Probab. Theory Relat. Fields (2013) 157:107–156
DOI 10.1007/s00440-012-0451-2

Asymptotic expansions in the CLT in free probability

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Received: 26 January 2012 / Accepted: 1 September 2012 / Published online: 26 September 2012

Abstract We prove Edgeworth type expansions for distribution functions of sums of free random variables under minimal moment conditions. The proofs are based on the analytic definition of free convolution.

Keywords Free random variables · Cauchy transform · Free convolutions · Central Limit Theorem · Asymptotic expansion

Mathematics Subject Classification (1991) Primary 46L50 · 60E07; Secondary 60E10

1 Introduction

In recent years a number of papers are devoted to limit theorems for the free convolutions of probability measures. Free convolutions were introduced by Voiculescu [42, 43]. The key concept is the notion of freeness, which can be interpreted as a kind of independence for noncommutative random variables. As in the classical probability where the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the probability measures, the free

G. P. Chistyakov and F. Götze's research supported by SFB 701.

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convolution. Many classical results in the theory of addition of independent random variables have their counterparts in Free Probability, such as the Law of Large Numbers, the Central Limit Theorem, the Lévy–Khintchine formula and others. We refer to Voiculescu et al. [44] and Hiai and Petz [27] for an introduction to these topics. Bercovici and Pata [11] established the distributional behavior of sums of free identically distributed random variables and described explicitly the correspondence between limit laws for free and classical additive convolution. Using subordination functions for the definition of the additive free convolution, Chistyakov and Götze [22] generalized the results of Bercovici and Pata to the case of free non-identically distributed random variables. It was shown that the parallelism found by Bercovici and Pata holds in the general case of free non-identically distributed random variables (see [13] as well). This approach allowed us to obtain estimates of the rate of convergence of distribution functions of free sums. An analog of the Berry–Esseen inequality was proved for the semicircle approximation in [22]. For related results see [28].

In this paper we obtain an analogue of Edgeworth expansion in the Central Limit Theorem (CLT for short) for free identically distributed random variables, based on the method of subordination functions. In addition we shall give a bound for the remainder term in this expansion. In order to deduce this expansion we establish an approximation of distribution of normalized sums of free random variables by the free Meixner distributions. In classical probability asymptotic expansions have a different form for lattice and non-lattice distributions. An interesting feature of our expansions is that they have the same form for all distributions.

The paper is organized as follows. In Sect. 2 we formulate and discuss the main results of the paper. In Sect. 3 and 4 we formulate auxiliary results. In Sect. 5 we describe a formal expansion in the Free CLT and in Sects. 6 and 7 we prove Edgeworth's expansion in the CLT for free identically distributed random variables. Since the proofs of Theorem 2.1 and Theorem 2.3 (see Sect. 2) are similar, we give a proof of Theorem 2.3 in details in Sect. 5 and an outline of the proof of Theorem 2.1 in Appendix.

2 Results

Denote by $\mathcal{M}$ the family of all Borel probability measures defined on the real line $\mathbb{R}$. Define on $\mathcal{M}$ the compositions laws denoted $*$ and $\boxplus$ as follows. For $\mu, \nu \in \mathcal{M}$, let $\mu * \nu$ denote the classical convolution of μ and ν. In probabilistic terms, $\mu * \nu = \mathcal{L}(X+Y)$, where X and Y are independent random variables with $\mu = \mathcal{L}(X)$ and $\nu = \mathcal{L}(Y)$, respectively. Let $\mu \boxplus \nu$ be the free (additive) convolution of μ and ν introduced by Voiculescu [42] for compactly supported measures. Free convolution was extended by Maassen [32] to measures with finite variance and by Bercovici and Voiculescu [9] to the class $\mathcal{M}$. Thus, $\mu \boxplus \nu = \mathcal{L}(X+Y)$, where X and Y are free random variables such that $\mu = \mathcal{L}(X)$ and $\nu = \mathcal{L}(Y)$. There are free analogues of multiplicative convolutions as well; these were first studied in Voiculescu [43].

Henceforth $X, X_1, X_2, \dots$ stands for a sequence of identically distributed random variables with distribution $\mu = \mathcal{L}(X)$. Define

$$m_k := \int_{\mathbb{R}} u^k \, \mu(du) \quad \text{and} \quad \beta_q := \int_{\mathbb{R}} |u|^q \, \mu(du),$$

where $k = 0, 1, \dots$ and $q > 0$.

The classical CLT says that if $X_1, X_2, \dots$ are independent and identically distributed random variables with a probability distribution μ such that $m_1 = 0$ and $m_2 = 1$, then the distribution function $F_n(x)$ of

$$Y_n := \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \tag{2.1}$$

tends to the standard Gaussian law $\Phi(x)$ as $n \to \infty$ uniformly in x.

A free analogue of this classical result was proved by Voiculescu [41] for bounded free random variables and later generalized by Maassen [32] to unbounded random variables. Other generalizations can be found in [10,11,22,28–30,37,47,48]. When the assumption of independence is replaced by the freeness of the noncommutative random variables $X_1, X_2, \dots, X_n$, the limit distribution function of (2.1) is the semicircle law $w(x)$, i.e., the distribution function with the density $p_w(x) := \frac{1}{2\pi}\sqrt{(4 - x^2)_+}$, $x \in \mathbb{R}$, where $a_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. Denote by μ_w the probability measure with the distribution function $w(x)$.

Write $\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and denote by $H_m(x) := (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}$ the Hermite polynomial of degree m.

Assume that the random variables X_j are independent and have moments of all orders. For the distribution function $F_n(x)$ of Y_n there exists a formal expansion in a power series in $1/\sqrt{n}$ (see [26,38]):

$$F_n(x) = \Phi(x) + \varphi(x) \sum_{p=1}^{\infty} \frac{Q_p(x)}{n^{p/2}}, \tag{2.2}$$

where

$$Q_p(x) = -\sum H_{p+2s-1}(x) \prod_{m=1}^{p} \frac{1}{k_m!} \Big(\frac{\gamma_{m+2}}{(m+2)!} \Big)^{k_m}$$

and γ_m is the cumulant of order m of random variable X. In the last equality the summation on the right-hand side is carried out over all nonnegative integer solutions $(k_1, \dots, k_m)$ of the equations

$$k_1 + 2k_2 + \cdots + pk_p = p \quad \text{and} \qquad s = k_1 + \cdots + k_p. \tag{2.3}$$

Note that $Q_1(x) = -m_3 H_2(x)/6$.

In terms of characteristic functions (2.2) has the form

$$\int_{-\infty}^{\infty} e^{itx}\, dF_n(x) = e^{-t^2/2} + \sum_{m=1}^{\infty} \frac{P_m(t)}{n^{m/2}} e^{-t^2/2}, \tag{2.4}$$

where

$$\int_{-\infty}^{\infty} e^{itx}\, dQ_m(x) = P_m(t) e^{-t^2/2}.$$

Esseen [25] proved that if the random variables X_j are independent, non-lattice distributed and $\beta_3 < \infty$, then $F_n(x)$ admits the following asymptotic expansion

$$F_n(x) = \Phi(x) - \frac{m_3}{6\sqrt{n}} H_2(x)\varphi(x) + o(1/\sqrt{n}) \tag{2.5}$$

which holds uniformly in x.

If the random variables X_j are independent and are lattice distributed, that is they take values in an arithmetic progression $\{a + kh;\ k = 0, \pm 1, \dots\}$ (h being maximal), and $\beta_3 < \infty$, then

$$F_n(x) = \Phi(x) + \frac{1}{\sqrt{n}} \varphi(x)\Big(- \frac{m_3}{6} H_2(x) + hT\Big(\frac{x\sqrt{n}}{h} - \frac{an}{h}\Big)\Big) + o(1/\sqrt{n}), \tag{2.6}$$

uniformly in x, where $T(x) := [x] - x + 1/2$.

If absolute moments β_k of order $k > 3$ exist, then generalizations of the asymptotic expansions (2.5) and (2.6) hold under additional conditions on the characteristic function of X [38].

An analytical approach using subordination functions allowed us to give explicit estimates for the rate of convergence of distribution functions of Y_n in the case of free random variables. We demonstrated this (see [22]) by proving a semicircle approximation theorem (an analogue of the Berry–Esseen inequality [38, p. 111]). In this paper we shall establish Edgeworth expansion in the semicircle approximation theorem and a complete analogue of the Berry–Esseen inequality for identically distributed free random variables.

We now formulate the main results of the paper. As before we denote by $F_n(x)$ the distribution function of Y_n where X_j are free random variables with the same distribution μ. Assume as well that X_j have moments of arbitrary order and $m_1 = 0,\ m_2 = 1$. We denote by μ_n the distribution of Y_n. Denote by $U_m(x)$ the Chebyshev polynomial of the second kind of degree m, i. e.,

$$U_m(x) = U_m(\cos\theta) := \frac{\sin(m+1)\theta}{\sin\theta}, \quad m = 1, 2, \dots.$$

It is easy to see $U_1(x) = 2x,\ U_2(x) = 4x^2 - 1,\ U_3(x) = 4x(2x^2 - 1)$.

It turns out that there exists an analogue of the formal expansion (2.4) for $F_n(x)$. To formulate it we need the following notation. Define the Cauchy transform of $\mu \in \mathcal{M}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z-x}, \qquad z \in \mathbb{C}^+, \tag{2.7}$$

where $\mathbb{C}^+$ denotes the open upper half of the complex plane. The formal expansion has the form

$$G_{\mu_n}(z) = G_{\mu_w}(z) + \sum_{k=1}^{\infty} \frac{B_k(G_{\mu_w}(z))}{n^{k/2}}, \tag{2.8}$$

where

$$B_k(z) = \sum c_{p,m} \frac{z^p}{(1/z - z)^m} \tag{2.9}$$

with real coefficients $c_{p,m}$ which depend on the free cumulants $\alpha_3, \dots, \alpha_{k+2}$ and do not depend on n. The free cumulants will be defined in Sect. 3, (3.8). Here we note that $\alpha_3 = m_3$ and $\alpha_4 = m_4 - 2$. The summation on the right-hand side of (2.9) is taken over a finite set of non-negative integer pairs (p, m). The coefficients $c_{p,m}$ can be calculated explicitly. For the cases $k = 1, 2$ we have

$$B_1(z) = \alpha_3 \frac{z^3}{1/z - z}$$

and

$$B_2(z) = \big(\alpha_4 - \alpha_3^2\big)\frac{z^4}{1/z - z} + \alpha_3^2\Big(\frac{z^5}{(1/z - z)^2} + \frac{z^2}{(1/z - z)^3}\Big). \tag{2.10}$$

Note that

$$B_1(G_{\mu_w}(z)) = \frac{\alpha_3}{\sqrt{z^2-4}} G^3_{\mu_w}(z) = -\alpha_3 \int_{-2}^{2} \frac{1}{z-x}\, d\Big(\frac{1}{3} U_2(x/2) p_w(x)\Big), \quad z \in \mathbb{C}^+. \tag{2.11}$$

If $\alpha_3 = 0$, then

$$B_2(G_{\mu_w}(z)) = \frac{\alpha_4}{\sqrt{z^2-4}} G^4_{\mu_w}(z) = -\alpha_4 \int_{-2}^{2} \frac{1}{z-x}\, d\Big(\frac{1}{4} U_3(x/2) p_w(x)\Big), \quad z \in \mathbb{C}^+. \tag{2.12}$$

Now we can formulate a counterpart of Edgeworth expansion in the Free CLT. We obtain this counterpart from the following results in which we establish an approximation of the measures μ_n by the free Meixner measures. Consider the three-parameter family of probability measures $\{\mu_{a,b,d} : a \in \mathbb{R}, b < 1, d < 1\}$ with the reciprocal Cauchy transform

$$\frac{1}{G_{\mu_{a,b,d}}(z)} = a + \frac{1}{2}\Big((1+b)(z-a) + \sqrt{(1-b)^2(z-a)^2 - 4(1-d)}\Big), \quad z \in \mathbb{C}, \tag{2.13}$$

which we will call the free centered Meixner measures (i.e. with mean zero). In this formula we choose the branch of the square root determined by the condition $\Im z > 0$ implies $\Im(1/G_{\mu_{a,b,d}}(z)) \geq 0$. These measures are counterparts of the classical measures discovered by Meixner [36]. The free Meixner type measures occurred in many places in the literature, see [3,17–19,31,35,39].

Assume that $m_4 < \infty, m_1 = 0, m_2 = 1$ and denote

$$a_n := \frac{m_3}{\sqrt{n}}, \quad b_n := \frac{m_4 - m_3^2 - 1}{n}, \quad d_n := \frac{m_4 - m_3^2}{n}, \quad n \in \mathbb{N}. \tag{2.14}$$

In the sequel we will use the free Meixner measures of the form $\mu_{0,0,0} = w$, $\mu_{a_n,0,0}$ if $\beta_3 < \infty, m_1 = 0, m_2 = 1$ and μ_{a_n,b_n,d_n} if $m_4 < \infty, m_1 = 0, m_2 = 1$ and $n > m_4$.

Recall that a probability measure μ is $\boxplus$-infinitely divisible if for every $n \in \mathbb{N}$ there exists $\nu_n \in \mathcal{M}$ such that $\mu = \nu_n \boxplus \nu_n \boxplus \cdots \boxplus \nu_n$ (n times).

Using the results of Saitoh and Yoshida [39], we will show in Sect. 4 that under the assumptions $\beta_3 < \infty$ and $n \geq m_3^2$ the free Meixner measure $\mu_{a_n,0,0}$ is absolute continuous with a density of the form (4.1), where $a = a_n, b = 0, d = 0$, and $\mu_{a_n,0,0}$ is $\boxplus$-infinitely divisible. Under the assumptions $m_4 < \infty$ and $n \geq 3m_4$ the free Meixner measure μ_{a_n,b_n,d_n} is absolute continuous with a density of the form (4.1), where $a = a_n, b = b_n, d = d_n$, and μ_{a_n,b_n,d_n} is $\boxplus$-infinitely divisible.

We now introduce some further notations. Assume that $\beta_q < \infty$ for some $q \geq 2$. Introduce the Lyapunov fractions

$$L_{qn} := \frac{\beta_q}{n^{(q-2)/2}} \quad \text{and let} \quad \rho_q(\mu, t) := \int\limits_{|u|>t} |u|^q \, \mu(du), \ \ t > 0. \tag{2.15}$$

Write

$$q_1 := \min\{q, 3\}, \quad q_2 := \min\{q, 4\}, \quad q_3 := \min\{q, 5\}.$$

Then denote, for $n \in \mathbb{N}$,

$$\eta_{qs}(n) := \inf_{0<\varepsilon\leq 10^{-1/2}} g_{qns}(\varepsilon), \quad \text{where} \quad g_{qns}(\varepsilon) := \varepsilon^{s+2-q_s} + \frac{\rho_{q_s}(\mu, \varepsilon\sqrt{n})}{\beta_{q_s}}\varepsilon^{-q_s} \tag{2.16}$$

provided that $\beta_q < \infty,\ q \geq s+1$, for $s = 1, 2, 3$, respectively. It is easy to see that $0 < \eta_{qs}(n) \leq 10^{1+s/2} + 1$ for $s + 1 \leq q_s \leq s + 2$ and $\eta_{qs}(n) \to 0$ monotonically as $n \to \infty$ if $s + 1 \leq q_s < s + 2$, and $\eta_{qs}(n) \geq 1,\ n \in \mathbb{N}$, if $q_s = s + 2$.

By agreement the symbols $c, c_1, c_2, \dots$ and $c(\mu), c_1(\mu), c_2(\mu), \dots$ shall denote absolute positive constants and positive constants depending on μ only, respectively. By c and $c(\mu)$ we denote generic constants in different (or even in the same) formulae. The symbols $c_1, c_2, \dots$ and $c_1(\mu), c_2(\mu), \dots$ are applied for explicit constants.

Theorem 2.1 *Assume that $X_j,\ j = 1, \dots,$ are free, $\beta_q < \infty$ with some $q \geq 2$ and $m_1 = 0,\ m_2 = 1$. Then, for $n \in \mathbb{N}$,*

$$\sup_{x\in\mathbb{R}} |F_n(x) - w(x)| \leq c \begin{cases} \eta_{q1}(n)L_{qn} + n^{-1}, & \textit{if} \quad \beta_q < \infty,\ 2 \leq q < 3 \\ L_{3n}, & \textit{if} \quad \beta_q < \infty,\ q \geq 3. \end{cases} \tag{2.17}$$

In the case $m_2 < \infty$, Theorem 2.1 yields a type of Free CLT with the error bound

$$\sup_{x\in\mathbb{R}} |F_n(x) - w(x)| \leq c\Big(\eta_{q1}(n) + n^{-1}\Big), \quad n \in \mathbb{N}.$$

Since $\eta_{q1}(n) \leq 10^{3/2} + 1,\ n \in \mathbb{N}$, in the case $\beta_q < \infty,\ 2 \leq q \leq 3$, we obtain from (2.17) the complete analogue of the Berry–Esseen inequality as well.

Corollary 2.2 *Assume that $X_j,\ j = 1, \dots,$ are free, $\beta_q < \infty$ with $2 < q \leq 3$ and $m_1 = 0,\ m_2 = 1$. Then, for $n \in \mathbb{N}$,*

$$\sup_{x\in\mathbb{R}} |F_n(x) - w(x)| \leq c\, L_{qn}. \tag{2.18}$$

In the case $\beta_3 < \infty$ the inequality (2.18) has the form

$$\sup_{x\in\mathbb{R}} |F_n(x) - w(x)| \leq c\, L_{3n}, \qquad n \in \mathbb{N}. \tag{2.19}$$

The upper bound (2.19) sharpens previous results obtained by the authors [22] and Kargin [28].

Theorem 2.1 and Corollary 2.2 are free analogues of Esseen's inequality in classical probability theory (see [38, pp. 112–120]).

