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Invariance principle for the random conductance model

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Abstract We study a continuous time random walk X in an environment of i.i.d. random conductances $\mu_e \in [0, \infty)$ in $\mathbb{Z}^d$. We assume that $\mathbb{P}(\mu_e > 0) > p_c$, so that the bonds with strictly positive conductances percolate, but make no other assumptions on the law of the μ_e. We prove a quenched invariance principle for X, and obtain Green's functions bounds and an elliptic Harnack inequality.

Keywords Random conductance model · Heat kernel · Invariance principle · Ergodic · Corrector

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1 Introduction

We consider the Euclidean lattice $\mathbb{Z}^d$ with $d \geq 2$. Let E_d be the set of non oriented nearest neighbour bonds: $E_d = \{e = \{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$. The random environment is given by i.i.d. random variables $(\mu_e, e \in E_d)$ on $[0, \infty)$, defined on a probability space $(\Omega, \mathbb{P})$. We write $\mu_{xy} = \mu_{\{x,y\}} = \mu_{yx}$, and $\mu_{xy} = 0$ if $\{x, y\} \notin E_d$, and set

$$\mu_x = \sum_y \mu_{xy}, \qquad P(x, y) = \frac{\mu_{xy}}{\mu_x}. \tag{1.1}$$

We will study continuous time random walks on $\mathbb{Z}^d$ which jump according to the transitions $P(x, y)$. There are two natural choices of this. The first $X = (X_t, t \geq 0, P_\omega^x, x \in \mathbb{Z}^d)$ (the *constant speed random walk or CSRW*) waits at x for an exponential time with mean 1, while the second, $Y = (Y_t, t \geq 0, P_\omega^x, x \in \mathbb{Z}^d)$ (the *variable speed random walk or VSRW*) waits at x for an exponential time with mean $1/\mu_x$. Write $\mathcal{L}_C$ and $\mathcal{L}_V$ for their generators, given by:

$$\begin{aligned}
\mathcal{L}_C f(x) &= \mu_x^{-1} \sum_y \mu_{xy}(f(y) - f(x)), \\
\mathcal{L}_V f(x) &= \sum_y \mu_{xy}(f(y) - f(x)).
\end{aligned} \tag{1.2}$$

If $\mu_x = 0$ we write $\mathcal{L}_C f(x) = \mathcal{L}_V f(x) = 0$.

If $\mu_e = 0$ then X never jumps across e. So if $p_+ = \mathbb{P}(\mu_e > 0)$ is less than $p_c = p_c(E_d)$, the critical probability for bond percolation on $\mathbb{Z}^d$, then X and Y are $\mathbb{P}$-a.s. confined to a finite set. Thus it is very natural to assume that

$$\mathbb{P}(\mu_e > 0) > p_c. \tag{1.3}$$

We define $\mathcal{O}_1 = \{e : \mu_e > 0\}$, and write $\mathcal{C}_1 = \mathcal{C}_\infty(\mathcal{O}_1)$ for the $\mathbb{P}$ almost surely unique infinite connected supercritical cluster with open edges $\mathcal{O}_1$. Let

$$\mathbb{P}_1(\cdot) = \mathbb{P}(\cdot \,|0 \in \mathcal{C}_1). \tag{1.4}$$

This model, of a reversible (or symmetric) random walk in a random environment, is known in the literature as the *random conductance model* or RCM. We are interested in the $\mathbb{P}_1$ almost sure or quenched long range behavior, and in particular in obtaining a quenched functional central limit theorem (QFCLT) or invariance principle for the processes X and Y starting at 0. Our first main result is the following QFCLT. Let

$$X_t^{(\varepsilon)} = \varepsilon X_{t/\varepsilon^2}, \quad Y_t^{(\varepsilon)} = \varepsilon Y_{t/\varepsilon^2}, \quad t \geq 0; \tag{1.5}$$

more generally, given any process $(V_t, t \geq 0)$ we define $V^{(\varepsilon)}$ in an analogous fashion.

Theorem 1.1 *Let $d \geq 2$ and suppose that $(\mu_e, e \in E_d)$ are i.i.d., $\mu_e \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(\mu_e > 0) > p_c$.*

(a) *Let Y be the VSRW with $Y_0 = 0$. Then, $\mathbb{P}_1$-a.s. $Y^{(\varepsilon)}$ converges (under P^0_ω) in law to a Brownian motion on $\mathbb{R}^d$ with covariance matrix $\sigma_V^2 I$, where $\sigma_V > 0$ is non-random.*

(b) *Let X be the CSRW with $X_0 = 0$. Then, $\mathbb{P}_1$-a.s. $X^{(\varepsilon)}$ converges (under P^0_ω) in law to a Brownian motion on $\mathbb{R}^d$ with covariance matrix $\sigma_C^2 I$, where*

$$\sigma_C^2 = \begin{cases} \sigma_V^2/(\mathbb{E}_1\mu_0), & \text{if } \mathbb{E}\mu_e < \infty, \\ 0, & \text{if } \mathbb{E}\mu_e = \infty. \end{cases}$$

If $d \geq 3$ we also have the following bounds on the Green's function of Y, defined by:

$$g^Y(x, y) = E^x_\omega \int_0^\infty 1_{(Y_s = y)} ds. \tag{1.6}$$

(We remark that g^Y is also the Green's function for X.)

Theorem 1.2 *Let $d \geq 3$.*

(a) *There exist constants $\delta, c_1, \ldots c_4$, depending only on d and the law of μ_e, and r.v. $R_x, x \in \mathbb{Z}^d$ satisfying*

$$\mathbb{P}(R_x \geq n | x \in \mathcal{C}_1) \leq c_1 e^{-c_2 n^\delta}, \tag{1.7}$$

such that

$$\frac{c_3}{|x - y|^{d-2}} \leq g^Y(x, y) \leq \frac{c_4}{|x - y|^{d-2}} \qquad \text{if } |x - y| \geq R_x \wedge R_y,\ x, y \in \mathcal{C}_1. \tag{1.8}$$

(b) *There exists a constant $C = \Gamma(d/2 - 1)(2\pi^{d/2}\sigma_V^2 \mathbb{P}(0 \in \mathcal{C}_1))^{-1}$ such that for any $\varepsilon > 0$ and $x \in \mathbb{Z}^d$ there exists a $\mathbb{P}_1$-a.s. finite r.v. $N_{\varepsilon,x}$ such that on $\{x \in \mathcal{C}_1\}$,*

$$\frac{(1 - \varepsilon)C}{|x - y|^{d-2}} \leq g^Y(x, y) \leq \frac{(1 + \varepsilon)C}{|x - y|^{d-2}} \quad \text{for } |x - y| > N_{\varepsilon,x}(\omega),\ y \in \mathcal{C}_1. \tag{1.9}$$

(c) *For each $x \in \mathbb{Z}^d$ we have $\mathbb{P}$–a.s. on $\{x \in \mathcal{C}_1\}$,*

$$\lim_{|y-x| \to \infty, y \in \mathcal{C}_1} |y - x|^{2-d} g^Y(x, y) = C. \tag{1.10}$$

(d) *For each $x \in \mathbb{Z}^d$ we have*

$$\lim_{|y-x|\to\infty} |y-x|^{2-d} \mathbb{E}(g^Y(x,y)|x, y \in \mathcal{C}_1) = C. \tag{1.11}$$

The random conductance model has been studied by a number of different authors under various restrictions on the law of μ_e. When $\mathbb{E}\mu_e < \infty$ a weak FCLT was obtained by [21] for general ergodic environments. To explain the difference between this and the QFCLT, let $T > 0$ and F be a bounded continuous function on the Skorohod space $D_T = D([0,T], \mathbb{R}^d)$. For $\omega \in \{0 \in \mathcal{C}_\infty(\mathcal{O}_1)\}$ set $\Psi_\varepsilon = E^0_\omega F(Y^{(\varepsilon)})$, and let $\Psi_0 = E_{BM} F(\sigma_V W)$, where (W, P_{BM}) is a Brownian motion started at 0. Then the weak FCLT states that $\Psi_\varepsilon \to \Psi_0$ in $\mathbb{P}_1$-probability, while the QFCLT states that this convergence occurs $\mathbb{P}_1$-a.s.

Quenched results have already been derived for the RCM in the following settings:

1. If $\mu_e \in \{0, 1\}$ then this problem reduces to that of a random walk on (supercritical) percolation clusters—see [2,31] for heat kernel bounds, [11,32,39] for a QFCLT, and [5] for a local limit theorem.
2. In the uniformly elliptic case where

$$\mathbb{P}(c^{-1} \leq \mu_e \leq c) = 1$$

   for some $c \geq 1$, heat kernel bounds follow from the results in [20], and a QFCLT is proved in [39] for i.i.d. $(\mu_e, e \in E_d)$. (See also [4] for an extension to ergodic environments).
3. The case with conductances bounded from above

$$\mathbb{P}(0 < \mu_e \leq 1) = 1 - \mathbb{P}(\mu_e = 0) > p_c,$$

   is treated in [12,14,33]. (The papers [12,14] consider a discrete time random walk.) A QFCLT for the CSRW is proved in [14,33], with a strictly positive diffusion constant σ_C^2. Further [12] shows that Gaussian upper heat kernel bounds do not hold in general in this case for $d \geq 5$ (see also [13] for $d = 4$).
4. The case when μ_e is bounded from below:

$$\mathbb{P}(1 \leq \mu_e < \infty) = 1,$$

   is studied in [4], and quenched heat kernels estimates for the VSRW, and a QFCLT for both VSRW and CSRW are derived in the i.i.d. setting.

Diffusions in random environment with a generator L^ω in divergence form:

$$L^\omega f(x) = \sum_{i,j=1}^{d} \partial_{x_i} \big(a_{i,j}(\omega, x) \partial_{x_j} f(x)\big), \quad f \in C^2(\mathbb{R}^d),$$

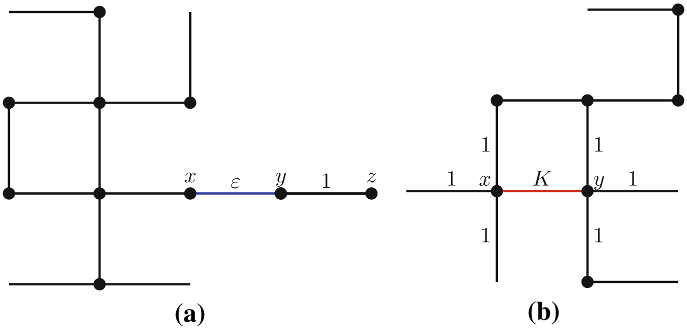

Fig. 1 **a** Trap of the first kind. **b** Trap of the second kind

where the matrix $a_{i,j}(\omega, x) = a_{i,j}(\tau_x\omega, 0)$ and τ_x, $x \in \mathbb{R}^d$, is an ergodic shift, have similar behavior to the random conductance model. In particular, assuming uniform ellipticity:

$$\sum_{i,j=1}^{d} \xi_i a_{i,j}(\omega, x)\xi_j \geq \varepsilon \sum_i \xi_i^2,$$

and bounded coefficients

$$|a_{i,j}(\omega, x)| \leq C,$$

Gaussian estimates for the heat kernel are well known and QFCLT holds—see [35]. For unbounded coefficients under suitable higher moments: $\mathbb{E}|a_{i,j}(x)|^p < \infty$, for some $p > d$, a QFCLT has been shown using analytical tools in [24]. Note that this result is quite different from ours since it holds for every ergodic environment, and it is an interesting question whether we could also show QFCLT for the unbounded general ergodic random conductance model under moment conditions.

The main difficulty in studying the general RCM is the possibility of 'traps', which may be due to either edges with small positive conductance, or very large conductance.

For the first kind of trap, consider points x, y, z, with $0 < \varepsilon = \mu_{xy} \ll 1$, and $\mu_{yz} = O(1)$, and such that the only connection from $\{y, z\}$ to the rest of $\mathcal{C}_1$ is through x. Starting at y, both CSRW and VSRW will be trapped for a time $O(\varepsilon^{-1})$ before they hit x and move on into the rest of $\mathcal{C}_1$. However, if the processes start outside the trap $\{y, z\}$ then they are unlikely to enter it. (Except when $d = 2$ and very long time scales are considered.)

The second kind of trap is associated with points $x, y \in \mathcal{C}_1$ with $\mu_{xy} = K \gg 1$, and with $\mu_e = O(1)$ for all other bonds e with an endpoint in $\{x, y\}$. In this case the CSRW will be trapped in the set $\{x, y\}$ for time $O(K)$, but the VSRW will not be trapped. (This explains why the VSRW has in general better properties than the CSRW.)

It should be noted that, one cannot expect a FCLT for unbounded conductances for a general ergodic environment: see Remark 6.6 of [4] for an example of a VSRW which explodes in finite time.

While there is not a great difference between the CSRW and VSRW in case of bounded conductances, in the situation when $\mathbb{E}\mu_e = \infty$, the VSRW and CRSW do have quite different long time behaviour. In particular due to the traps of the second kind the limiting variance of the CSRW vanishes and it is therefore natural to ask further about the behaviour of the CSRW. If the tail distribution $\mathbb{P}(\mu_e > t) \sim t^{-\alpha}$ then [3] show that $\varepsilon^\alpha X_{t/\varepsilon^2}$ converges to the 'fractional kinetic' motion with parameter α (see [8–10,16] for a connection with aging phenomena).

Our proof of the QFCLT is in essence similar to the one given in [14] or [33], however the presence of unbounded conductances introduces some new technical difficulties. Instead of the original VSRW Y on the cluster $\mathcal{C}_\infty(\mathcal{O}_1)$, we consider for fixed $K > 1$ its trace Z^K on the smaller cluster $\mathcal{C}_\infty(\mathcal{O}_2)$ resulting from the deletion of "bad" conductances: ones which are either too small ($e \in E_d$ with $\mu_e < 1/K$), too large ($e \in E_d$ with $\mu_e > K$), or adjacent to the previous ones. The process Z^K is then the time change of Y onto the set $\mathcal{C}_\infty(\mathcal{O}_2)$. Since we need the jump rate of Z^K to be bounded, it is necessary to delete not just the bonds $e = \{x, y\}$ with $\mu_e > K$, but also all bonds with endpoints x or y.

The process Z^K is again a symmetric process but with conductances which are bounded from above, and also from below on any bond which is in $\mathcal{O}_2$. However it can jump across holes of deleted connections. Using percolation estimates, the size of these "holes" can be well controlled. This allows us to show that both process Y and Z^K are close to each other for large enough K. Moreover using a method of Grigoryan (see [17,25,26]) we can derive Gaussian heat kernel estimates for Z^K.

We obtain the QFCLT for the process Z^K using the well known Kipnis-Varadhan technique based on the environment viewed from the particle, and the method of the 'corrector' due to Kozlov [29]. We write

$$Z_t^K(\omega) = M_t(\omega) + \chi\left(\omega, Z_t^K(\omega)\right)$$

where M_t is a martingale and $\chi : \Omega \times \mathcal{C}_\infty(\mathcal{O}_2) \longrightarrow \mathbb{R}^d$ is the corrector. The QFCLT for the martingale part $M^{(\varepsilon)}$ is standard, while we use our heat kernel estimate to control the corrector: for $\mathbb{P}_1$ almost all ω

$$\lim_{\varepsilon \to 0} \varepsilon \chi\left(\omega, Z^K_{t/\varepsilon^2}\right) = 0 \quad \text{in } P^0_\omega\text{-probability.}$$

The (quenched) heat kernel estimates also yield the tightness of both $Z^{K,(\varepsilon)}$ and $Y^{(\varepsilon)}$.

When $E\mu_e = \infty$ the existence of the corrector for the process Y does not follow from a simple projection argument. In [4], this problem was solved by first constructing the corrector for the time discretized process. This agrees with the corrector of the time continuous process—see [4, Remark 5.15]. In this paper we follow [14,33], and construct the corrector via the projection argument for the trace process Z^K, which has bounded conductances. Once we have the corrector for Z^K, we can obtain the corrector for the original process Y using harmonic extension—see Remark 7.4.

Our paper is organized as follows: in Sect. 2, we construct the different percolation clusters, which are not necessarily of i.i.d. type, but with finite range dependence and control their shape and size using the Liggett–Schonmann–Stacey coupling to i.i.d. percolation, cf. [30]. The upper bound estimates play a crucial role in Sect. 4, for the time changed process introduced in Sect. 3. The proof of the heat kernel upper bound first follows the argument of [2] in its derivation of on-diagonal bounds, though some care is needed in order to control the long range jumps of Z^K. The off-diagonal estimate is based on an argument introduced by Grigoryan [26] for diffusions on manifolds, and adapted to graphs in [17,25]. Although not explicitly needed for our QFCLT, we also derive the corresponding lower bounds for the heat kernel of Z^K using a weighted Poincaré inequality, and the method of Fabes and Stroock [23]. Of course due to irregularity of the environment one cannot expect uniform estimates, but Theorem 4.10 below summarizes our heat kernels bounds, and shows that whenever either time or distance is large enough, the standard Gaussian estimates are available.

Equipped with these heat kernel estimates, the QFCLT follows in Sect. 5 using the corrector technique as in [11,14,32,33], while in Sect. 6 the invariance principle for the original processes Y and X are derived via coupling to Z^K and time change.

Finally in Sect. 7 we use the heat kernel bounds to obtain a parabolic Harnack inequality, local limit theorem and Green's function bounds for Z^K. Using the fact that harmonic functions for Y can be obtained from harmonic functions for Z by 'filling in the holes', we obtain an elliptic Harnack inequality for Y, and prove Theorem 1.2.

We write c, c', c_i, C_i to denote constants which will depend on the dimension d, the law of (μ_e), and the large constant K chosen in Sect. 2—which can be chosen so that it just depends on d and the law of μ_e.

2 Percolation estimates

Let E_d be the set of edges of $\mathbb{Z}^d$. We write $x \sim y$ if $\{x, y\} \in E_d$. Given $\mathcal{O} \subset E_d$, let $\mathcal{C}_\infty(\mathcal{O})$ denote the infinite connected component of the graph $(\mathbb{Z}^d, \mathcal{O})$, provided it exists and is unique. (Otherwise we take $\mathcal{C}_\infty(\mathcal{O}) = \emptyset$.)

