Probab. Theory Relat. Fields (2008) 140:505–522
DOI 10.1007/s00440-007-0073-2

Hypercontractivity for a quantum Ornstein–Uhlenbeck semigroup

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Received: 2 June 2006 / Revised: 30 March 2007 / Published online: 10 May 2007

Abstract We prove hypercontractivity for a quantum Ornstein–Uhlenbeck semigroup on the entire algebra $\mathcal{B}(h)$ of bounded operators on a separable Hilbert space h. We exploit the particular structure of the spectrum together with hypercontractivity of the corresponding birth and death process and a proper decomposition of the domain. Then we deduce a logarithmic Sobolev inequality for the semigroup and gain an elementary estimate of the best constant.

Mathematics Subject Classification (2000) 81Q10 · 47D03 · 60J27

1 Introduction

Semigroups of completely positive, identity preserving maps on an operator algebra are the fundamental tool in the study of quantum open systems (see [1,2,23,29] and the references therein). They have been known in the physical literature since the 1970s as quantum dynamical semigroups and they can be viewed as a generalization of classical Markov semigroups on the Abelian algebra of functions on some space.

The most interesting examples of such semigroups arise in a canonical way in the Markovian limit of the evolution of a small system coupled with a quasi-free reservoir as, for instance, in the spontaneous decay of an atom, the spin–boson interaction, the Pauli–Fierz model, the one-mode electromagnetic field, resonance fluorescence and so on. These semigroups act on the von Neumann algebra $\mathcal{B}(h)$ of all bounded

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operators on the Hilbert space h (the Hilbert space of the small system) and have several interesting properties. Indeed, they leave some Abelian subalgebra invariant and their restriction to this algebra yields a classical Markov semigroup. They admit explicit representation formulae that are natural non-commutative extensions of well-known classical formulae. Moreover, they enjoy deepest properties, like exponential ergodicity or hypercontractivity (see [4,6,7,9,26]) which are essentially equivalent to the positivity of the spectral gap and to a non-commutative version of logarithmic Sobolev inequalities. They are also relevant from a physical point of view because, if the semigroup is interpreted as a quantum channel, a relationship between the contraction rate and the entropy production can be found [27].

In this paper we are concerned with the completely positive semigroup describing a free damped quantum harmonic oscillator. This semigroup acts on the algebra $\mathcal{B}(h)$ of all bounded operators on the Hilbert space $h = l^2(\mathbb{N})$ of complex-valued, square summable sequences indexed by the set $\mathbb{N}$ of natural numbers. The Hilbert space h is isometrically isomorphic to the Fock space on $\mathbb{C}$ arising in the canonical representation of 1D CCR. Denote by A, A^* the annihilation and creation operators on h. (If we introduce the canonical basis $(e_n)_{n\geq 0}$, we remember that A and A^* are described by $A^* e_n = \sqrt{n+1}e_{n+1}$, for $n \geq 0$; $Ae_0 = 0$, $Ae_n = \sqrt{n}e_{n-1}$ for $n \geq 1$.)

We shall call quantum Ornstein–Uhlenbeck (qOU) semigroup a collection of operators $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$, acting on $\mathcal{B}(h)$ and verifying the evolution equation

$$\frac{d}{\mathrm{d}t}\mathcal{T}_t(x) = \mathcal{L}\mathcal{T}_t(x), \quad \mathcal{T}_0 = id$$

where, at least for $x \in \mathcal{B}(h)$ with finite rank,

$$\mathcal{L}(x) = -\frac{\mu^2}{2}(A^*Ax - 2A^*xA + xA^*A) - \frac{\lambda^2}{2}(AA^*x - 2AxA^* + xAA^*),$$

with λ and μ real constants such that $0 < \lambda < \mu$.

The operator $\mathcal{L}$ is a Lindblad operator and the minimal associated semigroup $\mathcal{T}$ (whose existence is guaranteed by Davies' theorem [13]) is conservative and has a unique diagonal invariant state (see [9,17] and references therein)

$$\rho = (1-\nu)\sum_n \nu^n |e_n\rangle\langle e_n|, \quad \text{where } \nu = \lambda^2/\mu^2$$

(we use the Dirac notation: for u and v in h, $|u\rangle\langle v|$ denotes the operator on h defined by $|u\rangle\langle v|(w) = \langle v, w\rangle u$ for any w in h). For x and y in $\mathcal{B}(h)$, we will use the scalar product associated with the invariant state ρ, $\langle x, y\rangle_\rho = \mathrm{tr}(\rho^{1/2}x^*\rho^{1/2}y)$; we will also denote by $\|x\|_p = (\mathrm{tr}|\rho^{1/(2p)}x\rho^{1/(2p)}|^p)^{1/p}$, $p \geq 1$, the usual L^p norms associated with ρ. An interpolating family of L^p spaces is then generated considering $L_\infty = \mathcal{B}(h)$ and, for $p \in [1, +\infty)$, taking the space $L_p(h)$ as the closure of $\mathcal{B}(h)$ with respect to the norm $\|\cdot\|_p$. Then $L_2(h)$ will be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_\rho$. Notice that the definition of these norms is possible due to the faithfulness of ρ.

The restriction of the operator $\mathcal{L}$ to the algebra of diagonal operators (in the above basis) is the infinitesimal generator of a birth and death process with linear rates (see

Biane [3], (11) here); more details about this aspect can be found in the last section. Further, $\mathcal{L}$ is usually called the qOU generator since, by Schrodinger representation, one can see that its restriction to the space of multiplication operators by a function of a quadrature operator $\bar{z}A + zA^*$ ($z \in \mathbb{C}$) is the classical Ornstein–Uhlenbeck generator on smooth functions (see relations (7.5) in [10]). This commutative restriction of $\mathcal{L}$ is well-known and has been deeply studied; in particular hypercontractivity was proved by Gross [18] through log-Sobolev inequalities. The complexified version of this result was obtained by Epperson [16] by using a discretization technique and a simple application of the central limit theorem.

So the operator $\mathcal{L}$ is the natural extension of well-known generators on abelian algebras, but it has been extensively studied since it appears in quantum optics models. A complete study of the spectral properties of the qOU generator is made in [10] and some other properties, like L^p-contractivity and uniform exponential convergence, are studied in [9].

A natural question is now whether the qOU semigroup is hypercontractive, that is whether, for all $p, q \geq 1$, the operators $\mathcal{T}_t$ are contractive from L_q to L_p for t large enough, formally: for all $p, q \geq 1$, there exists t_{pq} such that $\|\mathcal{T}_t(x)\|_p \leq \|x\|_q$ for all x and all $t \geq t_{pq}$. We will briefly discuss some weaker and equivalent conditions for hypercontractivity in Sect. 4. Notice that the previous definition can result different in some papers, since some authors call this property strict hypercontractivity. Biane [4] and Bożejko [6] studied hypercontractivity for a (more general) qOU semigroup, associated with the representation of q-commutation relations, on the algebra generated by the position operator; an extension to general commutation relations has been recently analysed by Kròlak [20].

