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Concentration under scaling limits for weakly pinned Gaussian random walks

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Abstract We study scaling limits for d-dimensional Gaussian random walks perturbed by an attractive force toward a certain subspace of $\mathbb{R}^d$, especially under the critical situation that the rate functional of the corresponding large deviation principle admits two minimizers. We obtain different type of limits, in a positive recurrent regime, depending on the co-dimension of the subspace and the conditions imposed at the final time under the presence or absence of a wall. The motivation comes from the study of polymers or $(1+1)$-dimensional interfaces with δ-pinning.

Keywords Large deviation · Minimizers · Random walks · Pinning · Scaling limit · Concentration

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1 Introduction and main results

The present paper deals with Gaussian random walks on $\mathbb{R}^d$ perturbed by an attractive force toward a subspace M of $\mathbb{R}^d$, especially under the critical situation that the rate functional of the corresponding large deviation principle admits exactly two minimizers. The macroscopic time, observed after scaling, runs over the interval $D = [0, 1]$. The starting point of the (macroscopically scaled) walks at $t = 0$ is always specified, while we will or will not specify the arriving point at $t = 1$. We thus consider four different cases, in addition to the conditions at $t = 1$, depending whether a wall is located at the boundary of the upper half space of $\mathbb{R}^d$ or not, and study how the macroscopic scaling limits differ in these four cases.

1.1 Weakly pinned Gaussian random walks

In this subsection, we introduce (temporally inhomogeneous) Markov chains called the weakly pinned Gaussian random walks. Let $D_N = ND \cap \mathbb{Z} \equiv \{0, 1, 2, \dots, N\}$ be the range of (microscopic) time for the Markov chains corresponding to the macroscopic one D. The state spaces of the Markov chains are $\mathbb{R}^d$ or the upper half space $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$ according as we do not or do put a wall at $\partial\mathbb{R}^d_+$, where $\mathbb{R}_+ = [0, \infty)$. Let M be an m-dimensional subspace of $\mathbb{R}^d$ for $0 \le m \le d - 1$ and let $M^\perp$ be its orthogonal complement. We consider the measure $\nu(dy) = dy^{(1)}\delta_0(dy^{(2)})$ on $\mathbb{R}^d$ obtained by extending the surface measure $dy^{(1)}$ on M under the decomposition $y = (y^{(1)}, y^{(2)}) \in \mathbb{R}^d \cong M \times M^\perp$; in particular, if $M = \{0\}$, $y = y^{(2)}$ and $\nu(dy) = \delta_0(dy)$. The co-dimension of M will be denoted by $r \equiv \operatorname{codim} M = d - m$. We assume $M \subset \partial\mathbb{R}^d_+$ when the state space of the Markov chains is $\mathbb{R}^d_+$.

Given $a, b \in \mathbb{R}^d$ (or $\in \mathbb{R}^d_+$), the starting point of the Markov chains $\phi = (\phi_i)_{i\in D_N}$ is always $aN \in \mathbb{R}^d$ (or $\in \mathbb{R}^d_+$), while, for the arriving point at $i = N$, we consider two cases: $\phi_N = bN$ (we call Dirichlet case) or without giving any condition on ϕ_N (we call free case). The distributions of the Markov chains ϕ on $(\mathbb{R}^d)^{N+1}$ or $(\mathbb{R}^d_+)^{N+1}$ with a strength $\varepsilon \ge 0$ of the pinning force toward M, imposing the Dirichlet or free conditions at N and putting or without putting a wall at $\partial\mathbb{R}^d_+$, are described by the following four probability measures $\mu_N^{D,\varepsilon}$, $\mu_N^{D,\varepsilon,+}$, $\mu_N^{F,\varepsilon}$ and $\mu_N^{F,\varepsilon,+}$, respectively:

$$\mu_N^{D,\varepsilon,(+)}(d\phi) = \frac{1}{Z_N^{D,\varepsilon,(+)}} e^{-H_N(\phi)}\delta_{aN}(d\phi_0) \prod_{i\in D_N\setminus\{0,N\}} \left(\varepsilon\, \nu(d\phi_i) + d\phi_i^{(+)}\right) \delta_{bN}(d\phi_N), \tag{1.1}$$

$$\mu_N^{F,\varepsilon,(+)}(d\phi) = \frac{1}{Z_N^{F,\varepsilon,(+)}} e^{-H_N(\phi)}\delta_{aN}(d\phi_0) \prod_{i\in D_N\setminus\{0\}} \left(\varepsilon\, \nu(d\phi_i) + d\phi_i^{(+)}\right), \tag{1.2}$$

where $d\phi_i^{(+)}$ denotes the Lebesgue measure on $\mathbb{R}^d$ (or on $\mathbb{R}^d_+$), and $Z_N^{D,\varepsilon,(+)}$ and $Z_N^{F,\varepsilon,(+)}$ are the normalizing constants, respectively. The function $H_N(\phi)$ called the

Hamiltonian is given by

$$H_N(\phi) = \frac{1}{2} \sum_{i=0}^{N-1} |\phi_{i+1} - \phi_i|^2,$$

in which $|\cdot|$ stands for the Euclidean norm of $\mathbb{R}^d$. Note that, if $\varepsilon = 0$ (i.e., without pinning), ϕ under $\mu_N^{F,0}$ is a d-dimensional Brownian motion viewed at integer times.

We sometimes denote the partition functions as $Z_N^{D,\varepsilon,(+)} = Z_N^{a,b,\varepsilon,(+)}$ and $Z_N^{F,\varepsilon,(+)} = Z_N^{a,F,\varepsilon,(+)}$ to clarify the specific conditions at $i = 0$ and N. The Markov chain ϕ satisfies the condition $\phi_0 = aN$ (a.s.) at $i = 0$ under these four measures. At $i = N$, the Dirichlet condition $\phi_N = bN$ is satisfied under $\mu_N^{D,\varepsilon}$ and $\mu_N^{D,\varepsilon,+}$, while the free condition (i.e., no specific condition) is fulfilled under $\mu_N^{F,\varepsilon}$ and $\mu_N^{F,\varepsilon,+}$. The superscripts D and F are put to indicate the conditions at $i = N$. Both $\mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$ are probability measures on $(\mathbb{R}^d)^{N+1}$ defined under the absence of wall, while $\mu_N^{D,\varepsilon,+}$ and $\mu_N^{F,\varepsilon,+}$ are those on $(\mathbb{R}_+^d)^{N+1}$ defined under the presence of a wall at $\partial\mathbb{R}_+^d$. The following table exhibits the difference of these four measures in short:

at $i = N$	No wall	Wall at $\partial\mathbb{R}_+^d$
Dirichlet condition	$\mu_N^{D,\varepsilon}$	$\mu_N^{D,\varepsilon,+}$
Free condition	$\mu_N^{F,\varepsilon}$	$\mu_N^{F,\varepsilon,+}$

When $d = 1$ and $m = 0$, the Markov chain $(\phi_i \in \mathbb{R}\,(\text{or} \in \mathbb{R}_+))_{i \in D_N}$ may be interpreted as the heights of interfaces located in a plane measured from the position i on a reference line (x-axis), so that the system is called $(1+1)$-dimensional interface model with δ-pinning at 0, see [3,5,8,15]. See [14] for a relation to the polymer models.

1.2 Scaling limits and large deviation rate functionals

We will sometimes drop the superscripts ε if there is no confusion.

Let $h^N = \{h^N(t), t \in D\}$ be the macroscopic path of the Markov chain determined from the microscopic one ϕ under a proper scaling, namely, it is defined through a polygonal approximation of $\big(h^N(i/N) = \phi_i/N\big)_{i \in D_N}$ so that

$$h^N(t) = \frac{[Nt] - Nt + 1}{N} \phi_{[Nt]} + \frac{Nt - [Nt]}{N} \phi_{[Nt]+1}, \quad t \in D.$$

Then, the sample path large deviation principle holds for h^N under $\mu_N^D, \mu_N^{D,+}, \mu_N^F$ and $\mu_N^{F,+}$, respectively, on the space $\mathcal{C} = C([0,1], \mathbb{R}^d)$ equipped with the uniform topology as $N \to \infty$, see Theorem 4.1 in Sect. 4 (or Theorem 2.2 of [12] for μ_N^D when $d = 1$ and $m = 0$, and [4,16] when $\varepsilon = 0$). The speeds are always N and the unnormalized rate functionals are given by Σ^D, $\Sigma^{D,+}$, Σ^F and $\Sigma^{F,+}$, respectively, all of which are of the form:

$$\Sigma(h) = \frac{1}{2}\int_D |\dot{h}(t)|^2 dt - \xi |\{t \in D; h(t) \in M\}|, \tag{1.3}$$

for $h \in H^1_{a,b}(D) = \{h \in H^1(D); h(0) = a, h(1) = b\}$ in the Dirichlet case respectively $h \in H^1_{a,F}(D) = \{h \in H^1(D); h(0) = a\}$ in the free case with certain non-negative constants ξ, where $|\cdot|$ stands for the Lebesgue measure on D and $H^1(D) = H^1(D, \mathbb{R}^d)$ is the usual Sobolev space. We define $\Sigma(h) = +\infty$ for h's outside of these spaces, and also for h such that $h(t) \notin \mathbb{R}^d_+$ for some $t \in D$ under the presence of a wall. The constants ξ differ depending on the absence or presence of a wall as explained below.

We determine two non-negative constants $\xi^\varepsilon = \xi^\varepsilon_r$ and $\xi^{\varepsilon,+} = \xi^{\varepsilon,+}_r$ by the thermodynamic limits:

$$\xi^\varepsilon = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z^{0,0,\varepsilon}_{N,r}}{Z^{0,0}_{N,r}}, \quad \xi^{\varepsilon,+} = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z^{0,0,\varepsilon,+}_{N,r}}{Z^{0,0,+}_{N,r}}, \tag{1.4}$$

and another two constants $\xi^{F,\varepsilon}$ and $\xi^{F,\varepsilon,+}$ by

$$\xi^{F,\varepsilon} = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z^{0,F,\varepsilon}_{N,r}}{Z^{0,F}_{N,r}}, \quad \xi^{F,\varepsilon,+} = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z^{0,F,\varepsilon,+}_{N,r}}{Z^{0,F,+}_{N,r}}, \tag{1.5}$$

where the partition functions in the numerators are associated with the random walks in $\mathbb{R}^r$ with pinning at $M' = \{0\} \subset \mathbb{R}^r$ (i.e., $m = 0$) taking $a = b = 0 \in \mathbb{R}^r$ in the Dirichlet case and $a = 0 \in \mathbb{R}^r$ in the free case, while the denominators $Z^{0,0}_{N,r}$, $Z^{0,0,+}_{N,r}$, $Z^{0,F}_{N,r}$ and $Z^{0,F,+}_{N,r}$ are defined without pinning effect and equal to their corresponding numerators with $\varepsilon = 0$. See (2.1) for $Z^{0,0,\varepsilon}_{N,r}$, (2.13) for $Z^{0,F,\varepsilon}_{N,r}$ and others. As we will state in Theorem 1.1, the constants ξ defined for two different cases actually coincide with each other, i.e., $\xi^\varepsilon = \xi^{F,\varepsilon}$ and $\xi^{\varepsilon,+} = \xi^{F,\varepsilon,+}$ hold.

The constants ξ in (1.3) are defined by $\xi = \xi^\varepsilon_{\text{codim}\,M}$ for the functionals $\Sigma = \Sigma^D$, Σ^F and $\xi = \xi^{\varepsilon,+}_{\text{codim}\,M}$ for $\Sigma = \Sigma^{D,+}$, $\Sigma^{F,+}$, respectively, with the choice of $r = \text{codim}\,M$.

The non-positive constants $\tau^\varepsilon = -\xi^\varepsilon$ and $\tau^{\varepsilon,+} = -\xi^{\varepsilon,+}$ are sometimes called the pinning free energy and the wall (more precisely, wall+pinning or wetting) free energy, respectively. Explicit formulae determining ξ^ε and $\xi^{\varepsilon,+}$ are found in (2.4) and (2.12). In particular, we will see that $\xi^\varepsilon > \xi^{\varepsilon,+} \ge 0$ for all $\varepsilon \ge 0$ unless $\xi^\varepsilon = 0$, see Remark 2.1-(1). Furthermore, we have the following result on the phase transition (localization/delocalization transition) in ε, which is called pinning or wetting transitions in the framework of the interface model under the absence or presence of a wall, respectively.

Theorem 1.1 1. *The limits in* (1.4) *and* (1.5) *exist for every* $\varepsilon \ge 0$, *and we have that* $\xi^\varepsilon = \xi^{F,\varepsilon}$ *and* $\xi^{\varepsilon,+} = \xi^{F,\varepsilon,+}$.

2. (Absence of wall) *If* $r \geq 3$, *there exists* $\varepsilon_c > 0$ *determined by* (2.3) *such that* $\xi^\varepsilon > 0$ *if and only if* $\varepsilon > \varepsilon_c$ *and* $\xi^\varepsilon = 0$ *if and only if* $0 \leq \varepsilon \leq \varepsilon_c$. *If* $r = 1$ *and* 2, *the above statement holds with* $\varepsilon_c = 0$.
3. (Presence of wall) *For all* $r \geq 1$, *there exists* $\varepsilon_c^+ > 0$ (*in fact*, $\varepsilon_c^+ > \varepsilon_c$) *determined by* (2.11) *such that* $\xi^{\varepsilon,+} > 0$ *if and only if* $\varepsilon > \varepsilon_c^+$ *and* $\xi^{\varepsilon,+} = 0$ *if and only if* $0 \leq \varepsilon \leq \varepsilon_c^+$.

In short, the pinning transition occurs if $r \geq 3$, while the wetting transition occurs for all dimensions. The Markov chain, being transient at $\varepsilon = 0$, turns to be recurrent when the strength ε of the attractive force toward 0 increases and exceeds the critical value ε_c or ε_c^+; see [14] for random walks with discrete values. The asymptotic behavior of the free energies ξ^ε and $\xi^{\varepsilon,+}$ for ε close to their critical values is studied in Appendix A. This gives, in particular, the critical exponents for the free energies.

The large deviation principle (Theorem 4.1) immediately implies the concentration properties for $\mu_N = \mu_N^D, \mu_N^{D,+}, \mu_N^F$ and $\mu_N^{F,+}$:

$$\lim_{N\to\infty} \mu_N(\mathrm{dist}_\infty(h^N, \mathcal{H}) \leq \delta) = 1, \tag{1.6}$$

for every $\delta > 0$, where $\mathcal{H} = \{h^*; \text{minimizers of } \Sigma\}$ with $\Sigma = \Sigma^D, \Sigma^{D,+}, \Sigma^F, \Sigma^{F,+}$, respectively, and dist_∞ denotes the distance on $\mathcal{C}$ under the uniform norm $\|\cdot\|_\infty$. More precisely, for any $\delta > 0$ there exists $c(\delta) > 0$ such that

$$\mu_N(\mathrm{dist}_\infty(h^N, \mathcal{H}) > \delta) \leq e^{-c(\delta)N}$$

for large enough N.

1.3 Minimizers of the rate functionals

By rotation, we may assume without loss of generality

$$M = \left\{x = (x^{(1)}, 0) \in \mathbb{R}^m \times \mathbb{R}^r\right\} \subset \mathbb{R}^d. \tag{1.7}$$

Under such coordinate of $\mathbb{R}^d$, we decompose $a = (a^{(1)}, a^{(2)})$ and $b = (b^{(1)}, b^{(2)}) \in M \times M^\perp$.

There are at most two possible minimizers of Σ^D. One is $\bar{h}^D$ defined by linearly interpolating between a and b: $\bar{h}^D(t) = (1-t)a + tb$, $t \in D$. If $|a^{(2)}| + |b^{(2)}| < \sqrt{2\xi^\varepsilon}$ (i.e., $t_1 + t_2 < 1$ for t_1 and t_2 defined below), we define $\hat{h}^D (\equiv \hat{h}^{D,\varepsilon})$ by $\hat{h}^D(t) = \left(\hat{h}^{D,(1)}(t), \hat{h}^{D,(2)}(t)\right)$, where $\hat{h}^{D,(1)}(t) = (1-t)a^{(1)} + tb^{(1)}$,

$$\hat{h}^{D,(2)}(t) = \begin{cases} (t_1 - t)a^{(2)}/t_1, & t \in [0, t_1], \\ 0, & t \in [t_1, 1-t_2], \\ (t + t_2 - 1)b^{(2)}/t_2, & t \in [1-t_2, 1], \end{cases} \tag{1.8}$$

and $t_1 = |a^{(2)}|/\sqrt{2\xi^\varepsilon}$ and $t_2 = |b^{(2)}|/\sqrt{2\xi^\varepsilon}$. The last relation is sometimes called Young's relation. Those for $\Sigma^{D,+}$ are two functions $\bar{h}^D$ and $\hat{h}^{D,+}(\equiv \hat{h}^{D,\varepsilon,+})$ defined similarly to $\hat{h}^D$ with ξ^ε replaced by $\xi^{\varepsilon,+}$.

In the free case, first for Σ^F, we set $\bar{h}^F(t) = a,\ t \in D$, and if $|a^{(2)}| < \sqrt{2\xi^\varepsilon}$ (i.e., $t_1 < 1$), $\hat{h}^F(t) = (\hat{h}^{F,(1)}(t), \hat{h}^{F,(2)}(t))$ with $\hat{h}^{F,(1)}(t) = a^{(1)}$,

$$\hat{h}^{F,(2)}(t) = \begin{cases} (t_1 - t)a^{(2)}/t_1, & t \in [0, t_1], \\ 0, & t \in [t_1, 1], \end{cases}$$

where $t_1 = |a^{(2)}|/\sqrt{2\xi^\varepsilon}$. Those for $\Sigma^{F,+}$ are $\bar{h}^F$ and $\hat{h}^{F,+}(\equiv \hat{h}^{F,\varepsilon,+})$ which is defined similarly to $\hat{h}^F$ with ξ^ε replaced by $\xi^{\varepsilon,+}$.

Then, the following lemma can be shown similarly to the case where $d = 1$ and $m = 0$, cf. Sects. 6.3 and 6.4 of [8].

