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Convolution equivalence and distributions of random sums

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Abstract A serious gap in the Proof of Pakes's paper on the convolution equivalence of infinitely divisible distributions on the line is completely closed. It completes the real analytic approach to Sgibnev's theorem. Then the convolution equivalence of random sums of IID random variables is discussed. Some of the results are applied to random walks and Lévy processes. In particular, results of Bertoin and Doney and of Korshunov on the distribution tail of the supremum of a random walk are improved. Finally, an extension of Rogozin's theorem is proved.

Keywords Convolution equivalence · Subexponentiality · O-subexponentiality · Infinite divisibility · Random sum · IID

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1 Introduction

The subexponentiality of one-sided infinitely divisible distributions was completely characterized by Embrechts et al. [10]. See Theorem A below. Following this, the study on the convolution equivalence of those distributions was started by Embrechts and Goldie [8,9]. To complete their real analytic approach, they tried to solve the problem on the closedness of convolution roots in the class of one-sided convolution equivalent distributions. They conjectured that the class is closed under convolution roots. However, it is a difficult problem and is not yet completely solved. See Remark 5.3 below. Later, Sgibnev [32] gave an assertion on the convolution equivalence of

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two-sided infinitely divisible distributions. See Theorem B below. His Proof uses Banach algebraic arguments and contains a slightly obscure part in the Proof that (1) implies (2) in Theorem B. Indeed, the assertions on line 5 to 11 on p. 117 are not clear at least to the author. Recently, Pakes [23] asserted in Theorem 3.1 that he proved the same theorem. His "Proof" employs real analytic methods without depending on the closedness problem of convolution roots. However, unfortunately, there is a serious gap in his "Proof" that (1) implies (2) in Theorem B. Specifically, there are two points in the gap. The first point is that, in p. 416, he wrongly used Corollary 2.14 of Cline [4] in a stronger sense than the original statement. See Remark 1.2 below. The second point is that this corollary of Cline is incomplete, as was later pointed out in Remark 4.2 of Shimura and Watanabe [34]. Pakes [23] is cited by many papers without noticing the critical gap. Just in the middle of the review of the present paper, Pakes [24] and Wang et al. [38] tried to rescue Pakes's "Proof" of [23]. After all their efforts, they overcame the second point, but they did not notice the first point in the gap. In this paper, we definitely restore Pakes's "Proof" of [23] and complete the real analytic approach to Theorem B. It should be noted that Cline [4] is an interesting paper but contains many wrong or doubtful results; their influences to papers of other people are not small. It is also explained in the introductions of [24, 38]. We gave in Remark 4.2 of [34] a counterexample for (iv) of Lemma 2.1 and pointed out that in its influence there are many doubtful results in [4] and in the subsequent papers. Among them, it can be shown that (iii) of Lemma 3.1, (i) of Corollary 3.2, and Theorem 3.4 of [4] are erroneous. On the other hand, Corollary 2.14 of [4] was proved to be true under condition (i) or (ii) there by [38] or [24], respectively. Thus Corollary 3.3, Theorem 4.1, and Corollary 4.2 of [17] are recovered, although the original Proofs depended on Corollary 2.14 of [4]. Simultaneously, Theorem 1.1 of [37] is rescued, although Corollary 3.3 of [17] was used in the Proof. However, there still remain doubtful results in [4] other than those mentioned above. Some of them are explained in Remarks 1.4 and 5.1 below.

In what follows, we denote by $\mathbb{R}$ the real line and by $\mathbb{R}_+$ the half line $[0, \infty)$. We denote by $\delta_a(dx)$ the delta measure at $a \in \mathbb{R}$. Let η and ρ be probability measures on $\mathbb{R}$. We denote the convolution of η and ρ by $\eta * \rho$ and denote nth convolution power of ρ by ρ^{n*} with the understanding that $\rho^{0*}(dx) = \delta_0(dx)$. The right-tail of a measure ξ on $\mathbb{R}$ is denoted by $\bar{\xi}(x)$, that is, $\bar{\xi}(x) := \xi(x, \infty)$ for $x \in \mathbb{R}$. Let $\gamma \geq 0$. The γ-exponential moment of ξ is denoted by $\widehat{\xi}(\gamma)$, namely, $\widehat{\xi}(\gamma) := \int_{-\infty}^{\infty} e^{\gamma x}\xi(dx)$. For positive measurable functions $f(x)$ and $g(x)$ on $\mathbb{R}$, we define the relation $f(x) \sim g(x)$ by $\lim_{x\to\infty} f(x)/g(x) = 1$ and the relation $f(x) \asymp g(x)$ by $0 < \liminf_{x\to\infty} f(x)/g(x) \leq \limsup_{x\to\infty} f(x)/g(x) < \infty$. A distribution ρ on $\mathbb{R}$ is said to belong to the class $\mathcal{L}(\gamma)$ if $\bar{\rho}(x) > 0$ for every $x \in \mathbb{R}$ and if

$$\bar{\rho}(x + a) \sim e^{-\gamma a}\bar{\rho}(x) \qquad \text{for every} \quad a \in \mathbb{R}. \tag{1.1}$$

A distribution ρ on $\mathbb{R}$ is said to belong to the class $\mathcal{S}(\gamma)$ if $\rho \in \mathcal{L}(\gamma)$ and if

$$\overline{\rho * \rho}(x) \sim 2\widehat{\rho}(\gamma)\bar{\rho}(x). \tag{1.2}$$

Distributions which belong to $\mathcal{S}(\gamma)$ for some $\gamma \geq 0$ are called convolution equivalent. Among them distributions in $\mathcal{S}(0)$ are called subexponential. Note that if $\gamma = 0$ and ρ is one-sided on $\mathbb{R}_+$, then (1.2) implies (1.1). Concerning details and examples of distributions in $\mathcal{S}(0)$; see [12]. Let μ be an infinitely divisible distribution on $\mathbb{R}$. Then the characteristic function of μ is represented as follows:

$$\int_{-\infty}^{\infty} \exp(izx)\mu(dx) = \exp(\psi(z)), \qquad z \in \mathbb{R} \tag{1.3}$$

with

$$\psi(z) = \int_{-\infty}^{\infty} \left(e^{izx} - 1 - 1_{\{|x|\leqslant 1\}}(x)izx\right) \nu(dx) + iaz - \frac{1}{2}bz^2, \tag{1.4}$$

where $a \in \mathbb{R}$, $b \geq 0$ and ν is a measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty}(1 \wedge |x|^2)\nu(dx) < \infty$. The number b and the measure ν are called Gaussian variance and Lévy measure of μ, respectively. See [28]. For $c > 0$, define the right-hand normalization ν_c of ν by

$$\nu_c(dx) := \frac{1}{\bar{\nu}(c)} 1_{(c,\infty)}(x)\nu(dx), \tag{1.5}$$

whenever $\bar{\nu}(c) > 0$. Denote by μ^{t*} tth convolution power of μ for $t > 0$.

As an earlier result, Embrechts et al. [10] proved the following theorem by using real analytic methods.

Theorem A *Let μ be an infinitely divisible distribution on $\mathbb{R}_+$ with Lévy measure ν. Then the following are equivalent.*

(1) $\mu \in \mathcal{S}(0)$.
(2) $\nu_1 \in \mathcal{S}(0)$.
(3) $\bar{\mu}(x) \sim \bar{\nu}(x)$.

Theorem A was extended by Embrechts and Goldie [9] to an assertion involving $\mathcal{S}(\gamma)$ for a class of infinitely divisible distributions on $\mathbb{R}_+$. Then, treating distributions on $\mathbb{R}$, Sgibnev [32] asserted that he extended the above theorem in the following way.

Theorem B *Let $\gamma \geq 0$. Let μ be an infinitely divisible distribution on $\mathbb{R}$ satisfying* (1.3) *and* (1.4). *Then the following are equivalent.*

(1) $\mu \in \mathcal{S}(\gamma)$.
(2) $\nu_1 \in \mathcal{S}(\gamma)$.
(3) $\nu_1 \in \mathcal{L}(\gamma)$, $\widehat{\mu}(\gamma) < \infty$, *and* $\bar{\mu}(x) \sim \widehat{\mu}(\gamma)\bar{\nu}(x)$.

Theorems A and B are important and useful in many applications for Lévy processes and other processes related to infinite divisibility. In fact, Theorem B is used by [6,14,18,24,29,34]. See also [7]. Results analogous to Theorems A and B for O-subexponentiality and the dominated variation of infinitely divisible distributions on $\mathbb{R}_+$ are found in [34] and [39]. As mentioned above, in order to restore Pakes's Proof of Theorem B, it is enough to prove the first assertion of the following theorem.

Theorem 1.1 *It is true that* (1) *implies* (2) *in Theorem B. Further* (1) *is equivalent to the following statement.*

(4) $\nu_1 \in \mathcal{L}(\gamma)$ *and* $\bar{\mu}(x) \sim d\bar{\nu}(x)$ *with some* $d \in (0, \infty)$.

Remark 1.2 There are three papers [23,24,38] which claim to prove Theorem B by using real analytic arguments. However their "Proofs" that (1) implies (2) have the same serious mistake. They used Corollary 2.14 of [4] or its restoration by [24,38]. But the additional condition that $\nu_1 \in \mathcal{L}(\gamma)$ is assumed in the corollary. Thus what they have exactly proved is that if $\nu_1 \in \mathcal{L}(\gamma)$ and $\mu \in \mathcal{S}(\gamma)$, then $\nu_1 \in \mathcal{S}(\gamma)$. In our theorem above, such an additional assumption does not exist.

We can apply Theorem B to the distribution of a Lévy process $\{X(t)\}$ on $\mathbb{R}$ with μ being the distribution of $X(1)$. Assertion (i) below with $\gamma = 0$ is called in [10] a conjecture of F. W. Steutel.

Corollary 1.3 *Let* $\gamma \geq 0$. *Let* μ *be an infinitely divisible distribution on* $\mathbb{R}$.

(i) *If* $\mu^{t*} \in \mathcal{S}(\gamma)$ *for some* $t > 0$, *then* $\mu^{t*} \in \mathcal{S}(\gamma)$ *for all* $t > 0$ *and*

$$\overline{\mu^{t*}}(x) \sim t\widehat{\mu}(\gamma)^{t-1}\bar{\mu}(x) \quad \text{for all } t > 0. \tag{1.6}$$

(ii) *If* $\mu \in \mathcal{L}(\gamma)$ *and*

$$\overline{\mu^{t*}}(x) \sim t\widehat{\mu}(\gamma)^{t-1}\bar{\mu}(x) \quad \text{for some } t \in (0,1) \cup (1,\infty), \tag{1.7}$$

then $\mu \in \mathcal{S}(\gamma)$.

Next, to the end of this section, we discuss the convolution equivalence of the distributions of random sums of IID random variables. In particular, the distributions of random sums of IID random variables naturally appear in those of continuous time random walks (CTRW, in short) introduced by [22]. A CTRW is a random walk subordinated to a renewal process. It is also called a renewal model in insurance theory; see [12]. A compound Poisson process is an instance of CTRW. Let $\{p_n\}_{n=0}^{\infty}$ be a nonnegative sequence satisfying $\sum_{n=0}^{\infty} p_n = 1$ and $p_0 + p_1 < 1$. For a distribution ρ on $\mathbb{R}$, we define a compound distribution η on $\mathbb{R}$ by

$$\eta := \sum_{n=0}^{\infty} p_n \rho^{n*}. \tag{1.8}$$

Let $\{X_n\}_{n=1}^{\infty}$ be IID random variables with distribution ρ on $\mathbb{R}$ and let τ be a nonnegative integer-valued random variable independent of $\{X_n\}_{n=0}^{\infty}$ with $P(\tau = n) = p_n$. Let $\{S_n\}_{n=0}^{\infty}$ be a random walk on $\mathbb{R}$ defined by $S_0 := 0$ and $S_n := \sum_{k=1}^{n} X_k$ for $n \geq 1$.

