

A Berry-Esseen type inequality for convex bodies with an unconditional basis

Bo'az Klartag

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Abstract Suppose $X = (X_1, \dots, X_n)$ is a random vector, distributed uniformly in a convex body $K \subset \mathbb{R}^n$. We assume the normalization $\mathbb{E}X_i^2 = 1$ for $i = 1, \dots, n$. The body K is further required to be invariant under coordinate reflections, that is, we assume that $(\pm X_1, \dots, \pm X_n)$ has the same distribution as (X_1, \dots, X_n) for any choice of signs. Then, we show that

$$\mathbb{E}(|X| - \sqrt{n})^2 \leq C^2,$$

where $C \leq 4$ is a positive universal constant, and $|\cdot|$ is the standard Euclidean norm in \mathbb{R}^n . The estimate is tight, up to the value of the constant. It leads to a Berry-Esseen type bound in the central limit theorem for unconditional convex bodies.

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1 Introduction

Let X_1, \dots, X_n be random variables. We assume that the random vector $X = (X_1, \dots, X_n)$ is distributed according to a density $f : \mathbb{R}^n \rightarrow [0, \infty)$, and that the following hold:

- (A) The joint density f is log-concave. That is, the function f has the form $f = e^{-H}$ with $H : \mathbb{R}^n \rightarrow (-\infty, \infty]$ being a convex function.

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B. Klartag (✉)
Department of Mathematics, Princeton University,
Princeton, NJ 08544, USA
e-mail: bklartag@princeton.edu

- (B) The joint density f is “unconditional”. That is, for any point $(x_1, \dots, x_n) \in \mathbb{R}^n$ and a sign vector $(\delta_1, \dots, \delta_n) \in \{\pm 1\}^n$,

$$f(x_1, \dots, x_n) = f(\delta_1 x_1, \dots, \delta_n x_n).$$

Equivalently, the random vector (X_1, \dots, X_n) has the same distribution as $(\pm X_1, \dots, \pm X_n)$ for any choice of signs.

- (C) The isotropic normalization $\mathbb{E}X_i^2 = 1$ holds for $i = 1, \dots, n$.

A particular case is when X is distributed uniformly in a convex set $K \subset \mathbb{R}^n$, which is normalized so that $\mathbb{E}X_i^2 = 1$ for all i , and is also “unconditional”, i.e., for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and for any choice of signs,

$$(x_1, \dots, x_n) \in K \Rightarrow (\pm x_1, \dots, \pm x_n) \in K.$$

We prove the following Berry-Esseen type theorem:

Theorem 1 *Under assumptions (A), (B) and (C),*

$$\sup_{\alpha \leq \beta} \left| \mathbb{P} \left(\alpha \leq \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \leq \beta \right) - \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt \right| \leq \frac{C}{n}, \quad (1)$$

where $C > 0$ is a universal constant. Moreover, for any $\theta_1, \dots, \theta_n \in \mathbb{R}$ with $\sum_i \theta_i^2 = 1$,

$$\sup_{\alpha \leq \beta} \left| \mathbb{P} \left(\alpha \leq \sum_{i=1}^n \theta_i X_i \leq \beta \right) - \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt \right| \leq C \sum_{i=1}^n \theta_i^4. \quad (2)$$

The log-concavity requirement (A) is crucial. A simple example may be described as follows: Denote by e_1, \dots, e_n the standard orthonormal basis in \mathbb{R}^n . Let T be a random variable, distributed uniformly in the set $\{1, \dots, n\}$. Let U be a random variable, independent of T , distributed uniformly in the interval $[-\sqrt{3n}, \sqrt{3n}]$. Consider the random vector $X = Ue_T$. Then $(\pm X_1, \dots, \pm X_n)$ has the same distribution as (X_1, \dots, X_n) for any choice of signs, and also $\mathbb{E}X_i^2 = 1$ for all i . However, $\sum_i X_i = U$ is distributed uniformly in an interval, and hence its distribution is far from normal. This demonstrates that assumptions (B) and (C) alone cannot guarantee gaussian approximation.

The bound in (1) is optimal, up to the precise value of the constant, as shown by the example of X_1, \dots, X_n being independent random variables, with each X_i distributed, say, uniformly in a symmetric interval (see, e.g., [14, Vol. II, Sect. XVI.4]). A central element in the proof of Theorem 1 is the sharp estimate

$$\text{Var} \left(\frac{|X|^2}{n} \right) = \mathbb{E} \left(\frac{|X|^2}{n} - 1 \right)^2 \leq \frac{C}{n}, \quad (3)$$

for a positive universal constant $C \leq 16$. Inequality (3) implies that most of the mass of the random vector X is concentrated in a *thin spherical shell* of radius \sqrt{n} , centered at the origin in \mathbb{R}^n , whose width has the order of magnitude of a universal constant. The bound (3) was established by Wojtaszczyk [41] in the case of Orlicz balls following a result of Anttila, Ball and Perissinaki [1] regarding ℓ_p -balls. We say that a random vector $X = (X_1, \dots, X_n)$ in \mathbb{R}^n is isotropically-normalized if $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i X_j = \delta_{i,j}$ for all i, j , where $\delta_{i,j}$ is Kronecker's delta. A conjecture going back to Anttila, Ball and Perissinaki (see [1,5]) is that the thin spherical shell inequality (3) actually holds whenever X is an isotropically-normalized random vector in \mathbb{R}^n with a log-concave density. We were able to verify this conjecture under the additional assumption that the density of X is unconditional.

Theorem 1 ought to be understood in the context of the *central limit theorem for convex bodies*. The central limit theorem for convex bodies is the following high-dimensional effect, suggested in the works of Brehm and Voigt [8] and Anttila, Ball and Perissinaki [1], and proven in [22,23]: Whenever $X = (X_1, \dots, X_n)$ is an isotropically-normalized random vector in \mathbb{R}^n , for large n , with a log-concave density, then for "most" choices of coefficients $\theta_1, \dots, \theta_n \in \mathbb{R}$, the random variable $\sum_i \theta_i X_i$ is approximately gaussian. (In the context of Theorem 1, note that if the vector of coefficients $(\theta_1, \dots, \theta_n)$ is distributed uniformly on the unit sphere in \mathbb{R}^n , then the right-hand side of (2) is at most C/n with probability greater than $1 - C \exp(-c\sqrt{n})$. Here $C, c > 0$ are universal constants.) There is an intimate relation between the central limit theorem for convex bodies and thin spherical shell estimates like (3). This connection is well-known, beginning with the work of Sudakov [39]. The reader is referred to, e.g., [22] for more background on the central limit theorem for convex bodies and to, e.g., [1,4,5] for the relation to thin shell estimates.

Previous techniques for obtaining thin spherical shell estimates under convexity assumptions relied almost entirely on concentration of measure ideas, either on the sphere (see [15,22]), or on the orthogonal group (see [23]). The quantitative estimates that these techniques have yielded so far are sub-optimal. Inequality (3) was previously known to hold with the bound C/n^κ in place of C/n , where the exponent κ is slightly smaller than $1/5$, see [22,23]. The latter result is applicable for all isotropically-normalized random vectors with a log-concave density.

In this article we suggest a different approach. Rather than employing concentration of measure inequalities, our proof of the optimal inequality (3) is based on analysis of the Neumann Laplacian on convex domains, the so-called L^2 -method in convexity, going back to Hörmander [18] and to Helffer and Sjöstrand [17]. The argument is further simplified by using the theory of optimal transportation of measures. We expect this technique to be useful also in the study of other problems in convex geometry, such as central limit theorems for convex bodies with various types of symmetries. The argument leading to the thin shell estimate occupies Sects. 2, 3 and 5. In Sect. 6 we apply this estimate and complete the proof of Theorem 1.

Readers who are interested only in the proof of inequality (3) and Theorem 1 may skip Sect. 4. This section is devoted to several results, that were obtained as by-products, regarding the first non-zero eigenvalue and the corresponding eigenfunctions of the Neumann Laplacian on n -dimensional convex bodies. In particular, we

show that the eigenfunctions are all “biased” towards some direction in space. This rules out, for instance, the possibility of an even eigenfunction.

As the reader has probably figured out by now, we denote expectation by \mathbb{E} and probability by \mathbb{P} . We write Var for variance, and $Vol_n(A)$ for the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. The scalar product of $u, v \in \mathbb{R}^n$ is denoted by $u \cdot v$. The letters c, C, C', \tilde{c} etc. stand for various positive universal constants, whose value may change from one line to the next.

2 Convexity and the Neumann Laplacian

In this section we analyze some convexity related properties of the Neumann Laplacian, most of which are standard. A convex body in \mathbb{R}^n is a compact, convex set with a non-empty interior. Let $K \subset \mathbb{R}^n$ be a convex body with a C^∞ -smooth boundary, to be fixed throughout this section. We say that a function $\varphi : K \rightarrow \mathbb{R}$ belongs to $C^\infty(K)$ if all of its derivatives of all orders exist and are bounded in the interior of K . When φ is a $C^\infty(K)$ -smooth function, the boundary values of φ and its derivatives are well defined, and are C^∞ -smooth on the boundary ∂K . For $u \in C^\infty(K)$ define

$$\|u\|_{H^{-1}(K)} = \sup \left\{ \int_K \varphi u; \varphi \in C^\infty(K), \int_K |\nabla \varphi|^2 \leq 1 \right\}.$$

Note that necessarily $\|u\|_{H^{-1}(K)} = \infty$ when $\int_K u \neq 0$. For a function f in n variables and for $i = 1, \dots, n$ we write $\partial^i f$ for the derivative of f with respect to the i^{th} coordinate. When $f : K \rightarrow \mathbb{R}$ is a square-integrable function, set

$$Var_K(f) = \int_K (f(x) - E)^2 dx$$

with $E = Vol_n(K)^{-1} \int_K f$. The main result of this section reads as follows:

Lemma 1 *Let $K \subset \mathbb{R}^n$ be a convex body with a C^∞ -smooth boundary. Let $f : K \rightarrow \mathbb{R}$ be a $C^\infty(K)$ -smooth function. Then,*

$$Var_K(f) \leq \sum_{i=1}^n \|\partial^i f\|_{H^{-1}(K)}^2. \quad (4)$$

One may verify that the right-hand side of (4) does not depend on the choice of orthogonal coordinates in \mathbb{R}^n . See [13] for an analog of Lemma 1 for non-convex domains. Let $\rho : K \rightarrow \mathbb{R}$ be a convex function which is C^∞ -smooth with bounded derivatives of all orders in a neighborhood of ∂K , such that

$$\rho(x) = 0, \quad |\nabla \rho(x)| = 1 \quad \text{for } x \in \partial K$$

and $\rho(x) \leq 0$ for $x \in K$. For instance, we may select $\rho(x) = -d(x, \partial K) = -\inf_{y \in \partial K} |x - y|$. Note that for any $x \in \partial K$, the vector $\nabla \rho(x)$ is the outer unit normal to ∂K at x .

Denote by \mathcal{D} the space of all $C^\infty(K)$ -smooth functions $u : K \rightarrow \mathbb{R}$ that satisfy the following Neumann boundary condition:

$$\nabla u(x) \cdot \nabla \rho(x) = 0 \quad \text{for } x \in \partial K.$$

The following lemma is a standard Bochner-Weitzenböck type integration by parts formula, going back at least to Lichnerowicz [25], to Hörmander [18] and to Kadlec [21]. We write $\nabla^2 u$ for the hessian matrix of the function u .

