-1}}(\theta'', \theta') \right] \right| \\ & \leq \mathbb{E} \left[\frac{1}{N} \int_{T_{k-1}}^{T_k \wedge \tilde{\tau}_{\sigma_N}} ds \|\tilde{v}_s^k\|_{-1} \|H_s^k\|_{H_1} \right] \tag{4.22} \end{aligned}$$

where

$$\begin{aligned} H_s^k(\theta') &= \int_{\mathbb{S}} d\theta \int_{\mathbb{S}} d\theta'' \left(1 - \frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)} \right) \\ & \quad \times \partial_{\theta'} \mathcal{G}_{T_k-s}^{\psi_{k-1}}(\theta, \theta') J(\theta - \theta'') \partial_{\theta'} \mathcal{G}_{T_k-s}^{\psi_{k-1}}(\theta'', \theta'). \tag{4.23} \end{aligned}$$

We now plug in the explicit representation for the kernels:

$$\begin{aligned} H_s^k(\theta') &= \sum_{l_1, l_2=0}^{\infty} e^{-(\lambda_{l_1} + \lambda_{l_2})(T_k-s)} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'' \left(1 - \frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)} \right) \\ & \quad \times e_{\psi_{k-1}, l_1}(\theta) J(\theta - \theta'') e_{\psi_{k-1}, l_2}(\theta'') f'_{\psi_{k-1}, l_1}(\theta') f'_{\psi_{k-1}, l_2}(\theta'). \tag{4.24} \end{aligned}$$

We obtain

$$\begin{aligned} \|H_s^k\|_1 &\leq \sum_{l,m=0}^{\infty} e^{-(\lambda_{l_1} + \lambda_{l_2})(T_k - s)} \left| \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta'' \left(1 - \frac{1}{2\pi I_0^2(2Kr)q_{\psi_{k-1}}(\theta)} \right) \right. \\ &\quad \left. \times e_{\psi_{k-1}, l_1}(\theta) J(\theta - \theta'') e_{\psi_{k-1}, l_2}(\theta'') \right| \left(\|f''_{\psi_{k-1}, l_1}\|_2 \|f'_{\psi_{k-1}, l_2}\|_{\infty} + \|f'_{\psi_{k-1}, l_1}\|_{\infty} \|f''_{\psi_{k-1}, l_2}\|_2 \right). \end{aligned} \tag{4.25}$$

We aim at proving the convergence of this sum. For the integral term, thanks to the rotation symmetry, we can limit the study to $\psi_{k-1} = 0$. Since $J(\theta - \theta'') = -K \sin(\theta - \theta'') = -K \sin(\theta) \cos(\theta'') + K \cos(\theta) \sin(\theta'')$, we can split these double integrals into products of two simple ones. Corollary 8.5 implies that there exists $l_0 \in \mathbb{N}$ such that e_{0, l_0+2p} and e_{0, l_0+2p+1} can be written as

$$e_{0, l_0+2p} = pq_0^{1/2} (c_{1, l_0+2p} v_{1, l_0+p} + c_{2, l_0+2p} v_{2, l_0+p}) + O\left(\frac{1}{p}\right), \tag{4.26}$$

$$e_{0, l_0+2p+1} = pq_0^{1/2} (c_{1, l_0+2p+1} v_{1, l_0+p} + c_{2, l_0+2p+1} v_{2, l_0+p}) + O\left(\frac{1}{p}\right), \tag{4.27}$$

where

$$\sup_{l \geq l_0} \{|c_{1, l}|, |c_{2, l}|\} < \infty \tag{4.28}$$

and the functions $v_{i, l}$ are defined in Proposition 8.4. The $v_{i, l}$ are sums and products of sines and cosines, and there exists $h \in \mathbb{N}$ such that the only non-zero Fourier coefficients of v_{i, l_0+p} are of index included between $h + p - 2$ and $h + p + 2$ and are bounded with respect to p . We deduce that the simple integral terms containing e_{0, l_0+2p} , which are of the form

$$C \int_0^{2\pi} q_0^{\pm 1/2}(\theta) e_{0, l_0+2p}(\theta) g(\theta) d\theta, \tag{4.29}$$

where g is sine or cosine and C is a constant independent of p , are up to a correction of order $1/p$ a bounded linear combination of the Fourier coefficients of $q_0^{1/2}$ or $q_0^{-1/2}$ of index taken between $h + p - 3$ and $h + p + 3$. The same argument applies for e_{0, l_0+2p+1} . Since these Fourier terms decrease faster than exponentially (this can be seen by observing that $\int_{\mathbb{S}} \exp(a \cos \theta) d\theta = 2\pi I_n(a)$ and that $\exp(a \cos(\cdot))$ is an entire function), these simple integral terms are of order $1/p$. Using Remark 8.3 and Corollary 8.6 we deduce the following bound for $\|H_s^k\|_1$:

$$\|H_s^k\|_1 \leq C + C \sum_{l_1, l_2=1}^{\infty} \frac{l_1 + l_2}{l_1 l_2} e^{(T_k - s) \frac{l_1^2 + l_2^2}{C}} + C \sum_{l=1}^{\infty} e^{(T_k - s) \frac{l^2}{C}}, \tag{4.30}$$

where the first term of the right hand side corresponds to the case $l_1 = 0, l_2 = 0$ in (4.25), the second term corresponds to $l_1 > 0, l_2 > 0$ and the third term to $l_1 = 0, l_2 > 0$ or $l_2 = 0, l_1 > 0$. Applying (4.22) and Proposition 3.3, we get:

$$\begin{aligned} |\gamma_k| &\leq C \frac{T^{1/2}}{N^{3/2}} N^{2\zeta} \int_{T_{k-1}}^{T_k} ds \|H_s^k\|_1 \\ &\leq C \frac{T^{1/2}}{N^{3/2}} N^{2\zeta} \left(T + \sum_{h, l=1}^{\infty} \frac{h + l}{hl(h^2 + l^2)} + \sum_{l=1}^{\infty} \frac{1}{l^2} \right) \leq C \frac{T^{3/2}}{N^{3/2}} N^{2\zeta}, \end{aligned} \tag{4.31}$$

and thus for N large enough

$$\sum_{k=1}^n |\gamma_k| \leq \sqrt{\frac{T}{N}} N^{3\zeta}. \tag{4.32}$$

On the other hand, applying Doob Inequality, (4.9) and Proposition 3.3, it comes

$$\begin{aligned} \mathbb{P} \left[\sup_{1 \leq m \leq n} |M_m| \geq \sqrt{\frac{T}{N}} N^{3\zeta} \right] &\leq \frac{N}{T N^{6\zeta}} \mathbb{E}[\langle M \rangle_n] \\ &\leq \frac{N^{1-6\zeta}}{T} \sum_{m=1}^n \mathbb{E} [A_{k,3}^2] \leq \frac{T^3}{N^{1+2\zeta}}. \end{aligned} \tag{4.33}$$

Since $T N^{-1-2\zeta} \rightarrow 0$ (see Remark 2.6), the combination of (4.32) and (4.33) leads to

$$\mathbb{P} \left[\left| \sum_{i=1}^n A_{k,3} \right| \geq \sqrt{\frac{T}{N}} N^{3\zeta} \right] \rightarrow 0, \tag{4.34}$$

and the proof is complete. □

5 Approach to M

The long time behavior of the solutions to (1.4) is rather well understood, so, in particular, we know that if p_0 is not on the set attracted to the unstable solution $\frac{1}{2\pi}$, then it converges to one probability density in M (cf. Proposition 5.2) and, in particular, it reaches a given (N independent) neighborhood of M in finite time: this is directly extracted from [6, 21]). This takes care of the first stage of the evolution, because the deterministic result directly extends to the empirical measure by standard arguments since the time horizon is finite. But we do need to get to distances of about

$N^{-1/2}$ and this requires a more attentive control of the dynamics. In fact, we exploit the *approximate* contracting properties of the dynamics when the empirical measure is close to q_ψ . We talk about *approximate* contracting properties because the noise plays against getting to M and limits the contraction effect of the linearized operator. Nevertheless, the proof mimics the deterministic proof of nonlinear stability, to which the control of the noise is added. In principle the argument is straightforward: one exploits the spectral gap of the linearized evolution. In practice, one has to set up an iterative procedure similar to the one developed in Sect. 3, because the center of synchronization may change somewhat over long times. This procedure is however substantially easier than the one presented in Sect. 3, mostly because here the control required on the noise is for substantially shorter times ($\log N$ versus $N!$), so we will not go through the arguments in full detail again.

Proposition 5.1 *Choose $p_0 \in \mathcal{M}_1 \setminus U$ such that (1.9) is satisfied. Then there exists ψ_0 (non random!), that depends on K and $p_0(\cdot)$, C , that depends only on K , and a random variable Ψ_N such that*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\left\| \mu_{N, \tilde{\varepsilon}_N N} - q_{\Psi_N} \right\|_{-1} \leq \frac{N^{2\zeta}}{\sqrt{N}} \right) = 1, \tag{5.1}$$

where $\tilde{\varepsilon}_N := \lfloor C \log N \rfloor / N$, and $\lim_N \Psi_N = \psi_0$ in probability. Moreover for ε and ε_N as in Theorem 1.1 we have

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{t \in [\varepsilon_N N, \tilde{\varepsilon}_N N]} \left\| \mu_{N,t} - q_{\psi_0} \right\|_{-1} \leq \varepsilon \right) = 1, \tag{5.2}$$

Proof The proof is divided in two parts. First we prove, using the convergence of $\mu_t = \mu_{N,t}$ to the deterministic solution p_t , that for a given $h > 0$ (arbitrarily small), there exists t_0 such that for ε small enough, $\mathbb{P}(\text{dist}(\mu_{N,t_0}, M) \leq h) \rightarrow 1$ when $N \rightarrow \infty$. Then we show that after a time of order $\log N$, the empirical measure μ_t moves to a distance $N^\zeta^{-1/2}$ from M , without a macroscopic change of the phase.

The first part of the proof relies on the following result:

Proposition 5.2 *If $p_0 \in \mathcal{M}_1 \setminus U$ then there exists $\psi \in \mathbb{S}$ such that $\lim_{t \rightarrow \infty} p_t = q_\psi$ in $C^k(\mathbb{S}; \mathbb{R})$ (for every k).*

Proposition 5.2 is essentially taken from [21], in the sense that it follows by piecing together some results taken from [21]. We give below a proof that of course relies on [21]. We point out that the very same result can be proven also by adapting entropy production arguments, like in [2].