Here we study hypercontractivity for the previously introduced qOU semigroup on the whole algebra $\mathcal{B}(h)$ and this paper is basically devoted to the proof of the following main result

Theorem 1.1 *The quantum Ornstein–Uhlenbeck semigroup is hypercontractive.*

For the Ornstein–Uhlenbeck semigroup on the spaces $L^p(\mathbb{R}^n)$, the hypercontractivity was proved in the 1970s [18] but similar problems have been only recently solved also for finite Markov chains or birth and death semigroups (see [11, 12, 14, 15, 24, 25, 28], where relations between hypercontractivity and other convergence properties are emphasized). The extension of the result to the quantum case is not surprising, but the non-commutative context usually requires additional technical efforts. The usual approach by logarithmic-Sobolev inequalities (introduced in [18]) has further been developed by many other authors in non-commutative context (see e.g. [4, 8, 19, 21, 22]) and has been finally extended to the semigroups acting on the operator algebras we consider here by Olkiewicz and Zegarlinsky [26] in 1999. But this kind of approach would lead to quite difficult computations in our context, so we choose an alternative solution of the problem, deducing log-Sobolev inequalities by hypercontractivity.

Three main ingredients have been used to tackle the problem: (i) the spectral structure of the operator $\mathcal{L}$ (described in [10]), (ii) a particular diagonal decomposition of the space which is preserved by $\mathcal{L}$ and (iii) the analysis of the associated birth and death process. The properties (i) and (ii) are described in Sect. 2, where the equivalence of hypercontractivity to some norms inequalities for the semigroup is also clarified (see (2), (5)).

Then, in Sect. 3, we study the previous inequalities and move to similar relations for norms of commutative functions, defined on the set of natural numbers.

Finally, in Sect. 4, we study some properties of the (birth and death) semigroup P obtained by restricting $\mathcal{T}$ to the algebra of diagonal operators. This allows us to conclude the proof of Theorem 1.1. Some elementary estimates of the log-Sobolev constants for the semigroups P and $\mathcal{T}$ are also obtained.

2 Preliminaries and orthogonal decomposition of the domain

We want to prove hypercontractivity via spectral theory. Our starting points are

(a) The spectral structure of the OU generator: the operator $\mathcal{L}$ (see [10]) has eigenvalues $n(\lambda^2-\mu^2)/2$ with eigenspaces E_n,

$$E_n = \mathrm{Lin}\{p_n(Q_z) : |z| = 1\} = \mathrm{Lin}\{p_n(Q_z) : z \text{ such that } z^{2(n+1)} = 1\},$$

where p_n is the Hermite polynomial defined by $p_n(t) = \sum_{2r\le n}\left(-\frac{\mu^2+\lambda^2}{4(\mu^2-\lambda^2)}\right)^r \frac{n!}{r!(n-2r)!}t^{n-2r}$ and $Q_z = (\bar{z}A + zA^*)/\sqrt{2}$, for z in $\mathbb{C}$. In particular the subspaces $\{E_n : n \ge 0\}$ are mutually orthogonal and $\bigoplus_{n\ge 0} E_n = L_2(h)$. Moreover, we emphasize

$$U_n = \mathrm{Lin}\{\cup_{m=0}^n E_m\} = \mathrm{Lin}\{Q_z^m : |z| = 1, m \le n\} = \mathrm{Lin}\{A^{*i}A^j : i + j \le n\}.$$

(b) This spectral structure of $\mathcal{L}$ gives us the necessary conditions to apply a result in [5] (Theorem 2.1, p. 17), which guarantees that the semigroup generated by $\mathcal{L}$ is hypercontractive if

$$\text{there exists } C > 0 \text{ such that} \quad \|x\|_4 \le C^n\|x\|_2 \quad \text{for all } x \text{ in } E_n, \tag{1}$$

where C is independent of n and x.

We immediately remark two facts.

(i) Condition (1) is equivalent to

$$\text{there exists } C > 0 \text{ such that} \quad \|x\|_4 \le C^n\|x\|_2 \quad \text{for all } x \text{ in } U_n. \tag{2}$$

Notice that the constants C involved in (1) and (2) are different in general, but both are not less than 1 since, for the identity operator $\mathbb{1} \in E_0 = U_0$, we have $\|\mathbb{1}\|_2 = \|\mathbb{1}\|_4 = 1$.

It is quite obvious that the second condition implies the first and, for the inverse implication, just consider that any x in U_n has the orthogonal decomposition (in $L^2(h)$) $x = \sum_{k=0}^{n} y_k$, where $y_k \in E_k$ and so, if (1) holds, we have

$$\|x\|_4 \leq \sum_{k=0}^{n} \|y_k\|_4 \leq \sum_k C^k \|y_k\|_2 \leq C^n (n \sum_k \|y_k\|_2^2)^{1/2} \leq C_1^n \|x\|_2, \quad (3)$$

where we choose C_1 such that $C_1^n \geq C^n \sqrt{n}$ for all n.

(ii) Each condition ((1) or (2)) is also equivalent to hypercontractivity since, if $\mathcal{T}$ is hypercontractive, then there exists a $\bar{t}$ such that $\|\mathcal{T}_t(x)\|_4 \leq \|x\|_2$ for all $t \geq \bar{t}$. So, if x_n is in E_n, one has

$$\|\mathcal{T}_{\bar{t}}(x_n)\|_4 = e^{-\bar{t}n(\mu^2-\lambda^2)/2} \|x_n\|_4 \leq \|x_n\|_2,$$

and (1) follows choosing $C = e^{\bar{t}(\mu^2-\lambda^2)/2}$.

The proof of hypercontractivity for $\mathcal{T}$ can so be reduced to the research of a constant C (not necessarily the optimal one) verifying (2). The explicit computation of the norms in (1) or (2) are in general quite difficult, so the choice of a proper orthogonal basis of the space U_n will help us; notice that it will not be convenient to use the elements $p_n(Q_z)$, $|z| = 1$, generating the eigenspaces, since they are generally not mutually orthogonal. In this section, we shall introduce a "diagonal" orthogonal decomposition of the space; then, in Sect. 3, we shall see how this decomposition allows us to simplify the problem by considering norms' inequalities involving commutative functions instead of operators in $\mathcal{B}(h)$.

An element x of $\mathcal{B}(h)$ can always be written as $x = \sum_{n,m\geq 0} x_{nm} |e_n\rangle\langle e_m|$, where $x_{n,m} = \langle e_n, x(e_m)\rangle_h$; so x can be identified with the "matrix" $(x_{nm})_{n,m}$. We point out that the sum we have just considered for x (and similarly in what follows) is weak in $\mathcal{B}(h)$ and strong in $L^p(h)$ for all p.