Now let $\mu_e = \mu_{xy}$, $e = \{x, y\} \in E_d$, be i.i.d. with $\mu_e \in [0, \infty)$. We assume

$$\mathbb{P}(\mu_e > 0) = p_1 > p_c, \tag{2.1}$$

where $p_c = p_c(\mathbb{Z}^d)$ is the critical probability for bond percolation in $\mathbb{Z}^d$. Let

$$\mathcal{O}_1 = \{e : \mu_e > 0\}, \quad \mathcal{C}_1 = \mathcal{C}_\infty(\mathcal{O}_1). \tag{2.2}$$

We write $\mathcal{O}[p]$ for the edges of bond percolation with probability p in $\mathbb{Z}^d$. Then $\mathcal{O}_1$ is equal in law to $\mathcal{O}[p_1]$. Also, given a subset $I \subset [0, \infty)$ let

$$\mathcal{O}_I = \{e : \mu_e \in I\}. \tag{2.3}$$

Now choose $K < \infty$ (large) and set

$$q = q(K) = \mathbb{P}(0 < \mu_e < K^{-1}) + \mathbb{P}(\mu_e > K). \tag{2.4}$$

We will assume that $q(K)$ is small; initially we can suppose just that $q(K) < p_1 - p_c$, but we will need more than this later. We have that $\mathcal{O}_{[K^{-1},K]} \subset \mathcal{O}_1$, $\mathcal{C}_\infty(\mathcal{O}_{[K^{-1},K]}) \subset \mathcal{C}_\infty(\mathcal{O}_1) = \mathcal{C}_1$ and $\mathcal{O}_{[K^{-1},K]}$ is equal in law to $\mathcal{O}[p_1 - q(K)]$. Now let $\mathcal{O}_R = \mathcal{O}_{(0,K^{-1})\cup(K,\infty)}$, and

$$\mathcal{O}_S = \{e \in \mathcal{O}_1 : e \cap e' \neq \emptyset \text{ for some } e' \in \mathcal{O}_R\}, \tag{2.5}$$

$$\mathcal{O}_2 = \mathcal{O}_1 - \mathcal{O}_S. \tag{2.6}$$

(We write $e \cap e'$ for the set of vertices in both e and e'.) We write $\mathcal{C}_2 = \mathcal{C}_\infty(\mathcal{O}_2)$. We will use the results of [30] to prove that if K is large enough then $\mathcal{O}_2$ stochastically dominates a supercritical bond percolation process.

Remark 2.1 For our use of the set $\mathcal{O}_2$, it will be necessary that $\mu_e \in [K^{-1}, K]$ for all $e \in \mathcal{O}_2$, and that no vertex in $\mathcal{C}_2$ should be adjacent to a bond e with $\mu_e > K$. Thus, while we had to exclude the edges e such that $\mu_e \in (0, K^{-1})$, we did not have to exclude their neighbours. However, it is simpler to treat all the exceptional edges (that is, with large and small conductivities) in the same fashion.

Proposition 2.2 *Let $p_1 > p_c$. There exist positive constants c_1, c_2, δ_1, depending only on d, such that if $q = q(K) < c_2$ and $p_3 = p_1(1 - c_1 q^{\delta_1})$ then $\mathcal{O}_2$ stochastically dominates $\mathcal{O}[p_3]$.*

Proof We will build on the same probability space $(\Omega, \mathbb{P})$ i.i.d. r.v. (μ_e), and sets of edges

$$\mathcal{O}_3 \subset \mathcal{O}_2 \subset \mathcal{O}_1, \tag{2.7}$$

such that $\mathcal{O}_3 \overset{(d)}{=} \mathcal{O}[p_3]$, and $\mathcal{O}_1$ and $\mathcal{O}_2$ are given by (2.2) and (2.6). Let $q > 0$. We proceed in a number of steps. We write $\widehat{\mu}$ for a generic random variable with the same law as μ_e.

1. First, we define a set of edges $\mathcal{O}_1 \overset{(d)}{=} \mathcal{O}[p_1]$. Let $\mathcal{G} = (\mathbb{Z}^d, \mathcal{O}_1)$.
2. Next, we perform independent bond percolation with probability q/p_1 on $\mathcal{G}$, and write $\mathcal{O}_R$ for the set of edges we obtain: we have $\mathbb{P}(e \in \mathcal{O}_R) = q$.
3. Conditional on the sets $\mathcal{O}_1$ and $\mathcal{O}_R$ we define μ_e with the right conditional law. Thus (μ_e) are independent, $\mu_e = 0$ if $e \notin \mathcal{O}_1$, and
$$\mathbb{P}(\mu_e \in \cdot | e \in \mathcal{O}_1 - \mathcal{O}_R) = \mathbb{P}(\widehat{\mu} \in \cdot | \widehat{\mu} \in [K^{-1}, K]),$$
with an analogous definition for μ_e when $e \in \mathcal{O}_R$.
4. Define $\mathcal{O}_S$, $\mathcal{O}_2$ from $\mathcal{O}_R$ via (2.5) and (2.6). Then
$$\mathbb{P}(e \in \mathcal{O}_1 - \mathcal{O}_S | e \in \mathcal{O}_1) = (1 - q/p_1)^{4d-1}.$$
5. We now work conditionally on the graph $\mathcal{G}$. The bond percolation process $\mathcal{O}_2 = \mathcal{O}_1 - \mathcal{O}_S$ is finite range, so using [30, Theorem 1.3], provided q is small enough

$\mathcal{O}_1 - \mathcal{O}_S$ stochastically dominates an independent i.i.d. bond percolation process with probability $p' = p'(q,d) \geq 1 - c_1 q^{\delta_1}$. So by coupling we can define a percolation process $\mathcal{O}_3$ on the graph $\mathcal{G}$, such that $\mathcal{O}_3 \subset \mathcal{O}_2$ and

$$\mathbb{P}(e \in \mathcal{O}_3 | e \in \mathcal{O}_1) = 1 - c_1 q^{\delta_1}, \tag{2.8}$$

and for edges $e_1 \dots e_n \in E_d$ the events $\{e_i \in \mathcal{O}_3\}$ are independent conditional on $\{e_i \in \mathcal{O}_1, i = 1, \dots, n\}$.

It remains to verify that this construction has the required properties. It is clear that (2.7) holds and that (μ_e) are independent. Also, by (2.8) we have $\mathbb{P}(e \in \mathcal{O}_3) = p_3$, while the conditional independence of $\{e_i \in \mathcal{O}_3\}$ given $\mathcal{O}_1$ implies that $\mathcal{O}_3 \overset{(d)}{=} \mathcal{O}[p_3]$. □

For the remainder of this section we fix a probability space $(\Omega, \mathbb{P})$ as constructed in the Proposition above. We take $p_3 = p_3(p_1, q)$ to be as given in Proposition 2.2. We choose q small enough so that $p_3 > p_c$. Therefore the infinite cluster $\mathcal{C}_3 = \mathcal{C}_\infty(\mathcal{O}_3)$ exists $\mathbb{P}$—a.s., and by (2.7) we have

$$\mathcal{C}_3 \subset \mathcal{C}_2 \subset \mathcal{C}_1. \tag{2.9}$$

Note that while $\mathcal{C}_1$ and $\mathcal{C}_3$ have exactly the law of a supercritical percolation cluster, in general $\mathcal{C}_2$ will not have this law. Write $d_i = d_i(\omega)$ for the graph metric in $(\mathcal{C}_i, \mathcal{O}_i)$, for $i = 1, 2, 3$, and $B_i(x,r) = \{y \in \mathcal{C}_i : d_i(x,y) \leq r\}$ for balls in the d_i metric. We use $B_E(x,r)$ to denote balls in the Euclidean metric.

As explained in the introduction, we will ultimately study a time change of the VSRW Y on $\mathcal{C}_2$, and we now prove the properties of the cluster $\mathcal{C}_2$ that will be needed. These properties hold for supercritical percolation clusters, and we will use the fact that $\mathcal{C}_2$ is sandwiched between two supercritical clusters (with probabilities p_1 and p_3 and $p_1 - p_3 \ll 1$) to establish them for $\mathcal{C}_2$.

Let $\mathcal{H} = \mathcal{C}_1 - \mathcal{C}_2$, and $\mathcal{H}_3 = \mathcal{C}_1 - \mathcal{C}_3$. For $x \in \mathcal{C}_1$ let $\mathcal{H}(x)$ be the connected component of $\mathcal{C}_1 - \mathcal{C}_2$ containing x. (Note that $\mathcal{H}(x) = \emptyset$ if $x \in \mathcal{C}_2$.) We call the sets $\mathcal{H}, \mathcal{H}_3$ the 'holes'.

Lemma 2.3 *There exists $\delta_2 = \delta_2(d) > 0$ such that if $q(K) < \delta_2$ then the following holds.*

(i) *All the connected components $\mathcal{H}$ are finite. Further there exist constants c_i such that for each $x \in \mathbb{Z}^d$,*

$$\mathbb{P}(x \in \mathcal{C}_1,\ \operatorname{diam} \mathcal{H}(x) \geq n) \leq c_1 e^{-c_2 n}. \tag{2.10}$$

(Here diam *is the diameter in the ℓ_∞ distance in $\mathbb{Z}^d$.)*

(ii) *There exists a constant α_H such that, $\mathbb{P}$-a.s., for large enough n, the volume of any hole intersecting the box $[-n,n]^d$ is bounded from above by $(\log n)^{\alpha_H}$.*

Proof This result is proved for the set $\mathcal{H}_3$ in [14, Proposition 2.3] and in [33, Lemma 3.1], provided $p_1 - p_3$ is small enough. The lemma is then immediate since $\mathcal{H} \subset \mathcal{H}_3$. $\square$

Let $\mathbb{P}_2$ be the conditioned measure

$$\mathbb{P}_2(\cdot) = \mathbb{P}(\cdot\,|0 \in \mathcal{C}_2). \tag{2.11}$$

and $\mathbb{E}_2$ be the associated expectation operator. Let $b \in \mathbb{Z}^d$ with $|b| = 1$, let $N_2 = \min\{k > 0 : kb \in \mathcal{C}_2(\omega)\}$, and

$$\zeta = bN_2. \tag{2.12}$$

Lemma 2.4 (See [11, Lemma 4.3]). *Let $q(K) < \delta_2$. Then there exists a constant c_1 such that*

$$\mathbb{P}_2(|\zeta| > n) \le e^{-c_1 n}. \tag{2.13}$$

Proof Since $\mathbb{P}(0 \in \mathcal{C}_2) \ge \mathbb{P}(0 \in \mathcal{C}_3) = c > 0$, it is enough to prove that

$$\mathbb{P}(N_2 > n) \le e^{-c_1 n}.$$

Let N_3 be the r.v. defined in the same way for the cluster $\mathcal{C}_3$. Then $N_2 \le N_3$, and the proof of [11, Lemma 4.3] gives $\mathbb{P}(N_3 > n) \le e^{-c_1 n}$. $\square$

The remaining results on $\mathcal{C}_2$ will require the use of static renormalization arguments. These can be quite intricate, but fortunately all the hard work has already been done in [2,14,33]. We will follow [14] for Lemma 2.5, and [2] for Lemma 2.6.

Now assume that p_3 and K satisfy the hypotheses of Lemma 2.3. We define a set of edges E'_Z as follows. Let $x, y \in \mathcal{C}_2$. Then $\{x, y\} \in E'_Z$ if $\{x, y\} \notin \mathcal{O}_2$ and there exists a path $x = z_0, z_1, \dots z_k = y$ with $z_1, \dots, z_{k-1} \in \mathcal{H}$, and $\{z_{i-1}, z_i\} \in \mathcal{O}_1$ for $i = 1, \dots k$. If Z is the time change of Y with time in $\mathcal{H}$ cut out then the jumps of Z will be either on edges in $\mathcal{O}_2$ or E'_Z. Set $E_Z = \mathcal{O}_2 \cup E'_Z$. Let d_Z be graph distance on the graph $(\mathcal{C}_2, E_Z)$: clearly we have $d_Z(x, y) \le d_2(x, y)$ and also $|x - y| \le d_2(x, y)$ for $x, y \in \mathcal{C}_2$. The next Lemma gives that, with high probability, d_2, d_Z and the Euclidean metric are comparable.

Lemma 2.5 *There exists $\delta_3 = \delta_3(d) > 0$, and constants c_i such that if K is chosen so that $q(K) < \delta_3$ then for each $x, y \in \mathbb{Z}^d$*

$$\mathbb{P}(x, y \in \mathcal{C}_2, \quad \text{and} \quad d_Z(x, y) \le c_1|x - y|) \le c_2 e^{-c_3|x-y|}, \tag{2.14}$$
$$\mathbb{P}(x, y \in \mathcal{C}_2, \quad \text{and} \quad d_2(x, y) \ge c_1^{-1}|x - y|) \le c_2 e^{-c_3|x-y|}. \tag{2.15}$$

Proof As in [14] we define the lattice cubes

$$Q_L(x) = x + [0, L]^d \cap \mathbb{Z}^d, \quad \widetilde{Q}_{3L}(x) = x + [-L, 2L]^d \cap \mathbb{Z}^d.$$

For each of the percolation processes $(\mathbb{Z}^d, \mathcal{O}_i)$ we define a 'good event' $G_L^{(i)}(x)$, related to the cube $Q_L(Lx)$. The event $G_L^{(i)}(x)$ holds if:

(i) For each neighbour y of x, the side of the block $Q_L(Ly)$ adjacent to $Q_L(Lx)$ is connected to the opposite side of $Q_L(Ly)$ by a path (inside $Q_L(Ly)$) of bonds in $\mathcal{O}_i$.

(ii) Any two paths in $\widetilde{Q}_{3L}(Lx) \cap \mathcal{O}_i$ which connect $Q_L(Lx)$ to the boundary of $\widetilde{Q}_{3L}(Lx)$ are connected by an $\mathcal{O}_i$-occupied path inside $\widetilde{Q}_{3L}(Lx)$.

By [38, Theorem 3.1] (for $d \geq 3$) and [37, Theorem 5] (for $d = 2$) we have

$$\mathbb{P}(G_L^{(i)}(x)^c) \leq ce^{-cL}, \quad \text{for } i = 1, 3.$$

(Easier arguments, as in [14], give that $\mathbb{P}(G_L^{(i)}(x)^c) \to 0$ as $L \to \infty$, which is in fact all we need.)

The key property of the good events $G_L^{(i)}(Lx)$ is that if two adjacent boxes $Q_L(Lx)$ and $Q_L(Ly)$ are 'good' (that is the event $G_L^{(i)}(x) \cap G_L^{(i)}(y)$ occurs), then the clusters inside the two boxes have to connect. Let $G_L^*(x)$ be the event that no bond in $\mathcal{O}_1 - \mathcal{O}_3$ is in $\widetilde{Q}_{3L}(Lx)$.

Now let $\delta' > 0$. We first choose L large enough so that $\mathbb{P}(G_L^{(1)}(x)^c) < \frac{1}{2}\delta'$. Next we choose $\delta_3 \in (0, \delta_2)$ (where δ_2 is as in Lemma 2.3) such that if $q < \delta_3$ then

$$\mathbb{P}(G_L^*(x)^c) \leq \tfrac{1}{2}\delta'. \tag{2.16}$$

Set

$$G_L(x) = G_L^{(1)}(x) \cap G_L^*(x);$$

note that if $G_L(x)$ occurs then each of $G_L^{(i)}(x)$ occurs, and there are no holes in $\widetilde{Q}_{3L}(Lx)$.

Let $\eta(x) = 1_{G_L(x)}, x \in \mathbb{Z}^d$. Then $\eta(x)$ are not independent, but the process does have finite range. Therefore by [30, Theorem 0.0] the process η stochastically dominates i.i.d. Bernoulli random variables $(\xi(x), x \in \mathbb{Z}^d)$ with $\mathbb{P}(\xi(x) = 0) \to 0$ as $\mathbb{P}(\eta(x) = 0) \to 0$. Thus we can choose δ' small enough so that the site percolation process ξ has a unique infinite cluster $\mathcal{C}_\infty^\eta$, and all the connected components of $\mathbb{Z}^d - \mathcal{C}_\infty^\eta$ are finite.

As in [14, Lemma 3.1] we define a metric $d'(x', y')$ on $\mathbb{Z}^d$ from the site process η by wiring the holes in $\mathcal{C}_\infty^\eta$ – that is we place an edge between any x', y' which lie on the external boundary of the same connected component of $\mathbb{Z}^d - \mathcal{C}_\infty^\eta$.

It is enough to prove (2.14) when $x = 0$. Given $y \in \mathbb{Z}^d$, let y' be such that $y \in Q_L(Ly')$. Then (see [14, (3.10)]) we have $d_Z(0, x) \geq d'(0, x')$, and $|x'| \geq L^{-1}|x| - 1$. We can now proceed as in [14], and choose δ' small enough so that

$$\mathbb{P}(d'(0, x') \leq \tfrac{1}{2}|x'|) \leq ce^{-|x'|};$$

(2.14) then follows.