Theorem 2.3 *Assume that $X_j,\ j = 1, \dots,$ are free, $\beta_q < \infty$ with some $q \geq 3$ and $m_1 = 0,\ m_2 = 1$. Then, for $n \in \mathbb{N}$,*

$$\sup_{x\in\mathbb{R}} |F_n(x) - \mu_{a_n,0,0}((-\infty, x))| \leq c \begin{cases} \eta_{q2}(n)L_{qn} + L_{3n}^2 & \textit{if} \quad \beta_q < \infty,\ 3 \leq q < 4 \\ L_{4n} & \textit{if} \quad \beta_q < \infty,\ q \geq 4. \end{cases} \tag{2.20}$$

Corollary 2.4 *Under the assumptions of Theorem 2.3 the following expansion holds*

$$F_n(x) = w(x) - \frac{1}{3} a_n U_2(x/2) p_w(x) + \rho_{n1}(x), \quad x \in \mathbb{R}, \tag{2.21}$$

where the remainder term $\rho_{n1}(x)$ admits the bound, for $x \in \mathbb{R},\ n \in \mathbb{N}$,

$$|\rho_{n1}(x)| \le c \begin{cases} \eta_{q2}(n) L_{qn} + L_{3n}^2 + |a_n|^{3/2} & \text{if} \quad \beta_q < \infty,\ 3 \le q < 4 \\ L_{4n} + |a_n|^{3/2} & \text{if} \quad \beta_q < \infty,\ q \ge 4. \end{cases} \tag{2.22}$$

Note that in the case $\beta_3 < \infty$ the estimate (2.22) yields the bound

$$|\rho_{n1}(x)| \le c\Big(\eta_{q2}(n) + L_{3n} + |a_n|^{1/2}\Big) L_{3n}, \tag{2.23}$$

where $\eta_{q2}(n) \to 0$ as $n \to \infty$, and we obtain an analogue of Edgeworth expansion.

Since $\eta_{q2}(n) \le 101,\ 3 \le q \le 4, n \in \mathbb{N}$, the results (2.21) and (2.22) again yield the free Berry–Esseen inequality (2.19) as well.

In addition we obtain from Theorem 2.3 the following bounds.

Corollary 2.5 *Under the assumptions of Theorem 2.3*

$$\sup_{x\in\mathbb{R}} |F_n(x) - \mu_{a_n,0,0}((-\infty, x))| \le c\, L_{qn} \quad \text{for} \quad n \in \mathbb{N} \quad \text{if} \quad \beta_q < \infty,\ 3 \le q \le 4. \tag{2.24}$$

Before formulating the next result, denote by $\varsigma_n,\ n > m_4$, a signed measure with the density

$$p_{\varsigma_n}(x) := (e_n^2(x - a_n)^2 - 1) p_w(e_n(x - a_n)), \quad x \in \mathbb{R}, \tag{2.25}$$

where $e_n := (1 - b_n)/\sqrt{1 - d_n}$. Denote by $\kappa_n,\ n > m_4$, the *signed* measure $\kappa_n := \mu_{a_n,b_n,d_n} + \frac{1}{n}\varsigma_n$. It is easy to see from results of Sect. 4 that κ_n is a *probability* measure for $n \ge m_4/c$ with some sufficiently small c.

Theorem 2.6 *Assume that $X_j,\ j = 1, \dots,$ are free random variables, that $\beta_q < \infty$ with some $q \ge 4$ and that $m_1 = 0,\ m_2 = 1$. Then, for $n > m_4$,*

$$\sup_{x\in\mathbb{R}} |F_n(x) - \kappa_n((-\infty, x))| \le c \begin{cases} \eta_{q3}(n) L_{qn} + L_{4n}^{3/2} & \text{if} \quad \beta_q < \infty,\ 4 \le q < 5 \\ L_{5n} & \text{if} \quad \beta_q < \infty,\ q \ge 5. \end{cases} \tag{2.26}$$

Corollary 2.7 *Assume that the assumptions of Theorem 2.6 are satisfied. Then*

$$F_n(x+a_n) = w(x) + \Big(-\frac{a_n^2}{2} U_1(x/2) + \frac{a_n}{3}(3 - U_2(x/2)) - \frac{b_n - a_n^2 - 1/n}{4} U_3(x/2)\Big) p_w(x) + \rho_{n2}(x), \tag{2.27}$$

for all real x, where

$$|\rho_{n2}(x)| \le c \begin{cases} \eta_{q3}(n)L_{qn} + L_{4n}^{3/2} & \text{if} \quad \beta_q < \infty,\ 4 \le q < 5 \\ L_{5n} & \text{if} \quad \beta_q < \infty,\ q \ge 5 \end{cases} \quad \text{for} \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{2.28}$$

If $m_3 = 0$ *this formula has the following simple form*

$$F_n(x) = w(x) - \frac{m_4 - 2}{4n} U_3(x/2) p_w(x) + \rho_{n3}(x), \tag{2.29}$$

where $\rho_{n3}(x)$ *admits the bound* (2.28).

If $m_3 \neq 0$, we obtain from (2.27) the following expansion for $F_n(x)$:

$$\begin{aligned} F_n(x) = w(x) &- \frac{1}{3} a_n U_2(x/2) p_w(x) \\ &+ \Big(\frac{a_n^2}{6} U_1(x/2) - \frac{b_n - a_n^2 - 1/n}{4} U_3(x/2) \Big) p_w(x) \\ &+ Q_1(x, a_n) + Q_2(x, a_n, b_n, 1/n) + \rho_{n4}(x), \quad x \in \mathbb{R}, \end{aligned}$$

where

$$\begin{aligned} Q_1(x, a_n) &= w(x - a_n) - w(x) + a_n p_w(x) \\ &\quad + \frac{a_n}{3}(3 - U_2(x/2))(p_w(x - a_n) - p_w(x)), \\ Q_2(x, a_n, b_n, 1/n) &= \Big(\frac{a_n^2}{6} U_1(x/2) - \frac{b_n - a_n^2 - 1/n}{4} U_3(x/2) \Big) \\ &\quad \times (p_w(x - a_n) - p_w(x)) \end{aligned}$$

and the function $\rho_{n4}(x)$ admits the bound (2.28). The function $Q_1(x, a_n)$ is a function of bounded variation and it is not difficult to verify that

$$\begin{aligned} \frac{1}{c}|a_n|^{3/2} &\le \sup_{x \in \mathbb{R}} |Q_1(x, a_n)| \le c|a_n|^{3/2} \quad \text{and} \\ \frac{1}{c}|a_n|^{3/2} &\le ||Q_1(x, a_n)||_{TV} \le c|a_n|^{3/2}, \end{aligned} \tag{2.30}$$

with some $c \ge 1$. This means that $Q_1(x, a_n)$ is actually of order $n^{-3/4}$. We shall see that $Q_1(x, a_n)$ can not be cast by Taylor expansion around x into an expansion in powers of $n^{-1/2}$ like (2.27) in terms of $p_w(x)$ and the Chebyshev polynomials which is continuous up to the boundary ± 2 with finite total variation. Moreover $|Q_2(x, a_n, b_n, 1/n)| \le c L_{4n} \sqrt{|a_n|}$, $x \in \mathbb{R}$.

Indeed from the formal expansion of μ_n in (2.8) and (2.11) it follows that the first two summands on the right-hand side of (2.8) are Cauchy transforms of the finite signed measure on the right-hand side of (2.21). Moreover, in the case $m_3 = 0$ the first three summands on the right-hand side of (2.8) are Cauchy transforms of the finite signed measure on the right-hand side of (2.29). But in the case $m_3 \neq 0$ the third summand on the right-hand side of (2.8) can not be a Cauchy transform of a signed measure ζ which is finite on every bounded interval and $\int_{\mathbb{R}} |\zeta(du)|/(1+|u|) < \infty$. This will be proved in Sect. 4. Therefore, taking into account the formal expansion (2.8), we can not expect an expansion of type (2.27) for the function $F_n(x)$ without shift.

Remark 2.8 The methods used in the proof of Theorems 2.3 and 2.6 still do not yield a free analogue of Edgeworth asymptotic expansions under the assumption $\beta_q < \infty, q > 5$, with a remainder term of order $O\big(n^{-3/2-\gamma}\big)$ with $\gamma > 0$. This problem remains open.

Remark 2.9 It is known, see for example [10,21], that there is semigroup $\mu_t \in \mathcal{M},\ t \geq 1$, such that $\phi_{\mu_t}(z) = t\phi_{\mu_1}(z)$, where $\phi_{\mu_t}(z)$ are Voiculesku transforms of the probability measures μ_t. For the definition of Voiculesku's transform, see in Sect. 3. As before let $m_1 = 0$ and $m_2 = 1$. Define a probability measure $\hat{\mu}_t$ in the following way: $\hat{\mu}_t((-\infty, x)) = \mu((-\infty, x\sqrt{t})),\ x \in \mathbb{R}$. Theorems 2.1, 2.3, 2.6 and their Corollaries remain valid for $\hat{\mu}_t$ if the integers n are replaced by $t \geq 1$. One can prove these results exactly by the same proof.

Remark 2.10 Recall that, if the random variable X has density f, then the classical entropy of a distribution of X is defined as $h(X) = -\int_{\mathbb{R}} f(x) \log f(x)\, dx$, provided the positive part of the integral is finite. Thus we have $h(X) \in [-\infty, \infty)$.

A much stronger statement than the classical CLT—the entropic central limit theorem—indicates that, if for some n_0, or equivalently, for all $n \geq n_0$, Y_n from (2.1) have absolutely continuous distributions with finite entropies $h(Y_n)$, then there is convergence of the entropies, $h(Y_n) \to h(Y)$, as $n \to \infty$, where Y is a standard Gaussian random variable. This theorem is due to Barron [4]. Recently Bobkov et al. [16] found the rate of convergence in the classical entropic CLT.

Let ν be a probability measure on $\mathbb{R}$. The quantity

$$\chi(\nu) = \int\int_{\mathbb{R}\times\mathbb{R}} \log|x-y|\, \nu(dx)\nu(dy) + \frac{3}{4} + \frac{1}{2}\log 2\pi,$$

called free entropy of ν, was introduced by Voiculescu in [45]. Free entropy χ behaves like the classical entropy h. In particular, the free entropy is maximized by the standard semicircle measure μ_w with the value $\chi(\mu_w) = \frac{1}{2}\log 2\pi e$ among all probability measures with variance one, see [27,46]. Wang [48] has proved the free analogue of Barron's result.

It was proved in [5] that if the distribution μ of X_1 is not a Dirac measure, then $F_n(x)$ is Lebesgue absolutely continuous when $n \geq c_1(\mu)$ is sufficiently large. Denote by $p_n(x)$ the density of $F_n(x)$. Our method allows to obtain an expansion for the density $p_n(x)$ and yields an expansion in the entropic free CLT, see [23], Theorem 2.10 and Corollaries 2.11–2.13, and [24], Theorem 2.1 and Corollaries 2.2–2.4. These results will be published elsewhere.

3 Auxiliary results

We need results about some classes of analytic functions (see [1], Section 3, and [2], Section 6, §59).

The class $\mathcal{N}$ (Nevanlinna, R.) is the class of analytic functions $f(z) : \mathbb{C}^+ \to \{z : \Im z \geq 0\}$. For such functions there is the integral representation

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1+uz}{u-z}\,\tau(du) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{u-z} - \frac{u}{1+u^2}\Big)(1+u^2)\,\tau(du), \quad z \in \mathbb{C}^+, \tag{3.1}$$

where $b \geq 0$, $a \in \mathbb{R}$, and τ is a nonnegative finite measure. Moreover, $a = \Re f(i)$ and $\tau(\mathbb{R}) = \Im f(i) - b$. From this formula it follows that

$$f(z) = (b + o(1))z \tag{3.2}$$

for $z \in \mathbb{C}^+$ such that $|\Re z|/\Im z$ stays bounded as $|z|$ tends to infinity (in other words $z \to \infty$ nontangentially to $\mathbb{R}$). Hence if $b \neq 0$, then f has a right inverse $f^{(-1)}$ defined on the region

$$\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ : |\Re z| < \alpha \Im z,\ \Im z > \beta\}$$

for any $\alpha > 0$ and some positive $\beta = \beta(f, \alpha)$.

A function $f \in \mathcal{N}$ admits the representation

$$f(z) = \int_{\mathbb{R}} \frac{\sigma(du)}{u-z}, \quad z \in \mathbb{C}^+, \tag{3.3}$$

where σ is a finite nonnegative measure, if and only if $\sup_{y \geq 1} |yf(iy)| < \infty$.

For $\mu \in \mathcal{M}$, consider its Cauchy transform $G_\mu(z)$ (see (2.7)). The measure μ can be recovered from $G_\mu(z)$ as the weak limit of the measures

$$\mu_y(dx) = -\frac{1}{\pi}\Im G_\mu(x+iy)\,dx, \quad x \in \mathbb{R},\ y > 0,$$

as $y \downarrow 0$. If the function $\Im G_\mu(z)$ is continuous at $x \in \mathbb{R}$, then the probability distribution function $D_\mu(t) = \mu((-\infty, t))$ is differentiable at x and its derivative is given by

$$D'_\mu(x) = -\Im G_\mu(x)/\pi. \tag{3.4}$$

This inversion formula allows to extract the density function of the measure μ from its Cauchy transform.

Following Maassen [32] and Bercovici and Voiculescu [9], we shall consider in the following the *reciprocal Cauchy transform*

$$F_\mu(z) = \frac{1}{G_\mu(z)}. \tag{3.5}$$

The corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$ will be denoted by $\mathcal{F}$. This class coincides with the subclass of Nevanlinna functions f for which $f(z)/z \to 1$ as $z \to \infty$ nontangentially to $\mathbb{R}$. Indeed, reciprocal Cauchy transforms of probability measures have obviously such property. Let $f \in \mathcal{N}$ and $f(z)/z \to 1$ as $z \to \infty$ nontangentially to $\mathbb{R}$. Then, by (3.2), f admits the representation (3.1) with $b = 1$. By (3.2) and (3.3), $-1/f(z)$ admits the representation (3.3) with $\sigma \in \mathcal{M}$.

The functions f of the class $\mathcal{F}$ satisfy the inequality

$$\Im f(z) \geq \Im z, \qquad z \in \mathbb{C}^+. \tag{3.6}$$

The function $\phi_\mu(z) = F_\mu^{(-1)}(z) - z$ is called the Voiculescu transform of μ and $\phi_\mu(z)$ is an analytic function on $\Gamma_{\alpha,\beta}$ with the property $\Im\phi_\mu(z) \leq 0$ for $z \in \Gamma_{\alpha,\beta}$, where $\phi_\mu(z)$ is defined. On the domain $\Gamma_{\alpha,\beta}$, where the functions $\phi_{\mu_1}(z)$, $\phi_{\mu_2}(z)$, and $\phi_{\mu_1 \boxplus \mu_2}(z)$ are defined, we have

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z). \tag{3.7}$$

This relation for the distribution $\mu_1 \boxplus \mu_2$ of $X + Y$, where X and Y are free random variables, is due to Voiculescu [42] for the case of compactly supported measures. The result was extended by Maassen [32] to measures with finite variance; the general case was proved by Bercovici and Voiculescu [9].

Assume that $\beta_k < \infty$ for some $k \in \mathbb{N}$. Then

$$G_\mu(z) = \frac{1}{z} + \frac{m_1}{z^2} + \cdots + \frac{m_k}{z^{k+1}} + o\Big(\frac{1}{z^{k+1}}\Big), \quad z \to \infty, \ z \in \Gamma_{\alpha,1}.$$

It follows from this relation (see for example [29]) that

$$\phi_\mu(z) = \alpha_1 + \frac{\alpha_2}{z} + \cdots + \frac{\alpha_k}{z^{k-1}} + o\Big(\frac{1}{z^{k-1}}\Big), \quad z \to \infty, \ z \in \Gamma_{\alpha,1}. \tag{3.8}$$

We call the coefficients α_m, $m = 1, \dots, k$, the free cumulants of the probability measure μ. It is easy to see that $\alpha_1 = m_1$, $\alpha_2 = m_2 - m_1^2$, $\alpha_3(\mu) = m_3 - 3m_1m_2 + 2m_1^3$. In the case $m_1 = 0$ and $m_2 = 1$ we have $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = m_3$ and $\alpha_4 = m_4 - 2$.

If $\mu \in \mathcal{M}$ has moments of any order, that is $\beta_k < \infty$ for any $k \in \mathbb{N}$, then there exist cumulants α_m, $m = 1, \dots,$ and we can consider the formal power series

$$\phi_\mu(z) = \sum_{m=1}^{\infty} \frac{\alpha_m}{z^{m-1}}. \tag{3.9}$$

In addition $\phi_\mu(z)$ satisfies (3.8) for any fixed $k \in \mathbb{N}$. If μ has a bounded support, $\phi_\mu(z)$ is an analytic function on the domain $|z| > R$ with some $R > 0$ and the series (3.9) converges absolutely and uniformly for such z.

Voiculescu [45] showed for compactly supported probability measures that there exist unique functions $Z_1, Z_2 \in \mathcal{F}$ such that $G_{\mu_1 \boxplus \mu_2}(z) = G_{\mu_1}(Z_1(z)) = G_{\mu_2}(Z_2(z))$ for all $z \in \mathbb{C}^+$. Using Speicher's combinatorial approach [40] to freeness, Biane [15] proved this result in the general case.

Chistyakov and Götze [21], Bercovici and Belinschi [6], Belinschi [7], proved, using complex analytic methods, that there exist unique functions $Z_1(z)$ and $Z_2(z)$ in the class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)). \tag{3.10}$$

The function $F_{\mu_1}(Z_1(z))$ belongs again to the class $\mathcal{F}$ and there exists $\mu \in \mathcal{M}$ such that $F_{\mu_1}(Z_1(z)) = F_\mu(z)$, where $F_\mu(z) = 1/G_\mu(z)$ and $G_\mu(z)$ is the Cauchy transform as in (2.7). We can define the additive free convolution in the following way $\mu_1 \boxplus \mu_2 := \mu$. The measure μ depends on μ_1 and μ_2 only. The relation (3.7) follows immediately from (3.10) and we see that this definition coincides with the Voiculescu, Bercovici, Maassen definition. Hence we have the equivalence of a "characteristic function" approach and a probabilistic approach to the definition of the additive free convolution.

Specializing to $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$ write $\mu_1 \boxplus \cdots \boxplus \mu_n = \mu^{n\boxplus}$. The relation (3.10) admits the following consequence (see for example [21]).