Proposition 5.2 guarantees that the deterministic solution p_t converges to a element q_{ψ_0} of M . Therefore for $t \geq t_0$, we have that p_t is no farther than $h/2$ from q_{ψ_0} (this is a statement that can be made for example in C^k , but here we just need it in H_{-1}). Actually, it is not difficult to see that one can choose $t_0 = -\frac{2}{\lambda_1} \log h$, for h sufficiently small (λ_1 is the spectral gat of $L_{q_{\psi_0}}$), but this is of little relevance here. Applying the Itô formula

$$\mu_t - p_t = e^{t \frac{\Delta}{2}} (\mu_0 - p_0) - \int_0^t e^{(t-s) \frac{\Delta}{2}} [\mu_s J * \mu_s - p_s J * p_s] ds + z_t, \tag{5.3}$$

where

$$z_t = \frac{1}{N} \sum_{j=1}^N \partial_{\theta'} \mathcal{H}(\theta, \phi_s^{j,N}) dW_s^j, \tag{5.4}$$

$e^{t \frac{\Delta}{2}}$ is the semi-group of the Laplacian and \mathcal{H} is the kernel of $e^{s \frac{\Delta}{2}}$ in \mathbb{L}^2 . Define $W_N = \{w, \|\mu_0 - p_0\|_{-1} \leq \varepsilon\}$. Using the classical estimate $\|e^{t \frac{\Delta}{2}} u\|_{-1} \leq \frac{C}{\sqrt{t}} \|u\|_{-2}$ and similar argument as in Sect. 3, we deduce that there exist events $\tilde{W}_N \subset W_N$ such that $\mathbb{P}(\tilde{W}_N) \rightarrow 1$ and that for all outcomes in \tilde{W}_N we have

$$\sup_{0 \leq t \leq t_0} \|z_t\|_{-1} \leq \sqrt{\frac{t_0}{N}} N^\zeta. \tag{5.5}$$

From now, we restrict ourselves to \tilde{W}_N . From (5.3) we get for all $t \in [0, t_0]$

$$\|\mu_t - p_t\|_{-1} \leq \varepsilon + C \int_0^t \frac{1}{\sqrt{t-s}} \|\mu_s - p_s\|_{-1} ds + \sqrt{\frac{t_0}{N}} N^\zeta. \tag{5.6}$$

The Gronwall–Henry inequality (see [32]) implies that there exists $\gamma > 0$ (independent of ε and N) such that

$$\sup_{t \leq t_0} \|\mu_{t_0} - p_{t_0}\|_{-1} \leq \left(\varepsilon + \sqrt{\frac{t_0}{N}} N^\zeta \right) e^{\gamma t_0}. \tag{5.7}$$

So for $\varepsilon = h/4$ and N large enough, $\|\mu_{t_0} - q_{\psi_0}\|_{-1} \leq h$ on the event \tilde{W}_N .

To show that we enter a neighborhood of size *slightly larger* than $N^{-1/2}$, it will be $N^{2\zeta-1/2}$, we set up an iterative scheme. It is very similar to the one given in Sect. 2.4, but with times t_i bounded with respect to N . This times are chosen such that after each iteration, the distance between the empirical measure and M is at least divided by 2. We define $h_0 := h$ and for $m \geq 1$

$$t_m := t_{m-1} + \frac{1}{\lambda_1} |\log \alpha|, \tag{5.8}$$

$$h_m := \frac{1}{2} h_{m-1}, \tag{5.9}$$

until the index m_f defined by

$$m_f := \inf \left\{ m \geq 1, h_m \leq N^{2\zeta-1/2} \right\}. \tag{5.10}$$

The constant α above does not depends on N and will be chosen below. It is now easy

to check that m_f is of order $\log N$. Then we define $\tilde{\tau}_0 := t_0$, and for $1 \leq m \leq m_f + 1$

$$\tilde{\psi}_{m-1} := \mathbb{P}(\mu_{t_{m-1}}), \tag{5.11}$$

$$\tilde{v}_{t_{m-1}}^m := \mu_{t_{m-1}} - q\tilde{\psi}_{m-1} \tag{5.12}$$

if $\text{dist}(\mu_{t_{m-1}}, M) \leq \sigma$ (see Lemma 2.4). We consider for $1 \leq m \leq m_f$ the stopping times

$$\begin{aligned} \tilde{\tau}_m &:= \tilde{\tau}_{m-1} \mathbf{1}_{\{\tilde{\tau}_{m-1} < t_{m-1}\}} \\ &+ \inf\{s \in [t_{m-1}, t_m], \|\mu_s - q\tilde{\psi}_{m-1}\|_{-1} \geq \sigma\} \mathbf{1}_{\{\tilde{\tau}_{m-1} \geq t_{m-1}\}}. \end{aligned} \tag{5.13}$$

and the process solution of

$$\begin{aligned} \tilde{v}_t^m &= \mathbf{1}_{\{\tilde{\tau}_m < t_{m-1}\}} \tilde{v}_{\tilde{\tau}_m}^m + \mathbf{1}_{\{\tilde{\tau}_m \geq t_{m-1}\}} \\ &\times \left(e^{-(t \wedge \tilde{\tau}_m - t_{m-1})L_{\tilde{\psi}_{m-1}}} \tilde{v}_{t_{m-1}}^m - \int_{t_{m-1}}^{t \wedge \tilde{\tau}_m} e^{-(t \wedge \tilde{\tau}_m - s)L_{\tilde{\psi}_{m-1}}} \partial_\theta [\tilde{v}_s^m J * \tilde{v}_s^m] ds + \tilde{Z}_{t \wedge \tilde{\tau}_m}^m \right), \end{aligned} \tag{5.14}$$

where

$$\tilde{Z}_t^m = \frac{1}{N} \sum_{j=1}^N \int_{t_{m-1}}^t \partial_{\theta^j} \mathcal{G}_{t-s}^{\tilde{\psi}_{m-1}} \left(\theta, \varphi_s^{j,N} \right) dW_s^j. \tag{5.15}$$

With the same arguments as given in Lemma 3.1, we can prove (recall that m_f is of order $\log N$) that the probability of the event

$$\Omega_N := \left\{ \sup_{1 \leq m \leq m_f} \sup_{t_{m-1} \leq t \leq t_m} \|\tilde{Z}_t^m\|_{-1} \leq \sqrt{\frac{t_m - t_{m-1}}{N}} N^\zeta \right\} \tag{5.16}$$

tends to 1 when $N \rightarrow \infty$. From now, we assume that Ω_N is verified. We insist on the fact that the generic constants C appearing in the following do not depend on N , and if not mentioned do not depend on α . From Lemma 7.2 and (5.14) we get that for all $1 \leq m \leq m_f$,

$$\|\tilde{v}_t^m\|_{-1} \leq Ch_{m-1} + C(t + \sqrt{t}) \sup_{s \in [t_{m-1}, t]} \|\tilde{v}_s^m\|_{-1} + \sqrt{\frac{t_m - t_{m-1}}{N}} N^\zeta. \tag{5.17}$$

We now prove that for $1 \leq m \leq m_f - 1$, $\|\tilde{v}_{t_{m-1}}^m\|_{-1} \leq h_{m-1}$ implies $\|\tilde{v}_{t_m}^{m+1}\|_{-1} \leq h_m$, and that $\|\tilde{v}_{t_{m_f-1}}^{m_f}\|_{-1} \leq h_{m_f-1}$ implies $\|\tilde{v}_{t_{m_f}}^{m_f+1}\|_{-1} \leq N^{2\zeta-1/2}$. Define

$$s_m^* := \sup\{s \in [t_{m-1}, t_m], \|\tilde{v}_s^m\|_{-1} \leq h_{m-1}^{3/4}\}. \tag{5.18}$$

Then for $s < s_m^*$, if $\|v_{t_{m-1}}^m\|_{-1} \leq h_{m-1}$, we get using (5.17)

$$\|\tilde{v}_s^m\|_{-1} \leq Ch_{m-1} + C(s + \sqrt{s})h_{m-1}^{3/2} + \sqrt{\frac{t_m - t_{m-1}}{N}}N^\zeta. \tag{5.19}$$

Since $N^{2\zeta-1/2} \leq h_{m-1}$, we deduce that $s_m^* = t_m$ if h_0 is small enough. Then using (5.14) we get

$$\|\tilde{v}_{t_m}^m\|_{-1} \leq C\alpha h_{m-1} + Ch_{m-1}^{3/2} + \sqrt{\frac{t_m - t_{m-1}}{N}}N^\zeta. \tag{5.20}$$

Since $h_{m-1}^{3/2} \leq \alpha h_{m-1}$ for h_0 small enough, it leads us to (recall that $h_{m-1} = 2h_m$)

$$\|\tilde{v}_{t_m}^m\|_{-1} \leq 4C\alpha h_m + \sqrt{\frac{t_m - t_{m-1}}{N}}N^\zeta. \tag{5.21}$$

If $m < m_f$, $\sqrt{\frac{t_m - t_{m-1}}{N}}N^\zeta \leq C\alpha h_m$ and thus $\|\tilde{v}_{t_m}^m\|_{-1} \leq 5C\alpha h_m$. If $m = m_f$, $h_m \leq N^{2\zeta-1/2}$ and thus $\|\tilde{v}_{t_m}^m\|_{-1} \leq 5C\alpha N^{2\zeta-1/2}$. We now have a good control on $\mu_{t_m} = q_{\tilde{\psi}_{m-1}} + \tilde{v}_{t_m}^m$, and project it with respect to $\tilde{\psi}_m$ (writing $\mu_{t_m} = q_{\tilde{\psi}_m} + \tilde{v}_{t_m}^{m+1}$) to get a bound for $\|\tilde{v}_{t_m}^{m+1}\|_{-1}$. We use the same decomposition as the proof of Proposition 3.3:

$$\begin{aligned} \tilde{v}_{t_m}^{m+1} &= q_{\tilde{\psi}_{m-1}} + \tilde{v}_{t_m}^m - q_{\tilde{\psi}_m} = P_{\tilde{\psi}_m}^\perp [q_{\tilde{\psi}_{m-1}} + \tilde{v}_{t_m}^m - q_{\tilde{\psi}_m}] \\ &= \left(P_{\tilde{\psi}_m}^\perp - P_{\tilde{\psi}_{m-1}}^\perp \right) [q_{\tilde{\psi}_{m-1}} + \tilde{v}_{t_m}^m - q_{\tilde{\psi}_m}] + P_{\tilde{\psi}_{m-1}}^\perp [q_{\tilde{\psi}_{m-1}} - q_{\tilde{\psi}_m}] + P_{\tilde{\psi}_{m-1}}^\perp \tilde{v}_{t_m}^m. \end{aligned} \tag{5.22}$$

Since the projection \mathfrak{p} is smooth, we get the bound

$$|\tilde{\psi}_m - \tilde{\psi}_{m-1}| = |\mathfrak{p}(\mu_{t_m}) - \mathfrak{p}(\mu_{t_{m-1}})| \leq C\|\mu_{t_m} - \mu_{t_{m-1}}\|_{-1} \leq C\|\tilde{v}_{t_m}^m - \tilde{v}_{t_{m-1}}^m\|_{-1}. \tag{5.23}$$

But (5.21) implies in particular that

$$\|\tilde{v}_{t_m}^m\|_{-1} \leq C(1 + 4\alpha)h_{m-1}, \tag{5.24}$$

which implies, using also (5.23),

$$|\tilde{\psi}_m - \tilde{\psi}_{m-1}| \leq 2C(1 + 4\alpha)h_{m-1}. \tag{5.25}$$

Using similar arguments as in the proof of Proposition 3.3 (using in particular the smoothness of the projection $P_{\tilde{\psi}}^\perp$), we see that the two first terms of the right hand side

in (5.22) are of order h_{m-1}^2 . More precisely, there exists a constant $C'[\alpha]$ depending in α (increasing in α) such that

$$\|\tilde{v}_m^{m+1}\|_{-1} \leq C'[\alpha]h_{m-1}^2 + C\|\tilde{v}_m^m\|_{-1}. \tag{5.26}$$

So, since $\|\tilde{v}_m^m\|_{-1} \leq 5C\alpha h_m$ for $m < m_f$ and $\|\tilde{v}_{m_f}^{m_f}\|_{-1} \leq 5C\alpha N^{2\zeta-1/2}$, if h_0 and α are small enough we get $\|\tilde{v}_m^{m+1}\|_{-1} \leq h_m$ for $m < m_f$ and $\|\tilde{v}_{m_f}^{m_f+1}\|_{-1} \leq N^{2\zeta-1/2}$.