The proof of (2.15) is similar, except that instead of wiring the holes in $\mathcal{C}_\infty^\eta$ we find a path which avoids them, as in [1, Proposition 3.1]. □

The next Lemma summarizes volume bounds and an isoperimetric inequality for $\mathcal{C}_2$ in a finite box. We remark that [36] has given a proof of the isoperimetric inequality which is much quicker than that in [2,31]. Let

$$\beta = 1 - \frac{2}{1+d} < \frac{d-1}{d}. \tag{2.17}$$

Lemma 2.6 *There exists $\delta_4 \in (0, \delta_3)$ so that if $q(K) < \delta_4$ then there exist constants c_i such that the following holds. Let Q be a cube side n in $\mathbb{Z}^d$, and let $\mathcal{C}^+(Q)$ be the largest connected component of the graph $(Q, \mathcal{O}_2)$. Let $G_1(Q)$ be the event that $|\mathcal{C}^+(Q)| \geq \frac{1}{2}\theta(p_3)|Q|$, where $\theta(p_3) = \mathbb{P}(0 \in \mathcal{C}_\infty(\mathcal{O}[p_3])$. Let $G_2(Q)$ be the event that if A is any subset of $\mathcal{C}^+(Q)$ such that A and $\mathcal{C}^+(Q) - A$ are connected (in the graph $(\mathcal{C}^+(Q), \mathcal{O}_2)$), and $|A| \leq \frac{1}{2}\mathcal{C}^+(Q)$ then*

$$\left|\left\{\{x, y\} : x \in A, y \in \mathcal{C}^+(Q) - A\right)\right\}\right| \geq \frac{c_1|A|}{n}. \tag{2.18}$$

Then

$$\mathbb{P}(G_1(Q)^c \cup G_2(Q)^c) \leq c_2 \exp(-c_3 n^\beta). \tag{2.19}$$

Proof As in the previous Lemma we consider a block renormalization of the processes $\mathcal{O}_i$. Let L be large, and $j \in \{1, 2, 3\}$. We consider a tiling of $\mathbb{Z}^d$ by cubes $T(x)$, $x \in \mathbb{Z}^d$ with L^d points. Then [2] identifies a 'good event' $R_j(T(x))$, related to $\mathcal{O}_j$ in a region around $T(x)$, which is similar to (but a bit more complicated than) the events $G_L(x)$ defined in Lemma 2.5—see [2, p. 3040].

Let $\eta_j(x) = 1_{R_j(T(x))}$, let $\widetilde{Q}$ be a cube in $\mathbb{Z}^d$, and $Q = \cup_{x' \in \widetilde{Q}} T(x)$; let n be the side length of Q. [2] defines events $\widetilde{K} = \widetilde{K}(\widetilde{Q}, 7/8)$ and $\widetilde{F} = \widetilde{F}(\widetilde{Q}, \varepsilon_0)$ such that if $\widetilde{K}(\widetilde{Q}, 7/8) \cap \widetilde{F}(\widetilde{Q}, \varepsilon_0)$ occurs for η_2 then $G_1(Q) \cap G_2(Q)$ occurs—see the definitions on p. 3036, and Lemma 2.9 and Proposition 2.11.

As in the previous proof we define a new event $R^*(T(x))$ that no edge in $\mathcal{O}_1 - \mathcal{O}_3$ lies in $T(x)$ or any of its neighbours. Let $R(T(x)) = R_1(T(x)) \cap R^*(T(x))$, and $\eta(x) = 1_{R(T(x))}$. By first choosing L large, so that $\mathbb{P}(R_1(T(x))^c)$ is small, and then choosing δ_3 small enough so that $\mathbb{P}(R^*(T(x))^c) \leq \mathbb{P}(R_1(T(x))^c)$, we can ensure that $\mathbb{P}(\eta(x) = 1)$ is close to 1.

Again using [30, Theorem 0.0] we have that η dominates an independent i.i.d. site percolation process ξ, with $\mathbb{P}(\xi(x) = 1)$ close to 1. The events $\widetilde{F}$ and $\widetilde{K}$ are monotone (see p. 3036 of [2]), and so we can use Lemmas 2.2 and 2.5 of [2] to obtain

$$\mathbb{P}((\widetilde{K} \cap \widetilde{F})^c) \leq c \exp(-cn^\beta). \tag{2.20}$$

Since $\widetilde{K} \cap \widetilde{F}$ then implies $G_1(Q) \cap G_2(Q)$ we are done. □

Now fix K large enough so that $q(K) < \delta_4$. Define

$$\mu^0_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in \mathcal{O}_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

Let $\mu^0_x = \sum_y \mu^0_{xy}$, and extend μ^0 to a measure on $\mathbb{Z}^d$.

Definition 2.7 Let C_V, C_P, C_R and $C_W \geq 1$ be fixed strictly positive constants. We say a ball $B_2(x, r)$ in the graph $(\mathcal{C}_2, \mathcal{O}_2)$ is *good* if:

$$|x' - y| \geq C_R^{-1} r, \quad \text{if } x' \in B_2(x, r/2), \quad y \in \mathcal{C}_2 - B_2(x, 8r/9), \tag{2.22}$$

$$d_Z(x', y) \geq C_R^{-1} r, \quad \text{if } x' \in B_2(x, r/2), \quad y \in \mathcal{C}_2 - B_2(x, 8r/9), \tag{2.23}$$

$$C_V r^d \leq \mu^0(B_2(x, r)), \tag{2.24}$$

$$\operatorname{diam} \mathcal{H}(y) \leq r^\beta, \quad y \in B_E(x, r), \tag{2.25}$$

and the weak Poincaré inequality

$$\sum_{y \in B_2(x,r)} \left(f(y) - \overline{f}_{B_2(x,r)}\right)^2 \mu^0_y \leq C_P r^2 \sum_{y,z \in B_2(x, C_W r), z \sim y} |f(y) - f(z)|^2 \mu^0_{yz} \tag{2.26}$$

holds for every $f : B_2(x, C_W r) \to \mathbb{R}$. (Here $\overline{f}_{B_2(x,r)}$ is the value which minimizes the left hand side of (2.26)). Strictly speaking, because of condition (2.25) ‘good’ is a property of (x, r) in the environment (μ_e) rather than the ball $B_2(x, r)$ in the graph $(\mathcal{C}_2, \mathcal{O}_2)$. Note that since $(\mathcal{C}_2, \mathcal{O}_2)$ is a subgraph of $\mathbb{Z}^d$, and μ_e is bounded on $\mathcal{C}_2$, we always have the upper bound $\mu^0(B_2(x, r)) \leq C_0 r^d$ for $r \geq 1$.

We say $B_2(x, R)$ is *M-very good* if $B_2(y, r)$ is good whenever $y \in B_2(x, R)$ and $M \leq r \leq R$. We can always assume that $M \geq 2$.

Let $\alpha \in (0, 1]$. For $x \in \mathbb{Z}^d$ define $R^{(\alpha)}_x$ as follows. If $x \in \mathcal{C}_2$ let $R^{(\alpha)}_x$ be the smallest integer M such that $B_2(x, R)$ is R^α—very good for all $R \geq M$. If $x \in \mathcal{C}_1 - \mathcal{C}_2$ then let

$$R^{(\alpha)}_x = \max_{y \in \partial \mathcal{H}(x)} R^{(\alpha)}_y \vee (\operatorname{diam} \mathcal{H}(x))^{1/\alpha\beta}.$$

Finally, let $R^{(\alpha)}_x = 0$ if $x \notin \mathcal{C}_1$.

Proposition 2.8 *Let β be defined as in (2.17). There exist C_V, C_P, C_W, C_R (depending on K, the law μ_e and the dimension d) such that the following holds. For $x \in \mathbb{Z}^d$, $R \geq 1$, $\alpha \in (0, 1]$,*

$$\mathbb{P}(x \in \mathcal{C}_2,\ B_2(x, R) \text{ is not good}) \leq c_1 \exp(-c_2 R^\beta), \tag{2.27}$$

$$\mathbb{P}(x \in \mathcal{C}_2,\ B_2(x, R) \text{ is not } R^\alpha\text{–very good}) \leq c_1 \exp(-c_2 R^{\alpha\beta}). \tag{2.28}$$

Hence

$$\mathbb{P}(R_x^{(\alpha)} \geq n, x \in \mathcal{C}_1) \leq \exp(-c_2 n^{\alpha\beta}). \tag{2.29}$$

Proof Given Lemmas 2.3, 2.5 and 2.6, (2.27) and (2.28) follow by the same argument as Theorem 2.18 and Lemma 2.19 of [2]. Note that using (2.14) to compare the Euclidean metric with d_2, we have that if $x \in \mathcal{C}_2$ then $B_2(x, R)$ is contained in a cube Q of side cR with high probability. It is well known that the isoperimetric inequality (2.18) implies a Poincaré inequality for the graph $(\mathcal{C}^+(Q), \mathcal{O}_2)$—see for example [2, Proposition 1.4].

Summing (2.28) over $R \geq n$ gives

$$\mathbb{P}(R_x^{(\alpha)} \geq n, x \in \mathcal{C}_2) \leq c_1 \exp(-c_2 n^{\alpha\beta}). \tag{2.30}$$

So, writing $D = \text{diam}\,(\mathcal{H}(x))$,

$$\begin{aligned}
\mathbb{P}(R_x^{(\alpha)} \geq n, x \in \mathcal{C}_1) &\leq \mathbb{P}\Big(\max_{y \in \partial\mathcal{H}(x)} R_y^{(\alpha)} \geq n,\, D^{1/\alpha\beta} < n\Big) + \mathbb{P}\big(D^{1/\alpha\beta} \geq n\big) \\
&\leq \mathbb{P}\big(\max_{y \in B_E(0,n^{\alpha\beta}) \cap \mathcal{C}_2} R_y^{(\alpha)} \geq n\big) + \mathbb{P}\big(D > n^{\alpha\beta}\big) \\
&\leq c n^{\alpha\beta d} \exp(-c_2 n^{\alpha\beta}) \leq c \exp(-c_3 n^{\alpha\beta});
\end{aligned}$$

here we used (2.30) and (2.10) in the last line. □

Corollary 2.9 *Let* $\alpha \in (0, 1]$ *and* $\theta > 0$. *Then* $\mathbb{P}$*-a.s.*

$$\lim_{n \to \infty} n^{-\theta} \max_{y \in B_E(0,n)} R_y^{(\alpha)} = 0.$$

Proof By (2.29) we have

$$\mathbb{P}\left(\max_{y \in B_E(0,n)} R_y^{(\alpha)} \geq n^{\theta/2}\right) \leq c n^d \exp\left(-c_2 n^{\theta\alpha\beta/2}\right),$$

so by Borel–Cantelli $\max_n n^{-\theta/2} \max_{y \in B_E(0,n)} R_y^{(\alpha)} < \infty$. □

3 The time changed process

We continue with the notation of the previous section, and now fix for the rest of this paper a K large enough so that the results of Sect. 2 hold. We define $Z = Z^K$ to be the trace of Y on $\mathcal{C}_2$, that is the time change of Y by the inverse of the additive functional

$$A_t = \int_0^t 1_{(Y_s \in \mathcal{C}_2)} ds. \tag{3.1}$$

So, writing $\mathfrak{a}_t = \inf\{s : A_s > t\}$ for the right-continuous inverse of A,

$$Z_t = Y_{\mathfrak{a}_t}, \quad t \geq 0. \tag{3.2}$$

Thus Z is obtained by suppressing in the trajectory of Y all the visits to the holes. Consequently, unlike Y, the process Z may perform long jumps in $\mathbb{Z}^d$ by jumping over the holes of $\mathcal{C}_2$. We abuse notation slightly by writing P_ω^x for the law of Z when $Y_0 = x$, and $x \in \mathcal{C}_1(\omega)$. If $x \in \mathcal{C}_2(\omega)$ then we have $Z_0 = Y_0 = x$, P_ω^x-a.s., but otherwise $Z_0 = Y_{\mathfrak{a}_0}$ will be the first point in $\mathcal{C}_2$ hit by Y.

Proposition 3.1 *For $\mathbb{P}$–a.e. ω, and $x \in \mathcal{C}_2(\omega)$, the random process Z under P_ω^x is a symmetric Markov process on $\mathcal{C}_2(\omega)$. Moreover, the reversible measure is given by the counting measure on $\mathcal{C}_2$.*

Proof See Proposition 2.1 in [33]. □

Write $\nu^{(i)}$, $i = 1, 2$ for counting measure on $\mathcal{C}_i$, $i = 1, 2$. We recall that the Dirichlet form for the process Y is

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x,y \in \mathcal{C}_1} (f(x) - f(y))(g(x) - g(y))\mu_{xy},$$

on the space $L^2(\mathcal{C}_1, \nu^{(1)})$. Using the definition of the generators in (1.2) we have

$$\mathcal{E}(f, g) = -\langle \mathcal{L}_V f, g\rangle_{\nu^{(1)}} = -\langle \mathcal{L}_C f, g\rangle_\mu.$$

The Dirichlet form for the time changed process Z is

$$\mathcal{E}_Z(f, g) = \frac{1}{2} \sum_{x,y \in \mathcal{C}_2} (f(x) - f(y))(g(x) - g(y))\mu'_{xy}, \tag{3.3}$$

on the space $(\mathcal{C}_2, \nu^{(2)})$. Writing $\mathcal{L}_Z = \mathcal{L}_Z^\omega$ for the the generator of Z, since $\mathcal{E}_Z(f, g) = -\langle \mathcal{L}_Z f, g\rangle_{\nu^{(2)}}$ for f, g with finite support, we have

$$\mathcal{L}_Z f(x) = \sum_{y \in \mathcal{C}_2} \mu'_{xy}(f(y) - f(x)). \tag{3.4}$$

Here $\mu'_{xy} = \mu_{xy} + \mu''_{xy}$ is the new weight which gives the rate of jumps by Z from x to y, and is composed of μ_{xy}, the original weight from the direct edge between the two vertices, and μ''_{xy}, the weight induced by paths across the holes $\mathcal{H}$ which may connect x and y. We let $\mu'_{xy} = 0$ if either x or y is not in $\mathcal{C}_2$, and set $\mu'_x = \sum_{y \in \mathcal{C}_2} \mu'_{xy}$. For $x, y \in \mathcal{C}_2$

$$\frac{\mu'_{xy}}{\mu'_x} = P_x^\omega(y \text{ is the next point in } \mathcal{C}_2 \text{ visited by the random walk } Y).$$

It is clear from this, and the definition of the metric d_Z in the previous section, that

$$\mu'_{xy} > 0 \text{ if and only if } d_Z(x, y) = 1.$$

Further, $\mu'_{xy} = \mu'_{yx}$ as follows from the reversibility of Z. In what follows we will find it most convenient to regard the process Z as a random walk on the graph $(\mathcal{C}_2, \mathcal{O}_2)$, but one which may make 'long range' jumps. We will use the notation $x \sim y$ to mean that x and y are neighbours in $(\mathcal{C}_2, \mathcal{O}_2)$ (and hence in $\mathbb{Z}^d$).

Lemma 3.2 (a) *$\mu'_x \leq \mu_x$ for all $x \in \mathcal{C}_2$. In particular,* $\sup_{x\in\mathcal{C}_2} \mu'_x \leq 2dK$.

(b) *$\mu'_{xy} \leq 2dK$ for all $x, y \in \mathcal{C}_2$.*

(c) *$\mu'_{xy} \geq K^{-1}$ for all $x, y \in \mathcal{C}_2$ such that $x \sim y$.*

Proof (a) Write τ_Y and τ_Z for the first jumps of Y and Z. Then if $Y_0 = x \in \mathcal{C}_2$, the construction of Z gives that $Z_s = Y_s$ for $s \in [0, \tau_Y)$, so that $\tau_Z \geq \tau_Y$. We therefore have $\mu_x^{-1} = E_\omega^x \tau_Y \leq E_\omega^x \tau_Z = (\mu'_x)^{-1}$. The second assertion is immediate, since the construction of $\mathcal{O}_2$ gives that $\mu_e \leq K$ for every $e \in \mathcal{O}_2$.

(b) Since $\mu'_{xy} \leq \mu'_x$, this follows from (a).

(c) This is also clear from the construction of $\mathcal{O}_2$. □

Note that we have no lower bound for μ'_{xy} for x, y which are neighbours with respect to the d_Z metric but not the d_2 metric.

4 Heat kernel estimates for the process *Z*

We now establish heat kernel estimates for the time changed process Z. Many of the arguments follow the same lines as in [2], and we will only give details where they differ in significant ways.

Define the Dirichlet form

$$\mathcal{E}_0(f, f) = \tfrac{1}{2} \sum_{x,y\in\mathcal{C}_2} (f(y) - f(x))^2 \mu^0_{xy}, \tag{4.1}$$

where μ^0 is as in (2.21). Since $\mu'_{xy} \geq K^{-1}$ if $x \sim y$, we have

$$\mathcal{E}_Z(f, f) \geq K^{-1}\mathcal{E}_0(f, f) \quad \text{for all } f. \tag{4.2}$$

Write

$$q_t^Z(x, y) = P_\omega^x(Z_t = y) \tag{4.3}$$

for the transition density of Z, or the heat kernel on the graph $(\mathcal{C}_2, E_Z)$. Standard long range estimates due to Carne, Varopolous and Davies (see [15,19,40]) give that if $d_Z(x, y) = D$ then

$$q_t^Z(x, y) \le \begin{cases} c_1 \exp(-c_2 D(1 + \log(D/t))) & \text{if } D \ge t \ge 1, \\ c_1 \exp(-c_2 D^2/t) & \text{if } D \le t,\ t \ge 1. \end{cases} \tag{4.4}$$

Note that if $c^{-1} t \le D \le ct$ then both terms in (4.4) are of the form $c_1 \exp(-cD)$. Note also that if $\varepsilon > 0$ and $t < c_3 D^{2(1-\varepsilon)}$ then

$$\begin{aligned} \exp\left(-c_2 D^2/t\right) &\le \exp\left(-\tfrac{1}{2} c_2 D^2/t\right) \exp\left(-\tfrac{1}{2} c_2 c_3^{-1} D^{2\varepsilon}\right) \\ &\le \exp\left(-\tfrac{1}{2} c_2 D^2/t\right) \exp\left(-c t^{\varepsilon/(1-\varepsilon)}\right) \le c' t^{-d/2} \exp\left(-\tfrac{1}{2} c_2 D^2/t\right). \end{aligned} \tag{4.5}$$

4.1 Upper bounds

Our first step is to establish an on-diagonal bound. As we have truncated the edge weights above and below on $\mathcal{C}_2$ we are close to the random walk on a supercritical bond percolation cluster, and so can follow the proof of [2] Proposition 3.1 quite closely. Note though that we need to control the long range jumps of Z, and that by better 'initialization' we can weaken the condition of the size of N_B.