Lemma 2 *Let $u \in \mathcal{D}$ and denote $f = -\Delta u$. Then,*

$$\int_K f^2 = \int_K \sum_{i=1}^n |\nabla \partial^i u|^2 + \int_{\partial K} \nabla^2 \rho(\nabla u) \cdot \nabla u. \tag{5}$$

Proof The function $x \mapsto \nabla u(x) \cdot \nabla \rho(x)$ vanishes on ∂K . Since ∇u is tangential to ∂K , the derivative of the function $x \mapsto \nabla u(x) \cdot \nabla \rho(x)$ in the direction of ∇u vanishes on ∂K . That is,

$$\nabla u(x) \cdot \nabla (\nabla u(x) \cdot \nabla \rho(x)) = 0 \quad \text{for } x \in \partial K.$$

Equivalently,

$$(\nabla^2 u)(\nabla \rho) \cdot \nabla u + (\nabla^2 \rho)(\nabla u) \cdot \nabla u = 0 \text{ on } \partial K. \tag{6}$$

By Stokes theorem,

$$\int_K f^2 = \int_K (\Delta u)^2 = - \int_K \nabla(\Delta u) \cdot \nabla u + \int_{\partial K} (\Delta u \nabla u) \cdot \nabla \rho. \tag{7}$$

The boundary term vanishes, since $\nabla u \cdot \nabla \rho = 0$ on ∂K . We conclude from (7) and from an additional application of Stokes theorem that

$$\int_K f^2 = - \sum_{i=1}^n \int_K \partial^i u \Delta(\partial^i u) = \sum_{i=1}^n \int_K |\nabla \partial^i u|^2 - \int_{\partial K} \sum_{i=1}^n (\partial^i u \nabla \partial^i u) \cdot \nabla \rho.$$

Note that the integrand in the integral over ∂K is exactly $\nabla^2 u(\nabla \rho) \cdot \nabla u$. Hence, from (6),

$$\int_K f^2 = \sum_{i=1}^n \int_K |\nabla \partial^i u|^2 + \int_{\partial K} \nabla^2 \rho(\nabla u) \cdot \nabla u,$$

and the lemma is proven. □

The convexity of K will be used next. Recall that ρ is a convex function, and hence its hessian $\nabla^2\rho(x)$ is a positive semi-definite matrix for any $x \in \partial K$. Therefore, Lemma 2 implies that for any $u \in \mathcal{D}$,

$$\sum_{i=1}^n \int_K |\nabla \partial^i u|^2 \leq \int_K f^2 \quad (8)$$

where $f = \Delta u$. Lemma 1 will be proven by dualizing inequality (8), in a way which is very much related to the approach taken by Hörmander [18] and by Helffer and Sjöstrand [17].

Proof of Lemma 1 We are given $f \in C^\infty(K)$ and we would like to prove (4). We may assume that $\int_K f = 0$ (otherwise, subtract $\frac{1}{\text{Vol}_n(K)} \int_K f$ from the function f).

Since $f \in C^\infty(K)$ and $\int_K f = 0$, there exists $u \in \mathcal{D}$ with

$$-\Delta u = f.$$

The existence of such $u \in \mathcal{D}$ is a consequence of the classical existence and regularity theory of the Neumann problem for the Laplacian on domains with a C^∞ -smooth boundary (see, e.g., Folland's book [16, Chap. 7]). Stokes theorem yields

$$\int_K f^2 = - \int_K f \Delta u = \int_K \nabla f \cdot \nabla u - \int_{\partial K} f \nabla u \cdot \nabla \rho = \sum_{i=1}^n \int_K \partial^i f \partial^i u,$$

where the boundary term vanishes since $u \in \mathcal{D}$. From the definition of the $H^{-1}(K)$ -norm and the Cauchy-Schwartz inequality,

$$\begin{aligned} \int_K f^2 &= \sum_{i=1}^n \int_K \partial^i f \partial^i u \leq \sum_{i=1}^n \|\partial^i f\|_{H^{-1}(K)} \sqrt{\int_K |\nabla \partial^i u|^2} \\ &\leq \sqrt{\sum_{i=1}^n \|\partial^i f\|_{H^{-1}(K)}^2} \cdot \sqrt{\sum_{i=1}^n \int_K |\nabla \partial^i u|^2}. \end{aligned} \quad (9)$$

Combine (9) and (8) to conclude that

$$\int_K f^2 \leq \sum_{i=1}^n \|\partial^i f\|_{H^{-1}(K)}^2.$$

□

3 Transportation of measure

Suppose μ_1 and μ_2 are finite Borel measures on \mathbb{R}^m and \mathbb{R}^n respectively, and $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a measurable map. We say that T pushes forward, or transports, μ_1 to μ_2 if

$$\mu_1(T^{-1}(A)) = \mu_2(A)$$

for all Borel sets $A \subseteq \mathbb{R}^n$. In this case we write $\mu_2 = T\#\mu_1$, and we call T the transportation map. Note that $\int(\varphi \circ T)d\mu_1 = \int \varphi d(T\#\mu_1)$ for any bounded, measurable function φ .

For example, let γ be a Borel measure on $\mathbb{R}^n \times \mathbb{R}^n$. For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we write $P^1(x, y) = x$ and $P^2(x, y) = y$. A measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ with $P^1\#\gamma = \mu_1$ and $P^2\#\gamma = \mu_2$ is called a ‘‘coupling’’ of μ_1 and μ_2 .

Suppose μ_1 and μ_2 are two finite Borel measures on \mathbb{R}^n . If T pushes forward μ_1 to μ_2 , then the map

$$x \mapsto (x, Tx)$$

transports the measure μ_1 to a measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ which is a coupling of μ_1 and μ_2 . The L^2 -Wasserstein distance between μ_1, μ_2 is defined as

$$W_2(\mu_1, \mu_2) = \inf_{\gamma} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 d\gamma(x, y) \right)^{1/2},$$

where the infimum runs over all couplings γ of μ_1 and μ_2 . If there is no coupling, then $W_2(\mu_1, \mu_2) = \infty$. Let μ be a finite, compactly supported Borel measure on \mathbb{R}^n . For a C^∞ -smooth function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, set

$$\|u\|_{H^{-1}(\mu)} = \sup \left\{ \int_{\mathbb{R}^n} u\varphi d\mu ; \varphi \in C^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} |\nabla\varphi|^2 d\mu \leq 1 \right\}.$$

This definition fits with the one given in Sect. 2; We have $\|u\|_{H^{-1}(\lambda_K)} = \|u\|_{H^{-1}(K)}$ where λ_K denotes the restriction of the Lebesgue measure to K .

The next theorem is an extension of a remark by Yann Brenier [9] that we learned from Robert McCann. For the convenience of the reader, we provide in the appendix a detailed exposition of the elegant proof from Villani [40, Sect. 7.6].

Theorem 2 *Let μ be a finite, compactly supported Borel measure on \mathbb{R}^n . Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded, measurable function with*

$$\int hd\mu = 0.$$

For a sufficiently small $\varepsilon > 0$, let μ_ε be the measure whose density with respect to μ is the non-negative function $1 + \varepsilon h$. Then,

$$\|h\|_{H^{-1}(\mu)} \leq \liminf_{\varepsilon \rightarrow 0^+} \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon}.$$

See [9] and [40] for the intuition behind Theorem 2. We write e_1, \dots, e_n for the standard orthonormal basis in \mathbb{R}^n . Let $K \subset \mathbb{R}^n$ be a convex body. Fix a point $x \in K$ and $i = 1, \dots, n$. Consider the line $x + \mathbb{R}e_i$, that is, the line in the direction of e_i that passes through x . This line meets K with a closed segment (or a single point). The two endpoints of this segment in \mathbb{R}^n will be denoted by $\mathcal{B}_i^-(x)$ and $\mathcal{B}_i^+(x)$, where $\mathcal{B}_i^-(x) \cdot e_i \leq \mathcal{B}_i^+(x) \cdot e_i$. Thus,

$$K \cap (x + \mathbb{R}e_i) = [\mathcal{B}_i^-(x), \mathcal{B}_i^+(x)],$$

the line segment from $\mathcal{B}_i^-(x)$ to $\mathcal{B}_i^+(x)$. See Fig. 1.

For $i = 1, \dots, n$ consider the projection

$$\pi_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

defined for $(x_1, \dots, x_n) \in \mathbb{R}^n$. Then $\pi_i(K)$ is a convex body in \mathbb{R}^{n-1} . For $y \in \pi_i(K)$, we define $q_i^-(y) \in \mathbb{R}$ to be the minimal i^{th} coordinate among all points $x \in K$ with $\pi_i(x) = y$. Similarly, we define $q_i^+(y)$ to be the maximal i^{th} coordinate.

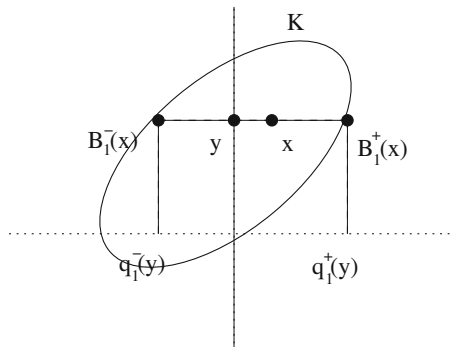


Fig. 1

Lemma 3 Let $K \subset \mathbb{R}^n$ be a convex body with a C^∞ -smooth boundary. Fix $i = 1, \dots, n$. Let $\Psi : K \rightarrow \mathbb{R}$ be a $C^\infty(K)$ -smooth function such that for any $x \in K$,

$$\Psi(\mathcal{B}_i^-(x)) = \Psi(\mathcal{B}_i^+(x)). \quad (10)$$

For a sufficiently small $\varepsilon > 0$ denote by μ_ε the measure whose density with respect to μ is $1 + \varepsilon \partial^i \Psi$. Then,

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon} \leq \sqrt{\int_K [\Psi(x) - \Psi(\mathcal{B}_i^+(x))]^2 dx}.$$

Proof Without loss of generality, assume that $i = 1$. For a sufficiently small $\varepsilon > 0$, the function $1 + \varepsilon \partial^1 \Psi$ is positive on K , and hence μ_ε is a non-negative measure. Fix such a sufficiently small $\varepsilon > 0$.

