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Smoothness of scale functions for spectrally negative Lévy processes

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Abstract Scale functions play a central role in the fluctuation theory of spectrally negative Lévy processes and often appear in the context of martingale relations. These relations are often require excursion theory rather than Itô calculus. The reason for the latter is that standard Itô calculus is only applicable to functions with a sufficient degree of smoothness and knowledge of the precise degree of smoothness of scale functions is seemingly incomplete. The aim of this article is to offer new results concerning properties of scale functions in relation to the smoothness of the underlying Lévy measure. We place particular emphasis on spectrally negative Lévy processes with a Gaussian component and processes of bounded variation. An additional motivation is the very intimate relation of scale functions to renewal functions of subordinators. The results obtained for scale functions have direct implications offering new results concerning the smoothness of such renewal functions for which there seems to be very little existing literature on this topic.

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1 Spectrally negative Lévy processes and scale functions

Suppose that $X = \{X_t : t \geq 0\}$ is a spectrally negative Lévy process with probabilities $\{P_x : x \in \mathbb{R}\}$. For convenience we shall write P in place of P_0. That is to say a real valued stochastic process whose paths are almost surely right continuous with left limits and whose increments are stationary and independent. Let $\{\mathcal{F}_t : t \geq 0\}$ be the natural filtration satisfying the usual assumptions and denote by ψ its Laplace exponent so that

$$E(e^{\theta X_t}) = e^{t\psi(\theta)}$$

where E denotes expectation with respect to P.

It is well known that ψ is finite for all $\theta \geq 0$, is strictly convex on $(0, \infty)$ and satisfies $\psi(0+) = 0$ and $\psi(\infty) = \infty$. Further, from the Lévy-Khintchin formula, it is known that

$$\psi(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \left(e^{\theta x} - 1 - \theta x \mathbf{1}_{(x>-1)}\right) \Pi(\mathrm{d}x)$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π satisfies $\int_{(-\infty,0)}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$ and is called the Lévy measure.

Suppose that for each $q \geq 0$, $\Phi(q)$ is the largest root of the equation $\psi(\theta) = q$ in $[0, \infty)$ (there are at most two). We recall from [2,3] that for each $q \geq 0$ there exits a function $W^{(q)} : \mathbb{R} \to [0, \infty)$, called the *q-scale function* defined in such a way that $W^{(q)}(x) = 0$ for all $x < 0$ and on $[0, \infty)$ its Laplace transform is given by

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\theta) - q} \quad \text{for} \quad \theta > \Phi(q). \tag{1}$$

For convenience we shall write W in place of $W^{(0)}$ and call this the *scale function* rather than the 0-scale function.

The importance of q-scale functions appears in a number of one and two sided exit problems for (reflected) spectrally negative Lévy processes. See for example, [1–3,5,7,16,19,21,22]. Notably however the q-scale function takes its name from the identity

$$E_x\left(e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)}\right) = \frac{W^{(q)}(x)}{W^{(q)}(a)} \tag{2}$$

where $\tau_a^+ = \inf\{t > 0 : X_t > a\}$, $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ and E_x is expectation with respect to P_x. The latter identity provides an analogue to a similar situation for scale functions of diffusions.

Set $q \geq 0$. Under the exponential change of measure

$$\left.\frac{\mathrm{d}P_x^{\Phi(q)}}{\mathrm{d}P_x}\right|_{\mathcal{F}_t} = e^{\Phi(q)(X_t - x) - qt}$$

it is well known that $(X, P^{\Phi(q)})$ is again a spectrally negative Lévy process whose Laplace exponent is given by

$$\psi_{\Phi(q)}(\theta) = \psi(\theta + \Phi(q)) - q \tag{3}$$

for $\theta \geq -\Phi(q)$ and whose Lévy measure, $\Pi_{\Phi(q)}$, satisfies

$$\Pi_{\Phi(q)}(dx) = e^{\Phi(q)x}\Pi(dx)$$

on $(-\infty, 0)$. Note in particular that $\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q))$ which is strictly positive when $q > 0$ or when $q = 0$ and $\psi'(0+) < 0$. Using the latter change of measure, we may deduce from (1) and (3) that

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x) \tag{4}$$

where $W_{\Phi(q)}(x)$ is the scale function for the process $(X, P^{\Phi(q)})$.

There exists an excursion theory argument given in [2] from which it is known that for any $0 < x < \infty$,

$$W(x) = W(a)\exp\left\{-\int_x^a n(\overline{\epsilon} > t)\mathrm{d}t\right\} \tag{5}$$

for any arbitrary $a > x$ where n is the excursion measure of the local-time-indexed process of excursions $\{\epsilon_t : t \geq 0\}$ of the reflected Lévy process $\{\sup_{s \leq t} X_s - X_t : t \geq 0\}$ and $\overline{\epsilon}$ is the excursion height of the generic excursion. Note that when $\psi'(0+) > 0$ the scale function may also be represented in the form $W(x) = P_x(\inf_{s \leq t} X_s \geq 0)/\psi'(0+)$. From (5) it is immediate that on $(0, \infty)$ the function W is monotone and almost everywhere differentiable with left and right derivatives given by

$$W'(x-) = n(\overline{\epsilon} \geq x)W(x) \quad \text{and} \quad W'(x+) = n(\overline{\epsilon} > x)W(x) \tag{6}$$

so that W is continuously differentiable on $(0, \infty)$ if and only if $n(\overline{\epsilon} = t) = 0$ for all $t > 0$ in which case $W'(x) = n(\overline{\epsilon} \geq x)W(x)$. In [13] it is shown that the latter is the case if for example the process X has paths of unbounded variation (and in particular if it possesses a Gaussian component) or if X has paths of bounded variation and the Lévy measure is absolutely continuous with respect to Lebesgue measure. More recently, it has been shown in [11] that, in the case of bounded variation paths, W is continuously differentiable if and only if Π has no atoms.

The principle objective of this paper is to investigate further the smoothness properties of scale functions. In particular we are interested in providing conditions on the Lévy measure Π and Gaussian coefficient σ such that, for all $q \geq 0$, the restriction of $W^{(q)}$ to $(0, \infty)$ belongs to $C^k(0, \infty)$ for $k = 2, 3, \ldots$ (henceforth we shall write the latter with the slight abuse of notation $W^{(q)} \in C^k(0, \infty)$).

