From Gumbel to Tracy-Widom

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Abstract   The Tracy-Widom distribution that has been much studied in recent years can be thought of as an extreme value distribution. We discuss interpolation between the classical extreme value distribution $\exp(-\exp(-x))$, the Gumbel distribution, and the Tracy-Widom distribution. There is a family of determinantal processes whose edge behaviour interpolates between a Poisson process with density $\exp(-x)$ and the Airy kernel point process. This process can be obtained as a scaling limit of a grand canonical version of a random matrix model introduced by Moshe, Neuberger and Shapiro. We also consider the deformed GUE ensemble, $M = M_0 + \sqrt{2S}V$, with M_0 diagonal with independent elements and V from GUE. Here we do not see a transition from Tracy-Widom to Gumbel, but rather a transition from Tracy-Widom to Gaussian.

1 Introduction and results

1.1 Introduction

In the random matrix litterature there has been alot of discussion about the transition from Poissonian to random matrix eigenvalue statistics, see for example [5,6,10,22,24]. One motivation comes from disordered systems, and another from quantum chaos where Poissonian statistics is expected to describe the eigenvalue statistics of classically integrable systems (Berry–Tabor conjecture), and random matrix statistics should describe eigenvalue statistics of systems whose classical dynamics is fully chaotic (Bohigas–Gianonni–Schmidt

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conjecture). Hence it has been natural to look at transitions between Poissonian and random matrix statistics. In general there could be many ways to go between different ensembles, but it is nevertheless interesting to find natural interpolating ensembles and investigate their properties. In particular interpolating ensembles which are themselves scaling limits. Mathematically it is easiest to consider Hermitian (unitary) ensembles. Previous papers on the problem have been mainly concerned with the transition statistics for eigenvalues in the bulk of the spectrum. In the bulk we should see a transition from a Poisson process to a sine kernel determinantal point process, and for the nearest neigbour spacing statistics we should see a transition from the exponential distribution to the Gaudin distribution.

In this paper we will discuss the edge behaviour of the eigenvalues. In a finite random matrix ensemble we look at a scaling limit around the largest eigenvalue instead of in the bulk of the spectrum. If we take a diagonal matrix with independent Gaussian entries, the largest eigenvalue will, as the size of the matrix grows, fluctuate according to the Gumbel extreme value distribution. If we take a full Gaussian matrix from GUE, then the largest eigenvalue will fluctuate according to the Tracy-Widom distribution. Can we find interesting distributions that interpolate between Gumbel and Tracy-Widom? Is there a family of determinantal processes that interpolates? Should we typically expect to see a transition from Tracy-Widom to Gumbel? To shed some light on these questions we will discuss two interpolating random matrix ensembles on Hermitian matrices.

1.2 The ensembles

(1) *Deformed GUE*. Let $\text{diag}(y_1, \ldots, y_N)$ denote the diagonal matrix with elements $y_1, \ldots, y_N$. We take $y_1, \ldots, y_N$ to be independent Gaussian, $N(0, 1/2)$ say, random variables. Let V be an independent $N \times N$ GUE matrix with density $Z_N^{-1} \exp[-\text{Tr } V^2] dV$. Consider the random matrix

$$M = \text{diag}(y_1, \ldots, y_N) + \sqrt{2S} V, \tag{1}$$

where $S \geq 0$ is a parameter. When $S = 0$ we have a diagonal matrix with independent entries and when $S \to \infty$, the matrix $M/\sqrt{S}$ approaches a GUE matrix.

(2) *MNS-model*. This model was introduced by Moshe et al. [21], and we will call it the *MNS-model*. Let H be a Hermitian matrix and U a fixed unitary matrix. A probability measure on the space of Hermitian matrices is defined by

$$P_{N,U}(H) dH = \frac{1}{Z_N} e^{-\text{Tr } H^2} e^{-b\text{Tr}([U,H][U,H]^*)} dH,$$

where $b > 0$, dH is Lebesgue measure on the Euclidean space of Hermitian matrices, $[U, H] = UH - HU$ and the star denotes Hermitian conjugate.

The weight is maximal when $[U, H] = 0$ and then U and H can be simultaneously diagonalized. The unitary matrix selects a preferred basis. We can get a unitarily invariant measure by averaging over the unitary group with respect to the Haar measure, so that we consider a random preferred basis. We obtain the probability measure

$$P_N(H)dH = \frac{1}{Z'_N} e^{-\text{Tr} H^2} \left(\int_{U(N)} e^{-b\text{Tr}\,([U,H][U,H]^*)} dU \right) dH. \qquad (2)$$

The integral over the unitary group can be evaluated using the Harish–Chandra or Itzykson/Zuber integral and this makes it possible to compute the eigenvalue measure induced by (2), [21]. This gives the probability measure

$$p_N(x)d^N x = \frac{1}{Z''_N} \det \left(e^{-(b+1/2)(x_i^2+x_j^2)+2bx_ix_j} \right)_{1 \leq i,j \leq N} d^N x \qquad (3)$$

on $\mathbb{R}^N$, where $x_1, \ldots, x_N$ are the eigenvalues of H. Actually we will consider a grand canonical version of the model, see below, as was also done in [21]. There is a generalization of the MNS-model to Laguerre/Chiral type ensembles that we will not discuss here, see [7].

Both of the models above have interpretations in terms of non-interesecting paths.

(1) *Deformed GUE*. Consider N standard Brownian motions on the real line, we think of them as particles, started at $y_1, \ldots, y_N$ at time 0 and conditioned never to intersect. Let $x_1, \ldots, x_N$ be the positions of the particles at time S. The probability distribution of $x_1, \ldots, x_N$ is the same as the eigenvalue distribution of M in (1), see for example [12].

(2) *MNS-model*. Consider N standard Brownian motions on the real line started at $x_1, \ldots, x_N$ at time 0, conditioned to come back to $x_1, \ldots, x_N$ at time t and without having had any collisions during this time. Put an initial density $\prod_{i=1}^{N} e^{-x_i^2}$ on the points $x_1, \ldots, x_N$. By a theorem of Karlin and McGregor [17], we get a probability density

$$\frac{1}{\mathcal{Z}_N} \det \left(e^{-\frac{1}{2}(x_i^2+x_j^2)-\frac{1}{2t}(x_i-x_j)^2} \right)_{1 \leq i,j \leq N} d^N x, \qquad (4)$$

on the x_js. This is the same as (3) if we take $b = 1/2t$. We can think of this as a model of non-intersecting paths on a cylinder. As stated above we will be interested in the transition at the edge of the spectrum. The transition in the bulk of the spectrum in the the MNS-model occurs when $b/N^2 \sim c$ or $t \sim 1/2cN^2, c > 0$ a constant, as $N \to \infty$. (The bulk transition in deformed GUE occurs when $S \sim C/N^2$.) It is remarked, but not discussed further, in [21] that when the bulk transition occurs, the behaviour at the edge is

still like that of independent eigenvalues. Below, we will see that there is a transition at the edge when $b/N^{2/3} \sim c$ as $N \to \infty$.

1.3 The Gumbel and Tracy-Widom distributions

Consider N independent random variables $X_1, \ldots, X_N$ with distribution $N(0, 1/2)$. Then it is well known that, [20],

$$\mathbb{P}\left[\frac{\max(X_1, \ldots, X_N) - a_N}{b_N} \leq x\right] \to F_G(x) \doteq \mathrm{e}^{-\mathrm{e}^{-x}} \tag{5}$$

as $N \to \infty$, where

$$a_N = \sqrt{\log N} - \frac{\log(4\pi \log N)}{4\sqrt{\log N}}, \tag{6}$$

$$b_N = \frac{1}{2\sqrt{\log N}}. \tag{7}$$

The distribution function F_G is often called the *Gumbel distribution*. If we think of $X_1, \ldots, X_N$ as a point process on the real line with N points and we take the appropriate scaling limit around the rightmost point we get a Poisson process on $\mathbb{R}$ with density e^{-x}. Its correlation functions are

$$\rho_k(x_1, \ldots, x_k) = \prod_{j=1}^{k} \mathrm{e}^{-x_j}, \quad k \geq 1. \tag{8}$$

The *Tracy-Widom distribution*, F_{TW}, is defined by the Fredholm determinant

$$F_{\mathrm{TW}}(x) = \det(I - K_{\mathrm{Airy}})_{L^2(x, \infty)}, \tag{9}$$

where

$$K_{\mathrm{Airy}}(x, y) = \int_0^\infty \mathrm{Ai}\,(x + \lambda)\,\mathrm{Ai}\,(y + \lambda)\mathrm{d}\lambda, \tag{10}$$

is the *Airy kernel*, [30]. This distribution occurs in several different places and has been much studied in recent years, see [14,31] for reviews.

If we have a point process on $\mathbb{R}$ then its correlation functions, $\rho_k(x_1, \ldots, x_k)$, are characterized by

$$\mathbb{E}\left[\prod_j (1 + \phi(x_j))\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^k} \prod_{j=1}^{k} \phi(x_j)\rho_k(x_1, \ldots, x_k)\mathrm{d}^k x, \tag{11}$$

for any measurable bounded function on $\mathbb{R}$ with compact support. Here the product in the left hand side is over all particles in the process.

A point process on $\mathbb{R}$ is called determinantal if all its correlation functions, $\rho_k(x_1, \ldots, x_k), k \geq 1$, exist and are given by

$$\rho_k(x_1, \ldots, x_k) = \det(K(x_i, x_j))_{1 \leq i,j \leq k} \tag{12}$$

for some function $K : \mathbb{R}^2 \to \mathbb{R}$, the *correlation kernel*. A Poisson process on $\mathbb{R}$ with density e^{-x} can be viewed as a, somewhat degenerate, determinantal process with correlation kernel,

$$K_{\mathrm{ext}}(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ e^{-x} & \text{if } x = y \end{cases}. \tag{13}$$

Here we could replace e^{-x} with an arbitrary density $\rho(x)$.

1.4 The interpolating process

Before we discuss the asymptotics of the MNS-model and the deformed GUE model we will consider a determinantal process which interpolates between the Poisson process with density e^{-x} and the Airy kernel point process, i.e. the determinantal process that has kernel (10). We will see later that this process can be obtained as a scaling limit of the (grand canonical) MNS-model, and we will call it the *interpolating process*. Also, we will see that we do not have a transition between the Tracy-Widom and the Gumbel distribtions in the deformed GUE ensemble. Rather we will see a transition from Tracy-Widom to Gaussian. This will be discussed further below.

Define

$$M_\alpha(x, y) = \int\limits_{-\infty}^{\infty} \frac{e^{\alpha\lambda}}{e^{\alpha\lambda} + 1} \mathrm{Ai}\,(x + \lambda)\mathrm{Ai}\,(y + \lambda)\mathrm{d}\lambda. \tag{14}$$

That the integral is convergent follows for example from (44) and the Cauchy–Schwarz inequality.

Proposition 1.1 *The kernel M_α defines a trace class operator in $L^2(a, \infty)$ for any real a, and there is a determinantal process, the interpolating process with correlation kernel M_α.*

That the kernel M_α interpolates between the correlation kernels for the Poisson process with density e^{-x} and the Airy kernel point process is seen in the next theorem. The theorem will be proved in Sect. 2.

Theorem 1.2 *We have the following scaling limits*

$$\lim_{\alpha \to 0+} \frac{1}{\alpha} M_\alpha \left(\frac{u}{\alpha} - \frac{1}{2\alpha} \log(4\pi\alpha^3), \frac{v}{\alpha} - \frac{1}{2\alpha} \log(4\pi\alpha^3) \right) = K_{\text{ext}}(u, v) \quad (15)$$

and

$$\lim_{\alpha \to \infty} M_\alpha(u, v) = K_{\text{Airy}}(u, v). \quad (16)$$

It is not hard to see that $\int_t^\infty M_\alpha(x, x) dx < \infty$ for any t, see (45) below, and hence the interpolating process has a last particle almost surely. The distribution function F_α for this last particle will interpolate between the Gumbel and the Tracy-Widom distributions.