Proposition 3.1 *Let $\mu \in \mathcal{M}$. There exists a unique function $Z \in \mathcal{F}$ such that*

$$z = nZ(z) - (n-1)F_\mu(Z(z)), \quad z \in \mathbb{C}^+, \tag{3.11}$$

and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z(z))$.

The next lemma was proved in [21].

Lemma 3.2 *Let $g : \mathbb{C}^+ \to \mathbb{C}^-$ be analytic with*

$$\liminf_{y \to +\infty} \frac{|g(iy)|}{y} = 0. \tag{3.12}$$

Then the function $f : \mathbb{C}^+ \to \mathbb{C}$ defined via $z \mapsto z + g(z)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is in the class $\mathcal{F}$.

This lemma generalizes a result of Maassen [32] (see Lemma 2.3). Maassen proved Lemma 3.2 under the additional restriction $|g(z)| \le c(g)/\Im z$ for $z \in \mathbb{C}^+$, where $c(g)$ is a constant depending on g.

Using the representation (3.1) for $F_\mu(z)$ we obtain

$$F_\mu(z) = z + \Re F_\mu(i) + \int_{\mathbb{R}} \frac{(1+uz)\,\tau(du)}{u-z}, \quad z \in \mathbb{C}^+, \tag{3.13}$$

where τ is a nonnegative measure such that $\tau(\mathbb{R}) = \Im F_\mu(i) - 1$. Denote $z = x + iy$, where $x, y \in \mathbb{R}$. We see that, for $\Im z > 0$,

$$\Im\Big(nz - (n-1)F_\mu(z)\Big) = y\Big(1 - (n-1)I_\mu(x,y)\Big),$$
$$\text{where} \quad I_\mu(x,y) := \int\limits_{\mathbb{R}} \frac{(1+u^2)\,\tau(du)}{(u-x)^2 + y^2}.$$

For every real fixed x, consider the equation

$$y\Big(1 - (n-1)I_\mu(x,y)\Big) = 0, \quad y > 0. \tag{3.14}$$

Since $y \mapsto I_\mu(x,y),\ y > 0$, is positive and monotone, and decreases to 0 as $y \to \infty$, it is clear that the Eq. (3.14) has at most one positive solution. If such a solution exists, denote it by $y_n(x)$. Note that (3.14) does not have a solution $y > 0$ for any given $x \in \mathbb{R}$ if and only if $I_\mu(x,0) \le 1/(n-1)$. Consider the set $S := \{x \in \mathbb{R} : I_\mu(x,0) \le 1/(n-1)\}$. We put $y_n(x) = 0$ for $x \in S$. By Fatou's lemma, $I_\mu(x_0,0) \le \liminf_{x \to x_0} I_\mu(x,0)$ for any given $x_0 \in \mathbb{R}$, hence the set S is closed. Therefore $\mathbb{R} \setminus S$ is the union of finitely or countably many intervals $(x_k, x_{k+1}),\ x_k < x_{k+1}$. The function $y_n(x)$ is continuous on the interval (x_k, x_{k+1}). Since the set $\{z \in \mathbb{C}^+ : n\Im z - (n-1)\Im F_\mu(z) > 0\}$ is open, we see that $y_n(x) \to 0$ if $x \downarrow x_k$ and $x \uparrow x_{k+1}$. Hence the curve γ_n given by the equation $z = x + iy_n(x),\ x \in \mathbb{R}$, is continuous and simple.

Consider the open domain $D_n := \{z = x + iy,\ x, y \in \mathbb{R} : y > y_n(x)\}$.

Lemma 3.3 *Let $Z \in \mathcal{F}$ be the solution of the equation (3.11). The function $Z(z)$ maps $\mathbb{C}^+$ conformally onto D_n. Moreover the function $Z(z),\ z \in \mathbb{C}^+$, is continuous up to the real axis and it establishes a homeomorphism between the real axis and the curve γ_n.*

Proof We obtain from (3.11) the formula

$$Z^{(-1)}(z) = nz - (n-1)F_\mu(z) \tag{3.15}$$

for $z \in \Gamma_{\alpha,\beta}$ with some $\alpha, \beta > 0$. By this formula we may continue the function $Z^{(-1)}(z)$ as an analytic function to $\mathbb{C}^+$. Using the representation (3.13) for the function $F_\mu(z)$, we note that $Z^{(-1)}(z) = z + g(z),\ z \in \mathbb{C}^+$, where $g(z)$ is analytic on $\mathbb{C}^+$ and satisfies the assumptions of Lemma 3.2. By Lemma 3.2, we conclude that the function $Z^{(-1)}(z)$ takes every value in $\mathbb{C}^+$ precisely once. Moreover, as it is easy to see, $Z^{(-1)}(D_n) = \mathbb{C}^+$ and $\Re Z^{(-1)}(x + iy_n(x)) \to \pm\infty$ as $x \to \pm\infty$. The inverse $Z(z)$ gives us the conformal mapping of $\mathbb{C}^+$ onto D_n. By the well-known results of the theory of analytic functions on boundary behavior of conformal mapping (see [34], Section 2, no. 8, pp. 66–77), $Z(z)$ is continuous up to the real axis and it establishes a homeomorphism between the real axis and the curve γ_n.

Lemma 3.4 *Let μ be a probability measure such that $m_1 = 0, m_2 = 1$. Assume that $\rho_2(\mu, \sqrt{(n-1)/8}) \le 1/10$ for some positive integer $n \ge 10^3$. Then the following inequality holds*

$$|Z(z)| \ge \sqrt{(n-1)/8}, \qquad z \in \mathbb{C}^+, \tag{3.16}$$

where $Z \in \mathcal{F}$ is the solution of the Eq. (3.11).

Proof It is enough to prove that if $|\Re Z(z)| < \sqrt{(n-1)/8}$ then it follows that $|\Im Z(z)| \ge \sqrt{(n-1)/8}$. Since the values of $Z(z)$ lay in the domain D_n, it is enough to prove the corresponding statement for the solutions of (3.14).

Write

$$F_\mu(z) = \Big(\frac{1}{z} + \frac{r(z)}{z^2}\Big)^{-1}, \quad z \in \mathbb{C}^+ \quad \text{where} \quad r(z) := \int_{\mathbb{R}} \frac{u^2\,\mu(du)}{z-u}. \tag{3.17}$$

It is obvious that $|r(z)| \le 1/y,\ z \in \mathbb{C}^+$. Rewrite (3.14) in the form

$$y\Big(1 + (n-1)\Big(1 - \frac{1}{y}\Im F_\mu(z)\Big)\Big) = 0.$$

Let us show that

$$y_n(x) > \sqrt{(n-1)/8} \quad \text{for} \quad |x| \le \sqrt{(n-1)/8}. \tag{3.18}$$

In order to prove this inequality we shall establish that

$$(n-1)\Big(1 - \frac{1}{y}\Im F_\mu(z)\Big) < -1 \tag{3.19}$$

for $|z| = \frac{1}{2}\sqrt{(n-1)}$ and $|x| \le \sqrt{(n-1)/8}$. Indeed, since the function

$$-I_\mu(x,y) = (n-1)\Big(1 - \frac{1}{y}\Im F_\mu(z)\Big), \quad y > 0,$$

is negative and monotone, and increases to 0 as $y \to \infty$, (3.18) follows from (3.19).

We have, for the z considered above, $F_\mu(z) = z - r(z) + r_1(z)$, where $r_1(z)$ admits the upper bound $|r_1(z)| \le 2(|z|y^2)^{-1} \le 32/(n-1)^{3/2}$.

Using the previous formula, we easily obtain the relation, for the same z,

$$-I_\mu(x,y) = (n-1)\frac{\Im r(z)}{y} + r_2(z) = -(n-1)\int_{\mathbb{R}} \frac{u^2\,\mu(du)}{(u-x)^2 + y^2} + r_2(z), \tag{3.20}$$

where $r_2(z)$ admits the upper bound $|r_2(z)| \le 32\sqrt{8}/(n-1) < 1/6$. Hence we have, for the same z,

$$\begin{aligned}
-I_\mu(x,y) &\le -(n-1)\int\limits_{[-y,y]} \frac{u^2\,\mu(du)}{(u-x)^2+y^2} + r_2(z) \\
&\le -\frac{n-1}{3|z|^2}(1-\rho_2(\mu,y)) + r_2(z) \\
&\le -\frac{n-1}{3|z|^2}(1-\rho_2(\mu,\sqrt{(n-1)/8})) + r_2(z) \\
&< -\frac{6}{5}+\frac{1}{6} < -1
\end{aligned}$$

and (3.19) is proved.

The assertion of the lemma follows immediately from (3.18). □

Denote by $\Delta(\kappa',\kappa'')$ the Kolmogorov distance between the finite signed measures κ' and κ'' such that $\kappa'((-\infty,x)) \to 0$ and $\kappa''((-\infty,x)) \to 0$ as $x \to -\infty$, i.e.,

$$\Delta(\kappa',\kappa'') := \sup_{x\in\mathbb{R}} |\kappa'((-\infty,x)) - \kappa''((-\infty,x))|.$$

We need the following result of Bercovici–Voiculescu [9].

Proposition 3.5 *If μ, μ', ν and ν' are probability measures, then*

$$\Delta(\mu \boxplus \nu, \mu' \boxplus \nu') \le \Delta(\mu,\mu') + \Delta(\nu,\nu').$$

In addition the following proposition holds (see [38, p. 139]).

Proposition 3.6 *If $3 \le m \le k$, then the Lyapunov fractions L_{mn} and L_{kn} satisfy the inequality: $L_{mn}^{1/(m-2)} \le L_{kn}^{1/(k-2)}$.*

4 Properties of free Meixner measures

Saitoh and Yoshida [39] have proved that the absolutely continuous part of the free Meixner measure $\mu_{a,b,d}$, $a \in \mathbb{R}$, $b < 1$, $d < 1$, is given by

$$\frac{\sqrt{4(1-d)-(1-b)^2(x-a)^2}}{2\pi f(x)}, \tag{4.1}$$

when $a - 2\sqrt{1-d}/(1-b) \le x \le a + 2\sqrt{1-d}/(1-b)$, where

$$f(x) := bx^2 + a(1-b)x + 1 - d;$$

the measure may have a discrete part μ_D in the following cases:

1. if $f(x)$ has two real roots $y_1 \neq y_2$, then

$$\mu_D := \lambda_1 \delta_{y_1} + \lambda_2 \delta_{y_2}, \tag{4.2}$$

where

$$\lambda_j := \frac{1}{\sqrt{a^2(1-b)^2 - 4b(1-d)}} \Big(\frac{1-d}{|y_j|} - |y_j| \Big)_+, \quad j = 1, 2, \tag{4.3}$$

2. if $b = 0$ and $a \neq 0$, then

$$\mu_D := \Big(1 - \frac{1-d}{a^2}\Big)_+ \delta_y, \quad \text{where} \quad y := -\frac{1-d}{a}. \tag{4.4}$$

Recall that δ_y with $y \in \mathbb{R}$ is a Dirac measure concentrated at the point y.

Saitoh and Yoshida proved as well that for $0 \le b < 1$ the (centered) free Meixner measure $\mu_{a,b,d}$ is $\boxplus$-infinitely divisible. Note (see Bożejko and Bryc [17]) that $\mu_{a,b,d} = \mu_w$ if $a = b = d = 0$; $\mu_{a,b,d}$ is the free Poisson type measure, which is also known as Marchenko–Pastur measure [33], if $b = d = 0$ and $a \neq 0$, and $\mu_{a,0,d}$ with $a \neq 0$, $d \neq 0$ is the shifted free Poisson type measure ; $\mu_{a,b,d}$ is the free Pascal (negative binomial) type measure if $b > 0$ and $a^2(1-b)^2 > 4b(1-d)$; $\mu_{a,b,d}$ is the free gamma type measure if $b > 0$ and $a^2(1-b)^2 = 4b(1-d)$; $\mu_{a,b,d}$ is the pure free Meixner type measure if $b > 0$ and $a^2(1-b)^2 < 4b(1-d)$.

Now assume that $m_4 < \infty, m_1 = 0, m_2 = 1$ and $n \ge 3m_4$. By the well-known moment inequality

$$\begin{vmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{vmatrix} \ge 0$$

(see [1]), we conclude that $m_4 - 1 - m_3^2 \ge 0$. Therefore the lower bounds $b_n \ge 0$ and $d_n > 0$ hold. In addition we have $|a_n| \le 1/\sqrt{3}$, $b_n \le 1/3$ and $d_n \le 1/3$. Consider the measures μ_{a_n,b_n,d_n}. These measures may be the free Pascal, the free gamma and the pure free Meixner type measures.

Let $b_n > 0$. Note that the polynomial $f_n(x) = b_n x^2 + a_n(1-b_n)x + 1 - d_n$ has two real roots y_{1n} and y_{2n} in the case $a_n^2(1-b_n)^2 - 4b_n(1-d_n) > 0$ and these roots have the same sign. By the relation

$$\frac{1}{|y_1|} + \frac{1}{|y_2|} = \frac{|a_n|(1-b_n)}{1-d_n} \le \frac{\sqrt{3}}{2} < 1,$$

one can deduce the inequalities $|y_{jn}| \ge 1,\ j = 1, 2$. Using (4.2) and (4.3) we see that the discrete part of μ_{a_n,b_n,d_n} is equal to zero. Let $b_n = 0$ and $a_n \neq 0$. We see from (4.4) that in this case the discrete part of μ_{a_n,b_n,d_n} is equal to zero as well.

Thus, in the considered case it follows from Saitoh and Yoshida's results that the probability measure μ_{a_n,b_n,d_n} is $\boxplus$-infinitely divisible and it is absolutely continuous with a density of the form (4.1) where $a = a_n,\ b = b_n,\ d = d_n$.

Assume that $\beta_3 < \infty, m_1 = 0, m_2 = 1$ and $n \geq m_3^2$, i.e., $|a_n| \leq 1$. In this case the probability measure $\mu_{a_n,0,0}$ is absolute continuous with a density of the form (4.1) where $a = a_n, b = 0, d = 0$. In addition, by Saitoh and Yoshida's results, $\mu_{a_n,0,0}$ is $\boxplus$-infinitely divisible.

5 Formal asymptotic expansion in the free CLT

In this section we deduce formula (2.8).

By Proposition 3.1, there exists $Z(z) \in \mathcal{F}$ such that (3.11) holds, and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z(z))$. Hence $F_{\mu_n}(z) = F_\mu(\sqrt{n}S_n(z))/\sqrt{n}$, $z \in \mathbb{C}^+$, where $S_n(z) := Z(\sqrt{n}z)/\sqrt{n}$. Using the Voiculescu transform $\phi_\mu(z)$ (see (3.7), this relation implies that

$$S_n(z) = F_{\mu_n}(z) + \phi_\mu(\sqrt{n}F_{\mu_n}(z))/\sqrt{n}$$

for $z \in \Gamma_{\alpha,\beta}$ with some $\alpha, \beta > 0$. On the other hand we conclude from (3.11) that

$$S_n(z) = \frac{z}{n} + \frac{n-1}{n}F_{\mu_n}(z), \qquad z \in \mathbb{C}^+.$$

The last two equations give us

$$F_{\mu_n}(z) + \sqrt{n}\phi_\mu(\sqrt{n}F_{\mu_n}(z)) = z, \qquad z \in \Gamma_{\alpha,\beta}. \tag{5.1}$$

Consider the function $f(z) := z + \sqrt{n}\phi_\mu(\sqrt{n}z)$, $z \in \Gamma_{\alpha,\beta'}$ with some $\beta' \geq \beta$, and define the function

$$g(z) := \frac{1}{2}\Big(f(z) + \sqrt{f^2(z) - 4}\Big), \quad z \in \Gamma_{\alpha,\beta'}, \tag{5.2}$$

where we choose the branch of the square root by the condition $\Im g(z) > 0$ for $z \in \Gamma_{\alpha,\beta'}$. It is easy to see that $g(z) = z(1 + o(1))$ as $z \to \infty$ nontangentially to $\mathbb{R}$. In addition, by (5.1), $g(z)$ satisfies the relation

$$g(F_{\mu_n}(z)) + \frac{1}{g(F_{\mu_n}(z))} = f(F_{\mu_n}(z)) = z, \qquad z \in \Gamma_{\alpha,\beta'}. \tag{5.3}$$

We deduce from (5.3) that

$$g(F_{\mu_n}(z)) = \frac{1}{2}\Big(z + \sqrt{z^2 - 4}\Big) = F_{\mu_w}(z), \qquad z \in \Gamma_{\alpha,\beta'}.$$

Recall that we denote by μ_w the semicircle measure.

Since the function $g(z)$ has a right inverse $g^{(-1)}(z)$ in $\Gamma_{\alpha,\beta''}$ with some $\beta'' \geq \beta'$, we have

$$F_{\mu_n}(z) = g^{(-1)}(F_{\mu_w}(z)), \qquad z \in \Gamma_{\alpha,\beta'''}, \quad \text{where} \quad \beta''' \geq \beta''. \tag{5.4}$$

Let $\mu \in \mathcal{M}$ such that all moments of μ exist. In addition let $m_1 = 0$ and $m_2 = 1$. Consider the formal power series in z

$$\sqrt{n}\phi_\mu(\sqrt{n}z) := \sum_{k=1}^{\infty} \frac{\alpha_{k+1}}{n^{(k-1)/2} z^k}, \tag{5.5}$$

where $\alpha_k,\ k = 1, 2, \ldots,$ are free cumulants of the measure μ and the formal power series of g:

$$g(z) = z + \sum_{k=0}^{\infty} \frac{a_k}{z^k}. \tag{5.6}$$

In our case $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = m_3$ and $\alpha_4 = m_4 - 2$. Using (5.3) and (5.5), (5.6) we obtain the following relation for the considered formal power series

$$z + \sum_{k=0}^{\infty} \frac{a_k}{z^k} + \frac{1}{z}\left(1 - \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}} + \left(\sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}}\right)^2 - \ldots\right) = z + \sum_{k=1}^{\infty} \frac{\alpha_{k+1}}{n^{(k-1)/2} z^k}.$$

It follows from this relation that $a_0 = 0$, $a_1 = 0$ and

$$a_k - a_{k-2} + \sum_{s=0}^{k-3} a_s a_{k-s-3} - \cdots + (-1)^{k-1} a_0^{k-1} = \frac{\alpha_{k+1}}{n^{(k-1)/2}}, \quad k = 2, 3, \ldots. \tag{5.7}$$

We have from (5.7) the relations $a_2 = \alpha_3/\sqrt{n}$, $a_3 = \alpha_4/n$. In addition we obtain from (5.7) by induction that

$$a_{2s} = \frac{\alpha_3}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right), \quad \text{and} \quad a_{2s+1} = \frac{\alpha_4}{n} - \frac{(s-1)(s-2)}{2} \frac{\alpha_3^2}{n} + O\left(\frac{1}{n^{3/2}}\right) \tag{5.8}$$

as $n \to \infty$ for $s = 2, \ldots$.