2 Motivation and main results

Before moving to the main results, let us motivate further the reason for studying the smoothness of scale functions. From (2) it is straightforward to deduce by applying the Strong Markov Property that for $a > 0$

$$\left\{ e^{-qt} W^{(q)}(X_t) : t < \tau_a^+ \wedge \tau_0^- \right\}$$

is a martingale. In the spirit of the theory of stochastic representation associated with one dimensional diffusions, this fact may in principle be used to solve certain boundary value problems by affirming that $W^{(q)}$ solves the integro-differential equation

$$(\Gamma - q) W^{(q)}(x) = 0 \quad \text{on} \quad (0, a)$$

where Γ is the infinitesimal generator of X.

The latter equation would follow by an application of Itô's formula to the aforementioned martingale providing that one can first assert sufficient smoothness of $W^{(q)}$. In general, a comfortable sufficient condition to do this would be that $W^{(q)} \in C^2(0, \infty)$, although this is not strictly necessary. For example, when X is of bounded variation, knowing that $W^{(q)} \in C^1(0, \infty)$ would suffice.

In general it is clear from (1) that imposing conditions on the Lévy triple (a, σ, Π), in particular the quantities σ and Π, is one way to force a required degree of smoothness on $W^{(q)}$. It is also possible to get some a priori feeling for what should be expected by way of results by looking briefly at the intimate connection between scale functions when $\psi'(0+) \geq 0$ and $q = 0$ and renewal functions.

Suppose that X is such that $\psi'(0+) \geq 0$, then thanks to the Wiener–Hopf factorisation we may write $\psi(\theta) = \theta\phi(\theta)$ where ϕ is the Laplace exponent of the descending ladder height subordinator (see for example Chapter VI of [2] for definitions of some of these terms). Note that in this exposition, we understand a subordinator in the broader sense of a Lévy process with non-decreasing paths which is possibly killed at an independent and exponentially distributed time. A simple integration by parts in (1) shows that

$$\int_{[0,\infty)} e^{-\theta x} W(\mathrm{d}x) = \frac{1}{\phi(\theta)} \tag{7}$$

which in turn uniquely identifies $W(\mathrm{d}x)$ as the renewal measure associated with the descending ladder height subordinator. Recall that if $H = \{H_t : t \geq 0\}$ is the subordinator associated with the Laplace exponent ϕ then a very first view on smoothness properties could easily incorporate known facts concerning smoothness properties of renewal measures.

For example, [8] formalise several facts which are implicit in the Wiener–Hopf factorisation. Specifically it is shown that given any Laplace exponent of a subordinator, ϕ, there exists a spectrally negative Lévy process with Laplace exponent $\psi(\theta) = \theta\phi(\theta)$ if and only if the Lévy measure associated to ϕ is absolutely continuous with non-increasing density. In that case the aforementioned density is necessarily equal to

$$\overline{\Pi}(x) := \Pi(-\infty, -x).$$

A sub-class of such choices for ϕ are the so called complete Bernstein functions. That is to say, Laplace exponents of subordinators whose Lévy measure is absolutely continuous with completely monotone density. A convenience of this class of ϕ is that, from Theorem 2.3 of [18] and Theorem 2.1 and Remark 2.2 of [20], the potential density associated to ϕ, that is to say W', is completely monotone and hence belongs to $C^\infty(0, \infty)$. This way, one easily reaches the conclusion that whenever $\psi'(0+) \geq 0$ and $\overline{\Pi}(x)$ is completely monotone, then W' is completely monotone.

The above arguments show that a strong smoothness condition on the Lévy measure Π, namely complete monotonicity of $\overline{\Pi}$, yields a strong degree of smoothness on the scale functions; at least for some particular parameter regimes. One would hope then that a weaker smoothness condition on the Lévy measure Π, say $\overline{\Pi} \in C^n(0, \infty)$ for $n = 1, 2, \ldots$, might serve as a suitable condition in order to induce a similar degree of smoothness for the associated scale functions. Another suspicion one might have given the theory of scale functions for diffusions is that, irrespective of the jump structure, the presence of a Gaussian component is always enough to guarantee that the scale functions are $C^2(0, \infty)$.

Let us now turn to our main results which do indeed reflect these intuitions. We deal first with the case of when a Gaussian component is present.

Theorem 1 *Suppose that X has a Gaussian component. For each fixed $q \geq 0$ the function $W^{(q)}$ belongs to the class $C^2(0, \infty)$.*

Theorem 2 *Suppose that X has a Gaussian component and its Blumenthal–Getoor index belongs to* $[0, 2)$*, that is to say*

$$\inf\left\{\beta \geq 0 : \int_{|x|<1} |x|^\beta \Pi(\mathrm{d}x) < \infty\right\} \in [0, 2).$$

Then for each $q \geq 0$ and $n = 0, 1, 2, \ldots$, $W^{(q)} \in C^{n+3}(0, \infty)$ if and only if $\overline{\Pi} \in C^n(0, \infty)$.

Next we have a result concerning the case of bounded variation paths.

Theorem 3 *Suppose that X has paths of bounded variation and $-\overline{\Pi}$ has a derivative $\pi(x)$, such that $\pi(x) \leq Cx^{-1-\alpha}$ in the neighbourhood of the origin, for some $\alpha < 1$*

and $C > 0$. *Then for each* $q \geq 0$ *and* $n = 1, 2, \ldots$, $W^{(q)} \in C^{n+1}(0, \infty)$ *if and only if* $\overline{\Pi} \in C^n(0, \infty)$.

Remark 4 Note that if $\overline{\Pi}(0) < \infty$ (that is to say X has a compound Poisson jump structure), then the Blumenthal–Getoor index is zero. As a consequence Theorem 2 implies, without further restriction on Π, that, if in addition a Gaussian component is present in X, then, for $n = 0, 1, 2, \ldots$, $W^{(q)} \in C^{n+3}(0, \infty)$ if and only if $\overline{\Pi} \in C^n(0, \infty)$.

