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On the central limit theorem for $f(n_k x)$

Christoph Aistleitner · István Berkes

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Abstract By a classical observation in analysis, lacunary subsequences of the trigonometric system behave like independent random variables: they satisfy the central limit theorem, the law of the iterated logarithm and several related probability limit theorems. For subsequences of the system $(f(nx))_{n\geq 1}$ with 2π-periodic $f \in L^2$ this phenomenon is generally not valid and the asymptotic behavior of $(f(n_k x))_{k\geq 1}$ is determined by a complicated interplay between the analytic properties of f (e.g., the behavior of its Fourier coefficients) and the number theoretic properties of n_k. By the classical theory, the central limit theorem holds for $f(n_k x)$ if $n_k = 2^k$, or if $n_{k+1}/n_k \to \alpha$ with a transcendental α, but it fails e.g., for $n_k = 2^k - 1$. The purpose of our paper is to give a necessary and sufficient condition for $f(n_k x)$ to satisfy the central limit theorem. We will also study the critical CLT behavior of $f(n_k x)$, i.e., the question what happens when the arithmetic condition of the central limit theorem is weakened "infinitesimally".

Keywords Lacunary series · Central limit theorem · Diophantine equations

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C. Aistleitner (✉)
Institute of Mathematics A, Graz University of Technology,
Steyrergasse 30, 8010 Graz, Austria
e-mail: aistleitner@finanz.math.tugraz.at

I. Berkes
Institute of Statistics, Graz University of Technology,
Münzgrabenstrasse 11, 8010 Graz, Austria
e-mail: berkes@tugraz.at

1 Introduction

By classical results of Salem and Zygmund [21,22] and Erdős and Gál [8], if a sequence $(n_k)_{k\geq 1}$ of positive integers satisfies the Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \ldots), \tag{1.1}$$

then $(\cos 2\pi n_k x)_{k\geq 1}$ satisfies the central limit theorem and the law of the iterated logarithm, i.e.,

$$\lim_{N\to\infty} \mathbb{P}\left\{x \in (0,1) : \sum_{k=1}^{N} \cos 2\pi n_k x \leq t\sqrt{N/2}\right\} = \Phi(t) \tag{1.2}$$

and

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k=1}^{N} \cos 2\pi n_k x = 1 \quad \text{a.e.} \tag{1.3}$$

where $\mathbb{P}$ denotes the Lebesgue measure and Φ is the standard normal distribution function. These results show that lacunary subsequences of the trigonometric system behave like sequences of independent random variables. Actually, much more than (1.2) and (1.3) is valid: Philipp and Stout [19] showed that the function

$$S(t) = S(t, x) = \sum_{k\leq t} \cos 2\pi n_k x \quad (t \geq 0)$$

considered as a stochastic process over the probability space $([0, 1], \mathcal{B}, \mathbb{P})$, is a small perturbation of a Wiener process $\{W(t),\ t \geq 0\}$ and thus it satisfies several refined asymptotic results for $W(t)$. Typical examples are Chung's law of the iterated logarithm

$$\liminf_{N\to\infty} (N/\log\log N)^{-1/2} \max_{k\leq N} \left|\sum_{\ell=1}^{k} \cos 2\pi n_\ell x\right| = \pi/4 \quad \text{a.e.}$$

and the arcsin law

$$\lim_{N\to\infty} \mathbb{P}\{x \in (0,1) : A_N(x) \leq tN\} = \frac{2}{\pi} \arcsin \sqrt{t}$$

where $A_N(x)$ denotes the number of positive partial sums $\sum_{j=1}^{k} \cos 2\pi n_j x$, $1 \leq k \leq N$.

In view of this remarkable behavior of lacunary trigonometric series, it is natural to ask if a similar result holds for the sequence $(f(n_k x))_{k\geq 1}$ for general periodic functions f. As Erdős and Fortet (see [16, p. 646]) showed, the answer is negative: if $f(x) = \cos 2\pi x + \cos 4\pi x$, then

$$\lim_{N\to\infty} \mathbb{P}\left\{x \in (0,1) : \sum_{k=1}^{N} f((2^k-1)x) \le t\sqrt{N}\right\} = \pi^{-1/2} \int_0^1 \int_{-\infty}^{t/2|\cos \pi s|} e^{-u^2} du\, ds$$

and

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k=1}^{N} f((2^k-1)x) = 2\cos \pi x \quad \text{a.e.}$$

On the other hand, Kac [15] proved that if $f : \mathbb{R} \to \mathbb{R}$ is of bounded variation on [0, 1] or it is Lipschitz continuous satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)\, dx = 0, \tag{1.4}$$

then

$$\lim_{N\to\infty} \mathbb{P}\left\{x \in (0,1) : \sum_{k=1}^{N} f(2^k x) \le t\sigma\sqrt{N}\right\} = \Phi(t) \tag{1.5}$$

provided

$$\sigma^2 = \int_0^1 f^2(x)dx + 2\sum_{k=1}^{\infty} \int_0^1 f(x) f(2^k x)dx \ne 0. \tag{1.6}$$

These results show that the probabilistic behavior of $(f(n_k x))_{k\ge 1}$ depends not only on the speed of growth of $(n_k)_{k\ge 1}$, but also on its number theoretic properties. The sequence $(f(n_k x))_{k\ge 1}$ is generally not orthogonal, and the asymptotic evaluation of the integral

$$\int_0^1 \left(\sum_{k\le N} c_k f(n_k x)\right)^2 dx \tag{1.7}$$

is a difficult problem, closely connected with the behavior of the Dirichlet series

$$\sum_{k=1}^{\infty} a_k k^{-s}, \quad \sum_{k=1}^{\infty} b_k k^{-s}$$

where

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos 2\pi kx + b_k \sin 2\pi kx)$$

is the Fourier series of f (see Wintner [26]). The asymptotics of (1.7) is a delicate problem even for lacunary $(n_k)_{k \geq 1}$, as is shown by the case $n_k = \theta^k$ where $\theta > 1$ (not necessarily an integer). Petit [17] and Fukuyama [9] showed that if f satisfies (1.4), $\int_0^1 f^2(x)\,dx = 1$ and mild regularity conditions, then

$$\int_0^1 \left(\sum_{k \leq N} f(\theta^k x) \right)^2 dx \sim c_{f,\theta} N \tag{1.8}$$

where $c_{f,\theta}$ is a constant depending sensitively on f, θ. If θ^r is irrational for $r = 1, 2, \ldots$, then $c_{f,\theta} = 1$, which is exactly the constant we would get if $f(\theta^k x)$ were independent random variables with mean 0 and variance 1. For other delicate phenomena for $\sum_{k \leq N} f(\theta^k x)$, see Fukuyama [10]. For a series representation of $c_{f,\theta}$, see Petit [17].

The growth properties of $\sum_{k=1}^N f(n_k x)$, in particular, estimates for

$$\sup_{f \in \mathcal{C}} \left| \sum_{k=1}^N f(n_k x) \right|$$

for some classes $\mathcal{C}$ of functions f satisfying (1.4) play an important role in metric discrepancy theory, see Baker [2], Philipp [18] for strong bounds for general, resp. lacunary sequences (n_k).