The linear spaces

$$\mathcal{F}_k = \{x \in \mathcal{B}(h) \,|\, x_{nm} = 0 \text{ for } m - n \neq k\}, \qquad k \in \mathbb{Z}$$

($\mathcal{F}_k$ contains the elements which can have only the k-th diagonal different from zero) are an orthogonal decomposition of the space $\mathcal{B}(h) \subset L_2(h)$, with respect to $\langle\, ,\, \rangle_\rho$. So, for all x, we can write the orthogonal decomposition $x = \sum_k \xi^k$, where $\xi^k = \sum_{n\geq 0} x_{n,n+k} |e_n\rangle\langle e_{n+k}| \in \mathcal{F}_k$, for k in $\mathbb{Z}$.

This "diagonal" orthogonal decomposition will fit our approach to the problem and some of its good properties are evident:

- first, a simple direct computation shows that the spaces $\mathcal{F}_k$ are preserved by the qOU generator, in the sense that $\mathcal{L}(\mathcal{F}_k) \subset \mathcal{F}_k$;
- second, the elements of $\mathcal{F}_k$, having only one significant diagonal, are simple operators, someway similar to commutative functions defined on $\mathbb{N}$ (we will work on this feature in next section);
- finally, the decomposition of an operator in its diagonal components is very easy to compute explicitly; in particular this is true for the elements of the spaces U_n.

Indeed, an element of the form $A^{*i}A^j$ belongs to $\mathcal{F}_{j-i}$, so $\langle A^{*i}A^j, A^{*l}A^m\rangle_\rho = 0$ if and only if $m-l \neq j-i$. We consider x in U_n, then $x = \sum_{i,j:i+j\leq n} a_{ij}A^{*i}A^j$, for some real coefficients a_{ij}, we call ξ_k the k-th diagonal of x and write the orthogonal decomposition

$$x = \sum_{|k|\leq n} \xi_k, \qquad \text{where} \quad \xi_k = \sum_{i,j:i+j\leq n, j-i=k} a_{i,j}A^{*i}A^j \in \mathcal{F}_k. \tag{4}$$

If we introduce the spaces

$$W_k^n = \text{Lin}\{A^{*i}A^j : i, j \geq 0, i+j \leq n, j-i = k\} = \mathcal{F}_k \cap U_n,$$

for each $k \in \mathbb{Z}$, $|k| \leq n$, it is quite simple to see that $\xi_k \in W_k^n$, $U_n = \oplus_{|k|\leq n} W_k^n$ and $\mathcal{L}$ is stable also with respect to each W_k^n.

In addition, if the desired inequality is true for ξ in W_k^n, that is there exists $C_0 > 0$ such that

$$\|\xi\|_4 \leq C_0^n \|\xi\|_2 \quad \text{for all } n, k \leq n, \text{and } \xi \in W_k^n \ , \tag{5}$$

then (2) is true, since, for x in U_n, using the orthogonal decomposition (4), as in (3)

$$\|x\|_4 \leq \sum_k \|\xi_k\|_4 \leq (C_0 \vee 1)^n \sum_k \|\xi_k\|_2 \leq C_0^n((2n+1)\sum_k \|\xi_k\|_2^2)^{1/2}, \tag{6}$$

and we just have to choose C such that $C^n \geq C_0^n\sqrt{2n+1}$ for all n. Notice again that $C_0 \geq 1$ since, for the identity operator $\mathbb{1} \in W_0^0$, we have $\|\mathbb{1}\|_2 = \|\mathbb{1}\|_4 = 1$.

Moreover, since for any ξ in W_k^n, $\xi^* \in W_{-k}^n$ and $\|\xi\|_p = \|\xi^*\|_p$ for all p, it is sufficient to prove (5) only for $k \geq 0$.

Summing up, in this section, we have seen that the qOU semigroup is hypercontractive if and only if the following condition **(H)** holds

(H) there exists $C_0 > 0$ such that, for all n, $0 \leq k \leq n$, $\xi \in W_k^n \quad \|\xi\|_4 \leq C_0^n\|\xi\|_2$.

In the next section, we show that it is sufficient to verify a corresponding inequality for polynomials (relation (8) in Proposition 3.2) to prove **(H)**. The hypercontractivity of the qOU semigroup will consequently follow by the hypercontractivity of the associated birth and death process described by relation (11).

3 The equivalence with commutative inequalities

We underline that ρ can be thought of as a probability measure on the set $\mathbb{N}$ of natural numbers, giving mass $(1-\nu)\nu^m$ to the singleton $\{m\}$. So we can define a family of

L^p spaces for complex valued functions defined on $\mathbb{N}$

$$l^p(\mathbb{N},\rho)=\left\{f:\mathbb{N}\to\mathbb{C}\text{ such that }\sum_{m\geq 0}\nu^m|f(m)|^p<\infty\right\}$$

$$\text{with norm}\qquad \|f\|_{l^p(\mathbb{N},\rho)}=\left((1-\nu)\sum_{m\geq 0}\nu^m|f(m)|^p\right)^{1/p}.$$

We will denote by $l^\infty(\mathbb{N})$ the set of bounded functions defined on $\mathbb{N}$. Notice that, for a diagonal operator $x=\sum_{m\geq 0}f(m)|e_m\rangle\langle e_m|$, we will have $\|x\|_p=\|f\|_{l^p(\mathbb{N},\rho)}$. So these spaces have a clear connection with the $L_p(h)$ spaces introduced before. Our aim is to rewrite inequalities in condition **(H)**, involving non-commutative operators, as inequalities for particular functions of $l^p(\mathbb{N},\rho)$.

We need to introduce other function spaces. For $n\geq 0$, $0\leq k\leq n$, call $M:=[(n-k)/2]$ ($[\,\cdot\,]$ denotes the entire part) and define

$$\mathcal{V}_k^n=\left\{f:\mathbb{N}\to\mathbb{C}\text{ s.t. }f(m)=\left(\frac{m!}{(m-k)!}\right)^{1/2}p_M(m),\ p_M\right.$$
$$\left.\text{polynomial of degree }M\right\}.$$

It is obvious that the square of an element f of $\mathcal{V}_k^n$ is a polynomial of degree $2M+k\leq n$ and that $\mathcal{V}_k^n\subset l^p(\mathbb{N},\rho)$ for all p.

In the next two propositions we will prove that condition **(H)** can be written equivalently as an inequality for the norms (in $l^p(\mathbb{N},\rho)$) of functions in the sets $\mathcal{V}_k^n$ or, better, of polynomials, which are the elements of the spaces $\mathcal{V}_0^n$.