For $x = (t, x_2, \dots, x_n) \in \mathbb{R}^n$ we will use the coordinates $x = (t, y)$ where $y = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. Fix $y \in \pi_1(K)$ and denote $p = q_1^-(y)$ and $q = q_1^+(y)$. According to our assumption (10),

$$\int_p^q (1 + \varepsilon \partial^1 \Psi(t, y)) dt = (q - p) + \varepsilon \Psi(t, y)|_{t=p}^q = q - p.$$

Consequently, the densities $t \mapsto 1$ and $t \mapsto 1 + \varepsilon \partial^1 \Psi(t, y)$ have an equal amount of mass on the interval $[p, q]$. We consider the monotone transportation between these two densities. That is, we define a map $T = T^y : [p, q] \rightarrow [p, q]$ by requiring that for any $x_1 \in [p, q]$,

$$\int_p^{x_1} (1 + \varepsilon \partial^1 \Psi(t, y)) dt = \int_p^{T(x_1)} dt. \tag{11}$$

The unique map $T : [p, q] \rightarrow [p, q]$ that satisfies (11) transports the measure whose density is $1 + \varepsilon \partial^1 \Psi(t, y)$ on $[p, q]$ to the Lebesgue measure on $[p, q]$. We deduce from (11) that for $x_1 \in [p, q]$,

$$T(x_1) = x_1 + \varepsilon [\Psi(x_1, y) - \Psi(p, y)].$$

Therefore,

$$\int_p^q |T(t) - t|^2 \cdot (1 + \varepsilon \partial^1 \Psi(t, y)) dt = \varepsilon^2 \int_p^q [\Psi(t, y) - \Psi(p, y)]^2 dt + \varepsilon^3 R, \tag{12}$$

with $|R|$ bounded by a constant depending only on Ψ and K (and in particular, independent of ε or y). We now let $y \in \pi_1(K)$ vary, and we write

$$S(x_1, y) = (T^y(x_1), y) \quad \text{for } (x_1, y) \in K.$$

Note that S is well-defined (since x_1 belongs to the domain of definition of T^y when $(x_1, y) \in K$), one-to-one, continuous, and maps K onto K . Moreover, by Fubini, for

any continuous function $\varphi : K \rightarrow \mathbb{R}$,

$$\begin{aligned} \int_K \varphi(S(x)) d\mu_\varepsilon(x) &= \int_{\pi(K)} \left[\int_{q_1^-(y)}^{q_1^+(y)} \varphi(T^y(x_1), y) \cdot (1 + \varepsilon \partial^1 \Psi) dx_1 \right] dy \\ &= \int_{\pi(K)} \left[\int_{q_1^-(y)}^{q_1^+(y)} \varphi(x_1, y) dx_1 \right] dy = \int_K \varphi(x) d\mu(x). \end{aligned}$$

Therefore the map S transports μ_ε to μ . According to (12),

$$W_2(\mu, \mu_\varepsilon)^2 \leq \int_K |S(x) - x|^2 d\mu_\varepsilon(x) = \varepsilon^2 \int_K [\Psi(x) - \Psi(\mathcal{B}_1^-(x))]^2 dx + \varepsilon^3 R',$$

with $|R'|$ smaller than a constant depending only on K and Ψ , and in particular independent of ε . To complete the proof, let ε tend to zero. \square

4 A digression: Neumann eigenvalues and eigenfunctions

This section presents some additional relations between convexity and the Neumann Laplacian. We retain the setup and notation of Sect. 2. We write $L^2(K)$ for the Hilbert space that is the completion of $C^\infty(K)$ with respect to the norm

$$\|u\|_{L^2(K)} = \sqrt{\int_K u^2}.$$

The operator $-\Delta$, acting on the subspace $\mathcal{D} \subset L^2(K)$, is a symmetric, positive semi-definite operator. The classical theory implies that $-\Delta$ has a complete system of orthonormal Neumann eigenfunctions $\varphi_0, \varphi_1, \dots \in \mathcal{D}$ and Neumann eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \dots$ (see, e.g., [16, Chap. 7]). The first eigenvalue is $\lambda_0 = 0$, with the eigenfunction φ_0 being constant. It is well-known that $\lambda_1 > 0$ when K is convex (see, e.g., [34]). It is actually enough to assume that K is connected, see e.g., [11, Theorem 1]). We refer to λ_1 as the first non-zero Neumann eigenvalue of K . It is well-known that for any $C^\infty(K)$ -smooth function u with $\int_K u = 0$,

$$\lambda_1 \int_K u^2 \leq \int_K |\nabla u|^2. \quad (13)$$

Equality in (13) holds if and only if u is an eigenfunction corresponding to the eigenvalue λ_1 .

We say that the boundary of K is uniformly strictly convex if $\nabla^2\rho(x)$ is a positive definite matrix for any $x \in \partial K$. Equivalently, ∂K is uniformly strictly convex if the principal curvatures are all positive—and not merely non-negative—everywhere on the boundary. Our next corollary claims, loosely speaking, that any non-trivial eigenfunction corresponding to λ_1 cannot be “spatially isotropic”, but must have “preference” for a certain direction in space.

Corollary 1 *Suppose $K \subset \mathbb{R}^n$ is a convex body whose boundary is C^∞ -smooth and uniformly strictly convex. Let $0 \neq \varphi \in \mathcal{D}$ be an eigenfunction corresponding to the first non-zero Neumann eigenvalue. Then,*

$$\int_K \nabla\varphi \neq 0. \tag{14}$$

Consequently, the multiplicity of the first non-zero Neumann eigenvalue is at most n .

Proof: Assume the opposite. Then,

$$\int_K \partial^i\varphi = 0 \text{ for } i = 1, \dots, n. \tag{15}$$

We write λ_1 for the first non-zero eigenvalue, i.e., $\Delta\varphi = -\lambda_1\varphi$. Since $\varphi \in \mathcal{D}$, inequality (8) gives

$$\lambda_1^2 \int_K \varphi^2 = \int_K |\Delta\varphi|^2 \geq \sum_{i=1}^n \int_K |\nabla\partial^i\varphi|^2. \tag{16}$$

From (15) we know that $\int_K \partial^i\varphi = 0$ for all i . Thus (16) and (13) yield

$$\lambda_1^2 \int_K \varphi^2 \geq \sum_{i=1}^n \int_K |\nabla\partial^i\varphi|^2 \geq \lambda_1 \sum_{i=1}^n \int_K (\partial^i\varphi)^2 = \lambda_1 \int_K |\nabla\varphi|^2 = \lambda_1^2 \int_K \varphi^2.$$

Therefore, there must be equality in all steps and hence $\partial^1\varphi, \dots, \partial^n\varphi$ are all Neumann eigenfunctions with eigenvalue λ_1 . We necessarily have equality also in (16). According to Lemma 2 this means that

$$\int_{\partial K} \nabla^2\rho(\nabla\varphi) \cdot \nabla\varphi = 0.$$

Since the integrand is non-negative and continuous, necessarily

$$\nabla^2\rho(\nabla\varphi) \cdot \nabla\varphi = 0 \text{ on } \partial K. \tag{17}$$

So far we have only used the convexity of K . The uniform strict convexity of ∂K means that $\nabla^2 \rho > 0$ on ∂K . Equation (17) has the consequence that $\nabla \varphi = 0$ on ∂K , and therefore

$$\varphi \equiv \text{Const} \quad \text{on } \partial K. \tag{18}$$

This is well-known to be impossible for a Neumann eigenfunction corresponding to the first non-zero eigenvalue. We sketch the standard argument, see, e.g., [11] for more information. Denote

$$N = \{x \in K; \varphi(x) > 0\}.$$

The set N is non-empty since $\int_K \varphi = 0$. Moreover, φ vanishes on ∂N because of (18). Since $\Delta \varphi = -\lambda_1 \varphi$ in N , then φ is a Dirichlet eigenfunction of the domain N corresponding to the Dirichlet eigenvalue λ_1 . For a domain $\Omega \subset \mathbb{R}^n$, denote by $\lambda_0^D(\Omega)$ the minimal eigenvalue of $-\Delta$ with Dirichlet boundary conditions on Ω . Then $\lambda_0^D(N) \leq \lambda_1$, as is witnessed by φ . Furthermore, $\lambda_0^D(N) \geq \lambda_0^D(K)$ by domain monotonicity (see, e.g., [11]), hence $\lambda_0^D(K) \leq \lambda_1$. However, a standard comparison result between the Neumann and the Dirichlet eigenvalues is the strict inequality $\lambda_0^D(K) > \lambda_1$ (see, e.g., [24] for a much more accurate result). We thus arrive at a contradiction. Consequently our assumption that $\int_K \nabla \varphi = 0$ was absurd. The proof of (14) is complete.

The linear map $\varphi \mapsto \int_K \nabla \varphi$ from the eigenspace of λ_1 to \mathbb{R}^n is therefore injective, so the multiplicity of the eigenvalue cannot exceed n . □

Remark Leonid Friedlander explained to us how to eliminate the uniform strict convexity requirement from Corollary 1. His idea is to observe that since $\partial^1 \varphi, \dots, \partial^n \varphi$ are all eigenfunctions, then the restriction of φ to the boundary ∂K is actually an eigenfunction of the Laplacian associated with the Riemannian manifold ∂K . However, (17) entails that φ is constant in some open set in ∂K , which is known to be impossible for an eigenfunction. We omit the details.

For $i = 1, \dots, n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ write

$$\sigma_i(x) = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n),$$

i.e., we flip the sign of the i th coordinate. For a function f , we write $\sigma_i(f)(x) = f(\sigma_i(x))$. Our next corollary exploits the well-known relationship between the eigenfunctions and symmetry. Similar arguments appear, e.g., in [2].

Corollary 2 *Suppose $K \subset \mathbb{R}^n$ is a convex body with a C^∞ -smooth boundary. Denote by $E_{\lambda_1} \subset \mathcal{D}$ the eigenspace corresponding to the first non-zero Neumann eigenvalue of K .*

- (i) *If K is unconditional, then there exist $i = 1, \dots, n$ and an eigenfunction $0 \neq \varphi \in E_{\lambda_1}$, such that*

$$\sigma_i(\varphi) = -\varphi.$$

- (ii) If K is centrally-symmetric (i.e., $K = -K$), then there exists an eigenfunction $0 \neq \varphi \in E_{\lambda_1}$, such that

$$\varphi(-x) = -\varphi(x) \text{ for } x \in K.$$

Proof Begin with the proof of (i). We are given the unconditional convex body K . Since K is unconditional, then $f \in E_{\lambda_1}$ implies $\sigma_i(f) \in E_{\lambda_1}$ for $i = 1, \dots, n$. Start with any non-zero eigenfunction $f_0 \in E_{\lambda_1}$, and recursively define

$$f_i = f_{i-1} + \sigma_i(f_{i-1}).$$

Then $f_0, f_1, \dots, f_n \in E_{\lambda_1}$. If there exists $i = 1, \dots, n$ such that $f_i \equiv 0$ then we are done: Suppose i is the minimal such index. Then $0 \neq f_{i-1} \in E_{\lambda_1}$ with $\sigma_{i-1}(f_{i-1}) = -f_{i-1}$, and we found our desired eigenfunction.

It remains to deal with the case where $\psi = f_n$ is a non-zero eigenfunction. Note that $\sigma_i(\psi) = \psi$ and hence

$$\sigma_i(\partial^i \psi) = -\partial^i \psi \tag{19}$$

for $i = 1, \dots, n$. Therefore,

$$\int_K \nabla \psi = 0. \tag{20}$$

In the proof of Corollary 1 (the first part, which did not use the uniform strict convexity) we observed that (20) implies that $\partial^1 \psi, \dots, \partial^n \psi \in E_{\lambda_1}$. Since $\int_K |\nabla \psi|^2 > 0$, there exists $i = 1, \dots, n$ with $\partial^i \psi \neq 0$. We see from (19) that $\partial^i \psi \in E_{\lambda_1}$ is the eigenfunction we are looking for. This completes the proof of the first part of the lemma.