Returning to the CLT, in the lacunary case Gaposhkin [12] showed that

$$\lim_{N \to \infty} \mathbb{P} \left\{ x \in (0,1) : \sum_{k=1}^N f(n_k x) \leq t\sigma_N \right\} = \Phi(t) \tag{1.9}$$

holds provided that

$$\sigma_N^2 := \int_0^1 \left(\sum_{k=1}^N f(n_k x) \right)^2 dx \geq CN \tag{1.10}$$

for a positive constant $C > 0$ and $(n_k)_{k \geq 1}$ satisfies one of the following conditions:

$$\begin{aligned} &\text{(a)} \quad n_{k+1}/n_k \text{ is an integer for any } k \geq 1, \\ &\text{(b)} \quad \lim_{k \to \infty} n_{k+1}/n_k = \alpha \quad \text{where } \alpha^r \text{ is irrational for } r = 1, 2, \ldots \end{aligned} \tag{1.11}$$

Takahashi [23] proved that the CLT (1.9) also holds if $n_{k+1}/n_k \to \infty$ and $f \in \mathrm{Lip}\,(\alpha)$, $\alpha > 0$. The explanation of these phenomena was given in a profound paper of Gaposhkin [13], who showed the following remarkable result:

Theorem A *Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers satisfying the Hadamard gap condition (1.1) and assume that (1.10) holds for some constant $C > 0$. Assume further that for any fixed positive integers a, b, ν the number of solutions of the Diophantine equation*

$$an_k - bn_\ell = \nu \quad (k, \ell \geq 1) \tag{1.12}$$

is bounded by a constant $C(a, b)$, independent of ν. Then the central limit theorem (1.9) holds.

Let $(n_k^{(d)})$ denote the set-theoretic union of the sequences $(n_k)_{k\geq 1}, (2n_k)_{k\geq 1}, \ldots, (dn_k)_{k\geq 1}$. Clearly each element of $(n_k^{(d)})$ can be represented at most in d different ways in the form jn_k, $1 \leq j \leq d$, $k \geq 1$ and thus the Diophantine condition in Theorem A is equivalent to saying that for each $d \geq 1$ the number of solutions of the equation

$$n_k^{(d)} - n_\ell^{(d)} = \nu \quad (k, \ell \geq 1)$$

is at most $C = C(d)$, uniformly in $\nu > 0$. The last condition is trivially satisfied if $(n_k^{(d)})$ satisfies the Hadamard gap condition for each $d \geq 1$, and it is not hard to see that this is the case if $n_{k+1}/n_k \to \infty$ and in examples (a), (b) in (1.11) above.

Theorem A reveals the Diophantine background of the central limit theorem (1.9), but, as the example at the end of this section will show, its condition is far from necessary for the CLT. The purpose of the present paper is to improve Theorem A and to find the precise condition for the central limit theorem (1.9). Given a sequence (n_k) of positive integers, define for any $d \geq 1$, $\nu \in \mathbb{Z}$,

$$\begin{aligned} L(N, d, \nu) &= \#\{1 \leq a, b \leq d,\ 1 \leq k, \ell \leq N : an_k - bn_\ell = \nu\} \\ L(N, d) &= \sup_{\nu > 0} L(N, d, \nu). \end{aligned} \tag{1.13}$$

Our main result is the following.

Theorem 1.1 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition (1.1) and let f be a function of bounded variation satisfying (1.4) and (1.10). Assume that for any fixed $d \geq 1$ we have*

$$L(N, d) = o(N) \quad \text{as } N \to \infty. \tag{1.14}$$

Then the central limit theorem (1.9) holds. If f is a trigonometric polynomial of order r, it suffices to assume (1.14) for $d = r$.

As we will see, the Diophantine condition (1.14) in Theorem 1.1 is best possible: for any $d \geq 1$, $\delta > 0$ there exists a sequence $(n_k)_{k \geq 1}$ of positive integers satisfying the Hadamard gap condition such that $L(N, d) \leq \delta N$ for $N \geq N_0(d, \delta)$ and the CLT (1.9) fails for a trigonometric polynomial of order d. Condition (1.10) is inevitable, as is shown by the example $f(x) = \cos 2\pi x - \cos 4\pi x$, $n_k = 2^k$, for which the Diophantine condition of Theorem 1.1 is satisfied, but the CLT is not.

Of special interest is the following case, where we can calculate the variance σ_N^2 explicitly. Slightly modifying the definition of $L(N, d)$ in (1.13), let

$$L^*(N, d) = \sup_{\nu \geq 0} L(N, d, \nu).$$

For $\nu = 0$ we exclude the trivial solutions $a = b$, $k = \ell$ from $L(N, d, \nu)$. Put also $\|f\|_2 = (\int_0^1 f^2(x)\, dx)^{1/2}$.

Theorem 1.2 *Let $(n_k)_{k \geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition* (1.1) *and let f be a function of bounded variation satisfying* (1.4) *and $\|f\|_2 > 0$. Assume that for any fixed $d \geq 1$ we have*

$$L^*(N, d) = o(N) \quad \text{as } N \to \infty.$$

Then the central limit theorem (1.9) *holds with $\sigma_N = \|f\|_2 \sqrt{N}$.*

The difference between the conditions of Theorems 1.1 and 1.2 is that in Theorem 1.2 we bound the number of solutions of

$$an_k - bn_\ell = \nu, \quad 1 \leq k, \ell \leq N \tag{1.15}$$

also for $\nu = 0$. It is easily seen that Theorem 1.2 applies if $n_{k+1}/n_k \to \infty$, and in example (b) in (1.11) above. In contrast, in case (a) in (1.11), Theorem 1.2 may fail, as the example $n_k = 2^k$ shows. In this case Eq. (1.15) has too many (namely cN) solutions for $\nu = 0$ e.g., if $a = 1, b = 2$ and in Kac's theorem (1.5) the norming factor is $\sigma\sqrt{N}$ with σ^2 defined by (1.6) and not $\|f\|_2\sqrt{N}$. Finally, it is easy to see that Theorem 1.2 applies also if $n_k = 2^k + h(k)$, $k \geq k_0$ where h is a nonconstant polynomial with integer coefficients.

Theorems 1.1 and 1.2 show that the validity of the CLT for $f(n_k x)$ is determined by the Diophantine functions $L(N, d)$ and $L^*(N, d)$. Actually, much more than this is valid: even if the the CLT fails, the number of solutions of the Diophantine equation (1.15) is directly connected with the magnitude of the deviation

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(\sigma_N^{-1} \sum_{k=1}^{N} f(n_k x) \leq t \right) - \Phi(t) \right|$$

for large N. To make this precise, let

$$\varepsilon_d = \overline{\lim}_{N \to \infty} \frac{1}{N} L(N, d).$$

Clearly $L(N, d, \nu) \le d^2 N$ and consequently $L(N, d) \le d^2 N$ and $\varepsilon_d \le d^2$. Note that the quantity ε_d depends also on the sequence (n_k).

Theorem 1.3 *Let $(n_k)_{k\ge1}$ be a sequence of positive integers satisfying the Hadamard gap condition (1.1) and let f be a trigonometric polynomial of order d with nonnegative coefficients. Let $S_N = \sum_{k\le N} f(n_k x)$. Then*

$$\overline{\lim}_{N\to\infty} \sup_t |\mathbb{P}(S_N \le t\sigma_N) - \Phi(t)| \le B\varepsilon_d^{1/5}, \tag{1.16}$$

where B is a constant depending on f, d and q. On the other hand, for each $d \ge 1$ there exists a trigonometric polynomial of order d with nonnegative coefficients and a sequence $(n_k)_{k\ge1}$ satisfying the Hadamard gap condition (1.1) such that

$$\overline{\lim}_{N\to\infty} \sup_t |\mathbb{P}(S_N \le t\sigma_N) - \Phi(t)| \ge C(d)\varepsilon_d^3 \tag{1.17}$$

where $C(d)$ is a constant depending only on d.

Thus if ε_d is small, i.e., (n_k) "almost" satisfies the conditions of Theorems 1.1 and 1.2, then $f(n_k x)$ "almost" satisfies the central limit theorem.

As the proof of Theorem 1.3 will show, ε_d^3 in the lower bound in (1.17) can be improved to $\varepsilon_d^2(\log \frac{1}{\varepsilon_d})^{-1}$. However, closing the gap between the exponents 1/5 and 2 in the upper and lower bounds seems to be a very difficult problem and we did not pursue it in the present paper.

In case of trigonometric polynomials with nonnegative coefficients, Theorem 1.3 quantifies quite precisely the connection between the number of solutions of the Diophantine equation (1.15) and the distribution of S_N/σ_N for large N. Without the nonnegativity condition the situation is much more complicated: in this case the constant B in Theorem 1.3 should be replaced by B/C, where C is the constant in (1.10), which itself depends on (n_k). To decide for which f and (n_k) condition (1.10) holds is a difficult question not investigated in the present paper.