Remark 5 In [11,12] and [15] related results exist. In particular the latter two papers show that if $\overline{\Pi}$ is log-convex, then for all $q \geq 0$, $W^{(q)\prime}$ is log-convex and if moreover, Π is absolutely continuous with log-convex density (which implies that $\overline{\Pi}$ is log-convex) and there is a Gaussain component, then $W^{(q)} \in C^2(0, \infty)$. Also as an elaboration on some of the discussion in Sect. 2, it was shown in [14] that if $\overline{\Pi}$ has a completely monotone density, then for all $q \geq 0$, $W^{(q)\prime}$ is convex and $W^{(q)} \in C^\infty(0, \infty)$.

The various methods of proof we shall appeal to in order to establish the above three theorems reveal that the case that X has paths of unbounded variation but no Gaussian component is a much more difficult case to handle and unfortunately we are not able to offer any concrete statements in this regime.

We conclude this section with a brief summary of the remainder of the paper. In the next section we shall look at some associated results which concern smoothness properties of a certain family of renewal measures. These results will form the basis of one of two key techniques used in the proofs. Moreover, this analysis will also lead to a new result, extending a classical result of [9], concerning smoothness properties of renewal measures of subordinators with drift.

We then turn to the proofs of our results on renewal equations and finally use them, together with other probabilistic techniques, to prove the results on scale functions.

3 Renewal equations

Henceforth the convolution of two given functions, f and g mapping $[0, \infty)$ to $\mathbb{R}$, is defined as

$$f * g(x) = \int_0^x f(x - y)\mathrm{d}g(y).$$

In the subsequent analysis it will always be the case g is absolutely continuous with respect to Lebesgue measure. When appropriate, we shall also understand g^{*n} to be the n-fold convolution of g on $(0, \infty)$ where $n = 0, 1, \ldots$. In particular, on $[0, \infty)$, $g^{*0}(\mathrm{d}x) = \delta_0(\mathrm{d}x)$, $g^{*1} = g$ and for $n = 2, 3, \ldots$,

$$g^{*n}(x) = \int_0^x g^{*(n-1)}(x - y)g'(y)\mathrm{d}y.$$

We note that in the following theorems, the issue of uniqueness of solutions to the renewal equation is already well understood, see e.g. [6]. However we include the statement of uniqueness for completeness.

Theorem 6 *Let g be a negative, decreasing function on $(0,\infty)$ with $g(0)=0$ such that $|g'(x)|$ is continuous and decreasing with $g'(\infty)=0$. Then the solution on $(0,\infty)$ of the renewal equation*

$$f(x) = 1 + f * g(x) \tag{8}$$

is unique in the class of functions which are bounded on bounded intervals and has the usual form

$$f(x) = \sum_{n\geq 0} g^{*n}(x). \tag{9}$$

Also the series in (9) is uniformly and absolutely convergent on any finite interval. Moreover, the first derivative $f' \in C(0,\infty)$, $f'(x) = \sum_{n\geq 0} g^{*n\prime}(x)$ and $\sum_{n\geq 0} g^{*n\prime}(x)$ is uniformly and absolutely convergent on any finite interval, which does not contain 0*.*

Theorem 7 *Let g be as in Theorem 6. Also assume that $g \in C^2(0,\infty)$ and*

$$\max\left\{\sup\{x\in[0,b) : |g^{*k_0\prime}(x)|\};\ \sup\left\{x\in[0,b) : |g^{*k_0\prime\prime}(x)|\right\}\right\} < A(b) < \infty \tag{10}$$

for any interval $[0,b)$ and some positive integer k_0. Then, for any $k = 0,1,2,\ldots$, $f \in C^{k+2}(0,\infty)$ if and only if $g \in C^{k+2}(0,\infty)$.

Remark 8 In view of the fact that $g \in C^2(0,\infty)$, (10) follows by a straightforward application of the definition of convolution if $|g'(x)| \leq C_1 x^{-\alpha+1}$, $|g''(x)| \leq C_2 x^{-\alpha}$ on some neighbourhood of 0, where C_1 and C_2 are positive constants and $0 \leq \alpha < 2$.

Although the above two theorems will be used to address the issue of smoothness properties of scale functions, we will also deduce some new results concerning renewal functions of subordinators. Indeed this is the purpose of the next corollary which generalises a classical result of [9]. The latter says that whenever $U(\mathrm{d}x)$ is the renewal measure of a subordinator with drift coefficient $\delta > 0$ then U is absolutely continuous with respect to Lebesgue measure with a density, $u(x)$ which has a continuous version (which in turn may be identified as δ^{-1} multiplied by the probability that the underlying subordinator crosses the level x by creeping) for $x > 0$.

Corollary 9 *Suppose that U is the renewal measure of a subordinator with positive drift δ and a Lévy measure μ which is also assumed to have no atoms. Then the renewal density has the form*

$$u(x) = \sum_{n\geq 0} \frac{\eta^{*n}(x)}{\delta^{n+1}}, \tag{11}$$

where $\eta^{*(n+1)}(x) = -\int_0^x \eta^{*n}(x-y)\overline{\mu}(y)\mathrm{d}y$, $\eta(x) = -\int_0^x \overline{\mu}(y)\mathrm{d}y$ *and* $\overline{\mu}(x) = \mu(x,\infty)$ *and it is continuously differentiable with*

$$u'(x) = \sum_{n\geq 0} \frac{\eta^{*n\prime}(x)}{\delta^{n+1}}. \tag{12}$$

Proof Assume that the underlying subordinator is not subject to killing. Then the subordinator crosses a level x by creeping or jumping over this level. With the help of [9] this can be written as

$$\delta u(x) + \int_0^x u(x-y)\overline{\mu}(y)\mathrm{d}y = 1 \tag{13}$$

(see for example Chapter 3 of [2]). Let $f(x) = \delta u(x)$ and $g(x) = -\delta^{-1}\int_0^x \overline{\mu}(y)\mathrm{d}y$. Thus the conditions of Theorem 6 are satisfied and the statement follows.