Then η is the distribution of the random sum S_τ. Note that if the distribution $\sum_{n=0}^{\infty} p_n \delta_n(dx)$ of τ is infinitely divisible, then η is also infinitely divisible; see [28, E34.5] or [35, Proposition IV 3.1]. We define the convolution $p \otimes p$ of $p = \{p_n\}$ with itself by $(p \otimes p)_n := \sum_{k=0}^{n} p_{n-k} p_k$. Then $\eta^{2*} = \sum_{n=0}^{\infty} (p \otimes p)_n \rho^{n*}$. We consider the following problem.

Problem 1 Let $\gamma \geq 0$ and let ρ and η be as above. Are the following statements equivalent ?

(a) $\eta \in \mathcal{S}(\gamma)$.
(b) $\rho \in \mathcal{S}(\gamma)$.
(c) $\rho \in \mathcal{L}(\gamma)$, $\widehat{\rho}(\gamma) < \infty$, and $\bar{\eta}(x) \sim \sum_{n=1}^{\infty} n p_n \widehat{\rho}(\gamma)^{n-1} \bar{\rho}(x)$.

The following theorem gives a partial but substantial answer to the problem above except whether (a) implies (b). In Theorem 1.5 below we present a negative answer to the problem whether (a) implies (b). But, under what condition (a) implies (b) is an unsolved problem even in the case of assuming $\rho \in \mathcal{L}(\gamma)$.

Theorem C *Let $\gamma \geq 0$ and let the distribution η and the assertions* (a), (b), *and* (c) *be the same as those in Problem 1. Assume that there is $\varepsilon > 0$ such that*

$$\sum_{n=1}^{\infty} p_n \left((\widehat{\rho}(\gamma) + \varepsilon) \vee 1 \right)^n < \infty. \tag{1.9}$$

Then (b) *and* (c) *are equivalent, and moreover,* (b) *implies* (a).

Remark 1.4 It is obvious from Lemma 2.6 in the next section that (a) implies (b) under the assumption that $\rho \in \mathcal{L}(\gamma)$ and $\bar{\eta}(x) = O(\bar{\rho}(x))$. This assertion is also found in [24,38]. Theorem 5.1 of Pakes [23] asserted a two-sided extension of Theorem 2.13 of [4]. Theorem C is the correct part of Theorem 5.1 of [23]. In Theorem 5.1 of [23], it is stated that (a) implies (b) under the assumption that $\rho \in \mathcal{L}(\gamma)$ and $\bar{\rho}(x) \neq o(\bar{\eta}(x))$. However the Proof of this statement contains a gap since it depends on the yet unjustified part of Theorem 2.13 of [4]. So far, the statement is neither proved nor denied even in the case $\gamma = 0$.

It is not easy to solve Problem 1 completely. However, we can introduce a new idea for this problem and give a counterexample for the assertion that (a) implies (b). The following theorem (Theorem 1.5 below) is quite a bit more general than Theorem 1.2 of [38] as is seen from Remark 1.6 below. The subexponential case (that is, the case $\gamma = 0$) in assertion (ii) in Theorem 1.5 below was already given in [31] in a more general assertion by using the class Γ.

Theorem 1.5 *Let $\gamma \geq 0$ and let the assertions* (a) *and* (b) *be the same as those in Problem 1. Assume that p_n is positive for all sufficiently large n.*

(i) *Assume that $\rho \in \mathcal{L}(\gamma)$ and satisfies either*

$$\liminf_{n\to\infty} \frac{(p \otimes p)_n}{p_n} > 2\widehat{\eta}(\gamma), \tag{1.10}$$

or

$$\limsup_{n\to\infty} \frac{(p\otimes p)_n}{p_n} < 2\widehat{\eta}(\gamma). \tag{1.11}$$

Then (a) *implies* (b).

(ii) *Assume that, for some* $0 < r \le 1$*, it holds that* $0 < d := \sum_{n=0}^{\infty} r^{-n} p_n < \infty$ *and*

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = r \text{ and } \lim_{n\to\infty} \frac{(p\otimes p)_n}{p_n} = 2d. \tag{1.12}$$

If $r = d = 1$ *and* ρ *is one-sided on* $\mathbb{R}_+$ *with*

$$\lim_{x\to\infty} \frac{\overline{\rho^{(n+1)*}}(x)}{\overline{\rho^{n*}}(x)} = \infty \quad \text{for every } n \ge 1, \tag{1.13}$$

then $\eta \in \mathcal{S}(0)$ *on* $\mathbb{R}_+$. *If* $0 < r < 1$*, then, for any* $\gamma > 0$*, there is* ρ *such that* $\eta \in \mathcal{S}(\gamma)$ *with* $d = \widehat{\eta}(\gamma)$*, but* $\rho \in \mathcal{L}(\gamma+\delta)$ *for some* $\delta > 0$ *and hence* $\rho \notin \mathcal{S}(\gamma)$.

Remark 1.6 (i) If $\lim_{n\to\infty} p_{n+1}/p_n = 0$, then $\lim_{n\to\infty}(p\otimes p)_n/p_n = \infty$ and hence (1.10) holds.

(ii) Let $p_n = a_n r^n$ with $0 < r \le 1$, $a_n \ge 0$, and $\lim_{n\to\infty} a_{n+1}/a_n = 1$. Then it is obvious from $\liminf_{n\to\infty}(p\otimes p)_n/p_n \ge 2\sum_{n=0}^{\infty} a_n$ that the condition $\sum_{n=0}^{\infty} a_n > \widehat{\eta}(\gamma)$ implies (1.10).

(iii) The compound Poisson and negative binomial cases are, respectively, typical examples of (i) and (ii) above. Namely $\lim_{n\to\infty}(p\otimes p)_n/p_n = \infty$ provided that

$$p_n = e^{-c}\frac{c^n}{n!} \quad \text{with } c > 0, \tag{1.14}$$

or that

$$p_n = \binom{\alpha+n-1}{\alpha-1}(1-\lambda)^{\alpha}\lambda^n \qquad \text{with} \quad 0 < \lambda < 1 \text{ and } \alpha > 0. \tag{1.15}$$

(iv) If $p_n > 0$ for all sufficiently large n and

$$\limsup_{n\to\infty} \frac{p_{n+1}}{p_n} < \frac{1}{\widehat{\rho}(\gamma)}, \tag{1.16}$$

then (1.10) holds. Thus Theorem 1.5 contains Theorem 1.2 of [38], which is a corrected and two-sided version of Corollary 2.14 of [4] under condition (i) there.

(v) In the case (1.12) with $d = \widehat{\eta}(\gamma)$, we do not know whether there is a distribution ρ on $\mathbb{R}$ such that $\rho \in \mathcal{L}(\gamma) \setminus \mathcal{S}(\gamma)$ but $\eta \in \mathcal{S}(\gamma)$. If $\rho \in \mathcal{L}(\gamma + \delta)$ for some $\delta > 0$ and $\eta \in \mathcal{S}(\gamma)$, then ρ does not satisfy (1.9).

We present a corrected and two-sided version of Corollary 2.14 of [4] under condition (ii) as follows. The one-sided case was already found in Lemma 3.4 of [24].

Proposition 1.7 *Let $\gamma \geq 0$ and let ρ and η be the same as in (1.8). Assume that $\rho \in \mathcal{L}(\gamma)$ and $\eta \in \mathcal{S}(\gamma)$. Let $r := \sum_{n=1}^{\infty} p_n(\rho[0,\infty))^{n-1}$. If, for some $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} p_n \left(\left(r^{-1}\widehat{\eta}(\gamma) + \varepsilon \right) \vee 1 \right)^n < \infty, \tag{1.17}$$

then $\rho \in \mathcal{S}(\gamma)$ and (1.9) holds. In particular, if $p_0 = 0$, $\gamma = 0$, and ρ is one-sided on $\mathbb{R}_+$, then (1.17) is the same as (1.9).

Finally, we obtain the following application of Theorem 1.1 to compound negative binomial distributions on $\mathbb{R}$. Thus the statements (a)–(c) in Problem 1 are equivalent in the compound Poisson case by virtue of Theorem B and in the compound negative binomial case by virtue of Theorem 1.8 below. Note that (a) implies (b) without assuming $\rho \in \mathcal{L}(\gamma)$ in these cases. Theorem 4.1 of [24] states an analogous result, but the additional condition $\rho \in \mathcal{L}(\gamma)$ is needed when (a) implies (b). See [10] for $\gamma = 0$ and $\alpha = 1$. We define a function $\varphi_\alpha(s)$ on $(-1/\lambda, 1/\lambda)$ for $0 < \lambda < 1$ and $\alpha > 0$ by

$$\varphi_\alpha(s) := \left[\frac{1-\lambda}{1-\lambda s} \right]^\alpha = \sum_{n=0}^{\infty} \binom{\alpha+n-1}{\alpha-1} (1-\lambda)^\alpha \lambda^n s^n. \tag{1.18}$$

Denote by $\varphi'_\alpha(s)$ the derivative of $\varphi_\alpha(s)$.

Theorem 1.8 *Let $0 < \lambda < 1$, $\alpha > 0$, and $\gamma \geq 0$. Let ρ be a distribution on $\mathbb{R}$ satisfying $\lambda \widehat{\rho}(\gamma) < 1$. Define a distribution η_α on $\mathbb{R}$ by*

$$\eta_\alpha := \sum_{n=0}^{\infty} \binom{\alpha+n-1}{\alpha-1} (1-\lambda)^\alpha \lambda^n \rho^{n*}. \tag{1.19}$$

Then the following are equivalent.

(a) *$\eta_\alpha \in \mathcal{S}(\gamma)$ for some, equivalently for all, $\alpha > 0$.*
(b) *$\rho \in \mathcal{S}(\gamma)$.*
(c) *$\rho \in \mathcal{L}(\gamma)$ and $\bar{\eta}_\alpha(x) \sim \varphi'_\alpha(\widehat{\rho}(\gamma))\, \bar{\rho}(x)$.*

We define a distribution ρ_I on $\mathbb{R}_+$ for the distribution ρ on $\mathbb{R}_+$ satisfying $\int_0^\infty x\rho(dx) < \infty$ as

$$\rho_I(dx) := \frac{\bar{\rho}(x)}{\int_0^\infty x\rho(dx)} dx. \tag{1.20}$$

Note that

$$\widehat{\rho}_I(\gamma) = \frac{\widehat{\rho}(\gamma) - 1}{\gamma \int_0^\infty x\rho(dx)}. \tag{1.21}$$

The following corollary is an extension of Corollary 4.2 of [17].

Corollary 1.9 *Let* $0 < \lambda < 1$, $\alpha > 0$, *and* $\gamma > 0$. *Let* ρ *be a distribution on* $\mathbb{R}_+$ *satisfying* $\int_0^\infty x\rho(dx) < \infty$ *and* $\lambda\widehat{\rho}_I(\gamma) < 1$. *Define a distribution* η_α *on* $\mathbb{R}_+$ *by*

$$\eta_\alpha := \sum_{n=0}^{\infty} \binom{\alpha + n - 1}{\alpha - 1} (1-\lambda)^\alpha \lambda^n \rho_I^{n*}. \tag{1.22}$$

Then the following are equivalent.

(a) $\eta_\alpha \in \mathcal{S}(\gamma)$ *for some, equivalently for all,* $\alpha > 0$.
(b) $\rho \in \mathcal{S}(\gamma)$.
(c) $\rho_I \in \mathcal{S}(\gamma)$.
(d) $\rho \in \mathcal{L}(\gamma)$ *and* $\bar{\eta}_\alpha(x) \sim \varphi_\alpha'(\widehat{\rho}_I(\gamma))\bar{\rho}(x)/\left(\gamma \int_0^\infty x\rho(dx)\right)$.