A typical case for an almost CLT of the type (1.16) is when classical number-theoretic criteria for the CLT are infinitesimally weakened. For example, the CLT holds for $f(n_k x)$ if $n_{k+1}/n_k \to \alpha$ where α^r is irrational for $r = 1, 2, \ldots$ For rational α this criterion fails, but the CLT almost holds if in the reduced form $\alpha = p/q$ both p and q are large. More precisely, for any f of bounded variation satisfying (1.4) and any $\varepsilon > 0$ there exists a $K = K(\varepsilon, f)$ such that if $n_{k+1}/n_k \to \alpha$ where $\alpha = p/q$ where p and q are coprime integers exceeding K then the left hand side of (1.16) is at most ε. The same phenomenon holds if α^r is irrational for $1 \le r < s$ and $\alpha^s = p/q$ with p, q large. A further example for a near CLT is when $n_{k+1}/n_k \ge q$, $k = 1, 2, \ldots$ with q large. Such a result was proved earlier in Berkes [3], see Theorem 4.1 on p. 360.

The following example illustrates the difference between our Diophantine condition (1.14) and Gaposhkin's condition in Theorem A.

Example Let $(m_k)_{k\ge1}$ be a sequence of positive integers with $m_{k+1} - m_k \to \infty$ and let the sequence $(n_k)_{k\ge1}$ consist of the numbers $2^{m_k} - 1$, $k = 1, 2, \ldots$, plus the numbers $2^{m_k+1} - 1$ for the indices k of the form $k = [n^\alpha]$, $\alpha > 2$. Let f be a periodic

Lipschitz function with mean 0 and $\|f\|_2 = 1$. By a result of Takahashi [23], the central limit theorem holds for $f((2^{m_k}-1)x)$ with the norming sequence $\sqrt{N}$. Clearly $\sum_{k=1}^N f(n_k x) = \sum_{j=1}^M f((2^{m_j}-1)x) + O(N^{1/\alpha})$ where $N - 2N^{1/\alpha} \le M \le N$ for $N \ge N_0$, which implies that $f(n_k x)$ also satisfies the CLT. On the other hand, for infinitely many ℓ we have $n_\ell = 2^{m_k} - 1$, $n_{\ell+1} = 2^{m_k+1} - 1$ for some k and thus $n_{\ell+1} - 2n_\ell = 1$. The number of such ℓ's up to N is $\sim N^{1/\alpha}$ and thus the equation $2n_i - n_j = 1$ has at least $cN^{1/\alpha}$ solutions for the indices $1 \le i, j \le N$. Consequently, Gaposhkin's number theoretic condition fails for (n_k).

In conclusion we note that Gaposhkin's condition implies the validity of the CLT for all subsequences of $f(n_k x)$ as well, and for this stronger version of the CLT, Gaposhkin's condition is necessary. However, since different subsequences of $f(n_k x)$ can have totally different CLT behavior for arithmetic reasons, it is preferable to give conditions implying the CLT for $f(n_k x)$ for a specific sequence $(n_k)_{k\ge1}$, without referring to subsequences of $(n_k)_{k\ge1}$.

In the case when $(n_k)_{k\ge1}$ grows subexponentially, i.e., when $n_{k+1}/n_k \to 1$, the asymptotic behavior of $f(n_k x)$ becomes much more complicated than in the Hadamard lacunary case and the central limit theorem generally fails even for $f(x) = \cos 2\pi x$, $f(x) = \sin 2\pi x$. A precise condition for the CLT for $(\cos n_k x)_{k\ge1}$ was obtained by Erdős [7], see Takahashi [24,25] for additional information. For the CLT for trigonometric series with small gaps see Berkes [4] and Bobkov and Götze [6] introducing a completely new method in gap theory. For recent asymptotic results for $f(n_k x)$ for subexponential $(n_k)_{k\ge1}$ see e.g., Philipp [20], Fukuyama and Petit [11] and Aistleitner and Berkes [1].

2 Proof of Theorems 1.1 and 1.2

In the proof of our theorems, we will use the following theorem by Heyde and Brown [14]:

Theorem B *Let $(Y_n, \mathcal{F}_n, n \ge 1)$ be a martingale difference sequence with finite fourth moments, let $V_M = \sum_{i=1}^M \mathbb{E}(Y_i^2|\mathcal{F}_{i-1})$ and let $(b_M)_{M\ge1}$ be any sequence of positive numbers. Then*

$$\sup_t \left|\mathbb{P}((Y_1+\cdots+Y_M)/\sqrt{b_M} < t) - \Phi(t)\right| \le A\left(\frac{\sum_{i=1}^M \mathbb{E}Y_i^4 + \mathbb{E}\left((V_M - b_M)^2\right)}{b_M^2}\right)^{1/5},$$

where A is an absolute constant.

In [14] this result is only stated for $b_M = \sum_{i=1}^M \mathbb{E}Y_i^2$, but the proof remains valid for general b_M without any change (see [5, Theorem A]).

To simplify the formulas we will prove Theorems 1.1 and 1.2 only in the case when f is an even function; the general case requires only minimal changes. Let

$$f(x) \sim \sum_{j=1}^\infty a_j \cos 2\pi j x$$

be the Fourier expansion of f. Without loss of generality we may assume that $\|f\|_\infty \le 1$ and Var $f \le 1$, where Var f denotes the total variation of f on the interval [0, 1]. This implies

$$|a_j| \le j^{-1}, \quad j \ge 1$$

(see Zygmund [27, p. 48]). Let $\varepsilon > 0$ be given. We put $d = \lceil \varepsilon^{-3} \rceil + 1$,

$$p(x) = \sum_{j=1}^{d} a_j \cos 2\pi jx, \quad r(x) = \sum_{j=d+1}^{\infty} a_j \cos 2\pi jx.$$

Lemma 2.1

$$\left\| \sum_{k=1}^{N} f(n_k x) \right\|_2 \le C\sqrt{N}, \quad \left\| \sum_{k=1}^{N} p(n_k x) \right\|_2 \le C\sqrt{N}, \quad \left\| \sum_{k=1}^{N} r(n_k x) \right\|_2 \le C\sqrt{\varepsilon^3 N}$$

Here, and in the sequel, C denotes positive numbers, not always the same, depending only on q, while c denotes positive numbers depending on q and d (and therefore on ε as well).

Proof The second inequality is a special case of the first. The other two inequalities follow from

$$\int_0^1 \left(\sum_{k=1}^{N} \sum_{j=J+1}^{\infty} a_j \cos 2\pi n_k jx \right)^2 dx \le \sum_{1 \le k \le k' \le N} \sum_{j,j'=J+1}^{\infty} \mathbf{1}_{(jn_k = j'n_{k'})} \frac{1}{jj'}$$
$$\le \sum_{1 \le k \le k' \le N} \sum_{j'=J+1}^{\infty} \frac{n_k}{j'^2 n_{k'}} \le \sum_{1 \le k \le k' \le N} q^{k-k'} \sum_{j'=J+1}^{\infty} \frac{1}{j'^2} \le \begin{cases} CN & \text{for } J = 0 \\ Cd^{-1}N & \text{for } J = d. \end{cases}$$

□

Lemma 2.2 *For any function f satisfying (1.4) we have*

$$\left| \int_a^b f(\lambda x)\, dx \right| \le \frac{1}{\lambda} \int_0^1 |f(x)|\, dx \le \frac{1}{\lambda} \|f\|_\infty$$

for any real numbers $a < b$ and any $\lambda > 0$.