Proposition 4.1 *There exists a constant $C_0 > 1$ such that the following holds. Let $x_0 \in \mathcal{C}_2$, and let $B = B_2(x_0, R)$ be N_B– very good with $N_B = C_0^{-1} R/\log R$. Then writing $t_0 = C_0 N_B^2 \log N_B$, $t_1 = C_0^{-1} R^2/\log R$, for $x_1 \in B_2(x_0, R/2)$,*

$$q_t^Z(x_1, x_1) \le \begin{cases} c_1 \exp(-c_2 t/N_B^2) & \textit{if } 0 \le t \le t_0, \\ c_1 (t - t_0 + N_B^2)^{-d/2} & \textit{if } t_0 \le t \le t_1. \end{cases} \tag{4.6}$$

Remark 4.2 The bound on N_B is enough to ensure that $t_0 \le t_1$ when $R \ge 1$. Note that (4.6) gives $q_t^Z(x_1, x_1) \le c t^{-d/2}$ if $2t_0 \le t \le t_1$.

Proof Set $f_t(y) = q_t^\omega(x_1, y)$, and let

$$\psi(t) = \langle f_t, f_t \rangle_{\nu^{(2)}} = \sum_{y \in \mathcal{C}_2} f_t(y)^2 = q_{2t}^Z(x_1, x_1).$$

Then we have

$$-\psi'(t) = \sum_{x, y \in \mathcal{C}_2} (f_t(y) - f_t(x))^2 \mu'_{xy}.$$

Write

$$\varepsilon_B(t) = \sum_{x \in \mathcal{C}_2 - B} f_t(x)^2.$$

Since $\sum f_t(x) = 1$, we have

$$\varepsilon_B(t) \le \sup_{x \in \mathcal{C}_2 - B} f_t(x) \sum_{x \in \mathcal{C}_2 - B} f_t(x) \le \sup_{x \in \mathcal{C}_2 - B} f_t(x).$$

As $B_2(x_0, R)$ is good, we have $d_Z(x_1, x) \ge cR$ for all $x \in B^c$. So, by the long range bounds (4.4)

$$\varepsilon_B(t) \le R^{-d} \text{ provided } t \le cR^2/\log R. \tag{4.7}$$

Let $N_B \le r \le R$. Then we can choose $z_i \in B$ so that $B_2(z_i, r/2)$ are disjoint and $B_i = B_2(z_i, r)$ cover B. Write $B_i^* = B_2(z_i, C_W r)$. Since each B_i is good, $\mu^0(B_i) \ge cr^d$, and hence there exists a constant c' such that each $x \in B$ is in at most c' of the B_i^*. (Otherwise $\mu^0(B_2(x, 2C_W r))$ would be too large.)

Since $r \in [N_B, R]$ is good, the weak Poincaré inequality (2.26) holds for each B_i. As μ^0 and μ' are comparable on $\mathcal{C}_2$, this inequality also holds with respect to μ'. Therefore, applying the Poincaré inequality to each $B_i \subset B_i^*$, and writing $\overline{f}_{t,i}$ for the mean of f_t on B_i,

$$\begin{aligned}
-\psi'(t) &\ge c \sum_i \sum_{x,y \in B_i^*} (f_t(y) - f_t(x))^2 \mu'_{xy} \\
&\ge c \sum_i r^{-2} \sum_{x \in B_i} \left(f_t(x) - \overline{f}_{t,i}\right)^2 \\
&= cr^{-2} \sum_i \sum_{x \in B_i} f_t(x)^2 - cr^{-2} \sum_i \mu'(B_i)^{-1} \left(\sum_{x \in B_i} f_t(x) \right)^2 \\
&\ge cr^{-2} \sum_{x \in B} f_t(x)^2 - cr^{-2} \left(c' r^d\right)^{-1} \left(\sum_i \sum_{x \in B_i} f_t(x) \right)^2 \\
&\ge cr^{-2} \left(\psi(t) - \varepsilon_B(t) - cr^{-d} \right).
\end{aligned}$$

Using (4.7) then gives that for $0 < t \le cR^2/\log R$ and $N_B \le r \le R$,

$$\psi'(t) \le -2c_5 r^{-2} (\psi(t) - c_6 r^{-d}). \tag{4.8}$$

We now choose

$$r = r(t) = N_B \vee (2c_6/\psi(t))^{1/d}.$$

Let

$$t_0 = \inf\{t : r(t) > N_B\} = \inf\{t : \psi(t) < 2c_6 N_B^{-d}\}.$$

On $[0, t_0]$ we have

$$\psi'(t) \le -c_5 N_B^{-2} \psi(t).$$

Since $\psi(0) = 1/\mu'_{x_1} \le c$, it follows that

$$\psi(t) \le c \exp(-t/(c_5 N_B^2)), \quad t \in (0, t_0]. \tag{4.9}$$

Consequently,

$$t_0 \le c_7 N_B^2 \log N_B.$$

When $t_0 < t \le t_1 = cR^2/\log R$ we have

$$\psi'(t) \le -c\psi(t)^{1+2/d},$$

so that if $g(t) = \psi(t)^{-2/d}$ then $g'(t) \ge c$, and thus $g(t) - g(t_0) \ge c(t - t_0)$. As $g(t_0) = cN_B^2$, we obtain

$$\psi(t) \le c\left(N_B^2 + t - t_0\right)^{-d/2}, \quad t \ge t_0. \tag{4.10}$$

Combining (4.9) and (4.10) and adjusting the constants completes the proof □

Let $\varepsilon \in (0, \frac{1}{2})$, let $\alpha = \frac{1}{2} - \varepsilon$, and write R_x for $R_x^{(\alpha)}$, as defined in Definition 2.7.

Corollary 4.3 *Let $\varepsilon \in (0, \frac{1}{2})$. Then there exist constants c_1 and $c_2 = c_2(\varepsilon)$ such that if $x, y \in \mathcal{C}_2$ then*

$$q_t^Z(y, y) \le c_1 t^{-d/2} \text{ if } t \ge (c_2(\varepsilon) \vee 2d_2(x, y) \vee R_x)^{1-\varepsilon}. \tag{4.11}$$

Proof Let $R = t^{1/(1-\varepsilon)}$, so that the condition on t implies that $d_2(x, y) \le \frac{1}{2}R$ and $R \ge R_x$. Hence $B_2(x, R)$ is N_B—very good with $N_B \le R^{1/2-\varepsilon}$. By Proposition 4.1 the bound (4.11) holds provided

$$C_0 N_B^2 \log N_B \le t \le C_0^{-1} R^2/\log R. \tag{4.12}$$

However,

$$C_0 N_B^2 \log N_B \le C_0 t^{(1-2\varepsilon)/(1-\varepsilon)} \log t^{(1/2-\varepsilon)(1-\varepsilon)} \le t,$$

provided t is large enough. Similarly the right side of (4.12) holds once t is sufficiently large. □

We now turn to obtaining general Gaussian type upper bounds on $q_t^Z(x,y)$. [2] used the method of Nash and Bass—see [7,34], but we can obtain slightly sharper bounds with less work if we use an approach introduced by Grigoryan [26] for manifolds. This method has been adapted to graphs in [17,25].

For $T \geq 1$, $A \geq 1$, $\gamma > 1$ let $\mathcal{G}(A,\gamma,T)$ be the set of increasing functions g from $[T,\infty)$ to $\mathbb{R}_+$ which satisfy $\sup_{t\geq T} g(t)\exp(-t^{1/2}) \leq A$ and are '(A,γ) regular' on $[T,\infty)$: that is for $T \leq t_1 < t_2$,

$$\frac{g(\gamma t_1)}{g(t_1)} \leq A\frac{g(\gamma t_2)}{g(t_2)}. \tag{4.13}$$

Proposition 4.4 *Let $T \geq 1$, $A > 0$, $\gamma > 1$, $x_1, x_2 \in \mathcal{C}_2$ and suppose that there exist functions $g_i \in \mathcal{G}(A,\gamma,T)$ such that*

$$q_t^Z(x_i,x_i) \leq \frac{1}{g_i(t)}, \quad t \in [T,\infty). \tag{4.14}$$

Then there exists a constant $C = C(A,\gamma) < \infty$ such that if $t \geq C(T^2 \vee d_Z(x_1,x_2))$ then

$$q_t^Z(x_1,x_2) \leq \frac{C}{(g_1(t/C)g_2(t/C))^{1/2}}\exp\left(-\frac{d_Z(x_1,x_2)^2}{Ct}\right). \tag{4.15}$$

Proof See Theorem 1.3 of [25]. Note that since $\mu'_x \geq K^{-1}$ for all $x \in \mathcal{C}_2$, the condition there on the lower bound of vertex weights holds automatically, and also that we can take the d_θ there to be $d_Z(x,y)$ □

Theorem 4.5 *Let $x, y_1, y_2 \in \mathcal{C}_2$, $t \geq 1$. If either*

$$d_2(y_1,y_2) \geq R_x \quad or \quad t \geq c_0 R_x^{2-2\varepsilon}, \tag{4.16}$$

and

$$d_2(x,y_1) \leq (3d_2(y_1,y_2)) \vee ct^{1/(2-\varepsilon)}, \tag{4.17}$$

then

$$q_t^Z(y_1,y_2) \leq c_1 t^{-d/2}\exp(-c_2 d_2(y_1,y_2)^2/t), \quad \text{if } t > d_2(y_1,y_2), \tag{4.18}$$

$$q_t^Z(y_1,y_2) \leq c_1 \exp(-c_2 d_2(y_1,y_2)(1+\log(d_2(y_1,y_2)/t))), \quad \text{if } t \leq d_2(y_1,y_2). \tag{4.19}$$

Proof Let $D = d_2(y_1,y_2)$ and $D' = d_2(x,y_1)$. We have to consider two cases.

Case 1: $t < cD^{2-2\varepsilon}$. Both the conditions in (4.16) imply that $D \geq R_x$. Also, $t^{1/(2-\varepsilon)} \leq c'D^{(2-2\varepsilon)(2-\varepsilon)} < D$, so (4.17) implies that $D' \leq 3D$.

Thus $y_1 \in B_2(x, 3D)$, and so (as $D \ge R_x$), the ball $B_2(y_1, D)$ is good. Hence, using (2.23) we have $d_Z(y_1, y_2) \ge cD$. We can now use the long range bounds (4.4) and (4.5) to obtain (4.19).

Case 2: $cD^{2-2\varepsilon} < t$. Note that $d_2(x, y_2) \le D + D' \le 2D \vee 2D'$. Let

$$T = (C_1 \vee (4D) \vee (4D') \vee R_x)^{1-\varepsilon}.$$

Then Corollary 4.3 gives that $q_s(y_i, y_i) \le cs^{-d/2}$ for $s \ge T$, $i = 1, 2$. So by Proposition 4.4 the bound (4.18) holds if $t \ge cT^2$, and it remains to check that the conditions (4.16) and (4.17) imply that $t \ge cT^2$. We need therefore to show that

$$t \ge cD^{2-2\varepsilon}, \quad t \ge c(D')^{2-2\varepsilon}, \quad t \ge cR_x^{2-2\varepsilon}. \tag{4.20}$$

The first of these holds since we are in Case 2. Hence the second holds if $D' \le 3D$; if not then (4.17) implies that $t \ge c(D')^{2-\varepsilon} \ge c(D')^{2-2\varepsilon}$. If the first condition in (4.16) holds then $t \ge cD^{2-2\varepsilon} \ge cR_x^{2-2\varepsilon}$, so the third condition in (4.20) also holds. □

Corollary 4.6 *Let $x, y \in \mathcal{C}_2$. Then if either $|x-y| \ge R_x$ or $t \ge cR_x^{2-2\varepsilon}$,*

$$q_t^Z(x, y) \le \begin{cases} c_1 t^{-d/2} \exp(-c_2|x-y|^2/t), & \text{if } t > |x-y|, \\ c_1 \exp(-c_2|x-y|(1+\log(|x-y|/t))), & \text{if } t \le |x-y|. \end{cases} \tag{4.21}$$

Proof Write $D_2 = d_2(x, y)$ and $D_E = |x-y|$; we always have $D_2 \ge D_E$, while $D_2 \le cD_E$ provided $D_2 \ge R_x$. If $D_E \ge R_x$ then $D_2 \ge R_x$, so D_2 and D_E are comparable and (4.21) follows from (4.18) and (4.19). Now suppose that $t \ge cR_x^{2-2\varepsilon}$, but that $D_2 < R_x$. Then $t > D_2$ and so (4.21) follows from (4.18). □

Write

$$\Psi(R, t) = \begin{cases} e^{-R^2/t} & \text{if } t > e^{-1}R, \\ e^{-R\log(R/t)} & \text{if } t < e^{-1}R. \end{cases} \tag{4.22}$$

Proposition 4.7 (a) *Let $x \in \mathcal{C}_2$, $R \ge R_x$ and $y \in B_2(x, 3R)$. Then for $t > 0$,*

$$P_\omega^y\big(Z_t \notin B_2(y, R)\big) \le c_1\Psi(c_2R, t). \tag{4.23}$$

(b) *Write $\tau_A = \inf\{t : Z_t \notin A\}$. Let $x \in \mathcal{C}_2$ and $t > 0$. If $R \ge 2R_x$ then*

$$P_\omega^x(\tau_{B_E(x,R)} < t) \le P_\omega^x(\tau_{B_2(x,R)} < t) \le c_3\Psi(c_4R, t). \tag{4.24}$$

(c) *Write $\tau_A^Y = \inf\{t : Y_t \notin A\}$. Let $x \in \mathcal{C}_1$ and $t > 0$. If $R \ge 3R_x$ then*

$$P_\omega^x(\tau_{B_E(x,R)}^Y < t) \le c_3\Psi(c_4R, t), \tag{4.25}$$

$$P_\omega^x(\tau_{B_E(x,R)} < t) \le c_3\Psi(c_4R, t). \tag{4.26}$$

Proof (a) Let $y_2 \in B_2(y, R)^c$. Then $d_2(y, y_2) \geq R \geq \frac{1}{3} d_2(x, y)$, so (4.17) holds (with $y = y_1$). Since also $d_2(y, y_2) \geq R \geq R_x$, we can use (4.18) to bound $q_t^Z(y, y_2)$ for $y_2 \in B_2(y, R)^c$. For $n \geq 0$ let $D_n = B_2(x, 2^{n+1}R) - B_2(x, 2^n R)$, and $Q_n = \max_{z \in D_n} q_t^Z(y, z)$. Since d_2 dominates the Euclidean metric, $|D_n| \leq c(2^n R)^d$. Therefore,

$$P_\omega^y\big(Z_t \notin B_2(x, R)\big) = \sum_{n=0}^{\infty} \sum_{z \in D_n} q_t^Z(y, z) \leq \sum_{n=0}^{\infty} |D_n| Q_n \leq c R^d \sum_{n=0}^{\infty} 2^{nd} Q_n. \tag{4.27}$$

If $t \leq R$ then writing $A = \log(eR/t)$,

$$Q_n \leq c_1 \exp(-c_2(2^n R) \log(2^n Re/t)) \leq c_1 \exp(-2^n c_2 RA).$$

Hence (4.27) is bounded by $cR^d \exp(-cRA)$, so that (4.23) follows.
If $t > R$ then let m be the smallest integer so that $2^m R > t$. Then

$$Q_n \leq \begin{cases} c_1 \exp(-c_2 2^n R \log(2^n Re/t)) & \text{if } n \geq m, \\ c_1 \exp(-c_2(2^n R)^2/t) & \text{if } 0 \leq n < m. \end{cases} \tag{4.28}$$

Substituting these bounds into (4.27) gives (4.23).

(b) Since $|x - y| \leq d_2(x, y)$, the first inequality is immediate. For the second we have, writing $\tau = \tau_{B_2(x,R)}$,

$$P_\omega^x(\tau < t) \leq P_\omega^x\big(Z_t \notin B_2(x, R/2)\big) + P_\omega^x\big(\tau < t, Z_t \in B_2(x, R/2)\big). \tag{4.29}$$

By (a) the first term in (4.29) is bounded by $c\Psi(R/2, t)$. By the strong Markov property,

$$\begin{aligned} &P_\omega^x(\tau < t, Z_t \in B_2(x, R/2)) \\ &\quad = E_\omega^x 1_{(\tau<t)} P_\omega^{Z_\tau}\big(Z_{t-\tau} \in B_2(x, R/2)\big) \\ &\quad \leq P_\omega^x(\tau < t) \max_{y \in \partial B_2(x.R)} \sup_{0<s<t} P_\omega^y\big(Z_s \notin B_2(y, R/2)\big) \\ &\quad \leq P_\omega^x(\tau < t) c_1 \Psi(c_2 R/2, t), \end{aligned} \tag{4.30}$$

where for the final bound we used (a).
If the final term in (4.30) is less than $1/2$ then the second term in (4.29) is less than $\frac{1}{2} P_\omega^x(\tau < t)$, and so we obtain (4.24). If this term is greater than $1/2$ then $R^2/t = O(1)$, and so by again adjusting the constant c_3 we can make the right hand side of (4.24) greater than 1.

(c) This follows easily from (b): the only difficulty is that we have to take care of the possibility that Y might exit $B_E(x, R)$ via the set $\mathcal{C}_1 - \mathcal{C}_2$—that is through one of the holes. Write $\tau^Y = \tau^Y_{B_E(x,R)}$. Let z be the first point on the path of Y which is not in $B_E(x,R)$. If $\tau^Y < t$, then there exists $t_0 \in [0,t)$ such that $Y_{t_0} = z$. Let $s_0 = A_{t_0}$; note that $s_0 \le t_0 < t$.