Proposition 3.1 *Condition* **(H)** *is fulfilled if and only if there exists* $C_0>0$ *such that*

$$\|f\|_{l^4(\mathbb{N},\rho)}\leq C_0^n\nu^{-k/8}\|f\|_{l^2(\mathbb{N},\rho)}\tag{7}$$

for all n, $0\leq k\leq n$*, and all functions* f *in* $\mathcal{V}_k^n$*.*

Proof We denote by N the number operator on the space h, $Ne_n=ne_n$ for all n, with the notations of the Introduction. For the operators A, A^*, ρ, the following commutation relations hold

$$A\rho^s=\nu^s\rho^sA,\quad A^*\rho^s=\nu^{-s}\rho^sA^*,$$
$$A^mA^{*m}=(N+1)\cdots(N+m),\quad A^{*m}A^m=N(N-1)\cdots(N-m+1),$$

for all real s and for all m in $\mathbb{N}$, $m\geq 1$.

Fix n and consider $k \in \{0, \dots, n\}$, then any element ξ in W_k^n can be written in the form

$$\xi = \sum_{i=0}^{M} a_{i,i+k} A^{*i} A^{i+k}, \qquad M = [(n-k)/2],$$

so we can associate with ξ the function f_ξ in $\mathcal{V}_k^n$

$$f_\xi(m) = \left(\frac{m!}{(m-k)!}\right)^{1/2} \sum_{j=0}^{M} a_{j,j+k}(m-k)\cdots(m-k-j+1).$$

The map $\xi \in W_k^n \mapsto f_\xi \in \mathcal{V}_k^n$ is obviously one-to-one. Indeed, consider any function f in $\mathcal{V}_k^n$, $f(m) = \left(\frac{m!}{(m-k)!}\right)^{1/2} p_M(m)$, with p_M polynomial of degree M; then we can write p_M in the form $p_M(m) = \sum_{j=0}^{M} a_{j,j+k}(m-k)\cdots(m-k-j+1)$, for suitable (unique) $a_{j,j+k}$; so, choosing $\xi = \sum_{i=0}^{M} a_{i,i+k} A^{*i} A^{i+k}$, we obtain $f_\xi = f$. Now we can easily find a relation between the norms of ξ and f_ξ

$$\begin{aligned}
\|\xi\|_2^2 &= \mathrm{tr}\{\rho^{1/2}\bar{\xi}\rho^{1/2}\xi\} \\
&= \sum_{i,j=0}^{M} \bar{a}_{i,i+k} a_{j,j+k} \mathrm{tr}(\rho A^{*i+k} A^i A^{*j} A^{j+k}) \nu^{-k/2} \\
&= \nu^{-k/2} \mathrm{tr}\left\{\rho \frac{N!}{(N-k)!} \left|\sum_{j=0}^{M} a_{j,j+k}(N-k)\cdots(N-k-j+1)\right|^2\right\} \\
&= (1-\nu)\nu^{-k/2} \sum_{m\geq 0} \nu^m f_\xi^2(m) = \nu^{-k/2} \|f_\xi\|_{l^2(\mathbb{N},\rho)}^2,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\|\xi\|_4^4 &= \nu^{-k/2} \mathrm{tr}\left\{\rho \left(\frac{N!}{(N-k)!}\right)^2 \left|\sum_{j=0}^{M} a_{j,j+k}(N-k)\cdots(N-k-j+1)\right|^4\right\} \\
&= (1-\nu)\nu^{-k/2} \sum_{m\geq 0} \nu^m f_\xi^4(m) \\
&= \nu^{-k/2} \|f_\xi\|_{l^4(\mathbb{N},\rho)}^4.
\end{aligned}$$

Consequently the inequality for ξ given by **(H)** and (7) are the same. □

Remark 3.1 The map $\xi \in W_k^n \mapsto f_\xi \in \mathcal{V}_k^n$ is a kind of isometry (up to a constant depending on ν) for any L^p norm. This relationship describes how to pass our non-commutative context to a commutative one.

We can go further and show that it is sufficient to prove the inequality in the previous proposition only for $k = 0$. This means that our problem may be reduced to studying only particular diagonal operators since the spaces $\mathcal{V}_0^n$ correspond to subspaces of $\mathcal{F}_0$.

Proposition 3.2 *Condition* **(H)** *is fulfilled if and only if there exists* $C_1 > 0$ *such that*

$$\|p_n\|_{l^4(\mathbb{N},\rho)} \leq C_1^n \|p_n\|_{l^2(\mathbb{N},\rho)} \tag{8}$$

for all n and all polynomials p_n *of degree n.*

Proof It is obvious that condition (8) is equivalent to condition (7) with $k = 0$: just notice that p_n is in $\mathcal{V}_0^{2n}$ and choose $C_1 = C_0^2$. We have to prove that (8) implies (7) also for $k > 0$. So let us suppose that (8) holds, consider $n \geq 1$, $1 \leq k \leq n$, and f in $\mathcal{V}_k^n$, i.e. $f^2(m) = \frac{m!}{(m-k)!}|p_M(m)|^2$, for some p_M polynomial of degree $M = [(n-k)/2]$. Then, for $k \neq 0$, k even, we can write

$$\begin{aligned}
\|f\|_{l^4(\mathbb{N},\rho)}^4 = \; & (1-\nu)\sum_{m\geq k}\nu^m|p_M(m)|^4(m-k+1)^2(m-k+2)^2\cdots(m-1)^2m^2 \\
& \text{since } s+1 \leq 2s \text{ for any } s \geq 1 \\
\leq \; & 2^k(1-\nu)\sum_{m\geq k}\nu^m|p_M(m)|^4(m-k+1)^4(m-k+3)^4\cdots(m-1)^4 \\
\overset{by (8)}{\leq} \; & C_1^{4M+2k}2^k\Bigg((1-\nu)\sum_{m\geq k}\nu^m|p_M(m) \\
& \times(m-k+1)(m-k+3)\cdots(m-1)|^2\Bigg)^2 \\
\leq \; & C_1^{4M+2k}2^k\left((1-\nu)\sum_{m\geq k}\nu^m|p_M(m)|^2(m-k+1)(m-k+2)\cdots m\right)^2 \\
\leq \; & C_1^{4M+2k}2^k\|f\|_{l^2(\mathbb{N},\rho)}^4 \leq (2C_1^2)^n\|f\|_{l^2(\mathbb{N},\rho)}^4.
\end{aligned}$$

It remains to study the case k odd. Assume now $k = 2r+1$, with $r \geq 0$, and introduce the constant C_ν such that

$$C_\nu^2 := (1-\nu)\sum_{m\geq 0}\nu^m m^4 = \nu(1+\nu)(\nu^2+10\nu+1)(1-\nu)^{-4}.$$