Theorem 1.3 *The distribution function for the last particle in the interpolating process is*

$$F_\alpha(t) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_{(t,\infty)^n} \det(M_\alpha(x_i, x_j))_{1 \le i,j \le n} d^n x = \det(I - M_\alpha)_{L^2(t,\infty)}. \quad (17)$$

Furthermore

$$\lim_{\alpha \to 0+} F_\alpha \left(\frac{\xi}{\alpha} - \frac{1}{2\alpha} \log(4\pi\alpha^3) \right) = F_G(\xi) \quad (18)$$

and

$$\lim_{\alpha \to \infty} F_\alpha(\xi) = F_{TW}(\xi). \quad (19)$$

We postpone the proof to Sect. 2.

There is a different way of obtaining the distribution F_α that is given in the next proposition, which will be proved in Sect. 2. The construction in the theorem will not give us the whole interpolating process though.

Proposition 1.4 *Let $x_1 > x_2 > \cdots$ be a realization of the Airy kernel point process. Let $y_1, y_2, \ldots$ be independent random variables with common distribution function*

$$G_\alpha(x) = \frac{e^{\alpha x}}{1 + e^{\alpha x}}, \quad (20)$$

which are also independent of $\{x_i\}$. Define a new point process by $z_j = x_j + y_j$, $j \ge 1$. Then,

$$\mathbb{P}\left[\max_{j \ge 1} z_j \le \xi \right] = F_\alpha(\xi). \quad (21)$$

Note that the point process $\{z_j\}$ in the theorem is not the interpolating process, it is only the last particle distribution that is the same. Processes with Poissonian edge behaviour constructed in a similar way have recently been studied in [25].

1.5 Grand canonical determinantal processes

The probability measure (4) on $\mathbb{R}^N$ does not define a finite determinantal point process on $\mathbb{R}$. To get a determinantal point process we have to consider a grand canonical ensemble with varying N, see for example [4,13] for related constructions. Let us first consider a general model with the same structure.

Let X be a complete separable metric space with a reference measure μ. Assume that $\psi_j, j \geq 0$, is an orthonormal family of complex-valued functions in $L^2(X, \mu)$. Also, let $a_n \geq 0$ be a sequence such that $\sum_{n=0}^{\infty} a_n < \infty$. Set

$$\phi(x, y) = \sum_{n=0}^{\infty} a_n \psi_n(x) \overline{\psi_n(y)}. \tag{22}$$

The function $\phi(x, y)$ is well-defined in $L^2(\mu \times \mu)$ and $\phi(x, x)$ is well-defined in $L^1(\mu)$. We can define a probability measure on X^n by

$$p_N(x) \mathrm{d}^N \mu(x) = \frac{1}{Z_N} \det(\phi(x_i, x_j))_{1 \leq i,j \leq N} \mathrm{d}^N \mu(x), \tag{23}$$

where

$$Z_N = \int_{X^N} \det(\phi(x_i, x_j))_{1 \leq i,j \leq N} \mathrm{d}^N \mu(x). \tag{24}$$

Here we assume that $p_N(x) \geq 0$ and $Z_N > 0$. We construct a grand canonical point process, compare [2], p. 123, by letting

$$q_N = \frac{\lambda^N}{N!} \frac{Z_N}{Z(\lambda)} \tag{25}$$

be the probability of seeing exactly N particles, and $p_N(x) \mathrm{d}^N \mu(x)$ be the probability measure for finding particles at $x_1, \ldots, x_N$ given that there are exactly N particles. Here $Z(\lambda)$ is a normalization constant (grand canonical partition function),

$$Z(\lambda) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} Z_N, \tag{26}$$

where $Z_0 = 1$. If g is a function in L^∞ with bounded support, then

$$
\mathbb{E}\left[\prod_j (1 + g(x_j))\right] = \sum_{N=0}^{\infty} \frac{q_N}{Z_N} \int_{X^N} \prod_{j=1}^{N} (1 + g(x_j)) \det(\phi(x_i, x_j))_{1 \le i,j \le N} \mathrm{d}^N \mu(x),
$$

(27)

where the product in the left hand side is over all particles in the process. The next theorem, that will be proved in Sect. 3, says that this construction leads to a determinantal process.

Theorem 1.5 *The grand canonical point process defined above is a determinantal process on X with correlation kernel*

$$
K_\lambda(x, y) = \sum_{n=0}^{\infty} \frac{\lambda a_n}{1 + \lambda a_n} \psi_n(x) \overline{\psi_n(y)}.
$$

(28)

We call this type of process a grand canonical determinantal process.

1.6 The MNS-model

In the MNS-model we have $X = \mathbb{R}$ and μ is the Lebesgue measure, in the above construction. We take

$$
\phi(x, y) = \phi_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x^2 + y^2)/2 - (x-y)^2/2t}
$$

(29)

Then the probability measure (4) is exactly the measure (23). That we have an expansion of the form (22) follows from the next lemma, which is just a way of writing Mehler's formula. We will give the details in the beginning of Sect. 4.

Lemma 1.6 *Set $\beta_q = \sqrt{\frac{1+q}{1-q}}$. Then*

$$
\frac{\sqrt{q}}{(1-q)\sqrt{\pi}} \exp\left(-\frac{1}{2}(x^2 + y^2) - \frac{q}{(1-q)^2}(x-y)^2\right)
$$
$$
= \sum_{n=0}^{\infty} \beta_q q^{n+1/2} h_n(\beta_q x) h_n(\beta_q y) \exp\left(-\frac{\beta_q^2}{2}(x^2 + y^2)\right),
$$

(30)

where $h_n(x), n \ge 0$, are the normalized Hermite polynomials.

If we make the identification $1/2t = q/(1-q)^2$ and define

$$
\psi_n(x) = \sqrt{\beta_q} h_n(\beta_q x) e^{-\beta_q^2 x^2/2},
$$

(31)

then the $\psi_n, n \geq 1$ are orthonormal and $\phi = \phi_t$ can be expanded as in (22) with $a_n = q^{n+1/2}$. Theorem 1.5 then gives the next theorem.

Theorem 1.7 *The grand canonical MNS-model coming from* (3) *or* (4) *is a determinantal point process on* $\mathbb{R}$ *with correlation kernel*

$$K_\lambda(x, y) = \sum_{n=0}^{\infty} \frac{\lambda q^{n+1/2}}{1 + \lambda q^{n+1/2}} \psi_n(x) \psi_n(y), \tag{32}$$

where the three parameters are related by

$$b = \frac{1}{2t} = \frac{q}{(1 - q)^2}. \tag{33}$$

Write

$$q = e^{-\mu} \tag{34}$$

and fix a number $N \geq 0$. Note that $\mu \to \infty$ corresponds to $t \to \infty$ and $\mu \to 0+$ to $t \to 0+$. If we choose

$$\lambda = e^{\mu N} - 1, \tag{35}$$

then $\int_{\mathbb{R}} K_\lambda(x, x) dx \approx N$, so the expected number of particles in the process is approximately N.

The next proposition shows that the kernel K_λ interpolates between a point process defined by N independent Gaussian random variables and GUE as we should expect. We postpone the proof to Sect. 4.

Proposition 1.8 *If we choose q as in* (34) *and* λ *as in* (35), *then*

$$\lim_{\mu \to \infty} K_\lambda(x, y) = \sum_{n=0}^{N-1} h_n(x) h_n(y) e^{-(x^2+y^2)/2} \doteq K_{\mathrm{GUE}(N)} \tag{36}$$

uniformly for x, y *in a compact set, and*

$$\lim_{\mu \to 0+} K_\lambda(x, y) = \begin{cases} \frac{N}{\sqrt{\pi}} e^{-x^2} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{37}$$

pointwise.

As mentioned above the bulk transition occurs when $\mu \sim 1/cN$. This is the limit that was studied and discussed in [21].

Theorem 1.9 *Let $\mu = 1/cN$, with $c > 0$ fixed, and let λ be given by (35). In this case λ is a constant $\lambda = e^{1/c} - 1$. The following limit holds,*

$$\lim_{N\to\infty} \frac{\pi}{2N\sqrt{c}} K_\lambda \left(\frac{\pi x}{2N\sqrt{c}}, \frac{\pi y}{2N\sqrt{c}} \right) = L_c(x, y) \doteq \int_0^\infty \frac{\cos \pi (x - y) u}{\lambda^{-1} e^{u^2/c} + 1} du \quad (38)$$

uniformly for x, y in a compact set.

The theorem will be proved in Sect. 4.

Thus in this transition region in the bulk of the point pocess we will have a determinantal process with correlation kernel L_c. Suitable scaling limits will give the sine kernel as $c \to 0+$ and a uniform Poisson process as $c \to \infty$

In [21] only the following approximate expression

$$L_c(x, y) \approx \frac{\pi c}{2} \frac{\sin \pi (x - y)}{\sinh \pi^2 c(x - y)/2} \quad (39)$$

is given, valid when c is small. At the end of Sect. 4 we will sketch an argument leading to this approximate formula without discussing the error.

As briefly mentioned in [21], but not really discussed, when $\mu = 1/cN$, the behaviour at the edge is still like that of independent particles, i.e. we get a Poisson process with density e^{-x}. More precisely we have the following theorem, which will be proved in Sect. 4.

Theorem 1.10 *Let $\mu = 1/cN, c > 0$ fixed and $\lambda = e^{1/c} - 1$ as in the previous theorem. Set*

$$a_N(c) = \sqrt{\log N} - \frac{\log(4\pi \log N/\lambda^2 c^2)}{4\sqrt{\log N}}$$

and $b_N = (2\sqrt{\log N})^{-1}$ as in (7). Then,

$$\lim_{N\to\infty} b_N K_\lambda(a_N(c) + b_N \xi, a_N(c) + b_N \eta) = K_{\text{ext}}(\xi, \eta), \quad (40)$$

pointwise.

To get an intermediate process at the edge we have to pick a larger μ. In fact the intermediate process will be exactly the interpolating process with kernel M_α discussed above. The next theorem will be proved in Sect. 4.

Theorem 1.11 *Choose $\mu = \alpha/N^{1/3}, \lambda = e^{\alpha N^{2/3}} - 1$, where $\alpha > 0$ is fixed. Then,*

$$\lim_{N\to\infty} \frac{\sqrt{\alpha}}{2N^{1/3}} K_\lambda \left(N^{1/3}\sqrt{\alpha} + \frac{\sqrt{\alpha}}{2N^{1/3}} \xi, N^{1/3}\sqrt{\alpha} + \frac{\sqrt{\alpha}}{2N^{1/3}} \eta \right) = M_\alpha(\xi, \eta) \quad (41)$$

uniformly for ξ, η in a compact set.

Hence, in the grand canonical version of the MNS-model, we can see a transition between Gumbel statistics and Tracy-Widom statistics for the largest eigenvalue.

1.7 The deformed GUE model

We turn now to the deformed GUE model (1). The bulk transition in this and related models has been discussed for example in [5,10,24] and we will not discuss it here. It occurs for $S \sim c/N^2$, which is the same as for the MNS-model. When we look at the edge, the behaviour of the deformed GUE will be different from that of the MNS-model. We will not see a transition between Tracy-Widom and Gumbel. If we choose $S = \alpha^2/N^{2/3}$ we will see a change of the edge behaviour as we vary α, but the transition will be from Tracy-Widom as $\alpha \to \infty$ to Gaussian as $\alpha \to 0+$. Informally we can interpret this as follows. The eigenvalue distribution is approximately a semicircle and with $y_1, \ldots, y_N$ fixed we would see Tracy-Widom fluctuations. However, the fluctuations of $y_1, \ldots, y_N$ causes the semicircle to fluctuate, that is the position of the edge fluctuates like a Gaussian. We can think of the semicircle as fluctuating basically like $\frac{1}{N} \sum_{i=1}^{N} y_i$, i.e. like a Gaussian. The effect is that the largest eigenvalue will fluctuate like a Tracy-Widom random variable plus an independent Gaussian. There is some similarity between this problem and the random growth model with random parameters studied in [9].