Lemma 1.2 *The set of minimizers of Σ^D is contained in $\{\bar{h}^D, \hat{h}^D\}$. Similarly, the sets of minimizers of $\Sigma^{D,+}$, Σ^F and $\Sigma^{F,+}$ are contained in $\{\bar{h}^D, \hat{h}^{D,+}\}$, $\{\bar{h}^F, \hat{h}^F\}$ and $\{\bar{h}^F, \hat{h}^{F,+}\}$, respectively.*

The structure of the sets of minimizers is clarified in terms of a and b in Appendix B especially when $d = 1$ and $m = 0$.

1.4 Main results

We are concerned with the critical case where $\bar{h}$ and $\hat{h}$ are different and both are simultaneously the minimizers of Σ^D (or $\Sigma^{D,+}$), and similar situations for Σ^F (or $\Sigma^{F,+}$); especially when $d = 1$ and $m = 0$, this is equivalent to $\xi = \xi^\varepsilon$ (or $\xi^{\varepsilon,+}$) > 0 and $(a, b) \in \mathcal{C}_1$ (see Proposition B.1) in the Dirichlet case and $|a| = \sqrt{\xi/2}$ in the free case. Otherwise, h^N converges to the unique minimizer of Σ as $N \to \infty$ in probability, recall (1.6). We therefore assume the following conditions in each situation:

$$\begin{array}{llll} (C)_D & \varepsilon > \varepsilon_c & \text{and} & \Sigma^D(\bar{h}^D) = \Sigma^D(\hat{h}^D), \\ (C)_{D,+} & \varepsilon > \varepsilon_c^+ & \text{and} & \Sigma^{D,+}(\bar{h}^D) = \Sigma^{D,+}(\hat{h}^{D,+}), \\ (C)_F & \varepsilon > \varepsilon_c & \text{and} & \Sigma^F(\bar{h}^F) = \Sigma^F(\hat{h}^F), \\ (C)_{F,+} & \varepsilon > \varepsilon_c^+ & \text{and} & \Sigma^{F,+}(\bar{h}^F) = \Sigma^{F,+}(\hat{h}^{F,+}). \end{array}$$

Note that the second condition in $(C)_D$ (or $(C)_{D,+}$) is equivalent to

$$\sqrt{2\xi}\left(\left|a^{(2)}\right| + \left|b^{(2)}\right|\right) - \xi = \frac{1}{2}\left|a^{(2)} - b^{(2)}\right|^2,$$

while that in $(C)_F$ (or $(C)_{F,+}$) is equivalent to $|a^{(2)}| = \sqrt{\xi/2}$.

We are now in a position to state our main results. We say that the limit under μ_N is h^* if

$$\lim_{N\to\infty} \mu_N(\|h^N - h^*\|_\infty \le \delta) = 1$$

holds for every $\delta > 0$. We say that two functions $\bar{h}$ and $\hat{h}$ coexist in the limit under μ_N with probabilities $\bar{\lambda}$ and $\hat{\lambda}$ if $\bar{\lambda}, \hat{\lambda} > 0$, $\bar{\lambda} + \hat{\lambda} = 1$ and

$$\lim_{N\to\infty} \mu_N(\|h^N - \bar{h}\|_\infty \le \delta) = \bar{\lambda},$$
$$\lim_{N\to\infty} \mu_N(\|h^N - \hat{h}\|_\infty \le \delta) = \hat{\lambda}$$

hold for every $0 < \delta < |a^{(2)}| \wedge |b^{(2)}|$; it is evident from Lemma 1.2 and (1.6) that one has to check these properties only for arbitrary small $\delta > 0$.

Theorem 1.3 (Dirichlet case) 1. (No wall) *Under the condition* $(C)_D$, *the limit under* $\mu_N^{D,\varepsilon}$ *is* $\hat{h}^D$ *if* codim $M = 1$ *and* $\bar{h}^D$ *if* codim $M \ge 3$. *If* codim $M = 2$, $\bar{h}^D$ *and* $\hat{h}^D$ *coexist in the limit under* $\mu_N^{D,\varepsilon}$ *with probabilities* $\bar{\lambda}^{D,\varepsilon}$ *and* $\hat{\lambda}^{D,\varepsilon}$, *respectively, given by* (3.16).

2. (Wall at $\partial\mathbb{R}_+^d$) *Under the condition* $(C)_{D,+}$, *the limit under* $\mu_N^{D,\varepsilon,+}$ *is* $\hat{h}^{D,+}$ *if* codim $M = 1$ *and* $\bar{h}^D$ *if* codim $M \ge 3$. *If* codim $M = 2$, $\bar{h}^D$ *and* $\hat{h}^{D,+}$ *coexist in the limit under* $\mu_N^{D,\varepsilon,+}$ *with probabilities* $\bar{\lambda}^{D,\varepsilon,+}$ *and* $\hat{\lambda}^{D,\varepsilon,+}$, *respectively, given by* (3.21).

Theorem 1.4 (Free case) 1. (No wall) *Under the condition* $(C)_F$, *if* codim $M = 1$, $\bar{h}^F$ *and* $\hat{h}^F$ *coexist in the limit under* $\mu_N^{F,\varepsilon}$ *with probabilities* $\bar{\lambda}^{F,\varepsilon}$ *and* $\hat{\lambda}^{F,\varepsilon}$, *respectively, given by* (3.27). *If* codim $M \ge 2$, *the limit under* $\mu_N^{F,\varepsilon}$ *is* $\bar{h}^F$.

2. (Wall at $\partial\mathbb{R}_+^d$) *Under the condition* $(C)_{F,+}$, *if* codim $M = 1$, $\bar{h}^F$ *and* $\hat{h}^{F,+}$ *coexist in the limit under* $\mu_N^{F,\varepsilon,+}$ *with probabilities* $\bar{\lambda}^{F,\varepsilon,+}$ *and* $\hat{\lambda}^{F,\varepsilon,+}$, *respectively, given by* (3.28). *If* codim $M \ge 2$, *the limit under* $\mu_N^{F,\varepsilon,+}$ *is* $\bar{h}^F$.

The central limit theorem holds for the times when the Markov chains first or last hit M. Set

$$i_\ell = \min\{i \in D_N;\quad \phi_i \in M\},$$
$$i_r = \max\{i \in D_N;\quad \phi_i \in M\},$$

and consider them under a proper scaling:

$$X = \frac{1}{\sqrt{N}}(i_\ell - t_1 N) \quad \text{and} \quad Y = \frac{1}{\sqrt{N}}(i_r - (1-t_2)N),$$

where we set $\min\emptyset = N$ (in the Dirichlet case), $= N+1$ (in the free case), $\max\emptyset = 0$, and Y is considered only for the Dirichlet case.

Theorem 1.5 1. (Dirichlet case) *Under* $\mu_N^{D,\varepsilon}$ *or* $\mu_N^{D,\varepsilon,+}$, *conditioned on the event* $\{i_\ell \le N-1\}$ *if* codim $M \ge 2$, *the pair of random variables* (X, Y) *weakly converges to* (U_1, U_2) *as* $N \to \infty$, *where* $U_1 = N(0, |a^{(2)}|/(2\xi)^{3/2})$ *and* $U_2 = N(0, |b^{(2)}|/(2\xi)^{3/2})$ (*with* $\xi = \xi^\varepsilon$ *or* $\xi^{\varepsilon,+}$) *are mutually independent centered Gaussian random variables.*

2. (Free case) *Under* $\mu_N^{F,\varepsilon}$ *or* $\mu_N^{F,\varepsilon,+}$ *conditioned on the event* $\{i_\ell \le N\}$, X *weakly converges to* $U = N(0, |a^{(2)}|/(2\xi)^{3/2})$ *as* $N \to \infty$ (*with* $\xi = \xi^\varepsilon$ *or* $\xi^{\varepsilon,+}$).

The conditioning on the event $\{i_\ell \le N-1\}$ is unnecessary when codim $M = 1$, since the probability of such event converges to one as $N \to \infty$ in this case.

The proof of Theorems 1.3 and 1.4, together with Theorem 1.5, will be given in Sect. 3. The conditions $(C)_D$–$(C)_{F,+}$ guarantee that the leading exponential decay rates of the probabilities of the neighborhoods of the two different minimizers balance with each other. This enforces us to study their precise asymptotics, which can be obtained as an application of the renewal theory and discussed in Sect. 2. The proof of Theorem 1.1 is also given in Sect. 2. Section 4 is for the sample path large deviation principles. In Appendix A we study the critical exponents for the free energies, while in Appendix B we clarify the structure of the set of minimizers of Σ when $d = 1$ and $m = 0$. It is straightforward to generalize our results to $H_N(\phi)$ of the form

$$H_N(\phi) = \frac{1}{2}\sum_{i=0}^{N-1}(\phi_{i+1}-\phi_i)\cdot A(\phi_{i+1}-\phi_i)$$

with a positive symmetric $d \times d$ matrix A if M is an eigensubspace of A.

A dichotomy in concentrations on $\bar{h}$ or $\hat{h}$ is shown in the Dirichlet case for a model with the Hamiltonians perturbed by weak self potentials, see [9]. The scaling limits for the two-dimensional model (more precisely, a model with two-dimensional time parameters) under the volume conservation law are studied by [1]. Some related results are obtained by [17,18] for the one-dimensional discrete SOS model and the two-dimensional Ising model, respectively. The corresponding fluctuation limits are studied by [5] for general interaction potential and by [3,15] for a discrete model under the Dirichlet condition at $t = 0$ with $a = 0$.

2 Precise asymptotics for the partition functions

In this section, we will prove a number of results on the precise asymptotic behavior of the ratios of partition functions associated with the Gaussian random walks in $\mathbb{R}^r$ with pinning at $0 \in \mathbb{R}^r$ and starting at $0 \in \mathbb{R}^r$ (and reaching 0 in the Dirichlet case), which were mentioned in Sect. 1.2 to determine ξ's. In particular, these will imply the statements in Theorem 1.1. A similar method is used in [3,14]. We will omit the subscript r of the partition functions, for example, $Z_{N,r}^{0,0,\varepsilon}$ is simply denoted by $Z_N^{0,0,\varepsilon}$ in this section.

2.1 Dirichlet case without wall

We denote $D_N^\circ := D_N \setminus \{0, N\} (= \{1, 2, \ldots, N-1\})$. The partition function $Z_N^{0,0,\varepsilon}$ is given by

$$Z_N^{0,0,\varepsilon} = \int_{(\mathbb{R}^r)^{N+1}} e^{-H_N(\phi)}\delta_0(d\phi_0)\prod_{i\in D_N^\circ}(\varepsilon\delta_0(d\phi_i)+d\phi_i)\,\delta_0(d\phi_N), \tag{2.1}$$

and $Z_N^{0,0} = Z_N^{0,0,0}$, i.e., $\varepsilon = 0$. An explicit calculation shows that $Z_N^{0,0} = (2\pi)^{rN/2}/(2\pi N)^{r/2}$.

Lemma 2.1 *The renewal equation holds for* $Z_N^{0,0,\varepsilon}$, $N \ge 2$ *with* $Z_1^{0,0,\varepsilon} = Z_1^{0,0} = 1$:

$$Z_N^{0,0,\varepsilon} = Z_N^{0,0} + \varepsilon \sum_{i \in D_N^\circ} Z_i^{0,0} Z_{N-i}^{0,0,\varepsilon}.$$

Proof Expand the product measure in (2.1) by specifying i_ℓ as

$$\begin{aligned}&\prod_{i \in D_N^\circ} (\varepsilon \delta_0(d\phi_i) + d\phi_i) \\ &\quad = \prod_{j \in D_N^\circ} d\phi_j + \sum_{i \in D_N^\circ} \prod_{j \in D_{N,i,-}^\circ} d\phi_j \cdot \varepsilon \delta_0(d\phi_i) \cdot \prod_{j \in D_{N,i,+}^\circ} \left(\varepsilon \delta_0(d\phi_j) + d\phi_j\right),\end{aligned}$$

where $D_{N,i,-}^\circ = \{1, 2, \dots, i-1\}$ and $D_{N,i,+}^\circ = \{i+1, i+2, \dots, N-1\}$. The i on the right hand side represents the first i such that the factor $\varepsilon\delta_0(d\phi_i)$ appears in the expansion, i.e., $i = i_\ell$. If such i does not exist, we have the measure $\prod_{j \in D_N^\circ} d\phi_j$. This expansion immediately leads to the conclusion. □

Let us define the function

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{(2\pi n)^{r/2}}, \quad 0 \le x < 1. \tag{2.2}$$

Note that g is increasing, $g(0) = 0$, $g(1)\ (:= g(1-)) < \infty$ if $r \ge 3$ and $g(1-) = \infty$ if $r = 1, 2$. Set

$$\varepsilon_c = \begin{cases} 1/g(1) > 0, & r \ge 3, \\ 0, & r = 1, 2. \end{cases} \tag{2.3}$$

For each $\varepsilon > \varepsilon_c$, we determine $x = x^\varepsilon \in (0, 1)$ as the unique solution of $g(x) = 1/\varepsilon$ and introduce two positive constants:

$$\xi^\varepsilon = -\log x^\varepsilon \quad \text{and} \quad C^{D,\varepsilon} = \frac{(2\pi)^{r/2}}{\varepsilon^2 x^\varepsilon g'(x^\varepsilon)}. \tag{2.4}$$

Proposition 2.2 *For each* $\varepsilon > \varepsilon_c$, *we have the precise asymptotics for the ratio of two partition functions*:

$$\frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0}} \sim C^{D,\varepsilon} N^{r/2} e^{N\xi^\varepsilon},$$

as $N \to \infty$, *where* $\sim$ *means that the ratio of both sides tends to* 1.

Proof We set $u_0 = a_0 = b_0 = 0$ and, for $n = 1, 2, \ldots$, $u_n = x^n Z_n^{0,0,\varepsilon}/(2\pi)^{rn/2}$, $a_n = \varepsilon x^n Z_n^{0,0}/(2\pi)^{rn/2}$ and $b_n = x^n Z_n^{0,0}/(2\pi)^{rn/2} = x^n/(2\pi n)^{r/2}$, where $x = x^\varepsilon$. Then, Lemma 2.1 shows that

$$u_n = b_n + \sum_{i=0}^{n} a_i u_{n-i} \tag{2.5}$$

for every $n \geq 0$. However, the definition of $x = x^\varepsilon$ implies that

$$\sum_{n=0}^{\infty} a_n = \varepsilon \sum_{n=1}^{\infty} \frac{x^n}{(2\pi n)^{r/2}} = 1.$$

Thus, an application of the renewal theory (cf. Chapter XIII of [6]) shows that $\lim_{n\to\infty} u_n = B/A$, where

$$B = \sum_{n=0}^{\infty} b_n = g(x) = 1/\varepsilon,$$

and

$$A = \sum_{n=0}^{\infty} n a_n = \varepsilon \sum_{n=1}^{\infty} \frac{n x^n}{(2\pi n)^{r/2}} = \varepsilon x g'(x).$$

We therefore obtain

$$\lim_{n\to\infty} \frac{x^n}{(2\pi)^{rn/2}} Z_n^{0,0,\varepsilon} = \frac{1}{\varepsilon^2 x g'(x)}.$$

Finally, using $Z_N^{0,0} = (2\pi)^{rN/2}/(2\pi N)^{r/2}$ again, the conclusion is shown by

$$\frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0}} \sim \frac{(2\pi N)^{r/2}}{\varepsilon^2 x^\varepsilon g'(x^\varepsilon)} (x^\varepsilon)^{-N} = C^{D,\varepsilon} N^{r/2} e^{N\xi^\varepsilon}.$$

□

2.2 Dirichlet case with wall

We recall that

$$Z_N^{0,0,+} = \int_{(\mathbb{R}_+^r)^{N+1}} e^{-H_N(\phi)} \delta_0(d\phi_0) \prod_{i\in D_N^\circ} d\phi_i^+ \delta_0(d\phi_N), \tag{2.6}$$

where $d\phi_i^+$ is the Lebesgue measure on $\mathbb{R}_+^r = \mathbb{R}^{r-1} \times \mathbb{R}_+$, and this leads to a representation of the ratio of two partition functions:

$$q_N := \frac{Z_N^{0,0,+}}{Z_N^{0,0}} = P_{0,0}^{0,1}\left(B(i/N) \geq 0 \quad \text{for all } 1 \leq i \leq N-1\right), \tag{2.7}$$

by means of a one-dimensional Brownian bridge $\{B(t), t \in [0,1]\}$ satisfying $B(0) = B(1) = 0$ under $P_{0,0}^{0,1}$. It is known (see (20) in [5]) that q_N is given by

$$q_N = \frac{1}{N}. \tag{2.8}$$

The partition function $Z_N^{0,0,\varepsilon,+}$, given by (2.1) with $(\mathbb{R}^r)^{N+1}$ replaced by $(\mathbb{R}_+^r)^{N+1}$, satisfies the renewal equation:

$$Z_N^{0,0,\varepsilon,+} = Z_N^{0,0,+} + \varepsilon \sum_{i \in D_N^\circ} Z_i^{0,0,+} Z_{N-i}^{0,0,\varepsilon,+}, \tag{2.9}$$

for $N \geq 2$ with $Z_1^{0,0,\varepsilon,+} = Z_1^{0,0,+} = 1$. The proof of (2.9) is similar to Lemma 2.1. With the function

$$g^+(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(2\pi n)^{r/2}}, \quad 0 \leq x \leq 1, \tag{2.10}$$

noting that $g^+(1) < \infty$ for all $r \geq 1$, we define

$$\varepsilon_c^+ = 1/g^+(1) > 0. \tag{2.11}$$

We then determine, for each $\varepsilon > \varepsilon_c^+$, $x = x^{\varepsilon,+} \in (0,1)$ as the unique solution of $g^+(x) = 1/\varepsilon$ and introduce two positive constants:

$$\xi^{\varepsilon,+} = -\log x^{\varepsilon,+} \quad \text{and} \quad C^{D,\varepsilon,+} = \frac{(2\pi)^{r/2}}{\varepsilon^2 g(x^{\varepsilon,+})}. \tag{2.12}$$

Proposition 2.3 *We have the precise asymptotics*

$$\frac{Z_N^{0,0,\varepsilon,+}}{Z_N^{0,0,+}} \sim C^{D,\varepsilon,+} N^{1+r/2} e^{N\xi^{\varepsilon,+}}$$

as $N \to \infty$ *for each* $\varepsilon > \varepsilon_c^+$.