We have therefore shown that after a time of order $\log N$, the empirical measure comes at distance $N^{2\zeta-1/2}$ from $q_{\tilde{\psi}_{m_f}}$. This angle $\tilde{\psi}_{m_f}$ corresponds to the angle Ψ_N in the Proposition 5.1. So it remains to prove that $\tilde{\psi}_{m_f}$ converges to ψ_0 in probability as N goes to infinity. We decompose

$$|\tilde{\psi}_{m_f} - \psi_0| \leq |\tilde{\psi}_0 - \psi_0| + \sum_{m=1}^{m_f} |\tilde{\psi}_m - \tilde{\psi}_{m-1}|. \tag{5.27}$$

We restrict our study on the event $\Omega_N \cap \tilde{W}_N$, whose probability tends to 1. Since $\|\mu_{t_0} - q_{\psi_0}\|_{-1} \leq h$ and the projection \mathfrak{p} is smooth, we get

$$|\tilde{\psi}_0 - \psi_0| \leq Ch \tag{5.28}$$

and (5.25) implies (recall $h_0 = h$ and $h_{m-1} = 2h_m$)

$$|\tilde{\psi}_m - \tilde{\psi}_{m-1}| \leq C2^{1-m}h. \tag{5.29}$$

Consequently for C large enough $\mathbb{P}[|\tilde{\psi}_{m_f} - \psi_0| > Ch] \rightarrow_{N \rightarrow \infty} 0$, which completes the proof of (5.1). The bound (5.2) is much rougher and it follows directly from the argument we have used for establishing (5.1). This completes the proof of Proposition 5.1 \square

Proof of Proposition 5.2 The crucial issues are the gradient flow structure of (1.4) and its dissipativity properties. The gradient structure of (1.4) [6] implies that the functional

$$\mathcal{F}(p) := \frac{1}{2} \int_{\mathbb{S}} p(\theta) \log p(\theta) d\theta - \frac{K}{2} \int_{\mathbb{S}} \int_{\mathbb{S}} p(\theta) \cos(\theta - \theta') p(\theta') d\theta d\theta', \tag{5.30}$$

is non increasing along the time evolution. The dissipativity properties proven in [21, Theorem 2.1] show that for every $k \in \mathbb{N}$ and $a > 0$ we can find \tilde{t} such that $\|p_t\|_{C^k} < a$ for every $t \geq \tilde{t}$. Therefore for any k there exists $\{t_n\}_{n=1,2,\dots}$ such that $t_{n+1} - t_n > 1$ and $\lim_n p_{t_n}$ exists in C^k and we call it p_∞ . An immediate consequence is that $\lim_n \mathcal{F}(p_{t_n}) = \mathcal{F}(p_\infty)$. But we can go beyond by introducing the semigroup S_t associated to (1.4), by setting $S_{t'} p_t = p_{t+t'}$. [21, Theorem 2.2] implies the continuity of this semigroup in C^k , so that, since for $t \in [0, 1]$ we have $t_n \leq t_n + t < t_{n+1}$, we obtain $\mathcal{F}(S_t p_\infty) = \mathcal{F}(p_\infty)$. Therefore $\partial_t \mathcal{F}(S_t p_\infty) = 0$, but the condition $\partial_t \mathcal{F}(p_t) =$

0, for a solution of (1.4), directly implies that $\partial_{\theta}^2 p_t = 2\partial_{\theta}(p_t J * p_t)$, which is the stationarity condition for (1.4). Therefore p_t is either q_{ψ} , for some ψ , or it coincides with $\frac{1}{2\pi}$ [see (1.5)–(1.6)].

Let us point out that if p_{t_n} converges to $\frac{1}{2\pi}$ then $\{p_t\}_{t>0}$ itself converges to $\frac{1}{2\pi}$. This is just because $\mathcal{F}(\frac{1}{2\pi}) > \mathcal{F}(q_{\psi})$, so that if $\lim_n p_{t'_n} = q_{\psi}$ and $\lim_n p_{t_n} = \frac{1}{2\pi}$ then it suffices to choose n such that $\mathcal{F}(p_{t'_n}) < \mathcal{F}(\frac{1}{2\pi})$ and m such that $t_m > t'_n$ to get $\mathcal{F}(p_{t'_n}) \geq \mathcal{F}(p_{t_m}) \geq \mathcal{F}(\frac{1}{2\pi})$, which is impossible.

So we have seen that either $\lim_{t \rightarrow \infty} p_t = \frac{1}{2\pi}$ or all limit points are in M . The stronger result we need is the convergence also when the limit point is not $\frac{1}{2\pi}$. This result is provided by the nonlinear stability result [21, Theorem 4.6] which says that if p_0 is in a neighborhood of M (the result is proven for a \mathbb{L}^2 neighborhood, which is much more than what we need here), then there exists ψ such that $\lim_{t \rightarrow \infty} p_t = q_{\psi}$ in C^k .

To complete the proof we need to characterize the portion of \mathcal{M}_1 which is attracted by $\frac{1}{2\pi}$, that is we need to identify the stable manifold of the unstable point with the set U in (1.7). But this is the content of [21, Proposition 4.4]. □

6 Proof of Theorem 1.1

The proof of Theorem 1.1 relies on the results of the previous sections and on a convergence argument of the process in the tangent space that we give here.

Proof of Theorem 1.1 First of all Proposition 5.1 takes care of the evolution up to time $N\tilde{\varepsilon}_N = C \log N$ and provides an estimate on the closeness of the empirical measure to the manifold M that allows to apply directly Proposition 3.3 and then Proposition 4.1. Note that the iterative scheme that we have set up in Sect. 2.4 has been presented without asking ψ_0 not to be random or not to depend on N . In fact we start the iterative scheme at time $N\tilde{\varepsilon}_N$ and from the random phase Ψ_N of Proposition 5.1 that converges in probability to the (non random) value ψ_0 . Of course there is here an abuse of notation in the use of ψ_0 , but notice actually that, by the rotation invariance of the system, we can actually consider without loss of generality that the empirical measure $\mu_{N,C \log N}$ has precisely the phase ψ_0 . Moreover we make a time shift of $N\tilde{\varepsilon}_N$, so that the phase is ψ_0 at time $T_0 = 0$. The result in Theorem 1.1 is given for times starting from $N\varepsilon_N$ and not $N\tilde{\varepsilon}_N$, but as stated in Proposition 5.1, the empirical measure stays close to q_{ψ_0} in the time interval $[N\varepsilon_N, N\tilde{\varepsilon}_N]$. Therefore we have the finite sequence of times T_0, T_1, \dots, T_n , with the corresponding phases $\psi_0, \psi_1, \dots, \psi_n$ and we define ψ_t for every $t \in [0, T_n]$ by linear interpolation. We assume $T_n > \tau_f N$.

We then note that, in view of (3.43), the control on the phases, see Proposition 4.1, on the times T_1, T_2, \dots of our iteration scheme suffices not only to control the distance between the empirical measure $\mu_{N,t}$ and q_{ψ_t} , in the H_{-1} norm, for $t = T_k$, but for every $t \in [0, T_n]$. We are now ready to identify the process $W_{N,\cdot}$ of Theorem 1.1:

$$W_{N,\tau} := \frac{\psi_{\tau N} - \psi_0}{D_K}, \tag{6.1}$$

where we recall that $\tau \in [0, T_n/N]$. We are therefore left with showing that $W_{N,\cdot}$ converges to standard Brownian motion.

In proving the convergence to Brownian motion we apply Proposition 4.1 and replace the process ψ_\cdot with the cadlag process $\psi_0 + M_{N,\cdot} \in D([0, T_n/N]; \mathbb{R})$ defined by

$$M_{N,\tau} := \sum_{k \in \mathbb{N}: T_k \leq N\tau} \Delta M_{N,k}, \tag{6.2}$$

and

$$\Delta M_{N,k} := \frac{(Z_{T_k}^k, q'_{\psi_{k-1}})_{-1,1/q_{\psi_{k-1}}}}{(q', q')_{-1,1/q}}. \tag{6.3}$$

It is straightforward to see that $M_{N,\cdot}$ is a martingale with respect to the filtration $\tilde{\mathcal{F}}_\tau := \mathcal{F}_{[\tau T]_T}$, where \mathcal{F}_\cdot is the natural filtration of $\{W_{N,j}\}_{j=1,\dots,N}$: the martingale is actually in L^p , for every p , as the moment estimates in Sect. 3 show. We can now apply the Martingale Invariance Principle in the form given by [23, Corollary 3.24, Ch. VIII] to $M_{N,\cdot}$ for continuous time martingales: the hypotheses to verify in the case of piecewise constant cadlag martingales boil down to the variance convergence condition that for every $\tau \in [0, \tau_f]$

$$\lim_{N \rightarrow \infty} \sum_{k \in \mathbb{N}: T_k \leq \tau N} \mathbb{E} \left[(\Delta M_{N,T_k})^2 \mid \mathcal{F}_{T_{k-1}} \right] = \tau D_K^2, \tag{6.4}$$

in probability, and the Lindeberg condition that for every $\varepsilon > 0$ in probability we have

$$\lim_{N \rightarrow \infty} \sum_{k \in \mathbb{N}: T_k \leq \tau N} \mathbb{E} \left[(\Delta M_{N,T_k})^2 ; \Delta M_{N,T_k}^2 > \varepsilon \mid \mathcal{F}_{T_{k-1}} \right] = 0. \tag{6.5}$$

For what concerns (6.4) we have

$$\mathbb{E} \left[(\Delta M_{N,T_k})^2 \mid \mathcal{F}_{T_{k-1}} \right] = \frac{1}{N \|q'\|_{-1,1/q}^2} \int_{T_{k-1}}^{T_k} \int_{\mathbb{S}} (f'_{\psi_{k-1},0}(\theta))^2 \mu_{N,s}(d\theta) ds. \tag{6.6}$$

Now take the sum over k and use the uniform estimate (3.43) of Proposition 3.3 to replace the empirical measure with $q_{\psi_{T_{k-1}}}(\theta)d\theta$. Since a direct computation shows that $\int_{\mathbb{S}} (f'(\theta)_{\psi,0})^2 q_\psi(\theta)d\theta = 1$, (6.4) follows.

For what concerns (6.5) we remark that, by the Markov inequality, it suffices to show that

$$\lim_{N \rightarrow \infty} \sum_{k \in \mathbb{N}: T_k \leq \tau N} \mathbb{E} \left[(\Delta M_{N,T_k})^4 \mid \mathcal{F}_{T_{k-1}} \right] = 0. \tag{6.7}$$

Actually one can show that there exists a non random constant C such that almost surely

$$\mathbb{E} \left[(\Delta M_{N, T_k})^4 \middle| \mathcal{F}_{T_{k-1}} \right] \leq C \left(\frac{T}{N} \right)^2. \tag{6.8}$$

This is an immediate consequence of (3.32), but of course, since we are projecting on q' and since we are just considering the fourth moment, a similar estimate can be easily obtained explicitly by proceeding like for (6.4) and by using the fact that $\|f'_{\psi, 0}\|_\infty = \|f'_0\|_\infty < \infty$. Of course (6.7) follows from (6.8).