Now we turn to the case when the underlying subordinator is killed. In that case, suppose that it has the same law as an unkilled subordinator, say Y with Laplace exponent $\eta(\theta) = -\log E(e^{-\theta Y_1})$ for $\theta \geq 0$, killed at rate $q > 0$. The result of [9] tells us that

$$U(\mathrm{d}x) = \frac{1}{\delta} P\left(e^{-qT_x^+}; Y_{T_x^+} = x\right)\mathrm{d}x$$

where $T_x^+ = \inf\{t > 0 : Y_t = x\}$ (see for example Exercise 5.5 in [10]). We may thus write

$$U(\mathrm{d}x) = \frac{1}{\delta} e^{\eta^{-1}(q)x} P\left(e^{-\eta^{-1}(q)x - qT_x^+}; Y_{T_x^+} = x\right)\mathrm{d}x = \frac{1}{\delta} e^{\eta^{-1}(q)x} U_{\eta^{-1}(q)}(\mathrm{d}x)$$

where $U_{\eta^{-1}(q)}(\mathrm{d}x)$ is the renewal measure of the subordinator Y when seen under the exponential change of measure associated with the martingale $\{e^{-\eta^{-1}(q)Y_t - qt} : t \geq 0\}$ and η^{-1} is the right inverse of η. Note that the Laplace exponent of process Y under the aforementioned change of measure is given by $\eta(\theta + \eta^{-1}(q)) - q$ for $\theta \geq 0$ and it is straightforward to deduce that there is no killing term and the Lévy measure is given by $e^{-\eta^{-1}(q)x}\mu(\mathrm{d}x)$. Since $U_{\eta^{-1}(q)}(\mathrm{d}x)$ is the potential measure of an unkilled subordinator and since the behaviour at the origin of the measure $e^{-\eta^{-1}(q)x}\mu(\mathrm{d}x)$ is identical to that of μ with regard to the role it plays through the function g, the result for the case of killed subordinators follows from the first part of the proof. □

4 Renewal equation proofs

We start by the following auxiliary lemma.

Lemma 10 *Let g be a negative, decreasing function on $(0,\infty)$ with $g(0)=0$ and let $|g'(x)|$ be decreasing. Then $|g^{*n}(x)| = (-1)^n g^{*n}(x)$. Moreover $|g^{*n\prime}(x)| = (-1)^n g^{*n\prime}(x)$ and*

$$g^{*(n+1)\prime}(x) = \int_0^x g'(x-y)g^{*n\prime}(y)\mathrm{d}y = \int_0^x g^{*n\prime}(x-y)g'(y)\mathrm{d}y. \tag{14}$$

In conclusion $g^{*n}(x)$ is increasing for n even and decreasing otherwise.

Proof The first statement is obvious from the definition of convolution

$$g^{*(n+1)}(x) := \int_0^x g^{*n}(x-y)g'(y)\mathrm{d}y.$$

Next we prove $|g^{*n\prime}(x)| = (-1)^n g^{*n\prime}(x)$ and (14). Note that $g'(x)$ is absolutely integrable and write, for $h>0$,

$$|g^{*2}(x+h) - g^{*2}(x)| \le \int_0^x |g(x+h-y) - g(x-y)||g'(y)|\mathrm{d}y + |g(h)||g(x+h) - g(x)|.$$

Since $|g'(x)|$ decreases then $|g(x+h-y)-g(x-y)| \le h|g'(x-y)| = -hg'(x-y)$. Finally from the existence of $\int_0^x g'(x-y)g'(y)\mathrm{d}y$, the nonpositivity of $g'(x)$ and the dominated convergence theorem we deduce that

$$|g^{*2\prime}(x+)| = g^{*2\prime}(x+) = \lim_{h\to 0+} \frac{|g^{*2}(x+h) - g^{*2}(x)|}{h} = \int_0^x g'(x-y)g'(y)\mathrm{d}y.$$

Similarly we show that $|g^{*2\prime}(x-)| = \int_0^x g'(x-y)g'(y)\mathrm{d}y$.

Thus the induction hypothesis states that $|g^{*n\prime}(x)| = (-1)^n g^{*n\prime}(x)$ and the identity

$$g^{*n\prime}(x) = \int_0^x g'(x-y)g^{*(n-1)\prime}(y)\mathrm{d}y = \int_0^x g^{*(n-1)\prime}(x-y)g'(y)\mathrm{d}y.$$

To show these statements for $n+1$, we use the induction hypothesis, the preceding part of the proof and

$$|g^{*(n+1)}(x+h) - g^{*(n+1)}(x)| \le \int_0^x |g(x+h-y) - g(x-y)||g^{*n\prime}(y)|\mathrm{d}y + |g(h)||g^{*n}(x+h) - g^{*n}(x)|.$$

Finally an application of the dominated convergence theorem completes the proof. □

Proof of Theorem 6 Pick an interval $(0, a)$. Next we show that $\sum_{n\geq 0} g^{*n}(x)$ is uniformly convergent and therefore $\phi(x)$ is continuous on $(0, a)$.

From the conditions of the theorem, i.e. $g(0) = 0$, $|g(x)| < 1$ on some interval $(0, b]$. From Lemma 10, $|g^{*k}(x)|$ is increasing, for each $k \geq 1$. Hence

$$|g^{*n}(x)| = \left|\int_0^x g^{*(n-1)}(x-y)g'(y)\mathrm{d}y\right| \leq |g^{*(n-1)}(x)||g(x)| \leq |g(x)|^n, \tag{15}$$

and the series $\sum_{n\geq 0} g^{*n}(x)$ is uniformly convergent on $(0, b]$.

Uniformity can be extended to the interval $(0, 2b]$. Set up an induction hypothesis that, for each $n \geq 1$, $|g^{*n}(2b)| \leq n|g(b)|^{n-1}|g(2b)|$, which holds for $n = 1$. We prove this hypothesis by using Lemma 10 and (15). Indeed we have

$$\begin{aligned} |g^{*(n+1)}(2b)| &\leq \int_0^b |g^{*n}(2b-x)||g'(x)|dx + \int_b^{2b} |g^{*n}(2b-x)||g'(x)|dx \\ &\leq |g^{*n}(2b)||g(b)| + |g^{*n}(b)||g(2b)| \leq (n+1)|g(b)|^n|g(2b)|. \end{aligned}$$

Since $|g(b)| < 1$, $\sum_{n\geq 0} g^{*n}(x) < \infty$ is uniformly convergent on $(0, 2b]$.