If $z \in \mathcal{C}_2$ then $\mathfrak{a}_{s_0} = t_0$, so $Z_{s_0} = Y_{t_0} = z$, and therefore $\{\tau_{B_E(x,R)} < t\}$ holds. If $z \in \mathcal{C}_1 - \mathcal{C}_2$ then since $R \ge 2R_x$ we deduce that z is in a hole of size less than $R^{1/2}$, and therefore that the boundary of $\mathcal{H}(z)$ is outside $B_E(x, 2R/3)$. Hence $\{\tau_{B_E(x,3R/4)} < t\}$ holds. So we have in all cases that $\{\tau^Y < t\} \subset \{\tau_{B_E(x,3R/4)} < t\}$. In addition, if $x \in \mathcal{C}_1 - \mathcal{C}_2$ then the definition of R_x implies that $\mathcal{H}(x)$ has diameter less than $R^{1/2}$. Thus

$$\begin{aligned} P^x_\omega(\tau^Y < t) &\le P^x_\omega(\tau_{B_E(x,3R/4)} < t) \\ &\le E^x_\omega\big[P^{Z_{\mathfrak{a}_0}}_\omega(\tau_{B_E(x,2R/3)} < t)\big] \le \max_{y \in \partial\mathcal{H}(x)} P^y_\omega(\tau_{B_E(x,2R/3)} < t). \end{aligned}$$

Using (4.24) and replacing R by $3R/2$ we obtain (4.25).

A similar argument gives (4.26). □

4.2 Lower bounds

In this section we use the methods of [23], which in turn are based on Nash [34], to obtain a lower bound on $q^Z_t(x,y)$. Since the proofs are quite similar to those in [2], we do not give full details. The lower bound relies on two basic inputs: a bound which shows that Z does not escape too quickly from a ball (as in Proposition 4.7), and a weighted Poincaré inequality. Given these two inputs, the Fabes–Stroock–Nash argument gives a 'near diagonal lower bound'—that is a lower bound on $q^Z_t(x,y)$ when x and y are not too far apart. A standard chaining argument, as in [23], then gives a more general lower bound.

We begin by establishing the weighted Poincare inequality. Let $B = B_2(x_0, R)$ and

$$\varphi(y) = \left(\frac{R \wedge d_2(y, B_2(x_0,R)^c)}{R}\right)^2, \quad y \in \mathcal{C}_2.$$

Proposition 4.8 *Let $B = B_2(x_0,R)$ be N_B—very good with $N_B \le R^{1/(d+2)}$. Then*

$$\inf_\lambda \sum_{x\in B}(f(x)-\lambda)^2\varphi(x) \le CR^2 \sum_{x,y\in\mathcal{C}_2}(f(x)-f(y))^2(\varphi(x)\wedge\varphi(y))\mu'_{xy}.$$

Proof By [2, Theorem 4.8] we have

$$\inf_\lambda \sum_{x\in B}(f(x)-\lambda)^2\mu^0_x\varphi(x) \le CR^2 \sum_{x,y\in\mathcal{C}_2}(f(x)-f(y))^2(\varphi(x)\wedge\varphi(y))\mu^0_{xy}.$$

Since $\mu'_{xy} \ge K^{-1}\mu^0_{xy}$ and $\mu^0_x \asymp 1$, we have the result. □

Next we give the near diagonal lower bound. We write

$$q^{Z,B}_t(x,y) = P^x_\omega(Z_t = y, \tau_B < t)$$

for the heat kernel of Z killed on exiting from B.

Proposition 4.9 *Let $x_0 \in \mathcal{C}_2$ and suppose that $B_2(x_0, R_1)$ is $R_1^{1/(d+2)}$—very good ball for all $R_1 \geq R$. Then there exist constants c_i such that, writing $B = B_2(x_0, R)$,*

$$q_t^{Z,B}(y_1, y_2) \geq c_2 t^{-d/2}, \quad y_1, y_2 \in B_2\left(x, \tfrac{1}{4}R\right), \ c_1 R^2 \leq t \leq 2c_1 R^2. \qquad (4.31)$$

Proof The argument is almost the same as [2, Proposition 5.1]. The point in the proof where the long range jumps of Z could potentially cause a problem is that (as in [2, equation (5.9)]) we need, for $x \in B_2(x_0, \frac{1}{3}R)$, and $t \leq c_3 R^2$ that

$$\sum_{y \in B_2(x, 2R/3)} q_t^{Z,B}(x, y)\mu'_y \geq \tfrac{1}{2}.$$

However, this bound follows from Proposition 4.7a by taking the constant c_3 small enough. Note that the condition on B implies that $R \geq R_x$ for all $x \in B_2(x_0, R/2)$. □

Definition 4.10 For $x \in \mathcal{C}_2$ let S_x be the smallest integer R such that $B_2(x, n)$ is $n^{1/(3(d+2))}$—very good for all $n \geq R$.

Theorem 4.11 *There exist constants $\delta > 0$ and c such that the following holds. There exists a set $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ and $S_x, x \in \mathbb{Z}^d$ such that $S_x(\omega) < \infty$ for each $\omega \in \Omega_1$ and $x \in \mathcal{C}_2(\omega)$, and*

$$\mathbb{P}(S_x \geq n, x \in \mathcal{C}_2) \leq ce^{-cn^\delta}. \qquad (4.32)$$

(a) *For $x, y \in \mathcal{C}_2(\omega)$ the transition density of Z satisfies*

$$q_t^Z(x, y) \leq ct^{-d/2} \exp(-c|x-y|^2/t), \quad t \geq |x-y| \vee S_x, \qquad (4.33)$$
$$q_t^Z(x, y) \geq ct^{-d/2} \exp(-c|x-y|^2/t), \quad t \geq |x-y|^{3/2} \vee S_x. \qquad (4.34)$$

(b) *Further, if $x \in \mathcal{C}_2, t \geq S_x$ and $B = B_2(x, 2\sqrt{t})$ then*

$$q_t^{Z,B}(x, y) \geq ct^{-d/2}, \quad \textit{for } y \in B_2(x, \sqrt{t}).$$

Proof The upper bound in (a) is given in Corollary 4.6, while (b) follows from Proposition 4.9.

The lower bound in (a) is proved from Proposition 4.9 by a chaining argument—see Lemma 5.2 and Theorem 5.3 of [2]. □

Remark 4.12 Note that we only give Gaussian lower bounds in (4.34) when $|x-y| \leq t^{2/3}$. The power $2/3$ could be improved, but the arguments in Lemma 5.2 and Theorem 5.3 of [2] do not allow us to extend these bounds to $|x-y| \leq ct$. The reason is that the chaining argument works by connecting the points x and y by a chain of balls $B_2(z_i, r)$, where $r = O(t/|x-y|)$, and then using the lower bound (4.31) in each ball. For this we need (at least) that each ball $B_2(z_i, r)$ should be $r^{1/(d+2)}$ very good, and we cannot ensure this if r is too small.

In [2] a stronger result was obtained, by using the fact that the chaining argument does not require that every chain of balls connecting x and y is very good, but just that at least one such chain exists. By looking at a block percolation process of cubes side k (large but fixed) it was shown that there are enough 'good chains' so that Gaussian lower bounds can be obtained for $|x-y| \le ct$,

It is very likely that a similar argument could be made in this case. We do not do so because the improvement requires a considerable amount of extra work, and the lower bound (4.34) is already enough for most applications.

5 Invariance principle for the process Z

In this section we prove:

Theorem 5.1 (Quenched invariance principle for Z) *There exists $\delta > 0$ such that if K is large enough so that $q(K) < \delta$ the following holds. For $\mathbb{P}_2$-almost every environment, under P^0_ω, the process $(Z^{(\varepsilon)}_t, t \ge 0)$ converges in law as ε tends to zero to a non-degenerate Brownian motion with covariance matrix $\sigma^2_Z I$ where $\sigma^2_Z = \sigma^2_Z(K)$ is strictly positive and does not depend on ω.*

An invariance principle for a similar process, also jumping over holes with small conductances, has been proven in [33, Theorem 2.2]. However as we allow unbounded conductances, in general the process Z^K will jump over the holes in a different way to the process considered in [33]. Thus we cannot deduce Theorem 5.1 directly from Theorem 2.2 of [33].

5.1 Construction of the corrector

We assume that the conductances μ_e are defined on the space $(\Omega, \mathbb{P})$, where

$$\Omega = [0,\infty)^{E_d}.$$

We write $\mu_e(\omega) = \omega(e)$ for the coordinate maps as well as $\omega = (\omega(e), e \in E_d)$ and $\omega(x,y) = \omega(\{x,y\})$. For $x \in \mathbb{Z}^d$ define $T_x : \Omega \to \Omega$ by

$$T_x(\omega)(z,w) = \omega(z+x, w+x).$$

Recall from Sect. 3 the definition of μ'_{xy}: we have

$$\mu'_{xy} \circ T_z = \mu'_{x+z,y+z}.$$

The process $(T_{Z_t}(\omega), t \in [0,\infty))$ then gives the 'environment seen from the particle'. For $F \in L^2(\Omega, \mathbb{P})$ write $F_x = F \circ T_x$. Then (T_{Z_t}) has generator

$$\widehat{L} F(\omega) = \sum_{x \in \mathbb{Z}^d} \mu'_{0x}(\omega)(F_x(\omega) - F(\omega)).$$

Set

$$\widehat{\mathcal{E}}(F,G)=\mathbb{E}\sum_{x\in\mathbb{Z}^d}\mu'_{0x}(F-F_x)(G-G_x).$$

Lemma 5.2 (i) *For $F\in L^1(\Omega,\mathbb{P})$,*

$$\mathbb{E}F=\mathbb{E}F_x,$$
$$\mathbb{E}(\mu'_{0x}F_x)=\mathbb{E}(\mu'_{0,-x}F).$$

(ii) *For $F,G\in L^2(\Omega,\mathbb{P})$, $\widehat{\mathcal{E}}(F,F)<\infty$, $\widehat{\mathcal{E}}(F,G)$ is defined, $\widehat{L}F\in L^2(\Omega,\mathbb{P})$ and $\mathbb{E}(G\widehat{L}F)=-\widehat{\mathcal{E}}(F,G)$.*

Proof This follows by the same arguments as in Lemmas 5.2–5.4 in [4]. □

Now we look at 'vector fields'. We define for $G=G(\omega,x):\Omega\times\mathbb{Z}^d\to\mathbb{R}$,

$$\overline{\mathbb{E}}G=\sum_x\mathbb{E}_2\mu'_{0x}G(\cdot,x)=\mathbb{E}_2\sum_{x\in\mathcal{C}_2}\mu'_{0x}G(\cdot,x).$$

Note that $\overline{\mathbb{E}}G$ is not affected by $G(\omega,x)$ if $x\notin\mathcal{C}_2(\omega)$.

Definition We say $G(\omega,x)$ has the *cocycle property* if $\mathbb{P}_2$-a.s.,

$$G(T_x\omega,y-x)=G(\omega,y)-G(\omega,x),\quad\text{for all } x,y\in\mathcal{C}_2(\omega).\tag{5.1}$$

Let $\overline{L}^2$ be the set of vector fields G with the cocycle property and $||G||^2=\overline{\mathbb{E}}G^2<\infty$.

Lemma 5.3 *Let $G=G(\omega,x)\in\overline{L}^2$.*

(i) *For $\mathbb{P}_2$-a.e. ω, $G(\omega,0)=0$ and $G(T_x\omega,-x)=-G(\omega,x)$ for all $x\in\mathcal{C}_2$.*

(ii) *If $x_0,x_1,\dots,x_n\in\mathcal{C}_2$ then*

$$\sum_{i=1}^n G(T_{x_{i-1}}\omega,x_i-x_{i-1})=G(\omega,x_n)-G(\omega,x_0).\tag{5.2}$$

Proof (i) follows immediately from the definition. For (ii), as G has the cocycle property

$$G(T_{x_{i-1}}\omega,x_i-x_{i-1})=G(\omega,x_i)-G(\omega,x_{i-1}),$$

giving (5.2). □

It is easy to check:

Lemma 5.4 $\overline{L}^2$ *is a Hilbert space.*

For $F \in L^2$ we set

$$\nabla F(\omega, x) = F(T_x\omega) - F(\omega), \qquad x \in \mathbb{Z}^d.$$

Lemma 5.5 *If $F \in L^2(\Omega, \mathbb{P})$ then $\nabla F \in \overline{L}^2$.*

Proof First,

$$\overline{\mathbb{E}}|\nabla F|^2 = \sum_x \mathbb{E}_2 \mu'_{0x}(F_x - F)^2 \le \frac{\widehat{\mathcal{E}}(F, F)}{\mathbb{P}(0 \in \mathcal{C}_2)} < \infty.$$

Also, for $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} \nabla F(T_x\omega, y - x) &= F(T_{y-x}T_x\omega) - F(T_x\omega) \\ &= F(T_y\omega) - F(T_x\omega) = \nabla F(\omega, y) - \nabla F(\omega, x), \end{aligned}$$

so ∇F has the cocycle property. □

Lemma 5.6 *For every $G \in \overline{L}^2$ we have for all $x, y \in \mathbb{Z}^d$,*

$$\mathbb{E}_2\left[\mu'_{xy}|G(\cdot, y) - G(\cdot, x)|^2\right] \le ||G||^2.$$

Proof Recall that $\mu'_{xy} \neq 0$ only if $x, y \in \mathcal{C}_2$. Using the cocycle property and the shift-invariance of $\mathbb{P}$ we get

$$\begin{aligned} \mathbb{E}_2\left[\mu'_{xy}|G(\cdot, y) - G(\cdot, x)|^2\right] &= \frac{\mathbb{E}\left[\mu'_{xy}|G(T_x\omega, y - x)|^2 1_{\{x\in\mathcal{C}_2\}}\, 1_{\{0\in\mathcal{C}_2\}}\right]}{\mathbb{P}[0 \in \mathcal{C}_2]} \\ &\le \frac{\mathbb{E}\left[\mu'_{0,y-x}(T_x\omega)|G(T_x\omega, y - x)|^2 1_{\{0\in\mathcal{C}_2(T_x\omega)\}}\right]}{\mathbb{P}[0 \in \mathcal{C}_2]} \\ &= \frac{\mathbb{E}\left[\mu'_{0,y-x}|G(\cdot, y - x)|^2 1_{\{0\in\mathcal{C}_2\}}\right]}{\mathbb{P}[0 \in \mathcal{C}_2]} \\ &\le \sum_z \mathbb{E}_2\left[\mu'_{0,z}|G(\cdot, z)|^2\right] = ||G||^2. \end{aligned}$$

□

Proposition 5.7 (Polynomial growth) *Let $G \in \overline{L}^2$, and $\theta > d$. Then, $\mathbb{P}_2$-a.s.,*

$$\lim_{n\to\infty} \max_{|x|\le n, x\in\mathcal{C}_2} \frac{|G(\omega, x)|}{n^\theta} = 0. \tag{5.3}$$

Proof We use the same argument as in Theorem 4.1 (4) in [14]. By Proposition 2.8 we have that $B_2(0, n)$ is good for all sufficiently large n, $\mathbb{P}_2$-a.s. So, using the property (2.22) of good balls it is sufficient to prove that $\lim_{n\to\infty} n^{-\theta} R_n(G) = 0$, where

$$R_n(G) = \max_{x\in B_2(0,n)} |G(\omega, x)|. \tag{5.4}$$

If $x \in B_2(0, n)$ then there exists a path $0 = y_0, y_1, \dots, y_k = x$ connecting 0 and x in $\mathcal{C}_2$. By Lemma 5.3

$$G(\omega, x) \le \sum_{i=1}^{k} |G(\omega, y_i) - G(\omega, y_{i-1})| \le \sum_{y\in B_2(0,n)} \sum_{z\sim y} |G(\omega, y) - G(\omega, z)|. \tag{5.5}$$

Thus since $\mu'_{yz} \ge K^{-1}$ for $y, z \in \mathcal{C}_2$ with $y \sim z$,

$$\begin{aligned} R_n(G) &\le \sum_{y\in B_2(0,n)} \sum_{z\sim y} \left(\mu'_{yz} K\right)^{1/2} |G(\omega, y) - G(\omega, z)| \\ &\le K^{1/2} \sum_{y\in B_2(0,n)} \sum_{z\sim y} \left(\mu'_{yz}\right)^{1/2} |G(\omega, y) - G(\omega, z)| \\ &\le K^{1/2} \left(\sum_{y\in B_2(0,n)} \sum_{z\sim y} \mu'_{yz} |G(\omega, y) - G(\omega, z)|^2 \right)^{1/2} \left(cn^d\right)^{1/2}; \end{aligned} \tag{5.6}$$

here we used Cauchy–Schwarz in the final line. We take expectations and use Lemma 5.6 and the fact that $B_2(0, n) \subset B_E(0, n)$ to obtain

$$\mathbb{E}_2 R_n(G)^2 \le c_1 n^d\, \mathbb{E}_2 \sum_{y\in B_E(0,n)} \sum_{z\sim y} \mu'_{yz}\, |G(\omega, y) - G(\omega, x)|^2 \le c_2\, n^{2d}\, ||G||^2.$$

Applying Chebyshev's inequality and summing n over powers of 2 a Borel–Cantelli argument now gives $R_n(G)/n^\theta \to 0$ a.s. □

Following [32] we introduce an orthogonal decomposition of the space $\overline{L}^2$. Set

$$\overline{L}^2_p = \mathrm{cl}\, \{\nabla F,\ F \in L^2\} \text{ in } \overline{L}^2,$$

and let $\overline{L}^2_s$ be the orthogonal complement of $\overline{L}^2_p$ in $\overline{L}^2$. (Here p stands for 'potential' and s for 'solenoidal'.)

Fix $b \in \mathbb{Z}^d$ with $|b| = 1$ and recall the definition of ζ from (2.12). Let $\sigma_b(\omega) = T_{\zeta(\omega)}\omega$. A key fact is that by Theorem 3.2 in [11] the shift σ_b is $\mathbb{P}_2$-preserving and ergodic with respect to $\mathbb{P}_2$. We define the iterates $\zeta_k : \Omega \to \mathcal{C}_2$ by $\zeta_1 = \zeta$,

$$\zeta_{k+1}(\omega) := \zeta(T_{\zeta_k(\omega)}(\omega)), \quad k \ge 2.$$

Lemma 5.8 *Let $G \in \overline{L}^2_p$. Then*

(i) $\mathbb{E}_2|G(\cdot,\zeta(\cdot))| < \infty$,
(ii) $\mathbb{E}_2 G(\cdot,\zeta(\cdot)) = 0$.