Theorem 1.12 *Let* $d\mu(t)$ *be a probability measure on* $\mathbb{R}$ *satisfying* $\int t d\mu(t) = 0$, $\int t^2 d\mu(t) = \sigma^2$ *and* $\int |t|^7 d\mu(t) < \infty$. *Let* $y_1, \ldots, y_N$ *be independent random variables with distribution* $d\mu(t)$ *and consider the random* $N \times N$ *matrix*

$$M = \text{diag}(y_1, \ldots, y_N) + \sqrt{2S}V, \tag{42}$$

where $S = \alpha^2/N^{2/3}$ *and* V *is an independent GUE matrix with density* Z_N^{-1} $\exp(-\text{Tr} \, V^2)dV$. *Let* $\lambda_{\max}^{(N)}$ *be the largest eigenvalue of* M. *There is a number* $R(N) \sim 2\alpha N^{1/6}$, *given by (77) below, which depends on* $d\mu, \alpha$ *and* N, *so that*

$$\lim_{N \to \infty} \mathbb{P}\left[\frac{\lambda_{\max}^{(N)} - R(N)}{\alpha/\sqrt{N}} \le t \right] = \mathbb{P}[X + Y \le t], \tag{43}$$

where X *and* Y *are independent,* X *has the Tracy-Widom distribution and* Y *has distribution* $N(0, \sigma^2/\alpha^2)$.

If we want to compare with proposition 1.4 we can let $x_1 > x_2 > \cdots$ be a realization of the Airy kernel point process and y be an independent random variable with distribution $N(0, \sigma^2/\alpha^2)$. Set $z_j = x_j + y$. Then $\max_{j \ge 1} x_j = x_1 + y$, will be distributed according to the right hand side of (43).

Remark 1.13 Another model for the transition between independent eigenvalues and GUE random matrix eigenvalues is a band Hermitian matrix with Gaussian elements. Let m_{ii} be independent $N(0,1/2)$, $1 \leq i \leq N$ and Re m_{ij}, Im m_{ij}, $1 \leq i < j \leq N$, and $j - i < b$, be independent $N(0,1/4)$, for some given $b, 1 \leq b \leq N$. Set $m_{ij} = 0$ for $1 \leq i < j \leq N$, and $j - i \geq b$ and $m_{ij} = m_{ji}$. Then $M = (m_{ij})_{1 \leq i,j \leq N}$ is a diagonal matrix when $b = 1$ and a GUE matrix when $b = N$. It is conjectured, see for example [19] and references therein, that the local bulk statistics, in the limit $N \to \infty$, shows a transition from a Poissonian to a determinantal sine-kernel point process when $b \sim cN^{1/2}, 0 < c < \infty$. When is there a transition at the edge? Based on the results above one might guess that the edge transition takes place for a larger b. Do we see a transition from Gumbel to Tracy-Widom or is there something else happening in between? It is not easy to approach these problems. Since we are dealing with the edge and not the bulk it could be that the method of moments, used with great success in [27] for Wigner matrices, is useful here also.

Remark 1.14 The comparison of the Tracy-Widom distribution with the Gumbel distribution suggests that we are thinking of the Tracy-Widom distribution as a kind of extreme value distribution. One way to motivate this is as follows. Let $w(i,j), (i,j) \in \mathbb{Z}_+^2$, be i.i.d. geometric random variables, and let $\pi_k^{(N)}, k = 1, \ldots, \binom{2N}{N}$, be all up/right paths from $(1,1)$ to (N,N). Set

$$X_k^{(N)} = \sum_{(i,j) \in \pi_k^{(N)}} w(i,j).$$

For N large each $X_k^{(N)}$ is approximately normal. Clearly, the $X_k^{(N)}$ are not independent. The random variable $G(N,N) = \max_k X_k^{(N)}$ is thus a maximum over dependent random variables each of which is approximately normal. We know, [11], that $G(N,N)$, appropriately rescaled converges to the Tracy-Widom distribution, which thus arises as an extreme value distribution for certain dependent random variables. We are not aware of any last-passage percolation problems that would interpolate between Tracy-Widom and Gumbel.

In measures on partitions both the Gumbel and the Tracy-Widom distribution appear, [32]. Are there any natural measures on partitions that interpolate in the way that the MNS-model does?

2 The interpolating model

In this section we will give the proofs of the results for the interpolating determinantal process with correlation kernel M_α. A basic identity that is useful is

$$\int\limits_{-\infty}^{\infty} e^{\alpha t} \mathrm{Ai}\,(x+t)\mathrm{Ai}\,(y+t)\mathrm{d}t = \frac{1}{\sqrt{4\pi\alpha}} e^{-(x-y)^2/4\alpha - \alpha(x+y)/2 + \alpha^3/12} \qquad (44)$$

for $\alpha > 0$ and all x, y, see for example [23].

Proof (Proposition 1.1) We first prove that m_α defined by (14) is a trace class operator on $L^2(a, b)$ for $-\infty < a < b \leq \infty$. Note that M is symmetric and

$$\sum_{i,j=1}^{n} z_i \bar{z}_j M_\alpha(x_i, x_j) = \int\limits_{-\infty}^{\infty} \frac{e^{\alpha\lambda}}{e^{\alpha\lambda} + 1} \left| \sum_{i=1}^{n} z_i \mathrm{Ai}\,(x_i + \lambda) \right|^2 \mathrm{d}\lambda,$$

for any complex numbers $z_1, \ldots, z_N$ and all x_i, so $M(x, y)$ is a Hermitian positive definite function. Hence, see e.g. [26], it suffices to show that

$$\int\limits_{a}^{\infty} M_\alpha(x, x)\mathrm{d}x < \infty. \qquad (45)$$

It then follows that M_α defines a trace class operator on $L^2(a, b)$ with $\mathrm{Tr}\, M_\alpha = \int_a^b M_\alpha(x, x)\mathrm{d}x$. The inequality (45) follows from the estimate

$$\begin{aligned} M_\alpha(x, x) &= \int\limits_{-\infty}^{\infty} \frac{e^{\alpha\lambda}}{e^{\alpha\lambda} + 1} \mathrm{Ai}\,(x+\lambda)^2 \mathrm{d}\lambda \\ &\leq \int\limits_{-\infty}^{\infty} e^{\alpha\lambda} \mathrm{Ai}\,(x+\lambda)^2 \mathrm{d}\lambda = \frac{1}{\sqrt{4\pi\alpha}} e^{-\alpha x + \alpha^3/12}, \qquad (46) \end{aligned}$$

by (44).

If we can show that $0 \leq M_\alpha \leq I$, it follows that there is a determinantal process with correlation kernel M_α, see [28]. Let f be a continuous function on the real line with compact support. Then,

$$\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} M_\alpha(x, y)f(x)f(y)\mathrm{d}x\mathrm{d}y = \int\limits_{-\infty}^{\infty} \frac{e^{\alpha\lambda}}{e^{\alpha\lambda} + 1} \left(\int\limits_{-\infty}^{\infty} \mathrm{Ai}\,(x+\lambda)f(x)\mathrm{d}x \right)^2 \mathrm{d}\lambda$$

by Fubini's theorem and hence $M_\alpha \geq 0$. Fix $\epsilon > 0, 0 < \epsilon < \alpha$, and note that

$$\frac{e^{\alpha\lambda}}{e^{\alpha\lambda} + 1} \leq e^{\epsilon\lambda} \qquad (47)$$

for all real λ. Thus

$$\int_{-\infty}^{\infty} \frac{e^{\alpha\lambda}}{e^{\alpha\lambda}+1} \left(\int_{-\infty}^{\infty} \mathrm{Ai}\,(x+\lambda)f(x)dx \right)^2 d\lambda$$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{\epsilon\lambda} \mathrm{Ai}\,(x+\lambda)\mathrm{Ai}\,(y+\lambda) \right) f(x)f(y)dx\,dy$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\epsilon}} e^{-(x-y)^2/4\epsilon-\epsilon(x+y)+\epsilon^3/12} f(x)f(y)dx\,dy.$$

Since f is continuous and has compact support this last integral $\to \|f\|_2^2$ as $\epsilon \to 0+$. Since $\epsilon > 0$ can be taken arbitrarily small we obtain $M_\alpha \leq I$. □

Next we turn to the scaling limits of the kernel M_α.

Proof (Theorem 1.2) Set $f(\alpha) = \frac{1}{2\alpha} \log(4\pi\alpha^3)$. Then

$$M_\alpha^*(u,v) \doteq \frac{1}{\alpha} M_\alpha \left(\frac{u}{\alpha} - f(\alpha), \frac{v}{\alpha} - f(\alpha) \right)$$

$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} \frac{e^{\alpha(t+f(\alpha))}}{e^{\alpha(t+f(\alpha))}+1} \mathrm{Ai}\,(t+u/\alpha)\mathrm{Ai}\,(t+v/\alpha)dt$$

$$= \sqrt{4\pi\alpha} \int_{-\infty}^{\infty} \frac{e^{\alpha t}}{\sqrt{4\pi\alpha^3}e^{\alpha t}+1} \mathrm{Ai}\,(t+u/\alpha)\mathrm{Ai}\,(t+v/\alpha)dt. \qquad (48)$$

Using the identity (44) this can be written as $M_\alpha^*(u,v) = A - B$, where

$$A = e^{-(u-v)^2/4\alpha^3-(u+v)/2+\alpha^3/12}$$

$$B = 4\pi\alpha^2 \int_{-\infty}^{\infty} \frac{e^{2\alpha t}}{\sqrt{4\pi\alpha^3}e^{\alpha t}+1} \mathrm{Ai}\,(t+u/\alpha)\mathrm{Ai}\,(t+v/\alpha)dt.$$

It is clear that $A \to 0$ as $\alpha \to 0+$ if $u \neq v$ and $A \to e^{-u}$ as $\alpha \to 0+$ if $u = v$. Hence, we have to show that $B \to 0$ as $\alpha \to 0+$. We can assume that $u \geq v$ without loss of generality and write

$$B = B_1 + B_2 + B_3$$

$$= \left(\int_{-v}^{\infty} + \int_{-u}^{-v} + \int_{-\infty}^{-u} \right) \frac{4\pi\alpha e^{2y}}{\sqrt{4\pi\alpha^3}e^y+1} \mathrm{Ai}\left(\frac{y+u}{\alpha} \right) \mathrm{Ai}\left(\frac{y+v}{\alpha} \right) dy.$$

Now $x \to \mathrm{Ai}(x)$ is a bounded function and we have the estimates

$$|\mathrm{Ai}(x)| \leq \frac{C}{|x|^{1/4}} \tag{49}$$

for $x < 0$, and

$$|\mathrm{Ai}(x)| \leq \frac{Ce^{-2x^{3/2}/3}}{x^{1/4}} \tag{50}$$

for $x > 0$, where C is a numerical constant. It follows from these estimates that

$$|B_1| \leq C\alpha^{3/2}e^{-2v} \int_0^\infty \frac{e^{2y}}{\sqrt{y}} e^{-4y^{3/2}/3\alpha^{3/2}} \, dy$$

Clearly, $B_1 \to 0$ as $\alpha \to 0+$. Similarly,

$$|B_2| \leq C\alpha^{3/2} \int_{-u}^{-v} \frac{e^{2y}}{|y+u|^{1/4}|y+v|^{1/4}} e^{-(y+u)^{3/2}/3\alpha^{3/2}} \, dy$$

and hence $B_2 \to 0$ as $\alpha \to 0+$. Finally,

$$|B_3| \leq C\alpha^{3/2}e^{-2u} \int_{-\infty}^0 \frac{e^{2y}}{\sqrt{|y|}} dy,$$

which goes to 0 as $\alpha \to 0+$. This proves (i) in the theorem.