Proof The lemma follows from

$$\int_a^b f(\lambda x)\, dx = \frac{1}{\lambda} \int_{\lambda a}^{\lambda b} f(x)\, dx = \frac{1}{\lambda} \left[\int_{\lambda a}^{\lambda a + k} f(x)\, dx + \int_{\lambda a + k}^{\lambda b} f(x)\, dx \right] = \frac{1}{\lambda} \int_{\lambda a + k}^{\lambda b} f(x)\, dx,$$

where $k \ge 0$ is the integer with $\lambda a + k \le \lambda b < \lambda a + k + 1$. □

We begin with the proof of Theorem 1.1. By the assumptions of the theorem, for any $d \geq 1$ there exists a function $g(N) = g_d(N) \to \infty$ such that

$$L(N, d, \nu) \leq N/g(N) \quad \text{for any } \nu > 0. \tag{2.1}$$

We divide the set of positive integers into consecutive blocks $\Delta_1', \Delta_1, \Delta_2', \Delta_2, \dots, \Delta_i'$, $\Delta_i, \dots$ of lenghts $\lceil 4 \log_q i \rceil$ and $\lfloor i^{1/2} \rfloor$, respectively. Let i^- and i^+ denote the smallest, respectively largest integer in Δ_i. Clearly

$$\frac{n_{(i-1)^+}}{n_{i^-}} \leq q^{-4 \log_q i} \leq i^{-4}. \tag{2.2}$$

For every $k \in \cup_{i \geq 1} \Delta_i$ let $i = i(k)$ be defined by $k \in \Delta_i$, put $m(k) = \lceil \log_2 n_k + 2 \log_2 i \rceil$ and approximate $p(n_k x)$ by a discrete function $\varphi_k(x)$ such that the following properties are satisfied:

(P1) $\varphi_k(x)$ is constant for $\frac{v}{2^{m(k)}} \leq x < \frac{v+1}{2^{m(k)}}, \quad v = 0, 1, \dots, 2^{m(k)} - 1$

(P2) $\|\varphi_k(x) - p(n_k x)\|_\infty \leq c i^{-2}$

(P3) $\mathbb{E}(\varphi_k(x) | \mathcal{F}_{i-1}) = 0$

where $\mathcal{F}_i$ denotes the σ-field generated by the intervals

$$\left[\frac{v}{2^{m(i^+)}}, \frac{v+1}{2^{m(i^+)}}\right), \quad v = 0, 1, \dots, 2^{m(i^+)} - 1.$$

Since $p(x)$ is a trigonometric polynomial, it is Lipschitz-continuous, and thus

$$\left| p(n_k x) - p(n_k x') \right| \leq c n_k 2^{-m(k)} \leq c i^{-2} \quad \text{for}$$
$$\frac{v}{2^{m(k)}} \leq x, x' < \frac{v+1}{2^{m(k)}} \quad \text{and} \quad 0 \leq v < 2^{m(k)}.$$

Thus it is possible to approximate $p(n_k x)$ by discrete functions $\hat{\varphi}_k(x)$ that satisfy (P1) and (P2). For $k \in \Delta_i$ and any atom I of the σ-field $\mathcal{F}_{i-1}$ (an interval of length $2^{-m((i-1)^+)}$) we get, letting $|I|$ denote the length of I,

$$\begin{aligned}
\frac{1}{|I|} \left| \int_I \hat{\varphi}_k(x)\, dx \right| &\leq \frac{1}{|I|} \left| \int_I p(n_k x)\, dx \right| + \frac{1}{|I|} \int_I \frac{c}{i^2}\, dx \\
&\leq \frac{\|p\|_\infty 2^{m((i-1)^+)}}{n_{i^-}} + \frac{c}{i^2} \\
&\leq \frac{2 \cdot 2^{1 + 2 \log_2 i + \log_2 n_{(i-1)^+}}}{n_{i^-}} + \frac{c}{i^2} \\
&= \frac{4 i^2 n_{(i-1)^+}}{n_{i^-}} + \frac{c}{i^2} \\
&\leq \frac{c}{i^2}
\end{aligned}$$

by Lemma 2.2, (2.2) and since $\|p\|_\infty \le \|f\|_\infty + \mathrm{Var}\, f \le 2$ by (4.12) of Chapter II and (1.25) and (3.5) of Chapter III of Zygmund [27]. Every $x \in [0,1)$ is contained in an interval of type I for some v, so we put $\varphi_k(x) = \hat{\varphi}_k(x) - |I|^{-1} \int_I \hat{\varphi}_k(t)\, dt$ for $x \in I$ and have functions that satisfy (P1), (P2) and (P3).

We put

$$Y_i = \sum_{k\in\Delta_i} \varphi_k(x), \quad T_i = \sum_{k\in\Delta_i} p(n_k x), \quad T_i' = \sum_{k\in\Delta_i'} p(n_k x), \quad V_M = \sum_{i=1}^{M} \mathbb{E}(Y_i^2 | \mathcal{F}_{i-1}).$$

Clearly $\varphi_k(x)$, $k \in \Delta_i$ are $\mathcal{F}_i$ measurable and thus Y_i is also $\mathcal{F}_i$ measurable. Let also

$$w_i = \int_0^1 \left(\sum_{k\in\Delta_i} p(n_k x) \right)^2 dx \quad \text{and} \quad s_M = \left(\sum_{i=1}^{M} w_i \right)^{1/2}.$$

We observe that

$$\begin{aligned}
w_i &= \int_0^1 \left(\sum_{k\in\Delta_i} \sum_{j=1}^{d} a_j \cos 2\pi j n_k x \right)^2 dx \\
&= \sum_{k,k'\in\Delta_i} \sum_{1\le j,j'\le d} \frac{a_j a_{j'}}{2} \int_0^1 (\cos 2\pi (j n_k - j' n_{k'}) x) + (\cos 2\pi (j n_k + j' n_{k'}) x)\, dx \\
&= |\Delta_i| \|p\|_2^2 + \sum_{k,k'\in\Delta_i, k'>k} \sum_{1\le j'<j\le d} \mathbf{1}_{(j n_k = j' n_{k'})} \cdot a_j a_{j'}
\end{aligned}$$

and get

$$\begin{aligned}
T_i^2 - w_i &= \left(\sum_{k\in\Delta_i} p(n_k x) \right)^2 - w_i \\
&= \left(\sum_{k\in\Delta_i} \sum_{j=1}^{d} a_j \cos 2\pi j n_k x \right)^2 - w_i \\
&= \sum_{\substack{1\le j,j'\le d,\ k,k'\in\Delta_i \\ 0<|j n_k - j' n_{k'}| < i^{-2}\cdot n_{(i-1)+}}} \frac{1}{2} a_j a_{j'} \cos 2\pi (j n_k - j' n_{k'}) x \\
&\quad + \sum_{\substack{1\le j,j'\le d,\ k,k'\in\Delta_i \\ i^{-2}\cdot n_{(i-1)+} \le |j n_k - j' n_{k'}| < n_{i-}}} \frac{1}{2} a_j a_{j'} \cos 2\pi (j n_k - j' n_{k'}) x + R_i(x) \\
&= U_i(x) + W_i(x) + R_i(x).
\end{aligned} \tag{2.3}$$

Here R_i is a sum of at most $2d^2|\Delta_i|^2$ trigonometric functions with coefficients at most 1 and frequencies at least n_{i^-}. Therefore by Lemma 2.2 with $f(x) = \cos 2\pi x$,

$$|\mathbb{E}(R_i|\mathcal{F}_{i-1})| \le 4d^2|\Delta_i|^2 \frac{2^{m((i-1)^+)}}{n_{i^-}} \le 8d^2 i \frac{(i-1)^2 n_{(i-1)^+}}{n_{i^-}} \le c i^{-1}. \tag{2.4}$$