Therefore $M_{N, \cdot} \in D([0, \tau_f]; \mathbb{R})$ converges in law to $W./\|q'\|_{-1, 1/q}$, where W . is a standard Brownian motion. This is almost the result we want (recall that $D_K = 1/\|q'\|_{-1, 1/q}$), since $M_{N, \cdot}/D_K$ differs from $W_{N, \cdot}$ just for the fact that they interpolate in a different way between the times T_k (where they coincide) and that in the case of $W_{N, \cdot}$ the convergence is in $C^0([0, \tau_f]; \mathbb{R})$. But (6.8) guarantees that the sum of the fourth power of the jumps of $M_{N, \cdot}$ adds up to $O(T^2/N) = o(1)$ in probability, so the supremum of the jumps is $o(1)$, and therefore the convergence for $M_{N, \cdot} \in D([0, \tau_f]; \mathbb{R})$ implies the convergence of $W_{N, \cdot} \in C^0([0, \tau_f]; \mathbb{R})$. The proof of Theorem 1.1 is therefore complete. \square

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Appendix A: The evolution in H_{-1}

In what follows we fix q in the invariant manifold M (see (1.6)). Unlike the rest of the paper here we do not identify q with q_ψ and then with ψ , so in particular we write L_q (and not L_ψ), $\mathcal{G}_t^q(\cdot)$ (and not $\mathcal{G}_t^\psi(\cdot)$ like in (2.12)), and so on. We work with the signed measure

$$\nu_{N, t}(d\theta) := \mu_{N, t}(d\theta) - q(\theta)d\theta, \tag{7.1}$$

which can be seen as an element of H_{-1} . This is simply because it is the difference of two probability measures. In fact, if $\mu \in \mathcal{M}_1, \theta \mapsto \mu([0, \theta])$ is a primitive of μ and, by Remark 2.1, $\|\mu - \nu\|_{-1}^2 \leq \int_{\mathbb{S}} (\mu([0, \theta]) - \nu([0, \theta]))^2 d\theta \leq 2\pi$. Therefore $\|\mu - \nu\|_{-1} \leq \sqrt{2\pi}$: of course this quick argument needs to be cleaned up by first *smoothing* the measures. That is, we introduce an approximate identity $\phi_n \in C^\infty$ ($\phi_n \geq 0, \phi_n(\theta) = 0$ for $\theta \in [1/n, 2\pi - 1/n]$, $\int_{\mathbb{S}} \phi_n = 1$ and $\lim_n \int_{\mathbb{S}} F \phi_n = F(0)$ for every $F \in C^0$). We then introduce the probability density $\theta \mapsto \mu_n(\theta) := \int_{\mathbb{S}} \phi_n(\theta - \theta') \mu(d\theta')$ and verify that

$$\|\mu_n - \mu_m\|_{-1}^2 \leq \frac{4}{\min(n, m)}, \tag{7.2}$$

so that $\lim_n \mu_n$ exists in H_{-1} (of course the limit exists also weakly and it is μ).

We aim at proving:

Proposition 7.1 *If $\{\varphi_t^{j,N}\}_{t \geq 0, j=1, \dots, N}$ solves (1.1) then $v_{N, \cdot} \in C^0([0, \infty); H_{-1})$ and we have*

$$v_{N,t} = \exp(tL_q)v_{N,0} - \int_0^t \exp((t-s)L_q)\partial((J * v_{N,s})v_{N,s}) ds + Z_{N,t}, \tag{7.3}$$

where $Z_{N,t}$ is the limit in H_{-1} as $\tau \nearrow t$ of $Z_{N,t,\tau}$, where

$$Z_{N,t,\tau}(\theta) := \frac{1}{N} \sum_{j=1}^N \int_0^\tau \partial_{\theta'} \mathcal{G}_{t-s}^q(\theta, \varphi_s^{j,N}) dW_s^j \tag{7.4}$$

Moreover all the terms appearing in the right-hand side of (7.3), as functions of time, are in $C^0([0, \infty); H_{-1})$.

Proof For $(t, \theta) \mapsto F_t(\theta)$ in $C^{1,2}(\mathbb{R}^+ \times \mathbb{S}; \mathbb{R})$, from (1.1) we directly obtain

$$\begin{aligned} \int_{\mathbb{S}} F_t(\theta)v_{N,t}(d\theta) &= \int_{\mathbb{S}} F_0(\theta)v_{N,0}(d\theta) + \int_0^t \int_{\mathbb{S}} (L_q^* F_s)(\theta)v_{N,s}(d\theta) ds \\ &+ \int_0^t \int_{\mathbb{S}} \partial_s F_s(\theta)v_{N,s}(d\theta) ds + \int_0^t \int_{\mathbb{S}} \partial_{\theta} F_s(\theta)(J * v_{N,s})(\theta)v_{N,s}(d\theta) ds + Z_{N,t}^F, \end{aligned} \tag{7.5}$$

where

$$Z_{N,t}^F = \frac{1}{N} \int_0^t \sum_{j=1}^N \partial_{\theta} F_s(\theta) \Big|_{\theta=\varphi_s^{j,N}} dW_s^j, \tag{7.6}$$

and L_q^* is the adjoint in \mathbb{L}_0^2 of L_q , that is

$$L_q^*v = \frac{1}{2}v'' + (J * q)v' - J * (qv') - \int_{\mathbb{S}} (J * q)v', \tag{7.7}$$

for $v \in C^2(\mathbb{S}; \mathbb{R})$ such that $\int_{\mathbb{S}} v = 0$

We sum up here some useful properties of L_q^* :

1. In [6] it is shown that the \mathbb{L}_0^2 -norm is equivalent to the Dirichlet form norm of L_q : the squared Dirichlet form norm of u is $\|u\|_{-1,1/q}^2 + (u, (-L_q)u)_{-1,1/q}$. On the other hand it is straightforward to see that the properties of L_q in $H_{-1,1/q}$, notably the fact that it is self-adjoint and that it has compact resolvent, still hold true in the space of the Dirichlet form. So L_q has compact resolvent in \mathbb{L}_0^2 , which

directly implies that L_q^* has compact resolvent and the very same spectrum (see e.g. [31, VI.5]).

- Recall that we denote by $\{e_j\}_{j=0,1,\dots}$ a complete set of eigenvectors of L_q which is orthonormal in $H_{-1,1/q}$ and observe that there is a unique solution f_j to

$$A_q f_j(\theta) := -\partial_\theta (q(\theta)\partial_\theta f_j(\theta)) = e_j(\theta), \tag{7.8}$$

such that $\int_{\mathbb{S}} f_j = 0$. More generally, A_q is a bijection from $\{u \in C^\infty : \int_{\mathbb{S}} u = 0\}$ to itself: in fact, $v = A_q u$ is equivalent to $u' = -\mathcal{V}/q$ in our standard notations, which determines u since $\int_{\mathbb{S}} u = 0$. In particular $f'_j = -\mathcal{E}_j/q$ and $f_j \in C^\infty$, since e_j is C^∞ , and one obtains

$$(f_i, e_j)_2 = \int_{\mathbb{S}} f_i e_j = - \int_{\mathbb{S}} f'_i \mathcal{E}_j = \int_{\mathbb{S}} \frac{\mathcal{E}_i \mathcal{E}_j}{q} = \delta_{i,j}. \tag{7.9}$$

By using the fact that $q(\cdot)$ is even, one verifies directly also that if e_j is even (respectively, odd)—recall from Sect. 2.1 that e_j is either even or odd—the f_j is even (respectively, odd) too.

- By observing also that $L_q A_q = A_q L_q^*$ one verifies that $\{f_j\}_{j=0,1,\dots}$ is a complete set of eigenfunctions for L_q^* and, of course, $L_q^* f_j = -\lambda_j f_j$.

Therefore for every $t > 0$ and $s \leq t$ we can define $F_s(\theta) = (\exp((t - s)L_q^*)F)(\theta)$ for $F \in \mathbb{L}_0^2$ and standard parabolic regularity results imply that $F_s(\cdot)$ is C^∞ for $s < t$ (in our case this can be proven directly by using the Fourier transform, like in [21], but for what follows we choose $F \in C^2$ and the regularity result is even more straightforward). By plugging this choice into (7.5) we obtain

$$\begin{aligned} \int_{\mathbb{S}} F(\theta)v_{N,t}(\mathrm{d}\theta) &= \int_{\mathbb{S}} (\exp(tL_q^*)F)(\theta)v_{N,0}(\mathrm{d}\theta) \\ &+ \int_0^t \int_{\mathbb{S}} \partial_\theta (\exp((t - s)L_q^*)F)(\theta)(J * v_{N,s})(\theta)v_{N,s}(\mathrm{d}\theta)\mathrm{d}s + Z_{N,t}^F. \end{aligned} \tag{7.10}$$

At this point we step to looking at $v_{N,t}$ as an element of H_{-1} and we reconsider (7.10) with this novel viewpoint.

First of all $\int_{\mathbb{S}} F(\theta)v_{N,t}(\mathrm{d}\theta) = \langle F, v_{N,t} \rangle_{1,-1}$, where $\langle \cdot, \cdot \rangle_{1,-1}$ is the duality between H_1 and H_{-1} (cf. Sect. 2.1). For the first term in the right-hand side we observe that, for $v \in H_{-1}$ we have $\langle \exp(tL_q^*)F, v \rangle_{1,-1} = \langle F, \exp(tL_q)v \rangle_{1,-1}$: this is because this relation holds when $v \in \mathbb{L}_0^2$ (in this case the duality can be replaced by the \mathbb{L}^2 scalar product) and because one can choose a sequence $\{v_n\}_{n=1,2,\dots}$, $v_n \in \mathbb{L}_0^2$ such that $v_n \rightarrow v$ in H_{-1} (one can choose $v_n = \phi_n * v$) so that

$$\langle \exp(tL_q^*)F, v \rangle_{1,-1} = \lim_n \langle F, \exp(tL_q)v_n \rangle_2 = \langle F, \exp(tL_q)v \rangle_{1,-1}, \tag{7.11}$$

where we have used the continuity properties of the duality and of the semigroup operator.

For the second term in the right-hand side of (7.10) we write

$$\begin{aligned} & \int_0^t \int_{\mathbb{S}} \partial_\theta (\exp((t-s)L_q^*)F)(\theta)(J * v_{N,s})(\theta) v_{N,s} (d\theta) ds \\ &= \int_0^t \langle (J * v_{N,s}) \partial \exp((t-s)L_q^*)F, v_{N,s} \rangle_{1,-1} ds, \end{aligned} \tag{7.12}$$

We now introduce $v_{n,s} := \phi_n * v_{N,s}$ so that for every $s \in [0, t)$

$$\begin{aligned} & \langle (J * v_{N,s}) \partial \exp((t-s)L_q^*)F, v_{N,s} \rangle_{1,-1} \\ &= - \lim_n \langle F, \exp((t-s)L_q) \partial ((J * v_{N,s})v_{n,s}) \rangle_2 \\ &= - \langle F, \exp((t-s)L_q) \partial ((J * v_{N,s})v_{N,s}) \rangle_{1,-1}, \end{aligned} \tag{7.13}$$

where in the last step we have used the fact that $\exp((t-s)L_q)$ is a continuous operator from H_{-2} to H_{-1} (Lemma 7.2). Notice moreover that we have

$$\begin{aligned} & | \langle F, \exp((t-s)L_q) \partial ((J * v_{N,s})v_{n,s}) \rangle_2 | \\ & \leq \| \phi_n \|_1 \| \partial ((J * v_{N,s}) \partial \exp((t-s)L_q^*)F) \|_2 \| v_{N,s} \|_{-1}, \end{aligned} \tag{7.14}$$

and, since $J(\cdot) = -K \sin(\cdot)$, one sees that this expression is bounded by a constant times $\|F''\|_2$, uniformly in n and $s \leq t$. Such a bound tells us that one can exchange limit and integration in

$$\int_0^t \lim_n \langle F, \exp((t-s)L_q) \partial ((J * v_{N,s})v_{n,s}) \rangle_2 ds, \tag{7.15}$$

and then, for fixed n one can of course exchange integral in ds and integral in $d\theta$. At this point we appeal again to Lemma 7.2 that guarantees that $\int_0^t \exp((t-s)L_q) \partial ((J * v_{N,s})v_{n,s}) ds$ converges, in H_{-1} , to $\int_0^t \exp((t-s)L_q) \partial ((J * v_{N,s})v_{N,s}) ds$: note in fact that $\| \partial ((J * v_{N,s})v) \|_{-2} \leq c_J \|v\|_{-1}$ so that (by Lemma 7.2)

$$\begin{aligned} & \left\| \int_0^t \exp((t-s)L_q) \partial ((J * v_{N,s})(v_{n,s} - v_{n',s})) ds \right\|_{-1} \\ & \leq c_J C \int_0^t \left(1 + \frac{1}{\sqrt{t-s}} \right) \|v_{n,s} - v_{n',s}\|_{-1} ds, \end{aligned} \tag{7.16}$$

and the right-hand side vanishes for $\min(n, n') \rightarrow \infty$. Therefore we obtain

$$\int_0^t \int_{\mathbb{S}} \partial_\theta (\exp((t-s)L_q^*)F)(\theta) (J * \nu_{N,s})(\theta) \nu_{N,s}(d\theta) ds = \left\langle F, \int_0^t \exp((t-s)L_q) \partial((J * \nu_{N,s})\nu_{N,s}) ds \right\rangle_{1,-1}. \tag{7.17}$$