The fact that $M_\alpha(u,v) \to K_{Airy}(u,v)$ as $\alpha \to \infty$ follows from the estimates (49), (50) and the dominated convergence theorem. □

It follows from the estimate (45) that the interpolating process has a last particle almost surely. Its distribution function is given by Theorem 1.3 which we now prove.

Proof (Theorem 1.3) It follows from Hadamard's inequality and (49) that

$$\sum_{n=0}^\infty \frac{1}{n!} \int_{(t,\infty)^n} \det(M_\alpha(x_i, x_j)) d^n x \leq \sum_{n=0}^\infty \frac{1}{n!} \left(\int_t^\infty M_\alpha(x,x) dx \right)^n < \infty.$$

Hence, the first equality in (17) holds, see for example [16]. The second inequality follows since M_α is trace class on $L^2(t,\infty)$ for any t and $\mathrm{Tr}\, M_\alpha = \int_t^\infty M_\alpha(t,t) dt$, [8].

Next, we turn to the proof of (18). Write $f(\alpha) = \frac{1}{2\alpha} \log(4\pi\alpha^3)$. Then

$$F_\alpha(\xi/\alpha - f(\alpha)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int\limits_{(\xi,\infty)^n} \det(M_\alpha^*(x_i, x_j)) \mathrm{d}^n x, \qquad (51)$$

where M_α is as in (48). It follows from the estimate (46) that

$$M_\alpha^*(x, x) \leq \frac{1}{\sqrt{4\pi\alpha^3}} \mathrm{e}^{-x + \alpha f(\alpha) + \alpha^3/12} = \mathrm{e}^{-x + \alpha^3/12}.$$

Hence, by Hadamard's inequality,

$$\det(M_\alpha^*(x_i, x_j))_{1 \leq i,j \leq n} \leq \mathrm{e}^{n\alpha^3/12} \mathrm{e}^{-\sum_{j=1}^{n} x_j},$$

and it follows from (15), (51) and the dominated convergence theorem that

$$\lim_{\alpha \to 0+} F_\alpha(\xi/\alpha - f(\alpha)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int\limits_{(\xi,\infty)^n} \prod_{i=1}^{n} \mathrm{e}^{-x_i} \mathrm{d}^n x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{e}^{-n\xi} = F_G(\xi),$$

which proves (18).

If we use the estimate (47) with $\epsilon = 1$ and (44) we see that for $\alpha \geq 1$

$$M_\alpha(x, x) \leq \mathrm{e}^{-x}. \qquad (52)$$

Hence, for α large, we have

$$\det(M_\alpha(x_i, x_j))_{1 \leq i,j \leq n} \leq \mathrm{e}^{-\sum_{j=1}^{n} x_j}.$$

Consequently, we can use (16), (17) and the dominated convergence theorem to conclude that (19) holds. □

We will now prove Proposition 1.4 which gives an alternative representation of the F_α-distribution.

Proof (Proposition 1.4) We have

$$\mathbb{P}\left[\max_{j \geq 1} z_j \leq \xi\right] = \mathbb{E}\left[\prod_{j=1}^{\infty}(1 - \chi_{(\xi,\infty)}(z_j))\right] = \mathbb{E}\left[\prod_{j=1}^{\infty}(1 - \chi_{(\xi,\infty)}(x_j + y_j))\right]$$

$$= \mathbb{E}_x\left[\prod_{j=1}^{\infty}\left(\int_{-\infty}^{\infty}(1 - \chi_{(\xi,\infty)}(x_j - y))\mathrm{d}G_\alpha(y)\right)\right],$$

where $\mathbb{E}_x$ denotes expectation with respect to the Airy kernel point process. Here we have used the fact that the y_js are independent with distribution G_α and that they are independent of the Airy kernel point process. The last equality then follows from Fubini's theorem. We have also used the fact that the G_α-distribution is symmetric to replace y with $-y$. Note that $\chi_{(\xi,\infty)}(x_j - y) = 0$ if and only if $y \geq x_j - \xi$ and thus the last expression can be written

$$\mathbb{E}_x \left[\prod_{j=1}^\infty (1 - G_\alpha(x_j - \xi)) \right]$$

$$= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_{\mathbb{R}^k} \prod_{i=1}^k G_\alpha(x_i - \xi) \det \left(\int_0^\infty \mathrm{Ai}\,(x_i + t) \mathrm{Ai}\,(x_j + t) \mathrm{d}t \right) \mathrm{d}^k x$$

since we have the Airy kernel point process with correlation kernel (10). We now make the shift $x_j \to x_j + \xi$ and manipulate the expressions as follows

$$= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_{\mathbb{R}^k} \prod_{i=1}^k G_\alpha(x_i) \det \left(\int_\xi^\infty \mathrm{Ai}\,(x_i + t) \mathrm{Ai}\,(x_j + t) \mathrm{d}t \right) \mathrm{d}^k x$$

$$= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_{\mathbb{R}^k} \mathrm{d}^k x \prod_{i=1}^k G_\alpha(x_i) \int_{(\xi,\infty)^k} \mathrm{d}^k t \det(\mathrm{Ai}\,(x_i + t_i) \mathrm{Ai}\,(x_j + t_i))$$

$$= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_{(\xi,\infty)^k} \mathrm{d}^k t \int_{\mathbb{R}^k} \mathrm{d}^k x \left(\prod_{i=1}^k G_\alpha(x_i) \mathrm{Ai}\,(x_i + t_i) \right) \det(\mathrm{Ai}\,(x_i + t_j)),$$

where we have used the fact that the determinant is unchanged under transposition. Now, this last expression can be written

$$\sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_{(\xi,\infty)^k} \det \left(\int_{-\infty}^\infty G_\alpha(x) \mathrm{Ai}\,(x + t_i) \mathrm{Ai}\,(x + t_j) \mathrm{d}x \right) \mathrm{d}^k t$$

$$= \det(I - M_\alpha)_{L^2(\xi,\infty)} = F_\alpha(\xi).$$

$\square$

3 The grand canonical point process

In this section we will show that the grand canonical point process defined in Sect. 1 using (23) and (25) is a determinantal process with correlation kernel given by (28). The proof is based on the identity (27).

Proof (Theorem 1.5) We want to prove that

$$
\mathbb{E}\left[\prod_j (1 + g(x_j))\right] = \frac{1}{Z(\lambda)} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int_{X^N} \prod_{j=1}^{N} (1 + g(x_j)) \det(\phi(x_i, x_j))_{1 \le i,j \le N} \mathrm{d}^N \mu(x)
$$

$$
= \sum_{N=0}^{\infty} \frac{1}{N!} \int_{X^N} \prod_{j=1}^{N} g(x_j) \det(K_\lambda(x_i, x_j))_{1 \le i,j \le N} \mathrm{d}^N \mu(x). \tag{53}
$$

The first equality is just (25) and (27). The identity (53) implies the theorem, see e.g. [16]. To prove (53) we will use some facts on von Koch determinants, see e.g. [8]. Let $(a_{ij})_{i,j=0}^{\infty}$ be an infinite matrix and assume that

$$
\sum_{i=0}^{\infty} |a_{ii}| < \infty, \quad \sum_{i,j=0}^{\infty} |a_{ij}|^2 < \infty. \tag{54}
$$

Then,

$$
\det(I + A) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m \in \mathbb{N}^n} \det(a_{m_i m_j})_{i,j=1}^{n} \tag{55}
$$

is well-defined. Furthermore, if we have two such matrices A and B, then

$$
\det(I + A) \det(I + B) = \det(I + A + B + AB). \tag{56}
$$

Inserting (22) into the left hand side of (53) we get

$$
\frac{1}{Z(\lambda)} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int_{X^N} \prod_{j=1}^{N} (1 + g(x_j)) \frac{1}{N!} \sum_{m \in \mathbb{N}^N} \prod_{j=1}^{N} a_{m_j} \det(\psi_{m_i}(x_j)) \det(\overline{\psi_{m_i}(x_j)}) \mathrm{e}^N x
$$

$$
= \frac{1}{Z(\lambda)} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \sum_{m \in \mathbb{N}^N} \prod_{j=1}^{N} a_{m_j} \det\left(\int_X (1 + g(x)) \psi_{m_i}(x) \overline{\psi_{m_j}(x)} \mathrm{d}x\right)
$$

$$
= \frac{1}{Z(\lambda)} \det(I + D_g),
$$

where

$$
D_g(i, j) = \lambda a_i^{1/2} \int_X (1 + g(x)) \psi_i(x) \overline{\psi_j(x)} \mathrm{d}\mu(x) a_j^{1/2}.
$$

Here we have used the determinatal identity

$$\det\left(\int_X \phi_i(x)\psi_j(x)\mathrm{d}\mu(x)\right) = \frac{1}{N!}\int_{X^N} \det(\phi_i(x_j))\det(\psi_i(x_j))\mathrm{d}^N\mu(x), \quad (57)$$

where all the determinants are of size $N \times N$, see e.g. [16]. Clearly, $D_0(i,j) = \lambda a_i \delta_{ij}$, and hence $Z(\lambda) = \det(I + A)$, where

$$A(i,j) = \lambda a_i \delta_{ij}.$$

Set

$$B_g(i,j) = \frac{\lambda a_i^{1/2}}{1 + \lambda a_i}\int_X g(x)\psi_i(x)\overline{\psi_j(x)}\mathrm{d}\mu(x)a_j^{1/2}.$$

Note that since g is bounded we have $|D_g(i,j)| \le Ca_i^{1/2}a_j^{1/2}$, $|B_g(i,j)| \le Ca_i^{1/2}a_j^{1/2}$, so the conditions (54) are satisfied. Note also that,

$$\delta_{ij} + \lambda a_i \delta_{ij} + B_g(i,j) + (AB_g)_{ij} = \delta_{ij} + D_g(i,j).$$

Hence, by (56)

$$\frac{1}{Z(\lambda)}\det(I + D_g) = \frac{1}{\det(I + A)}\det(I + A)\det(I + B_g)$$

$$= \det(I + B_g) = \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{m\in\mathbb{N}^N}\det(B_g(m_i,m_j))_{1\le i,j\le N}$$

$$= \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{m\in\mathbb{N}^N}\prod_{i=1}^{N}\frac{\lambda a_{m_i}}{1 + \lambda a_{m_i}}\frac{1}{N!}\int_{X^N}\det(\psi_{m_i}(x_j))\det(\overline{\psi_{m_i}(x_j)})\prod_{j=1}^{N}g(x_j)\mathrm{d}^N\mu(x)$$

$$= \sum_{N=0}^{\infty}\frac{1}{N!}\int_{X^N}\prod_{j=1}^{N}g(x_j)\det\left(\sum_{m=0}^{\infty}\frac{\lambda a_m}{1 + \lambda a_m}\psi_m(x_i)\overline{\psi_m(x_j)}\right)\mathrm{d}^N\mu(x),$$

which is the right hand side of (53). Here we have used the identity (57) again. □

4 The MNS-model

In this section we will give the proofs for the results on the MNS-model stated in Sect. 1. First we must prove Lemma 1.6 which makes it possible to use the formalism for a grand canonical determinantal process and obtain Theorem 1.6, which is the starting point for the asymptotic analysis.