Proof Define three sequences u_n, a_n and b_n as in the proof of Proposition 2.2 with x, $Z_n^{0,0,\varepsilon}$ and $Z_n^{0,0}$ replaced by $x^{\varepsilon,+}$, $Z_n^{0,0,\varepsilon,+}$ and $Z_n^{0,0,+}$, respectively. Then, we have the relation (2.5) from (2.9) and also $\sum_{n=0}^{\infty} a_n = 1$ from $Z_n^{0,0,+} = Z_n^{0,0}/n$, recall (2.7) and (2.8). Thus, relying on the renewal theory again (and noting $x(g^+)'(x) = g(x)$), one obtains

$$Z_N^{0,0,\varepsilon,+} \sim \frac{(2\pi)^{rN/2}}{x^N} \frac{1}{\varepsilon^2 g(x)},$$

as $N \to \infty$. Since $Z_N^{0,0,+} = (2\pi)^{rN/2}/N(2\pi N)^{r/2}$, the conclusion is shown as

$$\frac{Z_N^{0,0,\varepsilon,+}}{Z_N^{0,0,+}} \sim \frac{N(2\pi N)^{r/2}}{\varepsilon^2 g(x^{\varepsilon,+})} \left(x^{\varepsilon,+}\right)^{-N} = C^{D,\varepsilon,+} N^{1+r/2} e^{N\xi^{\varepsilon,+}}.$$

□

Remark 2.1 1. Comparing (2.3), (2.11) with $g(1) > g^+(1)$, we see $0 \le \varepsilon_c < \varepsilon_c^+$. Set $x^{\varepsilon,(+)} = 1$ and $\xi^{\varepsilon,(+)} = 0$ for $0 \le \varepsilon \le \varepsilon_c^{(+)}$. Then, since $g(x) > g^+(x)$ for $0 < x < 1$, we have $x^\varepsilon < x^{\varepsilon,+}$ and therefore $\xi^{\varepsilon,+} < \xi^\varepsilon$ for every $\varepsilon > \varepsilon_c$. Indeed, $\xi^{\varepsilon,(+)}$ defined through the thermodynamic limit in Sect. 1.2 is equal to 0 for every $0 \le \varepsilon \le \varepsilon_c^{(+)}$.
2. Propositions 2.2 and 2.3 combined with (2.7), (2.8) imply that

$$\mu_N^{0,0,\varepsilon}(\phi_i \in \mathbb{R}_+^r \text{ for all } i \in D_N) = \frac{Z_N^{0,0,\varepsilon,+}}{Z_N^{0,0,\varepsilon}} \sim \frac{C^{D,\varepsilon,+}}{C^{D,\varepsilon}} e^{-N(\xi^\varepsilon - \xi^{\varepsilon,+})}$$

as $N \to \infty$, if $\varepsilon > \varepsilon_c^+$.

2.3 Free case without wall

We now move to the case with the free condition at $t = 1$ (or microscopically at $i = N$), and denote $D_N^{\circ,F} := D_N \setminus \{0\} (= \{1, 2, \dots, N\})$. The partition function $Z_N^{0,F,\varepsilon}$ is given by

$$Z_N^{0,F,\varepsilon} = \int_{(\mathbb{R}^r)^{N+1}} e^{-H_N(\phi)} \delta_0(d\phi_0) \prod_{i \in D_N^{\circ,F}} (\varepsilon\delta_0(d\phi_i) + d\phi_i), \tag{2.13}$$

and we have $Z_N^{0,F} (= Z_N^{0,F,0}) = (2\pi)^{rN/2}$.

Lemma 2.4 *The renewal equation holds for* $Z_N^{0,F,\varepsilon}$, $N \ge 1$ *with* $Z_0^{0,F,\varepsilon} = 1$:

$$Z_N^{0,F,\varepsilon} = Z_N^{0,F} + \varepsilon \sum_{i \in D_N^{\circ,F}} Z_i^{0,0} Z_{N-i}^{0,F,\varepsilon}.$$

Proof The proof is concluded, similarly to Lemma 2.1, by expanding the product measure in (2.13) as

$$\prod_{i\in D_N^{\circ,F}} (\varepsilon\delta_0(d\phi_i)+d\phi_i)$$
$$=\prod_{j\in D_N^{\circ,F}} d\phi_j + \sum_{i\in D_N^{\circ,F}} \prod_{j\in D_{N,i,-}^{\circ}} d\phi_j\cdot\varepsilon\delta_0(d\phi_i)\cdot \prod_{j\in D_{N,i,+}^{\circ,F}} \big(\varepsilon\delta_0(d\phi_j)+d\phi_j\big),$$

where $D_{N,i,+}^{\circ,F}=\{i+1,i+2,\dots,N\}$. □

Recall the function g defined by (2.2), the unique solution $x=x^\varepsilon\in(0,1)$ of $g(x)=1/\varepsilon$ and $\xi^\varepsilon=-\log x^\varepsilon>0$ in (2.4) for each $\varepsilon>\varepsilon_c$. We then define a positive constant:

$$C^{F,\varepsilon}=\frac{1}{\varepsilon x^\varepsilon(1-x^\varepsilon)g'(x^\varepsilon)}. \tag{2.14}$$

Proposition 2.5 *We have the precise asymptotics*

$$\frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F}}\sim C^{F,\varepsilon}e^{N\xi^\varepsilon}$$

as $N\to\infty$ for each $\varepsilon>\varepsilon_c$.

Proof We set $u_0=b_0=1$, $a_0=0$ and, for $n=1,2,\dots$, $u_n=x^nZ_n^{0,F,\varepsilon}/(2\pi)^{rn/2}$, $a_n=\varepsilon x^nZ_n^{0,0}/(2\pi)^{rn/2}$ and $b_n=x^nZ_n^{0,F}/(2\pi)^{rn/2}=x^n$, where $x=x^\varepsilon$. Then, Lemma 2.4 shows that (2.5) holds for every $n\ge 0$. However, the definition of $x=x^\varepsilon$ implies that $\sum_{n=0}^\infty a_n=1$. Thus, an application of the renewal theory shows that $\lim_{n\to\infty}u_n=B/A$, where

$$B=\sum_{n=0}^\infty b_n=\sum_{n=0}^\infty x^n=\frac{1}{1-x},$$

and

$$A=\sum_{n=0}^\infty na_n=\varepsilon xg'(x).$$

We therefore obtain

$$\lim_{n\to\infty}\frac{x^n}{(2\pi)^{rn/2}}Z_n^{0,F,\varepsilon}=\frac{1}{\varepsilon x(1-x)g'(x)}.$$

Finally, using $Z_N^{0,F}=(2\pi)^{rN/2}$ again, the conclusion is shown by

$$\frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F}}\sim\frac{1}{\varepsilon x^\varepsilon(1-x^\varepsilon)g'(x^\varepsilon)}(x^\varepsilon)^{-N}=C^{F,\varepsilon}e^{N\xi^\varepsilon}.$$

□

2.4 Free case with wall

The partition function $Z_N^{0,F,\varepsilon,+}$, given by (2.13) with $(\mathbb{R}^r)^{N+1}$ replaced by $(\mathbb{R}^r_+)^{N+1}$, satisfies the renewal equation:

$$Z_N^{0,F,\varepsilon,+} = Z_N^{0,F,+} + \varepsilon \sum_{i\in D_N^{\circ,F}} Z_i^{0,0,+} Z_{N-i}^{0,F,\varepsilon,+}, \tag{2.15}$$

for $N \geq 1$ with $Z_0^{0,F,\varepsilon,+} = 1$. The proof of (2.15) is similar to Lemma 2.4. Recall the function $g^+(x)$ defined by (2.10), the unique solution $x = x^{\varepsilon,+} \in (0,1)$ of $g^+(x) = 1/\varepsilon$ and $\xi^{\varepsilon,+} = -\log x^{\varepsilon,+} > 0$ in (2.12) for $\varepsilon > \varepsilon_c^+$. We then define a positive constant:

$$C^{F,\varepsilon,+} = \frac{y^{\varepsilon,+}}{\varepsilon C^{F,+} g(x^{\varepsilon,+})}, \tag{2.16}$$

where $y^{\varepsilon,+} = \sum_{N=0}^\infty q_N^F (x^{\varepsilon,+})^N$, and q_N^F and $C^{F,+}$ are determined by $q_0^F = 1$ and

$$q_N^F := \frac{Z_N^{0,F,+}}{Z_N^{0,F}} = P_0\left(B(i/N) \geq 0 \quad \text{for all } 1 \leq i \leq N\right) \sim C^{F,+}/\sqrt{N}, \tag{2.17}$$

for $N \geq 1$ with a one-dimensional standard Brownian motion $\{B(t), t \in [0,1]\}$ satisfying $B(0) = 0$ under P_0. See (16) in [5] for the asymptotic behavior of q_N^F in (2.17) as $N \to \infty$.

Proposition 2.6 *We have the precise asymptotics*

$$\frac{Z_N^{0,F,\varepsilon,+}}{Z_N^{0,F,+}} \sim C^{F,\varepsilon,+} N^{1/2} e^{N\xi^{\varepsilon,+}}$$

as $N \to \infty$ *for each* $\varepsilon > \varepsilon_c^+$.

Proof We set $u_0 = b_0 = 1$, $a_0 = 0$ and, for $n = 1, 2, \ldots, u_n = x^n Z_n^{0,F,\varepsilon,+}/(2\pi)^{rn/2}$, $a_n = \varepsilon x^n Z_n^{0,0,+}/(2\pi)^{rn/2}$ and $b_n = x^n Z_n^{0,F,+}/(2\pi)^{rn/2}$, where $x = x^{\varepsilon,+}$. Then, (2.15) shows that (2.5) holds for every $n \geq 0$. However, the definition of $x = x^{\varepsilon,+}$ implies that $\sum_{n=0}^\infty a_n = 1$. Thus, relying on the renewal theory, one obtains that

$$Z_N^{0,F,\varepsilon,+} \sim \frac{(2\pi)^{rN/2}}{x^N} \frac{y^{\varepsilon,+}}{\varepsilon g(x)}.$$

Since $Z_N^{0,F,+} \sim C^{F,+} N^{-1/2} Z_N^{0,F} = C^{F,+} N^{-1/2} (2\pi)^{rN/2}$, we have

$$\frac{Z_N^{0,F,\varepsilon,+}}{Z_N^{0,F,+}} \sim \frac{y^{\varepsilon,+}}{\varepsilon C^{F,+} g(x^{\varepsilon,+})} N^{1/2} (x^{\varepsilon,+})^{-N} = C^{F,\varepsilon,+} N^{1/2} e^{N\xi^{\varepsilon,+}}.$$

□

3 Proof of Theorems 1.3, 1.4 and 1.5

We assume the conditions $(C)_D$–$(C)_{F,+}$ in this section and give the proof of Theorems 1.3 and 1.4 together with Theorem 1.5. Our first immediate observation is that, under the coordinates introduced in (1.7), the two components $\phi^{(1)} = (\phi_i^{(1)})_{i\in D_N} \in M^{N+1}$ and $\phi^{(2)} = (\phi_i^{(2)})_{i\in D_N} \in (M^{\perp})^{N+1}$ of the Markov chain $\phi = (\phi_i = (\phi_i^{(1)}, \phi_i^{(2)}))_{i\in D_N} \in (\mathbb{R}^d)^{N+1}$ are independent. In fact, for instance in the Dirichlet case without wall, the distribution of ϕ and its normalizing constant are decomposed into the products:

$$\mu_N^{a,b,\varepsilon} = \mu_{N,m}^{a^{(1)},b^{(1)},0} \times \mu_{N,r}^{a^{(2)},b^{(2)},\varepsilon} \quad \text{and} \quad Z_N^{a,b,\varepsilon} = Z_{N,m}^{a^{(1)},b^{(1)},0} \times Z_{N,r}^{a^{(2)},b^{(2)},\varepsilon}. \tag{3.1}$$

The subscripts m and r indicate that the objects are defined for $\mathbb{R}^m$ and $\mathbb{R}^r$, respectively.

We may assume without loss of generality $d = r$ and $m = 0$. To see this, we choose the norm $\|h\|_\infty = \max_{t\in D} |h(t)|$ for $h \in \mathcal{C}\ (= C([0,1],\mathbb{R}^d))$ with $|h(t)| = \max\{|h^{(1)}(t)|, |h^{(2)}(t)|\}$ for $h(t) = (h^{(1)}(t), h^{(2)}(t)) \in \mathbb{R}^m \times \mathbb{R}^r$, which is equivalent to the Euclidean norm of $h(t)$ in $\mathbb{R}^d$. As we are only concerned with the ratio of probabilities of neighborhoods of $\hat{h}$ and $\bar{h}$, the factor coming from the first component $\phi^{(1)}$ cancels. Thus, the proof can be reduced to the Markov chains on $\mathbb{R}^r$ (or $\mathbb{R}^r_+$) with pinning at $M' = \{0\}$. We will omit the subscript r: $\mu_{N,r}^{D,\varepsilon}$ and $Z_{N,r}^{D,\varepsilon}$ are simply denoted by $\mu_N^{D,\varepsilon}$ and $Z_N^{D,\varepsilon}$, respectively, and $a^{(2)}, b^{(2)}, \bar{h}^{(2)}, \hat{h}^{(2)}$ are denoted by $a, b, \bar{h}, \hat{h}$ and others.

3.1 Proof of Theorems 1.3-(1) and 1.5 for $\mu_N^{D,\varepsilon}$

If $0 \le j < k \le N$, we write $\mu_{j,k}^{a,b}$ for the measure on $(\mathbb{R}^r)^{\{j,\dots,k\}} = \{\phi = (\phi_i)_{j\le i\le k};\ \phi_i \in \mathbb{R}^r\}$ without pinning, under the Dirichlet conditions $\phi_j = aN$ and $\phi_k = bN$:

$$\mu_{j,k}^{a,b}(d\phi) = \frac{1}{Z_{j,k}^{a,b}} e^{-H_{j,k}(\phi)} \delta_{aN}(d\phi_j) \prod_{i=j+1}^{k-1} d\phi_i\, \delta_{bN}(d\phi_k), \tag{3.2}$$

where $Z_{j,k}^{a,b} = Z_{k-j}^{a,b}$ is the normalizing constant and $H_{j,k}(\phi) := \frac{1}{2}\sum_{i=j}^{k-1} |\phi_{i+1} - \phi_i|^2$. The corresponding measure with pinning is denoted by $\mu_{j,k}^{a,b,\varepsilon}(d\phi)$. Clearly

$$\begin{aligned} Z_n^{a,b} &= e^{-N^2|a-b|^2/2n} Z_n^{0,0}, \\ Z_n^{0,0} &= \frac{(2\pi)^{rn/2}}{(2\pi n)^{r/2}}. \end{aligned} \tag{3.3}$$

Under the measure $\mu_{j,k}^{a,b}$, the macroscopic path determined from $(\phi_i)_{j\le i\le k}$ concentrates on the straight line between $(j/N, a)$ and $(k/N, b)$:

$$g^{a,b}_{[j/N,k/N]}(t) := \left(1 - \frac{Nt-j}{k-j}\right) a + \frac{Nt-j}{k-j} b, \quad \frac{j}{N} \le t \le \frac{k}{N},$$

in particular, $g^{a,b}_{[0,1]} = \overline{h}$. More precisely

Lemma 3.1 *For any $\delta' > 0$, there exists $c\left(\delta'\right) > 0$ and $N_0\left(\delta'\right) \in \mathbb{N}$ such that for any $a, b \in \mathbb{R}^r$, $0 \le j < k \le N$:*

$$\mu^{a,b}_{j,k}\left(\left\{\phi;\ \max_{i:j\le i\le k}\left|\frac{\phi_i}{N} - g^{a,b}_{[j/N,k/N]}\left(\frac{i}{N}\right)\right| \ge \delta'\right\}\right) \le e^{-c(\delta')N}$$

for $N \ge N_0\left(\delta'\right)$.

Proof This is straightforward from the fact that for any i with $j \le i \le k$, ϕ_i is normally distributed under $\mu^{a,b}_{j,k}$ with mean $(1-(i-j)/(k-j))\,Na + ((i-j)/(k-j))\,Nb$, and standard deviation bounded by const $\times\sqrt{N}$. □

We write

$$\gamma^{a,b}_{j,k}(\delta) := \mu^{a,b}_{j,k}\left(\left\|h^N_{[j/N,k/N]} - \hat{h}_{[j/N,k/N]}\right\|_\infty \le \delta\right)$$

where $\hat{h} = \hat{h}^{D,(2)}$ in this subsection, and $f_{[u,v]}$ is the restriction of a function f: $[0,1] \to \mathbb{R}^d$ to the subinterval $[u,v]$ of $[0,1]$. Also $\gamma^{a,b,\varepsilon}_{j,k}(\delta)$ is the similarly defined quantity with pinning. We sometimes also write $U_\delta(\hat{h}_{[u,v]})$ for the δ-neighborhood with respect to $\|\cdot\|_\infty$ in the space of functions on $[u,v]$ of $\hat{h}_{[u,v]}$; when the subscript $[u,v]$ is dropped, it is considered on $[0,1]$. We similarly write $U_\delta(\bar{h})$ for $\bar{h} = \bar{h}^{D,(2)}$.

We remind the reader that it suffices to evaluate

$$\lim_{N\to\infty} \frac{\mu^{D,\varepsilon}_N\left(h^N \in U_\delta\left(\hat{h}\right)\right)}{\mu^{D,\varepsilon}_N\left(h^N \in U_\delta\left(\overline{h}\right)\right)}$$

for arbitrarily small $\delta > 0$.