Next, we extend uniformity to the interval $(0, 4b]$. Since $\sum_{n\geq 0} g^{*n}(x)$ is uniformly convergent on $(0, 2b]$, we find k such that $|g^{*k}(2b)| < 1$ and similarly as before we deduce that $\sum_{l\geq 1} g^{*(lk)}(x)$ is uniformly convergent on $(0, 4b]$. For a series of the type $\sum_{l\geq 0} g^{*(lk+j)}(x)$, for $0 \leq j \leq k-1$, we get uniform convergence using the trivial estimates $|g^{*(lk+j)}(x)| \leq |g^{*(lk)}(x)||g(x)|^j$, $|g(x)|^j \leq |g(4b)|^j$ and the uniform convergence of $\sum_{l\geq 0} g^{*(lk)}(x)$. Finally note that $\phi(x) = \sum_{j=1}^{k-1}\sum_{l\geq 0} g^{*(lk+j)}(x)$. This process can be continued *ad infinitum*.

Next we wish to show that $\phi'(x) = \sum_{n\geq 1} g^{*n\prime}(x)$. Note that since g' is integrable and monotone then $|xg'(x)| < C(a) < \infty$ on any finite interval $[0, a]$. Pick an interval $(\epsilon, b]$, where b is chosen in a way that $|g(b)| = \zeta < 1/4$. From (14) in Lemma 10 and monotonicity of $|g'| = -g'$, we obtain, for each $x > 0$,

$$|g^{*2\prime}(x)| \leq 2\left|g'\left(\frac{x}{2}\right)\right|\left|g\left(\frac{x}{2}\right)\right|.$$

This allows us to set up the following induction hypothesis :

$$|g^{*n\prime}(x)| \leq n\left|g\left(\frac{x}{2}\right)\right|^{n-1}\left|g'\left(\frac{x}{2^{n-1}}\right)\right|, \quad \text{for each } x > 0.$$

From (14), the induction hypothesis and (15), we check that

$$\begin{aligned}
\left|g^{*(n+1)\prime}(x)\right| &\leq \int_0^{\frac{x}{2}} |g^{*n\prime}(x-y)||g'(y)|\mathrm{d}y + \int_{\frac{x}{2}}^{x} |g^{*n\prime}(x-y)||g'(y)|\mathrm{d}y \\
&\leq \sup_{\frac{x}{2}\leq s\leq x} \left|g^{*n\prime}(s)\right| \left|g\left(\frac{x}{2}\right)\right| + \left|g'\left(\frac{x}{2}\right)\right| \left|g^{*n}\left(\frac{x}{2}\right)\right| \\
&\leq n\left|g\left(\frac{x}{2}\right)\right|^{n-1} \left|g'\left(\frac{x}{2^n}\right)\right| \left|g\left(\frac{x}{2}\right)\right| + \left|g\left(\frac{x}{2}\right)\right|^{n} \left|g'\left(\frac{x}{2^n}\right)\right| \\
&= (n+1)\left|g\left(\frac{x}{2}\right)\right|^{n} \left|g'\left(\frac{x}{2^n}\right)\right|. \qquad (16)
\end{aligned}$$

From the conditions of Theorem 6 and the choice of b, we have $|g'(x)| \leq C(b)/x$ and $|g(x)| \leq \zeta$, for each $x \in (\epsilon, b)$. Therefore

$$\sum_{n\geq 1} |g^{*n\prime}(x)| \leq \frac{C(b)}{\epsilon} \sum_{n\geq 1} (n+1)2^n \zeta^n < \infty.$$

Thus, we conclude that, for each $\epsilon > 0$, the series $\sum_{n\geq 1} g^{*n\prime}(x)$ is uniformly convergent on (ϵ, b) and hence by the dominated convergence theorem $\phi'(x)$ exists and equals $\sum_{n\geq 1} g^{*n\prime}(x)$. Since ϵ is arbitrary, the latter conclusion holds for $x \in (0, b]$.

Next we extend this identity to $(0, 2b]$. Fix an interval $(\epsilon, 2b]$. Due to the uniform convergence of $\sum_{n\geq 0} g^{*n}(x)$ on $(0, b]$ we choose k such that

$$\xi = \max_{0\leq j<k} \sup_{s\in(0,b]} |g^{*(k+j)}(s)| < \frac{1}{4}.$$

It can directly be estimated using (16) that, for each $x > 0$,

$$\begin{aligned}
\left|g^{*(2k+j)\prime}(x)\right| &\leq \int_0^{\frac{x}{2}} \left|g^{*k\prime}(x-y)\right| \left|g^{*(k+j)\prime}(y)\right| \mathrm{d}y \\
&\quad + \int_{\frac{x}{2}}^{x} \left|g^{*k\prime}(x-y)\right| \left|g^{*(k+j)\prime}(y)\right| \mathrm{d}y \\
&\leq \sup_{\frac{x}{2}\leq s\leq x} \left|g^{*k\prime}(s)\right| \left|g^{*(k+j)}\left(\frac{x}{2}\right)\right| + \sup_{\frac{x}{2}\leq s\leq x} \left|g^{*(k+j)\prime}(s)\right| \left|g^{*k}\left(\frac{x}{2}\right)\right| \\
&\leq k\left|g'\left(\frac{x}{2^k}\right)\right| \left|g\left(\frac{x}{2}\right)\right|^{k-1} \left|g^{*(k+j)}\left(\frac{x}{2}\right)\right| \\
&\quad + (k+j)\left|g'\left(\frac{x}{2^{k+j}}\right)\right| \left|g\left(\frac{x}{2}\right)\right|^{k+j-1} \left|g^{*k}\left(\frac{x}{2}\right)\right|
\end{aligned}$$

$$\leq 2(k+j)\left|g'\left(\frac{x}{2^{k+j}}\right)\right|\left(\max_{0\leq j<k}\left|g^{*(k+j)}\left(\frac{x}{2}\right)\right|\right)$$
$$\left(\max\left\{\left|g\left(\frac{x}{2}\right)\right|^{k+j-1},\left|g\left(\frac{x}{2}\right)\right|^{k-1}\right\}\right).$$

Therefore, for each $x \in (0, 2b]$, we get

$$\left|g^{*(2k+j)\prime}(x)\right| \leq 2(k+j)Q(x)\xi\left|g'\left(\frac{x}{2^{k+j}}\right)\right|,$$

where $Q(x) = Q(k, j, x) = \max\{|g(\frac{x}{2})|^{k+j-1}, 0 \leq j < k\}$ is a non-decreasing function in x.