Proof (Lemma 1.6) We will use Mehler's formula,

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} q^n = \frac{1}{\sqrt{1-q^2}} \exp\left(-\frac{q^2}{1-q^2}(x^2+y^2) + \frac{2q}{1-q^2}xy\right), \quad (58)$$

where $0 < q < 1$ and H_n are the standard Hermite polynomials. If we use instead the normalized Hermite polynomials,

$$h_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n(x) \tag{59}$$

and rewrite the exponent we obtain

$$\sum_{n=0}^{\infty} q^n h_n(x) h_n(y) e^{-(x^2+y^2)/2}$$

$$= \frac{1}{\pi\sqrt{1-q^2}} \exp\left(-\frac{1-q}{2(1+q)}(x^2+y^2) + \frac{q}{1-q^2}(x-y)^2\right). \tag{60}$$

The change of variables $x \to \beta_q x, y \to \beta_q y$ and multiplication by $\beta_q q^{1/2}$ now gives (30). $\qquad\square$

The choice of the parameter λ in (35) with q given by (34) is motivated by the fact that the expected number of particles becomes

$$\int_{\mathbb{R}} K_\lambda(x,x)dx = \sum_{n=0}^{\infty} \frac{\lambda q^{n+1/2}}{1+\lambda q^{n+1/2}} \approx \int_0^{\infty} \frac{\lambda e^{-\mu x}}{1+\lambda e^{-\mu x}} dx$$

$$= \frac{1}{\mu} \log(1+\lambda) = N.$$

We turn to the proof of proposition 1.8

Proof (Proposition 1.8) With q as in (34) and λ as in (35) we have

$$K_\lambda(x,y) = \sum_{n=0}^{\infty} \frac{1}{1+(1-e^{-\mu N})^{-1}e^{(n+1/2-N)\mu}} \psi_n(x)\psi_n(y),$$

where ψ_n is given by (31). We split this into two sums, one from $n=0$ to $N-1$, called Σ_1, and one from N to infinity, called Σ_2. Since $\beta_q \to 1$ as $\mu \to \infty$ we see that Σ_1 converges to the right hand side of (36). We have to prove that $\Sigma_2 \to 0$ as $\mu \to \infty$. A useful bound is

$$|h_n(x)e^{-x^2/2}| \le \frac{C}{n^{1/12}}, \tag{61}$$

for all x, see [18]. Hence,

$$\Sigma_2 \le C \sum_{n=N}^{\infty} e^{(N-n-1/2)\mu} \frac{1}{n^{1/6}},$$

which goes to 0 as $\mu \to \infty$. This proves (36).

To prove (37) consider first the case $x = y$. We have

$$1 \le 1 + (e^{\mu N} - 1)e^{-(n+1/2)\mu} \le e^{\mu N}.$$

Since

$$K_\lambda(x, y) = \sum_{n=0}^{\infty} \frac{(e^{\mu N} - 1)e^{-(n+1/2)\mu}}{1 + (e^{\mu N} - 1)e^{-(n+1/2)\mu}} \psi_n(x)\psi_n(y)$$

we obtain

$$\frac{1 - e^{-\mu N}}{\mu N} \mu N \sum_{n=0}^{\infty} e^{-(n+1/2)\mu} \psi_n(x)^2 \le K_\lambda(x, x)$$

$$\frac{e^{\mu N} - 1}{\mu N} \mu N \sum_{n=0}^{\infty} e^{-(n+1/2)\mu} \psi_n(x)^2 \le K_\lambda(x, x).$$

The formula (30) gives

$$\frac{1 - e^{-\mu N}}{\mu N} \frac{\mu N}{(1-e^{-\mu})\sqrt{\pi}} e^{-\mu/2} e^{-x^2} \le K_\lambda(x, x) \le \frac{e^{\mu N} - 1}{\mu N} \frac{\mu N}{(1-e^{-\mu})\sqrt{\pi}} e^{-\mu/2} e^{-x^2}.$$

By letting $\mu \to 0+$ we get the first part of (37).

Consider now the second case, $x \ne y$. Write

$$K_\lambda(x, y) = (e^{\mu N} - 1) \sum_{n=0}^{\infty} e^{-(n+1/2)\mu} \psi_n(x)\psi_n(y)$$

$$+(e^{\mu N} - 1) \sum_{n=0}^{\infty} \left[\frac{1}{1 + (e^{\mu N} - 1)e^{-(n+1/2)\mu}} - 1 \right]$$

$$\times e^{-(n+1/2)\mu} \psi_n(x)\psi_n(y) \doteq S_1 + S_2.$$

By (30),

$$S_1 = \frac{e^{\mu N} - 1}{1 - e^{-\mu}} e^{-\mu/2} e^{-(x^2+y^2)/2 - e^{-\mu}(1-e^{-\mu})^{-2}(x-y)^2},$$

which $\to 0$ as $\mu \to 0+$. Furthermore,

$$|S_2| \le (e^{\mu N} - 1)^2 \sum_{n=0}^{\infty} e^{-(n+1/2)\mu} |\psi_n(x)\psi_n(y)|$$

$$\le \frac{1}{2\sqrt{\pi}} (e^{\mu N} - 1)^2 \frac{e^{-\mu/2}}{1 - e^{-\mu}} (e^{-x^2} + e^{-y^2})$$

by the Cauchy–Schwarz' inequality and (30). We see that the last expression $\to 0$ as $\mu \to 0+$.   □

We turn now to the proof of the two theorems that concern the asymptotic behaviour of the kernel K_λ in the regime where we have a transition in the bulk.

Proof (Theorem 1.9) We will use the following asymptotic formula for the Hermite polynomials, [3], valid for $-1 + \delta \le x \le 1 - \delta, \delta > 0$ fixed,

$$h_n\left(\sqrt{2n}x\right) e^{-nx^2} = \frac{2^{1/4}}{n^{1/4}\sqrt{\pi}} \frac{1}{(1-x^2)^{1/4}} \left(\cos\left[2nF(x) - \frac{1}{2}\arcsin x\right] + O\left(\frac{1}{n}\right)\right), \tag{62}$$

where

$$F(x) = \int_x^1 \sqrt{1 - y^2} dy = \frac{1}{2}(\arccos x - x\sqrt{1 - x^2}).$$

Set $A_N = \lambda^{-1}e^{1/2cN}$. Note that $A_N = \lambda^{-1} + O(1/N)$ and $\beta_q = \sqrt{2cN} + O(1/N^{3/2})$ as $N \to \infty$. Write $f_n(x) = h_n(x)e^{-x^2/2}$ Using the asymptotic formula (62) we obtain

$$\frac{\pi}{2N\sqrt{c}}K_\lambda\left(\frac{\pi x}{2N\sqrt{c}}, \frac{\pi y}{2N\sqrt{c}}\right)$$

$$= \frac{\pi\beta_q}{N\sqrt{2c}} \sum_{n=0}^{\infty} \frac{1}{A_N e^{n/cN} + 1} f_n\left(\frac{\pi\beta_q x}{2N\sqrt{c}}\right) f_n\left(\frac{\pi\beta_q y}{2N\sqrt{c}}\right)$$

$$= \frac{\beta_q}{N\sqrt{2c}} \sum_{n=1}^{\infty} \frac{1}{A_N e^{n/cN} + 1} \left(1 - \left(\frac{\pi\beta_q x}{2N\sqrt{2cn}}\right)^2\right)^{-1/4}$$

$$\times \left(1 - \left(\frac{\pi\beta_q y}{2N\sqrt{2cn}}\right)^2\right)^{-1/4}$$

$$\times \frac{1}{n^{1/2}} \cos\left[2nF\left(\frac{\pi\beta_q x}{2N\sqrt{2cn}}\right) - \frac{1}{2}\arcsin\left(\frac{\pi\beta_q x}{2N\sqrt{2cn}}\right)\right]$$

$$\times \cos\left[2nF\left(\frac{\pi\beta_q y}{2N\sqrt{2cn}}\right) - \frac{1}{2}\arcsin\left(\frac{\pi\beta_q y}{2N\sqrt{2cn}}\right)\right] + o(1)$$

$$
= \frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} \frac{1}{\lambda^{-1} e^{n/cN} + 1} \frac{1}{n^{1/2}}
$$

$$
\times \cos \left[2nF \left(\frac{\pi \beta_q x}{2N\sqrt{2cn}} \right) - \frac{1}{2} \arcsin \left(\frac{\pi \beta_q x}{2N\sqrt{2cn}} \right) \right]
$$

$$
\times \cos \left[2nF \left(\frac{\pi \beta_q y}{2N\sqrt{2cn}} \right) - \frac{1}{2} \arcsin \left(\frac{\pi \beta_q y}{2N\sqrt{2cn}} \right) \right] + o(1).
$$

Note that the sum

$$
\frac{1}{\sqrt{N}} \sum_{n=1}^{\infty} \frac{1}{\lambda^{-1} e^{n/cN} + 1} \frac{1}{n^{1/2}}
$$

is bounded in N. We have

$$
\frac{\pi \beta_q x}{2N\sqrt{2cn}} = \frac{\pi x}{2\sqrt{nN}} + O\left(\frac{1}{\sqrt{n}N^{5/2}} \right)
$$

and

$$
2nF\left(\frac{\pi x}{2\sqrt{nN}} \right) = \frac{\pi n}{2} - \pi x \sqrt{\frac{n}{N}} + O\left(\frac{1}{\sqrt{n}N^{5/2}} \right)
$$

as $N \to \infty$, x in a compact set. Hence,

$$
\frac{\pi}{2N\sqrt{c}} K_\lambda \left(\frac{\pi x}{2N\sqrt{c}}, \frac{\pi y}{2N\sqrt{c}} \right)
$$

$$
= \frac{1}{2N} \sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda^{-1} e^{n/cN} + 1} \left(\frac{n}{N} \right)^{-1/2} \cos\left(\pi(x+y)\sqrt{\frac{n}{N}} \right)
$$

$$
+ \frac{1}{2N} \sum_{n=1}^{\infty} \frac{1}{\lambda^{-1} e^{n/cN} + 1} \left(\frac{n}{N} \right)^{-1/2} \cos\left(\pi(x-y)\sqrt{\frac{n}{N}} \right)
$$

$$
\to \frac{1}{2} \int_0^{\infty} \frac{\cos \pi(x-y)\sqrt{t}}{\lambda^{-1} e^{t/c} + 1} \frac{dt}{\sqrt{t}}
$$

uniformly for x, y in a compact set as $N \to \infty$. If we make the change of variables $t = u^2$ we obtain $L_c(x, y)$ in (38). □

When we are in the transition region in the bulk, the behaviour at the edge is still like that of independent random variables. This is the content of Theorem 1.10 which we prove next.

Proof (Theorem 1.10) We split the kernel K_λ as follows

$$K_\lambda(x,y) = \sum_{n=0}^{\infty} A_N^{-1} e^{-n/cN} \psi_n(x)\psi_n(y)$$

$$+ \left(\sum_{n=0}^{M_N} + \sum_{n=M_N+1}^{\infty}\right) \left(\frac{1}{A_N e^{n/cN}+1} - \frac{1}{A_N e^{n/cN}}\right) \psi_n(x)\psi_n(y)$$

$$\doteq S_1(x,y) + S_2(x,y) + S_3(x,y),$$

where $M_N = [(1-\delta)cN \log N]$ with $\delta > 0$ small. Here A_N has the same meaning as in the proof of Theorem 1.9. Note that we have the estimate

$$\left|\frac{1}{A_N e^{n/cN}+1} - \frac{1}{A_N e^{n/cN}}\right| \leq C e^{-2n/cN}. \tag{63}$$

By (30),

$$b_N S_1(a_N(c)+b_N\xi, a_N(c)+b_N\eta)$$

$$= \frac{b_N}{A_N(1-q)\sqrt{\pi}} \exp\left(-\frac{1}{2}(a_N(c)+b_N\xi)^2 - \frac{1}{2}(a_N(c)+b_N\eta)^2\right.$$

$$\left.-\frac{qb_N^2}{(1-q)^2}(\xi-\eta)^2\right)$$

$$= \exp\left(-\frac{\xi+\eta}{2} - (c^2N^2 + O(N))(\xi-\eta)^2 + o(1)\right)$$

as $N \to \infty$. The last identity explains the choice of $a_N(c)$ and b_N and we will get (40) if we can prove that S_2 and S_3 both tend to zero as N tends to infinity.