Now consider the formal power series for the right inverse $g^{(-1)}(z)$

$$g^{(-1)}(z) = z + \sum_{k=0}^{\infty} \frac{b_k}{z^k}.$$

Rewrite the relation $g(g^{(-1)}(z)) = z$ in the form

$$z + \sum_{m=0}^{\infty} \frac{b_m}{z^m} + \sum_{k=2}^{\infty} \frac{a_k}{z^k \left(1 + \sum_{m=0}^{\infty} \frac{b_m}{z^{m+1}}\right)^k} = z.$$

Using the formula

$$\frac{1}{(1+w)^k}=\sum_{s=0}^{\infty}(-1)^s\binom{k-1+s}{k-1}w^s,$$

we finally get

$$\sum_{m=0}^{\infty}\frac{b_m}{z^m}+\sum_{k=2}^{\infty}\frac{a_k}{z^k}\sum_{s=0}^{\infty}(-1)^s\binom{k-1+s}{k-1}\Big(\sum_{m=0}^{\infty}\frac{b_m}{z^{m+1}}\Big)^s=0.$$

We obtain from this equality that $b_0=b_1=0$, $b_2=-a_2$ and

$$\begin{aligned} b_m+a_m+\sum_{k=2}^{m-1}a_k\sum_{s=1}^{m-k}(-1)^s\binom{k-1+s}{k-1} \\ \times\sum_{m_1+\dots+m_s=m-k-s}b_{m_1}\dots b_{m_s}=0,\quad m=3,\dots. \end{aligned}\tag{5.9}$$

Moreover it is easy to deduce from (5.8) and (5.9) that

$$b_{2m}=-a_{2m}+O\Big(\frac{1}{n^{3/2}}\Big)=-\frac{\alpha_3}{\sqrt{n}}+O\Big(\frac{1}{n^{3/2}}\Big)\tag{5.10}$$

and

$$\begin{aligned} b_{2m-1}&=-a_{2m-1}+2\sum_{s=1}^{m-2}sa_{2s}b_{2m-2-2s}+O\Big(\frac{1}{n^{3/2}}\Big)\\ &=-\frac{\alpha_4}{n}-\frac{(m-2)(m+1)}{2}\frac{\alpha_3^2}{n}+O\Big(\frac{1}{n^{3/2}}\Big),\quad m=2,\dots. \end{aligned}\tag{5.11}$$

for $m=2,\dots$. It remains to note that

$$\frac{1}{g^{(-1)}(z)}=\frac{1}{z+\sum_{k=0}^{\infty}\frac{b_k}{z^k}}=\frac{1}{z}\left(1-\sum_{k=0}^{\infty}\frac{b_k}{z^{k+1}}+\Big(\sum_{k=0}^{\infty}\frac{b_k}{z^{k+1}}\Big)^2-\dots\right)$$

and we can write the formal power series in $1/\sqrt{n}$

$$\frac{1}{g^{(-1)}(z)}=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{B_k(1/z)}{n^{k/2}}.$$

Taking into account the relations (5.10) and (5.11), we easily conclude that

$$B_1(1/z) = \alpha_3 \sum_{m=2}^{\infty} \frac{1}{z^{2m}} = \alpha_3 \frac{1}{z^3} \cdot \frac{1}{z - 1/z}$$

and

$$B_2(1/z) = \alpha_4 \sum_{m=2}^{\infty} \frac{1}{z^{2m+1}} + \alpha_3^2 \left(\sum_{m=2}^{\infty} \frac{(m-2)(m+1)}{2} \frac{1}{z^{2m+1}} + \frac{1}{z^3} \left(\sum_{m=1}^{\infty} \frac{1}{z^{2m}} \right)^2 \right).$$

Since

$$\sum_{m=2}^{\infty} \frac{m}{z^{2m+1}} = -\frac{1}{2} \left(\sum_{m=2}^{\infty} \frac{1}{z^{2m}} \right)' = -\frac{1}{2} \Big(\frac{1}{z^2(z^2-1)} \Big)' = \frac{1}{z^3(z^2-1)} + \frac{1}{z(z^2-1)^2},$$

$$\sum_{m=2}^{\infty} \frac{m^2}{z^{2m+1}} = -\frac{1}{2} \left(\sum_{m=2}^{\infty} \frac{m}{z^{2m}} \right)' = \frac{1}{z^3(z^2-1)} + \frac{1}{z(z^2-1)^2} + \frac{2z}{(z^2-1)^3},$$

we finally obtain

$$B_2(1/z) = \left(\alpha_4 - \alpha_3^2 \right) \frac{1}{z^4} \cdot \frac{1}{z - 1/z} + \alpha_3^2 \left(\frac{1}{z^5} \cdot \frac{1}{(z-1/z)^2} + \frac{1}{z^2} \cdot \frac{1}{(z-1/z)^3} \right).$$

In view of (5.7) and (5.9), we see as well that $B_k(z)$ are functions of the form

$$B_k(1/z) = \sum c_{p,m} \frac{1}{z^p} \frac{1}{(z-1/z)^m}$$

with real coefficients $c_{p,m}$ which depend on the free cumulants $\alpha_3, \dots, \alpha_{k+2}$ and do not depend on n. The summation is carried out over a finite set of non-negative integer pairs (p, m). The coefficients $c_{p,m}$ can be calculated explicitly in the way described above for the coefficients $c_{p,m}$ of the functions $B_1(1/z)$ and $B_2(1/z)$.

Hence we deduce from (5.4) the formal expansion

$$G_{\mu_n}(z) = G_{\mu_w}(z) + \sum_{k=1}^{\infty} \frac{B_k(G_{\mu_w}(z))}{n^{k/2}}. \tag{5.12}$$

Using integration by parts, it is not difficult to verify that

$$B_1(G_{\mu_w}(z)) = \frac{\alpha_3}{\sqrt{z^2-4}} G_{\mu_w}^3(z) = \frac{\alpha_3}{2\pi} \int_{-2}^{2} \frac{x(x^2-3)}{\sqrt{4-x^2}} \frac{dx}{z-x}$$
$$= -\alpha_3 \int_{-2}^{2} \frac{1}{z-x} \, d\Big(\frac{1}{3} U_2(x/2) p_w(x) \Big), \quad z \in \mathbb{C}^+. \tag{5.13}$$

On the other hand we see that if $\alpha_3 \neq 0$ then the function $B_2(G_{\mu_w}(z))$ is not the Cauchy transform of some signed measure. Indeed, it is easy to see, using direct calculations, that

$$B_2(G_{\mu_w}(z)) = \frac{\alpha_3^2}{(z^2-4)^{3/2}} + g(z), \quad z \in \mathbb{C}^+, \tag{5.14}$$

where the function $g(z)$ is analytic on $\mathbb{C}^+$ and there exists finite limit, for every $-\infty < t_1 < t_2 < +\infty$,

$$\lim_{y\downarrow 0} \int_{t_1}^{t_2} \Im g(x+iy)\,dx. \tag{5.15}$$

In addition we note that

$$\lim_{y\downarrow 0} \int_{3/2}^{2} \Im \frac{1}{((x+iy)^2-4)^{3/2}}\,dx = \infty. \tag{5.16}$$

Assume now that $B_2(G_{\mu_w}(z))$ is a Cauchy transform of a real-valued function $\omega(x)$ of bounded variation on every bounded interval and such that

$$\int_{-\infty}^{\infty} \frac{|d\omega(x)|}{1+|x|} < \infty.$$

Then, by Stieltjes–Perron's inversion formula [1], we have

$$\frac{\omega(t_2+0)-\omega(t_2-0)}{2} - \frac{\omega(t_1+0)-\omega(t_1-0)}{2} = -\lim_{y\downarrow 0} \frac{1}{\pi} \int_{t_1}^{t_2} \Im B_2(G_{\mu_w}(x+iy))\,dx. \tag{5.17}$$

Assuming in (5.17) $t_1 := 3/2$ and $t_2 := 2$, and taking into account (5.14)–(5.16), we arrive at a contradiction. If $\alpha_3(\mu) = 0$, then

$$B_2(G_{\mu_w}(z)) = \frac{\alpha_4}{\sqrt{z^2-4}} G_{\mu_w}^4(z) = \frac{\alpha_4}{2\pi} \int_{-2}^{2} \frac{x^4-4x^2+2}{\sqrt{4-x^2}} \frac{dx}{z-x}$$
$$= -\alpha_4 \int_{-2}^{2} \frac{1}{z-x}\, d\Big(\frac{1}{4} U_3(x/2) p_w(x)\Big), \quad z \in \mathbb{C}^+. \tag{5.18}$$

6 Edgeworth expansion in the free CLT (the case $\beta_q < \infty,\ q \geq 3$)

In this section we prove Theorem 2.3 and Corollary 2.4.

Proof of Theorem 2.3 Recall that we denote by μ_n the distribution of Y_n in (2.1) for the free random variables X_j. Our first step is to reduce the problem to the case of bounded free random variables.

6.1 Passage to measures with bounded supports

Let $n \in \mathcal{N}$. Let $\varepsilon_n \in (0, 10^{-1/2}]$ be a point at which infimum of the function $g_{qn2}(\varepsilon)$ from (2.16) is attained. This means that

$$\eta_{q2}(n) := \varepsilon_n^{4-q_2} + \frac{\rho_{q2}(\mu, \varepsilon_n\sqrt{n})}{\beta_{q2}}\varepsilon_n^{-q_2}.$$

Without loss of generality we assume that

$$\eta_{q2}(n)L_{q2n} + L_{3n} < c_1, \tag{6.1}$$

where $c_1 > 0$ is a sufficiently small absolute constant. By Lyapunov's inequality $\beta_3 \geq m_2^{3/2} = 1$, we obtain from (6.1) that n in this case has to be sufficiently large, i.e., $n \geq c_1^{-2}\beta_3^2 \geq c_1^{-2}$.

Consider free random variables $\tilde{X}, \tilde{X}_1, \tilde{X}_2, \dots$ with distribution $\tilde{\mu} = \mathcal{L}(\tilde{X})$ such that $\tilde{\mu}([-\varepsilon_n\sqrt{n}, \varepsilon_n\sqrt{n}]) = 1$ and $\tilde{\mu}(B) = \mu(B)$ for all Borel sets $B \subseteq [-\varepsilon_n\sqrt{n}, \varepsilon_n\sqrt{n}] \setminus \{0\}$. Denote by $\tilde{\mu}_n$ distribution of the random variable $\tilde{Y}_n := (\tilde{X}_1 + \cdots + \tilde{X}_n)/\sqrt{n}$. In addition introduce random variables

$$X^* := \frac{\tilde{X} - A_n}{C_n},\ X_1^* := \frac{\tilde{X}_1 - A_n}{C_n},\ X_2^* := \frac{\tilde{X}_2 - A_n}{C_n}, \dots \quad \text{and} \quad Y_n^* := \frac{X_1^* + \cdots + X_n^*}{\sqrt{n}},$$

where

$$A_n := -\int\limits_{|u|>\varepsilon_n\sqrt{n}} u\,\mu(du) \quad \text{and} \quad C_n := \Bigg(1 - \int\limits_{|u|>\varepsilon_n\sqrt{n}} u^2\,\mu(du) - \Bigg(\int\limits_{|u|>\varepsilon_n\sqrt{n}} u\,\mu(du)\Bigg)^2\Bigg)^{1/2}.$$

Denote by μ^* and μ_n^* the distributions of the random variables X^* and Y_n^*, respectively. We denote by m_k^* and $\tilde{m}_k,\ k = 0, 1, \dots,$ the moments and by β_k^* and $\tilde{\beta}_k,\ k = 0, 1, \dots,$ the absolute moments of the distributions μ^* and $\tilde{\mu}$, respectively. It is obvious that $m_1^* = 0$ and $m_2^* = 1$. Using (6.1) we note that

$$|A_n| \le \varepsilon_n^{-(q_2-1)} n^{-(q_2-1)/2} \rho_{q_2}(\mu, \varepsilon_n\sqrt{n}) \le \frac{1}{\sqrt{n}}\eta_{q2}(n)L_{q_2n} \tag{6.2}$$

and

$$0 \le \frac{1}{C_n} - 1 \le 2(\rho_2(\mu, \varepsilon_n\sqrt{n}) + A_n^2) \le 3\eta_{q2}(n)L_{q_2n}. \tag{6.3}$$

By (6.1)–(6.3), we obtain

$$C_n^{-1}(\varepsilon_n\sqrt{n} + |A_n|) < \frac{1}{3}\sqrt{n}. \tag{6.4}$$

It follows from (6.4) that the support of μ^* is contained in $[-\frac{1}{3}\sqrt{n}, \frac{1}{3}\sqrt{n}]$.

By (6.1)–(6.3), we easily deduce as well that

$$\begin{aligned}|m_3^* - m_3| &\le C_n^{-3}|\tilde{m}_3 - m_3| + (C_n^{-3} - 1)|m_3| + C_n^{-3}(3|A_n|\tilde{m}_2 + 3A_n^2|\tilde{m}_1| + |A_n|^3)\\ &\le C_n^{-3}|\tilde{m}_3 - m_3| + 4|m_3|\eta_{q2}(n)L_{q_2n} + \frac{4}{\sqrt{n}}\eta_{q2}(n)L_{q_2n}\\ &\le C_n^{-3}\varepsilon_n^{-(q_2-3)} n^{-(q_2-3)/2}\rho_{q_2}(\mu, \varepsilon_n\sqrt{n}) + 4\Big(|m_3| + \frac{1}{\sqrt{n}}\Big)\eta_{q2}(n)L_{q_2n}\\ &\le 2\sqrt{n}\eta_{q2}(n)L_{q_2n},\end{aligned} \tag{6.5}$$

and, using similar arguments,

$$\beta_3^* \le C_n^{-3}\tilde{\beta}_3 + \frac{4}{\sqrt{n}}\eta_{q2}(n)L_{q_2n}, \quad m_4^* \le C_n^{-4}\tilde{m}_4 + 5L_{3n}\eta_{q2}(n)L_{q_2n}. \tag{6.6}$$

Let T be a random variable with distribution $\mu_{a_n,0,0}$. Denote by $\tilde{\mu}_{a_n,0,0}$ the distribution of $C_nT + \sqrt{n}A_n$.

By the triangle inequality, we have

$$\Delta(\mu_n, \mu_{a_n,0,0}) \le \Delta(\mu_n, \tilde{\mu}_n) + \Delta(\tilde{\mu}_n, \tilde{\mu}_{a_n,0,0}) + \Delta(\tilde{\mu}_{a_n,0,0}, \mu_{a_n,0,0}). \tag{6.7}$$

First we establish with the help of Proposition 3.5

$$\begin{aligned}\Delta(\mu_n, \tilde{\mu}_n) &\le n\Delta(\mu, \tilde{\mu}) \le n\mu(\{|u| > \varepsilon_n\sqrt{n}\}) \le \varepsilon_n^{-q_2} n^{-(q_2-2)/2}\rho_{q_2}(\mu, \varepsilon_n\sqrt{n})\\ &\le \eta_{q2}(n)L_{q_2n}.\end{aligned} \tag{6.8}$$

Recalling the definition of $\mu_{a_n,0,0}$ (see (2.13) and (4.1), (4.4)), we note that $\mu_{a_n,0,0}$ is an absolutely continuous measure with the support on $[a_n - 2, a_n + 2]$ and its density has the form

$$\sqrt{4 - (x - a_n)^2}/(2\pi(1 + a_nx)), \quad \text{when} \quad x \in [a_n - 2, a_n + 2]. \tag{6.9}$$

This density does not exceed 1 on the set $[a_n - 2, a_n + 2]$ and is equal to 0 outside of this set, therefore we easily deduce the following upper bound, using (6.2) and (6.3),

$$\Delta(\tilde{\mu}_{a_n,0,0}, \mu_{a_n,0,0}) \le c\Big(\frac{1}{C_n} - 1 + \frac{\sqrt{n}A_n}{C_n}\Big) \le c\eta_{q2}(n)L_{q2n}. \tag{6.10}$$

Finally we note that $\Delta(\tilde{\mu}_n, \tilde{\mu}_{a_n,0,0}) = \Delta(\mu_n^*, \mu_{a_n,0,0})$. Our next main aim is to estimate this quantity.

By Proposition 3.1, $G_{\mu_n^*}(z) = 1/F_{\mu_n^*}(z)$, $z \in \mathbb{C}^+$, where $F_{\mu_n^*}(z) := F_{\mu^*}(Z(\sqrt{n}z))/\sqrt{n}$. Here $Z(z) \in \mathcal{F}$ is the solution of the Eq. (3.11) with $\mu = \mu^*$.

Consider the functions

$$S(z) := \frac{1}{2}\Big(z + \sqrt{z^2 - 4}\Big), \quad S_n(z) := Z(\sqrt{n}z)/\sqrt{n},$$
$$S_{n1}(z) := a_n + \frac{1}{2}\Big(z - a_n + \sqrt{(z - a_n)^2 - 4}\Big), \qquad z \in \mathbb{C}^+.$$

Note that $1/S(z) = G_{\mu_w}(z)$, where μ_w denotes the semicircle measure. Since $S_n \in \mathcal{F}$, we saw in Sect. 3 that there exists $\hat{\mu}_n \in \mathcal{M}$ such that $1/S_n(z) = G_{\hat{\mu}_n}(z)$. In addition, it is easy to see, that $1/S_{n1}(z) = G_{\mu_{a_n,0,0}}(z)$.