An expansion of the product measure $\prod_{i\in D_N^\circ}(\varepsilon\delta_0(d\phi_i) + d\phi_i)$ in (1.1) by specifying $0 < i_\ell \le i_r < N$ gives rise to

$$\begin{aligned} p_N &:= \frac{Z^{D,\varepsilon}_N}{Z^{a,b}_N}\mu^{D,\varepsilon}_N\left(h^N \in U_\delta\left(\hat{h}\right)\right) \\ &= \gamma^{a,b}_{0,N}(\delta) + \sum_{j=1}^{N-1}\varepsilon\,\Xi^\varepsilon_{N,j,j}\gamma^{a,0}_{0,j}(\delta)\,\gamma^{0,b}_{j,N}(\delta) \\ &\quad + \sum_{0<j<k<N}\varepsilon^2\,\Xi^\varepsilon_{N,j,k}\gamma^{a,0}_{0,j}(\delta)\,\gamma^{0,0,\varepsilon}_{j,k}(\delta)\,\gamma^{0,b}_{k,N}(\delta) \\ &=: I^1_N + I^2_N + I^{3,}_N \end{aligned} \tag{3.4}$$

where

$$\Xi^{\varepsilon}_{N,j,k} = \frac{Z_j^{a,0} Z_{k-j}^{0,0,\varepsilon} Z_{N-k}^{0,b}}{Z_N^{a,b}} \tag{3.5}$$

for $0 < j \le k < N$. We set $Z_0^{0,0,\varepsilon} = 1$ to define $\Xi^{\varepsilon}_{N,j,j}$.

The first term I_N^1 covers all paths without touching 0: $i_\ell = N$, $i_r = 0$ and I_N^2 is for those touching 0 once: $0 < i_\ell = i_r (= j) < N$, while I_N^3 is for those touching 0 at least twice: $0 < i_\ell (= j) < i_r (= k) < N$.

If δ is chosen small enough, then $U_\delta(\hat{h}) \cap U_\delta(g^{a,b}_{[0,1]}) = \emptyset$. Using Lemma 3.1, it follows that I_N^1 is exponentially small in N. Similarly, noting that $\Xi^{\varepsilon}_{N,j,j}$ is bounded in N, for I_N^2 one has that either $U_\delta(\hat{h}_{[0,j/N]}) \cap U_\delta(g^{a,0}_{[0,j/N]}) = \emptyset$ or $U_\delta(\hat{h}_{[j/N,1]}) \cap U_\delta(g^{0,b}_{[j/N,1]}) = \emptyset$ and it follows that I_N^2 is exponentially small, i.e., we have

$$I_N^1 + I_N^2 \le e^{-cN} \tag{3.6}$$

for N sufficiently large, where $c > 0$.

By (3.3), the ratio of the partition functions in (3.5) can be rewritten for $j < k$ as

$$\Xi^{\varepsilon}_{N,j,k} = \alpha_{N,j,k} e^{-N\tilde{f}(s_1,s_2)} \frac{Z_{k-j}^{0,0,\varepsilon}}{Z_{k-j}^{0,0}} \tag{3.7}$$

where $s_1 = j/N$, $s_2 = (N-k)/N$,

$$\tilde{f}(s_1, s_2) := \frac{1}{2}\left(\frac{|a|^2}{s_1} + \frac{|b|^2}{s_2} - |a-b|^2\right), \tag{3.8}$$

and

$$\alpha_{N,j,k} = \left[\frac{N}{(2\pi)^2 j(k-j)(N-k)}\right]^{r/2}.$$

In the part I_N^3, we decompose the j-k-summation into the part over

$$A := \left\{(j,k)\,;\, |j - Nt_1| \le N^{3/5}, \quad |k - N(1-t_2)| \le N^{3/5}\right\}, \tag{3.9}$$

and over its complement. We always assume that N is large enough so that $Nt_1 + N^{3/5} < N(1-t_2) - N^{3/5}$. Using Proposition 2.2, we get

$$\begin{aligned}
&\sum_{(j,k)\notin A} \Xi^{\varepsilon}_{N,j,k} \gamma^{a,0}_{0,j}(\delta)\, \gamma^{0,0,\varepsilon}_{j,k}(\delta)\, \gamma^{0,b}_{k,N}(\delta) \\
&\quad \le \sum_{(j,k)\notin A} \Xi^{\varepsilon}_{N,j,k} \\
&\quad \le C \sum_{(j,k)\notin A} \alpha_{N,j,k} e^{-N\tilde{f}(s_1,s_2)} (k-j)^{r/2} e^{(k-j)\xi} \\
&\quad = C \sum_{(j,k)\notin A} \alpha_{N,j,k} (k-j)^{r/2} e^{-Nf(s_1,s_2)},
\end{aligned}$$

for some $C > 0$, where $\xi = \xi^{\varepsilon}$ and

$$\begin{aligned}
f(s_1,s_2) &= \tilde{f}(s_1,s_2) - \xi(1-s_1-s_2) \\
&= \frac{|a|^2}{2t_1^2 s_1}(s_1-t_1)^2 + \frac{|b|^2}{2t_2^2 s_2}(s_2-t_2)^2. \qquad (3.10)
\end{aligned}$$

In the second equation, we have used $|a-b|^2/2 = (|a|^2/t_1 + |b|^2/t_2)/2 - \xi(1-t_1-t_2)$ and $|a|/t_1 = |b|/t_2 = \sqrt{2\xi}$ from Condition $(C)_D$. On the complement A^c, we have

$$Nf(s_1,s_2) \ge CN^{1/5},$$

with some $C > 0$ and therefore

$$\sum_{(j,k)\notin A} \Xi^{\varepsilon}_{N,j,k} \le e^{-cN^{1/5}} \qquad (3.11)$$

for some $c > 0$, and large enough N.

For $(j,k) \in A$, we can expand $f(s_1,s_2)$:

$$f(s_1,s_2) = \frac{|a|^2}{2t_1^3}(s_1-t_1)^2 + \frac{|b|^2}{2t_2^3}(s_2-t_2)^2 + O\left(N^{-6/5}\right).$$

Furthermore, the straight lines $g^{a,0}_{[0,s_1]}$ and $g^{0,b}_{[1-s_2,1]}$ are within distance $\delta/2$ to the restrictions of $\hat{h}_{[0,s_1]}$ and $\hat{h}_{[1-s_2,1]}$, respectively, if N is large enough, and therefore, using Lemma 3.1 and Theorem 4.1 below (in fact, Proposition 4.3 is sufficient), we get

$$\begin{aligned}
\sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}\left(1-e^{-cN}\right) &\le \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k} \gamma^{a,0}_{0,j}(\delta)\, \gamma^{0,0,\varepsilon}_{j,k}(\delta)\, \gamma^{0,b}_{k,N}(\delta) \\
&\le \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}, \qquad (3.12)
\end{aligned}$$

for some $c > 0$. It therefore suffices to estimate $\sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}$. By using Proposition 2.2 and substituting $j - [Nt_1]$ and $k - [N(1-t_2)]$ into j and k, we have by a Riemann sum approximation

$$\varepsilon^2 \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k} \sim C_1 N^{-r/2} \sum_{|j|\le N^{3/5}} e^{-c_1(j/\sqrt{N})^2} \sum_{|k|\le N^{3/5}} e^{-c_2(k/\sqrt{N})^2}$$
$$\sim C_1 N^{1-r/2} \int_{-\infty}^{\infty} e^{-c_1 x^2}\, dx \int_{-\infty}^{\infty} e^{-c_2 x^2}\, dx$$
$$= \frac{C_1 \pi}{\sqrt{c_1 c_2}} N^{1-r/2}, \tag{3.13}$$

as $N \to \infty$, with $C_1 = \varepsilon^2 C^{D,\varepsilon}/(2\pi)^r (t_1 t_2)^{r/2}$ and $c_1 = |a|^2/2t_1^3 = (2\xi)^{3/2}/2|a|$, $c_2 = |b|^2/2t_2^3 = (2\xi)^{3/2}/2|b|$.

Summarizing, we get from (3.4), (3.6), (3.11) and (3.13), and for sufficiently large N

$$p_N = \frac{C_1\pi}{\sqrt{c_1c_2}} N^{1-r/2}\left(1 - O\left(e^{-cN}\right)\right) + O\left(e^{-cN^{1/5}}\right) + O\left(e^{-cN}\right)$$
$$\sim \frac{C_1\pi}{\sqrt{c_1c_2}} N^{1-r/2}. \tag{3.14}$$

On the other hand, the definition (1.1) of $\mu_N^{D,\varepsilon}$ implies for every $0 < \delta < |a| \wedge |b|$ that

$$\frac{Z_N^{D,\varepsilon}}{Z_N^{a,b}} \mu_N^{D,\varepsilon}\left(h^N \in U_\delta\left(\bar{h}\right)\right) = \mu_N^{D,0}\left(h^N \in U_\delta\left(\bar{h}\right)\right) \sim 1,$$

where $\bar{h} = \bar{h}^{D,(2)}$. Comparing with (3.14), we have the conclusion of Theorem 1.3-(1) by recalling that (1.6) implies

$$\lim_{N\to\infty} \left\{\mu_N^{D,\varepsilon}\left(h^N \in U_\delta\left(\hat{h}\right)\right) + \mu_N^{D,\varepsilon}\left(h^N \in U_\delta\left(\bar{h}\right)\right)\right\} = 1. \tag{3.15}$$

In particular, if $r = 2$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{D,\varepsilon}, \hat{\lambda}^{D,\varepsilon}) := \left(\frac{1}{1+C_2}, \frac{C_2}{1+C_2}\right), \tag{3.16}$$

where $C_2 = \varepsilon^2 C^{D,\varepsilon}/\{2\pi(2|a^{(2)}||b^{(2)}|\xi^{\varepsilon})^{1/2}\}\ (= C_1\pi/\sqrt{c_1c_2}) > 0$, and ξ^{ε} and $C^{D,\varepsilon}$ are the constants given in (2.4).

Proof of Theorem 1.5 for $\mu_N^{D,\varepsilon}$ For $x_1 < x_2$ and $y_1 < y_2$, let

$$A(x_1, x_2; y_1, y_2) := \left\{(j,k) \in A; \sqrt{N}x_1 \le j - t_1 N \le \sqrt{N}x_2,\ \sqrt{N}y_1 \le k - (1-t_2)N \le \sqrt{N}y_2\right\}.$$

By the same computation as that leading to (3.13) and (3.14), we obtain

$$\frac{Z_N^{D,\varepsilon}}{Z_N^{a,b}} \mu_N^{D,\varepsilon}\left((i_\ell, i_r) \in A(x_1, x_2; y_1, y_2),\ h^N \in U_\delta\left(\hat{h}\right)\right) \sim C_1 N^{1-r/2} \int_{x_1}^{x_2} e^{-c_1 x^2}\, dx \int_{y_1}^{y_2} e^{-c_2 x^2}\, dx.$$

Combining with (3.14), we obtain

$$\lim_{N\to\infty} \mu_N^{D,\varepsilon}\left((i_\ell, i_r) \in A(x_1, x_2; y_1, y_2) \middle| h^N \in U_\delta\left(\hat{h}\right)\right) = \frac{\sqrt{c_1 c_2}}{\pi} \int_{x_1}^{x_2} e^{-c_1 x^2}\, dx \int_{y_1}^{y_2} e^{-c_2 x^2}\, dx.$$

On the other hand, by the estimates leading to (3.14), we also have

$$\begin{aligned} &\mu_N^{D,\varepsilon}\left(\left\{h^N \in U_\delta\left(\hat{h}\right)\right\} \Delta \{i_\ell \le N-1\}\right) \\ &\le \frac{Z_N^{a,b}}{Z_N^{D,\varepsilon}} I_N^1 + \mu_N^{D,\varepsilon}\left(\{i_\ell \le N-1\} \Big\backslash \left\{h^N \in U_\delta\left(\hat{h}\right)\right\}\right) \\ &\le e^{-cN}, \end{aligned}$$

for some $c > 0$, where Δ denotes the symmetric difference, and where the estimate of the second summand comes from the fact that if h^N touches 0, but is not in $U_\delta(\hat{h})$, then it is outside $U_\delta(\hat{h}) \cup U_\delta(\bar{h})$. So by the large deviation estimate (cf. Theorem 4.1 below), the probability of the event that this happens is exponentially small. Therefore, we can replace the conditioning on $\{h^N \in U_\delta(\hat{h})\}$ by that on $\{i_\ell \le N-1\}$, and obtain

$$\lim_{N\to\infty} \mu_N^{D,\varepsilon}\left((i_\ell, i_r) \in A(x_1, x_2; y_1, y_2) \middle| i_\ell \le N-1\right) = \frac{\sqrt{c_1 c_2}}{\pi} \int_{x_1}^{x_2} e^{-c_1 x^2}\, dx \int_{y_1}^{y_2} e^{-c_2 x^2}\, dx,$$

which proves the claim. Remark that the conditioning on $\{i_\ell \le N-1\}$ is not needed for $r = 1$, as $\mu_N^{D,\varepsilon}(i_\ell \le N-1) \to 1$. □

3.2 Proof of Theorems 1.3-(2) and 1.5 for $\mu_N^{D,\varepsilon,+}$

For $a, b \in \mathbb{R}_+^r$ and $0 \le j < k \le N$, let $\mu_{j,k}^{a,b,+}$ be the measure on $(\mathbb{R}_+^r)^{\{j,\dots,k\}}$, defined similarly to $\mu_{j,k}^{a,b}$, with the normalizing constant $Z_{j,k}^{a,b,+} = Z_{k-j}^{a,b,+}$, i.e., the measure defined by the formula (3.2) with $Z_{j,k}^{a,b}$ and $d\phi_i$ replaced by $Z_{j,k}^{a,b,+}$ and $d\phi_i^+$, respectively. One can define the measure $\mu_{j,k}^{0,0,\varepsilon,+}$ on $(\mathbb{R}_+^r)^{\{j,\dots,k\}}$ with pinning and the Dirichlet conditions $\phi_j = \phi_k = 0$ having the normalizing constant $Z_{k-j}^{0,0,\varepsilon,+}$. Taking $\hat{h} = \hat{h}^{D,+,(2)}$ in this subsection, an expansion similar to (3.4) gives rise to

$$p_N^+ := \frac{Z_N^{D,\varepsilon,+}}{Z_N^{a,b,+}} \mu_N^{D,\varepsilon,+}\left(h^N \in U_\delta\left(\hat{h}\right)\right) = I_N^{1,+} + I_N^{2,+} + I_N^{3,+},$$

where $I_N^{\alpha,+}$ are the terms corresponding to I_N^α in (3.4) for $\alpha = 1, 2, 3$, in which we replace the measures $\mu_{j,k}^{a,b}$ by $\mu_{j,k}^{a,b,+}$, $\mu_{j,k}^{0,0,\varepsilon}$ by $\mu_{j,k}^{0,0,\varepsilon,+}$, and $\Xi_{N,j,k}^{\varepsilon}$ by $\Xi_{N,j,k}^{\varepsilon,+}$ defined as

$$\Xi_{N,j,k}^{\varepsilon,+} = \frac{Z_j^{a,0,+} Z_{k-j}^{0,0,\varepsilon,+} Z_{N-k}^{0,b,+}}{Z_N^{a,b,+}} \tag{3.17}$$

for $0 < j \le k < N$, where $Z_0^{0,0,\varepsilon,+} = 1$ as before. We prepare a lemma to find the asymptotic behavior of $\Xi_{N,j,k}^{\varepsilon,+}$. We will denote the rth coordinates of a and $b \in \mathbb{R}_+^r = \mathbb{R}^{r-1} \times \mathbb{R}_+$ by a^r and $b^r \in \mathbb{R}_+$, respectively.

Lemma 3.2 1. *If $a^r, b^r > 0$ (i.e., $a, b \in (\mathbb{R}_+^r)^\circ$), we have as $N \to \infty$*

$$\frac{Z_N^{a,b,+}}{Z_N^{a,b}} \sim 1.$$

2. *If $a^r = 0$ (i.e., $a \in \partial\mathbb{R}_+^r$) and $b^r > 0$, we have*

$$\frac{Z_N^{a,b,+}}{Z_N^{a,b}} \sim \beta\left(b^r\right) = \exp\left\{-\sum_{n=1}^{\infty} \frac{p(b^r\sqrt{n})}{n}\right\},$$

where $p(x) = \int_x^\infty e^{-y^2/2}\, dy/\sqrt{2\pi}$.

Proof 1. As we have observed in (2.7), the ratio of two partition functions has a representation and a bound:

$$\begin{aligned} 1 \ge \frac{Z_N^{a,b,+}}{Z_N^{a,b}} &= P_{a^r\sqrt{N},b^r\sqrt{N}}^{0,1}\left(B(i/N) \ge 0 \quad \text{for all } 1 \le i \le N-1\right) \\ &\ge P_{0,0}^{0,1}\left(B(t) \ge -\left(a^r(1-t) + b^r t\right)\sqrt{N} \quad \text{for all } t \in [0,1]\right) \to 1 \end{aligned}$$

as $N \to \infty$. The equality in the first line is by the scaling law of a (one-dimensional) Brownian bridge. The second line follows by noting that a Brownian bridge $B(t)$ satisfying $B(0) = a^r\sqrt{N}$ and $B(1) = b^r\sqrt{N}$ can be represented as $B(t) = \bar{B}(t) + (a^r(1-t) + b^r t)\sqrt{N}$ with another Brownian bridge $\bar{B}$ such that $\bar{B}(0) = \bar{B}(1) = 0$.