From the estimates (61), (63) and $\beta_q \sim \sqrt{2cN}$ we obtain

$$|b_N S_3(a_N(c)+b_N\xi, a_N(c)+b_N\eta)| \leq \frac{CN^{1/2}}{\sqrt{\log N}} \sum_{n=M_N+1}^{\infty} e^{-2n/cN}\frac{1}{n^{1/6}} \to 0$$

as $N \to \infty$ provided δ is sufficiently small.

We can write $\psi_n(a_N(c)+b_N\xi) = \psi_n(\sqrt{2n}y)$, where

$$y = \frac{1}{\sqrt{2n}}\left(\sqrt{2cN} + O\left(\frac{1}{N^{3/2}}\right)\right)$$

$$\times \left(\sqrt{\log N} - (4\sqrt{\log N})^{-1}\log\left(\frac{4\pi}{\lambda^2 c^2}\log N\right) + \xi(2\sqrt{\log N})^{-1}\right)$$

For a fixed ξ, we see that $y \geq 1 + \delta$ if N is sufficiently large and $1 \leq n \leq M_N$. We can then use the estimate,

$$|h_n(\sqrt{2n}x)e^{-nx^2}| \leq \frac{C_1}{n^{1/4}}e^{-nF(x)} \tag{64}$$

for $x \geq 1 + \delta$, [3], which gives

$$|h_n(\sqrt{2n}x)e^{-nx^2}| \leq \frac{C_1}{n^{1/4}}e^{-C_2 n\delta^{3/2}},$$

where C_2 is a numerical constant. If N is large enough, then $y \geq \sqrt{cNn^{-1}\log N}$ and we get

$$|b_N S_2(a_N(c) + b_N\xi, a_N(c) + b_N\eta)|$$
$$\leq \frac{1}{2\sqrt{N}} + \frac{C\sqrt{N}}{\sqrt{\log N}} \sum_{n=1}^{M_N} \frac{1}{n^{1/4}} e^{-C_2 n(\sqrt{cNn^{-1}\log N}-1)^{3/2}}$$

which $\to 0$ as $N \to \infty$. $\qquad\qquad\square$

The next result shows that the kernel M_α can be obtained as a scaling limit of the kernel K_λ

Proof (Theorem 1.11) Let A_N have the same meaning as in the proof of Theorem 1.9. We have

$$A_N \approx e^{-\alpha N^{2/3} + \alpha N^{-1/3}/2}$$

with a negligible error. Also, as $N \to \infty$,

$$\beta_q = \frac{\sqrt{2}N^{1/6}}{\sqrt{\alpha}} + O\left(\frac{1}{\sqrt{N}}\right).$$

Write $f_n(x) = h_n(x)e^{-x^2/2}$ as above. We have

$$\frac{\sqrt{\alpha}}{2N^{1/3}} K_\lambda\left(N^{1/3}\sqrt{\alpha} + \frac{\sqrt{\alpha}}{2N^{1/3}}\xi, N^{1/3}\sqrt{\alpha} + \frac{\sqrt{\alpha}}{2N^{1/3}}\eta\right)$$
$$= \left(\frac{1}{N^{1/6}\sqrt{2}} + O\left(\frac{1}{N^{5/6}}\right)\right) \sum_{n=0}^{\infty} \frac{1}{e^{\alpha N^{2/3}((n+1/2)/N-1)} + 1}$$
$$\times f_n\left(\sqrt{2N} + \frac{\xi}{N^{1/6}\sqrt{2}} + O\left(\frac{1}{N^{1/3}}\right)\right) f_n\left(\sqrt{2N} + \frac{\eta}{N^{1/6}\sqrt{2}} + O\left(\frac{1}{N^{1/3}}\right)\right)$$

$$= \left(\frac{1}{N^{1/6}\sqrt{2}} + O\left(\frac{1}{N^{5/6}}\right) \right) \left(\sum_{k=-\infty}^{-M_N-1} + \sum_{k=-M_N}^{M_N} + \sum_{k=M_N+1}^{N} \right) \frac{1}{e^{-\alpha(k-1/2)/N^{1/3}} + 1}$$

$$\times f_{N-k}\left(\sqrt{2N} + \frac{\xi}{N^{1/6}\sqrt{2}} + O\left(\frac{1}{N^{1/3}}\right)\right) f_{N-k}\left(\sqrt{2N} + \frac{\eta}{N^{1/6}\sqrt{2}} + O\left(\frac{1}{N^{1/3}}\right)\right)$$

$$\doteq \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where $M_N = \gamma N^{1/3} \log N$ with some fixed $\gamma > 0$ that can be chosen. The asymptotic contribution will come from Σ_2. Here we use the asymptotic formula

$$f_{N-k}\left(\sqrt{2N} + \frac{u}{N^{1/6}\sqrt{2}}\right) = \frac{2^{1/4}}{N^{1/12}} \text{Ai}\left(u + \frac{k-1/2}{N^{1/3}}\right)\left(1 + O\left(\frac{\log N}{N^{2/3}}\right)\right) \quad (65)$$

for $|k| \leq \gamma N^{1/3} \log N$ and u in a compact set. This formula folows from results in [3], see [1]. Using this we see that

$$\lim_{N\to\infty} \Sigma_2 = \lim_{N\to\infty} \frac{1}{N^{1/3}} \sum_{-M_N}^{M_N} \frac{1}{e^{-\alpha(k-1/2)/N^{1/3}} + 1} \text{Ai}\left(\xi + \frac{k-1/2}{N^{1/3}}\right) \text{Ai}\left(\eta + \frac{k-1/2}{N^{1/3}}\right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{e^{-\alpha x} + 1} \text{Ai}(x+\xi)\text{Ai}(x+\eta)\text{d}x = M_\alpha(\xi, \eta).$$

We still have to prove $\Sigma_1 \to 0$ and $\Sigma_3 \to 0$ as $N \to \infty$. For Σ_1 we use the estimate (61), which gives

$$|\Sigma_1| \leq \frac{C}{N^{1/6}} \sum_{k=-\infty}^{-M_N-1} \frac{1}{e^{-\alpha(k-1/2)/N^{1/3}} + 1} \frac{1}{(N-k)^{1/6}}$$

which goes to zero as N tends to infinity if we choose γ large enough.

For $1 < x \leq 1 + \delta$ we have an asymptotic formula for $f_n(\sqrt{2n}x)$ in terms of the Airy function, see [3]. Estimates of the Airy function then gives

$$\left| f_n(\sqrt{2n}x) \right| \leq \frac{C}{N^{1/12}} e^{-cn(x-1)^{3/2}} \quad (66)$$

for some constants c, C, when $1 < x \leq 1 + \delta$. Since $F(x) \geq c(x-1)^{3/2}$, we can combine this with (64) to see that (66) holds for all $x > 1$. For N sufficiently large this leads to an estimate

$$\left| f_{N-k}\left(\sqrt{2N} + \frac{\xi}{N^{1/6}\sqrt{2}} + O\left(\frac{1}{N^{1/3}}\right)\right) \right| \leq Ce^{-ck^{3/2}/\sqrt{N}}.$$

It follows that

$$|\Sigma_3| \leq C \sum_{k=M_N+1}^{N} e^{-ck^{3/2}/\sqrt{N}} \to 0$$

as $N \to \infty$ if we choose γ sufficiently large. $\square$

We give here a sketch of an argument for the approximate expression (39) for $L_c(x,y)$. Integration by parts gives

$$L_c(x,y) = \int_0^\infty \frac{\sin \pi(x-y)u}{\pi(x-y)} \frac{2u}{\lambda c} \frac{e^{u^2/c}}{(\lambda^{-1}e^{u^2/c}+1)^2} du$$

$$= \frac{2}{c} \int_0^\infty u \frac{\sin \pi(x-y)u}{\pi(x-y)} \frac{du}{\cosh^2(u^2/2c+a/2)},$$

where $a = \log(1/\lambda)$. When c is small $a \approx -1/c$. Make the change of variables $u = 1 + ct$. This gives

$$L_c(x,y) = \int_{-1/c}^\infty (1+ct) \frac{\sin \pi(x-y)(1+ct)}{\pi(x-y)} \frac{dt}{\cosh^2(t+ct^2/2)}$$

$$\approx \frac{1}{2} \int_{-\infty}^\infty (1+ct) \frac{\sin \pi(x-y)(1+ct)}{\pi(x-y)} \frac{dt}{\cosh^2 t}.$$

If we use the addition formula for the sine function and neglect terms containing c^2 we get

$$L_c(x,y) \approx \int_{-\infty}^\infty \frac{\cos \pi(x-y)ct}{\cosh 2t+1} \frac{\sin \pi(x-y)}{\pi(x-y)}$$

$$= \frac{\pi^2(x-y)c}{2\sin h(\pi^2 c(x-y)/2)} \frac{\sin \pi(x-y)}{\pi(x-y)} = \frac{\pi c}{2} \frac{\sin \pi(x-y)}{\sin h(\pi^2 c(x-y)/2)}.$$

We see that as $c \to 0+$ the kernel L_c approaches the sine kernel.

5 Largest eigenvalue for deformed GUE

This section contains the proof of Theorem 1.12. Consider N non-intersecting standard Brownian motions started at $y_1, \ldots, y_N$ and conditioned never to intersect. If we fix $y_1, \ldots, y_N$ the particle distribution at time S is a determinantal process with correlation functions

$$\rho_{m,N}(x_1,\ldots,x_m;y) = \det(K_N(x_i,x_j;y))_{1\leq i,j\leq m}, \tag{67}$$

where

$$K_N(u,v;y) = \frac{1}{(2\pi i)^2 S} \int_\gamma dz \int_\Gamma e^{(w-v)^2/2S-(z-u)^2/2S} \frac{1}{w-z} \prod_{j=1}^N \frac{w-y_j}{z-y_j}, \tag{68}$$

see e.g. [12]. Here γ is a positively oriented simple closed curve containing $y_1,\ldots,y_N$ and Γ a vertical line oriented upwards and not interesecting γ, we place it to the right of γ. The formula (67) also gives the correlation functions for the eigenvalues of the hermitian matrix M given by

$$M = \mathrm{diag}\,(y_1 + \cdots + y_N) + \sqrt{2S}V \tag{69}$$

with V a standard $N \times N$ GUE matrix. If $\lambda_{\max}^{(N)}$ is the largest eigenvalue of M then $\lambda_{\max}^{(N)}$ has the same distribution as $\max_{1\leq j\leq N} x_j$.

Let $\mathbb{P}_y$ denote the probability measure for $y_1, y_2, \ldots$ and let $\mathbb{P}_{x;y}$ denote the probability measure for the determinantal process with correlation kernel $K_N(u,v;y)$ given by (68). Furthermore, we let $\mathbb{P}_N = \mathbb{P}_y \otimes \mathbb{P}_{x;y}$ be the product measure. We are interested in the distribution function

$$F_N(t) = \mathbb{P}_N[\lambda_{\max}^{(N)} \leq t] = \mathbb{E}_N\left[\prod_{j=1}^N (1 - \chi_{(t,\infty)}(x_j))\right]$$

$$= \mathbb{E}_y\left[\mathbb{E}_{x;y}\left[\prod_{j=1}^N (1 - \chi_{(t,\infty)}(x_j))\right]\right]. \tag{70}$$

When computing the inner expectation we are considering $y_1,\ldots,y_N$ as fixed and hence we can work with the correlation functions (67).