As in the previous case, we have that

$$\begin{aligned}
\|f\|^4_{l^4(\mathbb{N},\rho)} \leq\ & 2^{2r}(1-\nu)\sum_{m\geq k}\nu^m|p_M(m)|^4(m-k+1)^4 \\
& \times(m-k+3)^4\cdots(m-2)^4m^2, \\
& \text{by the Schwarts inequality} \\
\leq\ & 2^{2r}C_\nu\Bigg((1-\nu)\sum_{m\geq k}\nu^m|p_M(m)(m-k+1) \\
& \times(m-k+3)\cdots(m-2)|^8\Bigg)^{1/2} \\
\overset{by(8)}{\leq}\ & 2^{2r}C_\nu C_1^{2(M+r)}(1-\nu)\sum_{m\geq k}\nu^m\big(|p_M(m)|(m-k+1) \\
& \times(m-k+3)\cdots(m-2)\big)^4 \\
\overset{by(8)}{\leq}\ & 2^{2r}C_\nu C_1^{6(M+r)}\Bigg((1-\nu)\sum_{m\geq k}\nu^m|p_M(m)|^2(m-k+1) \\
& \times(m-k+2)\cdots(m-1)\Bigg)^2 \\
=\ & 2^{2r}C_\nu C_1^{6(M+r)}\|f\|^4_{l^2(\mathbb{N},\rho)} \leq C_\nu(2C_1^3)^n\|f\|^4_{l^2(\mathbb{N},\rho)}.
\end{aligned}$$

By an apposite choice of C_0, we obtain the required estimates. □

In the next section we shall prove that relation (8) is verified (and so also condition **(H)**) by discussing the hypercontractivity of the associated classical birth and death process.

4 Logarithmic Sobolev inequality and associated birth and death process

Since hypercontractivity is often linked to log-Sobolev inequalities, it is worth considering the possible relations of the results of the previous section with the log-Sobolev inequality for the qOU semigroup. A rough estimate of the log-Sobolev constant can easily be found by searching for some constants (not necessarily the optimal ones) verifying inequalities (6–8). First, we recall some main definitions and results about hypercontractivity and log-Sobolev inequalities for semigroups (on eventually non-commutative algebras).

We consider an interpolating family of Banach spaces L_p, $p \in [0, +\infty]$, which contain a common dense Banach subspace $\mathcal{A}$; for our purposes, it will be sufficient to consider $\mathcal{A} = L_\infty = \mathcal{B}(h)$. For a linear operator R on $\mathcal{B}(h)$ and p, q in $[1, +\infty]$ we denote by $\|R\|_{p,q} = \sup_{x\neq 0}\frac{\|R(x)\|_q}{\|x\|_p}$ the norm of the operator $R : L_p(h) \to L_q(h)$.

We shall say that a semigroup $\mathcal{R} = (\mathcal{R}_t)_{t\geq 0}$ of linear maps on $\mathcal{B}(h)$ is *weakly hypercontractive* if,

(C1) for all $p, q \in [1, +\infty)$, there exist t_{pq} and $C_{p,q}$ s.t. $\|\mathcal{R}_t\|_{p,q} \leq C_{p,q}$ for all $t \geq t_{pq}$.

Recalling our definition of hypercontractivity given in the Introduction, a semigroup $\mathcal{R}$ is *hypercontractive* when **(C1)** holds with $C_{p,q} = 1$ for all p and q.

It is well-known (see [26] or [15], Ch.5) that, if the semigroup is L_∞-contractive (i.e. $\|\mathcal{R}_t\|_{\infty,\infty} \leq 1$) and there exist a time τ and a constant C such that $\|\mathcal{R}_\tau\|_{2,4} \leq C$, then

(C2) $\|\mathcal{R}_t\|_{2,q(t)} \leq \exp(1 - 2d(q(t))^{-1}), \quad$ for $q(t) = 1 + e^{t/(2\tau)}$, $d = \ln C$.

In this case, by interpolation results, **(C2)** is equivalent to **(C1)** (that is weak hypercontractivity) and we have hypercontractivity when **(C2)** holds with $d = 0$.

So, for L_∞-contractive semigroups (like the qOU semigroup $\mathcal{T}$), hypercontractivity is equivalent to the existence of a time τ such that $\|\mathcal{R}_\tau\|_{2,4} \leq 1$. This is a key-ingredient for the proof of Theorem 2.1 in [5], that we have cited at the beginning of Sect. 2.

We have already told in the Sect. 1 that, when we consider a semigroup $\mathcal{R}$ of linear maps on $\mathcal{B}(h)$, the norms of the interpolating L_p spaces can be chosen appropriately depending on the invariant state. Further, if $\mathcal{R}$ has an invariant state (or measure, in the commutative case) π and infinitesimal generator A, then we shall denote by $\mathcal{E}$ the associated quadratic form, $\mathcal{E}(x) = -\langle x, A(x)\rangle_\pi = \mathrm{tr}(\pi^{1/2}x^*\pi^{1/2}A(x))$ and, for any positive element x of $\mathcal{B}(h)$, we shall call entropy of x (w.r.t. π) the quantity

$$\mathrm{Entropy}(x) = \mathrm{tr}(\tilde{x}^*\tilde{x}\,\mathrm{lg}(\tilde{x})) - \frac{1}{4}\mathrm{tr}(\tilde{x}^*(\tilde{x}\,\mathrm{lg}\,\pi + \mathrm{lg}\,\pi\,\tilde{x})) - \|x\|_{2,\pi}^2\,\mathrm{lg}\,\|x\|_{2,\pi},$$

where $\tilde{x} = \pi^{1/4}x\pi^{1/4}$ and $\|x\|_{p,\pi}^p = \mathrm{tr}(|\pi^{1/(2p)}x\pi^{1/(2p)}|^p)$. The entropy and the quadratic form are the quantities involved in the so called logarithmic-Sobolev inequalities, that we want to discuss in this section. Here notations and terminology are similar to the ones used in [26], where we also find the proof of the following result.

Theorem 4.1 (R. Olkiewicz and B. Zegarlinski, Theorems 3.8 and 4.2 in [26])

(a) *Suppose $\mathcal{R} = (\mathcal{R}_t)_{t\geq 0}$ is a L^2-symmetric, positive and contractive (with respect to any $\|\cdot\|_p$, $1 \leq p \leq +\infty$) semigroup. If $\mathcal{R}$ is weakly hypercontractive, that is*

$$\|\mathcal{R}_t(x)\|_{q(t)} \leq \exp(d(1 - 2(q(t))^{-1}))\|x\|_2 \tag{9}$$

with $d \geq 0$, $q(t) = 1 + e^{2t/c}$ for some $c > 0$, then the following LS(c, d) (Logarithmic Sobolev inequality with constants c and d) is true

$$\mathrm{Entropy}(x) \leq c\mathcal{E}(x) + d\|x\|_2^2. \tag{10}$$

(b) *Moreover, if LS(c, d) holds and the semigroup has strictly positive spectral gap η, then LS$(c + (d + 1)\eta^{-1}, 0)$ holds.*

The value $\alpha = c^{-1}$, where c is the best constant verifying LS$(c, 0)$ is usually called the log-Sobolev constant of the semigroup $\mathcal{R}$.