The proof of the second part is similar. Begin with any $0 \neq f \in E_{\lambda_1}$ and set $\psi(x) = f(x) + f(-x)$. If $\psi \equiv 0$, then f is an odd function and we are done. Otherwise, ψ is an even function, hence $\int_K \nabla \psi = 0$. As before, this implies that $\partial^1 \psi, \dots, \partial^n \psi$ are all odd eigenfunctions corresponding to the same eigenvalue λ_1 . \square

Corollary 1 and 2 seem very much expected. Notably, Nadirashvili [29] has proved that in two dimensions, the multiplicity of the first non-zero Neumann eigenvalue is at most 2 for any simply-connected domain. Our simple proof of Corollary 1 is not applicable in such generality. Corollary 1 is related to the ‘‘hot spots’’ problem, see, e.g., Burdzy [10], Jerison and Nadirashvili [19] and references therein. A proof of Corollary 2 for the two-dimensional case—under much more general assumptions than convexity—can be found in [2, Theorem 4.3]. However, the proofs of the two-dimensional results mentioned do not seem to admit easy generalization to higher dimensions. As observed by Payne and Weinberger [33], Corollary 2 leads to the following comparison principle:

Corollary 3 *Let $K \subset \mathbb{R}^n$ be an unconditional convex body with a C^∞ -smooth boundary. Assume that $R > 0$ is such that*

$$K \subseteq [-R, R]^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n; |x_i| \leq R \text{ for } i = 1, \dots, n\}.$$

Denote by $\lambda_1 > 0$ the first non-zero Neumann eigenvalue of K . Then,

$$\lambda_1 \geq \frac{\pi^2}{R^2}.$$

Equality holds when $K = [-R, R]^n$, an n -dimensional cube.

Proof A well-known, elementary calculation shows that for any $0 < r \leq R$ and a smooth odd function $\psi : [-r, r] \rightarrow \mathbb{R}$,

$$\frac{\pi^2}{R^2} \int_{-r}^r \psi^2(x) dx \leq \frac{\pi^2}{r^2} \int_{-r}^r \psi^2(x) dx \leq \int_{-r}^r \left(\frac{d\psi}{dx}\right)^2 dx. \tag{21}$$

According to Corollary 2(i), there exists an index $1 \leq i \leq n$ and a non-zero eigenfunction φ corresponding to λ_1 such that $\sigma_i(\varphi) = -\varphi$. By Fubini’s theorem and (21),

$$\frac{\pi^2}{R^2} \int_K \varphi^2 \leq \int_K |\partial^i \varphi|^2 \leq \int_K |\nabla \varphi|^2 = \lambda_1 \int_K \varphi^2,$$

hence $\lambda_1 \geq \pi^2/R^2$. □

Remark 1. Corollary 3 shows that the cube satisfies a certain domain monotonicity principle for the Neumann Laplacian, in the category of unconditional, convex bodies. The Euclidean ball, for instance, does not satisfy a corresponding principle.

2. Suppose $K \subset \mathbb{R}^n$ is an unconditional convex body. Assume that K is isotropically normalized, i.e., the random vector X which is distributed uniformly in K is isotropically normalized. Corollary 3 implies the probably non-optimal bound

$$\lambda_1(K) \geq c/\log^2(n + 1), \tag{22}$$

where $\lambda_1(K) > 0$ is the first non-zero Neumann eigenvalue of K , and $c > 0$ is a universal constant. To establish (22), consider

$$K' = K \cap [-R, R]^n, \text{ for } R = 50 \log(n + 1).$$

Use Corollary 3 to deduce the bound $\lambda_1(K') > c/\log^2(n + 1)$. The body K' is a good approximation to the body K : It is easily proven that

$$Vol(K') \geq \left(1 - \frac{1}{n}\right) Vol_n(K).$$

We may thus apply E. Milman’s result [27, Theorem 1.7], which builds upon the Sternberg–Zumbrun concavity principle [38], to conclude that $\lambda_1(K) \geq c\lambda_1(K')$ and the bound (22) follows. See [20] for a conjectural better bound, without the logarithmic factor.

5 Unconditional convex bodies

We begin this section with a corollary to the theorems of Sect. 2 and Sect. 3.

Corollary 4 *Let $K \subset \mathbb{R}^n$ be an unconditional convex body.*

(i) *Let $\Psi : K \rightarrow \mathbb{R}$ be an unconditional, continuous function. Then,*

$$\text{Var}_K(\Psi) \leq \sum_{i=1}^n \int_K (\Psi(x) - \Psi(\mathcal{B}_i^+(x)))^2 dx.$$

(ii) *In particular, suppose $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$ are even, continuous functions. Denote $\Psi(x_1, \dots, x_n) = \sum_{i=1}^n f_i(x_i)$. Then,*

$$\text{Var}_K(\Psi) \leq \sum_{i=1}^n \int_K \sup_{s,t \in J_i(x)} (f_i(s) - f_i(t))^2 dx,$$

where $J_i(x) = [q_i^-(\pi_i(x)), q_i^+(\pi_i(x))] \subset \mathbb{R}$. That is, $J_i(x)$ is a symmetric interval about the origin with the same length as $[\mathcal{B}_i^-(x), \mathcal{B}_i^+(x)]$.

Proof Begin with (i). By approximation, we may assume that K has a C^∞ -smooth boundary, and that Ψ is a $C^\infty(K)$ -smooth function. Lemma 1 states that

$$\text{Var}_K(\Psi) \leq \sum_{i=1}^n \|\partial^i \Psi\|_{H^{-1}(K)}^2.$$

Fix $i = 1, \dots, n$. We may apply Theorem 2 for $h = \partial^i \Psi$ since $\int_K \partial^i \Psi = 0$, as implied by the symmetries of Ψ . We may apply Lemma 3, since clearly $\Psi(\mathcal{B}_i^+(x)) = \Psi(\mathcal{B}_i^-(x))$ for any $x \in K$. Theorem 2 and Lemma 3 entail the inequality

$$\|\partial^i \Psi\|_{H^{-1}(K)}^2 \leq \int_K (\Psi(x) - \Psi(\mathcal{B}_i^+(x)))^2 dx.$$

This proves (i). To deduce (ii), denote $\Psi_i(x_1, \dots, x_n) = f_i(x_i)$. Observe that $\Psi(x) = \sum_{i=1}^n \Psi_i(x)$ is unconditional and that for any $x \in K, i = 1, \dots, n$,

$$|\Psi(x) - \Psi(\mathcal{B}_i^+(x))| = |\Psi_i(x) - \Psi_i(\mathcal{B}_i^+(x))| \leq \sup_{s,t \in J_i(x)} |f_i(s) - f_i(t)|.$$

Thus (ii) follows from (i). □

We will use the following simple identities:

$$\int_{-r}^r (a|t|^p - ar^p)^2 dt = \frac{2p^2}{p+1} \int_{-r}^r (a|t|^p)^2 dt, \quad (23)$$

$$\int_{-r}^r (2ar^p)^2 dt = 8a^2r^{2p+1} = 4(2p+1) \int_{-r}^r (a|t|^p)^2 dt, \quad (24)$$

valid for all $a, p, r \geq 0$.

Lemma 4 *Let $X = (X_1, \dots, X_n)$ be a random vector in \mathbb{R}^n , that is distributed according to an unconditional, log-concave density. Let $p_1, \dots, p_n > 0$ and let $a_1, \dots, a_n \geq 0$. Then,*

- (i) $\text{Var} \left(\sum_{i=1}^n a_i |X_i|^{p_i} \right) \leq \sum_{i=1}^n \frac{2p_i^2}{p_i+1} a_i^2 \mathbb{E}|X_i|^{2p_i}$.
- (ii) *Furthermore, suppose $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$ are even, measurable functions with $|f_i(t)| \leq a_i |t|^{p_i}$ for all $t \in \mathbb{R}$, $i = 1, \dots, n$. Then,*

$$\text{Var} \left(\sum_{i=1}^n f_i(X_i) \right) \leq 4 \sum_{i=1}^n (2p_i+1) a_i^2 \mathbb{E}|X_i|^{2p_i}.$$

Proof Suppose first that X is distributed uniformly in an unconditional convex body $K \subset \mathbb{R}^n$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, denote

$$\Psi(x_1, \dots, x_n) = \sum_{i=1}^n a_i |x_i|^{p_i}.$$

The desired bound (i) is equivalent to

$$\text{Var}_K(\Psi) \leq \sum_{i=1}^n \frac{2p_i^2}{p_i+1} \int_K a_i^2 |x_i|^{2p_i} dx_1 \dots dx_n.$$

According to Corollary 4(i), it suffices to prove that for any $i = 1, \dots, n$,

$$\int_K (\Psi(x) - \Psi(\mathcal{B}_i^+(x)))^2 dx = \frac{2p_i^2}{p_i+1} \int_K a_i^2 |x_i|^{2p_i} dx_1 \dots dx_n. \quad (25)$$

Fix $i = 1, \dots, n$. We will prove (25) by Fubini's theorem. Fix a point

$$x' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \pi_i(K)$$

and denote $r = q_i^+(x') \geq 0$. In order to prove (25), it is enough to show that

$$\int_{-r}^r \left[\sum_{j=1}^n a_j |x_j|^{p_j} - \left(a_i r^{p_i} + \sum_{j \neq i} a_j |x_j|^{p_j} \right) \right]^2 dx_i = \frac{2p_i^2}{p_i + 1} \int_{-r}^r a_i^2 |x_i|^{2p_i} dx_i.$$

The equality we need is exactly the content of (23). The proof of (i) is thus complete, in the case where X is distributed uniformly in a convex body. The proof of (ii) is almost entirely identical. By approximation, we may assume that f_1, \dots, f_n are continuous. According to Corollary 4(ii), it is sufficient to prove that

$$\int_K \sup_{t,s \in J_i(x)} (f_i(s) - f_i(t))^2 dx \leq 4(2p_i + 1) \int_K a_i^2 |x_i|^{2p_i} dx_1 \dots dx_n.$$

This follows by Fubini’s theorem and (24). The lemma is thus proven, in the case where X is distributed uniformly in an unconditional convex body.

The general case follows via a standard argument. Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ stand for the unconditional, log-concave density of X . Next, we suppose that f is s -concave for some integer $s \geq 1$. That is, assume that

$$f^{1/s}(\lambda x + (1 - \lambda)y) \geq \lambda f^{1/s}(x) + (1 - \lambda) f^{1/s}(y)$$

for all $0 < \lambda < 1$ and $x, y \in \mathbb{R}^n$ for which $f(x), f(y) > 0$. Denote $N = n + s$. For $z \in \mathbb{R}^N$ we use the coordinates $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^s$. Let $K \subset \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^s$ be the unconditional convex body defined by

$$K = \left\{ (x, y) ; x \in \mathbb{R}^n, y \in \mathbb{R}^s, |y| \leq \kappa_s^{-1/s} f^{1/s}(x) \right\},$$

where $\kappa_s = \pi^{s/2} / \Gamma(s/2 + 1)$ is the volume of the s -dimensional Euclidean unit ball. Suppose that $Z = (Z_1, \dots, Z_N)$ is a random vector that is distributed uniformly in K . According to the case already considered, conclusions (i) and (ii) hold when the X_1, \dots, X_n are replaced by Z_1, \dots, Z_n . However, the random vector (Z_1, \dots, Z_n) has the same distribution as $X = (X_1, \dots, X_n)$. Thus (i) and (ii) hold also in the case where the density f is s -concave.