The number of summands in U_i and the number of summands in W_i (all of them trigonometric functions with coefficients at most 1) are bounded by $ci^{1/2}$, because the number of quadruples (j, j', k, k') with $1 \le j, j' \le d$, $k, k' \in \Delta_i$, for which $0 < |jn_k - j'n_{k'}| < n_{i^-}$, is at most $2d^2|\Delta_i| \left(1 + \log_q(d+1)\right)$. In fact, for fixed j, j' and k in the case $n_{k'} > (d+1)n_k$ we have $jn_k - j'n_{k'} < jn_k - j'(d+1)n_k = \left(j - j'(d+1)\right) n_k = (d - (d+1))\, n_k < -n_k \le -n_{i^-}$ and there are at most $1 + \log_q(d+1)$ indices $k' \ge k$ for which $n_{k'} \le (d+1)n_k$ (and similarly in case $n_{k'} < n_k/(d+1)$). In particular

$$\|U_i\|_\infty \le ci^{1/2} \quad \text{and} \quad \|W_i\|_\infty \le ci^{1/2}. \tag{2.5}$$

Clearly

$$\begin{aligned} |Y_i^2 - T_i^2| &\le \left(\sum_{k \in \Delta_i} |p(n_k x) - \varphi_k(x)|\right) \left(\sum_{k \in \Delta_i} |p(n_k x) + \varphi_k(x)|\right) \\ &\le \left(\sum_{k \in \Delta_i} ci^{-2}\right) \left(\sum_{k \in \Delta_i} c\right) \le ci^{-2}|\Delta_i|^2 \le ci^{-1}. \end{aligned} \tag{2.6}$$

Therefore by (2.3) and (2.6) we have

$$\begin{aligned} \|V_M - s_M^2\|_2 &= \|\sum_{i=1}^M E(Y_i^2|\mathcal{F}_{i-1}) - s_M^2\|_2 \qquad (2.7) \\ &\le \|\sum_{i=1}^M E(T_i^2|\mathcal{F}_{i-1}) - s_M^2\|_2 + c\log M \\ &= \|\sum_{i=1}^M E((T_i^2 - w_i)|\mathcal{F}_{i-1})\|_2 + c\log M \\ &\le \|\sum_{i=1}^M E(U_i|\mathcal{F}_{i-1})\|_2 + \|\sum_{i=1}^M E(W_i|\mathcal{F}_{i-1})\|_2 + \|\sum_{i=1}^M E(R_i|\mathcal{F}_{i-1})\|_2. \end{aligned}$$

By (2.4) we have

$$\|\sum_{i=1}^M E(R_i|\mathcal{F}_{i-1})\|_2 \le c\log M. \tag{2.8}$$

To estimate $\| \sum_{i=1}^{M} \mathbb{E}(W_i|\mathcal{F}_{i-1})\|_2$, we observe

$$\mathbb{E}\left(\sum_{i=1}^{M} \mathbb{E}(W_i|\mathcal{F}_{i-1})\right)^2 \le 2\mathbb{E}\left(\sum_{1\le i\le i'\le M} \mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i'-1})\right). \quad (2.9)$$

By (2.5),

$$\sum_{i=1}^{M} \mathbb{E}^2(W_i|\mathcal{F}_{i-1}) \le \sum_{i=1}^{M} ci \le cM^2. \quad (2.10)$$

For $i < i'$, since $\mathbb{E}(W_i|\mathcal{F}_{i-1})$ is $\mathcal{F}_{i-1}$-measurable,

$$\begin{aligned}\left|\mathbb{E}\left(\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i'-1})\middle|\mathcal{F}_{i-1}\right)\right| &= |\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})| \\ &\le \|W_i\|_\infty\, |\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})| \\ &\le ci^{1/2}\, |\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})|,\end{aligned}$$

whence by integration

$$|\mathbb{E}\left(\mathbb{E}(W_i|\mathcal{F}_{i-1})\mathbb{E}(W_{i'}|\mathcal{F}_{i'-1})\right)| \le ci^{1/2}\mathbb{E}\,|\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})|. \quad (2.11)$$

$W_{i'}$ can be written as a trigonometric polynomial of the form

$$\sum_{u=(i')^{-2}n_{(i'-1)^+}}^{n_{i'^-}} c_u \cos 2\pi u x,$$

where $\sum_u |c_u| \le c{i'}^{1/2}$. Thus using Lemma 2.2 with $f(x) = \cos 2\pi x$ we get

$$\begin{aligned}|\mathbb{E}(W_{i'}|\mathcal{F}_{i-1})| &\le \sum_{u=(i')^{-2}n_{(i'-1)^+}}^{n_{i'^-}} |c_u| u^{-1} 2^{m((i-1)^+)} \\ &\le 2^{m((i-1)^+)}(i')^2 \frac{1}{n_{(i'-1)^+}} \sum_{u=(i')^{-2}n_{(i'-1)^+}}^{n_{i'^-}} |c_u| \\ &\le ci^2(i')^{5/2}\frac{n_{(i-1)^+}}{n_{(i'-1)^+}} \\ &\le c\, i^2(i')^{5/2}\, q^{(i-1)^+-(i'-1)^+} \;\le\; c\, i^2(i')^{5/2}\, q^{-(i'-1)^{1/2}}. \quad (2.12)\end{aligned}$$

Combining the estimates (2.9)–(2.12), we get

$$\left\| \sum_{i=1}^{M} E(W_i|\mathcal{F}_{i-1}) \right\|_2 \leq \left(cM^2 + 2 \sum_{1 \leq i < i' \leq M} ci^{5/2} i'^{5/2} q^{-(i'-1)^{1/2}} \right)^{1/2} \leq cM. \quad (2.13)$$

Finally, we estimate $\| \sum_{i=1}^{M} E(U_i|\mathcal{F}_{i-1}) \|_2$. Note that U_i is a sum of trigonometric functions with frequencies at most $i^{-2} n_{(i-1)^+}$, i.e.,

$$U_i(x) = \sum_{u=1}^{i^{-2} n_{(i-1)^+}} c_u \cos 2\pi u x,$$

where $\sum_u |c_u| \leq c i^{1/2}$. Hence the fluctuation of U_i on any atom of $\mathcal{F}_{i-1}$ is at most

$$\begin{aligned} \sum_{u=1}^{i^{-2} n_{(i-1)^+}} |c_u| 2\pi u 2^{-m((i-1)^+)} &\leq 2\pi\, i^{-2} n_{(i-1)^+}\, 2^{-m((i-1)^+)} \sum_{u=1}^{i^{-2} \cdot n_{(i-1)^+}} |c_u| \\ &\leq c\, i^{1/2} \frac{n_{(i-1)^+}}{i^2} \frac{1}{i^2 n_{(i-1)^+}} = c\, i^{-7/2} \end{aligned}$$

and consequently,

$$|\mathbb{E}(U_i|\mathcal{F}_{i-1}) - U_i| \leq c i^{-7/2},$$

which gives

$$\left\| \sum_{i=1}^{M} \mathbb{E}(U_i|\mathcal{F}_{i-1}) \right\|_2 \leq \left\| \sum_{i=1}^{M} U_i \right\|_2 + c. \quad (2.14)$$

The largest frequency of the trigonometric functions in $\sum_{i=1}^{M} U_i$ is at most $M^{-2} n_{(M-1)^+}$, so we can write, grouping the terms with equal frequency,

$$\sum_{i=1}^{M} U_i(x) = \sum_{u=1}^{M^{-2} \cdot n_{(M-1)^+}} d_u \cos 2\pi u x,$$

where by (2.1)

$$|d_u| \leq 2 \frac{\sum_{i=1}^{M} (|\Delta_i| + |\Delta_{i'}|)}{g\left(\sum_{i=1}^{M} (|\Delta_i| + |\Delta_{i'}|)\right)} \leq c\, \frac{M^{3/2} + M \log M}{g(M)} \leq c\, \frac{M^{3/2}}{g(M)} \quad (2.15)$$

(without loss of generality we assume that g is nondecreasing) and consequenty $\sum_u |d_u| \le \sum_{i=1}^M c\, i^{1/2} \le c\, M^{3/2}$. Thus

$$\left\| \sum_{i=1}^M U_i \right\|_2^2 \le \sum_{u=1}^{M^{-2}\cdot n_{(M-1)^+}} d_u^2 \le c\, \frac{M^{3/2}}{g(M)} \sum_{u=1}^{M^{-2}\cdot n_{(M-1)^+}} |d_u|$$
$$\le c\, \frac{M^3}{g(M)}$$

and hence by (2.14) it follows that

$$\left\| \sum_{i=1}^M E(U_i|\mathcal{F}_{i-1}) \right\|_2 \le \left(c\, \frac{M^3}{g(M)} \right)^{1/2} + c. \tag{2.16}$$

Substituting the estimates (2.8), (2.13) and (2.16) into (2.7), we get

$$\left\| V_M - s_M^2 \right\|_2 \le c \log M + cM + \left(c\, \frac{M^3}{g(M)} \right)^{1/2} + c$$

and therefore

$$\mathbb{E}\left((V_M - s_M^2)^2 \right) \le c \frac{M^3}{g(M)} + c\, M^{5/2} \;\le\; c \frac{M^3}{g(M)},$$

since we can assume $g(x) \le x^{1/2}$.