In order to estimate $\Delta(\mu_n^*, \mu_{a_n,0,0})$ we will apply the Stieltjes–Perron inversion formula to the measures μ_n^* and $\mu_{a_n,0,0}$. For this we need further estimates for $|G_{\mu_n^*}(z) - G_{\mu_{a_n,0,0}}(z)|$ on $\mathbb{C}^+$.

6.2 The functional equation for the function $S_n(z)$

Using (2.7) with $\mu = \mu^*$, we write, for $z \in \mathbb{C}^+$,

$$\begin{aligned} Z(z)G_{\mu^*}(Z(z)) &= 1 + \frac{1}{Z^2(z)} + \frac{1}{Z^2(z)} \int_{\mathbb{R}} \frac{u^3\,\mu^*(du)}{Z(z) - u} \\ &= 1 + \frac{1}{Z^2(z)} + \frac{m_3^*}{Z^3(z)} + \frac{1}{Z^3(z)} \int_{\mathbb{R}} \frac{u^4\,\mu^*(du)}{Z(z) - u}. \end{aligned} \tag{6.11}$$

The Eq. (3.11) with $\mu = \mu^*$ may be rewritten as

$$G_{\mu^*}(Z(z))\Big(Z(z) - z\Big) = (n-1)(1 - Z(z)G_{\mu^*}(Z(z))), \quad z \in \mathbb{C}^+. \tag{6.12}$$

By (6.11) and the definition of $S_n(z)$, we represent (6.12) in the form

$$\left(1 + \frac{1}{nS_n^2(z)} + \frac{r_{n1}(z)}{nS_n^2(z)}\right)(S_n(z) - z) = -\frac{n-1}{n}\frac{1}{S_n(z)}\left(1 + \frac{m_3 + r_{n2}(z)}{\sqrt{n}S_n(z)}\right), \tag{6.13}$$

for $z \in \mathbb{C}^+$, where

$$r_{n1}(z) := \int_{\mathbb{R}} \frac{u^3\, \mu^*(du)}{Z(\sqrt{n}z) - u}, \qquad r_{n2}(z) := \int_{\mathbb{R}} \frac{u^4\, \mu^*(du)}{Z(\sqrt{n}z) - u} + m_3^* - m_3. \tag{6.14}$$

By (6.1) and (6.4), we obtain from Lemma 3.4 for $\mu = \mu^*$ the bound

$$|Z(\sqrt{n}z)| \geq \sqrt{(n-1)/8}, \quad z \in \mathbb{C}^+. \tag{6.15}$$

The functions $r_{nj}(z),\ j = 1, 2$, are analytic on $\mathbb{C}^+$ and with the help of the inequalities (6.3)–(6.6) and (6.15) admit the estimates, for $z \in \mathbb{C}^+$,

$$\begin{aligned}
|r_{n1}(z)| &\leq \int_{|u|\leq\frac{1}{3}\sqrt{n}} \frac{|u|^3\, \mu^*(du)}{||Z(\sqrt{n}z)| - |u||} \leq \frac{52\beta_3^*}{\sqrt{n}} \leq \frac{53}{\sqrt{n}}\Big(\tilde{\beta}_3 + \frac{4}{\sqrt{n}}\eta_{q2}(n)L_{q2n}\Big) \leq 54L_{3n},\\
|r_{n2}(z)| &\leq \int_{|u|\leq\frac{1}{3}\sqrt{n}} \frac{u^4\, \mu^*(du)}{||Z(\sqrt{n}z)| - |u||} + |m_3^* - m_3| \leq \frac{52m_4^*}{\sqrt{n}} + 2\sqrt{n}\eta_{q2}(n)L_{q2n}\\
&\leq \frac{53\tilde{m}_4}{\sqrt{n}} + 3\sqrt{n}\eta_{q2}(n)L_{q2n}.
\end{aligned} \tag{6.16}$$

We deduce from (6.13) the following relation

$$S_n^3(z) - zS_n^2(z) + (1 + \varepsilon_{n1}(z))S_n(z) + \varepsilon_{n2}(z) = 0, \quad z \in \mathbb{C}^+, \tag{6.17}$$

where

$$\varepsilon_{n1}(z) := \frac{1}{n}r_{n1}(z) \quad \text{and} \quad \varepsilon_{n2}(z) := \frac{m_3}{\sqrt{n}} + r_{n3}(z) := a_n + r_{n3}(z) \tag{6.18}$$

with

$$r_{n3}(z) := \Big(1 - \frac{1}{n}\Big)\frac{r_{n2}(z)}{\sqrt{n}} - \frac{z}{n}\Big(1 + r_{n1}(z)\Big) - \frac{m_3}{n\sqrt{n}}.$$

6.3 Estimates of $\varepsilon_{n1}(z)$ and $\varepsilon_{n2}(z)$

By (6.16), we obtain

$$|r_{n3}(z)| \leq \frac{53\tilde{m}_4}{n} + 3\eta_{q2}(n)L_{q2n} + \frac{|z|}{n}\Big(1 + 54L_{3n}\Big) + \frac{L_{3n}}{n}, \quad z \in \mathbb{C}^+. \tag{6.19}$$

Note that $\tilde{m}_4 \le \beta_{q_2}(\varepsilon_n^2 n)^{(4-q_2)/2}$. By (6.16), (6.19) and the last inequality, we have, for $z \in D_1 := \{z \in \mathbb{C}^+ : 0 < \Im z \le 3, |\Re z| \le 4\}$,

$$|\varepsilon_{n1}(z)| \le 54\frac{L_{3n}}{n} < \frac{1}{10}, \tag{6.20}$$

$$|r_{n3}(z)| \le 53\frac{\beta_{q_2}\eta_{q2}(n)}{n^{(q_2-2)/2}} + 3\eta_{q2}(n)L_{q_2n} + \frac{2|z| + L_{3n}}{n} \le 56\eta_{q2}(n)L_{q_2n} + \frac{11}{n} \tag{6.21}$$

and

$$|\varepsilon_{n2}(z)| \le 56\eta_{q2}(n)L_{q_2n} + 2L_{3n} < \frac{1}{10^4}. \tag{6.22}$$

6.4 Roots of the functional equation for $S_n(z)$

For every fixed $z \in \mathbb{C}^+$, consider the cubic equation

$$P(z, w) := w^3 - zw^2 + (1 + \varepsilon_{n1}(z))w + \varepsilon_{n2}(z) = 0.$$

Denote roots of this equation by $w_j = w_j(z),\ j = 1, 2, 3$.

We shall show that for $z \in D_1$ the equation $P(z, w) = 0$ has a root, say $w_1 = w_1(z)$, such that

$$w_1 = -a_n + r_{n4}(z), \tag{6.23}$$

where the quantity $r_{n4}(z)$ admits the following bound

$$|r_{n4}(z)| < 10^2 r, \quad \text{where} \quad r := \eta_{q2}(n)L_{q_2n} + L_{3n}^2. \tag{6.24}$$

In addition $|w_j + a_n| \ge 10^2 r,\ j = 2, 3$.

Indeed, introduce the polynomials

$$P_1(w) := w^3 - zw^2 \quad \text{and} \quad P_2(w) := (1 + \varepsilon_{n1}(z))w + \varepsilon_{n2}(z) = (1 + \varepsilon_{n1}(z))(w + \varepsilon_{n3}(z)),$$

where $\varepsilon_{n3}(z) := \varepsilon_{n2}(z)/(1+\varepsilon_{n1}(z))$. They admit the following estimates on the circle $|w + a_n| = 10^2 r$

$$\begin{aligned}|P_1(w)| &\le |w|^2(|w - z|) \le 2(|w + a_n|^2 + a_n^2)(|w + a_n| + |z + a_n|)\\ &\le 2(10^4 r^2 + a_n^2)(r + 11/2) \le 12(10^4 r^2 + a_n^2) \le 24\, r\end{aligned} \tag{6.25}$$

and

$$|P_2(w)| = |1 + \varepsilon_{n1}(z)||w + \varepsilon_{n3}(z)| \geq (1 - |\varepsilon_{n1}(z)|)|10^2 r - |\varepsilon_{n3}(z) - a_n||. \tag{6.26}$$

Since, by (6.20), $1 - |\varepsilon_{n1}(z)| \geq 9/10$ and, by (6.20)–(6.22),

$$|\varepsilon_{n3}(z) - a_n| \leq |r_{n3}(z)| + 2|\varepsilon_{n1}(z)||\varepsilon_{n2}(z)| \leq 57\, r,$$

we see from (6.26) that $|P_2(w)| \geq 36\, r$. This estimate and (6.25) gives us the inequality $|P_1(w)| < |P_2(w)|$ on the circle $|w + a_n| = 10^2 r$ and the desired result follows from Rouché's theorem.

6.5 The remaining roots w_2 and w_3 of the Eq. (6.17)

Now we shall investigate the behavior of the roots $w = w_2(z)$ and $w = w_3(z)$. As seen in Sect. 6.4 $w_2(z) \neq w_1(z)$ and $w_3(z) \neq w_1(z)$ for $z \in D_1$. We shall construct a set $D_2 \subset D_1$ where $w_2(z) \neq w_3(z)$, $z \in D_2$. Since $P(z, w) = P_3(z, w)(w - w_1)$, where

$$P_3(z, w) := w^2 - (z - w_1)w + 1 + \varepsilon_{n1}(z) - w_1(z - w_1),$$

we see that $w_2 = w_3$ for $z \in D_1$ such that

$$(z - w_1)^2 - 4(1 + \varepsilon_{n1}(z) - w_1(z - w_1)) = 0. \tag{6.27}$$

We conclude from this relation that $z = \pm 2\sqrt{1 + \varepsilon_{n1}(z) + w_1^2} - w_1$. Therefore, as it is easy to see from (6.20) and (6.23), (6.24), the relation (6.27) does not hold for $z \in D_2$, where $D_2 := \{z \in \mathbb{C} : 0 < \Im z \leq 3, |\Re z - a_n| \leq 2 - h_1\}$ and $h_1 := c_1^{-1/6} r$.

Hence the roots $w_1(z)$, $w_2(z)$ and $w_3(z)$ are distinct for $z \in D_2$.

Now we see that the roots w_2 and w_3 have the form

$$w_j := \frac{1}{2}\Big(z - w_1 + (-1)^{j-1}\sqrt{g(z)}\Big), \quad j = 2, 3, \tag{6.28}$$

where $g(z) := (z - w_1)^2 - 4 - 4\varepsilon_{n1}(z) + 4w_1(z - w_1) \neq 0$ for $z \in D_2$. In this formula we choose the branch of the square satisfying $\sqrt{g(i)} \in \mathbb{C}^+$.

Using (6.23), we rewrite (6.28) in the following way

$$\begin{aligned} w_j &:= \frac{1}{2}\Big(z + a_n + (-1)^{j-1}\sqrt{(z + a_n)^2 - 4 - 4a_n(z + a_n) + r_{n5}(z)}\Big) - \frac{1}{2} r_{n4}(z) \\ &= a_n + \frac{1}{2}\Big(z - a_n + (-1)^{j-1}\sqrt{(z - a_n)^2 - 4 - 4a_n^2 + r_{n5}(z)}\Big) - \frac{1}{2} r_{n4}(z), \end{aligned} \tag{6.29}$$

$j = 1, 2$, where

$$r_{n5}(z) := -4\varepsilon_{n1}(z) + (2z + 6a_n - 3r_{n4}(z))r_{n4}(z).$$

From (6.20) and (6.24) it follows that the following estimate holds, for $z \in D_2$,

$$|r_{n5}(z)| \le 4|\varepsilon_{n1}(z)| + (2|z| + 6|a_n| + 3|r_{n4}(z)|)|r_{n4}(z)|$$
$$\le 216\frac{L_{3n}}{n} + (10 + 6L_{3n} + 300r)10^2 r \le 1100\, r. \quad (6.30)$$

Using (6.30) we obtain

$$\Big|\frac{4a_n^2 - r_{n5}(z)}{(z - a_n)^2 - 4}\Big| \le \frac{1104\, r}{h_1} \le 1104\, c_1^{1/6} \le \frac{1}{10}, \quad z \in D_2. \quad (6.31)$$

By power series expansion of $(1 + z)^{1/2}$, $|z| < 1$, we obtain, for $z \in D_2$,

$$\sqrt{(z - a_n)^2 - 4 - 4a_n^2 + r_{n5}(z)} = \sqrt{(z - a_n)^2 - 4} + \frac{r_{n6}(z)}{\sqrt{(z - a_n)^2 - 4}},$$

where $|r_{n6}(z)| \le 1004r$. By this relation, we see that, for $z \in D_2$,

$$w_j = a_n + \frac{1}{2}\Big((z - a_n) + (-1)^{j-1}\sqrt{(z - a_n)^2 - 4}\Big)$$
$$-\frac{1}{2}r_{n4}(z) + \frac{(-1)^{j-1}}{2}\frac{r_{n6}(z)}{\sqrt{(z - a_n)^2 - 4}}, \quad j = 2, 3. \quad (6.32)$$

Let us show that $S_n(z) = w_3(z)$ for $z \in D_2$. By (6.1), (6.23) and (6.24), we see that $|w_1(z)| \le 1/6$ for $z \in D_2$. Since $S_n(z)$ satisfies the Eq. (6.17) and, by (6.15), $|S_n(z)| \ge 1/3$ for all $z \in \mathbb{C}^+$, we have $S_n(z) = w_2(z)$ or $S_n(z) = w_3(z)$ for $z \in D_2$.

First assume that, for every $z_0 \in D_2$, there exists $r_0 = r_0(z_0) > 0$ such that $S_n(z) = w_j(z)$ for all $z \in D_2 \cap \{|z - z_0| < r_0\}$, where $j = 2$ or $j = 3$. From this assumption it follows that $S_n(z) = w_j(z)$ for all $z \in D_2$, where $j = 2$ or $j = 3$. Furthermore, it is not difficult to see that the roots $w_2(z)$ and $w_3(z)$ admit the estimates: $|w_2(z)| \le 4/3$ for $z \in D_2$, and $|w_3(z)| \ge 3/2$ for $z \in D_2$ and $\Im z \ge 2$. The analytic function $S_n(z) \in \mathcal{F}$, by (3.6), satisfies the inequality $\Im S_n(z) \ge \Im z$. Hence, under the above assumption $S_n(z) = w_3(z)$ for all $z \in D_2$.

Now assume that the above assumption does not hold. Then there exists a point $z_0 \in D_2$ such that, for any $r_0 > 0$, there exist points $z' \in D_2$ and $z'' \in D_2$ in the disc $|z - z_0| < r_0$ such that $S_n(z') = w_2(z')$ and $S_n(z'') = w_3(z'')$. Let for definiteness $S_n(z_0) = w_2(z_0)$. By this assumption, there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to z_0$ and $S_n(z_k) = w_3(z_k)$. Therefore we have $w_2(z_0) = \lim_{z_k \to z_0} w_3(z_k)$. Using (6.32), rewrite this relation in the form

$$a_n+\frac{1}{2}\Big(z_0-a_n-\sqrt{(z_0-a_n)^2-4}\Big)-\frac{1}{2}\Big(r_{n4}(z_0)+\frac{r_{n6}(z_0)}{\sqrt{(z_0-a_n)^2-4}}\Big)$$
$$=a_n+\frac{1}{2}\Big(z_0-a_n+\sqrt{(z_0-a_n)^2-4}\Big)+\frac{1}{2}\lim_{z_k\to z_0}\Big(r_{n4}(z_k)-\frac{r_{n6}(z_k)}{\sqrt{(z_k-a_n)^2-4}}\Big). \tag{6.33}$$

From (6.33) we easily conclude with r as in (6.24)

$$c_1^{-1/12}\sqrt{r}\le|\sqrt{(z_0-a_n)^2-4}|\le 1004\left(c_1^{1/12}\sqrt{r}+r\right),$$

a contradiction for sufficiently small $c_1>0$. Hence, the first assumption holds only and $S_n(z)=w_3(z),\ z\in D_2$.

Denote by B_1 the set $[-2+h_1+a_n, 2-h_1+a_n]$. Recall that $h_1=c_1^{-1/6}r$ (see the definition of the set D_2).

6.6 Estimate of the integral $\int_{B_1}|G_{\mu_{a_n,0,0}}(x+i\varepsilon)-G_{\mu_n^*}(x+i\varepsilon)|\,dx$ for $0<\varepsilon\le 1$

We obtain an estimate of this integral, using the inequality

$$\int\limits_{B_1}|G_{\mu_{a_n,0,0}}(x+i\varepsilon)-G_{\mu_n^*}(x+i\varepsilon)|\,dx\le\int\limits_{B_1}|G_{\mu_{a_n,0,0}}(x+i\varepsilon)-G_{\hat\mu_n}(x+i\varepsilon)|\,dx$$
$$+\int\limits_{B_1}|G_{\hat\mu_n}(x+i\varepsilon)-G_{\mu_n^*}(x+i\varepsilon)|\,dx. \tag{6.34}$$

Therefore we need to evaluate the functions

$$G_{\mu_{a_n,0,0}}(z)-G_{\hat\mu_n}(z)\quad\text{and}\quad G_{\hat\mu_n}(z)-G_{\mu_n^*}(z) \tag{6.35}$$

for $z\in D_2$.

For $z\in D_2$, using the formula (6.29) with $j=3$ for $S_n(z)$, we write

$$G_{\hat\mu_n}(z)-G_{\mu_{a_n,0,0}}(z)=\frac{1}{S_n(z)}-\frac{1}{S_{n1}(z)}=\frac{S_{n1}(z)-S_n(z)}{S_{n1}(z)S_n(z)}$$
$$=\frac{1}{2S_{n1}(z)S_n(z)}\Big(r_{n4}(z)$$
$$+\frac{4a_n^2-r_{n5}(z)}{\sqrt{(z-a_n)^2-4}+\sqrt{(z-a_n)^2-4-4a_n^2+r_{n5}(z)}}\Big). \tag{6.36}$$

By (6.31), we have, for $z \in D_2$,

$$\begin{aligned}&\left|\sqrt{(z-a_n)^2-4}+\sqrt{(z-a_n)^2-4-4a_n^2+r_{n5}(z)}\right|\\&\quad=\left|\sqrt{(z-a_n)^2-4}\right|\left|1+\sqrt{1-(4a_n^2-r_{n5}(z))/((z-a_n)^2-4)}\right|\\&\quad\geq\left|\sqrt{(z-a_n)^2-4}\right|.\end{aligned} \tag{6.37}$$

In addition, we see from (6.15) that $|S_n(z)| \geq 1/3$ for $z \in \mathbb{C}^+$. The same estimate obviously holds for $|S_{n1}(z)|$.