Finally, an approximation argument eliminates the requirement that the density of f be s -concave: Write $f = e^{-\psi}$ for the unconditional, log-concave density of X . Then, for any $s > 0$, the function

$$x \mapsto \left(1 - \frac{\psi(x)}{s} \right)_+^s$$

is unconditional and s -concave, where $x_+ = \max\{x, 0\}$. This density clearly tends to $e^{-\psi}$ weakly (and also uniformly in \mathbb{R}^n) when $s \rightarrow \infty$. We thus deduce the general case as a limit of the s -concave case. □

Lemma 4 may be viewed as a substitute for the sub-independent coordinates idea of Anttila, Ball and Perissinaki [1]: Note the absence of cross terms from the right-hand side of Lemma 4(i). Suppose X is a real-valued random variable with an even, log-concave density. A classical inequality (see, e.g., [28], or [3, Theorem 12] and references therein) states that for any $p \geq 2$,

$$\left(\frac{\mathbb{E}|X|^p}{\Gamma(p+1)}\right)^{1/p} \leq \sqrt{\frac{\mathbb{E}|X|^2}{2}} \leq \mathbb{E}|X|, \tag{26}$$

where $\Gamma(p+1) = \int_0^\infty t^p e^{-t} dt$. For a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and for $p \geq 1$ we write

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

The following corollary contains a few obvious consequences of Lemma 4.

Corollary 5 *Let $X = (X_1, \dots, X_n)$ be a random vector in \mathbb{R}^n , with $\mathbb{E}X_i^2 = 1$ for $i = 1, \dots, n$, that is distributed according to an unconditional, log-concave density. Let $a_1, \dots, a_n \geq 0$. Then,*

$$\text{Var}\left(\sum_{i=1}^n a_i X_i^2\right) \leq C' \sum_{i=1}^n a_i^2, \tag{i}$$

where $C' \leq 16$ is a universal constant. Consequently,

$$\text{Var}(|X|^2) \leq C^2 n \quad \text{and} \quad \mathbb{E}(|X| - \sqrt{n})^2 \leq C^2, \tag{ii}$$

with $C \leq 4$, a positive universal constant. Moreover, for any $p \geq 1$,

$$\sqrt{\text{Var}(\|X\|_p)} \leq C_p n^{\frac{1}{p} - \frac{1}{2}} \tag{iii}$$

where $C_p > 0$ is a constant depending only on p .

Proof According to the Prékopa–Leindler inequality (see, e.g., the first pages of [35]), the random variable X_i has an even, log-concave density for all i . From Lemma 4(i) and (26) we see that

$$\text{Var}\left(\sum_{i=1}^n a_i X_i^2\right) \leq \frac{8}{3} \sum_{i=1}^n a_i^2 \mathbb{E}|X_i|^4 \leq 16 \sum_{i=1}^n a_i^2 \left(\mathbb{E}|X_i|^2\right)^2 = 16 \sum_{i=1}^n a_i^2.$$

This proves (i). By setting $a_i = 1$ ($i = 1, \dots, n$) in (30), we deduce that

$$\mathbb{E}(|X| - \sqrt{n})^2 \leq \frac{1}{n} \mathbb{E}(|X| - \sqrt{n})^2 \cdot (|X| + \sqrt{n})^2 = \frac{1}{n} \mathbb{E}(|X|^2 - n)^2 \leq 16,$$

and (ii) is proven. Denote $E = \mathbb{E}\|X\|_p^p$. From Lemma 4(i) and (26) we conclude that

$$\mathbb{E} (\|X\|_p^p - E)^2 = \text{Var} \left(\sum_{i=1}^n |X_i|^p \right) \leq 2^{1-p} p \Gamma(2p + 1)n.$$

For any $p \geq 2$, we have $\mathbb{E}|X_i|^p \geq (\mathbb{E}X_i^2)^{p/2} = 1$. For $1 \leq p \leq 2$,

$$\mathbb{E}|X_i|^p \geq (\mathbb{E}|X_i|)^p \geq 2^{-p/2} (\mathbb{E}X_i^2)^{p/2} = 2^{-p/2} \geq 2^{-1/2},$$

according to (26). Hence, $E = \sum_i \mathbb{E}|X_i|^p \geq n/\sqrt{2}$ for all $p \geq 1$ and

$$\text{Var} (\|X\|_p) \leq \mathbb{E} (\|X\|_p - E^{1/p})^2 \leq E^{-2\frac{p-1}{p}} \mathbb{E} (\|X\|_p^p - E)^2 \leq C_p n^{2/p-1},$$

where C_p is a constant depending solely on $p \geq 1$. This completes the proof. □

Schechtman and Zinn [36,37] provided estimates related to Corollary 5 for the case where X is distributed uniformly in the unit ball $\{x \in \mathbb{R}^n; \|x\|_q \leq 1\}$, for $q \geq 1$. More information regarding unconditional, log-concave densities in high dimension, especially in the large deviations scale, is available from Bobkov and Nazarov [6,7]. Under the assumptions of Corollary 5, they showed, for instance, that

$$\mathbb{P} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \geq t \right) \leq C \exp(-ct^2) \quad \forall 0 \leq t \leq \sqrt{n},$$

where $c, C > 0$ are universal constants. Another large-deviations estimate that was proved by Bobkov and Nazarov [6,7] is that

$$\mathbb{P} (|X| \geq t) \leq C \exp(-ct) \quad \text{for } t \geq C\sqrt{n}. \tag{30}$$

Paouris [31,32] was remarkably able to generalize inequality (30) to the class of all isotropically-normalized random vectors with a log-concave density in \mathbb{R}^n . Regarding smaller values of t in (30), the currently known bounds, which are valid for all isotropically-normalized, log-concave random vectors, are of the form

$$\mathbb{P} \left(\left| \frac{|X|}{\sqrt{n}} - 1 \right| \geq t \right) \leq C \exp(-cn^\alpha t^\beta) \quad \text{for } 0 < t < 1, \tag{31}$$

with, say, $\alpha = 0.33$ and $\beta = 3.33$ (see [23]).

Cordero-Erausquin, Fradelizi and Maurey [12] have recently proved the so-called (B)-conjecture in the unconditional case. This entails the following improvement over the Brunn-Minkowski theory:

- The function $t \mapsto \mathbb{P} (|X| \leq e^t)$ is log-concave in $t \in \mathbb{R}$.

(The Prékopa–Leindler inequality leads to the weaker statement in which the e^t is replaced by t). Corollary 5(ii) and Markov–Chebychev’s inequality yield

$$\mathbb{P}(|X| \leq \sqrt{n} - 8) \leq \frac{1}{4}, \quad \mathbb{P}(|X| \leq \sqrt{n} + 8) \geq \frac{3}{4}.$$

The log-concavity of the map $s \mapsto \mathbb{P}(|X| \leq e^s)$ thus implies that for any $t \geq 0$,

$$\mathbb{P}\left(|X| \leq (\sqrt{n} - 8) \cdot \left(\frac{\sqrt{n} - 8}{\sqrt{n} + 8}\right)^t\right) \leq \frac{1}{4 \cdot 3^t}.$$

After some simple manipulations, we deduce the inequality

$$\mathbb{P}(|X| \leq \sqrt{n} - t) \leq C \left(1 - \frac{t}{\sqrt{n}}\right)^{c\sqrt{n}} \leq C \exp(-ct), \tag{32}$$

valid for all $0 \leq t \leq \sqrt{n}$, for some universal constants $c, C > 0$. We currently do not know how to prove a bound as in (32) for the probability $\mathbb{P}(|X| \geq \sqrt{n} + t)$. The weaker estimate

$$\mathbb{P}(|X| \geq \sqrt{n} + t) \leq C \exp(-c\sqrt{t})$$

follows by combining Corollary 5(ii) with the distribution inequalities of Nazarov, Sodin and Volberg [30]. We omit the details.

6 Berry-Esseen type bounds

In previous sections we established sharp *thin shell estimates* for unconditional, log-concave densities. In the present section we complete the proof of Theorem 1. The argument we present is quite technical and is very much related to classical treatments of the central limit theorem for independent random variables. The reader may refer to, e.g., [14, Vol. II, Chap. XVI] for background on the rate of convergence in the classical central limit theorem. We are indebted to Sasha Sodin for many discussions, suggestions and simplifications that have lead to the proofs we present below.

Before proceeding to the actual proof, let us describe the general idea. Introduce independent, symmetric Bernoulli variables $\Delta_1, \dots, \Delta_n$. That is,

$$\mathbb{P}(\Delta_i = 1) = \mathbb{P}(\Delta_i = -1) = 1/2 \quad (i = 1, \dots, n).$$

These Bernoulli variables are also assumed to be independent of X . Write

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \text{ and } \Phi(t) = \int_t^\infty \varphi(s) ds$$

for all $t \in \mathbb{R}$. We condition on X , and apply the classical Berry–Esseen bound to obtain

$$\left| \mathbb{P} \left(\frac{\sum_i \Delta_i X_i}{\sqrt{n}} \geq t \right) - \Phi(t\sqrt{n}/|X|) \right| \leq C \frac{\sum_i |X_i|^3}{(\sum_i |X_i|^2)^{3/2}} \leq \frac{C'}{\sqrt{n}} \tag{33}$$

where the last inequality holds only for “typical” values of X . Since $|X|/\sqrt{n}$ is strongly concentrated around 1, as we learn from (3), we may substitute the $\Phi(t\sqrt{n}/|X|)$ term in (33) by $\Phi(t)$. Observe that since X is unconditional, the random variables

$$\sum_i X_i \quad \text{and} \quad \sum_i X_i \Delta_i$$

have exactly the same distribution. Hence, by considering the expectation over X in (33), we deduce a weaker version of (1) where the C/n is replaced with C/\sqrt{n} . In order to arrive at the optimal bound, we need to apply a smoothing technique: The estimate (33) will be replaced with a much better Berry-Esseen inequality which is available for the random variable $\Gamma + (\sum_i \Delta_i X_i) / \sqrt{n}$, for an appropriate “small” random variable Γ . The details will be described next.

Throughout this section, we fix a symmetric random variable Γ with $\mathbb{E}\Gamma^6 < \infty$, independent of all other random variables, such that the even function $\gamma(\xi) = \mathbb{E} \exp(-i\xi\Gamma)$ satisfies

$$\gamma(\xi) = 0 \quad \text{for } |\xi| \geq 1 \tag{34}$$

and

$$1 - 1000\xi^2 \leq \gamma(\xi) \leq 1 \quad \text{for } \xi \in \mathbb{R}. \tag{35}$$

For instance, Γ may be the random variable whose density is

$$x \mapsto \kappa_1 \sin^8(\kappa_2 x) / x^8,$$

for appropriate universal constants κ_1, κ_2 . (For this specific choice, γ is the 8-fold convolution of the characteristic function of an interval.) We shall use the standard O -notation in this section. The notation $O(x)$, for some expression x , is an abbreviation for some complicated quantity y with the property that

$$|y| \leq Cx$$

for some universal constant $C > 0$. All constants hidden in the O -notation in our proof are in principle explicit. The following lemma seems rather standard (see [14, Vol. II, Chap. XVI] for similar statements). For lack of a precise reference, we provide its proof.