Now we estimate $\sum_{i=1}^M \mathbb{E}Y_i^4$. By Lemma 2.1 and property (P2) we have $\mathbb{E}Y_i^2 \le C i^{1/2}$, and so

$$\mathbb{E}Y_i^4 \le (\|Y_i\|_\infty)^2 \mathbb{E}Y_i^2 \le C i^{3/2}$$

and

$$\sum_{i=1}^M \mathbb{E}Y_i^4 \le C M^{5/2}.$$

Hence by Theorem B we get, using again $g(x) \le x^{1/2}$,

$$\begin{aligned}
&\sup_t \left|\mathbb{P}\left((Y_1+\cdots+Y_M)/s_M < t\right) - \Phi(t)\right| \\
&\quad \le A\left(\frac{\sum_{i=1}^M \mathbb{E}Y_i^4 + \mathbb{E}\left((V_M - s_M^2)^2\right)}{s_M^4}\right)^{1/5} \\
&\quad \le cA\left(\frac{M^{5/2}+M^3/g(M)}{s_M^4}\right)^{1/5} \le cA\left(\frac{2M^3/g(M)}{s_M^4}\right)^{1/5}. \qquad (2.17)
\end{aligned}$$

Now let a positive integer N be given. There exists an $M = M(N)$ with $\sqrt{N} \le M \le CN^{2/3}$ such that $N \in (\Delta_{M+1} \cup \Delta'_{M+1})$ and therefore $N - \sum_{i=1}^M \left(|\Delta_i| + |\Delta'_i|\right) \le |\Delta_{M+1}| + |\Delta'_{M+1}| \le CN^{1/3}$. We put $\hat{N} = \sum_{i=1}^M \left(|\Delta_i| + |\Delta'_i|\right)$. Then

$$\sum_{k=1}^N f(n_k x) = \sum_{i=1}^M Y_i + \sum_{i=1}^M (T_i - Y_i) + \sum_{i=1}^M T'_i + \sum_{k=\hat{N}+1}^N p(n_k x) + \sum_{k=1}^N r(n_k x). \qquad (2.18)$$

We put

$$\sigma_N = \left\| \sum_{k=1}^N f(n_k x) \right\|_2 .$$

We observe that

$$\begin{aligned}
&\left| s_M^2 - \int_0^1 \left(\sum_{k \in \bigcup_{i=1}^M \Delta_i} p(n_k x) \right)^2 dx \right| = \left| \int_0^1 \left[\sum_{i=1}^M \left(\sum_{k\in\Delta_i} p(n_k x)\right)^2 - \left(\sum_{k \in \bigcup_{i=1}^M \Delta_i} p(n_k x) \right)^2 \right] dx \right| \\
&\qquad \le \sum_{\substack{k,\,k' \in \bigcup_{i=1}^M \Delta_i \\ (k,k') \notin \bigcup_{i=1}^M (\Delta_i \times \Delta_i)}} \ \sum_{1 \le j, j' \le d} \mathbf{1}_{(jn_k = j'n_{k'})} \frac{2}{jj'} \\
&\qquad \le 4 \sum_{i=1}^M \sum_{k \in \Delta_i} \sum_{k' \in \bigcup_{i'=1}^{i-1} \Delta_{i'}} \sum_{1 \le j \le d} \frac{n_{k'}}{j^2 n_k} \\
&\qquad \le 4 \sum_{j=1}^\infty \frac{1}{j^2} \sum_{i=1}^M \sum_{k \in \Delta_i} \sum_{k' \in \bigcup_{i'=1}^{i-1} \Delta_{i'}} q^{k'-k} \\
&\qquad \le C \sum_{i=1}^M \sum_{k \in \Delta_i} \sum_{k' \in \bigcup_{i'=1}^{i-1} \Delta_{i'}} q^{-4\log_q i} \\
&\qquad \le C \sum_{i=1}^M i^{1/2}\, i^{3/2}\, i^{-4} \le C.
\end{aligned}$$

Thus by Minkowski's inequality and Lemma 2.1

$$\begin{aligned}
\sigma_N &\le \left\| \sum_{k=\hat{N}+1}^{N} f(n_k x) \right\|_2 + \left\| \sum_{k\in\bigcup_{i=1}^{M}\Delta_i'} f(n_k x) \right\|_2 + \left\| \sum_{k\in\bigcup_{i=1}^{M}\Delta_i} r(n_k x) \right\|_2 + \left\| \sum_{k\in\bigcup_{i=1}^{M}\Delta_i} p(n_k x) \right\|_2 \\
&\le (CN^{1/3})^{1/2} + (CN^{2/3}\lceil 4\log_q CN^{2/3}\rceil)^{1/2} + \left(C\varepsilon^3 \sum_{i=1}^{M} |\Delta_i| \right)^{1/2} + (s_M^2 + C)^{1/2} \\
&\le s_M + CN^{1/3}\log N + C\sqrt{\varepsilon^3 N}.
\end{aligned}$$

A similar calculation yields

$$\sigma_N \ge s_M - CN^{1/3}\log N - C\sqrt{\varepsilon^3 N}.$$

Since by assumption $\sigma_N^2 \ge KN$, choosing ε so small that $\sqrt{K} - C\sqrt{\varepsilon^3} > 0$, we get

$$s_M^2 \ge \left(\sigma_N - CN^{1/3}\log N - C\sqrt{\varepsilon^3 N}\right)^2 \ge CN \tag{2.19}$$

and thus

$$1 - C\sqrt{\varepsilon^3} \le \sigma_N/s_M \le 1 + C\sqrt{\varepsilon^3}. \tag{2.20}$$

By (2.18) we have for any fixed t and $0 < \varepsilon < 1$

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=1}^{N} f(n_k x) \le t\sigma_N \right) &\le \mathbb{P}\left(\sum_{i=1}^{M} Y_i \le (t+\varepsilon)\sigma_N \right) + \mathbb{P}\left(\left| \sum_{k=1}^{N} r(n_k x) \right| > \varepsilon\sigma_N/4 \right) \\
&\quad + \mathbb{P}\left(\left| \sum_{i=1}^{M} T_i' \right| > \varepsilon\sigma_N/4 \right) + \mathbb{P}\left(\left| \sum_{i=1}^{M} (Y_i - T_i) \right| > \varepsilon\sigma_N/4 \right) \\
&\quad + \mathbb{P}\left(\left| \sum_{k=\hat{N}+1}^{N} p(n_k x) \right| > \varepsilon\sigma_N/4 \right).
\end{aligned} \tag{2.21}$$