Fix a number $\epsilon \in (1/7,1/6)$ and set

$$A_N = \{y \in \mathbb{R}^N\,;\,|y_i| \leq N^\epsilon,\,1 \leq i \leq N\}.$$

Define a cut-off measure $d\mu_N(t)$ by

$$d\mu_N(t) = \frac{1}{\mu([-N^\epsilon,N^\epsilon])}\chi_{[-N^\epsilon,N^\epsilon]}(t)d\mu(t), \tag{71}$$

and the function

$$G_N(z) = \int_\mathbb{R} \frac{d\mu_N(t)}{z-t} \tag{72}$$

for $z \in \mathbb{C} \setminus [-N^\epsilon, N^\epsilon]$.

Since by assumption μ has finite 7th moment it follows that

$$\mathbb{P}_y[A_N^c] \leq N \frac{C}{N^{7\epsilon}}$$

which $\to 0$ as $N \to \infty$. Hence, since the expression in the $\mathbb{E}_y$-expectation in (70) is bounded, we can restrict our attention to A_N and use μ_N instead of μ, so we regard $y_1, \ldots, y_N$ as independent random variables with distribution μ_N. Denote this probability measure by $\mathbb{P}_y^{(N)}$.

Lemma 5.1 *There is a real number $w_c = w_c(N)$, which is approximately $\sqrt{NS} = \alpha N^{1/6}$, such that*

$$G'_N(w_c) = -\frac{1}{\alpha^2 N^{1/3}} \tag{73}$$

for all sufficiently large N.

Proof We have

$$G'_N(z) = -\int_{-N^\epsilon}^{N^\epsilon} \frac{d\mu_N(t)}{(z-t)^2}.$$

The moment conditions on μ can be used to see that

$$\left| G'_N(z) - \frac{1}{z^2} \right| \leq \frac{C}{z^3} \tag{74}$$

for real $z \geq 2N^\epsilon$ say. We see that, for N sufficiently large, G'_N is a decreasing function in $[2N^\epsilon, \infty)$. Furthermore, $G'_N(z) \to 0$ as $z \to \infty$ and $G''_N(2N^\epsilon) \approx 1/4N^{2\epsilon} > 1/\alpha N^{1/3}$ Hence, when N is large enough, there is a $z = w_c$ such that (73) holds. From (74) we see that $1/w_c^2 \sim 1/\alpha^2 N^{1/3}$, which gives the asymptotic behaviour.   □

Set

$$r_N(y) = -\sum_{j=1}^{N} \frac{1}{(w_c - y_j)^2} + NG'_N(w_c), \tag{75}$$

and

$$v_c = w_c + S \sum_{j=1}^{N} \frac{1}{w_c - y_j}. \tag{76}$$

Furthermore, set

$$R(N) = w_c + \frac{\alpha^2 N^{1/3}}{w_c} + \frac{\alpha^2 N^{1/3}}{w_c} \int_{\mathbb{R}} \frac{y}{1 - y/w_c} d\mu_N(y) \tag{77}$$

and define

$$s_N(y) = \frac{\alpha}{w_c N^{1/6}} \left(\sum_{j=1}^{N} \frac{y_j}{w_c - y_j} - \int_{\mathbb{R}} \frac{y}{w_c - y} d\mu_N(y) \right). \tag{78}$$

Note that

$$v_c = R(N) + \frac{\alpha}{N^{1/2}} s_N(y). \tag{79}$$

Lemma 5.2 *(i) There is a constant C such that*

$$\mathbb{E}_y^{(N)}[r_N(y)] \le C. \tag{80}$$

(ii) We have the limit

$$\text{Var}_y^{(N)}[s_N(y)] \to \sigma^2/\alpha^2 \tag{81}$$

as $N \to \infty$.

(iii) The random variable $s_N(y)$ converges in distribution to $N(0, \sigma^2/\alpha^2)$ as $N \to \infty$

Proof Since the 7th moment is finite and $\int t d\mu(t) = 0$ we get $|\int t d\mu_N(t)| \le CN^{-6\epsilon} \le CN^{-6/7}$. The definition of w_c and G_N gives $\mathbb{E}_y^{(N)}[r_N(y)] = 0$ and thus

$$\mathbb{E}_y^{(N)}[r_N(y)^2] = \text{Var}_y^{(N)}[r_N(y)] = \text{Var}_y^{(N)} \left[\sum_{j=1}^{N} \frac{1}{(w_c - y_j)^2} \right]$$
$$= N \left[-\frac{1}{6} G_N^{(3)}(w_c) - G_N'(w_c)^2 \right] \le \frac{NC}{w_c^6},$$

where the last inequality follows from our moment condition. Since $w_c \sim \alpha N^{1/6}$ we see that the right hand side is bounded. To prove (ii) we compute

$$\text{Var}_y^{(N)}[s_N(y)] = \frac{\alpha^2 N^{2/3}}{w_c^4} \left(\int \frac{y^2}{(1 - y/w_c)^2} d\mu_N(y) - \left(\int \frac{y}{1 - y/w_c} d\mu_N(y) \right)^2 \right).$$
$$\tag{82}$$

Writing $y(1 - y/w_c)^{-1} = y + y^2/w_c + \cdots$, we get $\int y(1 - y/w_c)^{-1} d\mu_N(y) = o(1)$. Similarly, writing $y^2/(1 - y/w_c)^2 = y^2 + 2y^3/w_c + \cdots$, we get $\int y^2/(1 - y/w_c)^2 d\mu_N(y) = \sigma^2 + o(1)$. Using $w_c \sim \alpha N^{1/6}$, the identity (82) now yields (ii). The claim (iii) follows immediately from the central limit theorem, since s_N is a sum of independent random variables. □

Define B_N to be the set of all $y \in A_N$ such that $|r_N(y)| \le CN^\epsilon$ and $|s_N(y)| \le CN^\epsilon$. It follows from Lemma 5.2 (i) and (ii) that we can restrict ourselves to $y \in B_N$. We will study

$$F_N^*(t) = \mathbb{E}_y^{(N)}\left[\chi_{B_N}(y)\mathbb{E}_{x;y}\left[\prod_{j=1}^N (1 - \chi_{(t,\infty)}(x_j)) \right] \right] \tag{83}$$

instead of $F_N(t)$ given by (70). Hence it is enough to consider a fixed y in B_N and work with

$$F_N^*(t;y) = \mathbb{E}_{x;y}\left[\prod_{j=1}^N (1 - \chi_{(t,\infty)}(x_j)) \right]$$

$$= \sum_{k=0}^N \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \det(K_N(x_i,x_j;y))d^k x. \tag{84}$$

Our problem is then to investigate the asymptotics of $K_N(u,v;y)$ for $y \in B_N$. Set

$$f(w) = \frac{w^2}{2S} - \frac{v_c w}{S} + \sum_{j=1}^N \log(w - y_j),$$

where we choose the principal branch of the logarithm. The number v_c was defined so that

$$f'(w_c) = 0. \tag{85}$$

We consider

$$u = v_c + \xi\frac{\alpha}{\sqrt{N}}; \quad v = v_c + \eta\frac{\alpha}{\sqrt{N}}, \tag{86}$$

where ξ and η lie in a compact set.

To perform a saddle-point argument in the integral (68) we must specify appropriate contours. Let $C_1 : [0,\infty) \ni t \to w_c + t + it$, $C_2 : [0,\infty) \ni t \to w_c + t - it$, $C_3 : [0,\infty) \ni t \to w_c - t + it$ and $C_4 : [0,\infty) \ni t \to w_c - t - it$. We want to show that we can deform Γ to $C_1 - C_2$ and γ to $C_3 - C_4$ in the

contour integral (68). Let C_i^A be the parts of the contours where we restrict t to $[0, A + w_c]$ and let $\gamma_A : [-w_c - A, w_c + A] \ni t \to -A - it$, where $A > 0$. Then γ can be deformed to $C_3^A + \gamma_A - C_4^A$ in (68) if A is sufficiently large. From (77) we see that $R(N) \sim 2\alpha N^{1/6}$ and since ξ belongs to a compact set we see from (79), $|s_N(y)| \leq N^\epsilon$ and (86) that $u \geq \alpha N^{1/6}$ for all sufficiently large N. Since $\mathrm{Re}\,(-(z - u)^2) \leq -2uA$ we see that we can let $A \to \infty$ and conclude that the contribution from γ_A goes to zero.

Choose Γ to be a vertical line through w_c. We want to show that the part of this line that lies in the upper half plane can be deformed to C_1, and the in the lower half plane to C_2. Set $w = w_c + t + iA, 0 \leq t \leq A$. Then,

$$g(t) = \mathrm{Re}\,((w - v)^2) = t^2 + 2(w_c - v)t + (w_c - v)^2 - A^2.$$

For $0 \leq t \leq A/2$, we see that $g(t) \leq -A^2/2$ for A large, and when $A/2 \leq t \leq A$ we have

$$g(t) \leq (w_c - v)A + (w_c - v)^2 \leq -\frac{\alpha}{2}N^{1/6}A$$

for N and A large. Hence we can deform the upper part of Γ to C_1 in (68). The deformation to C_2 is analogous by symmetry.

Next, we want to localize the integration to a small neighbourhood of w_c.

Lemma 5.3 *Define* $g_i(t) = \mathrm{Re}\,(f(C_i(t)) - f(w_c))$ *for* $i = 1, 2$ *and* $g_i(t) = -\mathrm{Re}\,(f(C_i(t)) - f(w_c))$ *for* $i = 3, 4, t \geq 0$. *There is a positive constant* c *so that*

$$g_i(t) \leq \begin{cases} -cN^{1/2}t^3, & 0 \leq t \leq \alpha N^{1/6}/2 \\ -cN^{5/6}t & t \geq \alpha N^{1/6}/2 \end{cases}, \tag{87}$$

for all sufficiently large N.

Proof Consider $g_1(t)$ We have

$$g_1(t) = \frac{1}{2S}\left[(w_c + t)^2 - t^2\right] - \frac{1}{2}v_c(w_c + t)$$
$$+ \frac{1}{2}\sum_{j=1}^{N}\log((w_c + t - y_j)^2 + t^2) - \mathrm{Re}\,f(w_c).$$

Differentiation gives

$$g_1'(t) = \frac{w_c - v_c}{S} + \sum_{j=1}^{N}\frac{w_c - y_j + 2t}{(w_c - y_j + t)^2 + t^2}$$

and hence $g_1'(0) = 0$ by (76). Hence

$$g_1'(t) = g_1'(t) - g_1'(0) = -\sum_{j=1}^{N} \frac{2t^2}{(w_c - y_j)((w_c - y_j + t)^2 + t^2)}.$$

We know that $w_c \sim \alpha N^{1/6}$ and $|y_j| \le N^\epsilon$. If $|t| \le \alpha N^{1/6}/2$, we see that there is a positive constant c such that $g_1'(t) \le -cN^{1/2}t^2$ for $0 \le t \le \alpha N^{1/6}/2$. Since $g_1(0) = 0$ we obtain the first part of (87) for $i = 1$. If $t \ge \alpha N^{1/6}/2$, then $g_1'(t) \ge -cN^{5/6}$ for some postive constant c, and we obtain the second part of (87).