This allows us to set up the following induction hypothesis. For each $0 \leq j \leq k-1$ and $x \in (0, 2b]$,

$$\left|g^{*(nk+j)\prime}(x)\right| \leq n(k+j)Q(x)\xi^{n-1}\left|g'\left(\frac{x}{2^{k+j+n}}\right)\right|.$$

Then we write using the latter

$$\begin{aligned}\left|g^{*((n+1)k+j)\prime}(x)\right| &= \left|\int_0^x g^{*(nk+j)\prime}(x-y)g^{*k\prime}(y)\mathrm{d}y\right| \\ &\leq \int_0^{\frac{x}{2}} \left|g^{*(nk+j)\prime}(x-y)\right|\left|g^{*k\prime}(y)\right|\mathrm{d}y \\ &\quad + \int_{\frac{x}{2}}^{x} \left|g^{*(nk+j)\prime}(x-y)\right|\left|g^{*k\prime}(y)\right|\mathrm{d}y \\ &\leq \sup_{\frac{x}{2}\leq s\leq x}\left|g^{*(nk+j)\prime}(s)\right|\left|g^{*k}\left(\frac{x}{2}\right)\right| + \sup_{\frac{x}{2}\leq s\leq x}\left|g^{*k\prime}(s)\right|\left|g^{*(nk+j)}\left(\frac{x}{2}\right)\right| \\ &\leq (n+1)(k+j)Q(x)\xi^n\left|g'\left(\frac{x}{2^{k+j+n+1}}\right)\right|,\end{aligned}$$

where the second inequality uses $|g^{*n\prime}(x)| = (-1)^{n+1}g^{*n\prime}(x)$ (cf. Lemma 10) for the integration, and the third inequality (16) follows from the induction hypothesis and $|g^{*nk+j}(\frac{x}{2})| \leq |g^{*k}(\frac{x}{2})|^{n-1}|g^{*(k+j)}(\frac{x}{2})| \leq \xi^n$, for $x \leq 2b$. Finally, recall that $|g'(\frac{x}{2^{k+j+n}})| \leq C(2b)2^{k+j+n}/x$ holds on $x \in (0, 2b]$ with some $C(2b) > 0$ to deduce that

$$\sum_{n\geq 1}|g^{*(kn+j)\prime}(x)| \leq \frac{1}{\epsilon}\sum_{n\geq 1}(n+1)Q(2b)C(2b)\xi^n 2^{k+j+n} < \infty,$$

for each $x \in (\epsilon, 2b]$ (taking account that $|g'(0+)|$ can be infinity) and $j \leq k-1$. Hence we conclude that $\sum_{n\geq 1} g^{*n\prime}(x) = \sum_{j=0}^{k-1}\sum_{n\geq 0} g^{*(kn+j)\prime}(x)$ is uniformly convergent on $(\epsilon, 2b]$ and by the dominated convergence theorem it follows that $\phi'(x)$ equals $\sum_{n\geq 1} g^{*n\prime}(x)$. Since this procedure can be repeated forever and ϵ is arbitrary, then $\phi'(x) = \sum_{n\geq 1} g^{*n\prime}(x)$ on $(0, \infty)$. □

Proof of Theorem 7 Since $g(x)$ satisfies all the conditions in Theorem 6 we can readily use the conclusions therein.

Next we move on to the higher order smoothness properties. Set the following induction hypothesis: $g \in C^l(0, \infty)$ implies that $f \in C^r(0, \infty)$, for all $r \leq l$, $\mathrm{d}^r f(x)/\mathrm{d}x^r = \sum_{n\geq 0} \mathrm{d}^r g^{*n}(x)/\mathrm{d}x^r$ and each of these series is uniformly and absolutely convergent on any interval (ϵ, a). Note that Theorem 6 implies the induction hypothesis, for $l = 0$ when $g \in C(0, \infty)$. Assume that $g \in C^{l+1}(0, \infty)$. Next take an arbitrary interval $(2\epsilon, a)$, and recall that for a fixed index k_0, (10) holds with some constant $A(a)$. Then, for any $x \in (2\epsilon, a)$,

$$
\begin{aligned}
&\frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}} g^{*(n+k_0)}(x) \\
&= \frac{\mathrm{d}^l}{\mathrm{d}x^l} \int_0^x g^{*n\prime}(x-y) g^{*(k_0)\prime}(y)\mathrm{d}y \\
&= \int_0^{\epsilon} \frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}} g^{*n}(x-y) g^{*(k_0)\prime}(y)\mathrm{d}y + \int_0^{x-\epsilon} g^{*n\prime}(y) \frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}} g^{*(k_0)}(x-y)\mathrm{d}y \\
&\quad + \sum_{m=1}^{l} \frac{\mathrm{d}^m}{\mathrm{d}x^m} g^{*n}(\epsilon) \frac{\mathrm{d}^{l+1-m}}{\mathrm{d}x^{l+1-m}} g^{*(k_0)}(x-\epsilon) \\
&= \int_0^{\epsilon} \frac{\mathrm{d}^{l}}{\mathrm{d}x^{l}} g^{*n}(x-y) g^{*(k_0)\prime\prime}(y)\mathrm{d}y + \int_0^{x-\epsilon} g^{*n\prime}(y) \frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}} g^{*(k_0)}(x-y)\mathrm{d}y \\
&\quad + \sum_{m=2}^{l} \frac{\mathrm{d}^m}{\mathrm{d}x^m} g^{*n}(x-\epsilon) \frac{\mathrm{d}^{l+1-m}}{\mathrm{d}x^{l+1-m}} g^{*(k_0)}(\epsilon).
\end{aligned} \tag{17}
$$

Next, we sum the left and right hand side of (17) with a view to establishing the induction hypothesis, i.e. that

$$
\sum_{n\geq 1} \frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}} g^{*n}(x) = \sum_{j\leq k_0} \frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}} g^{*n}(x) + \sum_{n\geq 1} \frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}} g^{*(n+k_0)}(x) \tag{18}
$$

is uniformly and absolutely convergent on any interval (ϵ, a), which allows us to interchange differentiation and summation.