Also, a lower bound for $\mathbb{P}(\sum_{k=1}^{N} f(n_k x) \le t\sigma_N)$ is obtained if in the second line of (2.21) we replace $t+\varepsilon$ by $t-\varepsilon$ and change the sign of the four subsequent terms to negative. By property (P2) and $N - \hat{N} \le CN^{1/3}$ the last two summands in (2.21) are zero if N is large enough. By Minkowski's inequality

$$\begin{aligned}\left\| \sum_{i=1}^{M} T_i' \right\|_2 &= \left\| \sum_{i=1}^{M} \sum_{k\in\Delta_i'} \sum_{j=1}^{d} a_j \cos 2\pi j n_k x \right\|_2 \\ &\leq \sum_{j=1}^{d} |a_j| \left\| \sum_{i=1}^{M} \sum_{k\in\Delta_i'} \cos 2\pi j n_k x \right\|_2 \\ &\leq c \left(\sum_{i=1}^{M} |\Delta_i'| \right)^{1/2} \leq cN^{1/3} \log N\end{aligned}$$

and thus by Chebyshev's inequality

$$\mathbb{P}\left(\left| \sum_{i=1}^{M} T_i' \right| > \varepsilon\sigma_N/4 \right) \leq cN^{-1/3} (\log N)^2.$$

The third relation of Lemma 2.1 and another application of Chebyshev's inequality yield

$$\mathbb{P}\left(\left| \sum_{k=1}^{N} r(n_k x) \right| > \varepsilon\sigma_N/4 \right) \leq C\varepsilon.$$

Therefore by (2.17)–(2.21) we get for $0 < \varepsilon \leq \varepsilon_0$, $N \geq N_0(\varepsilon)$ and any $t \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{P}\left(\sum_{k=1}^{N} f(n_k x) \leq t\sigma_N \right) - \Phi(t) \\ &\leq \left| \mathbb{P}\left(\sum_{i=1}^{M} Y_i \leq s_M \left((t+\varepsilon)\sigma_N/s_M\right) \right) - \Phi((t+\varepsilon)\sigma_N/s_M) \right| \\ &\quad + |\Phi((t+\varepsilon)\sigma_N/s_M) - \Phi(t)| + C\varepsilon + cN^{-1/3}(\log N)^2 \\ &\leq c\, A \left(2M^3 g(M)^{-1} s_M^{-4} \right)^{1/5} + \left| \Phi\left((1 + C\theta\sqrt{\varepsilon^3})(t+\varepsilon) \right) - \Phi(t) \right| \\ &\quad + C\varepsilon + cN^{-1/3}(\log N)^2 \\ &\leq c(N^2 g(\sqrt{N})^{-1} N^{-2})^{1/5} + C\varepsilon + cN^{-1/3}(\log N)^2 \\ &\leq c\, g(\sqrt{N})^{-1/5} + C\varepsilon + \varepsilon \qquad (2.22)\end{aligned}$$

for some θ with $|\theta| \leq 1$. Here we used the fact that $|\Phi((1+C\theta\sqrt{\varepsilon^3})(t+\varepsilon)) - \Phi(t)| = O(\varepsilon)$ for any $t \in \mathbb{R}$ with an absolute constant in the O, which can be seen separately, using the mean value theorem, for $|t| \leq 2$ and $|t| > 2$, observing that in the case $|t| > 2$ and $0 < \varepsilon \leq \varepsilon_0$, any ξ between t and $(1 + C\theta\sqrt{\varepsilon^3})(t+\varepsilon)$ satisfies $|\xi| \geq |t|/2$ and thus $0 \leq \Phi'(\xi) \leq e^{-t^2/8}$. By the remark after (2.21), the difference

$\Phi(t) - \mathbb{P}(\sum_{k=1}^{N} f(n_k x) \le t\sigma_N)$ can be estimated similarly as in (2.22), except that $t+\varepsilon$ in the second, third and fourth line should be replaced by $t-\varepsilon$. Since $0 < \varepsilon \le \varepsilon_0$ was arbitrary, Theorem 1.1 is proved.

To prove Theorem 1.2, assume that $L^*(N, d) = o(N)$ for any $d \ge 1$. Then the function $g(N) = g_d(N) \to \infty$ in (2.1) can be chosen so that (2.1) remains valid also for $\nu = 0$. Letting $\varepsilon > 0$ and using the same notations as above, we get

$$
\begin{aligned}
\sigma_N &\ge \left\| \sum_{k=1}^{N} p(n_k x) \right\|_2 - \left\| \sum_{k=1}^{N} r(n_k x) \right\|_2 \\
&\ge \left(\int_0^1 \left(\sum_{k=1}^{N} p(n_k x) \right)^2 dx \right)^{1/2} - C\sqrt{\varepsilon^3 N} \\
&\ge \left(\sum_{k=1}^{N} \sum_{j=1}^{d} \frac{a_j^2}{2} - 2 \sum_{1 \le k < k' \le N} \sum_{1 \le j, j' \le d} \mathbf{1}_{(j n_k = j' n_{k'})} \right)^{1/2} - C\sqrt{\varepsilon^3 N} \\
&\ge \left(\|f\|_2^2 N - \|r\|_2^2 N - \frac{2N}{g(N)} \right)^{1/2} - C\sqrt{\varepsilon^3 N} \\
&\ge \|f\|_2 \sqrt{N} - C\sqrt{\varepsilon^3 N}
\end{aligned}
$$

for sufficiently large N. A similar argument yields

$$\sigma_N \le \|f\|_2 \sqrt{N} + C\sqrt{\varepsilon^3 N}$$

for sufficiently large N and since $\varepsilon > 0$ was arbitrary, it follows that

$$\sigma_N \sim \|f\|_2 \sqrt{N}.$$

Thus Theorem 1.2 follows from Theorem 1.1.

3 Proof of Theorem 1.3

The upper bound in Theorem 1.3 is implicit in the proof of Theorem 1.1. We prove the lower bound in the case $d = 2$; the modifications in the case $d \ge 3$ are straightforward. Let $(m_\ell)_{\ell \ge 1}$ be a sequence of integers satisfying $m_{\ell+1} - m_\ell \to \infty$. Fix $r \ge 1$ and let $(n_k)_{k \ge 1}$ consist of the blocks $H_\ell = \{2^{m_\ell+1}-1, 2^{m_\ell+2}-1, 2^{m_\ell+4}-1, \ldots, 2^{m_\ell+2^{r-1}}-1\}$, $\ell \ge \ell_0$, where ℓ_0 is chosen so large that all the above blocks are disjoint. Clearly $\liminf_{k\to\infty} n_{k+1}/n_k = 2$ and thus $(n_k)_{k \ge 1}$ satisfies the Hadamard gap condition.

Let $a, b \in \mathbb{N}$ and consider the equation

$$a n_i - b n_j = \nu \tag{3.1}$$

where $\nu > 0$, $i > j$. Since $m_{\ell+1} - m_\ell \to \infty$, there exists a constant $K = K(a, b)$ such that if n_i and n_j $(i > j)$ belong to different blocks H_ℓ, $H_{\ell'}$ with $\ell > \ell'$, $\ell \geq K$, then $an_i - bn_j \geq an_i/2$. Hence in this case Eq. (3.1) implies that $an_i/2 \leq \nu \leq an_i$ and thus $\nu/a \leq n_i \leq 2\nu/a$. Because of the large separation between the blocks caused by $m_{\ell+1} - m_\ell \to \infty$, the last relation determines uniquely the block H_ℓ to which n_i belongs and thus the number of choices for n_i is at most $|H_\ell| = r$. Once n_i is known, n_j is uniquely determined by (3.1). Thus the number of solutions (i, j) of (3.1) where n_i and n_j belong to different blocks is at most $r + K_1(a, b)$, where the second term is due to the number of solutions (i, j) where $n_i \in H_\ell$, $n_j \in H_{\ell'}$ with $1 \leq \ell, \ell' < K(a, b)$. If i and j belong to the same block H_ℓ, then (3.1) can be written as

$$a(2^{m_\ell+2^s} - 1) - b(2^{m_\ell+2^t} - 1) = \nu \quad 0 \leq s, t \leq r - 1 \tag{3.2}$$

i.e.,

$$2^{m_\ell}(a2^{2^s} - b2^{2^t}) = \nu + a - b. \tag{3.3}$$

For $c \neq 0$, the equation $a2^{2^s} - b2^{2^t} = c$ has at most one solution (s, t) with $s = t$. If e.g., $s > t$ and $t > \log_2 \log_2(2b/a)$, then $a2^{2^s}/b2^{2^t} = (a/b)\, 2^{2^t(2^{s-t}-1)} \geq (a/b)\, 2^{2^t} > 2$ and thus the equation $a2^{2^s} - b2^{2^t} = c$ implies $c \leq a2^{2^s} < 2c$ which is satisfied for at most one s. Since s determines t uniquely, and the number of solutions (s, t) with $t \leq \log_2 \log_2(2b/a)$ is clearly $\leq \log_2 \log_2(2b/a)$, it follows that the number of solutions (s, t) of the equation $a2^{2^s} - b2^{2^t} = c$ with $c \neq 0$, $s > t$ is at most $1 + \log_2 \log_2(2b/a)$. If $c = 0$, then we have $a2^{2^s} = b2^{2^t}$, whence $\log_2(b/a) = 2^s - 2^t = 2^t(2^{s-t} - 1) \geq 2^t$ provided e.g., $s > t$, i.e., the number of such solutions is at most $\log_2 \log_2(b/a)$.