Consider next $g_3(t)$. Again $g_3(0) = g_3'(0) = 0$ and we get

$$g_3'(t) = -\sum_{j=1}^{N} \frac{2t^2}{(w_c - y_j)((w_c - y_j + t)^2 + t^2)}$$

and we can proceed as above. The functions g_2 and g_3 are treated analogously. $\square$

We also need a local approximation of $f(w)$ in a neighbourhood of w_c. By (85) we have $f'(w_c) = 0$ and we also have

$$f''(w_c) = \frac{1}{S} - \sum_{j=1}^{N} \frac{1}{(w_c - y_j)^2} = r_N(y), \qquad (88)$$

by (73) and (75). Furthermore,

$$f^{(3)}(w_c) = \sum_{j=1}^{N} \frac{2}{(w_c - y_j)^3}. \qquad (89)$$

Lemma 5.4 *For $\zeta \in \mathbb{C}$ and $|\zeta| \le N^{1/18}$,*

$$f(w_c + \zeta \alpha N^{-1/6}) = f(w_c) + \frac{1}{3}\zeta^3 + o(1), \qquad (90)$$

as $N \to \infty$, where $o(1)$ is uniform for $|\zeta| \le N^{1/18}$.

Proof Define $R(\zeta)$ by

$$f(w_c + \zeta) = f(w_c) + f'(w_c)\zeta + \frac{1}{2}f''(w_c)\zeta^2 + \frac{1}{6}f^{(3)}(w_c)\zeta^3 + R(\zeta). \qquad (91)$$

Since $w_c \sim \alpha N^{1/6}$ and $|w_c - y_j|$ is much greater than 1 for N large enough a Taylor expansion gives

$$|R(\zeta)| \leq CN^{1/3}|\zeta|^4 \tag{92}$$

for $|\zeta| \leq 1$. It follows from (85), (88) and (89) that

$$f(w_c + i\zeta\alpha N^{-1/6}) = f(w_c) - \frac{\alpha^2}{2}r_N(y)\zeta^2 N^{-1/3}$$

$$+ \frac{1}{3}\sum_{j=1}^{N}\frac{\zeta^3 N^{-1/2}}{(w_c - y_j)^3} + R(\zeta\alpha N^{-1/6}). \tag{93}$$

Since $|r_N(y)| \leq CN^\epsilon$ and $|\zeta| \leq N^{1/18}$ we see that $|r_N(y)\zeta^2 N^{-1/3}| \leq CN^{-1/18}$. Furthermore, by (92), $|R(\zeta\alpha N^{-1/6})| \leq CN^{-1/9}$. We can write

$$\sum_{j=1}^{N}\frac{1}{(w_c - y_j)^3} = \frac{N}{w_c^3} + \sum_{j=1}^{N}\frac{w_c^3 - (w_c - y_j)^3}{w_c^3(w_c - y_j)^3} = \frac{N^{1/2}}{\alpha^3}(1 + o(1))$$

as $N \to \infty$. We see now that (90) follows from (93). $\qquad\square$

It follows from (68), the definition of f and the change of contours discussed above that

$$K_N(v_c + \xi\frac{\alpha}{\sqrt{N}}, v_c + \eta\frac{\alpha}{\sqrt{N}})$$

$$= \frac{e^{v^2 - u^2}N^{2/3}}{(2\pi i)^2\alpha^2}\int_{C_3 - C_4}dz\int_{C_1 - C_2}dw\frac{e^{f(w) - f(z)}}{w - z}e^{-\eta N^{1/6}w/\alpha + \xi N^{1/6}z/\alpha} \tag{94}$$

Consider z on C_3 and w on C_1. The other cases are similar. Set $z = w_c + (-t + it)\alpha N^{-1/6}$, $w = w_c + (-\tau + i\tau)\alpha N^{-1/6}$, $t, \tau \geq 0$. It follows from Lemma 5.3 that we can localize the evaluation of (94) to $t, \tau \leq N^{1/18}$. By Lemma 5.4

$$f(w) - f(z) = \frac{1}{3}(\tau + i\tau)^3 - \frac{1}{3}(-t + it)^3 + o(1)$$

uniformly for $0 \leq t, \tau \leq N^{1/18}$. Hence, the contribution to (94) from z on C_3 and w on C_1 is

$$\frac{N^{1/2}e^{v^2 - u^2 + (\xi - \eta)N^{1/6}w_c/\alpha}}{(2\pi i)^2\alpha}\int_0^{N^{1/18}}dt\int_0^{N^{1/18}}d\tau\frac{e^{\frac{1}{3}(\tau + i\tau)^3 - \frac{1}{3}(-t + it)^3 + \xi(-t + it) - \eta(\tau + i\tau)}}{(\tau + i\tau) - (-t + it)}.$$

Define

$$K_N^*(u, v; y) = e^{v^2 - u^2 + (u - v)w_c/S}K_N(u, v; y). \tag{95}$$

We can just as well use K_N^* as K_N. If we argue as above for all parts of the contours we get

$$\lim_{N\to\infty} \frac{\alpha}{\sqrt{N}} K_N^* \left(v_c + \xi \frac{\alpha}{\sqrt{N}}, v_c + \eta \frac{\alpha}{\sqrt{N}} \right)$$
$$= \frac{1}{(2\pi i)^2} \int_{\gamma'} dz \int_{\Gamma'} dw e^{w^3/3 - z^3/3 + \xi z - \eta w} \frac{1}{w - z}, \qquad (96)$$

where γ' is the contour given by $t + it$ for $t \leq 0$ and $-t + it$ for $t \geq 0$, and Γ' is the reflection of γ' in the imaginary axis.

We have to show that the right hand side of (96) is really the Airy kernel. Let $\mathcal{C}$ be the contour given by $t + i|t|, t \in \mathbb{R}$. If we change variables by $z = i\zeta, w = -i\omega$, then γ' maps to $\mathcal{C}$ and Γ' to $-\mathcal{C}$ and we see that the right hand side of (96) becomes

$$-\frac{1}{4\pi^2} \int_{\mathcal{C}} d\zeta \int_{\mathcal{C}} d\omega e^{i\omega^3/3 + i\eta\omega + i\zeta^3/3 + i\xi\zeta} \frac{1}{i(\zeta + \omega)},$$

which is the Airy kernel $K_{Airy}(\xi, \eta)$, [15].

We have proved

Lemma 5.5 *Define*

$$\tilde{K}_N(\xi, \eta; y) = \frac{\alpha}{\sqrt{N}} K_N^* \left(v_c + \xi \frac{\alpha}{\sqrt{N}}, v_c + \eta \frac{\alpha}{\sqrt{N}} \right) \qquad (97)$$

with K_N^* as in (95) and v_c given by (79). Then

$$\lim_{N\to\infty} \tilde{K}_N(\xi, \eta; y) = K_{Airy}(\xi, \eta) \qquad (98)$$

uniformly for ξ, η in a compact set and $y \in B_N$.

To control the convergence of (84) we need some more estimates.

Lemma 5.6 *Fix a constant A. There is a constant C, depending on A, such that for $\xi, \eta \geq -A$ and all sufficiently large N we have the estimate*

$$\tilde{K}_N(\xi, \eta; y)| \leq C e^{-\frac{2}{3}(|\xi|^{3/2} + |\eta|^{3/2})} \qquad (99)$$

for $y \in B_N$.

Proof Deform the contour $C_3 - C_4$ to $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ and $C_1 - C_2$ to $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where $\gamma_1 : (-\infty, -\alpha\delta N^{-1/6}) \ni t \to w_c + t + it$, $\gamma_2 : (-\delta, \delta) \ni t \to w_c - \alpha\delta N^{-1/6} + \alpha it N^{-1/6}$, $\gamma_3 : (\alpha\delta N^{-1/6}, \infty) \ni t \to w_c - t + it$ and $\Gamma_1, \Gamma_2, \Gamma_3$ are obtained by reflection in the line Re $z = w_c$.

Note that for z on γ and w on Γ,

$$\frac{1}{|w-z|} \le \frac{N^{1/6}}{2\alpha\delta}$$

From (94), (95) and (97) we obtain

$$
\tilde{K}_N(\xi,\eta;y)| \le \frac{N^{1/3}}{8\alpha^2\delta\pi^2} \left(\int_\gamma e^{-\mathrm{Re}\,(f(z)-f(w_c))+\xi N^{1/6}\alpha^{-1}\mathrm{Re}\,(z-w_c)}|dz| \right)
$$
$$
\times \left(\int_\gamma e^{-\mathrm{Re}\,(f(w)-f(w_c))+\eta N^{1/6}\alpha^{-1}\mathrm{Re}\,(w-w_c)}|dw| \right). \qquad (100)
$$

On Γ_2 we can use Lemma 5.4 to get

$$\mathrm{Re}\,[f(w_c+\alpha(\delta+it)N^{-1/6})-f(w_c)] = \frac{1}{3}\delta^3 - \delta t^2 + o(1).$$

This gives

$$
\int_{\Gamma_2} e^{-\mathrm{Re}\,(f(w)-f(w_c))+\eta N^{1/6}\alpha^{-1}\mathrm{Re}\,(w-w_c)}|dw| \le \frac{C}{N^{1/6}}e^{\delta^3/3-\eta\delta} \int_{-\delta}^{\delta} e^{-\delta t^2}\,dt
$$
$$
\le \frac{C}{N^{1/6}\delta^{1/2}}e^{\delta^3/3-\eta\delta}.
$$

On Γ_3 we can use Lemma 5.3 to get

$$
\int_{\Gamma_3} e^{-\mathrm{Re}\,(f(w)-f(w_c))+\eta N^{1/6}\alpha^{-1}\mathrm{Re}\,(w-w_c)}|dw|
$$
$$
\le \int_{-\delta\alpha N^{-1/6}}^{\alpha N^{1/6}/2} e^{-cN^{1/2}t^3-\eta N^{1/6}t/\alpha}\,dt + \int_{\alpha N^{1/6}/2}^{\infty} e^{-cN^{5/6}t-\eta N^{1/6}t/\alpha}\,dt
$$
$$
\le \frac{C}{N^{1/6}}e^{-\eta\delta}.
$$

The contribution from Γ_1 is analogous. Choosing $\delta = \sqrt{\eta}$ for $\eta \ge 1$ and $\delta = 1$ otherwise, we get

$$
\int_\Gamma e^{-\mathrm{Re}\,(f(w)-f(w_c))+\eta N^{1/6}\alpha^{-1}\mathrm{Re}\,(w-w_c)}|dw| \le \frac{C}{N^{1/6}}e^{-\frac{2}{3}|\eta|^{3/2}}.
$$

The estimate for the other integral in (100) is analogous and the estimate (99) follows.   □

Define the distribution function H_N by

$$H_N(t; y) = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \det(\tilde{K}_N(\xi_i, \xi_j; y)) \mathrm{d}^k \xi. \tag{101}$$

It follows from Lemmas 5.5 and 5.6 that

$$\lim_{N \to \infty} H_N(t; y) = F_{\mathrm{TW}}(t) \tag{102}$$

uniformly for t in a compact subset and $y \in B_N$.

If we change from K_N to K_N^* in (84) and make the change of variables $x_i = v_c + \alpha \xi_i / \sqrt{N}$, we see that

$$F_N^*(t) = \mathbb{E}_y^{(N)} \left[\chi_{B_N}(y) G_N(\sqrt{N}(t - v_c)/\alpha; y) \right].$$

Thus

$$F_N^*(R(N) + \xi\alpha/\sqrt{N}) = \mathbb{E}_y^{(N)} [\chi_{B_N}(y) G_N(\xi - s_N(y); y)].$$

We can now use Lemma 5.2 and (102) to see that

$$\lim_{N \to \infty} F_N^*(R(N) + \xi\alpha/\sqrt{N}) = \int_{\mathbb{R}} F_{\mathrm{TW}}(\xi - u) h(u) \mathrm{d}u,$$

where $h(u) = (2\pi\sigma^2/\alpha^2)^{-1/2} \exp(-u^2\alpha^2/2\sigma^2)$. This completes the proof of theorem 1.12.

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