Note that the derivatives in the finite sum on the right hand side of (18) are well defined since one may follow a similar procedure to the justification given above for the continuous $(l+1)$th order derivatives of $g^{*(n+k_0)}(x)$. Indeed in the aforementioned argument, the role of g^{*k_0} is played instead by g'. Uniform and absolute convergence on any interval (ϵ, a) for the finite sum in (18) thus follows.

Next we turn to the second sum on the right hand side of (18). Note that the first term on the right hand side of (17) is absolutely and uniformly summable with respect to n on $x \in (2\epsilon, a)$ due to the induction hypothesis, i.e. $g \in C^l(0,\infty)$ implies that $f \in C^l(0,\infty)$, $\mathrm{d}^l f(x)/\mathrm{d}x^l = \sum_{n\geq 0} \mathrm{d}^l(g^{*n}(x))/\mathrm{d}x^l$ and the series is uniformly and absolutely convergent on any interval (ϵ, a). Similarly the third term on the right hand side of (17) is uniformly and absolutely summable with respect to n on $(2\epsilon, a)$ and defines a continuous function on $(2\epsilon, a)$. Finally note that $f'(x) = \sum_{n\geq 1} g^{*n\prime}(x)$ is absolutely integrable at zero. The latter is a consequence of (10), the fact that

$$g^{*(n+k_0)\prime}(x) = \int_0^x g^{*n}(x-y) g^{*(k_0)\prime\prime}(y)\mathrm{d}y, \tag{19}$$

$$\begin{aligned}\sum_{n\geq 0} |g^{*(n+k_0)\prime}(x)| &\leq \int_0^x \sum_{n\geq 1} |g^{*n}(x-y)||g^{*(k_0)\prime\prime}(y)|\mathrm{d}y \\ &\leq \sum_{n\geq 1} |g^{*n}(x)| \sup_{y\leq x} |g^{*(k_0)\prime\prime}(y)| x\end{aligned}$$

and the uniform convergence of $\sum_{n\geq 1} |g^{*n}(x)|$ on $(0, a)$. Therefore

$$\sum_{n\geq 1} \int_0^{x-\epsilon} g^{*n\prime}(y) \frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}} g^{*(2k)}(x-y)\mathrm{d}y,$$

and hence the sum with respect to n of the second term on the right hand side of (17), is uniformly and absolutely convergent on $(2\epsilon, a)$. Thus we have completed the proof of the claim that $g \in C^{l+1}(0,\infty)$ implies that $f \in C^{l+1}(0,\infty)$. Since, in the above reasoning, ϵ may be taken arbitrarily small, we may finally claim the validity of the the induction hypothesis at the next iteration.

For the converse of the latter conclusion, assume that $f \in C^{l+2}(0,\infty)$. We know that g is at least $C^2(0,\infty)$ thanks to the conditions of the theorem and therefore from the proof so far $f \in C^2(0,\infty)$. Put $z(x) = f(x) - 1$ and check by rearrangement that $g = z + g * (-z)$ and therefore

$$g = z + \sum_{n\geq 1} (-1)^n z^{*(n+1)}.$$

Also note that $z'(x) = \sum_{n\geq 1} g^{*n\prime}(x)$. Then similarly to (19), for $m = k_0 + j$, on $(0, a)$,

$$|z^{*m\prime}(x)| \leq \sum_{l\geq m} p(l)|g^{*l\prime}(x)| \leq A(a) \sum_{n\geq j} p(n+k_0)|g^{*n}(x)|,$$

where $p(n)$ is the number of partitions of n in terms of sums of positive integers. Note that since $\sum_{n\geq 1} g^{*n}(x)$ is absolutely convergent, then there is an index n_0 such that on $(0, a)$, $|g^{*n_0}(x)| < 1/2$. Then, for $j \geq n_0$, we have, for some $C > 0$,

$$|z^{*m'}(x)| \leq C \sum_{n \geq j} p(n+j)|g^{*n_0}(x)|^{\frac{n}{n_0}}.$$

From [17] asymptotically $\lim_{n\to\infty} p(n)e^{-\sqrt{n}} = 0$. Therefore $|z^{*m'}(x)|$ is bounded on $[0, a)$. Similarly we can deduce the same for $|z^{*m''}(x)|$ and the arguments used for studying smoothness of f given smoothness of g, via the equation $f = 1 + f * g$, are applicable to the series above and we conclude that $f \in C^{l+2}(0, \infty)$ implies that $g \in C^{l+2}(0, \infty)$. □

5 General remark on scale function proofs

Before we proceed to the proofs of Theorems 1, 2 and 3, we make a general remark which applies to all three of the latter. Specifically we note that it suffices to prove these three theorems for the case that $q = 0$ and $\psi'(0+) \geq 0$. In the case that $\psi'(0+) < 0$ and/or $q \geq 0$, we know from (4) that smoothness properties of $W^{(q)}$ reduce to smoothness properties of $W_{\Phi(q)}$. Indeed, on account of the fact that $\Phi(q) > 0$ (and specifically $\Phi(0) > 0$ when $\psi'(0+) < 0$ and $q = 0$), it follows that $\psi'_{\Phi(q)}(0+) = \psi'(\Phi(q)) > 0$. The smoothness properties of $W_{\Phi(q)}$ would then be covered by the proofs of the smoothness properties of W under the assumption that $\psi(0+) \geq 0$ providing that the Lévy measure $\Pi_{\Phi(q)}(\mathrm{d}x) = e^{\Phi(q)x}\Pi(\mathrm{d}x)$ simultaneously respects the conditions of the Theorems 1, 2 and 3. However, it is clear that this is the case as it is the behaviour of $\Pi_{\Phi(q)}$ in the neighbourhood of the origin which matters. Note that any statement of equivalence for smoothness of $W^{(q)}$ in relation to the smoothness of Π valid for a fixed q will extend itself to any $q \geq 0$. Henceforth we assume that $q = 0$ and $\psi'(0+) \geq 0$.