Theorem 1.8 and Corollary 1.9 with $\alpha = 1$ are frequently used in queueing theory, ruin theory, and branching processes. In the above we have given statements of our main results.

In Sect. 2, we collect preliminaries for the Proofs of the above results. In Sect. 3, we prove our main results. In Sect. 4, we give an application of Theorem 1.8 to the supremum of a random walk. Namely, Theorem 1 of [1] and Theorem 2 of [20] are improved. We also give an application of our results to a compounding of a random walk. In Sect. 5, we present some future problems to be solved. In Sect. 6, we add an extension of Rogozin's theorem. In a separate paper, we shall discuss local subexponentiality and random sums of IID random variables.

2 Properties of the classes $\mathcal{L}(\gamma)$ and $\mathcal{S}(\gamma)$

In this section, we give several lemmas which were extended to two-sided case by [23] and add new preliminary results on $\mathcal{L}(\gamma)$ and $\mathcal{S}(\gamma)$. Proposition 2.7 below is of interest from the viewpoint of the distributions of random walks. It plays a key role in the Proof of Theorem 1.5 and Proposition 1.7. Throughout this section, we define μ_+ on $\mathbb{R}_+$ for a distribution μ on $\mathbb{R}$ by $\mu_+(dx) := 1_{[0,\infty)}(x)\mu(dx) + \mu(-\infty, 0)\delta_0(dx)$. If μ is the distribution of a random variable X, then μ_+ is that of $X \vee 0$. Let $q := \mu[0,\infty)$. Define $\rho(dx) := q^{-1}1_{[0,\infty)}(x)\mu(dx)$ in case $q > 0$ and $\rho := 0$ in case $q = 0$, and $\sigma(dx) := (1-q)^{-1}1_{(-\infty,0)}(x)\mu(dx)$ in case $q < 1$ and $\sigma := 0$ in case $q = 1$. Then $\mu(dx) = q\rho(dx) + (1-q)\sigma(dx)$ and $\mu_+(dx) = q\rho(dx) + (1-q)\delta_0(dx)$.

Lemma 2.1 (Lemma 2.4 and Corollary 2.1 of [23]) *Let* $\gamma \geq 0$. *Let* μ, μ_1, *and* μ_2 *be distributions on* $\mathbb{R}$. *If* $\mu_1 \in \mathcal{S}(\gamma)$ *and* $\bar{\mu}_2(x) \sim c\bar{\mu}_1(x)$ *with* $c \in (0,\infty)$, *then* $\mu_2 \in \mathcal{S}(\gamma)$. *In particular,* $\mu \in \mathcal{S}(\gamma)$ *if and only if* $\mu_+ \in \mathcal{S}(\gamma)$.

The following lemma is a nice and useful result of Pakes [23], which is crucial in the Proofs of Theorem 1.1 and Corollary 1.3. He used Bingham–Teugels's Tauberian theorem in the Proof. See Theorem 4.9.1 of [2]. In [23], the condition that $\widehat{\mu}_1(\beta) < \infty$ for some $\beta > 0$ was missing in assertion (i) below.

Lemma 2.2 (Lemma 2.5 of [23]) *Let μ, μ_1, and μ_2 be distributions on $\mathbb{R}$ satisfying $\mu = \mu_1 * \mu_2$.*

(i) *Assume that $\mu \in \mathcal{L}(0)$, and that $\overline{\mu_1}(x) = o(\overline{\mu}(x))$ and $\widehat{\mu}_1(\beta) < \infty$ for some $\beta > 0$. Then we have*

$$\overline{\mu}(x) \sim \overline{\mu_2}(x). \tag{2.1}$$

(ii) *Let $\gamma > 0$ and let μ_1 and μ_2 be distributions on $\mathbb{R}$. Assume that $\mu \in \mathcal{L}(\gamma)$, and that $\widehat{\mu}_1(\beta) < \infty$ for some $\beta > \gamma$ and*

$$\int_{-\infty}^{\infty} \exp((\gamma + iz)x)\mu_2(dx) \neq 0, \quad \text{for every} \quad z \in \mathbb{R}. \tag{2.2}$$

Then we have

$$\overline{\mu}(x) \sim \widehat{\mu}_1(\gamma)\overline{\mu_2}(x). \tag{2.3}$$

Moreover, if $\mu \in \mathcal{S}(\gamma)$, then $\mu_2 \in \mathcal{S}(\gamma)$.

Lemma 2.3 (Lemmas 5.2 and 5.3 of [23]) *Let $\gamma \geq 0$. Suppose that $\mu \in \mathcal{S}(\gamma)$. Then the following hold.*

(i) *For any $n \geq 1$, $\mu^{n*} \in \mathcal{S}(\gamma)$ and*

$$\overline{\mu^{n*}}(x) \sim n\widehat{\mu}(\gamma)^{n-1}\overline{\mu}(x). \tag{2.4}$$

(ii) *For any $\varepsilon > 0$, there exists $K > 0$ such that*

$$\overline{\mu^{n*}}(x) \leq K((\widehat{\mu}(\gamma) + \varepsilon) \vee 1)^n \overline{\mu}(x) \quad \text{for every} \ x \in \mathbb{R} \text{ and } n \geq 1. \tag{2.5}$$

Lemma 2.4 (Lemma 2.1 of [23]) *Let $\gamma \geq 0$ and let $\mu \in \mathcal{L}(\gamma)$ on $\mathbb{R}$. Assume that a distribution σ satisfies $\bar{\sigma}(x) = o(\bar{\mu}(x))$ and $\widehat{\sigma}(\beta) < \infty$ for some $\beta > \gamma$. Then $\mu * \sigma \in \mathcal{L}(\gamma)$.*

The following lemma shows that the class $\mathcal{L}(\gamma)$ is closed under convolution on $\mathbb{R}$. On the other hand, the class $\mathcal{S}(\gamma)$ is not closed under convolution even on $\mathbb{R}_+$. See [19,21].

Lemma 2.5 *Let $\gamma \geq 0$ and let $\mu_j \in \mathcal{L}(\gamma)$ on $\mathbb{R}$ for $j = 1, 2$. Then $\mu_1 * \mu_2 \in \mathcal{L}(\gamma)$. In particular, if $\mu \in \mathcal{L}(\gamma)$, then $\mu^{n*} \in \mathcal{L}(\gamma)$ for $n \geq 1$.*

Proof We show only the first assertion. For $j = 1, 2$, define q_j, ρ_j, and σ_j for μ_j similarly to q, ρ, and σ for μ. Then obviously $\rho_j \in \mathcal{L}(\gamma)$ for $j = 1, 2$, $\overline{\sigma_1}(x) = o(\overline{\rho_2}(x))$, $\overline{\sigma_2}(x) = o(\overline{\rho_1}(x))$, and $\widehat{\sigma}_j(\beta) < \infty$ for some $\beta > \gamma$ and $j = 1, 2$. Thus we have, by Lemma 2.4, $\rho_1 * \sigma_2 \in \mathcal{L}(\gamma)$, and $\rho_2 * \sigma_1 \in \mathcal{L}(\gamma)$. Note that $\rho_1 * \rho_2 \in \mathcal{L}(\gamma)$ by Theorem 3 of [8], $\overline{\sigma_1 * \sigma_2}(x) = 0$ for $x > 0$ and that

$$\mu_1 * \mu_2 = (q_1\rho_1 + (1 - q_1)\sigma_1) * (q_2\rho_2 + (1 - q_2)\sigma_2). \tag{2.6}$$

It follows that $\mu_1 * \mu_2 \in \mathcal{L}(\gamma)$. □

Lemma 2.6 *Let $\gamma \geq 0$. Let μ_j be distributions on $\mathbb{R}$ for $j = 1, 2$. If $\mu_1 \in \mathcal{S}(\gamma)$, $\mu_2 \in \mathcal{L}(\gamma)$, and $\bar{\mu}_2(x) \asymp \bar{\mu}_1(x)$, then $\mu_2 \in \mathcal{S}(\gamma)$.*

Proof Suppose that $\mu_1 \in \mathcal{S}(\gamma)$, $\mu_2 \in \mathcal{L}(\gamma)$, and $\overline{\mu_2}(x) \asymp \overline{\mu_1}(x)$. Then we see from Lemma 2.1 that $(\mu_1)_+ \in \mathcal{S}(\gamma)$, $(\mu_2)_+ \in \mathcal{L}(\gamma)$, and $\overline{(\mu_2)_+}(x) \asymp \overline{(\mu_1)_+}(x)$. Thus we find from Theorem 2.1 of [16] that $(\mu_2)_+ \in \mathcal{S}(\gamma)$ and again from Lemma 2.1 that $\mu_2 \in \mathcal{S}(\gamma)$. □

The following proposition is a two-sided extension of Theorem 2.10 of [9] and Theorem 2 of [10]. Assertion (i) states that the class $\mathcal{S}(\gamma)$ on $\mathbb{R}$ is closed under convolution roots in the class $\mathcal{L}(\gamma)$ on $\mathbb{R}$ for $\gamma > 0$, which is due to Theorem 5.1 of [24]. Assertion (ii) states that the class $\mathcal{S}(0)$ on $\mathbb{R}$ is closed under convolution roots without any additional assumptions. In Theorem 5.1 of [24], an additional condition that $\mu \in \mathcal{L}(0)$ on $\mathbb{R}$ is assumed. The class $\mathcal{L}(\gamma)$ is not closed under convolution roots for $\gamma \geq 0$. See [33]. Note from Theorem B that the class $\mathcal{S}(\gamma)$ on $\mathbb{R}$ is closed under convolution roots in the class of infinitely divisible distributions on $\mathbb{R}$.

Proposition 2.7 (i) (Theorem 5.1 of [24]) *Let $\gamma > 0$ and let $\mu \in \mathcal{L}(\gamma)$ on $\mathbb{R}$. If $\mu^{n*} \in \mathcal{S}(\gamma)$ for some $n \geq 2$, then $\mu \in \mathcal{S}(\gamma)$.*
(ii) *If $\mu^{n*} \in \mathcal{S}(0)$ for some $n \geq 2$, then $\mu \in \mathcal{S}(0)$.*

Proof of (ii) Suppose that $\mu^{n*} \in \mathcal{S}(0)$. Note that $\overline{\sigma^{n*}}(x) = 0$ for $x > 0$. We have for $x > 0$

$$\overline{(\mu_+)^{n*}}(x) - \overline{\mu^{n*}}(x) = \sum_{k=1}^{n-1} \binom{n}{k} q^k (1-q)^{n-k} \int_{-\infty}^{0-} (\overline{\rho^{k*}}(x) - \overline{\rho^{k*}}(x-y))\sigma^{(n-k)*}(dy) \geq 0. \tag{2.7}$$

Thus we see that

$$\liminf_{x\to\infty} \frac{\overline{(\mu_+)^{n*}}(x)}{\overline{\mu^{n*}}(x)} \geq 1. \tag{2.8}$$

Let $A > 0$. Then, noting that $\overline{\rho^{n*}}(x) - \overline{\rho^{n*}}(x-A) \le 0$, we find that for $x > A$

$$\begin{aligned}\overline{(\mu_+)^{n*}}(x) - \overline{\mu^{n*}}(x-A) &\le \sum_{k=1}^{n-1}\binom{n}{k}q^k(1-q)^{n-k}\\ &\quad \int_{-\infty}^{0-}(\overline{\rho^{k*}}(x) - \overline{\rho^{k*}}(x-A-y))\sigma^{(n-k)*}(dy)\\ &\le \sum_{k=1}^{n-1}\binom{n}{k}q^k(1-q)^{n-k}\overline{\rho^{k*}}(x)\sigma^{(n-k)*}(-\infty,-A]\\ &\le \overline{\rho^{n*}}(x)\sigma^{n*}(-\infty,-A]. \end{aligned} \tag{2.9}$$

Note that

$$\frac{\overline{\rho^{n*}}(x)}{\overline{\mu^{n*}}(x)} \le q^{-n} \quad \text{for } x > 0. \tag{2.10}$$

As $A \to \infty$, we obtain from (2.9) and $\mu^{n*} \in \mathcal{L}(0)$ that

$$\begin{aligned}\limsup_{x\to\infty}\frac{\overline{(\mu_+)^{n*}}(x)}{\overline{\mu^{n*}}(x)} &\le \lim_{A\to\infty}\limsup_{x\to\infty}\frac{\overline{\mu^{n*}}(x-A)}{\overline{\mu^{n*}}(x)}\\ &\quad +q^{-n}\lim_{A\to\infty}\sigma^{n*}(-\infty,-A] = 1. \end{aligned} \tag{2.11}$$

We see from (2.8) and (2.11) that

$$\overline{(\mu_+)^{n*}}(x) \sim \overline{\mu^{n*}}(x). \tag{2.12}$$

Thus we conclude from Lemma 2.1 that $(\mu_+)^{n*} \in \mathcal{S}(0)$ and by Theorem 2 of [10] that $\mu_+ \in \mathcal{S}(0)$, equivalently, $\mu \in \mathcal{S}(0)$. □

Finally we give an interesting result due to [36]. The definition of μ_I is similar to (1.20).