2. If $a^r = 0$, we have

$$\frac{Z_N^{a,b,+}}{Z_N^{a,b}} = P_{0,0}^{0,N}\left(\bar{B}(i) + b^r i \geq 0 \quad \text{for all } 1 \leq i \leq N-1\right) =: c_N^0(b^r),$$

with a Brownian bridge $\bar{B}$ such that $\bar{B}(0) = \bar{B}(N) = 0$. Replacing $\bar{B}$ with the standard Brownian motion B, one can prove that

$$c_N(b^r) := P_0^{0,N}\left(B(i) + b^r i \geq 0 \text{ for all } 1 \leq i \leq N-1\right) \longrightarrow \beta\left(b^r\right). \quad (3.18)$$

In fact, Theorem 1 (p. 413) of [7] shows that

$$\log \frac{1}{1-\tau(s)} = \sum_{n=1}^{\infty} \frac{s^n}{n} P_0(B(n) > b^r n),$$

for $\tau(s) = \sum_{n=1}^{\infty}(c_n(b^r) - c_{n+1}(b^r))\, s^n, 0 \leq s \leq 1$. This identity by taking $s = 1$ implies (3.18), since $1 - \tau(1) = \lim_{N\to\infty} c_N(b^r)$ noting that the limit exists by monotonicity. To complete the proof of (2), rewriting $c_N^0(b^r)$ into

$$c_N^0(b^r) = P_0^{0,N}\left(B(i) - \frac{i}{N}B(N) + b^r i \geq 0 \text{ for all } 1 \leq i \leq N-1\right),$$

one can compare it with $c_N(b^r)$ as

$$c_N(b^r - \theta) - P_0^{0,N}(B(N) > \theta N) \leq c_N^0(b^r) \leq c_N(b^r + \theta) + P_0^{0,N}(B(N) < -\theta N),$$

for every $\theta > 0$. The conclusion is shown by letting $N \to \infty$ and then $\theta \downarrow 0$.

□

The proof of Theorem 1.3-(2) can be given along the same line as Theorem 1.3-(1). Indeed, by Lemma 3.2 and then by (2.7), (2.8) and Proposition 2.3, if $\varepsilon > \varepsilon_c^+$, we have

$$\begin{aligned}
\Xi_{N,j,k}^{\varepsilon,+} &\sim \beta\left(a^r\right)\beta\left(b^r\right)\frac{Z_j^{a,0} Z_{k-j}^{0,0,\varepsilon,+} Z_{N-k}^{0,b}}{Z_N^{a,b}} \\
&= \beta\left(a^r\right)\beta\left(b^r\right)\alpha_{N,j,k} e^{-N\tilde{f}(s_1,s_2)} \cdot \frac{Z_{k-j}^{0,0,\varepsilon,+}}{Z_{k-j}^{0,0}} \\
&\sim \beta\left(a^r\right)\beta\left(b^r\right)\alpha_{N,j,k} C^{D,\varepsilon,+}(k-j)^{r/2} e^{-Nf^+(s_1,s_2)},
\end{aligned}$$

as $j, N-k, N$ and $k-j \to \infty$, where $f^+(s_1,s_2)$ is the function $f(s_1,s_2)$ in (3.10) with $\xi = \xi^{\varepsilon,+}$, so that

$$f^+(s_1,s_2) = \frac{|a|^2}{2t_1^2 s_1}(s_1-t_1)^2 + \frac{|b|^2}{2t_2^2 s_2}(s_2-t_2)^2,$$

by the condition $(C)_{D,+}$. If we define A as in (3.9), we get

$$\varepsilon^2 \sum_{(j,k)\in A} \Xi^{\varepsilon,+}_{N,j,k} \sim C_3 N^{1-r/2},$$

with $C_3 = \varepsilon^2 \beta(a^r)\beta(b^r) C^{D,\varepsilon,+}\pi/(2\pi)^r (t_1t_2)^{r/2}\sqrt{c_1c_2} > 0$, which is shown similarly to (3.13) (just replace $C^{D,\varepsilon}$ with $\beta(a^r)\beta(b^r)C^{D,\varepsilon,+}$), and this proves that

$$p_N^+ \sim C_3 N^{1-r/2}. \tag{3.19}$$

On the other hand, we have

$$\frac{Z_N^{D,\varepsilon,+}}{Z_N^{a,b,+}} \mu_N^{D,\varepsilon,+}\left(h^N \in U_\delta\left(\bar{h}\right)\right) = \mu_N^{D,0,+}\left(h^N \in U_\delta\left(\bar{h}\right)\right) \sim 1, \tag{3.20}$$

for $0 < \delta < |a| \wedge |b|$. The conclusion of Theorem 1.3-(2) follows from the combination of (3.19) and (3.20). In particular, if $r=2$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{D,\varepsilon,+}, \hat{\lambda}^{D,\varepsilon,+}) := \left(\frac{1}{1+C_3}, \frac{C_3}{1+C_3}\right), \tag{3.21}$$

where $C_3 = \varepsilon^2 \beta(a^r)\beta(b^r) C^{D,\varepsilon,+}/\left\{2\pi(2|a^{(2)}||b^{(2)}|\xi^{\varepsilon,+})^{1/2}\right\} > 0$, $\beta(a^r)$ is in Lemma 3.2-(2), and $\xi^{\varepsilon,+}$ and $C^{D,\varepsilon,+}$ are the constants given in (2.12).

The proof of Theorem 1.5 under $\mu_N^{D,\varepsilon,+}$ is parallel to that for $\mu_N^{D,\varepsilon}$ and omitted.

3.3 Proof of Theorems 1.4-(1) and 1.5 for $\mu_N^{F,\varepsilon}$

Let $\mu_N^{a,F} (= \mu_N^{F,0})$ be the measure defined on $(\mathbb{R}^r)^{D_N}$ without pinning and having the normalizing constant $Z_N^{a,F} (= Z_N^{a,F,0})$:

$$\mu_N^{a,F}(d\phi) = \frac{1}{Z_N^{a,F}} e^{-H_N(\phi)} \delta_{aN}(d\phi_0) \prod_{i \in D_N^{\circ,F}} d\phi_i. \tag{3.22}$$

For $0 \le j < k \le N$, one can define the measure $\mu_{j,k}^{0,F,\varepsilon}$ on $(\mathbb{R}^r)^{\{j,\dots,k\}}$ with pinning, the condition $\phi_j = 0$ at j, and the free condition (no specific condition) at k, having the normalizing constant $Z_{k-j}^{0,F,\varepsilon}$. The expansion of the product measure $\prod_{i\in D_N^{\circ,F}}(\varepsilon\delta_0(d\phi_i)+d\phi_i)$ in (1.2) by specifying $0<i_\ell\le N+1$ leads to

$$\begin{aligned}
p_N^F &:= \frac{Z_N^{F,\varepsilon}}{Z_N^{a,F}}\mu_N^{F,\varepsilon}\left(h^N\in U_\delta\left(\hat{h}\right)\right)\\
&= \mu_N^{a,F}\left(h^N\in U_\delta\left(\hat{h}\right)\right)\\
&\quad+\sum_{j\in D_N^{\circ,F}}\varepsilon\,\Xi_{N,j}^{F,\varepsilon}\mu_{0,j}^{a,0}\left(h_{[0,j/N]}^N\in U_\delta\left(\hat{h}_{[0,j/N]}\right)\right)\\
&\quad\times\mu_{j,N}^{0,F,\varepsilon}\left(h_{[j/N,1]}^N\in U_\delta\left(\hat{h}_{[j/N,1]}\right)\right)\\
&=: I_N^{1,F}+I_N^{2,F},
\end{aligned}\tag{3.23}$$

where $\hat{h}=\hat{h}^{F,(2)}$ in this subsection and

$$\Xi_{N,j}^{F,\varepsilon}=\frac{Z_j^{a,0}Z_{N-j}^{0,F,\varepsilon}}{Z_N^{a,F}}$$

for $j\in D_N^{\circ,F}$. Noting that $Z_n^{a,F}=Z_n^{0,F}=(2\pi)^{rn/2}$ and recalling (3.3) for $Z_j^{a,0}$, we see that

$$\Xi_{N,j}^{F,\varepsilon}=(2\pi j)^{-r/2}e^{-N\tilde{f}(s_1)}\cdot\frac{Z_{N-j}^{0,F,\varepsilon}}{Z_{N-j}^{0,F}},$$

where $s_1=j/N$ and $\tilde{f}(s_1)=|a|^2/2s_1$.

We put here

$$A:=\left\{j\in D_N^{\circ,F};\,|j-Nt_1|\le N^{3/5}\right\}$$

and arrive in the same way as in Sect. 3.1, using the large deviation estimate for $\mu_{0,j}^{a,0}$ and $\mu_{j,N}^{0,F,\varepsilon}$ (cf. Theorem 4.1 below), to

$$p_N^F=\varepsilon\sum_{j\in A}\Xi_{N,j}^{F,\varepsilon}\left(1-O\left(e^{-cN}\right)\right)+O\left(e^{-cN^{1/5}}\right)+O\left(e^{-cN}\right),\tag{3.24}$$

for some $c>0$. Furthermore, we get by Proposition 2.5,

$$\varepsilon\sum_{j\in A}\Xi_{N,j}^{F,\varepsilon}\sim\varepsilon C^{F,\varepsilon}(2\pi)^{-r/2}\sum_{j\in A}(Ns_1)^{-r/2}e^{-Nf^F(s_1)},$$

where $f^F(s) = \tilde{f}(s) - \xi(1-s)$ with $\xi = \xi^\varepsilon$. By the second condition of $(C)_F$, being equivalent to $\xi^\varepsilon = 2|a|^2$, one can rewrite f^F as

$$f^F(s) = \frac{2|a|^2}{s}(s-1/2)^2. \tag{3.25}$$

This finally proves, recalling $t_1 = 1/2$ and (3.24), that

$$p_N^F \sim \varepsilon C^{F,\varepsilon} \pi^{-r/2} N^{(1-r)/2} \int_{-\infty}^{\infty} e^{-4|a|^2 x^2}\, dx = \frac{\varepsilon C^{F,\varepsilon} \pi^{(1-r)/2}}{2|a|} N^{(1-r)/2}. \tag{3.26}$$

On the other hand, for every $0 < \delta < |a|$, we have that

$$\frac{Z_N^{F,\varepsilon}}{Z_N^{a,F}} \mu_N^{F,\varepsilon}\left(h^N \in U_\delta\left(\bar{h}\right)\right) = \mu_N^{F,0}\left(h^N \in U_\delta\left(\bar{h}\right)\right) \sim 1,$$

where $\bar{h} = \bar{h}^{F,(2)}$. Comparing this with (3.26), and recalling (1.6), the conclusion of Theorem 1.4-(1) is proved. In particular, if $r = 1$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{F,\varepsilon}, \hat{\lambda}^{F,\varepsilon}) := \left(\frac{2|a^{(2)}|}{\varepsilon C^{F,\varepsilon} + 2|a^{(2)}|}, \frac{\varepsilon C^{F,\varepsilon}}{\varepsilon C^{F,\varepsilon} + 2|a^{(2)}|}\right), \tag{3.27}$$

where $C^{F,\varepsilon}$ is the constant given in (2.14).

The proof of Theorem 1.5 under $\mu_N^{F,\varepsilon}$ conditioned on the event $\{i_\ell \le N\}$ is similar based on the computation like in (3.26), note that the variance of the limiting Gaussian distribution is $1/8|a^{(2)}|^2$ which is equal to $|a^{(2)}|/(2\xi)^{3/2}$.

3.4 Proof of Theorems 1.4-(2) and 1.5 for $\mu_N^{F,\varepsilon,+}$

For $a \in \mathbb{R}_+^r$, let $\mu_N^{a,F,+} (= \mu_N^{F,0,+})$ be the measure defined on $(\mathbb{R}_+^r)^{D_N}$ similarly to $\mu_N^{a,F}$ without pinning and having the normalizing constant $Z_N^{a,F,+} (= Z_N^{a,F,0,+})$, i.e., the measure defined by (3.22) with $Z_N^{a,F}$ and $d\phi_i$ replaced by $Z_N^{a,F,+}$ and $d\phi_i^+$, respectively. For $0 \le j < k \le N$, one can define the measure $\mu_{j,k}^{0,F,\varepsilon,+}$ on $(\mathbb{R}_+^r)^{\{j,\dots,k\}}$ with pinning and the normalizing constant $Z_{k-j}^{0,F,\varepsilon,+}$. Taking $\hat{h} = \hat{h}^{F,+,(2)}$ and $\bar{h} = \bar{h}^{F,(2)}$ in this subsection, a similar expansion to (3.23) leads to

$$p_N^{F,+} := \frac{Z_N^{F,\varepsilon,+}}{Z_N^{a,F,+}} \mu_N^{F,\varepsilon,+}\left(h^N \in U_\delta\left(\hat{h}\right)\right) = I_N^{1,F,+} + I_N^{2,F,+},$$

where $I_N^{\alpha,F,+}$ are the terms corresponding to $I_N^{\alpha,F}$ for $\alpha = 1, 2$, in which we replace the measures $\mu_N^{a,F}$, $\mu_{0,j}^{a,0}$ and $\mu_{j,N}^{0,F,\varepsilon}$ with $\mu_N^{a,F,+}$, $\mu_{0,j}^{a,0,+}$ and $\mu_{j,N}^{0,F,\varepsilon,+}$, and the constant $\Xi^{F,\varepsilon}_{N,j}$ with

$$\Xi^{F,\varepsilon,+}_{N,j} = \frac{Z^{a,0,+}_{j} Z^{0,F,\varepsilon,+}_{N-j}}{Z^{a,F,+}_{N}},$$

respectively. The next lemma can be shown similarly to Lemma 3.2-(1).

Lemma 3.3 *If* $a^r > 0$*, we have*

$$\frac{Z^{a,F,+}_{N}}{Z^{a,F}_{N}} \sim 1$$

as $N \to \infty$*.*

Using Lemmas 3.2-(2), 3.3, and then (2.17) and Proposition 2.6, if $\varepsilon > \varepsilon_c^+$, we have

$$\begin{aligned}\Xi^{F,\varepsilon,+}_{N,j} &\sim \beta\left(a^r\right) \frac{Z^{a,0}_{j} Z^{0,F,\varepsilon,+}_{N-j}}{Z^{a,F}_{N}} \\ &= \beta(a^r)(2\pi j)^{-r/2} e^{-N\tilde{f}(s_1)} \cdot q^F_{N-j} \frac{Z^{0,F,\varepsilon,+}_{N-j}}{Z^{0,F,+}_{N-j}} \\ &\sim \beta\left(a^r\right) C^{F,+} C^{F,\varepsilon,+} (2\pi j)^{-r/2} e^{-Nf^{F,+}(s_1)},\end{aligned}$$

where $f^{F,+}$ is the function f^F with $\xi = \xi^{\varepsilon,+}$, which can be rewritten as (3.25) by the second condition of $(C)_{F,+}$. Therefore, we obtain in the same way as in Sect. 3.3

$$\begin{aligned}p^{F,+}_{N} &\sim \varepsilon\beta\left(a^r\right) C^{F,+} C^{F,\varepsilon,+} (2\pi)^{-r/2} \sum_{|j-Nt_1|\le N^{3/5}} (Ns_1)^{-r/2} e^{-Nf^{F,+}(s_1)} \\ &\sim \frac{\varepsilon\beta\left(a^r\right) C^{F,+} C^{F,\varepsilon,+} \pi^{(1-r)/2}}{2|a|} N^{(1-r)/2}.\end{aligned}$$

In particular, if $r = 1$, the coexistence of $\bar{h}$ and $\hat{h}$ occurs in the limit with probabilities

$$(\bar{\lambda}^{F,\varepsilon,+}, \hat{\lambda}^{F,\varepsilon,+}) := \left(\frac{2|a^{(2)}|}{\varepsilon\beta(a^r) C^{F,+} C^{F,\varepsilon,+} + 2|a^{(2)}|}, \frac{\varepsilon\beta(a^r) C^{F,+} C^{F,\varepsilon,+}}{\varepsilon\beta(a^r) C^{F,+} C^{F,\varepsilon,+} + 2|a^{(2)}|} \right), \tag{3.28}$$

where $\beta\left(a^r\right)$ is in Lemma 3.2-(2), $C^{F,\varepsilon,+}$ is in (2.16) and $C^{F,+}$ is in (2.17), respectively; in fact, $a^{(2)} = a^r$ if $r = 1$.

The rest of the proof is essentially the same as Sect. 3.3.

4 Large deviation principle

This section is devoted to the sample path large deviation principle. Note that we do not require the conditions $(C)_D$–$(C)_{F,+}$.

4.1 Formulation of results

Theorem 4.1 *The large deviation principle* (LDP) *holds for* $h^N = \{h^N(t), t \in D\}$ *distributed under* $\mu_N = \mu_N^{D,\varepsilon}, \mu_N^{D,\varepsilon,+}, \mu_N^{F,\varepsilon}$ *and* $\mu_N^{F,\varepsilon,+}$ *on the spaces* $\mathcal{C}$ *or* $\mathcal{C}^+ = C([0,1], \mathbb{R}_+^d)$ *as* $N \to \infty$ *with the speed* N *and the good rate functionals* $I = I^{D,\varepsilon}, I^{D,\varepsilon,+}, I^{F,\varepsilon}$ *and* $I^{F,\varepsilon,+}$ *of the form*:

$$I(h) = \begin{cases} \Sigma(h) - \inf_H \Sigma, & \text{if } h \in H, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.1}$$

with $\Sigma = \Sigma^{D,\varepsilon}, \Sigma^{D,\varepsilon,+}, \Sigma^{F,\varepsilon}$ *and* $\Sigma^{F,\varepsilon,+}$ *given by* (1.3), *where* $H = H^1_{a,b}(D)$, $H^{1,+}_{a,b}(D) = H^1_{a,b}(D) \cap \mathcal{C}^+$, $H^1_{a,F}(D)$ *and* $H^{1,+}_{a,F}(D) = H^1_{a,F}(D) \cap \mathcal{C}^+$, *respectively. Namely, for every open set* $\mathfrak{O}$ *and closed set* $\mathfrak{C}$ *of* $\mathcal{C}$ *or* $\mathcal{C}^+$ *equipped with the uniform topology, we have that*

$$\liminf_{N\to\infty} \frac{1}{N} \log \mu_N(h^N \in \mathfrak{O}) \ge - \inf_{h\in\mathfrak{O}} I(h),$$
$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N(h^N \in \mathfrak{C}) \le - \inf_{h\in\mathfrak{C}} I(h),$$

in each of four situations.

The LDP for $\mu_N^{D,\varepsilon}$ is shown in [12], Theorem 2.2, when $d = 1$. Indeed, one can give the proof of Theorem 4.1 essentially just by copying the proof stated in [12] line by line. But, for completeness, we give another proof with slightly different flavor, which might be simpler in some aspect.

4.2 Preliminaries

4.2.1 The case without pinning

We start with the LDP for the case without pinning, which is actually standard.