Lemma 5 *Suppose $\Delta_1, \dots, \Delta_n$ are independent, symmetric Bernoulli random variables. Let $0 \neq \theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ and $\sigma > 0$. Assume that*

$$\sum_{i:|\theta_i|>\sigma} \theta_i^2 \leq \frac{1}{2}|\theta|^2. \tag{36}$$

Then, for any $t \in \mathbb{R}$,

$$\left| \mathbb{P}\left(\sigma\Gamma + \sum_{i=1}^n \theta_i \Delta_i \geq t\right) - \Phi\left(\frac{t}{|\theta|}\right) \right| \leq C\left(\frac{\sigma^2}{|\theta|^2} + \sum_{i=1}^n \frac{\theta_i^4}{|\theta|^4}\right), \tag{37}$$

where $C > 0$ is a universal constant.

Remark Note that when $\theta_i = 1/\sqrt{n} = \sigma$ for all i , the error term in Lemma 5 is $O(1/n)$. The addition of Γ/\sqrt{n} allows us to deduce a better bound than the $O(1/\sqrt{n})$ guaranteed by the Berry-Esseen inequality.

Proof of Lemma 5 The validity of both the assumptions and the conclusions of the lemma is not altered if we replace θ, σ with $r\theta, r\sigma$ for any $r > 0$. Normalizing, we may assume that $|\theta| = 1$. By symmetry, it is enough to prove (37) for non-negative t . Fix $t \geq 0$. Observe that for any $\xi \in \mathbb{R}$,

$$\mathbb{E} \exp\left(-i\xi \left[\sigma\Gamma + \sum_{i=1}^n \theta_i \Delta_i\right]\right) = \gamma(\sigma\xi) \prod_{i=1}^n \cos(\theta_i \xi).$$

Thus, from the Fourier inversion formula (see, e.g., [14, Vol. II, Chap. XVI]),

$$\begin{aligned} & \mathbb{P}\left(\sigma\Gamma + \sum_{i=1}^n \theta_i \Delta_i \leq t\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp(-s^2/2) ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\gamma(\sigma\xi) \prod_{i=1}^n \cos(\theta_i \xi) - e^{-\xi^2/2} \right] \frac{e^{it\xi} - 1}{i\xi} d\xi. \end{aligned} \tag{38}$$

Denote $\varepsilon = \sqrt{\sum_i \theta_i^4}$. To prove the lemma, it suffices to bound the absolute value of the integral in (38) by $C'(\varepsilon^2 + \sigma^2)$. We express the integral in (38) as $I_1 + I_2 + I_3$ where I_1 is the integral over $\xi \in [-\varepsilon^{-1/2}, \varepsilon^{-1/2}]$, I_2 is the integral over $\varepsilon^{-1/2} \leq |\xi| \leq \sigma^{-1}$ (when $\varepsilon^{-1/2} > \sigma^{-1}$, we set $I_2 = 0$) and I_3 is the integral over $|\xi| \geq \max\{\sigma^{-1}, \varepsilon^{-1/2}\}$.

Begin with estimating I_1 . We use the elementary inequality

$$e^{s^2/2} \cos s = e^{O(s^4)} \quad \text{for } |s| \leq 1.$$

Since $|\theta_i| \leq \varepsilon^{1/2}$ for all i , then for $|\xi| \leq \varepsilon^{-1/2}$,

$$\left| \prod_{i=1}^n e^{\xi^2 \theta_i^2 / 2} \cos(\theta_i \xi) - 1 \right| = \left| e^{O(\xi^4 \sum_{i=1}^n \theta_i^4)} - 1 \right| \leq C' \xi^4 \varepsilon^2. \tag{39}$$

Combine (39) with (35) to deduce that for $|\xi| \leq \varepsilon^{-1/2}$,

$$\begin{aligned} \gamma(\sigma \xi) \prod_{i=1}^n e^{\xi^2 \theta_i^2 / 2} \cos(\theta_i \xi) &= \left(1 + O(\sigma^2 \xi^2)\right) \left(1 + O(\xi^4 \varepsilon^2)\right) \\ &= 1 + O(\sigma^2 \xi^2 + \xi^4 \varepsilon^2). \end{aligned}$$

The latter estimate yields

$$\begin{aligned} |I_1| &= \left| \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} e^{-\xi^2 / 2} \left[\gamma(\sigma \xi) \prod_{i=1}^n e^{\xi^2 \theta_i^2 / 2} \cos(\theta_i \xi) - 1 \right] \frac{e^{it\xi} - 1}{i\xi} d\xi \right| \\ &\leq C' \int_{-\infty}^{\infty} e^{-\xi^2 / 2} (\sigma^2 \xi^2 + \xi^4 \varepsilon^2) \frac{2}{|\xi|} d\xi \leq \tilde{C} (\sigma^2 + \varepsilon^2). \end{aligned}$$

Next we estimate I_2 , in the case where $\varepsilon^{-1/2} \leq \sigma^{-1}$ (in the complementary case, $I_2 = 0$). Denote $\mathcal{I} = \{1 \leq i \leq n; |\theta_i| \leq \sigma\}$. Then, by (36),

$$\sum_{i \in \mathcal{I}} \theta_i^2 \geq 1/2. \tag{40}$$

We will use the elementary inequality $|\cos s| \leq e^{-cs^2}$ for $|s| \leq 1$. According to (40), whenever $|\xi| \leq \sigma^{-1}$,

$$\left| \prod_{i=1}^n \cos(\theta_i \xi) \right| \leq \prod_{i \in \mathcal{I}} |\cos(\theta_i \xi)| \leq e^{-c\xi^2 \sum_{i \in \mathcal{I}} \theta_i^2} \leq e^{-c\xi^2 / 2}.$$

Apply the well-known bound $\int_s^\infty e^{-u^2/2} \leq C e^{-cs^2}$ for $s \geq 0$, to deduce

$$\begin{aligned} |I_2| &\leq 2 \int_{\varepsilon^{-1/2}}^{\sigma^{-1}} \left[\left| \prod_{i=1}^n \cos(\theta_i \xi) \right| + e^{-\xi^2 / 2} \right] \frac{2}{|\xi|} d\xi \\ &\leq 4 \int_{\varepsilon^{-1/2}}^{\sigma^{-1}} \left[e^{-c\xi^2 / 2} + e^{-\xi^2 / 2} \right] d\xi \leq \tilde{C} e^{-\tilde{c}/\varepsilon} \leq \tilde{C} \varepsilon^2. \end{aligned} \tag{41}$$

The bound for I_3 is easy. From (34) we have $\gamma(\sigma\xi) = 0$ for $|\xi| \geq \sigma^{-1}$. Hence,

$$|I_3| \leq 2 \left| \int_{\max\{\sigma^{-1}, \varepsilon^{-1/2}\}}^{\infty} e^{-\xi^2/2} \frac{2}{|\xi|} d\xi \right| \leq C e^{-c/\sigma^2} \leq \bar{C} \sigma^2.$$

The lemma follows by combining the above bound for $|I_3|$ with the bound (41) for $|I_2|$ and the bound (40) for $|I_1|$. \square

Lemma 6 *Let $X = (X_1, \dots, X_n)$ be a random vector in \mathbb{R}^n , with $\mathbb{E}X_i^2 = 1$ for $i = 1, \dots, n$, that is distributed according to an unconditional, log-concave density. Let $(\theta_1, \dots, \theta_n) \in S^{n-1}$ and denote $\varepsilon = 10\sqrt{\sum_i \theta_i^4}$. Then,*

$$\mathbb{P} \left(\frac{1}{2} \leq \sum_{i=1}^n \theta_i^2 X_i^2 \leq \frac{3}{2} \quad \text{and} \quad \sum_{i:|\theta_i X_i| \geq \varepsilon} \theta_i^2 X_i^2 \leq \frac{1}{4} \right) \geq 1 - C\varepsilon^2,$$

where $C > 0$ is a universal constant.

Proof Note that $\mathbb{E} \sum_{i=1}^n \theta_i^2 X_i^2 = 1$. According to the Chebyshev’s inequality and Corollary 5,

$$\mathbb{P} \left(\left| \sum_{i=1}^n \theta_i^2 X_i^2 - 1 \right| \geq 1/2 \right) \leq 4 \text{Var} \left(\sum_{i=1}^n \theta_i^2 X_i^2 \right) \leq 64 \sum_{i=1}^n \theta_i^4 \leq \varepsilon^2. \tag{42}$$

Denote $Y = \sum_{i:|\theta_i X_i| \geq \varepsilon} \theta_i^2 X_i^2$. Clearly,

$$\varepsilon^2 Y = \varepsilon^2 \sum_{i:|\theta_i X_i| \geq \varepsilon} \theta_i^2 X_i^2 \leq \sum_{i=1}^n \theta_i^4 X_i^4.$$

Therefore

$$\mathbb{E}Y \leq \varepsilon^{-2} \sum_{i=1}^n \theta_i^4 \mathbb{E}X_i^4 \leq 6\varepsilon^{-2} \sum_{i=1}^n \theta_i^4 \leq \frac{1}{10},$$

where we used the inequality $\mathbb{E}X_i^4 \leq 6(\mathbb{E}X_i^2)^2 = 6$, quoted above as (26). Next, apply Lemma 4(ii) with $f_i(t) = \theta_i^2 t^2$ for $|t| \geq \varepsilon/\theta_i$ and $f_i(t) = 0$ otherwise. Then $Y = \sum_i f_i(X_i)$ and according to the conclusion of that lemma,

$$\text{Var}(Y) \leq 4 \sum_{i=1}^n 5\theta_i^4 \mathbb{E}X_i^4 \leq 120 \sum_{i=1}^n \theta_i^4 \leq C\varepsilon^2.$$

Denote $\mu = \mathbb{E}Y \leq 1/10$. Another application of the Chebyshev inequality yields

$$\mathbb{P}\left(Y \geq \frac{1}{4}\right) \leq \mathbb{P}\left(|Y - \mu| \geq \frac{1}{10}\right) \leq 100\text{Var}(Y) \leq C\varepsilon^2. \tag{43}$$

The lemma follows from (42) and (43). □

Lemma 7 *Let $X = (X_1, \dots, X_n)$ be a random vector in \mathbb{R}^n , with $\mathbb{E}X_i^2 = 1$ for $i = 1, \dots, n$, that is distributed according to an unconditional, log-concave density. Let $(\theta_1, \dots, \theta_n) \in S^{n-1}$ and denote $\varepsilon = 10\sqrt{\sum_i \theta_i^4}$. Then, for any $t \in \mathbb{R}$,*

$$\left| \mathbb{P}\left(\varepsilon\Gamma + \sum_{i=1}^n \theta_i X_i \geq t\right) - \Phi(t) \right| \leq C\varepsilon^2,$$

where $C > 0$ is a universal constant.