Lemma 2.8 (Lemma 3.1 of [36]) *Let $\gamma > 0$. Let μ be a distributions on $\mathbb{R}_+$ satisfying $\bar{\mu}(x) > 0$ on $(0,\infty)$ and $\int_0^\infty x\mu(dx) < \infty$. Then the following are equivalent.*

(a) $\mu \in \mathcal{L}(\gamma)$.
(b) $\lim_{x\to\infty}\bar{\mu}(x)/\int_x^\infty \bar{\mu}(u)du = \gamma$.
(c) $\mu_I \in \mathcal{L}(\gamma)$.

3 Proof of main results

In this section, we prove the main results given in Sect. 1.

Proof of Theorem 1.1 If $\gamma = 0$, then Pakes's Proof does not depend on Cline's lemma and it is correct. Let $\gamma > 0$. For a fixed $c > 1$, we decompose μ satisfying (1.3) and (1.4) as the convolution of two infinitely divisible distributions μ_1 and μ_2 in such a way that μ_2 is a compound Poisson distribution with Lévy measure $1_{(c,\infty)}(x)\nu(dx)$. Note from Theorem 26.8 of [28] that $\overline{\mu_1}(x) = o(\overline{\mu}(x))$ and $\widehat{\mu}_1(\beta) < \infty$ for all $\beta > \gamma$. The characteristic functions of μ_1 and μ_2 are expressed as, for $j = 1, 2$,

$$\int_{-\infty}^{\infty} \exp(izx)\mu_j(dx) = \exp(\psi_j(z)), \qquad z \in \mathbb{R} \tag{3.1}$$

with

$$\psi_1(z) = \int_{-\infty}^{c+} (e^{izx} - 1 - 1_{\{|x|\leqslant 1\}}(x)izx)\nu(dx) + iaz - \frac{1}{2}bz^2, \tag{3.2}$$

and

$$\psi_2(z) = \int_{c+}^{\infty} (e^{izx} - 1)\nu(dx). \tag{3.3}$$

Suppose that $\mu \in \mathcal{S}(\gamma)$. We prove that $\nu_c \in \mathcal{S}(\gamma)$ by developing the argument used in the Proof of Theorem 4.2 of [9]. Here ν_c is defined by (1.5). Since μ_2 is an infinitely divisible distribution with Lévy measure $1_{(c,\infty)}(x)\nu(dx)$, we see from Theorem 25.17 of [28] that (2.2) is always satisfied. Thus we see from Lemma 2.2 (ii) that

$$\bar{\mu}(x) \sim \widehat{\mu}_1(\gamma)\bar{\mu}_2(x) \quad \text{and } \mu_2 \in \mathcal{S}(\gamma). \tag{3.4}$$

Let $\delta := \bar{\nu}(c)$. Since $\delta \to 0$ and μ_2 converges weakly to $\delta_0(dx)$ as $c \to \infty$, we can take sufficiently large c such that $e^{\delta} - 1$ is close to 0 and

$$0 < e^{\delta}\widehat{\mu}_2(\gamma) + e^{\delta} - 2 < 1. \tag{3.5}$$

The distribution μ_2 is a compound Poisson on $\mathbb{R}_+$ and thus represented as

$$\mu_2 = e^{-\delta} \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \nu_c^{n*}. \tag{3.6}$$

Define a probability measure σ on $\mathbb{R}_+$ by

$$\sigma := \left(e^{\delta} - 1\right)^{-1} \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \nu_c^{n*}. \tag{3.7}$$

Since $\mu_2 \in \mathcal{S}(\gamma)$ and $\overline{\sigma}(x) = (1 - e^{-\delta})^{-1} \bar{\mu}_2(x)$, we have, by Lemma 2.1, $\sigma \in \mathcal{S}(\gamma)$. Noting that $0 < e^{\delta} - 1 < 1$, we see that

$$\delta \nu_c = -\sum_{n=1}^{\infty} \frac{\left(1 - e^{\delta}\right)^n}{n} \sigma^{n*}. \tag{3.8}$$

Since $\sigma \in \mathcal{S}(\gamma)$, it follows from Lemma 2.3 that, for any $\varepsilon \in (0, 1)$, there exists $K > 0$ such that

$$\frac{\overline{\sigma^{n*}}(x)}{\overline{\sigma}(x)} \leq K\left(\widehat{\sigma}(\gamma) + \varepsilon\right)^n \quad \text{for every} \quad x \geq 0 \text{ and } n \geq 1. \tag{3.9}$$

Noting that $\widehat{\sigma}(\gamma) = (e^{\delta}\widehat{\mu}_2(\gamma) - 1) / \left(e^{\delta} - 1\right)$, we find from (3.5) that, for each $x \geq 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\left(e^{\delta} - 1\right)^n}{n} \frac{\overline{\sigma^{n*}}(x)}{\overline{\sigma}(x)} &\leq K \sum_{n=1}^{\infty} \frac{\left(e^{\delta} - 1\right)^n}{n} \left(\widehat{\sigma}(\gamma) + \varepsilon\right)^n \\ &= K \sum_{n=1}^{\infty} \frac{1}{n} \left(e^{\delta}\widehat{\mu}_2(\gamma) - 1 + \varepsilon(e^{\delta} - 1)\right)^n < \infty. \end{aligned} \tag{3.10}$$

Thus, thanks to the dominated convergence theorem, we have by Lemma 2.3

$$\begin{aligned} \lim_{x \to \infty} \frac{\delta \overline{\nu_c}(x)}{\overline{\sigma}(x)} &= -\sum_{n=1}^{\infty} \left(1 - e^{\delta}\right)^n \widehat{\sigma}(\gamma)^{n-1} \\ &= \frac{1 - e^{-\delta}}{\widehat{\mu}_2(\gamma)} > 0. \end{aligned} \tag{3.11}$$

Hence, by Lemma 2.1, $\nu_c \in \mathcal{S}(\gamma)$ and thereby $\nu_1 \in \mathcal{S}(\gamma)$. Next we prove the second assertion. It is enough to show (4) implies (3) of Theorem B. Suppose that (4) holds. Then we see from Lemma 2.2 that

$$\overline{\mu_2}(x) \sim \frac{d}{\widehat{\mu}_1(\gamma)} \bar{\nu}(x). \tag{3.12}$$

Since μ_2 is a distribution on $\mathbb{R}_+$, we obtain from Theorem 1.2 of [34] that $d/\widehat{\mu}_1(\gamma) = \widehat{\mu}_2(\gamma) < \infty$, that is, $d = \widehat{\mu}(\gamma) < \infty$. Thus (3) holds. □

Proof of Corollary 1.3 Assertion (i) is clear from Theorem B.

Next we prove assertion (ii). Without loss of generality we can assume $t > 1$. Suppose that $\mu \in \mathcal{L}(\gamma)$ and

$$\overline{\mu^{t*}}(x) \sim t\widehat{\mu}(\gamma)^{t-1}\bar{\mu}(x) \quad \text{for some } t > 1. \tag{3.13}$$

Then obviously $\mu^{t*} \in \mathcal{L}(\gamma)$. We use the decomposition $\mu = \mu_1 * \mu_2$ as in the Proof of the above Theorem again. We see from Lemma 2.2 for μ, $\mu^{t*} \in \mathcal{L}(\gamma)$ that

$$\overline{\mu_2^{t*}}(x) \sim t\widehat{\mu_2}(\gamma)^{t-1}\overline{\mu_2}(x). \tag{3.14}$$

We can represent μ_2 and μ_2^{t*} as (3.6) and

$$\mu_2^{t*} = e^{-\delta t} \sum_{n=0}^{\infty} \frac{(\delta t)^n {\nu_c}^{n*}}{n!} \tag{3.15}$$

with $\delta = \delta(c) := \bar{\nu}(c)$. We shall show that

$$\lim_{c\to\infty} \limsup_{x\to\infty} \sum_{n=2}^{\infty} \frac{(\delta t)^n \overline{{\nu_c}^{n*}}(x)}{n!\delta\overline{\nu_c}(x)} = 0. \tag{3.16}$$

Suppose that, for any $N > 0$, there are $c = c_N > N$, $\{x_k\} = \{x_k(c)\}$ and $\delta_1 = \delta_1(c) > 0$ and an absolute constant $\delta_2 > 0$ such that $\lim_{k\to\infty} x_k = \infty$ and

$$\lim_{k\to\infty} \sum_{n=2}^{\infty} \frac{(\delta t)^n \overline{{\nu_c}^{n*}}(x_k)}{n!\delta\overline{\nu_c}(x_k)} = \delta_1(c) \geq \delta_2. \tag{3.17}$$

Note that $\lim_{c\to\infty} \delta(c) = 0$ and $\lim_{c\to\infty} \widehat{\mu_2}(\gamma) = 1$. Thus we can take sufficiently large $c = c_N$ such that

$$t\widehat{\mu_2}(\gamma)^{t-1} < e^{-\delta(t-1)} \frac{t(t^2 + \delta_2 t)}{t^2 + \delta_2}. \tag{3.18}$$

Define $I(k)$ and $J(k)$ for $k \geq 1$ as

$$I(k) := \sum_{n=2}^{\infty} \frac{\delta^n \overline{{\nu_c}^{n*}}(x_k)}{n!}, \tag{3.19}$$

and

$$J(k) := \sum_{n=2}^{\infty} \frac{(\delta t)^n \overline{{\nu_c}^{n*}}(x_k)}{n!}. \tag{3.20}$$

Thus noting that $J(k) \geq t^2 I(k)$, we obtain from (3.17) and (3.18) that for sufficiently large $c = c_N$

$$\begin{aligned} t\widehat{\mu_2}(\gamma)^{t-1} &= \lim_{k\to\infty} \frac{\overline{\mu_2^{t*}}(x_k)}{\overline{\mu_2}(x_k)} \\ &= e^{-\delta(t-1)} \lim_{k\to\infty} \frac{\delta t \overline{\nu_c}(x_k) + J(k)}{\delta \overline{\nu_c}(x_k) + I(k)} \\ &= e^{-\delta(t-1)} \lim_{k\to\infty} \frac{t + J(k)/(\delta\overline{\nu_c}(x_k))}{1 + I(k)/(\delta\overline{\nu_c}(x_k))} \\ &\geq e^{-\delta(t-1)} \lim_{k\to\infty} \frac{t + J(k)/(\delta\overline{\nu_c}(x_k))}{1 + J(k)/(t^2\delta\overline{\nu_c}(x_k))} \\ &\geq e^{-\delta(t-1)} \frac{t(t^2+\delta_2 t)}{t^2+\delta_2} > t\widehat{\mu_2}(\gamma)^{t-1}. \end{aligned} \tag{3.21}$$