Proposition 4.2 *The LDP holds for* h^N *under* $\mu_N = \mu_N^{a,b}, \mu_N^{a,b,+}, \mu_N^{a,F}$ *and* $\mu_N^{a,F,+}$ *on the spaces* $\mathcal{C}$ *or* $\mathcal{C}^+$ *as* $N \to \infty$ *with the speed* N *and the unnormalized rate functional*

$$\Sigma_0(h) = \frac{1}{2} \int_D |\dot{h}(t)|^2 \, dt.$$

Proof We first discuss the situation without a wall. The assertion for $\mu_N^{a,F}$ follows by Schilder's theorem (or Mogul'skii's theorem [16], [4]), while for $\mu_N^{a,b}$, we may employ the contraction principle for the LDP in addition as in the proof of Lemma 6.1 of [12].

We now put a wall at $\partial\mathbb{R}_+^d$. Assuming $a, b \in \mathbb{R}_+^d$, let us denote $\mu_N^{a,b}$ or $\mu_N^{a,F}$ by μ_N and $\mu_N^{a,b,+}$ or $\mu_N^{a,F,+}$ by μ_N^+, correspondingly. Then, μ_N^+ is a conditional distribution of μ_N on the event $A_N^+ = \{\phi_i \in \mathbb{R}_+^d \text{ for all } i \in D_N\}$. First, we consider the case where $a, b \in (\mathbb{R}_+^d)^\circ$. Then, the LDP for μ_N shown above proves that

$$\lim_{N\to\infty} \frac{1}{N} \log \mu_N(A_N^+) = 0. \tag{4.2}$$

Since a closed set $\mathfrak{C}$ of $\mathcal{C}^+$ is closed in $\mathcal{C}$, combined with (4.2), the LD upper bound for μ_N implies that for μ_N^+. The LD lower bound for μ_N^+ is also easy, since $\tilde{\mathfrak{O}} = \mathfrak{O} \cap \{h(t) \in (\mathbb{R}_+^d)^\circ \text{ for all } t \in D\}$ is open in $\mathcal{C}$ for every open set $\mathfrak{O}$ in $\mathcal{C}^+$ and $\mu_N(\mathfrak{O} \cap \mathcal{C}^+) = \mu_N(\tilde{\mathfrak{O}})$.

The case where a or/and $b \in \partial\mathbb{R}_+^d$ is more involved. The idea is to reduce the proof of the LDP for such case to the case where $a, b \in (\mathbb{R}_+^d)^\circ$. For $\mu_N^{a,F,+}$, we have a nice coupling $(h^{N,a}, h^{N,a'})$ for every pair of a and $a' \in \mathbb{R}_+^d$ realized on a common probability space distributed under $\mu_N^{a,F,+}$ and $\mu_N^{a',F,+}$ ($h^{N,a} \sim \mu_N^{a,F,+}$, $h^{N,a'} \sim \mu_N^{a',F,+}$), respectively, such that $\|h^{N,a} - h^{N,a'}\|_\infty \le |a - a'|$ a.s. (which is uniform in N). In fact, we may apply Lemma 2.2 of [11] in one dimension componentwisely noting that components $\{\phi^\alpha = (\phi_i^\alpha)_{i\in D_N}\}_{\alpha=1}^d$ are mutually independent under $\mu_N^{a,F,+}$. This coupling implies

$$\mu_N^{a,F,+}(\mathfrak{C}) \le \mu_N^{a^\gamma,F,+}(\mathfrak{C}^\gamma) \quad \text{and} \quad \mu_N^{a,F,+}(\mathfrak{O}) \ge \mu_N^{a^\gamma,F,+}(\mathfrak{O}^\gamma)$$

for every closed $\mathfrak{C}$ and open $\mathfrak{O}$ in $\mathcal{C}^+$ and $\gamma > 0$, where $a^\gamma = a + \gamma e^d \in (\mathbb{R}_+^d)^\circ$, $e^d = (0, \dots, 0, 1)$ is the dth unit vector, $\mathfrak{C}^\gamma = \{h \in \mathcal{C}^+; B(h,\gamma) \cap \mathfrak{C} \ne \emptyset\}$, $\mathfrak{O}^\gamma = \{h; B(h,\gamma) \subset \mathfrak{O}\}$ and $B(h,\gamma) = \{g; \|g - h\|_\infty \le \gamma\}$. Since $\mathfrak{C}^\gamma$ and $\mathfrak{O}^\gamma$ are closed and open in $\mathcal{C}^+$, respectively, we have the LDP for $\mu_N^{a,F,+}$ with $a \in \partial\mathbb{R}_+^d$ from that for $\mu_N^{a^\gamma,F,+}$ with $a^\gamma \in (\mathbb{R}_+^d)^\circ$ by noting that

$$\lim_{\gamma\downarrow 0} \inf_{h\in\mathfrak{C}^\gamma} \Sigma_0(h) = \inf_{h\in\mathfrak{C}} \Sigma_0(h) \quad \text{and} \quad \lim_{\gamma\downarrow 0} \inf_{h\in\mathfrak{O}^\gamma} \Sigma_0(h) = \inf_{h\in\mathfrak{O}} \Sigma_0(h).$$

Indeed, the first one is shown by the closedness of $\mathfrak{C}$ and the lower semicontinuity of Σ_0, while the second is from the openness of $\mathfrak{O}$. The proof of the LDP for $\mu_N^{a,b,+}$ with a or/and $b \in \partial\mathbb{R}_+^d$ is similar. □

4.2.2 Reduction to the case of $m = 0$

As we have seen in (3.1), the probability measure $\mu_N^{a,b,\varepsilon}$ is decomposed into the product:

$$\mu_N^{a,b,\varepsilon} = \mu_{N,m}^{a^{(1)},b^{(1)},0} \times \mu_{N,r}^{a^{(2)},b^{(2)},\varepsilon}.$$

Once Theorem 4.1 is shown for the second component $\mu_{N,r}^{a^{(2)},b^{(2)},\varepsilon}$, combining with Proposition 4.2 for the first component, Theorem 4.1 for $\mu_N^{a,b,\varepsilon}$ is shown. In fact, the LDP lower and upper bounds are shown first for products $\mathfrak{O} = \mathfrak{O}_1 \times \mathfrak{O}_2$ and $\mathfrak{C} = \mathfrak{C}_1 \times \mathfrak{C}_2$ of open and closed sets $\mathfrak{O}_1, \mathfrak{C}_1$ in $C(D, \mathbb{R}^m)$ and $\mathfrak{O}_2, \mathfrak{C}_2$ in $C(D, \mathbb{R}^r)$, respectively; note that $\{t \in D; h^{(2)}(t) = 0\} = \{t \in D; h(t) \in M\}$. Then, these estimates can be extended easily to general open set $\mathfrak{O}$ in $C(D, \mathbb{R}^d)$ and closet set $\mathfrak{C}$ in $C(D, \mathbb{R}^d)$. Other three measures $\mu_N^{a,b,\varepsilon,+}$, $\mu_N^{a,F,\varepsilon}$ and $\mu_N^{a,F,\varepsilon,+}$ can be treated similarly. We may thus assume that $d = r$ and $m = 0$. In particular, $M = \{0\}$ and, therefore, the unnormalized rate functional should have the form

$$\Sigma(h) = \frac{1}{2} \int_D |\dot{h}(t)|^2 dt - \xi |\{t \in D; h(t) = 0\}|. \tag{4.3}$$

4.2.3 Estimates via stochastic domination

The proof of the lower bound in Theorem 4.1 will be reduced to the following estimates for the measures with pinning starting at 0, see Sect. 4.3.1 below.

Proposition 4.3 *For every $\delta > 0$, there exist $C, c > 0$ such that*

$$\mu_N^{\varepsilon}(\|h^N\|_\infty \geq \delta) \leq Ce^{-cN}$$

for $\mu_N^{\varepsilon} = \mu_N^{0,0,\varepsilon}, \mu_N^{0,0,\varepsilon,+}, \mu_N^{0,F,\varepsilon}$ and $\mu_N^{0,F,\varepsilon,+}$.

The idea of the proof of Proposition 4.3 is simple. We will apply a coupling argument. For instance, under $\mu_N^{0,F,\varepsilon}$, the Markov chains $\phi^\varepsilon = (\phi_i^\varepsilon)_{i \in D_N}$ occasionally jump to the origin $0 \in \mathbb{R}^d$. It is therefore natural to expect to have a coupling, compared with the Markov chains $\phi^0 = (\phi_i^0)_{i \in D_N}$ without pinning distributed under $\mu_N^{0,F,0}$ (i.e., $\varepsilon = 0$), such that $|\phi_i^\varepsilon| \leq |\phi_i^0|, i \in D_N$ for Euclidean norms. This can be shown based on the FKG inequality, see Remark 4.1-(1) below. Once such coupling is established, the estimates stated in Proposition 4.3 are immediate from Proposition 4.2 for measures without pinning. We will actually establish the coupling not for the Euclidean norms of the Markov chains but for one-dimensional chains obtained by conditioning the original ones, in particular, to deal with the case with a wall.

Let $\mathcal{X}_N^\alpha$ and $\mathcal{X}_N^{\alpha,+}$, $1 \leq \alpha \leq d$, be the sets of all $\psi = \left(\psi^\beta = (\psi_i^\beta)_{i \in D_N}\right)_{\beta \neq \alpha} \in (\mathbb{R}^{d-1})^{D_N}$ respectively $\in (\mathbb{R}_+^{d-1})^{D_N}$ such that $\psi_i^\beta = 0$ for all $\beta \neq \alpha$ if $\psi_i^\gamma = 0$ for some $\gamma \neq \alpha$ and i. Note that $\mu_N^{0,0,\varepsilon}(\mathcal{X}_N^\alpha) = \mu_N^{0,F,\varepsilon}(\mathcal{X}_N^\alpha) = 1$ for $1 \leq \alpha \leq d$,

$\mu_N^{0,0,\varepsilon,+}(\mathcal{X}_N^{\alpha,+}) = \mu_N^{0,F,\varepsilon,+}(\mathcal{X}_N^{\alpha,+}) = 1$ for $1 \le \alpha \le d-1$ and $\mu_N^{0,0,\varepsilon,+}(\mathcal{X}_N^{d}) = \mu_N^{0,F,\varepsilon,+}(\mathcal{X}_N^{d}) = 1$.

For $1 \le \alpha \le d$ and $\psi \in \mathcal{X}_N^{\alpha,(+)}$ satisfying $\psi_0 = \psi_N = 0$ in the Dirichlet case and $\psi_0 = 0$ in the free case, let $\nu_{N,\psi}^{\varepsilon,\alpha}$ (more precisely, $\nu_{N,\psi}^{0,0,\varepsilon,\alpha}$, $\nu_{N,\psi}^{0,0,\varepsilon,\alpha,+}$, $\nu_{N,\psi}^{0,F,\varepsilon,\alpha}$ and $\nu_{N,\psi}^{0,F,\varepsilon,\alpha,+}$) be the conditional distribution on the space $\mathcal{Y}_N = \mathbb{R}^{D_N}$ (or $\mathcal{Y}_N^+ = \mathbb{R}_+^{D_N}$) of the αth coordinate $\phi^\alpha = (\phi_i^\alpha)_{i\in D_N}$ under μ_N^ε ($= \mu_N^{0,0,\varepsilon}$, $\mu_N^{0,0,\varepsilon,+}$, $\mu_N^{0,F,\varepsilon}$ and $\mu_N^{0,F,\varepsilon,+}$, respectively) under the condition that the other coordinates $\left(\phi^\beta = (\phi_i^\beta)_{i\in D_N}\right)_{\beta\neq\alpha}$ satisfy $\phi^\beta = \psi^\beta$ for $1 \le \beta \neq \alpha \le d$. For instance, we set $\nu_{N,\psi}^{0,0,\varepsilon,\alpha}(dx) = \mu_N^{0,0,\varepsilon}\left(\phi^\alpha \in dx \big| (\phi^\beta)_{\beta\neq\alpha} = \psi\right)$ for $x \in \mathcal{Y}_N$. For $\psi \in \mathcal{X}_N^{\alpha,(+)}$ satisfying the above conditions, we define a probability measure $\bar\nu_{N,\psi}$ on $\mathcal{Y}_N$, which describes a Markov chain in a random environment ψ, by

$$\bar\nu_{N,\psi}(dx) = \frac{1}{Z_{N,\psi}} e^{-\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2/2} \prod_{i\in \mathbf{i}(\psi)} \delta_0(dx_i) \prod_{i\in D_N\setminus \mathbf{i}(\psi)} dx_i, \tag{4.4}$$

where $\mathbf{i}(\psi) = \{i \in D_N; \psi_i = 0\}$. We will write $\bar\nu_{N,\psi}$ in two ways: $\bar\nu_{N,\psi}^{0,0}$ and $\bar\nu_{N,\psi}^{0,F}$ to clarify which case we are discussing; in particular, $\mathbf{i}(\psi) \supset \{0, N\}$ or $\mathbf{i}(\psi) \supset \{0\}$ in the Dirichlet or free cases, respectively. These measures are independent of ε and α. These probability measures restricted on $\mathcal{Y}_N^+$ and renormalized properly are denoted by $\bar\nu_{N,\psi}^{0,0,+}$ and $\bar\nu_{N,\psi}^{0,F,+}$, respectively. We also define the probability measures $\tilde\nu_N^{0,0}$, $\tilde\nu_N^{0,F}$, $\tilde\nu_N^{0,0,+}$ and $\tilde\nu_N^{0,F,+}$ by replacing $\mathbf{i}(\psi)$ on the right hand side of (4.4) with $\{0, N\}$ in the Dirichlet case and $\{0\}$ in the free case, respectively; note that these measures are independent of ψ. In fact, these are the same as $\mu_N^{0,0,0}$, $\mu_N^{0,F,0}$, $\mu_N^{0,0,0,+}$ and $\mu_N^{0,F,0,+}$ (i.e., μ_N^ε with $\varepsilon = 0$) in $d = 1$, respectively.

The following lemma gives the conditional distributions $\nu_{N,\psi}^{\varepsilon,\alpha}$ of μ_N^ε:

Lemma 4.4 *For $\varepsilon > 0$, we have that*

1. $\nu_{N,\psi}^{0,0,\varepsilon,\alpha} = \bar\nu_{N,\psi}^{0,0}$ ($\mu_N^{0,0,\varepsilon}$-*a.s.*ψ), $\nu_{N,\psi}^{0,F,\varepsilon,\alpha} = \bar\nu_{N,\psi}^{0,F}$ ($\mu_N^{0,F,\varepsilon}$-*a.s.*ψ), $1 \le \alpha \le d$,
2. $\nu_{N,\psi}^{0,0,\varepsilon,\alpha,+} = \bar\nu_{N,\psi}^{0,0}$ ($\mu_N^{0,0,\varepsilon,+}$-*a.s.*$\psi$), $\nu_{N,\psi}^{0,F,\varepsilon,\alpha,+} = \bar\nu_{N,\psi}^{0,F}$ ($\mu_N^{0,F,\varepsilon,+}$-*a.s.*$\psi$), $1 \le \alpha \le d-1$,
3. $\nu_{N,\psi}^{0,0,\varepsilon,d,+} = \bar\nu_{N,\psi}^{0,0,+}$ ($\mu_N^{0,0,\varepsilon,+}$-*a.s.*$\psi$), $\nu_{N,\psi}^{0,F,\varepsilon,d,+} = \bar\nu_{N,\psi}^{0,F,+}$ ($\mu_N^{0,F,\varepsilon,+}$-*a.s.*$\psi$).

Proof Conditioned by the σ-field $\mathcal{F}_0 = \sigma\{\phi_i = 0, i \in D_N\}$ of $\mathcal{X}_N = (\mathbb{R}^d)^{D_N}$, random variables ϕ^α and $(\phi^\beta)_{\beta\neq\alpha}$ are mutually independent under $\mu_N^{0,0,\varepsilon}$. Thus, for every $F = F(\phi^\alpha)$ and $G = G((\phi^\beta)_{\beta\neq\alpha})$, we have

$$\begin{aligned} E^{\mu_N^{0,0,\varepsilon}}[FG] &= E^{\mu_N^{0,0,\varepsilon}}\left[E^{\mu_N^{0,0,\varepsilon}}[F|\mathcal{F}_0]E^{\mu_N^{0,0,\varepsilon}}[G|\mathcal{F}_0]\right] \\ &= E^{\mu_N^{0,0,\varepsilon}}\left[E^{\bar\nu^{0,0}_{N,(\phi^\beta)_{\beta\neq\alpha}}}[F]G\right]. \end{aligned}$$

This completes the proof of the first identity in (1). The rest is similar. □

The space $\mathcal{Y}_N^+$ is equipped with a natural partial order $x \le y$ for $x = (x_i)_{i \in D_N}$, $y = (y_i)_{i \in D_N} \in \mathcal{Y}_N^+$ defined by $x_i \le y_i$ for every $i \in D_N$. For two probability measures ν_1 and ν_2 on $\mathcal{Y}_N^+$, we say that ν_2 stochastically dominates ν_1 and write $\nu_1 \le \nu_2$ if $E^{\nu_1}[F] \le E^{\nu_2}[F]$ holds for all bounded non-decreasing (in the above partial order) functions F on $\mathcal{Y}_N^+$. Note that $\nu_1 \le \nu_2$ is equivalent to the existence of two $\mathcal{Y}_N^+$-valued random variables X and Y, realized on a common probability space and distributed under ν_1 and ν_2 ($X \sim \nu_1, Y \sim \nu_2$), respectively, in such a manner that $X \le Y$ a.s., see [13,19]. Let $R : \mathcal{Y}_N \to \mathcal{Y}_N^+$ be the mapping defined by $Rx = (|x_i|)_{i \in D_N} \in \mathcal{Y}_N^+$ for $x = (x_i)_{i \in D_N} \in \mathcal{Y}_N$.