In this section we will also exploit a converse of the previous Theorem 4.1(a), which is true for classical Markov semigroups (see [14,15] for the proof):

Theorem 4.2 *Suppose $\mathcal{R} = (\mathcal{R}_t)_{t\geq 0}$ is a L^2-symmetric, positive and contractive semigroup acting on commutative L^p spaces. If $LS(c, d)$ holds, then (9) holds.*

Remark 4.1 We would like to point out that in [26] the authors prove a result similar to Theorem 4.2 also for semigroups acting on operators' algebras under an additional regularity condition on the quadratic form $\mathcal{E}$ associated with the infinitesimal generator. We do not write the details here, since this regularity condition is not immediate to describe and, moreover, our approach will require this "converse result" only in a commutative environment (so Theorem 4.2 will be sufficient), where the regularity condition on the quadratic form does not appear, since it is always satisfied. We also want to stress that, for the qOU semigroup, the structure of the spectrum guarantees the existence of the spectral gap, which gives a strong condition on $\mathcal{E}$ and someway "helps" the log-Sobolev inequalities (these aspects are also studied in [26], in particular in Sects. 4 and 5). So we do not need to study explicitly the regularity conditions of the quadratic form $\mathcal{E}$ here, but they are essentially hidden in the step where we use Theorem 4.2 of [5].

For the qOU semigroup $\mathcal{T}$, we can choose the L^p spaces associated with the invariant state ρ, that we have described in Sect. 1: $L_\infty = \mathcal{B}(h)$ and, for $p \in [1, \infty)$, $L_p(h)$ as the closure of the algebra $\mathcal{B}(h)$ with respect to the norm $\|\cdot\|_p = \|\cdot\|_{p,\rho}$. We will also have $\pi = \rho$ and $\mathcal{E}(x) = -\langle x, \mathcal{L}(x)\rangle_\rho$, for any x in the domain of $\mathcal{L}$. Then we know that $\mathcal{T}$ is a completely positive semigroup which is contractive with respect to any L^p norm (see [9]); moreover, it has spectral gap $\eta := \frac{\mu^2-\lambda^2}{2}$ (see [9,10]).

As we previously told, we can consider the restriction of $\mathcal{L}$ to the commutative algebra of diagonal operators; this restriction can be seen as an operator $\mathcal{G}$ acting on $l^\infty(\mathbb{N})$. For any $f = \sum_n f(n)e_n$ in $l^\infty(\mathbb{N})$, we define

$$\mathcal{G}f = \sum_n \Big(\mu^2 n(f(n-1) - f(n)) + \lambda^2(n+1)(f(n+1) - f(n))\Big)\, e_n;$$

then it is easy to see that any diagonal operator x in $\mathrm{dom}(\mathcal{L}) \subset \mathcal{B}(h)$ can be written $x = \sum_n f(n)|e_n\rangle\langle e_n|$ for some f in $l^\infty(\mathbb{N})$ and

$$\mathcal{L}(x) = \mathcal{L}\left(\sum_n f(n)|e_n\rangle\langle e_n|\right) = \sum_n (\mathcal{G}f)(n)|e_n\rangle\langle e_n|. \tag{11}$$

$\mathcal{G}$ is the infinitesimal generator of a semigroup $P = (P_t)_{t\geq 0}$ on $l^\infty(\mathbb{N})$ which can be similarly seen as the restriction of $\mathcal{T}$ to the algebra of diagonal operators. The measure on $\mathbb{N}$ induced by the state ρ is invariant for $\mathcal{G}$ and the corresponding L_p interpolating spaces are the spaces $l^p(\mathbb{N}, \rho)$ that we have introduced in the previous section. P is obviously a L^2-symmetric, positive and contractive semigroup, since $\mathcal{T}$ is.

Moreover, P is the commutative semigroup associated with a classical birth and death process with linear rates (birth rates $b_n = \lambda^2(n+1)$ and death rates $a_n = \mu^2 n$).

Let us call α_1 and α_0 the log-Sobolev constants of the qOU semigroup $\mathcal{T}$ and of its diagonal restriction P, respectively (obviously we will have $\alpha_0 \geq \alpha_1$).

We shall prove that $\mathcal{T}$ is hypercontractive, so Theorem 4.1 will assure us that a logarithmic Sobolev inequality holds, but we shall also ask for some indications about the value of the involved constant. We shall obtain some estimates of α_0 (which is unknown, as far as we are aware) and then some estimates of α_1 using α_0. More precisely, in this section, we will prove the following facts.

- P satisfies a log-Sobolev inequality $\mathrm{LS}(\alpha_0^{-1}, 0)$, where the best constant α_0 verifies

$$\frac{\lg(\nu^{-1})}{5\sqrt{5}\mu^2(1-\nu)^{3/2}} \leq \alpha_0^{-1} \leq \frac{255}{4}\,\frac{(1+\lg 2)(1-\nu)+\lg(\nu^{-1})}{\mu^2(1-\nu)^3}.$$

  As a result, P is hypercontractive by Theorem 4.2.
- By the hypercontractivity of P and the spectral properties of its infinitesimal generator $\mathcal{G}$, we deduce that condition (8) is verified with $C_1 = \sqrt{2}\, 3^{(\mu^2-\lambda^2)/(2\alpha_0)}$. So the qOU semigroup is hypercontractive.
- The qOU semigroup verifies a log-Sobolev inequality $\mathrm{LS}(\alpha_1^{-1}, 0)$ where the best constant α_1 verifies

$$\alpha_0^{-1} \leq \alpha_1^{-1} \leq \frac{4\,(5-\lg(1-\nu))}{\mu^2(1-\nu)} + (3\lg 3)\alpha_0^{-1}.$$

4.1 Hypercontractivity for the birth and death process

We are going to show that P is hypercontractive and we are interested, in particular, in proving that it satisfies a log-Sobolev inequality. This and other kinds of contractivity properties for semigroups on commutative spaces have been deeply investigated; we refer to [11] for a review of the existing results and to [12,14,24,25] for some recent developments on hypercontractivity. Here we will follow the technique described by Chen in [12], remembering that we are treating a birth and death process with birth rates $b_n = \lambda^2(n+1)$, death rates $a_n = \mu^2 n$, and so with invariant measure induced by ρ.