6 Proof of Theorem 1

Assume that X oscillates (equivalently $\psi'(0+) = 0$). Then from (7) $W(x)$ is the potential measure of the descending ladder height process. Then (13) holds with $u(x) = W'(x)$, $\delta = \sigma^2/2$ and $g(x) = -\int_0^x \overline{\mu}(y)\mathrm{d}y = -\int_0^x \overline{\overline{\Pi}}(y)\mathrm{d}y$, where

$$\overline{\overline{\Pi}}(x) := \int_x^\infty \overline{\Pi}(y)\mathrm{d}y.$$

On the other hand when X drifts to infinity, that is to say $\psi'(0+) > 0$, W is again the renewal function of the descending ladder height process and moreover, from the Wiener-Hopf factorization one also deduces that $W(x) = P(\inf_{s\geq 0} X_s \geq -x)/\psi'(0+)$, see for example Chapter 8 of [10]. The probability that the descending ladder height subordinator crosses level $x > 0$ is now equal to $1 - P(\inf_{s\geq 0} X_s \geq -x)$. Therefore

(13) is slightly transformed to

$$\frac{\sigma^2}{2} W'(x) + \int_0^x W'(x-y)\overline{\overline{\Pi}}(y)\mathrm{d}y = 1 - \psi'(0+) \int_0^x W'(y)\mathrm{d}y. \tag{20}$$

Putting $f(x) = \sigma^2 W'(x)/2$ and $g(x) = -2\sigma^{-2} \int_0^x \overline{\overline{\Pi}}(y)\mathrm{d}y - 2\psi'(0+)\sigma^{-2}x$ in (20) we get $f = 1 + f * g$. Note that the latter also agrees with the case $\psi'(0+) = 0$.

In either of the two cases (oscillating or drifting to infinity) we may now apply Theorem 6 in a similar way to the way it was used in the proof of Corollary 9 to deduce the required result.

7 Proof of Theorems 2, 3

Proof of Theorem 2 Recall from the proof of Theorem 1 the identity (20) and the choices $f(x) = \sigma^2 W'(x)/2$ and $g(x) = -2\sigma^{-2} \int_0^x \overline{\overline{\Pi}}(y)\mathrm{d}y - 2\psi'(0+)\sigma^{-2}x$ which transform it to the renewal equation $f = 1 + f * g$.

Our objective is to recover the required result from Theorem 7. Clearly all we need check is that (10) holds for the given $g(x)$. The assumption on the Blumenthal–Getoor index implies that there is a $\vartheta \in (1, 2)$ such that $\int_{-1}^0 |x|^\vartheta \Pi(dx) < \infty$. Therefore

$$|\epsilon|^\vartheta \left(\overline{\Pi}(\epsilon) - \overline{\Pi}(1)\right) \le \int_{-1}^{-\epsilon} |x|^\vartheta \Pi(dx) < \infty$$

and it follows that $|\epsilon|^\vartheta \overline{\Pi}(\epsilon) \le C$, for each $\epsilon \le 1$. This and the fact that $\lim_{x\to\infty} \overline{\Pi}(x) = 0$ show that $|g''(x)| \le C(a)x^{-\vartheta}$, for each interval $(0, a)$.

Next, from the definition of $\overline{\overline{\Pi}}(x)$ and the bound just obtained for $\overline{\Pi}(x)$, we get for $x \le 1$,

$$\overline{\overline{\Pi}}(x) \le \int_x^1 \frac{C(1)}{y^\vartheta}\mathrm{d}y + \overline{\Pi}(1) \le \frac{D(1)}{x^{\vartheta-1}}$$

for an appropriate constant $D(1) > 0$. Since $\lim_{x\to\infty} \overline{\overline{\Pi}}(x) = 0$ we conclude that $\overline{\overline{\Pi}}(x) \le C(a)x^{1-\vartheta}$ on every interval $(0, a)$ where $C(a) > 0$ plays the role of a generic appropriate constant. Then using these bounds on g' and g'', and Remark 8 we check that

$$\max\left\{\sup_{s<a}\{|g^{*l\prime}(s)|\}; \sup_{s<a}\{|g^{*l\prime\prime}(s)|\}\right\} \le D(l)a^{l-2-l(\vartheta-1)}, \quad \text{for some } D(l) < \infty.$$

Hence (10) holds for some l big enough and Theorem 7 is applicable. The proof is complete. □

Proof of Theorem 3 Suppose as usual that $\psi'(0+) \geq 0$. In this case it is well known that the Laplace exponent can be rewritten in the form

$$\psi(\theta) = \mathrm{c}\theta - \theta \int_0^\infty e^{-\theta x}\overline{\Pi}(x)\mathrm{d}x \quad \text{for} \quad \theta \geq 0 \tag{21}$$

where necessarily $\mathrm{c} > 0$ and $\int_{(-\infty,0)}(1 \wedge |x|)\Pi(\mathrm{d}x) < \infty$; see Chapter VII of [2]. On account of the fact that $\psi'(0+) \geq 0$, it follows from (21) that

$$\mathrm{c} \geq \int_0^\infty \overline{\Pi}(x)\mathrm{d}x > \int_0^\infty e^{-\theta x}\overline{\Pi}(x)\mathrm{d}x$$

for all $\theta > 0$. Using the above inequality to justify convergence, we may write

$$\frac{\theta}{\psi(\theta)} = \frac{1}{\mathrm{c}}\left(\frac{1}{1 - \frac{1}{\mathrm{c}}\int_0^\infty e^{-\theta x}\overline{\Pi}(x)\mathrm{d}x}\right) = \frac{1}{\mathrm{c}}\sum_{n\geq 0}\frac{1}{\mathrm{c}^n}\left(\int_0^\infty e^{-\theta x}\overline{\Pi}(x)\mathrm{d}x\right)^n. \tag{22}$$

From (7) we deduce that

$$W(x) = \frac{1}{\mathrm{c}}\sum_{n\geq 0} g^{*n}(x), \tag{23}$$

where $g(x) = \frac{1}{\mathrm{c}}\int_0^x \overline{\Pi}(x)\mathrm{d}x$. Taking $f(x) = \mathrm{c}W(x)$ we see that we are again reduced to studying the equation $f = 1 + f * g$. The proof now follows as a direct consequence of Theorem 7 with the conditions on g following as a straightforward consequence of the fact that $g''(x) = \mathrm{c}^{-1}\pi(x)$ together with the assumption on the latter density. □

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