Lemma 4.5 (*Stochastic domination*) *For all $\varepsilon > 0$ and ψ, we have that*

$$\bar{\nu}_{N,\psi}^{0,0} \circ R^{-1} \le \tilde{\nu}_N^{0,0} \circ R^{-1}, \quad \bar{\nu}_{N,\psi}^{0,F} \circ R^{-1} \le \tilde{\nu}_N^{0,F} \circ R^{-1},$$

$$\bar{\nu}_{N,\psi}^{0,0,+} \le \tilde{\nu}_N^{0,0,+}, \qquad \bar{\nu}_{N,\psi}^{0,F,+} \le \tilde{\nu}_N^{0,F,+},$$

where $\nu \circ R^{-1}$ stands for the image measure of ν under the mapping R.

Proof All four probability measures on the right hand side satisfy the FKG inequality. In fact, since $x = (x_i)_{i \in D_N}$ is a reflecting Brownian motion (i.e., one-dimensional Bessel process) viewed at integer times under $\tilde{\nu}_N^{0,F} \circ R^{-1}$ and a pinned reflecting Brownian motion under $\tilde{\nu}_N^{0,0} \circ R^{-1}$, the measures $\tilde{\nu}_N^{0,0} \circ R^{-1}$ and $\tilde{\nu}_N^{0,F} \circ R^{-1}$ satisfy the FKG inequality; see Sect. 5.3 of [10] for the FKG inequality for Bessel processes. On the other hand, the densities of $\tilde{\nu}_N^{0,0}$ and $\tilde{\nu}_N^{0,F}$ fulfill the Holley's condition on $\mathcal{Y}_N$ (since x is a Brownian motion or a pinned Brownian motion under these measures, see [10,13]), and therefore their restrictions $\tilde{\nu}_N^{0,0,+}$ and $\tilde{\nu}_N^{0,F,+}$ satisfy the same condition on $\mathcal{Y}_N^+$. This implies the FKG inequality for $\tilde{\nu}_N^{0,0,+}$ and $\tilde{\nu}_N^{0,F,+}$.

The four probability measures on the left hand side are given by the weak limits of probability measures having non-increasing densities with respect to the corresponding measures on the right hand side. For instance, we have

$$\bar{\nu}_{N,\psi}^{0,0} \circ R^{-1} = \lim_{\theta \downarrow 0} \nu_{\theta;N,\psi},$$

where $\nu_{\theta;N,\psi}(dx) = \prod_{i \in \mathbf{i}(\psi)} f_\theta(x_i)\, \tilde{\nu}_N^{0,0} \circ R^{-1}(dx)/Z_{\theta;N,\psi}$ with a suitable normalizing constant $Z_{\theta;N,\psi}$ and a non-negative non-increasing function f_θ on $\mathbb{R}_+$ such that $f_\theta(x)dx$ weakly converges to $\delta_0(dx)$ as $\theta \downarrow 0$. Since the FKG inequality for $\tilde{\nu}_N^{0,0} \circ R^{-1}$ implies the stochastic domination $\nu_{\theta;N,\psi} \le \tilde{\nu}_N^{0,0} \circ R^{-1}$, by taking the limit $\theta \downarrow 0$, we have that $\bar{\nu}_{N,\psi}^{0,0} \circ R^{-1} \le \tilde{\nu}_N^{0,0} \circ R^{-1}$. The other three stochastic dominations can be shown similarly. □

Proof of Proposition 4.3 The conclusion follows by

$$\begin{aligned}
\mu_N^{\varepsilon}(\|h^N\|_\infty \geq \delta) &\leq \sum_{\alpha=1}^{d} \mu_N^{\varepsilon}(\|h^{N,\alpha}\|_\infty \geq \delta/\sqrt{d}) \\
&= \sum_{\alpha=1}^{d} E^{\mu_N^{\varepsilon}}\left[\mu_N^{\varepsilon}\left(\|h^{N,\alpha}\|_\infty \geq \delta/\sqrt{d} \middle| (\phi^\beta)_{\beta\neq\alpha}\right)\right] \\
&\leq \sum_{\alpha=1}^{d} \tilde{\nu}_N(\|h^N\|_\infty \geq \delta/\sqrt{d}) \leq Ce^{-cN},
\end{aligned}$$

where $h^{N,\alpha}$ is the αth coordinate of $h^N \in \mathcal{C}$, and $\tilde{\nu}_N = \tilde{\nu}_N^{0,0}, \tilde{\nu}_N^{0,0,+}, \tilde{\nu}_N^{0,F}$ and $\tilde{\nu}_N^{0,F,+}$ according as $\mu_N^{\varepsilon} = \mu_N^{0,0,\varepsilon}, \mu_N^{0,0,\varepsilon,+}, \mu_N^{0,F,\varepsilon}$ and $\mu_N^{0,F,\varepsilon,+}$, respectively. In the third line, we have first used Lemmas 4.4 and 4.5, and then applied Proposition 4.2 with $a, b = 0$ and $d = 1$. □

Remark 4.1 1. At least under the absence of a wall, one can show the stochastic domination for the Euclidean norms of Markov chains:

$$\mu_N^{0,0,\varepsilon} \circ \bar{R}^{-1} \leq \mu_N^{0,0,0} \circ \bar{R}^{-1} \quad \text{and} \quad \mu_N^{0,F,\varepsilon} \circ \bar{R}^{-1} \leq \mu_N^{0,F,0} \circ \bar{R}^{-1}, \tag{4.5}$$

where $\bar{R} : (\mathbb{R}^d)^{D_N} \to \mathcal{Y}_N^+$ is defined by $R\phi = (|\phi_i|)_{i\in D_N} \in \mathcal{Y}_N^+$ for $\phi = (\phi_i)_{i\in D_N} \in (\mathbb{R}^d)^{D_N}$. In fact, $(|\phi_i|)_{i\in D_N}$ is a d-dimensional Bessel process viewed at integer times under $\mu_N^{0,F,0}$, and therefore $\mu_N^{0,F,0} \circ \bar{R}^{-1}$ and $\mu_N^{0,0,0} \circ \bar{R}^{-1}$ (i.e., $\varepsilon = 0$) satisfy the FKG inequality, see Sect. 5.3 of [10]. Then, (4.5) is shown by expressing $\mu_N^{0,0,\varepsilon} \circ \bar{R}^{-1}$ as a weak limit of a sequence of probability measures having non-increasing densities with respect to $\mu_N^{0,0,0} \circ \bar{R}^{-1}$ as in the proof of Lemma 4.5. The free case is similar.

2. Proposition 4.3 can be shown due to the renewal theory. This method is applicable to the situation that the FKG inequality does not work.
3. What we needed in Sect. 3 are, in fact except (3.15), only the estimates given in Proposition 4.3 rather than the full large deviation principle.

4.3 Proof of Theorem 4.1 for $\mu_N^{D,\varepsilon}$

We first note that, for the proof of Theorem 4.1 for $\mu_N^{D,\varepsilon}$, it is enough to show the following two estimates for every $g \in H^1_{a,b}(D)$:

$$\liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \geq -I^{D,\varepsilon}(g), \tag{4.6}$$

for every $\delta > 0$, and

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \leq -I^{D,\varepsilon}(g) + \theta, \tag{4.7}$$

for every $\theta > 0$ with some $\delta > 0$ (depending on θ), where $I^{D,\varepsilon}$ is defined by (4.1) with $\Sigma = \Sigma^{D,\varepsilon}$ and $H = H^1_{a,b}(D)$. This step of reduction is standard, for instance, see (6.6) and the estimate just above (6.11) in [12].

4.3.1 Lower bound

Let $\mathcal{J}_K$, $K \geq 1$ be the family of all $\mathbf{j} = \{j_1^p, j_2^p \in \mathbb{N}\}_{p=1}^K$ such that $0 < j_1^1 \leq j_2^1 < j_1^2 \leq j_2^2 < \cdots < j_1^K \leq j_2^K < N$. For $\mathbf{j} \in \mathcal{J}_K$, $K \geq 1$, we set

$$\Xi^{\varepsilon}_{N,\mathbf{j}} = \frac{1}{Z_N^{D,\varepsilon}} Z^{a,0}_{j_1^1} \cdot \prod_{p=1}^{K} Z^{0,0,\varepsilon}_{j_2^p - j_1^p} \cdot \prod_{p=1}^{K-1} Z^{0,0}_{j_1^{p+1} - j_2^p} \cdot Z^{0,b}_{N-j_2^K}$$

and

$$\begin{aligned}\Psi^{\varepsilon}_{N,\mathbf{j}}(g;\delta) = {} & \mu^{a,0}_{j_1^1}\left(\|h^N - g\|_{\infty,[0,j_1^1/N]} < \delta\right) \cdot \prod_{p=1}^{K} \mu^{0,0,\varepsilon}_{j_2^p - j_1^p}\left(\|h^N\|_{\infty,[j_1^p/N, j_2^p/N]} < \delta\right) \\ & \times \prod_{p=1}^{K-1} \mu^{0,0}_{j_1^{p+1}-j_2^p}\left(\|h^N - g\|_{\infty,[j_2^p/N, j_1^{p+1}/N]} < \delta\right) \\ & \times \mu^{0,b}_{N-j_2^K}(\|h^N - g\|_{\infty,[j_2^K/N,1]} < \delta),\end{aligned}$$

where $g \in H^1_{a,b}(D)$; note that g is not appearing in the second term of $\Psi^{\varepsilon}_{N,\mathbf{j}}(g;\delta)$. We say that a sequence $\mathbf{j}_N = \{j_1^{p,N}, j_2^{p,N}\}_{p=1}^K \in \mathcal{J}_K$ is macroscopically $\mathbf{t} = \{t_1^p, t_2^p\}_{p=1}^K \in \mathcal{T}_K$ if $\lim_{N\to\infty} j_\ell^{p,N}/N = t_\ell^p$ hold for every $1 \leq p \leq K$ and $\ell = 1, 2$, where $\mathcal{T}_K$ is a family of all $\mathbf{t}$ such that $0 < t_1^1 < t_2^1 < t_1^2 < t_2^2 < \cdots < t_1^K < t_2^K < 1$.

We now assume that $g \in H^1_{a,b}(D)$ satisfies the condition:

$$\{t \in D; g(t) = 0\} = \bigcup_{p=1}^{K} [t_1^p, t_2^p] \quad \text{with} \quad \mathbf{t} \in \mathcal{T}_K. \tag{4.8}$$

Lemma 4.6 *If a sequence* $\mathbf{j}_N$ *is macroscopically* $\mathbf{t}$ *and if* $g \in H^1_{a,b}(D)$ *satisfies* (4.8), *we have*

1. $\lim_{N\to\infty} \frac{1}{N} \log \Xi^{\varepsilon}_{N,\mathbf{j}_N} = \xi \sum_{p=1}^K (t_2^p - t_1^p) - \Sigma_0(a,b;t_1^1,t_2^K) + \inf_{H^1_{a,b}(D)} \Sigma(h)$,
2. $\liminf_{N\to\infty} \frac{1}{N} \log \Psi^{\varepsilon}_{N,\mathbf{j}_N}(g;\delta) \geq -\Sigma_0(g) + \Sigma_0(a,b;t_1^1,t_2^K)$,

for every $\delta > 0$, *where* $\xi = \xi^{\varepsilon}$ *and* $\Sigma_0(a,b;t_1^1,t_2^K) = \left\{|a|^2/t_1^1 + |b|^2/(1-t_2^K)\right\}/2$.

Proof The first task for (1) is to calculate the limit as $N \to \infty$ of ratio of two partition functions $Z_N^{a,b}$ and $Z_N^{D,\varepsilon}$ up to an exponential order. To this end, we recall the expansion

(3.4) which implies by letting $\delta \to \infty$

$$\frac{Z_N^{D,\varepsilon}}{Z_N^{a,b}} = 1 + \sum_{j \in D_N^\circ} \varepsilon\, \Xi_{N,j,j}^\varepsilon + \sum_{0<j<k<N} \varepsilon^2\, \Xi_{N,j,k}^\varepsilon.$$

However, from (3.7) and Proposition 2.2, $\Xi_{N,j,j}^\varepsilon$ and $\Xi_{N,j,k}^\varepsilon$ behave as

$$e^{-N\tilde{f}(s_1,s_2)+N\xi(1-s_1-s_2)}$$

except algebraic factors as $N \to \infty$, where $s_1 = j/N$, $s_2 = (N-k)/N$ (we regard $k = j$ for $\Xi_{N,j,j}^\varepsilon$) and $\tilde{f}(s_1, s_2)$ is given by (3.8). Let $\hat{h}_{s_1,s_2} \in \mathcal{C}$, $0 < s_1 \le 1 - s_2 < 1$ be the function $\hat{h}^{D,(2)}$ defined by (1.8) with $t_1, t_2, a^{(2)}, b^{(2)}$ replaced by s_1, s_2, a, b, respectively. Then, since

$$\tilde{f}(s_1, s_2) - \xi(1 - s_1 - s_2) = \Sigma(\hat{h}_{s_1,s_2}) - \Sigma_0(\bar{h}^D),$$

and also by Lemma 1.2, we obtain that

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{a,b}}{Z_N^{D,\varepsilon}} &= -\left[0 \vee \sup_{0<s_1\le 1-s_2<1} \left\{-\Sigma(\hat{h}_{s_1,s_2}) + \Sigma_0(\bar{h}^D)\right\}\right] \\ &= -\Sigma_0(\bar{h}^D) + \inf_{H_{a,b}^1(D)} \Sigma(h). \end{aligned} \tag{4.9}$$

The equality (1) follows from (4.9) and Proposition 2.2 recalling (3.3) (with $r = d$). The inequality (2) is a consequence of Propositions 4.2 and 4.3 noting the condition (4.8). □

We are now in a position to conclude the proof of the LD lower bound (4.6) for $\mu_N^{D,\varepsilon}$. We may assume the condition (4.8) for $g \in H_{a,b}^1(D)$, cf. [12]; if $K = 0$ (i.e., $g(t) \neq 0$ for all $t \in D$), (4.6) follows from Proposition 4.2 and (4.9). Determine $\mathbf{j}_N$ from $\mathbf{t}$ in (4.8) by $j_\ell^{p,N} = [Nt_\ell^p]$, $1 \le p \le K$, $\ell = 1, 2$, which is macroscopically $\mathbf{t}$. Then, we have

$$\mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \ge \varepsilon^{2K}\, \Xi_{N,\mathbf{j}_N}^\varepsilon\, \Psi_{N,\mathbf{j}_N}^\varepsilon(g; \delta), \tag{4.10}$$

by noting that $g = 0$ on $[t_1^p, t_2^p]$. In fact, this follows by restricting the probability on the left hand side on the event $\{\phi_{j_\ell^{p,N}} = 0 \text{ for all } 1 \le p \le K, \ell = 1, 2\} \cap \{\phi_i \neq 0 \text{ for all } i \in \cup_{p=0}^K [j_2^{p,N}, j_1^{p+1,N}] \cap \mathbb{Z}\}$, where $j_2^{0,N} = 0$ and $j_1^{K+1,N} = N$. The LD lower bound (4.6) is shown from Lemma 4.6 and (4.10) for g satisfying (4.8). The rest of the proof is similar to [12], Proof of Theorem 2.2, Step 1.

Remark 4.2 We have implicitly assumed that $a \neq 0$ and $b \neq 0$. The proof can be easily modified when $a = 0$ or $b = 0$. Indeed, if $a = 0$, we take $j_1^1 = 0$ for $\mathbf{j} \in \mathcal{J}_K$

and remove the first factors $Z^{a,0}_{j_1^1}$ from $\Xi^\varepsilon_{N,\mathbf{j}}$ and $\mu^{a,0}_{j_1^1}(\|h^N-g\|_{\infty,[0,j_1^1/N]}<\delta)$ from $\Psi^\varepsilon_{N,\mathbf{j}}(g;\delta)$, respectively. The factor $|a|^2/t_1^1$ does not appear in $\Sigma_0(a,b;t_1^1,t_2^K)$ in Lemma 4.6. Similar modification is possible when $b=0$, and the LD lower bound can be shown when $a=0$ or $b=0$.

4.3.2 Upper bound

Let $g\in H^1_{a,b}(D)$ be a function satisfying the condition:

$$\text{for every } \gamma>0 \text{ small enough,}\quad \{t\in D; |g(t)|\le\gamma\}=\bigcup_{p=1}^K[t_1^{p,\gamma},t_2^{p,\gamma}]\,\big(=: I^\gamma\big)\quad \text{with}\quad \mathbf{t}^\gamma=\{t_1^{p,\gamma},t_2^{p,\gamma}\}_{p=1}^K\in\mathcal{T}_K. \tag{4.11}$$

Then, if $0<\delta<\gamma$, since $|g(t)|>\gamma$ implies on the event $\|h^N-g\|_\infty<\delta$ that $|h^N(t)|>\gamma-\delta>0$ and therefore $\phi_i\ne 0$ for $i\in N(I^\gamma)^c\cap\mathbb{Z}$, we have

$$\mu_N^{D,\varepsilon}(\|h^N-g\|_\infty<\delta)\le\frac{Z_N^{a,b}}{Z_N^{D,\varepsilon}}\mu_N^{a,b}(\|h^N-g\|_\infty<\delta)+\sum_{k=1}^K\varepsilon^{2k}\sum_{\mathbf{j}\in\mathcal{J}_k(\mathbf{t}^\gamma)}\Xi^\varepsilon_{N,\mathbf{j}}\Psi^\varepsilon_{N,\mathbf{j}}(g;\delta+\gamma). \tag{4.12}$$

Here, for $\mathbf{t}\in\mathcal{T}_K$ and $1\le k\le K$, $\mathbf{j}=\{j_1^p,j_2^p\}_{p=1}^k\in\mathcal{J}_k(\mathbf{t})$ means that there exists $\mathbf{s}=\{s_1^p,s_2^p\}_{p=1}^k\in\mathcal{T}_k$ such that $\mathbf{s}\subset\mathbf{t}$ and $s_1^p\le j_1^p/N\le j_2^p/N\le s_2^p$ for every $1\le p\le k$.

We elaborate the results in Lemma 4.6 to some extent, i.e., we need uniform upper bounds for $\Xi^\varepsilon_{N,\mathbf{j}}$ and $\Psi^\varepsilon_{N,\mathbf{j}}(g;\delta)$. For $\tilde\gamma>0$, let $\mathcal{T}_{k,\tilde\gamma}$ be the set of all $\mathbf{t}\in\mathcal{T}_k$ such that $t_2^p-t_1^p\ge\tilde\gamma$ $(1\le p\le k)$ and $t_1^p-t_2^{p-1}\ge\tilde\gamma$ $(1\le p\le k+1)$, where $t_2^0=0$ and $t_1^{k+1}=1$. The function $g\in H^1_{a,b}(D)$ satisfying the condition (4.11) is fixed in the next lemma.