We introduce the constant

$$B_\nu = \sup_{n\geq 1}\sum_{j=1}^{n}\frac{1}{\mu^2 j \nu^j} M\left(\frac{\nu^n}{1-\nu}\right),$$

$$\text{where}\quad M(x) = \frac{\sqrt{4x+1}-1}{2} + x\lg\left(1+\frac{\sqrt{4x+1}+1}{2x}\right), \qquad \text{for } x > 0.$$

Adjusting some results in [12], we get the following estimate for α_0

Lemma 4.1 $\frac{2}{5}(M((1-\nu)^{-1}))^{-1}B_\nu \leq \alpha_0^{-1} \leq \frac{255}{4}B_\nu$.

Proof These inequalities are slightly different from the ones written in [12], but they can be easily obtained by Theorems 3.3 and 7.3 of [12] (and attentively reading the proof of the second theorem). □

Proposition 4.1 *P is hypercontractive and verifies a* $LS(\alpha_0^{-1}, 0)$ *inequality with*

$$\frac{\lg(\nu^{-1})}{5\sqrt{5}\mu^2(1-\nu)^{3/2}} \le \alpha_0^{-1} \le \frac{255}{4}\frac{(1+\lg 2)(1-\nu)+\lg(\nu^{-1})}{\mu^2(1-\nu)^3}.$$

Proof The proof consists in a direct application of Lemma 4.1, where we only have to estimate B_ν. So consider

$$M\left(\frac{\nu^n}{1-\nu}\right) = \frac{\nu^n}{1-\nu}\left[2\left(1+\sqrt{1+\frac{4\nu^n}{1-\nu}}\right)^{-1} + \lg\left(1+\frac{1-\nu}{2\nu^n}\left(1+\sqrt{1+\frac{4\nu^n}{1-\nu}}\right)\right)\right]$$

and notice that

$$\frac{\nu^n}{1-\nu}\lg\left(1+(1-\nu)\nu^{-n}\right) \le M\left(\frac{\nu^n}{1-\nu}\right) \le \frac{\nu^n}{1-\nu}(1+\lg(2\nu^{-n})).$$

We use these inequalities for M in order to obtain some bounds for B_ν.

Lower estimate for B_ν. For any ν, we denote by n_ν the smallest integer such that $(1-\nu)\nu^{-n} \ge \nu^{-n/2}$ for all $n \ge n_\nu$. Then

$$\begin{aligned} B_\nu &\ge \sup_{n\ge 1}\sum_{j=1}^{n}\frac{1}{\mu^2 j\nu^j}\frac{\nu^n}{1-\nu}\lg\left(1+(1-\nu)\nu^{-n}\right) \\ &\ge \frac{1}{\mu^2(1-\nu)}\sup_{n\ge n_\nu}\sum_{j=1}^{n}\frac{\nu^{n-j}}{j}\lg(\nu^{-n/2}) \\ &= \frac{1}{2\mu^2(1-\nu)}\sup_{n\ge n_\nu}\sum_{j=1}^{n}\nu^{n-j}\frac{n}{j}\lg(\nu^{-1}) \\ &\ge \frac{\lg(\nu^{-1})}{2\mu^2(1-\nu)}\sup_{n\ge n_\nu}\sum_{j=0}^{n-1}\nu^j = \frac{\lg(\nu^{-1})}{2\mu^2(1-\nu)^2}. \end{aligned}$$

Upper estimate for B_ν.

$$\begin{aligned}
B_\nu &\le \sup_{n\ge 1}\sum_{j=1}^{n}\frac{1}{\mu^2 j\nu^j}\frac{\nu^n}{1-\nu}(1+\lg(2\nu^{-n}))\\
&\le \frac{1}{\mu^2(1-\nu)}\sup_{n\ge 1}\sum_{j=1}^{n}\frac{\nu^{n-j}}{j}(1+\lg(2\nu^{-n}))\\
&\le \frac{1}{\mu^2(1-\nu)}\sup_{n\ge 1}\sum_{j=0}^{n-1}\nu^j\,\frac{1+\lg 2+n\lg(\nu^{-1})}{n-j}\\
&\le \frac{1}{\mu^2(1-\nu)}\sup_{n\ge 1}\sum_{j=0}^{n-1}\nu^j(1+\lg 2+\lg(\nu^{-1})(j+1))\\
&\le \frac{(1+\lg 2)(1-\nu)+\lg(\nu^{-1})}{\mu^2(1-\nu)^3}.
\end{aligned}$$

Using these bounds for B_ν in Lemma 4.1, we have

$$\frac{2}{5M((1-\nu)^{-1})}\frac{\lg(\nu^{-1})}{2\mu^2(1-\nu)^2}\le \alpha_0^{-1}\le \frac{255}{4}\frac{(1+\lg 2)(1-\nu)+\lg(\nu^{-1})}{\mu^2(1-\nu)^3}$$

and, since $M(x)\le\sqrt{4x+1}$ for any x, we finally get

$$\frac{\lg(\nu^{-1})}{5\sqrt{5}\mu^2(1-\nu)^{3/2}}\le \alpha_0^{-1}\le \frac{255}{4}\frac{(1+\lg 2)(1-\nu)+\lg(\nu^{-1})}{\mu^2(1-\nu)^3}.$$

Since a log-Sobolev inequality is verified, by Theorem 4.2, P is hypercontractive. □

Now we can go back to the study of hypercontractivity for the qOU semigroup and try to connect the formulation of the problem given in the previous section with the results we have just obtained.

It is known that the "diagonal" infinitesimal generator $\mathcal{G}$ has eigenvalues $(\lambda^2-\mu^2)n$, $n\ge 0$, each with eigenspace generated by a particular q_n, polynomial function of degree n (see Proposition 7.4 in [10]). We remember that $\{q_n, n\ge 0\}$ is a family of orthogonal polynomials in $l^2(\mathbb{N},\rho)$. This description of the spectrum of $\mathcal{G}$ allows us to complete the proof of Theorem 1.1 by the following.

Corollary 4.1 *The qOU semigroup is hypercontractive since condition* (8) *holds, for instance with* $C_1=\sqrt{2}\,3^{(\mu^2-\lambda^2)/(2\alpha_0)}$.