Here the last inequality is due to (3.18). This is a contradiction. Thus we have proved (3.16). It follows from Lemma 2.2 and (3.16) that

$$\begin{aligned} \limsup_{x\to\infty} \left| \frac{\overline{\mu}(x)}{\overline{\nu}(x)} - \widehat{\mu}(\gamma) \right| &= \lim_{c\to\infty} \limsup_{x\to\infty} \left| \widehat{\mu}_1(\gamma) \frac{\overline{\mu_2}(x)}{\delta\overline{\nu_c}(x)} - \widehat{\mu}(\gamma) \right| \\ &\leq \lim_{c\to\infty} \Bigg(\left| e^{-\delta}\widehat{\mu}_1(\gamma) - \widehat{\mu}(\gamma) \right| + \widehat{\mu}_1(\gamma) e^{-\delta} \\ &\qquad \limsup_{x\to\infty} \sum_{n=2}^{\infty} \frac{\delta^{n-1}\overline{\nu_c^{n*}}(x)}{n!\overline{\nu_c}(x)} \Bigg) = 0. \end{aligned} \tag{3.22}$$

Thus we conclude from $\mu \in \mathcal{L}(\gamma)$ and Theorem B that $\mu \in \mathcal{S}(\gamma)$. □

Proof of Theorem 1.5 First we prove (i) only in the case (1.10). The Proof of the other case is similar. We shall show that there is an integer $N \geq 2$ such that

$$\liminf_{x\to\infty} \frac{\overline{\rho^{N*}}(x)}{\overline{\eta}(x)} > 0. \tag{3.23}$$

On the contrary, suppose that for every integer $N \geq 2$

$$\liminf_{x\to\infty} \frac{\overline{\rho^{N*}}(x)}{\overline{\eta}(x)} = 0. \tag{3.24}$$

We see from (1.10) that there are $\delta > 0$ and an integer $N_0 \geq 2$ such that, for every $k \geq N_0 + 1$,

$$\frac{(p \otimes p)_k}{p_k} > 2\widehat{\eta}(\gamma) + \delta. \tag{3.25}$$

Further, we find that there is a sequence $\{x_n\}_{n=0}^{\infty}$ such that x_n is strictly increasing with $\lim_{n\to\infty} x_n = \infty$ and that

$$\lim_{n\to\infty} \frac{\overline{\rho^{N_0*}}(x_n)}{\overline{\eta}(x_n)} = 0. \tag{3.26}$$

Since $(\rho[0,\infty))^{N_0-1}\overline{\rho^{k*}}(x) \le \overline{\rho^{N_0*}}(x)$ for $1 \le k \le N_0$, we have

$$\lim_{n\to\infty} \frac{\overline{\rho^{k*}}(x_n)}{\overline{\eta}(x_n)} = 0 \quad \text{for } 1 \le k \le N_0. \tag{3.27}$$

Define $I_j(n)$ and $J_j(n)$ for $j = 1, 2$ as

$$I_1(n) = \sum_{k=0}^{N_0} p_k \overline{\rho^{k*}}(x_n), \tag{3.28}$$

$$I_2(n) = \sum_{k=N_0+1}^{\infty} p_k \overline{\rho^{k*}}(x_n), \tag{3.29}$$

$$J_1(n) = \sum_{k=0}^{N_0} (p \otimes p)_k \overline{\rho^{k*}}(x_n), \tag{3.30}$$

and

$$J_2(n) = \sum_{k=N_0+1}^{\infty} (p \otimes p)_k \overline{\rho^{k*}}(x_n). \tag{3.31}$$

Then we see from (3.27) that

$$\lim_{n\to\infty} \frac{I_1(n)}{\overline{\eta}(x_n)} = \lim_{n\to\infty} \frac{J_1(n)}{\overline{\eta}(x_n)} = 0. \tag{3.32}$$

Noting that $\eta^{2*} = \sum_{k=0}^{\infty}(p \otimes p)_k \rho^{k*}$, we obtain from (3.25) and (3.32) that

$$\begin{aligned} 2\widehat{\eta}(\gamma) &= \lim_{n\to\infty} \frac{\overline{\eta^{2*}}(x_n)}{\overline{\eta}(x_n)} \\ &= \lim_{n\to\infty} \frac{(J_1(n) + J_2(n))/\overline{\eta}(x_n)}{(I_1(n) + I_2(n))/\overline{\eta}(x_n)} \\ &= \lim_{n\to\infty} \frac{J_2(n)}{I_2(n)} \ge 2\widehat{\eta}(\gamma) + \delta. \end{aligned} \tag{3.33}$$

This is a contradiction. Thus we have proved (3.23) for some integer $N \ge 2$ and hence for all sufficiently large integer N. Since $\overline{\rho^{N*}}(x) \le (p_N)^{-1}\overline{\eta}(x)$ with $p_N > 0$ for sufficiently large integers N, it follows that

$$\overline{\rho^{N*}}(x) \asymp \overline{\eta}(x). \tag{3.34}$$

Note from Lemma 2.5 that $\rho \in \mathcal{L}(\gamma)$ implies $\rho^{N*} \in \mathcal{L}(\gamma)$. Thus we conclude from Lemma 2.6 that $\rho^{N*} \in \mathcal{S}(\gamma)$ and thereby from Proposition 2.7 that $\rho \in \mathcal{S}(\gamma)$.

Next we prove (ii). Suppose that (1.12) holds. First we show that if (1.13) holds, then

$$\lim_{x\to\infty} \frac{\overline{\eta^{2*}}(x)}{\overline{\eta}(x)} = 2d. \tag{3.35}$$

In fact, define $I_j(x)$ and $J_j(x)$ for $j = 1, 2$ and $N \geq 1$ as

$$I_1(x) = \sum_{k=0}^{N-1} p_k \overline{\rho^{k*}}(x), \tag{3.36}$$

$$I_2(x) = \sum_{k=N}^{\infty} p_k \overline{\rho^{k*}}(x), \tag{3.37}$$

$$J_1(x) = \sum_{k=0}^{N-1} (p \otimes p)_k \overline{\rho^{k*}}(x), \tag{3.38}$$

and

$$J_2(x) = \sum_{k=N}^{\infty} (p \otimes p)_k \overline{\rho^{k*}}(x). \tag{3.39}$$

Then we see from (1.13) that

$$\begin{aligned} \limsup_{x\to\infty} \frac{\overline{\eta^{2*}}(x)}{\overline{\eta}(x)} &= \limsup_{x\to\infty} \frac{(J_1(x) + J_2(x))/\overline{\rho^{N*}}(x)}{(I_1(x) + I_2(x))/\overline{\rho^{N*}}(x)} \\ &= \lim_{N\to\infty} \limsup_{x\to\infty} \frac{J_2(x)}{I_2(x)} = 2d \end{aligned} \tag{3.40}$$

and by the same way

$$\liminf_{x\to\infty} \frac{\overline{\eta^{2*}}(x)}{\overline{\eta}(x)} = 2d. \tag{3.41}$$

Thus we obtain (3.35) and the first assertion of (ii) is clear since $d = 1$ and ρ is one-sided. Next we fix $\rho(dx) := ae^{-ax}dx$ on $\mathbb{R}_+$ for some $a > 0$ and define

$$\begin{aligned} f_n(x) &:= \frac{a^n x^{n-1}}{(n-1)!} e^{-ax}, \\ g(x) &:= \sum_{n=1}^{\infty} p_n f_n(x), \quad \text{and} \\ g_1(x) &:= \sum_{n=1}^{\infty} p_{n+1} f_n(x). \end{aligned} \tag{3.42}$$

Note that ρ satisfies (1.13) and $g(x)$ is the density of $\eta(dx) - p_0\delta_0(dx)$. Suppose that $r = d = 1$. By the argument similar to the Proof of (3.35), we can prove from the first equation of (1.12) that $\lim_{x\to\infty} g_1(x)/g(x) = 1$. Then we find that $|g'(x)/g(x)|$ is bounded for $x \geq 1$ and that

$$\lim_{x\to\infty} \frac{g'(x)}{g(x)} = \lim_{x\to\infty} \frac{a(g_1(x) - g(x))}{g(x)} = 0. \tag{3.43}$$

Hence we see that

$$\lim_{x\to\infty} \frac{g(x+c)}{g(x)} = \lim_{x\to\infty} \exp\left(\int_0^c \frac{g'(x+u)}{g(x+u)} du \right) = 1 \quad \text{for every } c \in \mathbb{R}. \tag{3.44}$$

Next suppose that $0 < r < 1$ with $d = \sum_{n=0}^{\infty} r^{-n} p_n$. Then define $q_n := d^{-1} r^{-n} p_n$ and $h(x) := d^{-1} e^{a(1-r)x} g(x)$. Thus, replacing a, p_n, and $g(x)$ by ar, q_n, and $h(x)$, we see as in (3.44) that

$$\lim_{x\to\infty} \frac{h(x+c)}{h(x)} = 1 \quad \text{for every } c \in \mathbb{R}. \tag{3.45}$$

Therefore, $\eta \in \mathcal{L}(a(1-r))$ with $d = \widehat{\eta}(a(1-r))$. Hence, by (3.35), $\eta \in \mathcal{S}(a(1-r))$. Thus we have the conclusion of the second assertion of (ii) with $\gamma = a(1-r)$, which can be an arbitrarily positive value by taking appropriate a. □

Proof of Remark 1.6 We prove only (iv). The others are obvious. Suppose that

$$\limsup_{n\to\infty} \frac{p_{n+1}}{p_n} < \frac{1}{\widehat{\rho}(\gamma)}. \tag{3.46}$$

There is an integer $N \geq 1$ and a number $\delta > 0$ such that

$$p_{m-k} \geq (\widehat{\rho}(\gamma) + \delta)^k \, p_m \qquad \text{for} \quad m \geq 2n \geq N \quad \text{and} \quad 0 \leq k \leq n. \tag{3.47}$$

Thus we have, for $m \geq 2n \geq N$,

$$(p \otimes p)_m \geq 2 \sum_{k=0}^{n} p_k p_{m-k} \geq 2 p_m \sum_{k=0}^{n} p_k (\widehat{\rho}(\gamma) + \delta)^k. \tag{3.48}$$

Thus we see that

$$\liminf_{n\to\infty} \frac{(p \otimes p)_n}{p_n} \geq 2 \sum_{k=0}^{\infty} p_k \left(\widehat{\rho}(\gamma) + \delta\right)^k > 2\widehat{\eta}(\gamma). \tag{3.49}$$

Hence (1.10) holds true. □

Proof of Proposition 1.7 The idea of the Proof is similar to that of Theorem 1.5. The Proof of Lemma 3.4 of [24] used a corrected version of Lemma 2.1 of [4], whereas we apply Proposition 2.7. Let $q := \rho[0, \infty)$. Then $r = \sum_{n=1}^{\infty} p_n q^{n-1}$. Suppose that for some $\varepsilon > 0$

$$\sum_{n=1}^{\infty} p_n \left(\left(r^{-1}\widehat{\eta}(\gamma) + \varepsilon\right) \vee 1\right)^n < \infty. \tag{3.50}$$

Since $\overline{\rho^{n*}}(x) \geq \overline{\rho}(x) q^{n-1}$ for $n \geq 1$, note that $r^{-1}\overline{\eta}(x) \geq \overline{\rho}(x)$ and $r^{-n}\overline{\eta^{n*}}(x) \geq \overline{\rho^{n*}}(x)$ for $n \geq 1$. Thus it is clear from $r^{-1}\overline{\eta}(x) \geq \overline{\rho}(x)$ that (1.9) holds. Obviously, there is an integer $N \geq 1$ such that $p_N > 0$ and

$$\sum_{n=N+1}^{\infty} r^{-n} n p_n \widehat{\eta}(\gamma)^{n-1} < 1. \tag{3.51}$$

Suppose that

$$\liminf_{x\to\infty} \frac{\overline{\rho^{N*}}(x)}{\overline{\eta}(x)} = 0. \tag{3.52}$$