We are left with the last term in (7.10). It is now useful to use the kernel of the L_q -semigroup in \mathbb{L}_0^2

$$\mathcal{G}_s^q(\theta, \theta') := \sum_{l=0}^\infty \exp(-s\lambda_l) e_l(\theta) f_l(\theta'), \tag{7.18}$$

so that

$$(u, \exp(sL_q)v)_2 = (\exp(sL_q^*)u, v)_2 = \int_{\mathbb{S}} \int_{\mathbb{S}} u(\theta) \mathcal{G}_s^q(\theta, \theta') v(\theta') d\theta d\theta'. \tag{7.19}$$

Note also that, for $s > 0$, \mathcal{G}_s^q is C^∞ in both variables, by the standard parabolic regularity results we have mentioned above. So, for every $\tau < t, \theta \mapsto Z_{N,t,\tau}(\theta)$ [recall (7.4)] is well defined and smooth in θ . But Lemma 7.4 tells us that $\lim_{\tau \nearrow t} Z_{N,t,\tau}$ exists in H_{-1} . If we call the limit $Z_{N,t}$ we directly see that [recall (7.6)]

$$Z_{N,t}^F = \langle F, Z_{N,t} \rangle_{1,-1}. \tag{7.20}$$

Therefore we have shown that (7.10) implies the validity of (7.3) if we take the duality with respect to an arbitrary $F \in C^2$. But we have also shown that every term in (7.3) is in H_{-1} , therefore the equation extends to $F \in H_1$ and (7.3) is proven.

The continuity claimed in the statement follows by the continuity of the three terms in the right-hand side of (7.3). The continuity of the first term is immediate from the properties of the semigroup. The continuity of the second term follows from a direct estimate by applying both bounds in Lemma 7.2. Finally the continuity of the third term is claimed in Lemma 7.4. The proof of Proposition 7.1 is therefore complete. \square

Lemma 7.2 *For $\tau > 0$ the operator $\exp(\tau L_q)$ extends to a bounded operator from H_{-2} to H_{-1} and there exists $C > 0$ such that for every $\tau > 0$*

$$\|\exp(\tau L_q)u\|_{-1} \leq C \left(1 + \frac{1}{\sqrt{\tau}}\right) \|u\|_{-2}, \tag{7.21}$$

and such that for every $\varepsilon \in (0, 1/2)$ we have

$$\|\exp((\tau + \delta)L_q)u - \exp(\tau L_q)u\|_{-1} \leq C\delta^\varepsilon \left(1 + \frac{1}{\tau^{\varepsilon+1/2}}\right) \|u\|_{-2}, \tag{7.22}$$

for every $\tau > 0$ and $\delta \geq 0$.

Proof We introduce the interpolation spaces associated to L_q that is the (Hilbert) spaces

$$V^m := \left\{ u = \sum_{k=0}^\infty u_k e_k, \sum_{k=0}^\infty (1 + \lambda_k)^m u_k^2 < \infty \right\}, \tag{7.23}$$

associated with the norms

$$\|u\|_{V^m}^2 := \|(1 - L_q)^{m/2}u\|_{-1,1/q}^2 = \sum_{k=0}^\infty (1 + \lambda_k)^m u_k^2. \tag{7.24}$$

It is proven in [22, Remark A.1] that the norms $\|\cdot\|_{V^n}$ and $\|\cdot\|_{n-1}$ are equivalent. This equivalence can also be deduced from Remark 8.3. In particular $\|\cdot\|_{V^{-1}}$ and $\|\cdot\|_{-2}$ are equivalent, so we will prove (7.2) with $\|\cdot\|_{-1,1/q}$ and $\|\cdot\|_{V^{-1}}$. For all $u = \sum_{k=0}^\infty u_k e_k$, we extend $e^{\tau L_q}u$ as

$$e^{\tau L_q}u = \sum_{k=0}^\infty e^{-\lambda_k \tau} u_k e_k, \tag{7.25}$$

and we deduce

$$\|e^{\tau L_q}u\|_{-1,1/q}^2 = \sum_{k=0}^\infty (1 + \lambda_k) e^{-2\lambda_k \tau} \frac{u_k^2}{1 + \lambda_k}. \tag{7.26}$$

But if we define $f(y) := (1 + y)e^{-2y\tau}$, it is easy to see that for all $y \geq 0$, there exist C such that $f(y) \leq C^2(1 + \frac{1}{\sqrt{\tau}})^2$, which with (7.24). and (7.26) gives the first inequality.

For the second inequality we make a similar spectral decomposition and we obtain

$$\|e^{(\tau+\delta)L_q}u - e^{\tau L_q}u\|_{-1,1/q}^2 = \sum_{k=0}^\infty (1 + \lambda_k) e^{-2\lambda_k \tau} (1 - \exp(-\delta\lambda_k))^2 \frac{u_k^2}{1 + \lambda_k}. \tag{7.27}$$

We then use $(1 - \exp(-x)) \leq x^\varepsilon$ for $x \geq 0$ and $(1 + x)x^{2\varepsilon} \exp(-x\tau) \leq C^2(1 + \tau^{-\varepsilon-1/2})^2$, for a suitable C which can be chosen independent of $\varepsilon \in (0/1/2)$. \square

Lemma 7.3 For all $u, v \in H_{-1}$, there exists $C > 0$ such that

$$\|\partial_\theta(uJ * v)\|_{H_{-2}} \leq C \|u\|_{H_{-1}} \|v\|_{H_{-1}}. \tag{7.28}$$

Proof For this proof it is practical to write the H_{-1} -norms by using the Fourier coefficients. In fact if $u \in H_{-s}$, here $s = 1$ or $s = 2$, we can define $u_n = \langle u, b_n \rangle$, where $b_n(\theta) = \exp(in\theta)/2\pi$ (note that $u_0 = 0$) and $\theta \mapsto \sum_{n \in \mathbb{Z}; |n| \leq N} u_n \exp(in\theta)$ converges as $N \rightarrow \infty$ in H_s to u . Moreover we have

$$\|u\|_{-s} := \left(\frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \frac{u_m^2}{m^{2s}} \right)^{1/2}. \tag{7.29}$$

Since $J(\theta) = -K \sin(\theta)$, a direct calculation gives

$$\partial_\theta(uJ * v) = K\pi \left[(m - 1)v_{-1} \sum_{m \in \mathbb{Z}} e^{i(m-1)\theta} u_m - (m + 1)v_1 \sum_{m \in \mathbb{Z}} e^{i(m+1)\theta} u_m \right], \tag{7.30}$$

from which we extract

$$\begin{aligned} \|\partial_\theta(uJ * v)\|_{-2}^2 &= \frac{K^2\pi}{2} \sum_{m \in \mathbb{Z}, m \neq 0} m^{-4} |m(v_{-1}u_{m+1} - v_1u_{m-1})|^2 \\ &\leq K^2\pi \max(|v_{-1}|^2, |v_1|^2) \sum_{m \in \mathbb{Z}, m \neq 0} m^{-2}(u_{m-1}^2 + u_{m+1}^2) \\ &\leq 4K^2\pi \|v\|_{-1}^2 \|u\|_{-1}^2. \end{aligned} \tag{7.31}$$

□

Lemma 7.4 The almost sure limit of $Z_{N,t,\tau}$ as $\tau \nearrow t$ exists in H_{-1} and, if we call the limit point $Z_{N,t}$, we can choose a continuous version of $Z_{N,\cdot}$, that is $Z_{N,\cdot} \in C^0([0, \infty); H_{-1})$.

Proof The claim follows from the same estimates as the one that we have obtained for the proof of Lemma 3.1, which are however substantially more precise than what we need here: recall that now N is fixed, while in Sect. 3 one of the crucial points is to follow the N dependence of the results. Therefore we will not go through the arguments in detail, but we just point out that one goes from $Z_{N,t,\tau}$ to $Z_{t,t'}$, see (3.2) by making obvious changes. So, in particular, proceeding like for (3.25) we easily gets

$$\mathbb{E} \left[\|Z_{N,t,\tau} - Z_{N,t,\tau'}\|_{-1}^{2m} \right] \leq Ch_1^m (|\tau - \tau'|), \tag{7.32}$$

where C depends on N and m and $0 \leq \tau, \tau' < t$. An estimate like (7.32) implies almost sure Hölder continuity of $Z_{N,t,\cdot}$, by a direct application of Kolmogorov continuity

Lemma [33] or by using the Garsia–Rodemich–Rumsey Lemma (Lemma 3.2). That is, there exists a (positive) random variable X and a positive constant $c > 0$ such that

$$\|Z_{N,t,\tau} - Z_{N,t,\tau'}\|_{-1} \leq X |\tau - \tau'|^c, \tag{7.33}$$

for every $0 \leq \tau, \tau' < t$. Therefore the almost sure limit of $Z_{N,t,\tau}$, as $\tau \nearrow t$, exists.

The continuity of the limit follows in the same way, this time using also (3.31). Actually, in the proof of Lemma 3.1 we use Lemma 7.4 only to define Z_t^k as almost sure limit in H_{-1} : the proof of continuity is strictly contained in the argument that starts from (3.32) and goes till the end of that proof (but, once again, that proof is substantially more informative and involved, since it follows the N -dependence). \square

A.1 Second order estimates of the projection

As anticipated in Sect. 2.3 our approach requires a control up to and including the second order for the projection map $p(\cdot)$ (recall Sect. 2.2 for the definition). The expansion is with respect to the H_{-1} distance from the manifold M .