Proof As $G \in \overline{L}^2_p$ there exists a sequence of functions F_n in L^2 such that the sequence $G_n = \nabla F_n$ converges to G in $\overline{L}^2$. Since $\mathbb{P}_2$ is preserved by $\sigma_b = T_\zeta$ we have for all n

$$\mathbb{E}_2\left[G_n(\cdot,\zeta(\cdot))\right] = \mathbb{E}_2[F_n \circ T_\zeta] - \mathbb{E}_2[F_n] = 0.$$

Thus, it suffices to show that $G_n(\cdot,\zeta(\cdot)) \to G(\cdot,\zeta)$ in $L^1(\Omega,\mathbb{P}_2)$.

We begin with the following estimate. Let $R'_0 = R_0^{(1)}$ be as in Definition 2.7. Then if $R'_0 \le n$ and $0 \in \mathcal{C}_2$ and $d_2(0,\zeta) > n$ then by (2.22) we have $|\zeta| \ge cn$. So, with β as in Proposition 2.8, and using Lemma 2.4 to bound the tail of ζ,

$$\begin{aligned}\mathbb{P}(d_2(0,\zeta) > n, 0 \in \mathcal{C}_2) &\le \mathbb{P}(R'_0 > n, 0 \in \mathcal{C}_2) + \mathbb{P}(d_2(0,\zeta) > n, R'_0 \le n, 0 \in \mathcal{C}_2)\\ &\le c_1 \exp(-c'n^\beta) + \mathbb{P}(|\zeta| \ge cn, 0 \in \mathcal{C}_2) \le c_1 \exp(-c_2 n^\beta).\end{aligned}$$

By (5.5) and (5.6) we have writing $H_n = G - G_n$, $D = d_2(0,\zeta)$,

$$|H_n(\omega,\zeta)| \le cD^{d/2} S_D(H_n)^{1/2},$$

where for $k \ge 1$

$$S_k(H_n) = \sum_{y \in B_2(0,k)} \sum_{z \sim y} \mu'_{yz} |H_n(\omega,y) - H_n(\omega,z)|^2.$$

Then

$$\begin{aligned}\mathbb{E}_2|H_n(\omega,\zeta)| &\le c \sum_{k=1}^\infty \mathbb{E}_2(k^{d/2} S_k(H_n)^{1/2}; D = k)\\ &\le c \sum_{k=1}^\infty k^{d/2} (\mathbb{E}_2 S_k(H_n))^{1/2} \mathbb{P}_2(D = k)^{1/2}\\ &\le c \sum_{k=1}^\infty k^{d/2} (k^d ||H_n||^2)^{1/2} \exp(-ck^\beta) \le c_1 ||H_n||.\end{aligned}$$

Since $||H_n|| \to 0$ we have $H_n \to 0$ in $L^1(\mathbb{P}_2)$, which completes the proof. □

Lemma 5.9 *Let $G \in \overline{L}^2_p$. Then we have for $\mathbb{P}_2$-a.e. ω*

$$\lim_{k\to\infty} \frac{G(\cdot,\zeta_k)}{k} = 0.$$

Proof Let $F(\omega) = G(\omega, \zeta(\omega))$ and $\sigma_b(\omega) = T_{\zeta(\omega)}\omega$ be the induced shift. Then, by the cocycle property we can write

$$G(\omega, \zeta_k(\omega)) = \sum_{i=0}^{k-1} F \circ \sigma_b^i(\omega).$$

By Lemma 5.8 we have $F \in L^1(\Omega, \mathbb{P}_2)$ and $\mathbb{E}_2 F = 0$. Since σ_b is ergodic with respect to $\mathbb{P}_2$, the claim follows by the ergodic theorem. □

Proposition 5.10 (Sublinearity on average) *Let $G \in \overline{L}^2_p$. For each $\varepsilon > 0$,*

$$\lim_{n\to\infty} n^{-d} \sum_{|x|\le n, x\in\mathcal{C}_2} 1_{(|G(\omega,x)|>\varepsilon n)} = 0 \quad \textit{for } \mathbb{P}_2\textit{-a.e. } \omega.$$

Proof This follows from Lemma 5.9 exactly as Theorem 5.4 in [11]. □

Remark In [4, Theorem 5.12(d)] it was incorrectly stated that sublinearity followed from results in [28] – in fact one needs [11, Theorem 5.4].

Proposition 5.11 (Harmonicity) *Let $G \in \overline{L}^2_s$. Then, for $\mathbb{P}_2$-a.e. ω we have for all $x \in \mathcal{C}_2$*

$$\mathcal{L}_Z^\omega G(\omega, x) = \sum_{y\in\mathcal{C}_2} \mu'_{xy}(\omega)(G(\omega, y) - G(\omega, x)) = 0. \tag{5.7}$$

Hence $N_t = G(\omega, Z_t)$ is a P_ω^0-martingale for $\mathbb{P}_2$-a.e. ω. Further, writing

$$\|G(\omega, \cdot)\|_\omega^2 = \sum_x \mu'_{0x}(\omega) G(\omega, x)^2,$$

we have

$$\langle N\rangle_t = \int_0^t \|G(T_{Z_s}\omega, \cdot)\|_\omega^2 \, ds. \tag{5.8}$$

Proof We will first show that for $G \in \overline{L}^2_s$

$$\mathcal{L}_Z^\omega G(0) = \sum_{x\in\mathcal{C}_2} \mu'_{0x}(\omega) G(\omega, x) = 0, \qquad \mathbb{P}_2\text{-a.s.} \tag{5.9}$$

To that aim we proceed similarly to Lemma 5.11 in [4]. If $F \in L^2(\Omega, \mathbb{P})$ and $G \in \overline{L}^2$ then using Lemma 5.3 and the fact that $\mu'_{0x} = 0$ for all x if $0 \notin \mathcal{C}_2$ we get

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\mathbb{E}_2\mu'_{0x}G(\omega,x)F_x &= \mathbb{P}(0\in\mathcal{C}_2)^{-1}\sum_{x\in\mathbb{Z}^d}\mathbb{E}\mu'_{0x}G(\omega,x)F_x 1_{\{0\in\mathcal{C}_2\}}\\ &= \mathbb{P}(0\in\mathcal{C}_2)^{-1}\sum_{x\in\mathbb{Z}^d}\mathbb{E}\mu'_{0x}(T_{-x}\omega)G(T_{-x}\omega,x)F_x(T_{-x}\omega)\\ &= \mathbb{P}(0\in\mathcal{C}_2)^{-1}\sum_{x\in\mathbb{Z}^d}\mathbb{E}\mu'_{0,-x}(\omega)(-G(\omega,-x))F(\omega)\\ &= -\sum_{x\in\mathbb{Z}^d}\mathbb{E}_2\mu'_{0x}(\omega)G(\omega,x)F(\omega).\end{aligned}$$

Thus

$$\sum_{x\in\mathbb{Z}^d}\mathbb{E}_2\mu'_{0x}G(.,x)(F+F_x)=0.$$

If $G\in\overline{L}^2_s$ then

$$0=\overline{\mathbb{E}}(G\nabla F)=\sum_x\mathbb{E}_2\mu'_{0x}G(\cdot,x)(F_x-F),$$

and so $\mathbb{E}_2\sum\mu'_{0x}GF=0$. Since this holds for any $F\in L^2(\Omega,\mathbb{P})$ we obtain (5.9). Further, for any $x\in\mathcal{C}_2$

$$\begin{aligned}\mathbb{E}_2|\mathcal{L}_Z G(x)| &= \mathbb{E}_2\left|\sum_y \mu'_{xy}(\omega)(G(\omega,y)-G(\omega,x))\right|\\ &= \mathbb{P}(0\in\mathcal{C}_2)^{-1}\mathbb{E}\left|\sum_y\mu'_{0,y-x}(T_x\omega)G(T_x\omega,y-x)\right|1_{\{0\in\mathcal{C}_2\}}\\ &\le \mathbb{P}(0\in\mathcal{C}_2)^{-1}\mathbb{E}\left|\sum_z\mu'_{0z}(\omega)G(\omega,z)\right|\\ &= \mathbb{E}_2|\mathcal{L}_Z G(0)|=0,\end{aligned}$$

which implies (5.7). Thus, $N_t=G(\omega,Z_t)$ is a P^0_ω-martingale for $\mathbb{P}_2$-a.e. ω. To compute $\langle N\rangle$, which is the unique predictable process such that $N_t^2-\langle N\rangle_t$ is a martingale, note that the opérateur carré du champ is given by

$$\begin{aligned}\left[\mathcal{L}_Z G^2-2G\mathcal{L}_Z G\right](x) &= \sum_y\mu'_{xy}(\omega)\left(G(\omega,y)-G(\omega,x)\right)^2\\ &= \sum_y\mu'_{0,y-x}(T_x\omega)G(T_x\omega,y-x)^2=||G(T_x\omega,\cdot)||^2_\omega,\end{aligned}$$

for $\mathbb{P}_2$-a.e. ω and (5.8) follows. □

Let $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ be the identity, and write Π_j for the jth coordinate of Π. Then $\Pi_j(y-x) = \Pi_j(y) - \Pi_j(x)$, so Π_j has the cocycle property. Further

$$\overline{\mathbb{E}}|\Pi_j|^2 = \mathbb{E}_2 \sum_x \mu'_{0x}|x_j|^2 \le 2dK\,\mathbb{E}_2 \sum_x \frac{\mu'_{0x}}{\mu'_0}|x_j|^2 = 2dK\,\mathbb{E}_2 E^0_\omega(Z^j_{\tau_Z})^2 < \infty,$$

τ_Z denoting the first jump time of Z, so $\Pi_j \in \overline{L}^2$. So we can define $\chi_j \in \overline{L}^2_p$ and $\Phi_j \in \overline{L}^2_s$ by

$$\Pi_j = \chi_j + \Phi_j \in \overline{L}^2_p \oplus \overline{L}^2_s;$$

this gives our definition of the corrector $\chi = (\chi_1, \dots, \chi_d) : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$. We will sometimes write $\chi(x)$ for $\chi(\cdot, x)$ and $\Phi(x)$ for $\Phi(\cdot, x)$. Note that conventions about the sign of the corrector differ – compare [39] and [14]. As the environment process is invariant under isometries of $\mathbb{Z}^d$, $||\Phi_j|| = ||\Phi_1||$ for each $j = 1, \dots d$. We set

$$M_t = \Phi(\omega, Z_t) = Z_t - \chi(\omega, Z_t). \tag{5.10}$$

The following Proposition summarizes the properties of χ, Φ and M.

Proposition 5.12 (i) *For $\mathbb{P}_2$-a.e. ω and for every $v \in \mathbb{R}^d$, M and $v \cdot M$ are P^0_ω—martingales. The covariance process of the latter is given by*

$$\langle v \cdot M\rangle_t = \int_0^t \|v \cdot \Phi(T_{Z_s}\omega, \cdot)\|^2_\omega \, ds.$$

(ii) *For each $j = 1, \dots, d$*

$$\mathbb{E}_2 \sum_{x \in \mathcal{C}_2} \mu'_{0x}(\omega)|\Phi_j(\omega, x)|^2 = ||\Phi_1||^2 < \infty.$$

(iii) *χ has polynomial growth: for $\theta > d$*

$$\lim_{n\to\infty} \max_{\substack{|x|\le n\\ x\in\mathcal{C}_2}} \frac{|\chi(\omega, x)|}{n^\theta} = 0 \qquad \mathbb{P}_2\text{-}a.s.$$

(iv) *χ is sublinear on average: for each $\varepsilon > 0$*

$$\lim_{n\to\infty} n^{-d} \sum_{\substack{|x|\le n\\ x\in\mathcal{C}_2}} 1_{(|\chi(\omega,x)|>\varepsilon n)} = 0, \quad \mathbb{P}_2\text{-}a.s. \tag{5.11}$$

5.2 Proof of Theorem 5.1

We have

$$Z_t^{(\varepsilon)} = \varepsilon Z_{t/\varepsilon^2} = M_t^{(\varepsilon)} + \varepsilon\chi(\omega, \varepsilon^{-1} Z_t^{(\varepsilon)}). \tag{5.12}$$

To prove Theorem 5.1 it is sufficient to prove (1) that the processes $Z^{(\varepsilon)}$ are tight, (2) that the martingales $M^{(\varepsilon)}$ converges to a multiple of Brownian motion, and (3) that for $\mathbb{P}_2$-a.e. ω the final term in (5.12) converges in P_ω^0-probability to zero. We begin with tightness.

Proposition 5.13 (Tightness)

(a) *Let $T > 0, r > 0$. Then, for $\mathbb{P}_1$-a.e. ω,*

$$\lim_{R\to\infty} \sup_{0<\varepsilon\le 1} P_\omega^0\left(\sup_{s\le T} |Z_s^{(\varepsilon)}| > R\right) \to 0, \tag{5.13}$$

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} P_\omega^0\left(\sup_{|s_1-s_2|\le\delta, s_i\le T} |Z_{s_2}^{(\varepsilon)} - Z_{s_1}^{(\varepsilon)}| > r\right) = 0. \tag{5.14}$$

In particular, for $\mathbb{P}_1$-a.e. ω, under P_ω^0, the family of processes $(Z_t^\varepsilon)_{t\ge 0}$ is tight in the Skorohod topology.

(b) *The same statements hold for the processes $Y^{(\varepsilon)}$, for $\mathbb{P}_1$-a.e. ω.*

Proof (a) Recall the definition of $R_x^{(\alpha)}$ in Definition 2.7, and as in Proposition 4.7 let $\alpha \in (0, \frac{1}{2})$. Let $R/\varepsilon > 3R_0$. Then by Proposition 4.7c,

$$P_\omega^0\left(\sup_{s\le T} |Z_s^{(\varepsilon)}| > R\right) = P_\omega^0\left(\tau_{B_E(0,R/\varepsilon)} < T/\varepsilon^2\right) \le c_1\Psi\left(c_2 R/\varepsilon, T/\varepsilon^2\right).$$

Considering separately the cases $\varepsilon < T/R$ and $\varepsilon \ge T/R$ we deduce that

$$P_\omega^0\left(\sup_{s\le T} |Z_s^{(\varepsilon)}| > R\right) \le c_3 e^{-c_4 R^2/T} \vee e^{-R},$$

which gives (5.13)

The proof of (5.14) is similar to that in [4, Theorem 4.11]. Write

$$p(T, \delta, r) = P_\omega^0\left(\sup_{|s_1-s_2|\le\delta, s_i\le T} |Z_{s_2} - Z_{s_1}| > r\right), \tag{5.15}$$

so that

$$P_\omega^0\left(\sup_{|s_1-s_2|\le\delta, s_i\le T} |Z_{s_2}^{(\varepsilon)} - Z_{s_1}^{(\varepsilon)}| > r\right) = p(T/\varepsilon^2, \delta/\varepsilon^2, r/\varepsilon).$$

We begin by bounding $p(T,\delta,r)$ for fixed T, δ and r. Let

$$V_k = \sup_{0\le s\le\delta} |Z_{k\delta+s} - Z_{k\delta}|. \tag{5.16}$$

Then if $K = \lfloor T/\delta \rfloor$ and $V^* = \max_{0\le k\le K} V_k$, it is enough to control V^* since

$$\sup_{|s_1-s_2|\le\delta, s_i\le T} |Z_{s_2} - Z_{s_1}| \le 2V^*.$$

Let $R = T^{3/4}$ and write $\tau(y,r) = \tau_{B_E(y,r)}$. Then

$$P_\omega^0(V^* \ge r) \le P_\omega^0(\tau(0,R) \le T) + P_\omega^0(V^* \ge r, \tau(0,R) > T). \tag{5.17}$$

By Proposition 4.7c we have

$$P_\omega^0(\tau(0,R) \le T) \le c\exp(-c'R^2/T) = ce^{-c'T^{1/2}}, \text{ provided that } T^{3/4} \ge R_0. \tag{5.18}$$

Also,

$$\begin{aligned} P_\omega^0(V^* \ge r, \tau(0,R) > T) &\le \sum_{k=0}^{K} P_\omega^0(V_k \ge r, Z_{k\delta} \in B_E(0,R)) \\ &\le \sum_{k=0}^{K} \sum_{y\in B_E(0,R)} P_\omega^y(\tau(y,r) < \delta) P_\omega^0(Z_{k\delta} = y). \end{aligned}$$

Again by Proposition 4.7b, for $y \in B_E(0,R) \cap \mathcal{C}_2$,

$$P_\omega^y(\tau(y,r) < \delta) \le ce^{-cr^2/\delta}, \tag{5.19}$$

provided

$$r \ge \max_{y\in B_E(0,R)} R_y \quad \text{and} \quad \delta \ge r. \tag{5.20}$$

Combining (5.17), (5.18) (5.19), we obtain

$$p(T,\delta,2r) \le P_\omega^0(V^* \ge r) \le c\exp\left(-cT^{1/2}\right) + c(T/\delta)\exp\left(-cr^2/\delta\right), \tag{5.21}$$

provided $T \geq R_0(\omega)^{4/3}$, and (5.20) holds.
Hence

$$p\left(T/\varepsilon^2, \delta/\varepsilon^2, 2r/\varepsilon\right) \leq c \exp\left(-cT^{1/2}/\varepsilon\right) + c(T/\delta) \exp\left(-cr^2/\delta\right), \quad (5.22)$$

provided

$$T^{1/2} \geq \varepsilon R_0^{2/3}, \quad \delta > \varepsilon r, \quad r \geq \varepsilon \max_{y \in B_E(0, T^{3/4}\varepsilon^{-3/2})} R_y.$$

If r and δ are fixed, by Corollary 2.9 each of these conditions holds when ε is small enough. So, for $\mathbb{P}_2$-a.a. ω,

$$\limsup_{\varepsilon \to 0} p\left(T/\varepsilon^2, \delta/\varepsilon^2, 2r/\varepsilon\right) \leq c(T/\delta) \exp\left(-cr^2/\delta\right),$$

and (5.14) follows.

(b) The only property of Z that is used in the argument above is the estimate (4.26). The same arguments therefore give tightness for Y, using (4.25). □

Next, we show that the term in (5.12) involving the corrector converges to 0.