Then there is a strictly increasing sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = \infty$ such that

$$\lim_{n\to\infty} \frac{\overline{\rho^{N*}}(x_n)}{\overline{\eta}(x_n)} = 0. \tag{3.53}$$

Since $q^{N-1}\overline{\rho^{k*}}(x) \leq \overline{\rho^{N*}}(x)$ for $1 \leq k \leq N$, we have

$$\lim_{n\to\infty} \frac{\overline{\rho^{k*}}(x_n)}{\overline{\eta}(x_n)} = 0 \quad \text{for } 1 \leq k \leq N. \tag{3.54}$$

Thanks to (3.51), we can use the dominated convergence theorem and see from Lemma 2.3 and (3.54) that

$$
\begin{aligned}
1 = \lim_{n\to\infty} \frac{\overline{\eta}(x_n)}{\overline{\eta}(x_n)} &= \lim_{n\to\infty} \sum_{k=N+1}^{\infty} p_k \frac{\overline{\rho^{k*}}(x_n)}{\overline{\eta}(x_n)} \\
&\le \lim_{n\to\infty} \sum_{k=N+1}^{\infty} p_k \frac{r^{-k}\overline{\eta^{k*}}(x_n)}{\overline{\eta}(x_n)} \\
&= \sum_{k=N+1}^{\infty} r^{-k} k p_k \widehat{\eta}(\gamma)^{k-1} < 1.
\end{aligned} \tag{3.55}
$$

This is a contradiction. Thus, noting $\overline{\rho^{N*}}(x) \le (p_N)^{-1}\overline{\eta}(x)$ with $p_N > 0$, we have

$$
\overline{\eta}(x) \asymp \overline{\rho^{N*}}(x). \tag{3.56}
$$

Since $\rho \in \mathcal{L}(\gamma)$ and thus $\rho^{N*} \in \mathcal{L}(\gamma)$, we find from Lemma 2.6 that $\rho^{N*} \in \mathcal{S}(\gamma)$ and hence, by Proposition 2.7, $\rho \in \mathcal{S}(\gamma)$. □

Proof of Theorem 1.8 Thanks to Theorem C, it is enough to prove that (a) implies (b). Define $\delta := -\log(1-\lambda)$ and

$$
\nu_0 := \frac{1}{\delta} \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \rho^{n*}. \tag{3.57}
$$

By considering characteristic functions instead of Laplace transforms, we obtain the following as in the Proof of Corollary 3 of [10]. That is, we see that η_α is a compound Poisson distribution with Lévy measure $\delta\alpha\nu_0$ and the distributions η_α and ρ are represented as

$$
\eta_\alpha = e^{-\delta\alpha} \sum_{n=0}^{\infty} \frac{(\delta\alpha)^n}{n!} \nu_0^{n*} \tag{3.58}
$$

and

$$
\rho = -\lambda^{-1} \sum_{n=1}^{\infty} \frac{(-\delta)^n}{n!} \nu_0^{n*}. \tag{3.59}
$$

Suppose that $\eta_\alpha \in \mathcal{S}(\gamma)$ for some, or equivalently by Theorem B for all, $\alpha > 0$. Then we find from Theorem 1.1 that $\nu_0 \in \mathcal{S}(\gamma)$. Thus, by using the dominated convergence theorem, we obtain from Lemma 2.3 that

$$
\lim_{x\to\infty} \frac{\overline{\rho}(x)}{\overline{\nu_0}(x)} = \frac{\delta}{\lambda} \exp\left(-\delta\widehat{\nu}_0(\gamma)\right) > 0 \tag{3.60}
$$

and thereby from Lemma 2.1 that $\rho \in \mathcal{S}(\gamma)$. □

Proof of Corollary 1.9 The Proof of the corollary is clear from Lemma 2.8 and Theorem 1.8. □

4 Some applications

In this section, we add two applications of our results, which are useful in ruin theory. First we give an example of application of our results to a special case of a compound Poisson process. Second we prove an extension of theorems of [1,20] on the tail of the distribution of the supremum of a transient random walk. Let $\{X_n\}_{n=1}^{\infty}$ be IID random variables with ρ being the distribution of X_1. We define a random walk $\{S_n\}_{n=0}^{\infty}$ as $S_0 := 0$ and $S_n := \sum_{k=1}^{n} X_k$ for $n \geq 1$. Let $\{N(t)\}_{t\geq 0}$ be a non-negative integer-valued subordinator independent of $\{S_n\}$. The distribution of $N(t)$ is denoted by $\sum_{n=0}^{\infty} p_n(t)\delta_n(dx)$. In particular, denote $p_n := p_n(1)$. We consider the process $\{S(t)\}_{t\geq 0}$ defined by $S(t) := S_{N(t)}$. Then $\{S(t)\}$ is a compound Poisson process. Denote the Lévy measure of the process $\{N(t)\}$ by $\sum_{n=1}^{\infty} q_n\delta_n(dx)$. Then the distribution η_t of $S(t)$ is represented as

$$\eta_t = \sum_{n=0}^{\infty} p_n(t)\rho^{n*}. \tag{4.1}$$

The Lévy measure ν of the process $\{S(t)\}$ is given by

$$\nu = \sum_{n=1}^{\infty} q_n\rho^{n*}. \tag{4.2}$$

Denote $\eta := \eta_1$. Then obviously $\eta_t = \eta^{t*}$. The process $\{S(t)\}$ is called a compounding of $\{S_n\}$ by $\{N(t)\}$. See [28, E34.5]. The process of this type appears in a generalized Cramér–Lundberg model in ruin theory and in some applications to finance. See [12, 30]. The following is a direct application of Theorems 1.1, C, and 1.5 together with Theorem 25.17 of [28].

Theorem 4.1 *Let $\gamma \geq 0$. Assume that $\rho \in \mathcal{L}(\gamma)$ on $\mathbb{R}$ and (1.9) holds. Further, assume that, for some $t > 0$, either*

$$\liminf_{n\to\infty} \frac{p_n(2t)}{p_n(t)} > 2\widehat{\eta}(\gamma)^t \quad or \quad \limsup_{n\to\infty} \frac{p_n(2t)}{p_n(t)} < 2\widehat{\eta}(\gamma)^t \tag{4.3}$$

or assume that either

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} q_k q_{n-k}}{q_n} > 2\widehat{\nu}(\gamma) \quad or \quad \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} q_k q_{n-k}}{q_n} < 2\widehat{\nu}(\gamma). \tag{4.4}$$

Then the following statements are equivalent.

(1) $\eta_t \in \mathcal{S}(\gamma)$ *for some* $t > 0$, *equivalently for all,* $t > 0$.
(2) $\rho \in \mathcal{S}(\gamma)$.
(3) $\widehat{\rho}(\gamma) < \infty$ *and* $\overline{\eta_t}(x) \sim \sum_{n=1}^{\infty} np_n(t)\widehat{\rho}(\gamma)^{n-1}\overline{\rho}(x)$ *for some* $t > 0$, *equivalently for all,* $t > 0$.
(4) $\widehat{\rho}(\gamma) < \infty$ *and* $\overline{\nu}(x) \sim \sum_{n=1}^{\infty} nq_n\widehat{\rho}(\gamma)^{n-1}\overline{\rho}(x)$.

Proof First we note from Theorem 25.17 of [28] that (1.9) is equivalent to

$$\sum_{n=1}^{\infty} p_n(t)((\widehat{\rho}(\gamma)+\varepsilon)\vee 1)^n < \infty \quad \text{for all } t>0 \tag{4.5}$$

and also to

$$\sum_{n=1}^{\infty} q_n((\widehat{\rho}(\gamma)+\varepsilon)\vee 1)^n < \infty. \tag{4.6}$$

The equivalence of (2)–(4) and the fact that (2) implies (1) are due to Theorem C. We can prove that (1) implies (2) by using Theorem 1.5 under the assumption (4.3). We see from Theorem 1.1 that (1) implies $\nu_1 \in \mathcal{S}(\gamma)$. Thus, under another assumption (4.4), we can prove that (1) implies (2) by an argument similar to the Proof of Theorem 1.5. □

Next we show a second application. An analogous problem for Lévy processes is discussed in [18]. Define $\gamma_0 := \sup\{\varepsilon \geq 0 : \widehat{\rho}(\varepsilon) < \infty\}$. We shall assume that $0 < \gamma_0 < \infty$ and $\widehat{\rho}(\gamma_0) < 1$, which is called the intermediate case. See [13, 20]. Define λ as $\lambda := 1 - e^{-B}$ with $B := \sum_{n=1}^{\infty} n^{-1} P(S_n > 0) < \infty$. Let Z^+ be the first strictly ascending ladder height in the random walk $\{S_n\}$ and denote the defective distribution of Z^+ by $\lambda\mu$. This μ is a distribution on $\mathbb{R}_+$. Let ζ be the distribution of the supremum M of S_n, that is, $M := \sup_{n\geq 0} S_n$. It is well known that $\zeta = \sum_{n=0}^{\infty}(1-\lambda)\lambda^n \mu^{n*}$. The tail of the distribution ζ is known as the ruin probability in classical ruin theory. See [13]. Bertoin and Doney [1] showed the following.

Lemma 4.2 (Theorem 1 and Lemma 1 of [1]) *Suppose that* $0 < \gamma_0 < \infty$ *and* $\widehat{\rho}(\gamma_0) < 1$. *Then* $\lambda\widehat{\mu}(\gamma_0) < 1$ *and the following hold*:

(i) $\mu \in \mathcal{L}(\gamma_0)$ *if and only if* $\rho \in \mathcal{L}(\gamma_0)$. *If* $\rho \in \mathcal{L}(\gamma_0)$, *then*

$$\overline{\rho}(x) \sim \frac{\lambda(1-\widehat{\rho}(\gamma_0))}{1-\lambda\widehat{\mu}(\gamma_0)}\overline{\mu}(x). \tag{4.7}$$

(ii) *If* $\rho \in \mathcal{S}(\gamma_0)$, *then* $\zeta \in \mathcal{S}(\gamma_0)$ *and*

$$\overline{\zeta}(x) \sim \frac{1-\lambda}{(1-\widehat{\rho}(\gamma_0))(1-\lambda\widehat{\mu}(\gamma_0))}\overline{\rho}(x). \tag{4.8}$$

Later, Korshunov [20] proved the following.

Lemma 4.3 (Theorem 2 of [20]) *Suppose that* $0 < \gamma_0 < \infty$ *and* $\widehat{\rho}(\gamma_0) < 1$. *Assume that* $\rho \in \mathcal{L}(\gamma_0)$. *Then* $\rho \in \mathcal{S}(\gamma_0)$ *is equivalent to* (4.8) *and also to* $\overline{\zeta}(x) \sim c\overline{\rho}(x)$ *for some* $c \in (0, \infty)$.

We improve the above results as follows. Its Proof is clear from the above lemmas and Theorem 1.8 with $\alpha = 1$. The part that (1) implies (2) is new.

Theorem 4.4 *Suppose that* $0 < \gamma_0 < \infty$ *and* $\widehat{\rho}(\gamma_0) < 1$. *Then the following are equivalent.*

(1) $\zeta \in \mathcal{S}(\gamma_0)$.
(2) $\rho \in \mathcal{S}(\gamma_0)$.
(3) $\rho \in \mathcal{L}(\gamma_0)$, $\lambda\widehat{\mu}(\gamma_0) < 1$ *and* (4.8) *holds.*
(4) $\rho \in \mathcal{L}(\gamma_0)$, *and* $\overline{\zeta}(x) \sim c\overline{\rho}(x)$ *for some* $c \in (0, \infty)$.

5 Remarks and problems

There is an important theorem on the constant in the definition of the convolution equivalence.