Lemma 7.5 *For all $q = q_\psi \in M$ and $h \in H_{-1}$ with $\|h\|_{-1} < \sigma$, we have*

$$p(q + h) = \psi - \frac{(h, q')_{-1,1/q}}{(q', q')_{-1,1/q}} \left(1 - \frac{1}{2\pi I_0^2(2Kr)} \frac{(h, (\log q)'')_{-1,1/q}}{(q', q')_{-1,1/q}} \right) + O(\|h\|_{-1}^3). \tag{7.34}$$

Proof For h as in the statement we have that

$$(q_\psi + h - q_{\psi+\varepsilon}, q'_{\psi+\varepsilon})_{-1,1/q_{\psi+\varepsilon}} = 0, \tag{7.35}$$

for $\varepsilon := p(q_\psi + h) - \psi$. Since $p(\cdot)$ is smooth, we have $\varepsilon = O(\|h\|_{-1})$. By expanding $q_{\psi+\varepsilon}$ with respect to ε we see that (7.35) implies

$$\left(h + \varepsilon q'_\psi - \frac{\varepsilon^2}{2} q''_\psi, q'_{\psi+\varepsilon} \right)_{-1,1/q_{\psi+\varepsilon}} = O(\varepsilon^3). \tag{7.36}$$

Let us rewrite (7.36) more explicitly (recall Remark 2.3) as

$$\int_{\mathbb{S}} \left(\mathcal{H}(\theta) + \varepsilon q_\psi(\theta) - \frac{\varepsilon^2}{2} q'_\psi(\theta) \right) \left(1 - \frac{1}{2\pi I_0^2(2Kr)} \frac{1}{q_{\psi+\varepsilon}(\theta)} \right) d\theta = O(\varepsilon^3), \tag{7.37}$$

where \mathcal{H} is the primitive of h such that $\int_{\mathbb{S}} \frac{\mathcal{H}}{q_\psi} = 0$. At this point we expand also $q_{\psi+\varepsilon}$ with respect to ε and, using $\varepsilon = O(\|h\|_{-1})$, the parity of $q_\psi(\cdot + \psi)$ and Remark 2.3, we get to

$$(h, q'_\psi)_{-1,1/q_\psi} + \varepsilon (q'_\psi, q'_\psi)_{-1,1/q_\psi} + \varepsilon \frac{1}{2\pi I_0^2(2Kr)} (h, (\log q)'')_{-1,1/q_\psi} = O(\|h\|_{-1}^3). \tag{7.38}$$

Now it suffices to solve this equation for ε and perform one last Taylor expansion. \square

Appendix B: Spectral estimates

The aim of this section is to find approximations of the eigenvalues and eigenfunctions of the operators L_ψ for large eigenvalues. In such a regime we expect the Laplacian to dominate and the spectrum of L_ψ should get close to the one of the Laplacian (as long as we deal with large eigenvalues). These are standard estimates, developed for example in [28] that we follow, but we could not find in the literature the result for the non-local operators we consider. Without loss of generality, we can focus on L_0 . We have

$$\begin{aligned} L_0 u &= \frac{1}{2} u'' - (uJ * q_0 + q_0 J * u)' \\ &= \frac{1}{2} u'' - (J * q_0) u' - (J * q'_0) u - q'_0 J * u - q_0 J' * u. \end{aligned} \tag{8.1}$$

We make a change of variable to get rid of the coefficient of order 1: if we define

$$u = \sqrt{q_0} y, \tag{8.2}$$

and we observe that $\sqrt{q} = e^{\tilde{J} * q_0}$, with $\tilde{J}(\theta) := K \cos(\theta)$, then we get

$$u' = \sqrt{q_0} y' + (J * q_0) \sqrt{q_0} y, \tag{8.3}$$

$$u'' = \sqrt{q_0} y'' + 2(J * q_0) \sqrt{q_0} y' + (J * q'_0) \sqrt{q_0} y + (J * q_0)^2 \sqrt{q_0} y, \tag{8.4}$$

and these two last equations together with (8.1) give

$$\begin{aligned} L_0 \sqrt{q_0} y &= \frac{1}{2} [\sqrt{q_0} y'' + 2(J * q_0) \sqrt{q_0} y' + (J * q'_0) \sqrt{q_0} y + (J * q_0)^2 \sqrt{q_0} y] \\ &\quad - (J * q_0) [\sqrt{q_0} y' + (J * q_0) \sqrt{q_0} y] - (J * q'_0) \sqrt{q_0} y - q'_0 J * (\sqrt{q_0} y) \\ &\quad - q_0 J' * (\sqrt{q_0} y) \end{aligned} \tag{8.5}$$

which leads, after simplification, to the new operator

$$\tilde{L} y := \frac{1}{2} y'' - m(y), \tag{8.6}$$

where we have set

$$m(y) := \frac{1}{2} ((J * q_0)^2 + J * q'_0) y + \frac{q'_0}{\sqrt{q_0}} J * (\sqrt{q_0} y) + \sqrt{q_0} J' * (\sqrt{q_0} y). \tag{8.7}$$

Of course $m(y)$ is a function and when we want to make explicit the θ -dependence we use $m_\theta(y)$. Since the operator L_0 is negative, we are interested in couples (ρ, y) solution of

$$\tilde{L}y = -\rho^2 y, \tag{8.8}$$

where ρ is a positive real number. The method of variation of the parameters shows that such solutions exist (for all $\rho > 0$ if we do not restrict the study to the 2π -periodic eigenfunctions of \tilde{l}) and are of the form

$$y(\theta) = c_1 e^{\sqrt{2}\rho i\theta} + c_2 e^{-\sqrt{2}\rho i\theta} - \frac{1}{\sqrt{2}\rho} \int_0^\theta G(\theta, \theta', \rho) m_{\theta'}(y) d\theta' \tag{8.9}$$

where

$$G(\theta, \theta', \rho) = i e^{\sqrt{2}\rho i(\theta-\theta')} - i e^{-\sqrt{2}\rho i(\theta-\theta')}. \tag{8.10}$$

We define y_1 the solution such that $c_1 = 1, c_2 = 0$, and y_2 the solution such that $c_1 = 0, c_2 = 1$. In what follows we start by getting a first estimate of the eigenfunctions y_1 and y_2 with respect to $\rho \rightarrow \infty$. This estimate implies a first estimate of the eigenvalue $-\lambda = -\rho^2$, and this leads to a new approximation of the eigenfunctions, and thus a new approximation of $-\lambda$. This procedure can be repeated recursively, but for us two steps will suffice.

Lemma 8.1 *For each $\lambda > 0$, there exist y_1 and y_2 independent (non necessarily periodic) eigenfunctions of \tilde{L} associated to $-\lambda$ such that (recall that $\rho = \sqrt{\lambda}$):*

$$y_1(\theta, \rho) = e^{\sqrt{2}\rho i\theta} + O\left(\frac{1}{\rho}\right), \tag{8.11}$$

$$y_2(\theta, \rho) = e^{-\sqrt{2}\rho i\theta} + O\left(\frac{1}{\rho}\right), \tag{8.12}$$

$$y_1'(\theta, \rho) = \sqrt{2}\rho i e^{\sqrt{2}\rho i\theta} + O(1), \tag{8.13}$$

$$y_2'(\theta, \rho) = -\sqrt{2}\rho i e^{-\sqrt{2}\rho i\theta} + O(1), \tag{8.14}$$

where $\theta \in [0, 2\pi]$ and $O(\cdot)$ is as ρ tends to infinity (and we stress that here and below the $O(\cdot)$ term does not depend on θ or, equivalently, it is uniform in $\theta \in [0, 2\pi]$).

Proof We prove the result for y_1 . The proof for y_2 is similar. We define

$$A_0(\theta, \theta', v) = -\frac{1}{\sqrt{2}\rho} G(\theta, \theta', \rho) m_{\theta'}(v) \mathbf{1}_{\theta' < \theta} \tag{8.15}$$

so that for $\theta \in [0, 2\pi]$,

$$y_1(\theta) = e^{\sqrt{2}\rho i\theta} + \int_0^{2\pi} A_0(\theta, \theta', y_1) d\theta'. \tag{8.16}$$

The expression for y_1 ; cf. (8.9), can be iterated arbitrarily many times and it leads to a series expression for y_1 , at least for ρ sufficiently large. To see this set $f_0(\theta) := e^{\sqrt{2}\rho i\theta}$ and observe that

$$\begin{aligned}
 y_1(\theta_0) &= f_0(\theta_0) + \sum_{j=1}^m \int_0^{2\pi} \cdots \int_0^{2\pi} A_0(\theta_0, \theta_1, A_0(\theta_1, \theta_2, \cdots A_0(\theta_{i-1}, \theta_i, f_0) \cdots)) d\theta_1 \cdots d\theta_m \\
 &\quad + \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} A_0(\theta_0, \theta_1, A_0(\theta_1, \theta_2, \cdots A_0(\theta_m, \theta_{m+1}, y_1) \cdots)) d\theta_1 \cdots d\theta_{m+1}.
 \end{aligned}
 \tag{8.17}$$

One directly verifies that there exists $C = C(K)$ such that for $\theta, \theta' \in [0, 2\pi]$,

$$|A_0(\theta, \theta', v)| \leq \frac{C}{\rho} \|v\|,
 \tag{8.18}$$

where $\|v\| := \sup_{\theta \in [0, 2\pi]} |v(\theta)|$. From (8.17) and using $\|f_0(\cdot)\| \equiv 1$ we see that

$$\|y_1\| \leq 1 + \sum_{j=1}^m \left(\frac{2\pi C}{\rho}\right)^m + \left(\frac{2\pi C}{\rho}\right)^{m+1} \|y_1\|,
 \tag{8.19}$$

so for $\rho > 2\pi C$ we see that $\|y_1\| < \infty$ and we have a series expression for y_1 , from which we directly obtain (8.11).

To deal with y'_1 we take the derivative of both sides of (8.9) with $c_1 = 1$ and $c_2 = 0$, so that

$$y'_1(\theta) = \sqrt{2}\rho i e^{\sqrt{2}\rho i\theta} - \frac{1}{\sqrt{2}\rho} \int_0^\theta \partial_\theta G(\theta, \theta', \rho) m_{\theta'}(y_1) d\theta'.
 \tag{8.20}$$

We define the new kernel

$$A_1(\theta, \theta', v) := -\frac{1}{\sqrt{2}\rho^2} \partial_\theta G(\theta, \theta', \rho) m_{\theta'}(v) \mathbf{1}_{\theta' < \theta},
 \tag{8.21}$$

so we can write

$$\frac{1}{\rho} y'_1(\theta) = \sqrt{2}i e^{\sqrt{2}\rho i\theta} + \int_0^{2\pi} A_1(\theta, \theta', y_1) d\theta'.
 \tag{8.22}$$

Also A_1 verifies

$$|A_1(\theta, \theta', v)| \leq \frac{C}{\rho} \sup_{\theta \in [0, 2\pi]} |v(\theta)|,
 \tag{8.23}$$

for a suitable $C = C(K)$ and the same argument as above gives

$$\frac{1}{\rho}y_1'(\theta) = \sqrt{2}ie^{\sqrt{2}\rho i\theta} + O\left(\frac{1}{\rho}\right), \tag{8.24}$$

which is equivalent to (8.12). □

Lemma 8.2 *There exists $l_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ the eigenvalues of L_0 satisfy*

$$\lambda_{l_0+2p} = \frac{p^2}{2} + O(\sqrt{p}), \tag{8.25}$$

$$\lambda_{l_0+2p+1} = \frac{p^2}{2} + O(\sqrt{p}). \tag{8.26}$$

Remark 8.3 An immediate consequence of Lemma 8.2 and of the basic properties of L_0 is that there exist $C > 1$ such that for $j = 0, 1, \dots$

$$\frac{j^2}{C} \leq \lambda_j \leq Cj^2. \tag{8.27}$$

Proof Let y_1 and y_2 the eigenfunctions of \tilde{L} given by Lemma 8.1 associated to the eigenvalue $-\lambda = -\rho^2$. As a linear combination of y_1 and y_2 is 2π -periodic, the following determinant is equal to zero:

$$\begin{vmatrix} y_1(2\pi) - y_1(0) & y_2(2\pi) - y_2(0) \\ y_1'(2\pi) - y_1'(0) & y_2'(2\pi) - y_2'(0) \end{vmatrix} = 0. \tag{8.28}$$