Summarizing, we have proved that the number of solutions (s, t) of (3.1) such that i and j belong to the same fixed block H_ℓ is at most $C(a, b)$, and the number of solutions (s, t) such that i and j belong to different blocks is at most $r + K_1(a, b)$. Let now $N > r$ and choose $M \geq 1$ so that $Mr < N \leq (M+1)r$. Then the number of solutions of (3.1) for $i, j \leq N$ (which means that n_i and n_j are permitted to run in the first $M+1$ blocks) is at most $C(a, b)(M + 1) + r + K_1(a, b) \leq 2C(a, b)M + r + K_1(a, b) \leq 2C(a, b)N/r + N/r + K_1(a, b) \leq C_1(a, b)N/r$ if $N \geq r^2$. Thus $L(N, d, \nu) \leq C(d)N/r$, $L(N, d) \leq C(d)N/r$ for $N \geq r^2$ and thus

$$\varepsilon_d = \limsup_{N\to\infty} N^{-1} L(N, d) \leq C(d)/r. \tag{3.4}$$

Let now $f(x) = \cos 2\pi x + \cos 4\pi x$, then

$$\sum_{k\in H_\ell} f(n_k x) = \sum_{k\in H_\ell} \cos 2\pi n_k x + \sum_{k\in H_\ell} \cos 4\pi n_k x.$$

The frequencies in the first trigonometric sum on the right side are

$$2^{m_\ell+1} - 1,\ 2^{m_\ell+2} - 1,\ 2^{m_\ell+4} - 1,\ 2^{m_\ell+8} - 1, \ldots \tag{3.5}$$

and in the second sum are

$$2^{m_\ell+2}-2,\ 2^{m_\ell+3}-2,\ 2^{m_\ell+5}-2,\ 2^{m_\ell+9}-2,\dots \tag{3.6}$$

(The frequency of $\cos 2\pi\lambda x$ is meant as $|\lambda|$.) Note that the second frequency in (3.5) and the first frequency in (3.6) differ by 1 and thus the sum of the corresponding cosines is

$$2\cos(\pi x)\cos 2\pi(2^{m_\ell+2}-3/2)x.$$

The remaining frequencies in (3.5) and (3.6) give a trigonometric sum $g_\ell(x)$ with $2r-2$ terms, with frequencies between $2^{m_\ell+1}-1$ and $2^{m_\ell+2^{r-1}+1}-2$. Note that, given any $1<q<2$, the frequencies in g_ℓ satisfy the Hadamard gap condition with ratio q provided $\ell>\ell_0(q)$ and this property remains valid even if we include $2^{m_\ell+2}-3/2$ in the above set of frequencies. Thus discarding the first ℓ_0 blocks H_ℓ and shifting indices, we can assume that the above statements are valid for all $\ell\ge 1$. Thus letting $\psi(x)=2\cos(\pi x)$ we have

$$\sum_{k\in H_\ell} f(n_k x)=\psi(x)\cos 2\pi(2^{m_\ell+2}-3/2)x+g_\ell(x)$$

and consequently

$$\frac{1}{\sqrt{Nr}}\sum_{k\le Nr} f(n_k x)=\psi(x)\frac{1}{\sqrt{Nr}}\sum_{\ell=1}^{N}\cos 2\pi(2^{m_\ell+2}-3/2)x + \frac{1}{\sqrt{Nr}}\sum_{j=1}^{N(2r-2)}\cos 2\pi p_j x =: Z_N(x) \tag{3.7}$$

where $p_1<p_2<\dots$ is Hadamard lacunary, moreover it remains lacunary together with the frequencies $2^{m_\ell+1}-3/2$ in the first sum on the right hand side of (3.7). The limit distribution of $Z_N(x)$ is easy to determine for any continuous function ψ on $[0,1]$. Assume first that $\psi=\sum_{j=1}^{s} d_j I_{[v_{j-1},v_j)}$ is a stepfunction where $0=v_0<v_1<\dots<v_s=1$. Applying the CLT of Salem and Zygmund (see [21], statement (iii) on p. 333) for the interval $[v_{j-1},v_j)$, it follows that

$$\lim_{N\to\infty}\frac{1}{v_j-v_{j-1}}\int_{v_{j-1}}^{v_j}\exp(itZ_N(x))dx=\exp\left(-\frac{1}{2}\left(1-\frac{1}{r}+\frac{d_j^2}{2r}\right)t^2\right)$$

and thus

$$\lim_{N\to\infty}\int_0^1\exp(itZ_N(x))dx=\int_0^1\exp(h(x)t^2)dx \tag{3.8}$$

where

$$h(x) = -\frac{1}{2}\left(1 - \frac{1}{r} + \frac{\psi(x)^2}{2r}\right) = -\frac{1}{2} - \frac{\psi(x)^2 - 2}{4r}. \tag{3.9}$$

By a simple approximation argument, (3.8) and (3.9) remain valid for any continuous function ψ and thus $(Nr)^{-1/2}\sum_{k\le Nr} f(n_k x)$ has a limit distribution G whose characteristic function is

$$\varphi(t) = \int_0^1 \exp(h(x)t^2)dx \quad \text{with } h(x) = -\frac{1}{2} - \frac{2\cos^2\pi x - 1}{2r} = -\frac{1}{2} - \frac{\cos 2\pi x}{2r}.$$

By Taylor expansion we get for $0 \le t \le 1$

$$\begin{aligned}\varphi(t) &= e^{-t^2/2}\left(1 - \frac{t^2}{2r}\int_0^1 \cos 2\pi x\, dx + \frac{t^4}{8r^2}\int_0^1 \cos^2 2\pi x\, dx + O(r^{-3})\right)\\ &= e^{-t^2/2}\left(1 + \frac{t^4}{16r^2} + O(r^{-3})\right)\end{aligned}$$

where the constant in the O is absolute. Thus $|\varphi(t) - e^{-t^2/2}| \ge B/r^2$ for $1/2 \le t \le 1$, $r \ge r_0$ for some positive absolute constant B. On the other hand, for any two distribution functions F_1 and F_2 with characteristic functions φ_1 and φ_2 we have for any $T \ge 2$

$$|\varphi_1(1) - \varphi_2(1)| \le 3T \sup_x |F_1(x) - F_2(x)| + \int_{|x|\ge T} dF_1(x) + \int_{|x|\ge T} dF_2(x). \tag{3.10}$$

Using this for $F_1 = \Phi$, $F_2 = G$, the two integrals on the right side of (3.10) are $O(e^{-cT^2})$ and thus choosing $T = \log r$ we get for $r \ge r_0$ that

$$\sup_x |\Phi(x) - G(x)| \ge C/(r^2 \log r),$$

which, together with (3.4), completes the proof of Theorem 1.3. Choosing r large, we also proved the remark after Theorem 1.1, concerning the optimality of Theorem 1.1.

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