Theorem D *Let* $\gamma \geq 0$. *If* $\mu \in \mathcal{L}(\gamma)$ *on* $\mathbb{R}_+$ *and*

$$\overline{\mu^{2*}}(x) \sim 2d\overline{\mu}(x) \quad \text{with some } d < \infty, \tag{5.1}$$

then $d = \widehat{\mu}(\gamma)$ *and* $\mu \in \mathcal{S}(\gamma)$.

In this section, we present several fascinating problems in the following remarks. All these problems are concerned with the constants in the asymptotic relations of convolution tails as in Theorem D.

Remark 5.1 Chover et al. [3] and Cline [4] proved the above theorem. Later, Rogozin [25] also proved the same result and pointed out a gap in each Proof in [3,4]. Rogozin and Sgibnev [26] also explained a gap in the Proof in [3] and, by employing the results of [3], proved Theorem D in the case $\gamma > 0$. By using the above result of [4], Pakes [23] proved the same result for the distribution μ on $\mathbb{R}$. In [34], we claimed that Cline's Proof is correct because the gap pointed out by [25] can be closed. However, there is another gap in Lemma 2.3 (ii) of [4], which was used in the Proof of Theorem D. This gap was pointed out to Cline privately by E. Omey, and also recently mentioned by [15]. Cline [5] wrote a corrigendum for this gap. But it contains a serious mistake again on line 12 on p. 152. He claimed that, in (1.8), if $\rho \in \mathcal{L}(\gamma)$, then $\eta \in \mathcal{L}(\gamma)$ for $\gamma \geq 0$, but, in the case $\gamma > 0$, the statement is not true by virtue of the second assertion of Theorem 1.5 (ii) in the present paper. Thus we may call Theorem D Rogozin's theorem. Foss and Korshunov [15] proved a remarkable extension of Rogozin's theorem, which was expected by [9]. That is, Theorem 3 of [15] states that if we define $\gamma := \sup\{\varepsilon \geq 0 : \widehat{\mu}(\varepsilon) < \infty\}$ for μ on $\mathbb{R}_+$ and if (5.1) holds, then $\gamma < \infty$ and $d = \widehat{\mu}(\gamma) < \infty$. Further there is the following open problem on the random sums.

Problem 2 Let $\gamma_1 := \sup\{\varepsilon \geq 0 : \widehat{\rho}(\varepsilon) < \infty\}$. If $\bar{\eta}(x) \sim c\bar{\rho}(x)$ for η in (1.8) with some $c \in (0, \infty)$, then is it true that $\gamma_1 < \infty$, $\widehat{\rho}(\gamma_1) < \infty$ and $c = \sum_{n=1}^{\infty} np_n\widehat{\rho}(\gamma_1)^{n-1}$?

Remark 5.2 In relation to Corollary 1.3, we have the following two unsolved problems.

Problem 3 Let $\gamma \geq 0$ and let μ be an infinitely divisible distribution on $\mathbb{R}$. If $\mu \in \mathcal{L}(\gamma)$ and

$$\overline{\mu^{t*}}(x) \sim c\overline{\mu}(x) \quad \text{for some} \quad t \in (0,1) \cup (1,\infty) \quad \text{and} \quad c \in (0,\infty), \tag{5.2}$$

then is it true that $\widehat{\mu}(\gamma) < \infty$ and $c = t\widehat{\mu}(\gamma)^{t-1}$?

Problem 4 Let $\gamma \geq 0$ and let μ be a distribution on $\mathbb{R}$. If $\mu \in \mathcal{L}(\gamma)$ and, for some distinct positive integers m and n,

$$\overline{\mu^{n*}}(x) \sim c\overline{\mu^{m*}}(x) \quad \text{for some } c \in (0,\infty), \tag{5.3}$$

then is it true that $\widehat{\mu}(\gamma) < \infty$, $c = \frac{n}{m}\widehat{\mu}(\gamma)^{n-m}$ and $\mu \in \mathcal{S}(\gamma)$?

Remark 5.3 It is shown by [33] that, for every $\gamma \geq 0$, the class $\mathcal{L}(\gamma)$ is not closed under convolution roots, which denies the conjecture of [8,9]. Thus Proposition 2.7 with $\gamma > 0$ does not answer the following problem, which is difficult and not yet solved even in the one-sided case. Nevertheless, it is plausible that the answer is positive.

Problem 5 Let $\gamma > 0$. Is the class $\mathcal{S}(\gamma)$ on $\mathbb{R}$ closed under convolution roots? In other words, if $\mu^{n*} \in \mathcal{S}(\gamma)$ for some $n \geq 2$, then is it true that

$$\overline{\mu^{n*}}(x) \sim c\overline{\mu}(x) \quad \text{for some } c \in (0,\infty)? \tag{5.4}$$

Remark 5.4 Let $\gamma \geq 0$. Denote $a\mathbb{Z} = \{0, \pm a, \pm 2a, \ldots\}$ for $a > 0$. A distribution μ on $\mathbb{R}$ is said to belong to the class $\mathcal{L}_D(\gamma)$ if, for some $a > 0$, μ is a distribution on $a\mathbb{Z}$ and

$$\lim_{n\to\infty} \frac{\mu(\{(n+1)a\})}{\mu(\{na\})} = e^{-\gamma a}. \tag{5.5}$$

A distribution μ on $\mathbb{R}$ is said to belong to the class $\mathcal{S}_D(\gamma)$ if $\mu \in \mathcal{L}_D(\gamma)$ and

$$\lim_{n\to\infty} \frac{\mu^{2*}(\{na\})}{\mu(\{na\})} = 2\widehat{\mu}(\gamma), \tag{5.6}$$

where $a > 0$ is such that μ is a distribution on $a\mathbb{Z}$. Denote the class of all infinitely divisible distributions on $\mathbb{R}$ by $\mathcal{ID}$. Define the classes $\mathcal{WS}$ and $\mathcal{WIS}$ of distributions on $\mathbb{R}$ by

$$\mathcal{WS} := \{\mu : \overline{\mu^{2*}}(x) \sim c\overline{\mu}(x) \quad \text{for some } c \in (0,\infty)\}, \tag{5.7}$$

$$\mathcal{WIS} := \{\mu \in \mathcal{ID} : \overline{\mu}(x) \sim c\overline{\nu}(x) \quad \text{for some } c \in (0,\infty)\}. \tag{5.8}$$

Then there are three mysterious problems. Foss and Korshunov [15] also give a conjecture similar to Problem 6. Recall that, in the one-sided case, we gave some conjectures on the classes $\mathcal{WS}$ and $\mathcal{WIS}$ in [34], but we did not take $\mathcal{S}_D(\gamma)$ into account.

Problem 6 Is $\mathcal{WS} = \bigcup_{\gamma \geq 0}(\mathcal{S}(\gamma) \cup \mathcal{S}_D(\gamma))$?

Problem 7 Is $\mathcal{WIS} = \bigcup_{\gamma \geq 0}(\mathcal{S}(\gamma) \cup \mathcal{S}_D(\gamma)) \cap \mathcal{ID}$?

Problem 8 Is $\mathcal{WIS} = \mathcal{WS} \cap \mathcal{ID}$?

For example, we showed the following in the Proof of Theorem 1.5. If, for some $0 < d < \infty$, $\lim_{n\to\infty}(p \otimes p)_n/p_n = 2d$ and (1.13) holds in (1.8), then $\eta \in \mathcal{WS}$ with $c = 2d$. But, we do not know whether $\eta \in \bigcup_{\gamma \geq 0}(\mathcal{S}(\gamma) \cup \mathcal{S}_D(\gamma))$. We see from Theorem 1.1 and the two-sided extension of Theorem D that the answers to the above three problems are all "yes" in the class $\mathcal{L}(\gamma)$, but we do not know the answers in the class $\mathcal{L}_D(\gamma)$. See [3,11].

6 An extension of Rogozin's Theorem

We prove the following extension of Rogozin's Theorem (thus called in Remark 5.1), namely Theorem D.

Theorem 6.1 *Let $\gamma \geq 0$. Let η and ρ be distributions as in (1.8). Assume further that $\sum_{n=0}^{\infty} p_n x^n < \infty$ for every $x > 0$. If $\rho \in \mathcal{L}(\gamma)$ on $\mathbb{R}$ and*

$$\overline{\eta}(x) \sim c\overline{\rho}(x) \quad \text{for some } c \in (0,\infty), \tag{6.1}$$

then $c = \sum_{n=1}^{\infty} np_n\widehat{\rho}(\gamma)^{n-1}$ and $\rho \in \mathcal{S}(\gamma)$. In particular, if $\rho \in \mathcal{L}(\gamma)$ on $\mathbb{R}$ and, for some $n \geq 2$,

$$\overline{\rho^{n*}}(x) \sim c\overline{\rho}(x) \quad \text{for some } c \in (0,\infty) \tag{6.2}$$

then $c = n\widehat{\rho}(\gamma)^{n-1}$ and $\rho \in \mathcal{S}(\gamma)$.

Remark 6.2 The condition that $\sum_{n=0}^{\infty} p_n x^n < \infty$ for every $x > 0$ implies that either $\limsup_{n\to\infty}(p \otimes p)_n/p_n = \infty$ or $p_n = 0$ for infinitely many $n \geq 1$. See the Proof of Propositions 2.1 and 2.2 of [34].

For a distributin μ on $\mathbb{R}$, let μ_+ be the same as defined in Sect. 2. A distribution μ on $\mathbb{R}$ is said to belong to the class $\mathcal{OS}$ if $\bar{\mu}(x) > 0$ for every $x > 0$ and

$$\ell^*(\mu) := \limsup_{x\to\infty} \frac{\overline{\mu^{2*}}(x)}{\overline{\mu}(x)} < \infty. \tag{6.3}$$

Distributions which belong to the class $\mathcal{OS}$ are called O-subexponential. The class $\mathcal{OS}$ was introduced and studied in detail by [34]. It includes the classes $\bigcup_{\gamma \geq 0}(\mathcal{S}(\gamma) \cup \mathcal{S}_D(\gamma))$ and $\mathcal{WS}$. Let $\gamma > 0$. Let μ be a distribution on $\mathbb{R}$ with $\widehat{\mu}(\gamma) < \infty$. We define the exponential tilt (or γ-transform) μ^γ on $\mathbb{R}$ of μ by

$$\mu^\gamma(dx) := \frac{1}{\widehat{\mu}(\gamma)} e^{\gamma x}\mu(dx). \tag{6.4}$$

Note that the exponential tilt preserves convolutions, that is, $(\mu * \eta)^\gamma = \mu^\gamma * \eta^\gamma$.