Proof We may assume that ε is smaller than some given positive universal constant, as otherwise the conclusion is trivial. Let $\Delta_1, \dots, \Delta_n$ be independent, symmetric, Bernoulli random variables, that are independent also of X . For $t \in \mathbb{R}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ define

$$P(t, x) = \mathbb{P}\left(\varepsilon\Gamma + \sum_{i=1}^n \theta_i x_i \Delta_i \geq t\right).$$

Since the density of X is unconditional, the random variable $\sum_i \theta_i X_i$ has the same distribution as $\sum_i \theta_i X_i \Delta_i$. Fix $t \in \mathbb{R}$. Then,

$$\mathbb{P}\left(\varepsilon\Gamma + \sum_{i=1}^n \theta_i X_i \geq t\right) = \mathbb{P}\left(\varepsilon\Gamma + \sum_{i=1}^n \theta_i X_i \Delta_i \geq t\right) = \mathbb{E}P(t, X). \tag{44}$$

Write $\mathcal{A} \subset \mathbb{R}^n$ for the collection of all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ for which

$$\frac{1}{2} \leq \sum_{i=1}^n \theta_i^2 x_i^2 \leq \frac{3}{2} \quad \text{and} \quad \sum_{i:|\theta_i x_i| \geq \varepsilon} \theta_i^2 x_i^2 \leq \frac{1}{4} \leq \frac{1}{2} \sum_{i=1}^n \theta_i^2 x_i^2.$$

We may apply Lemma 5 for $(\theta_1 x_1, \dots, \theta_n x_n)$ and for $\sigma = \varepsilon$, and conclude that,

$$\left| P(t, x) - \Phi\left(\frac{t}{\sqrt{\sum_{i=1}^n \theta_i^2 x_i^2}}\right) \right| \leq C\left(\varepsilon^2 + \sum_{i=1}^n \theta_i^4 x_i^4\right) \quad \text{for all } x \in \mathcal{A}.$$

From Lemma 6 we have $\mathbb{P}(X \notin \mathcal{A}) \leq C\varepsilon^2$. Consequently,

$$\begin{aligned} & \left| \mathbb{E}P(t, X) - \mathbb{E}\Phi\left(\frac{t}{\sqrt{\sum_{i=1}^n \theta_i^2 X_i^2}}\right) \right| \\ & \leq 2\mathbb{P}(X \notin \mathcal{A}) + C\mathbb{E}\left(\varepsilon^2 + \sum_{i=1}^n \theta_i^4 X_i^4\right) \leq C'\varepsilon^2, \end{aligned} \tag{45}$$

where we used once more the bound $\mathbb{E}X_i^4 \leq 6(\mathbb{E}X_i^2)^2 = 6$. According to (44) and (45), in order to prove the lemma, all we need is to show that

$$\left| \mathbb{E}\Phi\left(\frac{t}{\sqrt{\sum_{i=1}^n \theta_i^2 X_i^2}}\right) - \Phi(t) \right| \leq C\varepsilon^2. \tag{46}$$

Write $Y = \sum_{i=1}^n \theta_i^2 X_i^2$. Then $\mathbb{P}(Y \geq 1/2) \geq 1 - C\varepsilon^2$, by Lemma 6. Therefore, to prove (46) and complete the proof of the lemma, it suffices to show that

$$\mathbb{E}\left[\Phi\left(\frac{t}{\sqrt{Y}}\right) - \Phi(t) \mid Y \geq 1/2\right] = O(\varepsilon^2). \tag{47}$$

We may assume that ε does not exceed a small positive universal constant, hence $\mathbb{P}(Y \geq 1/2)^{-1} \leq (1 - C\varepsilon^2)^{-1} \leq 1 + C'\varepsilon^2$. Therefore,

$$1 = \mathbb{E}Y \leq \mathbb{E}\left(Y \mid Y \geq \frac{1}{2}\right) \leq \mathbb{P}(Y \geq 1/2)^{-1} \leq 1 + C'\varepsilon^2. \tag{48}$$

Corollary 5(i) implies that $\mathbb{E}(Y - 1)^2 \leq C\varepsilon^2$. Hence,

$$\mathbb{E}\left((Y - 1)^2 \mid Y \geq \frac{1}{2}\right) \leq \mathbb{E}(Y - 1)^2 / \mathbb{P}(Y \geq 1/2) \leq \tilde{C}\varepsilon^2. \tag{49}$$

Denote $F(u) = \Phi(t/\sqrt{u})$. Clearly, $\varphi(s)s = O(1)$ and $\varphi'(s)s^2 = O(1)$ for any $s \in \mathbb{R}$. Consequently, for any $u \geq 1/2$,

$$F'(u) = \frac{1}{2u}\varphi\left(\frac{t}{\sqrt{u}}\right)\frac{t}{\sqrt{u}} = O(1)$$

and

$$F''(u) = -\frac{3}{4u^2}\varphi\left(\frac{t}{\sqrt{u}}\right)\frac{t}{\sqrt{u}} - \frac{1}{4u^2}\varphi'\left(\frac{t}{\sqrt{u}}\right)\frac{t^2}{u} = O(1).$$

By Taylor’s theorem,

$$\begin{aligned} \mathbb{E} \left[\Phi \left(t/\sqrt{Y} \right) - \Phi(t) \mid Y \geq 1/2 \right] &= \mathbb{E} [F(Y) - F(1) \mid Y \geq 1/2] \\ &= \mathbb{E} \left[F'(1)(Y - 1) + O \left((Y - 1)^2 \right) \mid Y \geq 1/2 \right] \\ &= F'(1) \left(\mathbb{E}(Y - 1) \mid Y \geq \frac{1}{2} \right) + O(\varepsilon^2) = O(\varepsilon^2), \end{aligned}$$

where we used the estimates for F' , F'' and the bounds (48) and (49). This completes the proof of (47). The lemma is proven. \square

Our next goal is to eliminate the “ $\varepsilon\Gamma$ ” term from the conclusion of Lemma 7. The following short computational lemma serves this purpose. We shall use the standard estimate

$$c \frac{\varphi(t_0)}{t_0 + 1} \leq \Phi(t_0) \leq C \frac{\varphi(t_0)}{t_0 + 1} \leq \tilde{C}\varphi(t_0) \tag{50}$$

for any $t_0 \geq 0$ (see, e.g., [14, Vol. I, Sect. VII.1]).

Lemma 8 *Let $t_0 \geq 0$ and denote $\delta = \Phi(t_0)$. Then,*

- (i) $\Phi \left(t_0 + 2\delta^{1/4} \right) \geq C_1^{-1}\delta$.
- (ii) $1 - \Phi \left(t_0 - 2\delta^{1/4} \right) \geq 1 - \Phi(-2) \geq C_1^{-1} \geq C_1^{-1}\delta$.
- (iii) *Suppose $x > 0$ satisfies $\left| \frac{1}{x} - \frac{1}{\varphi(t_0)} \right| \leq c_2\delta^{-3/4}$. Then $x^2 \leq C_1\delta$.*

Here, $C_1 > 1$ and $0 < c_2 < 1$ are universal constants.

Proof We have $t_0\delta^{1/4} \leq Ct_0(\varphi(t_0))^{1/4} \leq C'$ according to (50). Hence,

$$\frac{\Phi \left(t_0 + 2\delta^{1/4} \right)}{\Phi(t_0)} \geq c' \exp \left[\frac{t_0^2}{2} - \frac{\left(t_0 + 2\delta^{1/4} \right)^2}{2} \right] \geq \hat{c} \exp \left(-2t_0\delta^{1/4} \right) \geq c',$$

and (i) is proven. The statement (ii) is self-explanatory. Regarding (iii), it is readily verified that $\tilde{c}(t_0 + 1)^{3/4} \leq \varphi(t_0)^{-1/4}$ for any $t_0 \geq 0$. Therefore, by (50), for a sufficiently small $c_2 > 0$,

$$\frac{1}{\varphi(t_0)} - \frac{c_2}{\delta^{3/4}} \geq \frac{1}{\varphi(t_0)} - \frac{\tilde{c}(t_0 + 1)^{3/4}}{2\varphi(t_0)^{3/4}} \geq \frac{1}{\varphi(t_0)} - \frac{\varphi(t_0)^{-1/4}}{2\varphi(t_0)^{3/4}} = \frac{1}{2\varphi(t_0)}.$$

Note also that $\varphi(t_0) \leq C/(t_0 + 1)$. Consequently, for any $x > 0$,

$$\left| \frac{1}{x} - \frac{1}{\varphi(t_0)} \right| \leq \frac{c_2}{\delta^{3/4}} \quad \Rightarrow \quad x \leq 2\varphi(t_0) \leq C\sqrt{\frac{\varphi(t_0)}{t_0 + 1}} \leq \tilde{C}\sqrt{\delta},$$

where we used (50) again. \square

Lemma 9 *Let X be a real-valued random variable with an even, log-concave density. Let $0 < \varepsilon < 1, A \geq 1$. Suppose that for any $t \in \mathbb{R}$,*

$$|\mathbb{P}(\varepsilon\Gamma + X \geq t) - \Phi(t)| \leq A\varepsilon^2. \tag{51}$$

Then, for any $t \in \mathbb{R}$,

$$|\mathbb{P}(X \geq t) - \Phi(t)| \leq CA\varepsilon^2, \tag{52}$$

where $C > 0$ is a universal constant.

Proof By approximation, we may assume that the density of X is C^1 -smooth and positive everywhere (e.g., convolve X with a very small gaussian). We may also assume that $\varepsilon \leq c$ for a small universal constant $c > 0$. The function

$$E(t) = |\mathbb{P}(X \geq t) - \Phi(t)| \quad (t \in \mathbb{R})$$

is continuous and vanishes at $\pm\infty$. Consequently, there exists $t_0 \in \mathbb{R}$ where $E(t)$ attains its maximum. Since E is an even function, we may assume that $t_0 \geq 0$. Write $f : \mathbb{R} \rightarrow [0, \infty)$ for the density of X . As $E'(t_0) = 0$,

$$f(t_0) = \varphi(t_0) = \frac{1}{\sqrt{2\pi}}e^{-t_0^2/2}. \tag{53}$$

To prove the lemma, it suffices to show that $\max_t E(t) = E(t_0) \leq CA\varepsilon^2$.

Step 1: Suppose first that $\Phi(t_0) \leq 2C_1A\varepsilon^2$, for C_1 being the universal constant from Lemma 8. Then by (51),

$$\mathbb{P}(\varepsilon\Gamma + X \geq t_0) \leq \Phi(t_0) + A\varepsilon^2 \leq (2C_1 + 1)A\varepsilon^2,$$

hence,

$$\mathbb{P}(X \geq t_0) = 2\mathbb{P}(X \geq t_0, \Gamma \geq 0) \leq 2\mathbb{P}(\varepsilon\Gamma + X \geq t_0) \leq (4C_1 + 2)A\varepsilon^2.$$

Consequently, since $\Phi(t_0) \leq 2C_1A\varepsilon^2$,

$$\max_{t \in \mathbb{R}} E(t) = E(t_0) = |\mathbb{P}(X \geq t_0) - \Phi(t_0)| \leq (6C_1 + 2)A\varepsilon^2 \leq \bar{C}A\varepsilon^2.$$

The desired estimate (52) is therefore proven, in the case where $\Phi(t_0) \leq 2C_1A\varepsilon^2$.