Proposition 5.14 *Let $T > 0$. For $\mathbb{P}_2$-a.e. ω,*

$$\sup_{s \leq T} \varepsilon \left| \chi(\omega, \varepsilon^{-1} Z_s^{(\varepsilon)}) \right| \to 0 \quad \textit{in } P_\omega^0\textit{-probability}. \quad (5.23)$$

Proof We use [14, Theorem 2.4]. This results states that if the corrector χ has polynomial growth, and is sublinear on average, then Gaussian upper bounds on the heat kernel imply pointwise sublinearity of χ. Thus, using (4.21), (5.3) and (5.11) we have that for $\mathbb{P}_2$-a.e. ω,

$$\lim_{n \to \infty} \max_{|x| \leq n, x \in \mathcal{C}_2} \frac{|\chi(\omega, x)|}{n} = 0. \quad (5.24)$$

To prove the claim let $\eta > 0$ and $R > 0$. Then,

$$\begin{aligned}
&P_\omega^0\left(\sup_{s \leq T} \varepsilon |\chi\left(\omega, \varepsilon^{-1} Z_s^{(\varepsilon)}\right)| > \eta\right) \\
&\leq P_\omega^0\left(\sup_{s \leq T} \varepsilon \left|\chi\left(\omega, \varepsilon^{-1} Z_s^{(\varepsilon)}\right)\right| > \eta, \sup_{s \leq T} |Z_s^{(\varepsilon)}| \leq R\right) + P_\omega^0\left(\sup_{s \leq T} |Z_s^{(\varepsilon)}| > R\right) \\
&\leq P_\omega^0\left(\max_{|y| \leq R/\varepsilon, y \in \mathcal{C}_2} \varepsilon\, |\chi\, (\omega, y)| > \eta\right) + P_\omega^0\left(\sup_{s \leq T} |Z_s^{(\varepsilon)}| > R\right). \quad (5.25)
\end{aligned}$$

The tightness of $Z^{(\varepsilon)}$ (see (5.13)) implies that the second term in (5.25) converges to zero uniformly in ε as $R \to \infty$. The first term converges to zero as $\varepsilon \to 0$ by (5.24).

□

For the convergence of $M^{(\varepsilon)}$, we proceed as in [32].

Proposition 5.15 *For $\mathbb{P}_1$-a.e. ω, the sequence of processes $(M^{(\varepsilon)})$ converges in law in the Skorohod topology to a Brownian motion with covariance matrix $\sigma_Z^2 I$, where $\sigma_Z^2 := \overline{\mathbb{E}}[\Phi_1^2] \in (0, \infty)$.*

Proof The proof is based on the martingale convergence theorem by Helland [27, see Theorem 5.1a]. In particular, we will show that for every $v \in \mathbb{R}^d$ the family of martingales $(v \cdot M_t^{(\varepsilon)})_{t \geq 0}$ with associated covariance processes $\langle v \cdot M^{(\varepsilon)} \rangle$ satisfy the following two conditions for $\mathbb{P}_2$-a.e. ω:

(i) For any $t > 0$ we have that $\langle v \cdot M^{(\varepsilon)} \rangle_t$ converges in P_ω^0-probability to $t \cdot \mathbb{E}_2 \| v \cdot \Phi(\omega, \cdot)\|_\omega^2$ as ε tends to zero.

(ii) For any $t > 0$ and any $\eta > 0$, we have

$$\sum_{0 \leq s \leq t} \left(v \cdot M_s^{(\varepsilon)} - v \cdot M_{s-}^{(\varepsilon)} \right)^2 1_{\{|v \cdot M_s^{(\varepsilon)} - v \cdot M_{s-}^{(\varepsilon)}| \geq \eta\}} \to 0,$$

in P_ω^0-probability as ε tends to zero.

Then, by Helland's martingale convergence theorem the sequence of processes $(v \cdot M_t^{\varepsilon, j})_{t \geq 0}$ converges in law in the Skorohod topology to a Brownian motion with covariance $\mathbb{E}_2 \| v \cdot \Phi(\omega, \cdot)\|_\omega^2$.

In order to prove (i) and (ii) we will use the ergodicity of the processes $(T_{Z_t}\omega, t \geq 0)$ and $(T_{Z_{t-}}\omega, t \geq 0)$, respectively, w.r.t. $\mathbb{P}_2$—see Lemma 4.9 in [21]. Note that the functional $F(\omega) := \| v \cdot \Phi(\omega, \cdot)\|^2 \in L^1(\Omega, \mathbb{P}_2)$, so we obtain by the ergodic theorem that for any $t > 0$ and for $\mathbb{P}_2$-a.e. ω

$$\frac{1}{t} \langle v \cdot M^{(\varepsilon)} \rangle_t = \frac{\varepsilon^2}{t} \int_0^{t/\varepsilon^2} \| v \cdot \Phi(T_{Z_s}\omega, \cdot)\|_\omega^2 \, ds \to \mathbb{E}_2 \| v \cdot \Phi(\omega, \cdot)\|_\omega^2,$$

as ε tends to zero and (i) is proven. To prove (ii) we recall that for any function $f : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ that vanishes on the diagonal, the process

$$\sum_{0 \leq s \leq t} f(Z_{s-}, Z_s) - \int_{(0,t]} \sum_y \mu'_{Z_{s-}, y} f(Z_{s-}, y) \, ds$$

is a local P_ω^0-martingale for $\mathbb{P}_2$-a.e. ω. Let $L > 0$. Then choosing

$$f(x, y) = (v \cdot \Phi(y) - v \cdot \Phi(x))^2 \, 1_{\{|v \cdot \Phi(y) - v \cdot \Phi(x)| \geq L\}}$$

we obtain by the cocycle property and the ergodic theorem that for $\mathbb{P}_2$-a.e. ω

$$
\begin{aligned}
&E_\omega^0\left[\frac{1}{t}\sum_{0\le s\le t}(v\cdot M_s - v\cdot M_{s-})^2 1_{\{|v\cdot M_s - v\cdot M_{s-}|\ge L\}}\right]\\
&= E_\omega^0\left[\frac{1}{t}\sum_{0\le s\le t}(v\cdot\Phi(\omega,Z_s) - v\cdot\Phi(\omega,Z_{s-}))^2\, 1_{\{|v\cdot\Phi(\omega,Z_s)-v\cdot\Phi(\omega,Z_{s-})|\ge L\}}\right]\\
&= \frac{1}{t}\int_0^t ds\, E_\omega^0\left[\sum_y \mu'_{Z_{s-},y}(\omega)\,(v\cdot\Phi(\omega,y)-v\cdot\Phi(\omega,Z_{s-}))^2\, 1_{\{|v\cdot\Phi(\omega,y)-v\cdot\Phi(\omega,Z_{s-})|\ge L\}}\right]\\
&= \frac{1}{t}\int_0^t ds\, E_\omega^0\left[\sum_y \mu'_{0,y-Z_{s-}}(T_{Z_{s-}}\omega)\left(v\cdot\Phi(T_{Z_{s-}}\omega, y-Z_{s-})\right)^2 1_{\{|v\cdot\Phi(T_{Z_{s-}}\omega,y-Z_{s-})|\ge L\}}\right]\\
&\to \mathbb{E}_2\left[\sum_y \mu'_{0y}(\omega)(v\cdot\Phi(\omega,y))^2 1_{\{|v\cdot\Phi(\omega,y)|\ge L\}}\right] = \overline{\mathbb{E}}\left[(v\cdot\Phi)^2 1_{\{|v\cdot\Phi|\ge L\}}\right] < \infty,
\end{aligned}
$$

as t tends to infinity. Let $\eta > 0$, $L < \infty$ and take $\varepsilon < \eta/L$. Then

$$
\begin{aligned}
&E_\omega^0\sum_{0\le s\le t}\left(v\cdot M_s^{(\varepsilon)} - v\cdot M_{s-}^{(\varepsilon)}\right)^2 1_{\{|v\cdot M_s^{(\varepsilon)}-v\cdot M_{s-}^{(\varepsilon)}|\ge\eta\}}\\
&= \varepsilon^2 E_\omega^0\sum_{0\le s\le t/\varepsilon^2}(v\cdot M_s - v\cdot M_{s-})^2\, 1_{\{|v\cdot M_s-v\cdot M_{s-}|\ge\eta/\varepsilon\}}\\
&\le \varepsilon^2 E_\omega^0\sum_{0\le s\le t/\varepsilon^2}(v\cdot M_s - v\cdot M_{s-})^2\, 1_{\{|v\cdot M_s-v\cdot M_{s-}|\ge L\}}\\
&\to t\,\overline{\mathbb{E}}(v\cdot\Phi)^2 1_{\{|v\cdot\Phi|\ge L\}}
\end{aligned}
$$

as ε tends to zero. We let L tend to infinity and obtain ii). Hence $v\cdot M^{(\varepsilon)}$ converges to a real-valued Brownian motion with non-random covariance $\mathbb{E}_2\|v\cdot\Phi(\omega,\cdot)\|_\omega^2$, which can be written as $v\cdot Dv$, where D is the matrix with coefficients given by $D_{ij} = \overline{\mathbb{E}}\Phi_i\Phi_j$. By the Cramer–Wold Theorem (see e.g. Theorem 3.9.5 in [22]) we get that $M^{(\varepsilon)}$ converges in law to an $\mathbb{R}^d$-valued Brownian motion with covariance matrix D. Since the law of the random variables $\omega(e)$ is invariant under symmetries of $\mathbb{Z}^d$, we deduce that $D = \sigma_Z^2 I$ with $\sigma_Z = \overline{\mathbb{E}}\Phi_1^2$.

It remains to show that σ_Z is strictly positive. However, if $\sigma_Z^2 = 0$ then $\Phi = 0$, and therefore $\chi(x) = x$ which contradicts the pointwise sublinearity in (5.24). We remark that an alternative way to show that $\sigma_Z > 0$ would be to use the heat kernel upper bound in Proposition 4.1, as on page 271 of [4]. □

Remark 5.16 We can extend Theorem 5.1 to all initial points $x \in \mathcal{C}_2$. For each $x \in \mathbb{Z}^d$ let H_x be the set of ω such that $x \in \mathcal{C}_2(\omega)$ but the invariance principle fails for the process Z started at 0 in the environment $T_x(\omega)$. Then Theorem 5.1 gives that $\mathbb{P}(H_0) = 0$. However $\omega \in H_x$ if and only if $T_x(\omega) \in H_0$, so since T_x is measure preserving, we have $\mathbb{P}(H_x) = 0$ for all x, and thus $\mathbb{P}(\cup_{x\in\mathbb{Z}^d} H_x) = 0$.

It follows from this that the conclusion of Theorem 5.1 holds $\mathbb{P}_1$-a.s. Suppose $\omega \notin \cup_{x\in\mathbb{Z}^d} H_x$, and $0 \in \mathcal{C}_1(\omega) - \mathcal{C}_2(\omega)$. Then $Z_0 = Y_{\mathfrak{a}_0}$, and so Z_0 is on the boundary of the hole $\mathcal{H}_0$. Since the invariance principle holds P^y_ω-a.s. for all $y \in \mathcal{C}_2(\omega)$, it will also hold $P^{Z_0}_\omega$-a.s.

6 Invariance principles for the VSRW and CSRW

In this section we will deduce the invariance principles for the VSRW Y and CSRW X stated in our main result Theorem 1.1 from the invariance principle for the process Z. First recall that the tightness of $Y^{(\varepsilon)}$ has already been proven in Proposition 5.13. In order to identify the limit, we will show that the increments of $Y^{(\varepsilon)}$ converge, using arguments similar to Section 3 in [33]. Finally, we repeat the argument in [4] to obtain the invariance principle for the CSRW X.

Recall from Sect. 3 the definition of the processes A, $\mathfrak{a}$ and Z. In particular, we have

$$Z^{(\varepsilon)}_t = Y^{(\varepsilon)}_{\varepsilon^2 \mathfrak{a}_{t/\varepsilon^2}}, \qquad t \geq 0. \tag{6.1}$$

We start with a lemma dealing with the long-time behaviour of the additive functional A (cf. [33, Lemma 2.4]).

Lemma 6.1

$$\lim_{t\to\infty} \frac{A_t}{t} = \mathbb{P}_1(0 \in \mathcal{C}_2) =: C_0 > 0, \qquad \mathbb{P}_1 \times P^0_\omega\text{-}a.s. \tag{6.2}$$

Proof Consider the process $(T_{Y_t}\omega, t \geq 0)$ of the 'environment seen by the particle' associated with the VSRW Y. Then, by Lemma 4.9 in [21] the measure $\mathbb{P}_1$ is ergodic w.r.t. $T_{Y_\cdot}\omega$. Since

$$A_t = \int_0^t 1_{\{0\in\mathcal{C}_2(T_{Y_s}\omega)\}}\, ds,$$

(6.2) follows by the ergodic theorem. □

6.1 VSRW

In order to identify the limit of the sequence $Y^{(\varepsilon)}$ we write

$$Y^{(\varepsilon)}_t = \varepsilon\left(Y_{t/\varepsilon^2} - Z_{A_{t/\varepsilon^2}}\right) + \varepsilon\left(Z_{A_{t/\varepsilon^2}} - Z_{C_0 t/\varepsilon^2}\right) + Z^{(\varepsilon)}_{C_0 t}, \qquad t \geq 0, \tag{6.3}$$

where C_0 is as defined in (6.2). By the invariance principle for the process Z and Remark 5.16, $\mathbb{P}_1$-a.s. the last term converges in law to a Brownian motion with variance $\sigma_V^2 = C_0\sigma_Z^2$. To prove the invariance principle for Y it is therefore enough to prove that the first two terms converge to zero in probability. We remark that while both σ_Z and C_0 depend on the constant K chosen in Sect. 2, since the Y does not depend on K, σ_V must be independent of K.

Lemma 6.2 *For any $t > 0$ and $\eta > 0$, $\mathbb{P}_1$-a.s.,*

(i) $\limsup_{\varepsilon\to 0} P_\omega^0 \left[\varepsilon|Y_{t/\varepsilon^2} - Z_{A_{t/\varepsilon^2}}| > \eta\right] = 0,$

(ii) $\limsup_{\varepsilon\to 0} P_\omega^0 \left[\varepsilon|Z_{A_{t/\varepsilon^2}} - Z_{C_0 t/\varepsilon^2}| > \eta\right] = 0.$

Proof (i) Note that

$$\mathfrak{a}_{A_t} = \inf\{u > t : Y_u \in \mathcal{C}_2\},$$

so that $\mathfrak{a}_{A_t} = t$ if $Y_t \in \mathcal{C}_2$.
Now fix $t_0 > 0$, and let $\delta > 0$. By the tightness of of Y in Proposition 5.13b, there exists $R > 0$ such that

$$P_\omega^0 \left(\sup_{t\le t_0} |\varepsilon Y_{t/\varepsilon^2}| > R\right) \le \delta.$$

Let $s = t_0/\varepsilon^2$. Then $Z_{A_s} = Y_{A_{\mathfrak{a}_s}}$, so $Z_{A_s} = Y_s$ if $Y_s \in \mathcal{C}_2$. Otherwise we have that $|Z_{A_s} - Y_s|$ is less than the diameter of the hole containing Y_s: call this D_s. By Lemma 2.3 we have that $D_s \le (\log(R/\varepsilon))^{\alpha_H}$ if $|Y_s| \le R/\varepsilon$, and ε is small enough. So, for sufficiently small ε, we have

$$\varepsilon|Y_s - Z_{A_s}| \le \varepsilon(\log(R/\varepsilon))^{\alpha_H} \le \eta, \quad \text{provided that} \sup_{t\le t_0} |\varepsilon Y_{t/\varepsilon^2}| \le R.$$

So choosing ε small enough,

$$P_\omega^0 \left[\varepsilon|Y_{t_0/\varepsilon^2} - Z_{A_{t_0/\varepsilon^2}}| > \eta\right] \le \delta,$$

proving i).

(ii) For any $\delta > 0$,

$$\begin{aligned} P_\omega^0[\varepsilon|Z_{A_{t/\varepsilon^2}} - Z_{C_0 t/\varepsilon^2}| > \eta] &\le P_\omega^0[\varepsilon|Z_{A_{t/\varepsilon^2}} - Z_{C_0 t/\varepsilon^2}| > \eta,\ \varepsilon^2 A_{t/\varepsilon^2} - C_0 t| \le \delta] \\ &\quad + P_\omega^0\left[\left|\varepsilon^2 A_{t/\varepsilon^2} - C_0 t\right| > \delta\right]. \end{aligned}$$

The second term converges to zero as ε tends zero by (6.2). For the first term we get

$$\begin{aligned} &P_\omega^0\left[|Z^{(\varepsilon)}_{\varepsilon^2 A_{t/\varepsilon^2}} - Z^{(\varepsilon)}_{C_0 t}| > \eta,\ |\varepsilon^2 A_{t/\varepsilon^2} - C_0 t| \le \delta\right] \\ &\quad \le P_\omega^0\left[\sup_{|s_1 - s_2|\le\delta, s_i\le t} \left|Z^{(\varepsilon)}_{s_1} - Z^{(\varepsilon)}_{s_2}\right| > \eta\right], \end{aligned}$$

which becomes arbitrary small for ε and δ small enough by (5.14). □

To conclude the proof of Theorem 1.1a let $0 = t_0 < t_1 < \cdots < t_k$ be arbitrary. Since the tightness of the family $Y^{(\varepsilon)}$ has been established in Proposition 5.13, it suffices to show that the increments $(Y^{(\varepsilon)}_{t_1} - Y^{(\varepsilon)}_{t_0}, \dots, Y^{(\varepsilon)}_{t_k} - Y^{(\varepsilon)}_{t_{k-1}})$ converge in law to the increments of a Brownian motion. The increments are independent, so by (6.3) and Lemma 6.2 they converge if and only if the increments of $Z^{(\varepsilon)}_{C_0\cdot}$ converge, and in that case the limits are identical. But by the invariance principle for Z in Theorem 5.1 the latter converge in law to $(\sigma_V B_{t_1} - \sigma_V B_{t_0}, \dots, \sigma_V B_{t_k} - \sigma_V B_{t_{k-1}})$, where B is a Brownian motion and $\sigma_V^2 = C_0 \sigma_Z^2$.