Lemma 4.7 *For every $\tilde\gamma>0$ and $\theta>0$, there exist $\delta>0$, $N_0\ge 1$ and $\eta>0$ such that*

1. $\Xi^\varepsilon_{N,\mathbf{j}}\le\exp\left\{N\left(\xi\sum_{p=1}^k(t_2^p-t_1^p)-\Sigma_0(a,b;t_1^1,t_2^k)+\inf_{H^1_{a,b}(D)}\Sigma(h)+\theta\right)\right\}$,
2. $\Psi^\varepsilon_{N,\mathbf{j}}(g;\delta)\le\exp\left\{N\left(-\frac12\int_{D\setminus I}|\dot g(t)|^2dt+\Sigma_0(a,b;t_1^1,t_2^k)+\theta\right)\right\}$,

hold for all $N\ge N_0$ and $\mathbf{j}\in\mathcal{J}_k$, $\mathbf{t}\in\mathcal{T}_{k,\tilde\gamma}$, $k\ge 1$, satisfying $|j_\ell^p/N-t_\ell^p|\le\eta$ for each $1\le p\le k$ and $\ell=1,2$, where $I=\cup_{p=1}^k[t_1^p,t_2^p]$.

Proof The bound (1) is shown by looking over each step of the proof of Lemma 4.6-(1) attentively; we omit the details. To show the bound (2), since the second term (i.e., the product of probabilities under $\mu^{0,0,\varepsilon}_{j_2^p - j_1^p}$) of $\Psi^{\varepsilon}_{N,\mathbf{j}}(g;\delta)$ is estimated by 1 from above, we may deal with other terms. Since those terms can be treated essentially in a same way, we discuss only the first term denoting j_1^1 simply by j. Choose a sufficiently small $\eta > 0$ (in particular, $\eta < \delta \wedge 1$) in such a manner that $|g(s) - g(t)| \le \delta$ holds for $|s-t| \le \eta$. Then, we have

$$I_N^j(\delta) := \mu_j^{a,0}(\|h^N - g\|_{\infty,[0,j/N]} < \delta) = \mu_j^{a,0}\left(\left\|\frac{j}{N}h^j(\cdot) - g\left(\frac{j}{N}\cdot\right)\right\|_\infty < \delta\right),$$

where $h^N(t)$ on the left hand side is defined for $t \in [0, j/N]$ while $h^j(t)$ on the right is for $t \in D = [0,1]$, and this implies the following uniform estimate in j satisfying $|j/N - s| \le \eta$ with $s = t_1^1$:

$$\mu_j^{a,0}(\|h^N - g\|_{\infty,[0,j/N]} < \delta) \le \mu_j^{a,0}(h^j \in \mathcal{A}_\delta),$$

where $\mathcal{A}_\delta = \{h \in \mathcal{C}; \|h(\cdot) - g(s\cdot)/s\|_\infty < c\delta\}$ for some $c > 0$. Indeed, one can take $c = (2s + \|g\|_\infty)/s(s-\eta)$ by estimating

$$\begin{aligned}\|h(\cdot) - g(s\cdot)/s\|_\infty \le \|h(\cdot) - g(u\cdot)/u\|_\infty + \|g(u\cdot) - g(s\cdot)\|_\infty/u \\ +|u^{-1} - s^{-1}|\,\|g\|_\infty,\end{aligned}$$

for $u = j/N$. Since the event $\mathcal{A}_\delta$ is independent of j, from Proposition 4.2, this leads to the uniform upper bound for $I_N^j(\delta)$:

$$I_N^j(\delta) \le \exp\left\{N\left(-\frac{1}{2}\int_0^s |\dot{g}(t)|^2 dt + \frac{|a|^2}{2s} + \theta\right)\right\}$$

for N large enough, $\delta > 0$ small enough and all j satisfying $|j/N - s| \le \eta$. Thus, repeating this procedure for other terms, we obtain the upper bound (2). □

The LD upper bound (4.7) follows from (4.12) and Lemma 4.7 for $g \in H^1_{a,b}(D)$ satisfying the condition (4.11) by choosing $\gamma > 0$ sufficiently small. The rest of the proof is similar to [12], Proof of Theorem 2.2, Step 2.

4.4 Proof of Theorem 4.1 for $\mu_N^{D,\varepsilon,+}$, $\mu_N^{F,\varepsilon}$ and $\mu_N^{F,\varepsilon,+}$

For $\mu_N^{F,\varepsilon}$, we modify the definition of $\Xi^{\varepsilon}_{N,\mathbf{j}}$ and $\Psi^{\varepsilon}_{N,\mathbf{j}}(g;\delta)$ by replacing their first/last terms with $1/Z_N^{F,\varepsilon}$, $Z^{0,F}_{N-j_2^K}$ and $\mu^{0,F}_{N-j_2^K}(\|h^N - g\|_{\infty,[j_2^K/N,1]} < \delta)$, respectively. The modification is also clear under the presence of a wall. Then, one can follow the steps presented in Sect. 4.3 and obtain Theorem 4.1 for $\mu_N^{D,\varepsilon,+}$, $\mu_N^{F,\varepsilon}$ and $\mu_N^{F,\varepsilon,+}$.

Appendix A: Critical exponents for the free energies

Here we study the asymptotic behavior of the free energies ξ_r^ε and $\xi_r^{\varepsilon,+}$ near the critical values $\varepsilon_c = \varepsilon_{c,r}$ and $\varepsilon_c^+ = \varepsilon_{c,r}^+$, respectively. In general, when the physical order parameter exhibits a power law behavior in ε close to its critical value, the power is called the critical exponent. Our results give such critical exponents associated with the free energies.

Recall that the free energies ξ_r^ε and $\xi_r^{\varepsilon,+}$ are defined by the thermodynamic limits (1.4) and characterized by the equations: $g_r(e^{-\xi_r^\varepsilon}) = g_r^+(e^{-\xi_r^{\varepsilon,+}}) = 1/\varepsilon$. We put the subscripts r for g_r and g_r^+ to indicate their dependence on r. These functions are defined by (2.2) and (2.10), respectively, i.e., $g_r(x) = f_r(x)/(2\pi)^{r/2}$ and $g_r^+(x) = f_{r+2}(x)/(2\pi)^{r/2}$, where f_r is the so-called polylogarithm given by the power series:

$$f_r(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{r/2}} \tag{A.1}$$

for $0 \le x < 1$ (or $0 \le x \le 1$). The critical values $\varepsilon_{c,r}$ and $\varepsilon_{c,r}^+$ are determined by $\varepsilon_{c,r} = 1/g_r(1)$ and $\varepsilon_{c,r}^+ = 1/g_r^+(1)$ as in (2.3) and (2.11), respectively. Recall that $\varepsilon_{c,r} = 0$ for $r = 1, 2$, $\varepsilon_{c,r} = (2\pi)^{r/2}/\zeta(r/2) > 0$ for $r \ge 3$ and $\varepsilon_{c,r}^+ = (2\pi)^{r/2}/\zeta(r/2+1) > 0$ for all $r \ge 1$, where ζ is the Riemann's ζ-function.

The results of this section are summarized in the following proposition.

Proposition A.1 1. (Absence of wall) *As $\varepsilon \downarrow \varepsilon_{c,r}$, we have that*

$$\xi_r^\varepsilon \sim \begin{cases} C_r(\varepsilon - \varepsilon_{c,r})^2, & r = 1, 3, \\ e^{-2\pi/\varepsilon}, & r = 2, \\ C_4\, \varphi(\varepsilon - \varepsilon_{c,4}), & r = 4, \\ C_r(\varepsilon - \varepsilon_{c,r}), & r \ge 5, \end{cases}$$

where $C_1 = 1/2$, $C_3 = 2\pi^2/\varepsilon_{c,3}^4$, $C_4 = 4\pi^2/\varepsilon_{c,4}^2$, $C_r = 2\pi\varepsilon_{c,r-2}/\varepsilon_{c,r}^2$ for $r \ge 5$ and $\varphi(x) = -x/\log x$ for sufficiently small $x > 0$.

2. (Presence of wall) *As $\varepsilon \downarrow \varepsilon_{c,r}^+$, we have that*

$$\xi_r^{\varepsilon,+} \sim \begin{cases} C_1^+(\varepsilon - \varepsilon_{c,1}^+)^2, & r = 1, \\ C_2^+\, \varphi(\varepsilon - \varepsilon_{c,2}^+), & r = 2, \\ C_r^+(\varepsilon - \varepsilon_{c,r}^+), & r \ge 3, \end{cases}$$

where $C_1^+ = (2\pi)^2 C_3 \left(= 1/2(\varepsilon_{c,1}^+)^4\right)$, $C_2^+ = 2\pi C_4 \left(= 2\pi/(\varepsilon_{c,2}^+)^2\right)$ and $C_r^+ = 2\pi C_{r+2} \left(= 2\pi\varepsilon_{c,r-2}^+/(\varepsilon_{c,r}^+)^2\right)$ for $r \ge 3$.

Remark A.1 Proposition A.1 indicates that the critical exponents κ_r and κ_r^+ associated with the free energies ξ_r^ε and $\xi_r^{\varepsilon,+}$, respectively, are given by $\kappa_1 = 2$, $\kappa_2 = \infty$, $\kappa_3 = 2$, $\kappa_4 = 1+$ and $\kappa_r = 1$ for $r \ge 5$, while $\kappa_1^+ = 2$, $\kappa_2^+ = 1+$ and $\kappa_r^+ = 1$ for $r \ge 3$. Here $\kappa = \infty$ means that the free energy vanishes faster than any power of ε, and $\kappa = 1+$

means that the exponent is 1 with a logarithmic correction. The asymptotic behavior of ξ_1^ε is studied in [2].

Lemma A.2 *We have that* $2\pi\varepsilon_{c,r}^+ = \varepsilon_{c,r+2}$ *and* $\xi_r^{\varepsilon,+} = \xi_{r+2}^{2\pi\varepsilon}$ *for all* $\varepsilon \ge 0$ *and* $r \ge 1$.

Proof The conclusion is immediate from the relation $g_r^+(x) = 2\pi g_{r+2}(x)$. □

The following asymptotics for the functions f_r as $x \uparrow 1$ may be well-known, but we give the proof for the completeness.

Lemma A.3 *As* $x \uparrow 1$, *we have that*

$$\begin{aligned}
&f_1(x) \sim \sqrt{\pi}(1-x)^{-1/2},\\
&f_2(x) = -\log(1-x),\\
&f_3(1) - f_3(x) \sim 2\sqrt{\pi}(1-x)^{1/2},\\
&f_4(1) - f_4(x) \sim -(1-x)\log(1-x),\\
&f_r(1) - f_r(x) \sim \zeta(r/2-1)(1-x), \quad r \ge 5.
\end{aligned}$$

Proof The result for f_1 is a consequence of the Tauberian theorem, see [7], Theorem 5, p. 447. When $r = 2$, (A.1) is nothing but the Taylor expansion of $-\log(1-x)$ at $x = 0$. For $r = 3$ and 4, the relation $x f_r'(x) = f_{r-2}(x)$ for $0 < x < 1$ implies

$$f_r(1) - f_r(x) = \int_x^1 \frac{f_{r-2}(y)}{y}\,dy,$$

and this combined with the results for f_1 and f_2 shows the asymptotics for f_3 and f_4. If $r \ge 5$, f_r is differentiable at $x = 1$ from the left and $f_r'(1-) = f_{r-2}(1) = \zeta((r-2)/2)$. This shows the last asymptotic formula. □

Proof of Proposition A.1 The assertion (1) follows from Lemma A.3 recalling that $f_r(e^{-\xi_r^\varepsilon}) = (2\pi)^{r/2}/\varepsilon$. Note that $1 - e^{-\xi_r^\varepsilon} \sim \xi_r^\varepsilon$ as $\varepsilon \downarrow \varepsilon_{c,r}$ and, if $r \ge 3$ in addition, $f_r(1) = (2\pi)^{r/2}/\varepsilon_{c,r}$. Also note that $\psi^{-1}(x) \sim \varphi(x)$ holds as $x \downarrow 0$ for the inverse function ψ^{-1} of $\psi(\xi) = -\xi\log\xi$ defined for small enough $\xi > 0$. The assertion (2) follows from (1) combined with Lemma A.2. □

Appendix B: Structure of minimizers in $d = 1$

In this section, we consider the case where $d = 1$ and $m = 0$ so that $a, b \in \mathbb{R}$, and clarify the structure of the set of minimizers of $\Sigma = \Sigma^D, \Sigma^{D,+}, \Sigma^F$ and $\Sigma^{F,+}$. Indeed, for each $\xi > 0$, the minimizers of Σ^D (or $\Sigma^{D,+}$) are completely characterizable in terms of $(a,b) \in \mathbb{R}^2$ (or $(a,b) \in \mathbb{R}_+^2$), and those of Σ^F (or $\Sigma^{F,+}$) in $a \in \mathbb{R}$ (or $a \in \mathbb{R}_+$) as well. The result is summarized in the following proposition. In particular, if a and b have different signs, Σ^D (or $\Sigma^{D,+}$) admits a unique minimizer h^*. We simply write ξ, $\bar{h}$ and $\hat{h}$ omitting the superscripts D, F, ε and $+$.

Proposition B.1 *Assume that $\xi > 0$, namely, $\varepsilon > 0$ is arbitrary under the absence of wall and $\varepsilon > \varepsilon_c^+$ under the presence of wall.*
1. (Dirichlet case) *Let $\mathcal{O}$ be the bounded open region of $\mathbb{R}^2$ surrounded by its boundary $\mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1 = \{\sqrt{|a|} + \sqrt{|b|} = (2\xi)^{1/4}, ab > 0\}$ and $\mathcal{C}_2 = \{|a| + |b| = (2\xi)^{1/2}, ab \le 0\}$, which consists of four curves (see Fig. A). Then, the set $\mathcal{H}^D$ of all minimizers of Σ^D (or $\Sigma^{D,+}$) is given as follows: $\mathcal{H}^D = \{\hat{h}\}$ on $\mathcal{O}$, $\mathcal{H}^D = \{\bar{h}\}$ on $\mathbb{R}^2 \setminus \bar{\mathcal{O}}$, $\mathcal{H}^D = \{\hat{h}, \bar{h}\}$ on $\mathcal{C}_1$ and $\mathcal{H}^D = \{\bar{h}\}$ on $\mathcal{C}_2$. Note that $\hat{h} = \bar{h}$ on $\mathcal{C}_2 \cup \{(0,0)\}$ and $\hat{h} \ne \bar{h}$ on $\mathcal{C}_1$.*
2. (Free case) *Let $\mathcal{H}^F$ be the set of all minimizers of Σ^F (or $\Sigma^{F,+}$). Then, $\mathcal{H}^F = \{\hat{h}\}$ on $\{|a| < \sqrt{\xi/2}\}$, $\mathcal{H}^F = \{\bar{h}\}$ on $\{|a| > \sqrt{\xi/2}\}$ and $\mathcal{H}^F = \{\hat{h}, \bar{h}\}$ on $\{|a| = \sqrt{\xi/2}\}$. Note that $\hat{h} = \bar{h}$ at $a = 0$ and $\hat{h} \ne \bar{h}$ at $|a| = \sqrt{\xi/2}$.*

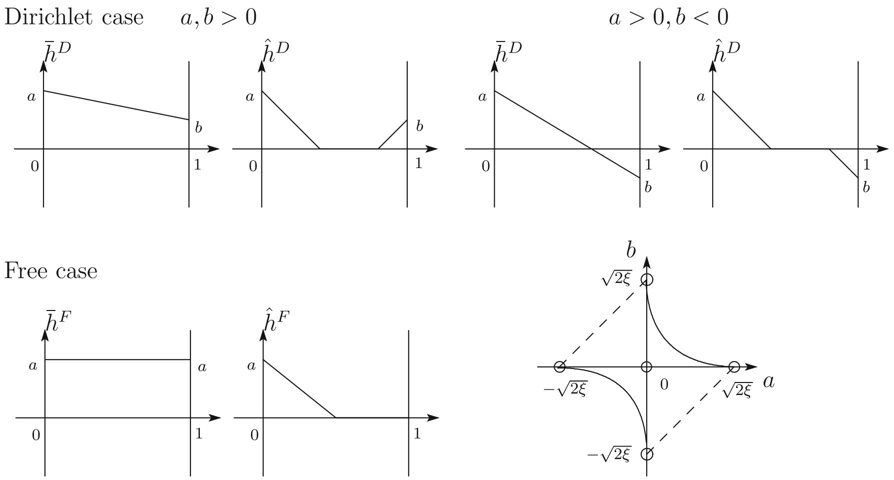

Fig. A

Proof We first give the proof of (1) assuming $a, b > 0$. If $a + b \ge \sqrt{2\xi}$ in addition, then $\bar{h}$ is the minimizer since it is the unique candidate in this case. If $a + b < \sqrt{2\xi}$, noting that

$$\Sigma(\hat{h}) = \frac{a^2}{2t_1} + \frac{b^2}{2t_2} - \xi(1 - t_1 - t_2) = \sqrt{2\xi}(a + b) - \xi$$

by Young's relation, we have

$$2\left\{\Sigma(\bar{h}) - \Sigma(\hat{h})\right\} = a^2 - 2a(b + 2\xi) + \left(b - \sqrt{2\xi}\right)^2.$$

Therefore, we easily see that $\Sigma(\bar{h}) = \Sigma(\hat{h})$ is equivalent to $\sqrt{a} + \sqrt{b} = (2\xi)^{1/4}$ (noting that $a + b < \sqrt{2\xi}$) and the conclusion of (1) follows when $a, b > 0$. The case where $a, b < 0$ can be reduced to this case by symmetry. The case where $a > 0$ and $b < 0$ is also a simple computation. The minimizers of Σ^F (or $\Sigma^{F,+}$) are easily studied so that the proof of (2) is immediate. □

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