Step 2: It remains to deal with the case where $t_0 \geq 0$ satisfies $\Phi(t_0) > 2C_1A\varepsilon^2$. Denote $\delta = \Phi(t_0) \geq 2C_1A\varepsilon^2 \geq A\varepsilon^2$. Note that

$$\mathbb{P}\left(|\varepsilon\Gamma| \geq \delta^{1/4}\right) \leq \frac{\varepsilon^6 \mathbb{E}\Gamma^6}{(\delta^{1/4})^6} \leq C \frac{\varepsilon^3}{A^{3/2}} \leq C\varepsilon\delta \leq \frac{\delta}{4C_1} \tag{54}$$

under the legitimate assumption that ε is smaller than a given universal constant. From Lemma 8(i) we have $\Phi(t_0 + 2\delta^{1/4}) \geq \delta/C_1$, hence by (51),

$$\mathbb{P}(\varepsilon\Gamma + X \geq t_0 + 2\delta^{1/4}) \geq \Phi(t_0 + 2\delta^{1/4}) - A\varepsilon^2 \geq \frac{\delta}{C_1} - A\varepsilon^2 \geq \frac{\delta}{2C_1}.$$

Consequently, from (54),

$$\mathbb{P}(X \geq t_0 + \delta^{1/4}) \geq \mathbb{P}(\varepsilon\Gamma + X \geq t_0 + 2\delta^{1/4}) - \mathbb{P}(\varepsilon\Gamma \geq \delta^{1/4}) \geq \delta/(4C_1).$$

A similar argument, using Lemma 8(ii) in place of Lemma 8(i), shows that

$$\mathbb{P}(X \leq t_0 - \delta^{1/4}) \geq \mathbb{P}(\varepsilon\Gamma + X \leq t_0 - 2\delta^{1/4}) - \mathbb{P}(|\varepsilon\Gamma| \geq \delta^{1/4}) \geq \delta/(4C_1).$$

We conclude that for any $t \in [t_0 - \delta^{1/4}, t_0 + \delta^{1/4}]$,

$$\min\{\mathbb{P}(X \geq t), \mathbb{P}(X \leq t)\} \geq \frac{\delta}{4C_1}. \tag{55}$$

Step 3: The density f is differentiable and positive everywhere. Fix $x_0 \in \mathbb{R}$. Since $\log f$ is concave, then

$$f(x) \leq f(x_0) \exp\left(\frac{f'(x_0)}{f(x_0)}(x - x_0)\right) \quad \forall x \in \mathbb{R}.$$

Consequently, when $f'(x_0) \neq 0$,

$$\begin{aligned} & \min\left\{\int_{x_0}^{\infty} f(x)dx, \int_{-\infty}^{x_0} f(x)dx\right\} \\ & \leq \int_{x_0}^{\infty} f(x_0) \exp\left(-\frac{|f'(x_0)(x - x_0)|}{f(x_0)}\right) dx = \frac{f(x_0)^2}{|f'(x_0)|}. \end{aligned}$$

We conclude from (55) that for any $t \in [t_0 - \delta^{1/4}, t_0 + \delta^{1/4}]$,

$$|f'(t)| \leq f^2(t) [\min\{\mathbb{P}(X \geq t), \mathbb{P}(X \leq t)\}]^{-1} \leq 4C_1\delta^{-1} f^2(t). \tag{56}$$

Equivalently, $|(1/f)'| \leq 4C_1\delta^{-1}$ in the interval $[t_0 - \delta^{1/4}, t_0 + \delta^{1/4}]$. Hence,

$$\left|\frac{1}{f(t)} - \frac{1}{f(t_0)}\right| \leq 4C_1\delta^{-1} \cdot \frac{c_2}{4C_1}\delta^{1/4} = c_2\delta^{-3/4} \quad \text{when } |t - t_0| \leq \frac{c_2}{4C_1}\delta^{1/4},$$

for $c_2 > 0$ being the universal constant from Lemma 8. Recall from (53) that $f(t_0) = \varphi(t_0)$. Lemma 8(iii) thus implies that

$$f^2(t) \leq C_1\delta \text{ for } t \in [t_0 - c\delta^{1/4}, t_0 + c\delta^{1/4}],$$

with $c = c_2/4C_1$. Returning to (56), we finally deduce the bound

$$|f'(t)| \leq \tilde{C} \text{ for } t \in [t_0 - \hat{c}\delta^{1/4}, t_0 + \hat{c}\delta^{1/4}].$$

Through Taylor’s theorem, the latter bound entails that

$$\mathbb{P}(X \geq t_0 + s) = \mathbb{P}(X \geq t_0) - f(t_0)s + O\left(s^2\right) \text{ for any } |s| \leq \hat{c}\delta^{1/4}. \tag{57}$$

Step 4: Let $\eta : \mathbb{R} \rightarrow [0, \infty)$ stand for the probability density of $\varepsilon\Gamma$. The function η is even. Recall that $\delta \geq \varepsilon^2$. Hence,

$$\int_{|s| \geq \hat{c}\delta^{1/4}} \eta(s)ds = \mathbb{P}\left(|\varepsilon\Gamma| \geq \hat{c}\delta^{1/4}\right) \leq \frac{\varepsilon^4 \mathbb{E}\Gamma^4}{\hat{c}^4\delta} \leq C\varepsilon^2, \tag{58}$$

where $\hat{c} > 0$ is the constant from (57). The crucial observation is that $s \mapsto f(t_0)s\eta(s)$ is an odd function, hence its integral on a symmetric interval about the origin vanishes. By (57) and (58),

$$\begin{aligned} & |\mathbb{P}(\varepsilon\Gamma + X \geq t_0) - \mathbb{P}(X \geq t_0)| \\ &= \left| \int_{-\infty}^{\infty} [\mathbb{P}(X \geq t_0 + s) - \mathbb{P}(X \geq t_0)] \eta(s)ds \right| \\ &\leq \left| \int_{-\hat{c}\delta^{1/4}}^{\hat{c}\delta^{1/4}} \left[-f(t_0)s + O\left(s^2\right)\right] \eta(s)ds \right| + 2 \int_{|s| \geq \hat{c}\delta^{1/4}} \eta(s)ds \\ &\leq \tilde{C} \int_{-\hat{c}\delta^{1/4}}^{\hat{c}\delta^{1/4}} s^2 \eta(s)ds + C\varepsilon^2 \leq \tilde{C}\mathbb{E}(\varepsilon\Gamma)^2 + C\varepsilon^2 \leq \check{C}\varepsilon^2, \end{aligned}$$

where $\hat{c} > 0$ is the constant from (57). We apply (51) and conclude that

$$E(t_0) = |\mathbb{P}(X \geq t_0) - \Phi(t_0)| \leq \check{C}\varepsilon^2 + |\mathbb{P}(\varepsilon\Gamma + X \geq t_0) - \Phi(t_0)| \leq \check{C}\varepsilon^2 + A\varepsilon^2.$$

Since $E(t_0) = \max_t E(t)$, the proof of the lemma is complete. □

Proof of Theorem 1 Let $\theta_1, \dots, \theta_n \in \mathbb{R}$ be such that $\sum_i \theta_i^2 = 1$. Denote $\varepsilon = 10\sqrt{\sum_{i=1}^n \theta_i^4}$. According to Lemma 7, the random variable $Y = \sum_{i=1}^n \theta_i X_i$ satisfies

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(\varepsilon \Gamma + Y \geq t) - \Phi(t)| \leq C\varepsilon^2, \tag{59}$$

with some universal constant $C \geq 1$. The random variable Y has an even, log-concave density by Prékopa–Leindler. We may thus apply Lemma 9, and conclude from (59) that

$$\sup_{\alpha \leq \beta} |\mathbb{P}(\alpha \leq Y \leq \beta) - [\Phi(\alpha) - \Phi(\beta)]| \leq 2 \sup_{t \in \mathbb{R}} |\mathbb{P}(Y \geq t) - \Phi(t)| \leq C'\varepsilon^2.$$

The theorem is thus proven. □

Appendix: Proof of Theorem 2

With Cédric Villani’s permission, we reproduce below the proof of Theorem 2 from his book [40, Sect. 7.6] with a few minor changes.

Proof of Theorem 2 We need to prove that for any C^∞ -smooth function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}^n} h\varphi d\mu \leq \sqrt{\int_{\mathbb{R}^n} |\nabla\varphi|^2 d\mu} \cdot \liminf_{\varepsilon \rightarrow 0^+} \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon}. \tag{60}$$

Since μ is compactly supported, it is enough to restrict attention to compactly supported functions φ . Fix such a test function φ . Then the second derivatives of φ are bounded on \mathbb{R}^n . By Taylor’s theorem, there exists a constant $R = R(\varphi)$ with

$$\varphi(y) - \varphi(x) \leq |\nabla\varphi(x)| \cdot |x - y| + R|x - y|^2 \quad \forall x, y \in \mathbb{R}^n. \tag{61}$$

We may assume that $\sup |h| > 0$ (otherwise, the theorem holds trivially), and let $\varepsilon > 0$ be smaller than $1/\sup |h|$. Then μ_ε is a non-negative measure on \mathbb{R}^n . Let γ be any coupling of μ and μ_ε . We see that

$$\int_{\mathbb{R}^n} h\varphi d\mu = \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \varphi d[\mu_\varepsilon - \mu] = \frac{1}{\varepsilon} \int_{\mathbb{R}^n \times \mathbb{R}^n} [\varphi(y) - \varphi(x)] d\gamma(x, y).$$

Write $W_2^\gamma(\mu, \mu_\varepsilon) = \sqrt{\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 d\gamma(x, y)}$. According to (61) and to the Cauchy-Schwartz inequality,

$$\begin{aligned} \int_{\mathbb{R}^n} h\varphi d\mu &\leq \frac{1}{\varepsilon} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla\varphi(x)| \cdot |x - y| d\gamma(x, y) + \frac{R}{\varepsilon} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 d\gamma(x, y) \\ &\leq \frac{1}{\varepsilon} \sqrt{\int_{\mathbb{R}^n} |\nabla\varphi(x)|^2 d\mu(x)} \cdot W_2^\gamma(\mu, \mu_\varepsilon) + \frac{R}{\varepsilon} W_2^\gamma(\mu, \mu_\varepsilon)^2. \end{aligned}$$

By taking the infimum over all couplings γ of μ and μ_ε , we obtain

$$\int_{\mathbb{R}^n} h\varphi d\mu \leq \sqrt{\int_{\mathbb{R}^n} |\nabla\varphi|^2 d\mu} \cdot \frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon} + R \frac{W_2(\mu, \mu_\varepsilon)^2}{\varepsilon}, \quad (62)$$

with R depending only on φ . We may assume that $\liminf_{\varepsilon \rightarrow 0^+} W_2(\mu, \mu_\varepsilon)/\varepsilon < \infty$; otherwise, there is nothing to prove. Consequently,

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{W_2(\mu, \mu_\varepsilon)^2}{\varepsilon} = \liminf_{\varepsilon \rightarrow 0^+} \varepsilon \left(\frac{W_2(\mu, \mu_\varepsilon)}{\varepsilon} \right)^2 = 0.$$

Hence by letting ε tend to zero in (62), we deduce (60). The proof is complete. \square

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