6.2 CSRW

We now consider the CSRW. Recall that $\mu_x(\omega) = \sum_y \mu_{xy}(\omega)$, set $F(\omega) = \mu_0(\omega)$ and

$$\tilde{A}_t = \int_0^t \mu_{Y_s} ds = \int_0^t F(T_{Y_s}\omega) ds. \tag{6.4}$$

Then if $\tilde{\mathfrak{a}}_t = \inf\{s \ge 0 : \tilde{A}_s \ge t\}$ is the inverse of $\tilde{A}$, the time changed process

$$X_t = Y_{\tilde{\mathfrak{a}}_t} \tag{6.5}$$

is the CSRW. By the ergodic theorem for the process $(T_{Y_t}\omega, t \ge 0)$

$$\lim_{t\to\infty} t^{-1}\tilde{A}_t = \mathbb{E}_1 F = 2d\mathbb{E}_1 \mu_e, \quad \mathbb{P}_1 \times P^0_\omega - \text{ a.s.},$$

where e is any edge adjacent to 0. So if $\mathbb{E}_1\mu_e < \infty$ then $\tilde{\mathfrak{a}}_t/t \to a$ a.s., where $a = 1/2d\mathbb{E}_1\mu_e > 0$. Then

$$X^{(\varepsilon)}_t = Y^{(\varepsilon)}_{at} + \big(X^{(\varepsilon)}_t - Y^{(\varepsilon)}_{at}\big). \tag{6.6}$$

As in Lemma 6.2, and using the tightness of $Y^{(\varepsilon)}$, we have that for any fixed $t_0 \ge 0$,

$$\sup_{0\le t\le t_0} |X^{(\varepsilon)}_t - Y^{(\varepsilon)}_{at}| \tag{6.7}$$

converges in P^0_ω-probability to 0. Thus $X^{(\varepsilon)}$ converges to $\sigma_C W'_t$, where W' is a Brownian motion and $\sigma_C^2 = a\sigma_V^2 > 0$.

In the case when $\mathbb{E}_1\mu_e = \infty$ we have that $\mathfrak{a}_t/t \to 0$, and hence $X^{(\varepsilon)}$ converges to a degenerate limit. □

We note that by conditioning on the σ-field $\sigma(1_{\{\mu_e>0\}}, e \in E_d)$ it is easy to see that for any edge e

$$\mathbb{E}_1\mu_e = \mathbb{E}\mu_e \frac{\mathbb{P}(\mu_e > 0, 0 \in \mathcal{C}_1)}{\mathbb{P}(\mu_e > 0)\mathbb{P}(0 \in \mathcal{C}_1)}. \tag{6.8}$$

In particular we have $\mathbb{E}_1\mu_e < \infty$ if and only if $\mathbb{E}\mu_e < \infty$.

7 Harnack inequalities and Green's function bounds

The heat kernel bounds Theorem 4.11 and the invariance principle allow us to obtain Harnack inequalities, local limit theorems and bounds on Green's functions, by the same methods as in [3,5,6].

We have a parabolic Harnack inequality (PHI) for the process Z, and begin with the definitions necessary to state this. Given $D \subset \mathcal{C}_2$ let $\partial_Z D = \{y \in \mathcal{C}_2 - D : d_Z(x, y) = 1 \text{ for some } x \in D_2\}$ be the external boundary of D in the graph $(\mathcal{C}_2, E_Z)$. Let $\mathrm{cl}_Z(D) = D \cup \partial_Z D$. For $x \in \mathcal{C}_2$ let

$$Q(x, R, T) = (0, T] \times B_2(x, R),$$

and

$$Q_-(x, R, T) = \left[\tfrac{1}{4}T, \tfrac{1}{2}T\right] \times B_2(x, \tfrac{1}{2}R), \quad Q_+(x, R, T) = \left[\tfrac{3}{4}T, T\right] \times B_2\left(x, \tfrac{1}{2}R\right).$$

We say that a function $u(t, x)$ is *caloric* on Q if u is defined on $\overline{Q} = [0, T] \times \mathrm{cl}_Z(B_2(x, R))$, and

$$\frac{\partial u}{\partial t}(t, x) = \mathcal{L}_Z u(t, x), \quad (t, x) \in Q(x, R, T). \tag{7.1}$$

We say the parabolic Harnack inequality (PHI) holds with constant C_H for $Q = Q(x, R, T)$ if whenever $u = u(t, x)$ is non-negative on $\overline{Q}$ and caloric on Q, then

$$\sup_{(t,x)\in Q_-} u(t, x) \le C_H \inf_{(t,x)\in Q_+} u(t, x). \tag{7.2}$$

Theorem 7.1 *Let $(S_x, x \in \mathbb{Z}^d)$ be as in Theorem 4.11. Then there exists a constant C_H such that if $R \ge S_x^2$ then the PHI holds with constant C_H for $Q(x, R, R^2)$.*

Proof This is proved as in [5, Section 3]. □

Since caloric functions are harmonic, we immediately obtain an elliptic Harnack inequality for Z-harmonic functions.

Combining the PHI and invariance principle for Z as in [5,18], we have a local limit theorem for q^Z; this will be used to obtain Green's function bounds for Y. Let $b_\omega : \mathbb{R}^d \to \mathcal{C}_2$ be defined so that $b_\omega(x)$ is a closest point in $\mathcal{C}_2$ to x, write

$$\widetilde{q}_t^Z(x, y) = q_t^Z(b_\omega(x), b_\omega(y)), \qquad a = 1/\mathbb{P}(0 \in \mathcal{C}_2),$$

and let

$$k_t(x) = (2\pi t\sigma_Z^2)^{-d/2} e^{-|x|^2/2\sigma_Z^2 t}$$

be the Gaussian heat kernel with diffusion constant σ_Z^2.

Proposition 7.2 *Let $T > 0$. Then $\mathbb{P}$-a.s. on the event $\{0 \in \mathcal{C}_2\}$,*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T} \left| n^{d/2} \widetilde{q}^{Z}_{nt} \left(0, \sqrt{n}x\right) - a k_t(x) \right| = 0. \tag{7.3}$$

Further, if $0 < \delta < T$ and $M > 0$, then $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} \inf_{\delta\leq t\leq T} \inf_{|x|,|y|\leq M} \frac{n^{d/2}\widetilde{q}^{Z}_{nt}(\sqrt{n}x, \sqrt{n}y)}{a k_t(x-y)}$$
$$= \lim_{n\to\infty} \sup_{\delta\leq t\leq T} \sup_{|x|,|y|\leq M} \frac{n^{d/2}\widetilde{q}^{Z}_{nt}(\sqrt{n}x, \sqrt{n}y)}{a k_t(x-y)} = 1.$$

Proof This is proved from the PHI and invariance principle as in [5], and [6, Theorem 3]. □

Counterexamples in [12,13] show that if $d \geq 4$ then the usual heat kernel upper bound may fail for the transition density $q_t(x, y)$ of X or Y. Thus, given the general equivalence between Gaussian heat kernel bounds and the PHI (see [20]) we cannot expect a PHI to hold in general for either X or Y. We do however, have an elliptic Harnack inequality, and bounds on the Green's functions of X and Y.

For $D \subset \mathcal{C}_1$ we define $\partial_1(D)$ to be the (exterior) boundary of D in the graph $(\mathcal{C}_1, \mathcal{O}_1)$, and set $\mathrm{cl}_1(D) = D \cup \partial_1(D)$. We say that a function h is Y–harmonic in $A \subset \mathcal{C}_1$ if h is defined on $\mathrm{cl}_1(A)$ and $\mathcal{L}_V h(x) = 0$ for $x \in A$. We now give a elliptic Harnack inequality for the process Y.

Theorem 7.3 *There exist r.v. $(R'_x, x \in \mathbb{Z}^d)$ with*

$$\mathbb{P}(R'_x \geq n | x \in \mathcal{C}_1) \leq c e^{-c' n^{\delta}}, \tag{7.4}$$

and a constant C_E such that if $x_0 \in \mathcal{C}_1$, $R \geq R'_{x_0}$ and $h : \mathrm{cl}_1(B_1(x_0, R)) \to \mathbb{R}_+$ is Y–harmonic on $B_1 = B_1(x_0, R)$, then writing $B'_1 = B_1(x_0, R/2)$,

$$\sup_{B'_1} h \leq C_E \inf_{B'_1} h. \tag{7.5}$$

Proof Our basic strategy is to use the fact that an elliptic Harnack inequality holds for Z-harmonic functions on $\mathcal{C}_2$, and the fact that all the holes (that is, connected components of $\mathcal{C}_1 - \mathcal{C}_2$) are small.

For $x \in \mathbb{Z}^d$ and $n \geq 1$ let $F_n(x)$ be the event that one of the 'holes' $\mathcal{H}(y)$, with $|y - x| \leq n^2$, has diameter greater than $n^{1/3d}$. Then by Lemma 2.3,

$$\mathbb{P}(F_n(x)) \leq c n^{2d} \exp(-2c' n^{1/3d}) \leq c \exp(-c' n^{1/3d}).$$

Let U_x be the smallest m such that $F_n(x)$ holds for all $n \geq m$; we have

$$\mathbb{P}(U_x \geq n) \leq c \exp(-c n^{1/3d}).$$

For $x \in \mathcal{C}_1$ let $g(x)$ be a closest point in $\mathcal{C}_2$ to x, and

$$R'_x = c_1(U_x \vee S_{g(x)} \vee R_{g(x)}),$$

where S_y is as in Theorem 7.1 and R_y as in Definition 2.7. Here $c_1 \geq 4$ is a constant chosen large enough to avoid 'small R' effects. Since

$$\mathbb{P}(R'_x \geq n) \leq \mathbb{P}(U_x \geq n/c_1) + \mathbb{P}\left(\max_{|y-x|\leq n} S_y \geq n/c_1\right),$$

the bound (7.4) is satisfied.

Now let $R \geq R'_{x_0}$. Write $y_0 = g(x_0)$, and note that since $R \geq U_{x_0}$ we have $|x_0 - y_0| \leq R^{1/3d}$. So if $y_0 \neq x_0$ then y_0 is in hole of diameter less than $R^{1/3d}$, and since this hole contains less than $c(R^{1/3d})^d$ points, $d_1(x_0, y_0) \leq cR^{1/3}$.

Let A be the set of y in $B_1 \cap \mathcal{C}_2$ such that $B_Z(y, 1) \subset B_1$. (So if $y \in A$ then there is no hole adjacent to y with a boundary point outside B_1.) Since the d_1–diameter of any holes intersecting B_1 is less than $cR^{1/3}$, we deduce that $B_1(y_0, R - cR^{1/3}) \subset A$. So, as $d_2 \geq d_1$, we have $B_2(y_0, 8R/9) \subset A$.

Now let h be Y–harmonic on B_1. Then $h(Y_t)$ is a local martingale up to the first exit of Y from B, and it follows that if $y \in A$ then $\mathcal{L}_Z h(y) = 0$. Thus h is Z-harmonic on $B_2(y_0, 8R/9)$, and so applying the elliptic Harnack inequality for Z-harmonic functions in the balls $B''_2 = B_2(y_0, 4R/9) \subset B_2(y_0, 8R/9)$, we have

$$\max_{B''_2} h \leq C \min_{B''_2} h. \tag{7.6}$$

Since $R \geq c_1 R_{y_0}$ the ball $B_2(y_0, R^{1/2})$ is good, and so using (2.22) it follows that there exists c_2 (depending only on the constants in Definition 2.7) such that

$$B_1(x_0, c_2 R) \subset \bigcup_{y \in B_2(y_0, R/3)} B_Z(y, 1).$$

Let $D = B_1(x_0, c_2 R)$. Now we show that $h(y) \leq \max_{B''_2} h$ for $y \in D$. If $y \in \mathcal{C}_2$ then since $y \in B''_2$ this is immediate, so suppose $y \in \mathcal{C}_1 - \mathcal{C}_2$. Then y is in some hole $\mathcal{H}(y)$. Since $\mathcal{H}(y)$ has diameter smaller than $R^{1/3d}$, the boundary of the hole is still contained in B''_2, and therefore by the maximum principle $h(y) \leq \max\{h(z) : z \in \partial\mathcal{H}(y)\} \leq \max_{B''_2} h$. Similarly we have $h(y) \geq \min_{B''_2} h$ for $y \in B'_1$, so (7.5) follows from (7.6). □

Remark 7.4 In Sect. 5 we defined function Φ and the corrector χ for Z so that $M_t = \Phi(Z_t) = Z_t - \chi(Z_t)$ was a martingale. Given ω such that $0 \in \mathcal{C}_2(\omega)$, we can use the same argument as above to extend the function $\Phi(\omega, x)$ on $\mathcal{C}_2$ to a Y-harmonic function $\Phi_Y(\omega, x)$ on $\mathcal{C}_1$. We can then define the corrector for Y (with law P^0_ω) by $\chi_Y(\omega, x) = x - \Phi_Y(\omega, x)$. Since the holes are all finite (and small), the pointwise sublinearity of χ in (5.24) then gives a similar pointwise sublinearity for χ_Y. If ω is such that $0 \in \mathcal{C}_1(\omega) - \mathcal{C}_2(\omega)$ then we can define $\chi_Y(\omega, \cdot)$ by first choosing $x \in \mathcal{C}_2(\omega)$, so that $0 \in \mathcal{C}_2(T_x\omega)$, constructing $\chi_Y(T_x\omega, \cdot)$, and finally using the cocycle property to obtain $\chi_Y(\omega, \cdot)$.

Let $d \geq 3$. Recall from Sect. 1 the definition of $g^Y(x, y)$, and define the Green's function for Z by

$$g^Z(x, y) = \int_0^\infty q_t^Z(x, y)dt = E_\omega^x \int_0^\infty 1_{(Z_s=y)}ds. \tag{7.7}$$

The function $g^Y(x, \cdot)$ is harmonic on $\mathcal{C}_1 - \{x\}$, and $g^Z(x, \cdot)$ is harmonic on $\mathcal{C}_2 - \{x\}$. Since the processes Y and Z agree on $\mathcal{C}_2$, it follows that

$$g^Y(x, y) = g^Z(x, y) \ \text{ if } x, y \in \mathcal{C}_2.$$

Lemma 7.5 *Let $d \geq 3$.*

(a) *There exist constants $\delta, c_1, \dots c_4$, depending only on d and the law of μ_e, and r.v. R''_x, $x \in \mathbb{Z}^d$ satisfying*

$$\mathbb{P}(R''_x \geq n | x \in \mathcal{C}_2) \leq c_1 e^{-c_2 n^\delta}, \tag{7.8}$$

such that

$$\frac{c_3}{|x-y|^{d-2}} \leq g^Z(x, y) \leq \frac{c_4}{|x-y|^{d-2}} \qquad \text{if } |x-y| \geq R''_x \wedge R''_y,\ x, y \in \mathcal{C}_2. \tag{7.9}$$

(b) *Let $C_Z = \Gamma(d/2-1)\big(2\pi^{d/2}\sigma_V^2\mathbb{P}(0 \in \mathcal{C}_2)\big)^{-1}$. Then for any $\varepsilon > 0$ there exists a r.v. N_ε such that on $\{0 \in \mathcal{C}_2\}$,*

$$\frac{(1-\varepsilon)C_Z}{|x|^{d-2}} \leq g^Z(0, x) \leq \frac{(1+\varepsilon)C_Z}{|x|^{d-2}} \quad \text{for } |x| > N_\varepsilon(\omega),\ x \in \mathcal{C}_2. \tag{7.10}$$

(c) *We have $\mathbb{P}$–a.s. on $\{\omega \in \mathcal{C}_2\}$,*

$$\lim_{|x|\to\infty, x\in\mathcal{C}_2} |x|^{2-d} g^Z(0, x) = \lim_{|x|\to\infty} |x|^{2-d}\mathbb{E}(g^Z(0, x)|0 \in \mathcal{C}_2) = C_Z. \tag{7.11}$$

Proof The bounds for $g^Z(x, y)$ (for $x, y \in \mathcal{C}_2$) follow from the bounds for q^Z and the local limit theorem as in [5, Section 6]. □

Proof of Theorem 1.2. If now $x \in \mathcal{C}_2$ but $y \in \mathcal{C}_1 - \mathcal{C}_2$ then provided $x \notin \mathcal{H}(y)$ the maximum principle, and the fact that g^Y and g^Z agree on $\mathcal{C}_2 \times \mathcal{C}_2$ implies that

$$\min_{z\in\partial_1\mathcal{H}(y)} g^Z(x, z) \leq g^Y(x, y) \max_{z\in\partial_1\mathcal{H}(y)} g^Z(x, z).$$

If R_x is chosen large enough then the diameter of $\mathcal{H}(y)$ is small compared with $|x-y|$, so (7.9) follows if $x \in \mathcal{C}_1$. Repeating the argument by considering $\mathcal{H}(x)$ then gives (a).

A similar approximation argument proves (b), (c) and (d); using translation invariance it is enough to prove these in the case $x = 0$. Note that the proof of (c) and (d) gives the same constant in (1.10) whichever choice of $Z = Z(K)$ is used. Using continuity, this constant must therefore be the same as that given by taking $K = \infty$. □

Remark 7.6 In addition as in [5, Proposition 6.2] we also have

$$\mathbb{E}(g^Z(x,x)^k | x \in \mathcal{C}_2) \le c(k), \quad k \ge 1. \tag{7.12}$$

We cannot expect such bounds for Y, since if x is in a 'hole' then x may be separated from the rest of $\mathcal{C}_1$ by a single bond with very low conductivity ε. The mean time Y then spends in x before leaving will be of order ε^{-1}.

Remark 7.7 While (7.8) gives good control of the tail of the random variables R_x in (7.9), we do not have any bounds on the tail of the r.v. N_ε in (7.10). This is because the proof of (7.10) relies on the invariance principle, where we do not have a rate of convergence.

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