Therefore we can conclude from (6.36) and (6.37) that

$$\begin{aligned}\int_{B_1}\left|G_{\hat{\mu}_n}(x+i\varepsilon)-G_{\mu_{a_n,0,0}}(x+i\varepsilon)\right|dx&=\int_{B_1}\left|\frac{1}{S_n(x+i\varepsilon)}-\frac{1}{S_{n1}(x+i\varepsilon)}\right|dx\\&\leq\frac{9}{2}\int_{B_1}\left(|r_{n4}(x+i\varepsilon)|+\frac{4a_n^2+|r_{n5}(x+i\varepsilon)|}{|\sqrt{(x-a_n+i\varepsilon)^2-4}|}\right)dx\end{aligned} \tag{6.38}$$

for $0 < \varepsilon \leq 1$.

From (6.24) it follows at once that

$$\int_{B_1}|r_{n4}(x+i\varepsilon)|\,dx \leq 4\cdot 10^2 r, \quad \varepsilon \in (0,1]. \tag{6.39}$$

From (6.30) we conclude that, for the same ε,

$$\int_{B_1}\frac{4a_n^2+|r_{n5}(x+i\varepsilon)|}{|\sqrt{(x-a_n+i\varepsilon)^2-4}|}\,dx \leq 1104\,r\int_{B_1}\frac{dx}{\sqrt{4-(x-a_n)^2}} \leq 4416\,r. \tag{6.40}$$

It follows from (6.38)–(6.40) that

$$\int_{B_1}\left|G_{\hat{\mu}_n}(x+i\varepsilon)-G_{\mu_{a_n,0,0}}(x+i\varepsilon)\right|dx \leq 3\cdot 10^4 r, \quad \varepsilon \in (0,1]. \tag{6.41}$$

Now we conclude from (6.11) that

$$G_{\mu_n^*}(z)-G_{\hat{\mu}_n}(z)=\frac{r_{n7}(z)}{S_n(z)}, \quad z \in \mathbb{C}^+, \tag{6.42}$$

where

$$r_{n7}(z) := \frac{1}{nS_n^2(z)}+\frac{r_{n1}(z)}{nS_n^2(z)}. \tag{6.43}$$

Since $|S_n(z)| \geq 1/3$ for $z \in \mathbb{C}^+$, we see from (6.16) that

$$|r_{n7}(z)| \leq \frac{9(1+54L_{3n})}{n}, \qquad z \in D_2. \tag{6.44}$$

Therefore, we deduce from (6.42) and (6.44) the upper bound

$$\int\limits_{B_1} |G_{\mu_n^*}(x+i\varepsilon) - G_{\hat{\mu}_n}(x+i\varepsilon)|\,dx \leq \int\limits_{B_1} \frac{|r_{n7}(x+i\varepsilon)|}{|S_n(x+i\varepsilon)|}\,dx \leq \frac{2\cdot 10^2}{n}, \quad \varepsilon \in (0,1]. \tag{6.45}$$

From (6.34), (6.41) and (6.45) we finally obtain

$$\int\limits_{B_1} |G_{\mu_{a_n,0,0}}(x+i\varepsilon) - G_{\mu_n^*}(x+i\varepsilon)|\,dx \leq 4\cdot 10^4\, r, \quad \varepsilon \in (0,1]. \tag{6.46}$$

6.7 Application of the Stieltjes–Perron inversion formula

By (6.9), we have the relation

$$\begin{aligned}\int\limits_{B_1} p_{\mu_{a_n,0,0}}(x)\,dx &= 1 - \left(\int\limits_{[2-h_1+a_n,2+a_n]} + \int\limits_{[-2+a_n,-2+h_1+a_n]}\right) \frac{\sqrt{4-(x-a_n)^2}}{2\pi(1+a_n x)}\,dx \\ &\geq 1 - h_1^{3/2}.\end{aligned} \tag{6.47}$$

From (6.46) and (6.47) we conclude, using the Stieltjes–Perron inversion formula,

$$\begin{aligned}\mu_n^*(B_1) \geq 1 - (4\cdot 10^4 + c_1^{-1/4} r^{1/2})\, r \ &\geq 1 - (4\cdot 10^4 + c_1^{1/4})\, r \\ &\geq 1 - (4\cdot 10^4 + 1)\, r.\end{aligned} \tag{6.48}$$

Finally we deduce from (6.46)–(6.48) and the Stieltjes–Perron inversion formula that

$$\Delta(\mu_n^*, \mu_{a_n,0,0}) \leq c\,r = c\left(\eta_{q2}(n) L_{q2n} + L_{3n}^2\right). \tag{6.49}$$

6.8 Completion of the proof of Theorem 2.3

The statement of the theorem follows immediately from (6.7), (6.8), (6.10) and (6.49). □

Proof of Corollary 2.4 It is easy to see that the assertion of Corollary 2.4 follows from Theorem 2.3 and from the following simple formula, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\mu_{a_n,0,0}((-\infty, x)) - \mu_w((-\infty, x)) = -\frac{m_3}{3\sqrt{n}}(x^2 - 1)p_w(x) + c\theta\Big(\frac{|m_3|}{\sqrt{n}}\Big)^{3/2}.$$

Proof of Corollary 2.5 The assertion of Corollary 2.5 follows immediately from (2.20) and Proposition 3.6. □

7 Edgeworth expansion in free CLT (the case $\beta_q < \infty$, $q \geq 4$)

In this section we prove Theorem 2.6 and Corollary 2.7. The proof of the theorem is similar to the proof of Theorem 2.3 but with some essential technical differences. Therefore we describe in detail those arguments which differ from the proof of Theorem 2.3 and omit arguments which directly repeat the arguments of Sect. 6. We preserve all notations of Sect. 6. Denote as well

$$S_{n2}(z) := a_n + \frac{1}{2}\Big(\big(1 + b_n\big)(z - a_n) + \sqrt{\big(1 - b_n\big)^2(z - a_n)^2 - 4\big(1 - d_n\big)}\Big), \quad z \in \mathbb{C}^+,$$

where a_n, b_n and d_n are defined in Sect. 2. The function $S_{n2}(z) \in \mathcal{F}$ and $1/S_{n2}(z) = G_{\mu_{a_n,b_n,d_n}}(z)$, where μ_{a_n,b_n,d_n} is the free Meixner measure with the parameters a_n, b_n and d_n, see (2.14).

The proof of Theorem 2.6 will be given in Sections 7.1–7.9. First we proceed to study the following:

7.1 The passage to measures with bounded supports

Let $n \in \mathcal{N}$. Let $\varepsilon_n \in (0, 10^{-1/2}]$ be a point at which the infimum of the function $g_{qn3}(\varepsilon)$ in (2.16) is attained. This means that

$$\eta_{q3}(n) := \varepsilon_n^{5-q_3} + \frac{\rho_{q3}(\mu, \varepsilon_n\sqrt{n})}{\beta_{q3}}\varepsilon_n^{-q_3}. \tag{7.1}$$

Using this parameter ε_n, we define free random variables $\tilde{X}, \tilde{X}_1, \tilde{X}_2, \dots$ and $X^*, X_1^*, X_2^*, \dots$ in the same way as in Sect. 6. We define probability measures $\tilde{\mu}_n, \mu^*, \mu_n^*$ in the same way as well.

Without loss of generality we assume that

$$\eta_{q3}(n)L_{q3n} + L_{4n} < c_2, \tag{7.2}$$

where $c_2 > 0$ is a sufficiently small absolute constant. From (7.2) it follows that n is sufficiently large $n > c_2^{-1}m_4 \geq c_2^{-1}$. Here and in the sequel we use Lyapunov's inequality $1 = m_2^{1/2} \leq \beta_3^{1/3} \leq m_4^{1/4}$.

Now we repeat the arguments of Sect. 6.1.

Using (7.2) we note that

$$|A_n| \le \varepsilon_n^{-(q_3-1)} n^{-(q_3-1)/2} \rho_{q_3}(\mu, \varepsilon_n\sqrt{n}) \le \frac{1}{\sqrt{n}} \eta_{q3}(n) L_{q_3 n} \tag{7.3}$$

and

$$0 \le \frac{1}{C_n} - 1 \le 2(\rho_2(\mu, \varepsilon_n\sqrt{n}) + A_n^2) \le 3\eta_{q3}(n) L_{q_3 n}. \tag{7.4}$$

By (7.3) and (7.4), we see that (6.4) holds and the support of μ^* is contained in $[-\frac{1}{3}\sqrt{n}, \frac{1}{3}\sqrt{n}]$.

Recalling (6.5) and (6.6), we easily deduce, by (7.2)–(7.4), that

$$\begin{aligned}
|m_3^* - m_3| &\le 2\sqrt{n}\eta_{q3}(n) L_{q_3 n}, \\
|m_4^* - m_4| & \\
&\le C_n^{-4}|\tilde{m}_4 - m_4| + (C_n^{-4} - 1)m_4 + C_n^{-4}(4|A_n||\tilde{m}_3| \\
&\quad +6A_n^2\tilde{m}_2 + 4|A_n|^3|\tilde{m}_1| + A_n^4) \\
&\le C_n^{-4}|\tilde{m}_4 - m_4| + 5m_4\eta_{q3}(n)L_{q_3 n} + \frac{5|\tilde{m}_3| + 1}{n}\eta_{q3}(n)L_{q_3 n} \\
&\le C_n^{-4}\varepsilon_n^{-(q_3-4)} n^{-(q_3-4)/2}\rho_{q_3}(\mu, \varepsilon_n\sqrt{n}) + 5m_4\Big(1 + \frac{2}{n}\Big)\eta_{q3}(n)L_{q_3 n} \\
&\le 2n\, \eta_{q3}(n)L_{q_3 n}
\end{aligned} \tag{7.5}$$

and

$$\beta_5^* \le C_n^{-5}\tilde{\beta}_5 + 6L_{4n}\eta_{q3}(n)L_{q_3 n}. \tag{7.6}$$

By the triangle inequality we have

$$\Delta(\mu_n, \kappa_n) \le \Delta(\mu_n, \tilde{\mu}_n) + \Delta(\tilde{\mu}_n, \tilde{\kappa}_n) + \Delta(\tilde{\kappa}_n, \kappa_n), \tag{7.7}$$

where, for $x \in \mathbb{R}$,

$$\begin{aligned}
\tilde{\kappa}_n((-\infty, x)) &:= \tilde{\mu}_{a_n,b_n,d_n}((-\infty, x)) + \frac{1}{n}\tilde{\varsigma}_n((-\infty, x)) \\
&:= \mu_{a_n,b_n,d_n}((-\infty, (x - \sqrt{n}A_n)/C_n)) + \frac{1}{n}\varsigma_n((-\infty, (x - \sqrt{n}A_n)/C_n)).
\end{aligned}$$

Note that $\Delta(\tilde{\mu}_n, \tilde{\kappa}_n) = \Delta(\mu_n^*, \kappa_n)$.

First we establish with the help of Proposition 3.5

$$\begin{aligned}
\Delta(\mu_n, \tilde{\mu}_n) &\le n\Delta(\mu, \tilde{\mu}) \le n\mu(\{|u| > \varepsilon_n\sqrt{n}\}) \le \varepsilon_n^{-q_3} n^{-(q_3-2)/2}\rho_{q_3}(\mu, \varepsilon_n\sqrt{n}) \\
&\le \eta_{q3}(n)L_{q_3 n}.
\end{aligned} \tag{7.8}$$

We saw in Sect. 4 that, for $n \geq 3m_4$, μ_{a_n,b_n,d_n} is an absolutely continuous measure with support on the set $B_2 := [a_n - 2/e_n, a_n + 2/e_n]$ and density of the form

$$p_{\mu_{a_n,b_n,d_n}}(x) := \frac{\sqrt{4(1-d_n) - (1-b_n)^2(x-a_n)^2}}{2\pi(b_n x^2 + a_n(1-b_n)x + 1 - d_n)}, \quad x \in B_2. \tag{7.9}$$

This density does not exceed 1 on the set B_2 and is equal 0 outside of this set.

The signed measure ς_n has density p_{ς_n}, see (2.25), which does not exceed 1 by modulus on the set B_2 and is equal to zero outside of B_2.

Therefore, in view of (7.2), a simple calculation shows that

$$\begin{aligned}\Delta(\tilde{\kappa}_n, \kappa_n) &\leq \Delta(\tilde{\mu}_{a_n,b_n,d_n}, \mu_{a_n,b_n,d_n}) + \frac{1}{n}\Delta(\tilde{\varsigma}_n, \varsigma_n) \leq c\Big(\frac{1}{C_n} - 1 + \frac{\sqrt{n}A_n}{C_n}\Big) \\ &\leq c\varepsilon_n^{-(q_3-1)} n^{-(q_3-2)/2} \rho_{q_3}(\mu, \varepsilon_n\sqrt{n}) \leq c\eta_{q3}(n) L_{q_3 n}.\end{aligned} \tag{7.10}$$

Our next aim is to estimate the quantity $\Delta(\mu_n^*, \kappa_n)$. In order to estimate $\Delta(\mu_n^*, \kappa_n)$ we need to apply the inversion formula to μ_n^* and κ_n. We shall now derive the necessary estimates for $|G_{\mu_n^*}(z) - G_{\kappa_n}(z)|$ on $\mathbb{C}^+$.

7.2 The functional equation for $S_n(z)$

Using (2.7) with $\mu = \mu^*$, we write, for $z \in \mathbb{C}^+$,

$$Z(z)G_{\mu^*}(Z(z)) = 1 + \frac{1}{Z^2(z)} + \frac{m_3^*}{Z^3(z)} + \frac{m_4^*}{Z^4(z)} + \frac{1}{Z^4(z)} \int_{\mathbb{R}} \frac{u^5\,\mu^*(du)}{Z(z) - u}. \tag{7.11}$$

By (7.11) and the definition of $S_n(z)$, the Eq. (3.11) with $\mu = \mu^*$ may be rewritten as

$$\begin{aligned}&\Big(1 + \frac{1}{nS_n^2(z)} + \frac{m_3^*}{n^{3/2}S_n^3(z)} + \frac{m_4^* + \zeta_{n1}(z)}{n^2 S_n^4(z)}\Big)(S_n(z) - z) \\ &= -\frac{n-1}{n}\frac{1}{S_n(z)}\Big(1 + \frac{m_3^*}{\sqrt{n}S_n(z)} + \frac{m_4^* + \zeta_{n1}(z)}{nS_n^2(z)}\Big)\end{aligned} \tag{7.12}$$

for $z \in \mathbb{C}^+$, where $\zeta_{n1}(z) := \int_{\mathbb{R}} \frac{u^5\,\mu^*(du)}{Z(\sqrt{n}z)-u}$.

We deduce from (7.12) the following relation, for $z \in \mathbb{C}^+$,

$$S_n^5(z) - zS_n^4(z) + S_n^3(z) + \frac{\zeta_{n2}(z)}{\sqrt{n}}S_n^2(z) + \frac{\zeta_{n3}(z)}{n}S_n(z) - \frac{\zeta_{n4}(z)z}{n^2} = 0, \tag{7.13}$$

where $\zeta_{n2}(z) := m_3^* - z/\sqrt{n}$, $\zeta_{n3}(z)(z) := m_4^* + \zeta_{n1}(z) - zm_3^*/\sqrt{n}$ and $\zeta_{n4}(z)(z) := m_4^* + \zeta_{n1}(z)$. Note that the functions $\zeta_{nj}(z)$, $j = 1, 2, 3, 4$, are analytic on $\mathbb{C}^+$.

7.3 Estimates of the functions $\zeta_{nj}(z),\ j=1,2,3$ on the set D_1

From (7.2) and (6.4) we deduce, using Lemma 3.4, that (6.15) holds. Therefore, in view of (7.4) and (7.6), we arrive at the estimate

$$
\begin{aligned}
|\zeta_{n1}(z)| &\le \int\limits_{|u|\le\frac{1}{3}\sqrt{n}} \frac{|u|^5\,\mu^*(du)}{||Z(\sqrt{n}z)|-|u||} \le \frac{52\beta_5^*}{\sqrt{n}} \le 53\,\frac{\tilde{\beta}_5}{\sqrt{n}} + 312\,\frac{L_{4,n}}{\sqrt{n}}\eta_{q3}(n)L_{q_3n} \\
&\le 53\,\frac{\beta_{q_3}(\varepsilon_n^2 n)^{(5-q_3)/2}}{\sqrt{n}} + 312\,\frac{L_{4n}}{\sqrt{n}}\eta_{q3}(n)L_{q_3n} \\
&\le 54\,n\,\eta_{q3}(n)L_{q_3n}, \quad z\in\mathbb{C}^+. \qquad (7.14)
\end{aligned}
$$

For $z\in D_1$, by (7.5) and (7.14), we get the bounds

$$
\begin{aligned}
\frac{|\zeta_{n2}(z)|}{\sqrt{n}} &\le 2(L_{3n}+\eta_{q3}(n)L_{q_3n}), \\
\frac{|\zeta_{n3}(z)|}{n} &\le \frac{m_4^*+|\zeta_{n1}(z)|}{n} + \frac{5|m_3^*|}{n^{3/2}} \le 2L_{4n}+56\,\eta_{q3}(n)L_{q_3n} \qquad (7.15)\\
\frac{|\zeta_{n4}(z)|}{n} &\le \frac{m_4^*+|\zeta_{n1}(z)|}{n} \le L_{4n}+56\,\eta_{q3}(n)L_{q_3n}.
\end{aligned}
$$

7.4 The roots of the functional Eq. (7.13) for $S_n(z)$

For every fixed $z\in\mathbb{C}^+$ consider the equation

$$
Q(z,w) := w^5 - zw^4 + w^3 + \frac{\zeta_{n2}(z)}{\sqrt{n}}w^2 + \frac{\zeta_{n3}(z)}{n}w - \frac{\zeta_{n4}(z)z}{n^2} = 0. \qquad (7.16)
$$

Denote the roots of the Eq. (7.16) by $w_j = w_j(z),\ j=1,\dots,5$. Let us show that for every fixed $z\in D_1$ the equation $Q(z,w)=0$ has three roots, say $w_j = w_j(z),\ j=1,2,3$, such that

$$
|w_j| < r' := 15(L_{4n}+\eta_{q3}(n)L_{q_3n})^{1/2}, \quad j=1,2,3, \qquad (7.17)
$$

two roots, say $w_j,\ j=4,5$, such that $|w_j|\ge r'$ for $j=4,5$. Recalling (7.2) we see that the bound $r' < \frac{1}{100}$ holds.