Lemma 8.1 implies

$$\begin{vmatrix} e^{2\sqrt{2}\pi\rho i} - 1 + O\left(\frac{1}{\rho}\right) & e^{-2\sqrt{2}\pi\rho i} - 1 + O\left(\frac{1}{\rho}\right) \\ \sqrt{2}\rho i(e^{2\sqrt{2}\pi\rho i} - 1) + O(1) & -\sqrt{2}\rho i(e^{-2\sqrt{2}\pi\rho i} - 1) + O(1) \end{vmatrix} = 0, \tag{8.29}$$

and thus we get

$$|e^{2\sqrt{2}\pi\rho i} - 1|^2 = O\left(\frac{1}{\rho}\right). \tag{8.30}$$

We deduce that there exists $k \in \mathbb{N}$ such that

$$\rho = \frac{k}{\sqrt{2}} + O\left(\frac{1}{\sqrt{k}}\right). \tag{8.31}$$

Reciprocally, all ρ satisfying (8.31) satisfies (8.29), so the Lemma follows. □

Proposition 8.4 *There exists $l_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ the eigenvalues of L_0 satisfy*

$$\lambda_{l_0+2p} = \frac{p^2}{2} - \frac{K^2 r^2}{8} + O\left(\frac{1}{p}\right), \tag{8.32}$$

$$\lambda_{l_0+2p+1} = \frac{p^2}{2} - \frac{K^2 r^2}{8} + O\left(\frac{1}{p}\right), \tag{8.33}$$

and any eigenfunction of L_0 associated to λ_{l_0+2p} or λ_{l_0+2p+1} is, up to a correction of order $1/p^2$, a linear combination of the two functions $q_0^{1/2} v_{1,l_0+p}$ and $q_0^{1/2} v_{2,l_0+p}$, where

$$\begin{aligned} v_{1,l_0+p}(\theta) &= \cos(p\theta) - \frac{\sin(p\theta)}{p} \left[\frac{Kr}{2} \sin(\theta) + \frac{K^2 r^2}{8} \sin(2\theta) \right], \\ v_{2,l_0+p}(\theta) &= \sin(p\theta) + \frac{\cos(p\theta)}{p} \left[\frac{Kr}{2} \sin(\theta) + \frac{K^2 r^2}{8} \sin(2\theta) \right]. \end{aligned} \tag{8.34}$$

From Proposition 8.4 one can directly extract some important conclusions: let us give them before the proof of the proposition.

Corollary 8.5 *There exists $l_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ and $\psi \in \mathbb{S}$, the unitary (in $H_{-1,1/q_\psi}$) eigenfunctions e_{ψ,l_0+2p} and e_{ψ,l_0+2p+1} of L_ψ are up to a correction of order $1/p$ a bounded (with respect to p) linear combination of $\theta \mapsto pq_\psi^{1/2}(\theta)v_{1,l_0+p}(\theta - \psi)$ and $\theta \mapsto pq_\psi^{1/2}(\theta)v_{2,l_0+p}(\theta - \psi)$ (see Proposition 8.4 for the definition of $v_{1,l}$ and $v_{2,l}$).*

Proof We set $\psi = 0$ without loss of generality. Proposition 8.4 tells us that the normalized eigenfunctions of L_0 can be written either as

$$c_p \left(\cos(p\theta) - \frac{\sin(p\theta)}{p} \left[\frac{Kr}{2} \sin(\theta) + \frac{K^2 r^2}{8} \sin(2\theta) \right] + r_p(\theta) \right) \tag{8.35}$$

where $r_p(\theta) = O(1/p^2)$ and c_p is the normalizing constant, or with the analogous expression coming from the second line in (8.34) (but we will deal only with (8.35) because the other case is treated analogously). To estimate c_p let us observe that the first two addends in (8.35) are in H^{-1} (since they are smooth, it suffices to remark that their integral from 0 to 2π is zero), so $r_p \in H_{-1}$, since the eigenfunction is: of course r_p is smooth, since the eigenfunction is. Now we claim that the $H_{-1,1/q}$ norm of $\cos(p \cdot)$, that is the first addendum, is proportional to $1/p$, apart for a correction that is beyond all orders in $1/p$, while the norm of the two other terms is $O(1/p^2)$. In fact if we set $u(\theta) := \cos(p\theta)$, then $\mathcal{U}(\theta) = \sin(p\theta)/p$ so

$$\|u\|_{-1,1/q} = \frac{1}{p} \sqrt{\int_{\mathbb{S}} \frac{1 - \cos(2p\theta)}{2q(\theta)} d\theta}. \tag{8.36}$$

If we use the standard estimate

$$I_k(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(k\theta) e^{x \cos(\theta)} d\theta = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2m+k} \leq \frac{C_x}{(k!)^{1/2}}. \tag{8.37}$$

we readily see that

$$\int_{\mathbb{S}} \frac{\cos(2p\theta)}{q(\theta)} d\theta = O\left(\frac{1}{\sqrt{(2p)!}}\right). \tag{8.38}$$

On the other hand

$$\int_{\mathbb{S}} \frac{1}{q(\theta)} d\theta = (2\pi I_0(2Kr))^2, \tag{8.39}$$

so $\|u\|_{-1, 1/q}$ is equal to $c(K)/p$, $c(K) := \sqrt{2\pi} I_0(2Kr)$, up to a correction that decays faster than any power of $1/p$.

For the second addendum it suffices to observe that it can be rewritten as a linear combination of terms of $\cos(p'\theta)$, with $|p - p'| = 1$ and 2 . But then the computation is very similar to the one that we have done for the first addendum (or, easier, one can explicitly compute the H_{-1} norm, without weight). Therefore this term is $O(1/p^2)$.

For the third addendum we recall that $|r_p(\theta)| \leq C/p^2$, so that if we set $\mathcal{R}(\theta) := \int_0^\theta r_p(\theta') d\theta'$, we have $\|\mathcal{R}(\theta)\| \leq C\theta/p^2$. Of course \mathcal{R} is not necessarily centered, but, by using Remark 2.1, we see that $\|r_p\|_{-1} \leq 2C^2\pi^2/p^2$.

By collecting the estimates of the three addends we see that

$$c_p = c(K)p(1 + O(1/p)), \tag{8.40}$$

and this completes the proof of Corollary 8.5. □

By putting Corollary 8.5 and (7.8) together we obtain

Corollary 8.6 *With $\{f_j\}_{j=0,1,\dots}$ defined as in Appendix A, we have $\sup_j \|f'_j\|_\infty < \infty$ and $\sup_j \|f''_j\|_\infty/j < \infty$.*

Proof From (7.8), see also the discussion right after that, we see that $f'_j = -\mathcal{E}_j/q$ and $f''_j = -e_j/q + \mathcal{E}_j q'/q^2$. Where e_j is the j th (normalized) eigenvector. Taking into account the normalization, see proof of Corollary 8.5, the claim is readily proven. □

Proof of Proposition 8.4 Injecting (8.11) in the integral term of (8.16) leads to

$$y_1(\theta) = e^{\sqrt{2}\rho i\theta} - \frac{1}{\sqrt{2}\rho} \left[i e^{\sqrt{2}\rho i\theta} \int_0^\theta e^{-\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot}) d\theta' - i e^{-\sqrt{2}\rho i\theta} \int_0^\theta e^{\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot}) d\theta' \right] + O\left(\frac{1}{\rho^2}\right). \tag{8.41}$$

Similarly, we obtain

$$y'_1(\theta) = \sqrt{2}\rho i e^{\sqrt{2}\rho i\theta} + \left[e^{\sqrt{2}\rho i\theta} \int_0^\theta e^{-\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot}) d\theta' + e^{-\sqrt{2}\rho i\theta} \int_0^\theta e^{\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot}) d\theta' \right] + O\left(\frac{1}{\rho}\right), \tag{8.42}$$

and similar expressions for y_2 and y'_2 , which actually are just the complex conjugate of y_1 and y'_1 . We define

$$H_1 = \int_0^{2\pi} e^{-\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot}) d\theta', \quad H_2 = \int_0^{2\pi} e^{\sqrt{2}\rho i\theta'} m_{\theta'}(e^{\sqrt{2}\rho i\cdot}) d\theta' \quad \text{and} \\ \Omega = e^{2\sqrt{2}\pi\rho i}. \tag{8.43}$$

With the higher estimates (8.41) and (8.42), we see that (8.28) becomes

$$\left| \begin{aligned} &\Omega - 1 - \frac{1}{\sqrt{2}\rho} [i\Omega H_1 - i\bar{\Omega} H_2] + O\left(\frac{1}{\rho^2}\right) & \bar{\Omega} - 1 - \frac{1}{\sqrt{2}\rho} [-i\bar{\Omega} \bar{H}_1 + i\Omega \bar{H}_2] + O\left(\frac{1}{\rho^2}\right) \\ &\Omega - 1 - \frac{1}{\sqrt{2}\rho} [i\Omega H_1 + i\bar{\Omega} H_2] + O\left(\frac{1}{\rho^2}\right) & -\bar{\Omega} + 1 - \frac{1}{\sqrt{2}\rho} [i\bar{\Omega} \bar{H}_1 + i\Omega \bar{H}_2] + O\left(\frac{1}{\rho^2}\right) \end{aligned} \right| = 0, \tag{8.44}$$

which implies

$$|\Omega - 1|^2 - \frac{\sqrt{2}}{\rho} \Im((\Omega - 1)H_1) = O\left(\frac{1}{\rho^2}\right). \tag{8.45}$$

We now use the expansion of ρ given by (8.31). In particular, the $O(1/\rho^2)$ above becomes a $O(1/k^2)$. The second term of the left hand side above is of order $1/k^2$. In fact, we get the first order of H_1 :

$$H_1 = \int_0^{2\pi} e^{-ki\theta'} m_{\theta'}(e^{ki\cdot}) d\theta' + O\left(\frac{1}{\sqrt{k}}\right), \tag{8.46}$$

where the non local terms in the integral are negligible, since we have

$$J * (\sqrt{q_0} e^{ki\cdot})(\theta) = \frac{iK}{2(2\pi I_0(2Kr))^{1/2}} (e^{i\theta} I_{k-1}(Kr) - e^{-i\theta} I_{k+1}(Kr)) \tag{8.47}$$

and we can apply (8.37). A similar bound apply for $J' * (\sqrt{q_0} e^{ki\cdot})$. So it remains the (real !!) first order (remark that $J * q_0(\cdot) = -Kr \sin(\cdot)$):

$$H_1 = \int_0^{2\pi} \frac{1}{2} ((J * q_0)^2 + J * q'_0)(\theta') d\theta' + O\left(\frac{1}{\sqrt{k}}\right) = \frac{\pi K^2 r^2}{2} + O\left(\frac{1}{\sqrt{k}}\right). \tag{8.48}$$

But since [using (8.31)]

$$\Omega - 1 = 2\pi i(\sqrt{2}\rho - k) + O\left(\frac{1}{k}\right), \tag{8.49}$$

where the first term of the right hand side is of order $1/\sqrt{k}$, we have improved the result of Lemma 8.2, since using (8.45), (8.48), (8.49) and (8.31) we obtain

$$|e^{2\sqrt{2}\pi\rho i} - 1|^2 - \frac{2\pi^2 K^2 r^2}{k}(\sqrt{2}\rho - k) = O\left(\frac{1}{k^2}\right) \tag{8.50}$$

which implies

$$\sqrt{2}\rho = k + O\left(\frac{1}{k}\right). \tag{8.51}$$

Taking (8.51) into account, (8.44) yields

$$|\Omega - 1|^2 - \frac{2}{k}\Im((\Omega - 1)H_1) + \frac{1}{k^2}(|H_1|^2 - |H_2|^2) = O\left(\frac{1}{k^3}\right). \tag{8.52}$$