Proof Inequality $\mathrm{LS}(\alpha_0^{-1},0)$ is true for P and P is hypercontractive, so, by Theorem 4.1(a), we have

$$\begin{aligned}
&\|P_t\|_{q(t),2}\le 1, &&\text{for } q(t)=1+\exp(2t\alpha_0);\\
\text{in particular}\quad &\|P_t\|_{4,2}\le 1, &&\text{for } t\ge\bar t:=(2\alpha_0)^{-1}\lg 3.
\end{aligned}$$

Repeating for P the same considerations made for $\mathcal{T}$ in Sect. 2 (just after Eq. (2)), we have the following inequalities for the eigenpolynomials' norms

$$\|q_n\|_{l^4(\mathbb{N},\rho)} \leq e^{\bar{t}n(\mu^2-\lambda^2)}\|q_n\|_{l^2(\mathbb{N},\rho)}, \qquad \text{for all } n.$$

So, if p_n is any polynomial of degree n, we can write $p_n = \sum_{k=0}^n c_k q_k$ for appropriate $(c_k)_{k=0\ldots n}$ and, similarly as in (6), we obtain

$$\begin{aligned}\|p_n\|_{l^4(\mathbb{N},\rho)} &\leq (n+1)^{1/2} e^{\bar{t}n(\mu^2-\lambda^2)}\|p_n\|_{l^2(\mathbb{N},\rho)} \\ &= (n+1)^{1/2} e^{n(\mu^2-\lambda^2)\lg 3/(2\alpha_0)}\|p_n\|_{l^2(\mathbb{N},\rho)},\end{aligned}$$

which is equivalent to condition (8), where a possible choice of the constant C_1 can be obtained asking

$$C_1 \geq (n+1)^{1/(2n)} 3^{(\mu^2-\lambda^2)/(2\alpha_0)} \quad \text{for all } n;$$

so we take $C_1 = \sqrt{2}\, 3^{(\mu^2-\lambda^2)/(2\alpha_0)}$. □

4.2 Log-Sobolev inequality for qOU semigroup

In order to get the aims of this section, we still have to study the log-Sobolev inequality for the qOU semigroup $\mathcal{T}$.

Proposition 4.2 *$\mathcal{T}$ is hypercontractive and verifies a LS(α_1^{-1}, 0) inequality with*

$$\alpha_0^{-1} \leq \alpha_1^{-1} \leq \frac{4\,(5-\lg(1-\nu))}{\mu^2(1-\nu)} + (3\lg 3)\alpha_0^{-1}.$$

Proof We have proved that $\mathcal{T}$ is hypercontractive, so the idea of this proof is that we can estimate the constants involved in Eq. (9), written for the qOU semigroup, and then obtain a $LS(c, 0)$ inequality. By this way, we get an estimate from above for α_1^{-1}.

In order to choose C_0 in (7), reading the proof of Proposition 3.2, it is clearly sufficient to ask

$$C_0^n \nu^{-k/8} \geq \begin{cases} (2C_1^2)^{n/4} & \text{for } k \text{ even}, \\ C_\nu^{1/4}(2C_1^3)^{n/4} & \text{for } k \text{ odd}. \end{cases}$$

Some elementary computations show that we can choose

$$C_0 = (2C_1^3(C_\nu \vee 1))^{1/4} = 2^{5/8}(C_\nu^{1/4} \vee 1)3^{3(\mu^2-\lambda^2)/(8\alpha_0)}.$$

Then inequality (6) is verified ($\|x\|_4 \leq C_0^n\sqrt{2n+1}\|x\|_2 \leq C^n\|x\|_2$ for all n and all x in U_n) if we take $C = C_0\sqrt{3} = \sqrt{3}\,2^{5/8}(C_\nu^{1/4} \vee 1)3^{3(\mu^2-\lambda^2)/(8\alpha_0)}$.

So, by Theorem 2.1 in [5] (or by direct computation, using the spectral decomposition of $\mathcal{L}$), we have

$$\|\mathcal{T}_t(x)\|_4 \leq (1 - Ce^{-\eta t})^{-1}\|x\|_2 \qquad \text{for any } x \in \mathcal{B}(h);$$

then, by Proposition 3.4 in [26] (alternatively see [15]), for all x,

$$\|\mathcal{T}_t(x)\|_{q(t)} \leq \exp(d(1-(q(t))^{-1}))\|x\|_2, \quad q(t) = 1 + e^{t/(2\tau)}, \quad d = -\lg(1 - Ce^{-\eta\tau})$$

for all τ such that $1 - Ce^{-\eta\tau} > 0$, or equivalently for all $\tau > \frac{2\lg C}{\mu^2-\lambda^2}$.

Therefore the inequalities $\mathrm{LS}(4\tau, -\lg(1 - Ce^{-\eta\tau}))$ are verified for the semigroup $\mathcal{T}$ for all $\tau > \frac{2\lg C}{\mu^2-\lambda^2}$. Moreover, since $\mathcal{T}$ has spectral gap $(\mu^2 - \lambda^2)/2$, by Theorem 4.1(b), also inequality $\mathrm{LS}(c', 0)$ is verified, where

$$c' = c'(\tau) = 4\tau + \frac{2}{\mu^2 - \lambda^2}\left[1 - \lg\left(1 - Ce^{-\tau(\mu^2-\lambda^2)/2}\right)\right].$$

We minimize c' as a function of $\tau > \frac{2\lg C}{\mu^2-\lambda^2}$ and obtain that the optimal τ is $\tau^* = \frac{2\lg(5C/4)}{\mu^2-\lambda^2}$ with corresponding

$$c'^* = c'(\tau^*) = \frac{2}{\mu^2 - \lambda^2}\lg(5^5 2^{-11/2} 3^2 e(C_\nu \vee 1)) + (3\lg 3)\alpha_0^{-1}.$$

Finally we obtain the estimate

$$\begin{aligned}\alpha_0^{-1} \leq \alpha_1^{-1} \leq c'^* &\leq \frac{2}{\mu^2(1-\nu)}\left(\lg(5^5 2^{-11/2} 3^2 e) + \frac{1}{2}\lg\frac{11}{(1-\nu)^4}\right) + (3\lg 3)\alpha_0^{-1}\\ &\leq \frac{2}{\mu^2(1-\nu)}(9.022 - 2\lg(1-\nu)) + (3\lg 3)\alpha_0^{-1}\end{aligned}$$

and the conclusion follows. □

Remark 4.2 Now by Proposition 4.1, we can control α_1^{-1}. In particular, there exist two positive constants A and B, such that

$$A(1-\nu)^{-1/2} \leq \alpha_1^{-1} \leq B(1-\nu)^{-2},$$

as $\nu \to 1$. Obviously these estimates are not accurate and could be easily improved, also with the same method, but here we are not concentrated on this aspect.

Remark 4.3 By proving the equivalence of different inequalities, we finally seem to deduce hypercontractivity for the quantum semigroup from the hypercontractivity of the associated birth and death process. Yet, we notice that this fact alone is not sufficient and, further, the involved constants get worse when considering the quantum case. This is quite natural and also happens for uniform exponential convergence and spectral gap (see once again [9]).

Acknowledgments We would like to thank Franco Fagnola for interesting discussion and an anonymous referee for useful comments.

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