Consider the polynomials

$$
Q_1(z,w) := w^5 - zw^4 + \frac{\zeta_2(z)}{\sqrt{n}}w^2 + \frac{\zeta_{n3}(z)}{n}w - \frac{\zeta_{n4}(z)z}{n^2} \quad\text{and}\quad Q_2(w) := w^3.
$$

The following estimates hold on the circle $|w|=r'$ for $z\in D_1$: $|w|^5 = (r')^2|w|^3 \le 10^{-4}|w|^3$, and $|zw^4| = |z|r'|w|^3 \le \frac{1}{20}|w|^3$. Since $n\,r' \ge 15\sqrt{m_4} \ge 15m_2 = 15$ and, by Proposition 3.6, $r' \ge 15L_{3n}$, we have as well, using (7.15),

$$\frac{|\zeta_{n2}(z)|}{\sqrt{n}}|w^2| = \frac{|\zeta_{n2}(z)|}{\sqrt{n}}\frac{1}{r'}|w|^3 \le \frac{|w|^3}{5},$$
$$\frac{|\zeta_{n3}(z)|}{n}|w| = \frac{|\zeta_{n3}(z)|}{n}\frac{1}{(r')^2}|w|^3 \le \frac{4|w|^3}{15},$$
$$\frac{|\zeta_{n4}(z)|}{n^2}|z| = \frac{|\zeta_{n4}(z)|}{n^2}\frac{|z|}{(r')^3}|w|^3 \le \frac{4|z|}{15nr'}|w|^3 \le \frac{4|w|^3}{45}.$$

We see from the last five inequalities that $|Q_1(z,w)| \le \frac{8}{9}|Q_2(w)|$ on the circle $|w| = r'$. Therefore, by Rouché's theorem, we obtain that the polynomial $Q_1(z,w)+Q_2(w)$ has only three roots which are less than r' in modulus, as claimed.

Represent $Q(z,w)$ in the form

$$Q(z,w) = (w^2 + s_1 w + s_2)(w^3 + g_1 w^2 + g_2 w + g_3),$$

where $w^3 + g_1 w^2 + g_2 w + g_3 = (w-w_1)(w-w_2)(w-w_3)$. From this formula we derive the relations

$$s_1 + g_1 = -z, \quad s_2 + s_1 g_1 + g_2 = 1, \quad s_2 g_1 + s_1 g_2 + g_3 = \frac{\zeta_{n2}(z)}{\sqrt{n}},$$
$$s_2 g_2 + s_1 g_3 = \frac{\zeta_{n3}(z)}{n}, \quad s_2 g_3 = -\frac{\zeta_{n4}(z) z}{n^2}. \tag{7.18}$$

By Vieta's formulae and (7.17), note that

$$|g_1| \le 3r', \quad |g_2| \le 3(r')^2, \quad |g_3| \le (r')^3. \tag{7.19}$$

Now we obtain from (7.18) and (7.19) the following bounds, for $z \in D_1$,

$$|s_1| \le 5 + 3r', \quad |1 - s_2| \le 3r'(4r'+5) \le 16r' \le \frac{1}{2}. \tag{7.20}$$

Then we conclude from (7.5), (7.15), (7.18)–(7.20) that, for the same z,

$$\Big|g_2 - \frac{\zeta_{n4}(z)}{n}\Big| \le \Big|g_2 - \frac{\zeta_{n3}(z)}{n}\Big| + \frac{|m_3^*||z|}{n^{3/2}} \le \frac{|s_1|}{|s_2|}|g_3| + \frac{|s_2 - 1|}{|s_2|}\frac{|\zeta_{n3}(z)|}{n} + (r')^3$$
$$\le 11(r')^3 + 8(r')^3 + (r')^3 = 20(r')^3. \tag{7.21}$$

Denote $a_n^* := m_3^*/\sqrt{n}$, $\rho_n^* := L_{4n}^* - 1/n := (m_4^* - 1)/n$ and $\rho_n := (m_4 - 1)/n$. By (7.5), it is easy to see that

$$|a_n - a_n^*| = \frac{|m_3 - m_3^*|}{\sqrt{n}} \le 2\eta_{q3}(n) L_{q3n} \text{ and}$$
$$|\rho_n - \rho_n^*| = \frac{|m_4 - m_4^*|}{n} \le 2\eta_{q3}(n) L_{q3n}. \tag{7.22}$$

From the first three relations in (7.18) it follows that

$$g_1 + zg_1^2 = a_n + \rho_n z + \zeta_{n5}(z), \tag{7.23}$$

where

$$\zeta_{n5}(z) := \frac{\zeta_{n1}(z)z}{n} + \Big(g_2 - \frac{\zeta_{n4}(z)}{n}\Big)z - g_1^3 + 2g_1g_2 - g_3 - (a_n - a_n^*) - (\rho_n - \rho_n^*)z.$$

By (7.14), (7.19), (7.21) and (7.22), we get the following estimate, for $z \in D_1$,

$$\begin{aligned}|\zeta_{n5}(z)| &\le \frac{|\zeta_{n1}(z)z|}{n} + \Big|g_2 - \frac{\zeta_{n4}(z)}{n}\Big||z| + |g_1^3| + 2|g_1g_2| + |g_3| \\ &\quad + |a_n - a_n^*| + |\rho_n - \rho_n^*| \\ &\le 274\eta_{q3}(n)L_{q3n} + 146(r')^3 \le 8\cdot 10^5\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big). \end{aligned}\tag{7.24}$$

Rewrite (7.23) in the form

$$g_1(1 + a_n z) = a_n + \rho_n z + (a_n + \rho_n z)\Big(\frac{1}{1 + g_1 z} - 1\Big) + a_n g_1 z + \frac{\zeta_{n5}(z)}{1 + g_1 z}.$$

Taking into account (7.19), (7.24) and Proposition 3.6 this relation leads us to the bound, for $z \in D_1$,

$$\begin{aligned}&|g_1 - a_n - (\rho_n - a_n^2)z| \\ &\le \frac{|a_n^3 z^2|}{|1 + a_n z|} + \frac{|a_n\rho_n z^2|}{|1 + a_n z|} + \frac{|a_n g_1^2 z^2|}{|1 + a_n z||1 + g_1 z|} \\ &+ \frac{|\rho_n g_1 z^2|}{|1 + a_n z||1 + g_1 z|} + \frac{|\zeta_{n5}(z)|}{|1 + a_n z||1 + g_1 z|} \\ &\le 50L_{3n}^3 + 50L_{3n}L_{4n} + 500L_{3n}(r')^2 + 150L_{4n}r' + 16\cdot 10^5\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big) \\ &\le 18\cdot 10^5\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big).\end{aligned}\tag{7.25}$$

To find the roots w_4 and w_5, we need to solve the equation $w^2 + s_1 w + s_2 = 0$. Using (7.18), we have, for $j = 4, 5$,

$$\begin{aligned}w_j &= \frac{1}{2}\Big(-s_1 + (-1)^j\sqrt{s_1^2 - 4s_2}\Big) \\ &= \frac{1}{2}\Big(z + g_1 + (-1)^j\sqrt{(z + g_1)^2 - 4(1 + (z + g_1)g_1 - g_2)}\Big) \\ &= \frac{1}{2}\Big(z + g_1 + (-1)^j\sqrt{(z - g_1)^2 - 4 - 4(g_1^2 - g_2)}\Big) = \frac{1}{2}\zeta_{n6}(z) + a_n \\ &\quad + \frac{1}{2}\Big(\big(1 + b_n\big)(z - a_n) + (-1)^j\sqrt{\big(1 - b_n\big)^2(z - a_n)^2 - 4\big(1 - d_n\big) + \zeta_{n7}(z)}\Big),\end{aligned}\tag{7.26}$$

where

$$\zeta_{n6}(z) := g_1 - a_n - b_n(z - a_n),$$
$$\zeta_{n7}(z) := -3\zeta_{n6}^2(z) - 2\zeta_{n6}(z)(4a_n + (1+3b_n)(z - a_n)) + 4(g_2 - L_{4n}) - 4b_n(z - a_n)(2a_n + b_n(z - a_n)). \quad (7.27)$$

We choose the branch of the analytic square root according to the condition $\Im w_4(i) \geq 0$. Note that the roots $w_4(z)$ and $w_5(z)$ are continuous functions in D_1.

7.5 Estimates of the functions $\zeta_{n6}(z)$ and $\zeta_{n7}(z)$ on the set D_1

We obtain, by (7.25),

$$|\zeta_{n6}(z)| \leq 18 \cdot 10^5 \Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big) + |a_n b_n| \leq (18 \cdot 10^5 + 1)\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big). \quad (7.28)$$

By (7.14), (7.21) and (7.22), we have

$$|g_2 - L_{4n}| \leq 56\eta_{q3}(n)L_{q3n} + 20(r')^3 \leq 10^5(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}). \quad (7.29)$$

Then, using (7.2), (7.28) and (7.29), we easily deduce from (7.27)

$$|\zeta_{n7}(z)| \leq 3|\zeta_{n6}(z)|^2 + 2|\zeta_{n6}(z)|(4|a_n| + (1+3b_n)(|z| + |a_n|)) + 4|g_2 - L_{4n}| + 4b_n(|z| + |a_n|)(2|a_n| + b_n(|z| + |a_n|)) \leq 3 \cdot 10^7\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big). \quad (7.30)$$

7.6 The roots w_4 and w_5

We saw in Sect. 7.4 that $w_4(z) \neq w_j(z),\ z \in D_1$, for $j = 1, 2, 3$. Returning to (7.26), it follows from (7.30) that $w_4(z) \neq w_5(z)$ for $z \in D_3$, where

$$D_3 := \Big\{z \in \mathbb{C} : 0 < \Im z \leq 3, |\Re z - a_n| \leq \frac{2}{e_n} - h_2\Big\}$$

where $e_n := (1 - b_n)/\sqrt{1 - d_n}$ and $h_2 := c_2^{-1/6}\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big)$.

Since the constant $c_2 > 0$ is sufficiently small, we have, by (7.30), for $z \in D_3$,

$$|\zeta_{n7}(z)|/|((1 - b_n)^2(z - a_n)^2 - 4(1 - d_n)| \leq 4 \cdot 10^7 c_2^{1/6} < 10^{-2}. \quad (7.31)$$

Therefore, using power series expansion of $(1+z)^{1/2}$, $|z|<1$, we obtain, for the same z,

$$\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)+\zeta_{n7}(z)} = \sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)} + \frac{\zeta_{n8}(z)}{\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)}},$$

where $|\zeta_{n8}(z)| \le 4\cdot 10^7\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big)$. By this relation, we see that

$$w_j(z) = a_n + \frac{1}{2}\Big(\big(1+b_n\big)(z-a_n) + (-1)^j\sqrt{\big(1-b_n\big)^2(z-a_n)^2-4\big(1-d_n\big)}\Big) + \frac{1}{2}\zeta_{n6}(z) + \frac{(-1)^j}{2}\frac{\zeta_{n8}(z)}{\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)}}, \quad j=4,5, \tag{7.32}$$

for $z\in D_3$.

Let us show that $S_n(z) = w_4(z)$ for $z\in D_3$. By (7.2) and (7.17), we see that $|w_j(z)|\le 1/6$ for $z\in D_3$. Since, in view of (6.15), $|S_n(z)|\ge 1/3$ for all $z\in\mathbb{C}^+$, we have $S_n(z)=w_4(z)$ or $S_n(z)=w_5(z)$ for $z\in D_3$.

Assume that, for every $z_0\in D_3$, there exists $r_0=r_0(z_0)>0$ such that $S_n(z) = w_j(z)$ for all $z\in D_3\cap\{|z-z_0|<r\}$, where $j=4$ or $j=5$. From this assumption it follows that $S_n(z)=w_j(z)$ for all $z\in D_3$, where $j=4$ or $j=5$. By (3.6), we have $|S_n(2i)|>1$. In addition it follows from (7.17) and (7.26), (7.28), (7.30) that $|w_4(2i)|>1$ and $|w_j(2i)|<1$, $j=1,2,3,5$. Hence in this case $S_n(z)=w_4(z)$ for $z\in D_3$.

If the above assumption is not true, there exists a point $z_0\in D_3$ such that, for any $r_0>0$, there exist points $z'\in D_3$ and $z''\in D_3$ from the disc $|z-z_0|<r_0$ such that $S_n(z')=w_4(z')$ and $S_n(z'')=w_5(z'')$. Let for definiteness $S_n(z_0)=w_5(z_0)$. By assumption there exists a sequence $\{z_k\}_{k=1}^\infty$ such that $z_k\to z_0$ and $S_n(z_k)=w_4(z_k)$. Therefore we have $w_5(z_0)=\lim_{z_k\to z_0} w_4(z_k)$. Rewrite this relation, using (7.32),

$$\begin{aligned} &a_n + \frac{1}{2}\Big(\big(1+b_n\big)(z_0-a_n) - \sqrt{\big(1-b_n\big)^2(z_0-a_n)^2-4\big(1-d_n\big)}\Big) \\ &\quad + \frac{1}{2}\Big(\zeta_{n6}(z_0) - \frac{\zeta_{n8}(z_0)}{\sqrt{(1-b_n)^2(z_0-a_n)^2-4(1-d_n)}}\Big) \\ &= a_n + \frac{1}{2}\Big(\big(1+b_n\big)(z_0-a_n) + \sqrt{\big(1-b_n\big)^2(z_0-a_n)^2-4\big(1-d_n\big)}\Big) \\ &\quad + \frac{1}{2}\lim_{z_k\to z_0}\Big(\zeta_{n6}(z_k) + \frac{\zeta_{n8}(z_k)}{\sqrt{(1-b_n)^2(z_k-a_n)^2-4(1-d_n)}}\Big). \end{aligned} \tag{7.33}$$

From (7.33) we easily conclude

$$c_2^{-1/12}\sqrt{\eta_{q3}(n)L_{q3n}+L_{4n}^{3/2}} \le \left|\sqrt{\big(1-b_n\big)^2(z_0-a_n)^2-4\big(1-d_n\big)}\right|$$
$$\le 4\cdot 10^7\sqrt{\eta_{q3}(n)L_{q3n}+L_{4n}^{3/2}}\Big(c_2^{1/12}$$
$$+\sqrt{\eta_{q3}(n)L_{q3n}+L_{4n}^{3/2}}\Big),$$

which leads to a contradiction for sufficiently small $c_2 > 0$. Hence our assumption holds and $S_n(z) = w_4(z),\ z \in D_3$.

Denote by B_3 the set $\Big[-\frac{2}{e_n}+h_2+a_n,\ \frac{2}{e_n}-h_2+a_n\Big]$.

7.7 Estimate of the integral $\int_{B_3}\big|G_{\mu_n^*}(x+i\varepsilon)-G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon)-\frac{1}{n}(G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon))^3\big|\,dx$ for $0<\varepsilon\le 1$

We obtain an estimate of this integral, using the inequality

$$\int\limits_{B_3}\Big|G_{\mu_n^*}(x+i\varepsilon)-G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon)-\frac{1}{n}(G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon))^3\Big|\,dx$$
$$\le\int\limits_{B_3}|G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon)-G_{\hat{\mu}_n}(x+i\varepsilon)|\,dx$$
$$+\int\limits_{B_3}\Big|G_{\hat{\mu}_n}(x+i\varepsilon)-G_{\mu_n^*}(x+i\varepsilon)-\frac{1}{n}(G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon))^3\Big|\,dx. \tag{7.34}$$

Therefore we need to evaluate the functions $G_{\mu_{a_n,b_n,d_n}}(z)-G_{\hat{\mu}_n}(z)$ and $G_{\hat{\mu}_n}(z)-G_{\mu_n^*}(z)-\frac{1}{n}(G_{\mu_{a_n,b_n,d_n}}(z))^3$ for $z\in D_3$.