The non local terms in H_2 are negligible as for H_1 (see above) and a direct calculation shows that the local terms are of order $1/k$, so from (8.52), (8.48) and (8.49) we get

$$(\sqrt{2}\rho - k)^2 - \frac{K^2 r^2}{2k}(\sqrt{2}\rho - k) + \frac{K^4 r^4}{16k^2} = \left(\sqrt{2}\rho - k - \frac{K^2 r^2}{4k}\right)^2 = O\left(\frac{1}{k^3}\right), \tag{8.53}$$

which implies

$$\sqrt{2}\rho = k + \frac{K^2 r^2}{4} \frac{1}{k} + O\left(\frac{1}{k^{3/2}}\right). \tag{8.54}$$

We now go further in the expansion to prove that the $O(1/k^{3/2})$ in (8.54) is in fact a $O(1/k^2)$. Using (8.17), we get the second order expansion of y_1 (recall (8.15) and $f_0 = e^{\sqrt{2}\rho i \cdot}$)

$$y_1(2\pi) = \Omega + \int_0^{2\pi} A_0(2\pi, \theta_1, f_0) d\theta_1 + \int_0^{2\pi} \int_0^{2\pi} A_0(2\pi, \theta_1, A_0(\theta_1, \theta_2, f_0)) d\theta_1 d\theta_2 + O\left(\frac{1}{\rho^3}\right). \tag{8.55}$$

From (8.51), we deduce

$$\begin{aligned} \int_0^{2\pi} A_0(\theta_1, \theta_2, f_0) d\theta_2 &= -\frac{1}{\sqrt{2}\rho} \left[i e^{\sqrt{2}\rho i \theta_1} \int_0^{\theta_1} e^{-\sqrt{2}\rho i \theta_2} m_{\theta_2}(e^{\sqrt{2}\rho i \cdot}) d\theta_2 \right. \\ &\quad \left. - i e^{-\sqrt{2}\rho i \theta_1} \int_0^{\theta_1} e^{\sqrt{2}\rho i \theta_2} m_{\theta_2}(e^{\sqrt{2}\rho i \cdot}) d\theta_2 \right] \\ &= -\frac{i}{k} \left[e^{ki\theta_1} \int_0^{\theta_1} e^{-ki\theta_2} m_{\theta_2}(e^{ki \cdot}) d\theta_2 \right. \\ &\quad \left. - e^{-ki\theta_1} \int_0^{\theta_1} e^{ki\theta_2} m_{\theta_2}(e^{ki \cdot}) d\theta_2 \right] + O\left(\frac{1}{k^2}\right), \tag{8.56} \end{aligned}$$

and since the non local terms are negligible (see (8.47)), we get

$$\int_0^{2\pi} A_0(\theta_1, \theta_2, f_0) d\theta_2 = \frac{iKr e^{ki\theta_1}}{2k} \left(\sin \theta_1 + \frac{Kr}{4} \sin(2\theta_1) - \frac{Kr}{2} \theta_1 \right) + O\left(\frac{1}{k^2}\right). \tag{8.57}$$

We deduce the following expansion for the third term of the right hand side of (8.55):

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} A_0(2\pi, \theta_1, A_0(\theta_1, \theta_2, f_0)) d\theta_1 d\theta_2 \\ &= \frac{Kr}{2k^2} \left(\Omega \int_0^{2\pi} e^{-ki\theta_1} m_{\theta_1} \left[e^{ki \cdot} \left(\sin \cdot + \frac{Kr}{4} \sin(2 \cdot) - \frac{Kr}{2} \cdot \right) \right] d\theta_1 \right. \\ & \quad \left. - \bar{\Omega} \int_0^{2\pi} e^{ki\theta_1} m_{\theta_1} \left[e^{ki \cdot} \left(\sin \cdot + \frac{Kr}{4} \sin(2 \cdot) - \frac{Kr}{2} \cdot \right) \right] d\theta_1 \right) + O\left(\frac{1}{k^3}\right). \end{aligned} \tag{8.58}$$

Using similar arguments as before, we get to

$$\int_0^{2\pi} e^{-ki\theta_1} m_{\theta_1} \left(e^{ki \cdot} \left(\sin(\cdot) + \frac{Kr}{4} \sin(2 \cdot) \right) \right) d\theta_1 = O\left(\frac{1}{k^3}\right), \tag{8.59}$$

$$\int_0^{2\pi} e^{ki\theta_1} m_{\theta_1} \left(e^{ki \cdot} \left(\sin(\cdot) + \frac{Kr}{4} \sin(2 \cdot) \right) \right) d\theta_1 = O\left(\frac{1}{k^3}\right). \tag{8.60}$$

Moreover, the non local terms of $m_{\theta_1}(e^{ki \cdot})$ are of order $1/k$. In fact, these non local terms are finite sums of the form

$$\int_0^{2\pi} e^{mi\theta} \sqrt{q_0}(\theta) \theta d\theta, \tag{8.61}$$

where $|m|$ is included in $[k - 1, k + 1]$, and it is easy to see that since the Fourier coefficients of $\sqrt{q_0}$ decay very quickly (see (8.37)), (8.61) is of order $1/k$. So (8.58) becomes

$$\int_0^{2\pi} \int_0^{2\pi} A_0(2\pi, \theta_1, A_0(\theta_1, \theta_2, f_0)) d\theta_1 d\theta_2 = -\frac{K^4 r^4 \pi^2}{8k^2} \Omega + O\left(\frac{1}{k^3}\right), \tag{8.62}$$

and we deduce from (8.55)

$$y_1(2\pi) - y_1(0) = \Omega - 1 - \frac{i}{k} (\Omega H_1 - \bar{\Omega} H_2) - \frac{K^4 r^4 \pi^2}{8k^2} \Omega + O\left(\frac{1}{k^3}\right). \tag{8.63}$$

Similarly, we obtain

$$\frac{y'_1(2\pi) - y'_1(0)}{\sqrt{2}\rho i} = \Omega - 1 - \frac{i}{k}(\Omega H_1 + i\bar{\Omega}H_2) - \frac{K^2 r^2 \pi^2}{8k^2} \Omega + O\left(\frac{1}{k^3}\right). \tag{8.64}$$

Using these new estimates, (8.44) becomes

$$\begin{vmatrix} \Omega - 1 - \frac{1}{\sqrt{2}\rho} [i\Omega H_1 - i\bar{\Omega}H_2] & \bar{\Omega} - 1 - \frac{1}{\sqrt{2}\rho} [-i\bar{\Omega}\bar{H}_1 + i\Omega\bar{H}_2] \\ -\frac{K^4 r^4 \pi^2}{8k^2} \Omega + O\left(\frac{1}{k^3}\right) & -\frac{K^4 r^4 \pi^2}{8k^2} \bar{\Omega} + O\left(\frac{1}{k^3}\right) \\ \Omega - 1 - \frac{1}{\sqrt{2}\rho} [i\Omega H_1 + i\bar{\Omega}H_2] & -\bar{\Omega} + 1 - \frac{1}{\sqrt{2}\rho} [i\bar{\Omega}\bar{H}_1 + i\Omega\bar{H}_2] \\ -\frac{K^4 r^4 \pi^2}{8k^2} \Omega + O\left(\frac{1}{k^3}\right) & +\frac{K^4 r^4 \pi^2}{8k^2} \bar{\Omega} + O\left(\frac{1}{k^3}\right) \end{vmatrix} = 0, \tag{8.65}$$

which leads to

$$\begin{aligned} |\Omega - 1|^2 - \frac{\sqrt{2}}{\rho} \Im((\Omega - 1)H_1) + \frac{1}{k^2} (|H_1|^2 - |H_2|^2) + \frac{K^4 r^4 \pi^2}{8k^2} (4 - 2\Omega - 2\bar{\Omega}) \\ + \frac{K^4 r^4 \pi^2}{2k^3} \Im(H_1) = O\left(\frac{1}{k^4}\right). \end{aligned} \tag{8.66}$$

The last term of (8.66) is of order $1/k^4$ since using (8.51) we get

$$\Omega = 1 + i2\pi(\sqrt{2}\rho - k) + O\left(\frac{1}{k^2}\right). \tag{8.67}$$

Moreover using (8.51) we have

$$\begin{aligned} H_1 &= \int_0^{2\pi} e^{-ki\theta} m_\theta(e^{ki\cdot}) d\theta + i(\sqrt{2}\rho - k) \\ &\times \left(-\int_0^{2\pi} e^{ki\theta} \theta m_\theta(e^{ki\cdot}) d\theta + \int_0^{2\pi} e^{ki\theta} m_\theta(e^{ki\cdot}) d\theta \right) + O\left(\frac{1}{k^2}\right). \end{aligned} \tag{8.68}$$

As before, the non local terms of $m_\theta(e^{ki\cdot})$ are of order $1/k$, so the last two integrals in (8.68) are equal up to a correction of order $1/k$, and thus (recall (8.48) for the first order term), using (8.51),

$$H_1 = \frac{\pi K^2 r^2}{2} + O\left(\frac{1}{k^2}\right). \tag{8.69}$$

We deduce that the first term of the second row of (8.66) is of order $1/k^4$, and that, using (8.67),

$$\frac{\sqrt{2}}{\rho} \Im((\Omega - 1)H_1) = \frac{2\pi^2 K^2 r^2}{k} (\sqrt{2}\rho - k) + O\left(\frac{1}{k^4}\right), \tag{8.70}$$

and

$$\frac{1}{k^2} |H_1|^2 = \frac{\pi^2 K^4 r^4}{4k^2} + O\left(\frac{1}{k^4}\right). \tag{8.71}$$

Since $|H_2|$ is of order $1/k$ and that (8.67) implies

$$|\Omega - 1|^2 = 4\pi^2 (\sqrt{2} - \rho)^2 + O\left(\frac{1}{k^4}\right), \tag{8.72}$$

(8.66) becomes

$$(\sqrt{2}\rho - k)^2 - \frac{K^2 r^2}{2k} (\sqrt{2}\rho - k) + \frac{K^4 r^4}{16k^2} = O\left(\frac{1}{k^4}\right), \tag{8.73}$$

and we deduce

$$\sqrt{2}\rho = k + \frac{K^2 r^2}{4} \frac{1}{k} + O\left(\frac{1}{k^2}\right). \tag{8.74}$$

Now we are able to get a second expansion of the eigenvectors: using (8.41), (8.57) and (8.74), we get the following expansion for y_1

$$y_1(\theta) = e^{ki\theta} \left(1 + \frac{Kri}{2k} \sin(\theta) + \frac{K^2 r^2 i}{8k} \sin(2\theta) \right) + O\left(\frac{1}{k^2}\right) \tag{8.75}$$

and y_2 is the complex conjugate. So if we define w_1 and w_2 the real and imaginary parts, we get

$$w_1(\theta) = \cos(k\theta) - \frac{\sin(k\theta)}{k} \left(\frac{Kr}{2} \sin \theta + \frac{K^2 r^2}{8} \sin(2\theta) \right) + O\left(\frac{1}{k^2}\right), \tag{8.76}$$

$$w_2(\theta) = \sin(k\theta) + \frac{\cos(k\theta)}{k} \left(\frac{Kr}{2} \sin \theta + \frac{K^2 r^2}{8} \sin(2\theta) \right) + O\left(\frac{1}{k^2}\right). \tag{8.77}$$

Therefore the proof of Proposition 8.4 is complete. □

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