Lemma 6.3 *Let μ be a distribution on $\mathbb{R}$ with $q := \mu[0, \infty)$.*

(i) *$\mu \in \mathcal{OS}$ if and only if $\mu_+ \in \mathcal{OS}$.*
(ii) *If $\mu \in \mathcal{OS}$ on $\mathbb{R}$, then for every $\varepsilon > 0$ there is $c_1 > 0$ such that*

$$\overline{\mu^{n*}}(x) \le c_1((\ell^*(\mu) - q + \varepsilon) \vee 1)^n \overline{\mu}(x) \quad \text{for every} \quad x \in \mathbb{R} \text{ and } n \ge 1. \tag{6.5}$$

Proof Define $\theta(dx) := q^{-1} 1_{[0,\infty)}(x)\mu(dx)$ and $\sigma(dx) := (1-q)^{-1} 1_{(-\infty,0)}(x)\mu(dx)$ for $0 < q < 1$. For the Proof of (i), see for $0 < q < 1$ and $x > 0$ that

$$\begin{aligned} &\left| \frac{\overline{\mu^{2*}}(x)}{\overline{\mu}(x)} - \frac{\overline{(\mu_+)^{2*}}(x)}{\overline{(\mu_+)}(x)} \right| \\ &= \left| \frac{2q(1-q)}{\overline{\mu}(x)} \left(\overline{(\theta * \sigma}(x) - \overline{\theta}(x) \right) \right| \\ &\le \frac{2}{\overline{\mu}(x)} \int_{x+}^{\infty} |\overline{\mu}(x-y) - 1| \, \mu(dy) \le 2(1-q). \end{aligned} \tag{6.6}$$

This is suggested by the equalities in the Proof of Lemma 2.2 of [23]. The Proof of (ii) is similar to that of Proposition 2.4 of [34] and is omitted. □

Lemma 6.4 *Let $\gamma > 0$. Let μ be a distribution on $\mathbb{R}$.*

(i) *If $\mu \in \mathcal{L}(\gamma)$ with $\widehat{\mu}(\gamma) < \infty$, then $\mu^\gamma \in \mathcal{L}(0)$ and*

$$\overline{\mu^\gamma}(x) \sim \frac{\gamma}{\widehat{\mu}(\gamma)} \int_x^\infty \overline{\mu}(u) e^{\gamma u} du. \tag{6.7}$$

(ii) *If $\mu \in \mathcal{OS} \cap \mathcal{L}(\gamma)$, then $\widehat{\mu}(\gamma) < \infty$ and $\mu^\gamma \in \mathcal{OS} \cap \mathcal{L}(0)$.*

Proof Assertion (i) and the fact that $\widehat{\mu}(\gamma) < \infty$ in assertion (ii) are proved as in the Proof of Theorem 1.2 of [34]. The other part in assertion (ii) is clear from (i). □

From now on, let ρ and η be the same as those in (1.8). Define $q := \rho[0, \infty) > 0$, $\rho_1(dx) := q^{-1} 1_{[0,\infty)}(x)\rho(dx)$, and $\rho_2(dx) := (1-q)^{-1} 1_{(-\infty,0)}(x)\rho(dx)$ for $q < 1$ and $\rho_2 := 0$ for $q = 1$. Let $c_0 := \int_{-\infty}^{0-} e^{\gamma y} \rho_2(dy)$, $r := q + (1-q)c_0 \le 1$ and $s := \sum_{n=0}^\infty r^n p_n$. Define distributions ξ and ζ on $\mathbb{R}_+$ by

$$\xi(dx) := r^{-1}(q\rho_1(dx) + (1-q)c_0\delta_0(dx)) \quad \text{and} \quad \zeta(dx) := s^{-1} \sum_{n=0}^\infty r^n p_n \xi^{n*}(dx). \tag{6.8}$$

Lemma 6.5 *Let $\gamma \geq 0$. Let η and ρ be distributions as in (1.8). Assume further that $\sum_{n=0}^{\infty} p_n x^n < \infty$ for every $x > 0$. If $\rho \in \mathcal{OS} \cap \mathcal{L}(\gamma)$, then we have*

$$\lim_{x\to\infty}\left(\frac{\overline{\eta}(x)}{\overline{\rho}(x)} - \frac{s\overline{\zeta}(x)}{r\overline{\xi}(x)}\right) = 0. \tag{6.9}$$

Proof We find from Lemma 6.3 that $\rho_+ \in \mathcal{OS} \cap \mathcal{L}(\gamma)$ and hence $\rho_1, \xi \in \mathcal{OS} \cap \mathcal{L}(\gamma)$. Note from Lemma 6.3 that there are c_j and R_j for $j = 1, 2$ such that for every $k \geq 1$ and every $x > 0$

$$\frac{\overline{\rho^{k*}}(x)}{\overline{\rho}(x)} \leq c_1 R_1^k \quad \text{and} \quad \frac{\overline{\xi^{k*}}(x)}{\overline{\xi}(x)} \leq c_2 R_2^k. \tag{6.10}$$

Thus, for any $\varepsilon > 0$, there is an integer $N > 0$ such that

$$\sum_{n=N+1}^{\infty} p_n (c_1 R_1^n + c_2 R_2^n) < \varepsilon. \tag{6.11}$$

We find from Lemma 6.3 that

$$\limsup_{x\to\infty} \frac{\overline{\rho_1^{k*}}(x)}{\overline{\rho_1}(x)} < \infty \quad \text{for } 1 \leq k \leq n-1. \tag{6.12}$$

We see that

$$\frac{\overline{\rho_1^{k*}}(x-y)}{\overline{\rho_1^{k*}}(x)} \leq 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{\overline{\rho_1^{k*}}(x-y)}{\overline{\rho_1^{k*}}(x)} = e^{\gamma y} \qquad \text{for } y \leq 0. \tag{6.13}$$

Noting that $\overline{\rho_2^{n*}}(x) = 0$ for $x > 0$, we obtain that for $x > 0$

$$\begin{aligned}
&\left|\frac{\overline{\rho^{n*}}(x)}{\overline{\rho}(x)} - \frac{r^{n-1}\overline{\xi^{n*}}(x)}{\overline{\xi}(x)}\right| \\
&= \left|\sum_{k=1}^{n-1} \binom{n}{k} q^k (1-q)^{n-k} \frac{1}{\overline{\rho}(x)} \left(\overline{\rho_1^{k*} * \rho_2^{(n-k)*}}(x) - \overline{\rho_1^{k*}}(x) c_0^{n-k}\right)\right| \\
&\leq \sum_{k=1}^{n-1} \binom{n}{k} q^{k-1} (1-q)^{n-k} \frac{\overline{\rho_1^{k*}}(x)}{\overline{\rho_1}(x)} \int_{-\infty}^{0-} \left|\frac{\overline{\rho_1^{k*}}(x-y)}{\overline{\rho_1^{k*}}(x)} - e^{\gamma y}\right| \rho_2^{(n-k)*}(dy)
\end{aligned} \tag{6.14}$$

with the understanding that $\sum_{k=1}^{0} = 0$. Thus we see from (6.12)–(6.14) that

$$\lim_{x\to\infty} \left| \frac{\overline{\rho^{n*}}(x)}{\overline{\rho}(x)} - \frac{r^{n-1}\overline{\xi^{n*}}(x)}{\overline{\xi}(x)} \right| = 0. \tag{6.15}$$

Therefore we have by (6.10) and (6.11)

$$\limsup_{x\to\infty} \left| \frac{\overline{\eta}(x)}{\overline{\mu}(x)} - \frac{s\overline{\zeta}(x)}{r\overline{\xi}(x)} \right| \le \limsup_{x\to\infty} \sum_{n=1}^{N} p_n \left| \frac{\overline{\rho^{n*}}(x)}{\overline{\rho}(x)} - \frac{r^{n-1}\overline{\xi^{n*}}(x)}{\overline{\xi}(x)} \right| + \limsup_{x\to\infty} \sum_{n=N+1}^{\infty} p_n \left(\frac{\overline{\rho^{n*}}(x)}{\overline{\rho}(x)} + \frac{r^{n-1}\overline{\xi^{n*}}(x)}{\overline{\xi}(x)} \right) < \varepsilon. \tag{6.16}$$

Since $\varepsilon > 0$ is arbitrary, we have proved (6.9). □

Lemma 6.6 *Let η and ρ be distributions as in* (1.8). *Assume further that $\sum_{n=0}^{\infty} p_n x^n < \infty$ for every $x > 0$. Then $\overline{\eta}(x) \asymp \overline{\rho}(x)$ if and only if $\rho \in \mathcal{OS}$.*

Proof Suppose that $\overline{\eta}(x) \asymp \overline{\rho}(x)$. There is $m \ge 2$ such that $p_m > 0$. Thus we have

$$\overline{\rho^{2*}}(x) \le q^{2-m}\overline{\rho^{m*}}(x) \le \frac{q^{2-m}}{p_m}\overline{\eta}(x) \asymp \overline{\rho}(x). \tag{6.17}$$

That is, $\rho \in \mathcal{OS}$. Next suppose that $\rho \in \mathcal{OS}$. We have

$$\overline{\rho}(x) \le q^{1-m}\overline{\rho^{m*}}(x) \le \frac{q^{1-m}}{p_m}\overline{\eta}(x). \tag{6.18}$$

We see from Lemma 6.3 that with some $c > 0$ and $R > 0$ we have

$$\overline{\eta}(x) \le \overline{\rho}(x) \sum_{n=1}^{\infty} p_n c R^n \quad \text{for } x > 0. \tag{6.19}$$

Thus we have established $\overline{\eta}(x) \asymp \overline{\rho}(x)$. □

The following proposition is essentially due to Proposition 3.3 of [34], for which results of [27] were crucial. See also Theorem 1 of [15].

Proposition 6.7 *Let η and ρ be distributions as in* (1.8). *Assume further that $\sum_{n=0}^{\infty} p_n x^n < \infty$ for every $x > 0$. If $\rho \in \mathcal{OS} \cap \mathcal{L}(0)$ on $\mathbb{R}$, then*

$$\liminf_{x\to\infty} \frac{\overline{\eta}(x)}{\overline{\rho}(x)} = \sum_{n=1}^{\infty} n p_n. \tag{6.20}$$

Proof Suppose that $\rho \in \mathcal{OS} \cap \mathcal{L}(0)$ on $\mathbb{R}$. Then, by Lemma 6.3, $\xi = \rho_+ \in \mathcal{OS} \cap \mathcal{L}(0)$ on $\mathbb{R}$ with $r = s = 1$. Thanks to Proposition 3.3 of [34], we have

$$\liminf_{x \to \infty} \frac{\overline{\zeta}(x)}{\overline{\xi}(x)} = \sum_{n=1}^{\infty} n p_n. \tag{6.21}$$

Thus we see from Lemma 6.5 that (6.20) holds. □

Proof of Theorem 6.1 Suppose that $\rho \in \mathcal{L}(\gamma)$ and (6.1) holds. We see from Lemma 6.6 that $\rho \in \mathcal{OS} \cap \mathcal{L}(\gamma)$. Let $\gamma = 0$. Then we find from Proposition 6.7 that

$$c = \liminf_{x \to \infty} \frac{\overline{\eta}(x)}{\overline{\rho}(x)} = \sum_{n=1}^{\infty} n p_n. \tag{6.22}$$

Thus, by Theorem C, we have $\rho \in \mathcal{S}(0)$. Next let $\gamma > 0$. Denote the exponential tilts of ρ and η by ρ^{γ} and η^{γ}. Then we obtain from Lemma 6.4 (ii) that $\widehat{\rho}(\gamma) < \infty$ and $\widehat{\eta}(\gamma) = \sum_{n=0}^{\infty} p_n \widehat{\rho}(\gamma)^n < \infty$, and

$$\eta^{\gamma}(dx) = \frac{1}{\widehat{\eta}(\gamma)} \sum_{n=0}^{\infty} p_n \widehat{\rho}(\gamma)^n (\rho^{\gamma})^{n*}. \tag{6.23}$$

Moreover, $\rho^{\gamma}, \eta^{\gamma} \in \mathcal{OS} \cap \mathcal{L}(0)$ and

$$\overline{\rho^{\gamma}}(x) \sim \frac{\gamma}{\widehat{\rho}(\gamma)} \int_x^{\infty} \overline{\rho}(u) e^{\gamma u} du, \tag{6.24}$$

and

$$\overline{\eta^{\gamma}}(x) \sim \frac{\gamma}{\widehat{\eta}(\gamma)} \int_x^{\infty} \overline{\eta}(u) e^{\gamma u} du. \tag{6.25}$$

Thus we see from Proposition 6.7 that

$$c \frac{\widehat{\rho}(\gamma)}{\widehat{\eta}(\gamma)} = \liminf_{x \to \infty} \frac{\overline{\eta^{\gamma}}(x)}{\overline{\rho^{\gamma}}(x)} = \frac{1}{\widehat{\eta}(\gamma)} \sum_{n=1}^{\infty} n p_n \widehat{\rho}(\gamma)^n. \tag{6.26}$$

That is, $c = \sum_{n=1}^{\infty} n p_n \widehat{\rho}(\gamma)^{n-1}$. Thus, by Theorem C, we have $\rho \in \mathcal{S}(\gamma)$. □

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