For $z\in D_3$, using the formula (7.26) with $j=4$ for $S_n(z)$, we write

$$S_{n2}(z)S_n(z)\Big(\frac{1}{S_n(z)}-\frac{1}{S_{n2}(z)}\Big)=S_{n2}(z)-S_n(z)=-\frac{1}{2}\zeta_{n6}(z)$$
$$-\frac{1}{2}\frac{\zeta_{n7}(z)}{\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)}+\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)+\zeta_{n7}(z)}}. \tag{7.35}$$

Using (7.31) we get, for $z\in D_3$,

$$\left|\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)}+\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)+\zeta_{n7}(z)}\right|$$
$$=\left|\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)}\right|\Big|1$$
$$+\sqrt{1+\zeta_{n7}(z)/((1-b_n)^2(z-a_n)^2-4(1-d_n))}\Big|$$
$$\ge|\sqrt{(1-b_n)^2(z-a_n)^2-4(1-d_n)}|. \tag{7.36}$$

It is easy to see that the bound $|S_n(z)| \ge 1/3$, $z \in \mathbb{C}^+$, holds for $S_{n2}(z)$ as well. Therefore in the same way as in the proof of (6.39) and (6.40) we conclude from (7.28), (7.30) and (7.35) that

$$\begin{aligned}\int\limits_{B_3} \Big|G_{\hat{\mu}_n}(x+i\varepsilon) - G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon)\Big|\,dx &= \int\limits_{B_3} \Big|\frac{1}{S_n(x+i\varepsilon)} - \frac{1}{S_{n2}(x+i\varepsilon)}\Big|\,dx \\ &\le \frac{9}{2}\int\limits_{B_2} \Big(|\zeta_{n6}(x+i\varepsilon)| + \frac{|\zeta_{n7}(x+i\varepsilon)|}{|\sqrt{(1-b_n)^2(x-a_n+i\varepsilon)^2 - 4(1-d_n)}|}\Big)\,dx \\ &\le c\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big). \end{aligned} \tag{7.37}$$

Now we deduce from (6.11) with the probability measure μ^*, involving ε_n introduced in (7.1), the relation

$$G_{\mu_n^*}(z) - G_{\hat{\mu}_n}(z) - \frac{1}{nS_n^3(z)} = \frac{r_{n1}(z)}{nS_n^3(z)}, \quad z \in \mathbb{C}^+, \tag{7.38}$$

where the function $r_{n1}(z)$ is defined in (6.14). Since (6.20) holds for $r_{n1}(z)$, we see that

$$\frac{|r_{n1}(z)|}{n|S_n^3(z)|} \le 1458\,\frac{L_{3n}}{n}, \qquad z \in D_3. \tag{7.39}$$

Since, for the same z,

$$\begin{aligned}\Big|\frac{1}{S_n^3(z)} - \frac{1}{S_{n2}^3(z)}\Big| &\le 2\Big|\frac{1}{S_n(z)} - \frac{1}{S_{n2}(z)}\Big|\Big(\frac{1}{|S_n(z)|^2} + \frac{1}{|S_{n2}(z)|^2}\Big) \\ &\le 36\Big|\frac{1}{S_n(z)} - \frac{1}{S_{n2}(z)}\Big|, \end{aligned} \tag{7.40}$$

we obtain, using (7.37)–(7.40) and Proposition 3.6, that, for $0 < \varepsilon \le 1$,

$$\begin{aligned}&\int\limits_{B_3} \Big|G_{\mu_n^*}(x+i\varepsilon) - G_{\hat{\mu}_n}(x+i\varepsilon) - \frac{1}{n}(G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon))^3\Big|\,dx \\ &\quad\le \int\limits_{B_3} \Big|G_{\mu_n^*}(x+i\varepsilon) - G_{\hat{\mu}_n}(x+i\varepsilon) - \frac{1}{n}(G_{\hat{\mu}_n}(x+i\varepsilon))^3\Big|\,dx \\ &\quad\quad + \frac{1}{n}\int\limits_{B_3} \Big|(G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon))^3 - (G_{\hat{\mu}_n}(x+i\varepsilon))^3\Big|\,dx \\ &\quad\le c\Big(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\Big). \end{aligned} \tag{7.41}$$

From (7.34), (7.37) and (7.41) we finally get, for $0 < \varepsilon \le 1$,

$$\int_{B_3} \left| G_{\mu_n^*}(x+i\varepsilon) - G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon) - \frac{1}{n}(G_{\mu_{a_n,b_n,d_n}}(x+i\varepsilon))^3 \right| dx$$
$$\le c\left(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\right). \tag{7.42}$$

7.8 Application of the Stieltjes–Perron inversion formula

Using (7.9), we have the relation

$$\int_{B_3} p_{\mu_{a_n,b_n,d_n}}(x)\,dx \ge 1 - h_2^{3/2}. \tag{7.43}$$

where $p_{\mu_{a_n,b_n,d_n}}(x)$ denotes the density $p_{\mu_{a_n,b_n,d_n}}(x)$ of the measure μ_{a_n,b_n,d_n} It is not difficult to verify that

$$(G_{\mu_{a_n,b_n,d_n}}(z))^3 = \int_{\mathbb{R}} \frac{\varsigma_{n1}(dx)}{z-x} = \int_{\mathbb{R}} \frac{p_{\varsigma_{n1}}(x)\,dx}{z-x}, \quad z \in \mathbb{C}^+,$$

where

$$p_{\varsigma_{n1}}(x) := \frac{1}{8\pi}\sqrt{(4(1-d_n)-(1-b_n)^2(x-a_n)^2)_+}$$
$$\times \frac{3((1+b_n)x+(1-b_n)a_n)^2 + (1-b_n)^2(x-a_n)^2 - 4(1-d_n)}{(b_n x^2 + (1-b_n)a_n x + 1 - d_n)^3}$$

for $x \in B_2$ and $p_{\varsigma_{n1}}(x) = 0$ for $x \notin B_2$. Therefore we easily deduce the obvious upper bounds

$$\left|\int_{B_3} p_{\varsigma_{n1}}(x)\,dx\right| \le ch_2^{3/2}, \quad \int_{\mathbb{R}\setminus B_3} |p_{\varsigma_{n1}}(x)|\,dx \le ch_2^{3/2} \text{ and } \Delta(\varsigma_{n1}, \varsigma_n)$$
$$\le c\left(|a_n| + L_{4n}\right). \tag{7.44}$$

From (7.42)–(7.44) and the Stieltjes–Perron inversion formula we conclude that

$$\mu_n^*(B_3) \ge 1 - c\left(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\right). \tag{7.45}$$

We finally conclude from (7.42), (7.44), (7.45) and the Stieltjes–Perron inversion formula that

$$\Delta(\mu_n^*, \kappa_n) \le c\left(\eta_{q3}(n)L_{q3n} + L_{4n}^{3/2}\right). \tag{7.46}$$

7.9 Completion of the proof of Theorem 2.6

The statement of the theorem follows immediately from (7.7), (7.8), (7.10), (7.46) and Proposition 3.6. □

Proof of Corollary 2.7 Recalling the definition of the density $p_{\mu_{a_n,b_n,d_n}}(x)$ of the measure μ_{a_n,b_n,d_n} for $n \ge c_2^{-1}m_4$, we see that

$$\begin{aligned} &p_{\mu_{a_n,b_n,d_n}}(x+a_n)\\ &\quad= \frac{1}{2\pi}\sqrt{(4(1-d_n)-(1-b_n)^2x^2)_+}(1+d_n-b_n-a_nx-(b_n-a_n^2)(x^2-1))\\ &\quad\quad+c\theta(L_{4n}+a_n^2)^{3/2}, \qquad x\in\mathbb{R}. \end{aligned}$$

In addition we have, for $x \in \mathbb{R}$,

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{x}\sqrt{(4(1-d_n)-(1-b_n)^2u^2)_+}\,du\\ &\quad= (1-d_n+b_n)\mu_w((-\infty,x))+\Big(\frac{1}{2}d_n-b_n\Big)x\frac{1}{2\pi}\sqrt{(4-x^2)_+}+c\,\theta L_{4n}^{3/2} \end{aligned}$$

and, for $x \in (-l_n, l_n)$, where $l_n := \max\{2, 2/e_n\}$,

$$\begin{aligned} &\Big|\sqrt{(4(1-d_n)-(1-b_n)^2x^2)_+}-\sqrt{(4-x^2)_+}\Big|\\ &\quad\le \frac{cL_{4n}}{\sqrt{(4(1-d_n)-(1-b_n)^2x^2)_+}+\sqrt{(4-x^2)_+}}. \end{aligned}$$

Using these formulae and the following obvious relations

$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{3}(4-x^2)^{3/2} \quad\text{and}\quad \int (x^2-1)\sqrt{4-x^2}\,dx = -\frac{1}{4}x(4-x^2)^{3/2},$$

we obtain from (2.26), using some simple calculations, the representation (2.27). □

Appendix 1: Proof of Theorem 2.1

In this Appendix we keep the notations of Sect. 6.

Passage to measures with bounded supports

Let $n \in \mathcal{N}$. Let $\varepsilon_n \in (0, 10^{-1/2}]$ be a point at which the infimum of the function $g_{qn1}(\varepsilon)$ from (2.16) is attained. This means that

$$\eta_{q1}(n) := \varepsilon_n^{3-q_1} + \frac{\rho_{q_1}(\mu, \varepsilon_n\sqrt{n})}{\beta_{q_1}}\varepsilon_n^{-q_1}.$$

Using this parameter ε_n, we define free random variables $\tilde{X}, \tilde{X}_1, \tilde{X}_2, \dots$ and $X^*, X_1^*, X_2^*, \dots$ in the same way as in Sect. 5. We define probability measures $\tilde{\mu}_n, \tilde{\mu}_w = \tilde{\mu}_{0,0,0}, \mu^*, \mu_n^*$ in the same way as well.

Without loss of generality we assume that

$$\eta_{q1}(n)L_{q_1 n} + 1/n < c_3, \tag{8.1}$$

where $c_3 > 0$ is a sufficiently small absolute constant.

Using (8.1) we note that

$$|A_n| \le \varepsilon_n^{-(q_1-1)} n^{-(q_1-1)/2}\rho_{q_1}(\mu, \varepsilon_n\sqrt{n}) \le \frac{1}{\sqrt{n}}\eta_{q1}(n)L_{q_1 n} \tag{8.2}$$

and

$$0 \le \frac{1}{C_n} - 1 \le 2\Big(\rho_2(\mu, \varepsilon_n\sqrt{n}) + A_n^2\Big) \le 3\eta_{q1}(n)L_{q_1 n}. \tag{8.3}$$

By (8.1)–(8.3), we obtain that (6.4) holds and the support of μ^* is contained in $[-\frac{1}{3}\sqrt{n}, \frac{1}{3}\sqrt{n}]$. By (8.1)–(8.3), we easily deduce as well that

$$\beta_3^* \le C_n^{-3}\tilde{\beta}_3 + \frac{4}{\sqrt{n}}\eta_{q1}(n)L_{q_1 n}. \tag{8.4}$$

By the triangle inequality, we have

$$\Delta(\mu_n, \mu_w) \le \Delta(\mu_n, \tilde{\mu}_n) + \Delta(\tilde{\mu}_n, \tilde{\mu}_w) + \Delta(\tilde{\mu}_w, \mu_w). \tag{8.5}$$

Furthermore, we have the following inequalities

$$\begin{aligned} \Delta(\mu_n, \tilde{\mu}_n) &\le \varepsilon_n^{-q_1} n^{-(q_1-2)/2}\rho_{q_1}(\mu, \varepsilon_n\sqrt{n}) \le \eta_{q1}(n)L_{q_1 n}, \\ \Delta(\tilde{\mu}_w, \mu_w) &\le c\varepsilon_n^{-(q_1-1)} n^{-(q_1-2)/2}\rho_{q_1}(\mu, \varepsilon_n\sqrt{n}) \le c\eta_{q1}(n)L_{q_1 n}. \end{aligned} \tag{8.6}$$

Our next aim is to estimate $\Delta(\tilde{\mu}_n, \tilde{\mu}_w) = \Delta(\mu_n^*, \mu_w)$.

As in Sect. 6, let $Z(z) \in \mathcal{F}$ be the solution of the equation (3.11) with $\mu = \mu^*$. Denote $S_n(z) := Z(\sqrt{n}z)/\sqrt{n}$.

The functional equation for $S_n(z)$

Using the formula

$$Z(z)G_{\mu^*}(Z(z)) = 1 + \frac{1}{Z^2(z)} + \frac{1}{Z^2(z)} \int_{\mathbb{R}} \frac{u^3\,\mu^*(du)}{Z(z)-u}, \tag{8.7}$$

and the Eq. (3.11) with $\mu = \mu^*$ we arrive at the following functional equation for $S_n(z)$

$$S_n^3(z) - zS_n^2(z) + (1 + r_n(z))S_n(z) - (1 + r_n(z))\frac{z}{n} = 0, \quad z \in \mathbb{C}^+, \tag{8.8}$$

where $r_n(z) := \int_{\mathbb{R}} \frac{u^3\,\mu^*(du)}{Z(\sqrt{n}z)-u}$. From (8.1) and (6.4) we deduce, using Lemma 3.4, that (6.15) holds. Therefore, in view of (8.4), we obtain

$$|r_n(z)| \le \frac{52\beta_3^*}{\sqrt{n}} \le 53\,\frac{\tilde{\beta}_3}{\sqrt{n}} + \frac{208}{n}\eta_{q1}(n)L_{q_1 n} \le 54\,\eta_{q1}(n)L_{q_1 n} < \frac{1}{10} \tag{8.9}$$

for $z \in \mathbb{C}^+$. Note that we obtain the functional equation (8.8) from (6.17) replacing $\varepsilon_{n1}(z)$ by $r_n(z)$ and $\varepsilon_{n2}(z)$ by $-(1 + r_n(z))z/n$.

The roots of the functional equation for $S_n(z)$

For every fixed $z \in \mathbb{C}^+$ consider the cubic equation

$$P(z, w) := w^3 - zw^2 + (1 + r_n(z))w - (1 + r_n(z))\frac{z}{n} = 0.$$

As in Sect. 6 denote the roots of this equation by $w_j = w_j(z),\ j = 1, 2, 3$. Repeating the arguments of Sect. 6.4 we prove that

$$w_1 = \frac{z}{n} + \hat{r}_n(z), \quad \text{where} \quad |\hat{r}_n(z)| < \frac{10^3}{n^2}, \tag{8.10}$$

and $|w_j - z/n| \ge 10^3/n^2,\ j = 2, 3$, for $z \in D_1$. Hence $w_1 \ne w_j$ for $j = 2, 3$ and $z \in D_1$.

As in Sect. 6.5 we obtain that $w_2 \ne w_3$ for $z \in D_4 := \{z \in \mathbb{C} : 0 < \Im z \le 3, |\Re z| \le 2 - h_3\}$, where $h_3 := c_3^{-1/6}(\eta_{q1}(n)L_{q_1 n} + 1/n)$. Hence the roots $w_1(z), w_2(z)$ and $w_3(z)$ are distinct for $z \in D_4$. Moreover $w_1(z)$ satisfies, by (8.1) and (8.10), the inequality

$$|w_1(z)| \le 6/n, \quad z \in D_4. \tag{8.11}$$

Using the arguments of Sect. 6.5 we deduce the formula (6.28) for $z \in D_4$ where $g(z) := (z - w_1)^2 - 4 - 4r_n(z) + 4w_1(z - w_1) \neq 0$, $z \in D_4$. Then we rewrite the formula (6.28) as follows

$$w_j := \frac{1}{2}\Big(z + (-1)^{j-1}\sqrt{z^2 - 4 + \tilde{r}_n(z)}\Big) - \frac{1}{2}w_1(z), \quad j = 1, 2, \tag{8.12}$$

where $\tilde{r}_n(z) := 2zw_1(z) - 3w_1^2(z) - 4r_n(z)$. By (8.9) and (8.11), this function admits the bound, for $z \in D_4$,

$$|\tilde{r}_n(z)| \le 10|w_1(z)| + 3|w_1(z)|^2 + 4|r_n(z)| \le 280\Big(\eta_{q1}(n)L_{q_1 n} + 1/n\Big). \tag{8.13}$$

In the same way as in Sect. 6.5 we obtain that $S_n(z) = w_3(z)$, $z \in D_4$. Denote $B_4 := [-2 + h_3, 2 - h_3]$.

Estimate of the integral $\int_{B_4} |G_{\mu_w}(x + i\varepsilon) - G_{\mu_n^*}(x + i\varepsilon)|\, dx$ for $0 < \varepsilon \le 1$

We obtain an estimate of this integral, using the inequality

$$\int_{B_4} |G_{\mu_w}(x + i\varepsilon) - G_{\mu_n^*}(x + i\varepsilon)|\, dx \le \int_{B_4} |G_{\mu_w}(x + i\varepsilon) - G_{\hat{\mu}_n}(x + i\varepsilon)|\, dx + \int_{B_4} |G_{\hat{\mu}_n}(x + i\varepsilon) - G_{\mu_n^*}(x + i\varepsilon)|\, dx. \tag{8.14}$$

Evaluating the function $G_{\mu_w}(z) - G_{\hat{\mu}_n}(z)$ for $z \in D_4$ in the same way as in Sect. 6.6, we arrive at the bound

$$\int_{B_4} |G_{\mu_w}(x + i\varepsilon) - G_{\hat{\mu}_n}(x + i\varepsilon)|\, dx \le c\Big(\eta_{q1}(n)L_{q_1 n} + 1/n\Big). \tag{8.15}$$

Now we conclude from (8.7) that

$$G_{\mu_n^*}(z) - G_{\hat{\mu}_n}(z) = \frac{1 + r_n(z)}{nS_n^3(z)}, \quad z \in \mathbb{C}^+. \tag{8.16}$$

Since $|S_n(z)| \ge 1/3$ for $z \in \mathbb{C}^+$, we see from (8.9) and (8.16) that

$$\int_{B_4} |G_{\mu_n^*}(x + i\varepsilon) - G_{\hat{\mu}_n}(x + i\varepsilon)|\, dx \le \int_{B_4} \frac{1 + |r_n(x + i\varepsilon)|}{n|S_n(x + i\varepsilon)|^3}\, dx \le \frac{120}{n}, \quad \varepsilon \in (0, 1]. \tag{8.17}$$

From (8.14), (8.15) and (8.17) we finally obtain

$$\int_{B_4} |G_{\mu_w}(x+i\varepsilon) - G_{\mu_n^*}(x+i\varepsilon)|\,dx \le c\Big(\eta_{q1}(n)L_{q_1 n} + 1/n\Big), \quad \varepsilon \in (0,1]. \tag{8.18}$$

Completion of the proof of Theorem 2.1

Note that

$$\int_{B_4} p_{\mu_w}(x)\,dx \ge 1 - h_3^{3/2}. \tag{8.19}$$

From (8.18) and (8.19) we deduce, using the Stieltjes–Perron inversion formula,

$$\mu_n^*(B_4) \ge 1 - c\Big(\eta_{q1}(n)L_{q_1 n} + 1/n\Big). \tag{8.20}$$

Finally we deduce from (8.18)– (8.20) and the Stieltjes–Perron inversion formula that

$$\Delta(\mu_n^*, \mu_w) \le c\Big(\eta_{q1}(n)L_{q_1 n} + 1/n\Big). \tag{8.21}$$

The statement of the theorem follows immediately from (8.5), (8.6) and (8.21). □

Proof of Corollary 2.2 The inequality (2.18) follows immediately from (2.17) and the Lyapunov inequality $1 = m_2^{1/2} \le \beta_q^{1/q}$ for $q \ge 2$. □

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