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Sandra Cerrai · Mark Freidlin

On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom

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Abstract. According to the Smolukowski-Kramers approximation, we show that the solution of the semi-linear stochastic damped wave equations $\mu u_{tt}(t,x) = \Delta u(t,x) - u_t(t,x) + b(x, u(t,x)) + Q\dot{W}(t)$, $u(0) = u_0$, $u_t(0) = v_0$, endowed with Dirichlet boundary conditions, converges as μ goes to zero to the solution of the semi-linear stochastic heat equation $u_t(t,x) = \Delta u(t,x) + b(x, u(t,x)) + Q\dot{W}(t)$, $u(0) = u_0$, endowed with Dirichlet boundary conditions. Moreover we consider relations between asymptotics for the heat and for the wave equation. More precisely we show that in the gradient case the invariant measure of the heat equation coincides with the stationary distributions of the wave equation, for any $\mu > 0$.

1. Introduction

The motion of a particle of a mass μ in the field $b(q) + \sigma(q)\dot{W}$ with the damping proportional to the speed (we put the coefficient equal to 1) is described, according to the Newton law, by the equation

$$\mu \ddot{q}_t^\mu = b(q_t^\mu) + \sigma(q_t^\mu)\dot{W}_t - \dot{q}_t, \quad q_0^\mu = q \in R^n, \quad \dot{q}_0^\mu = p \in R^n. \tag{1.1}$$

Here $b(q)$ is the deterministic component of the force and $\sigma(q)\dot{W}_t$, where $\dot{W}_t$ is the standard Gaussian white noise in R^n and $\sigma(q)$ is an $n \times n$-matrix, is the stochastic part. It is well known that, for $0 < \mu << 1$, q_t^μ can be approximated by the solution of the first order equation

$$\dot{q}_t = b(q_t) + \sigma(q_t)\dot{W}_t, \quad q_0 = q \in R^n, \tag{1.2}$$

in the sense that

$$\lim_{\mu \downarrow 0} P\{\max_{0 \le t \le T} |q_t^\mu - q_t| > \delta\} = 0, \tag{1.3}$$

S. Cerrai: Dip. di Matematica per le Decisioni, Università di Firenze, Via C. Lombroso 6/17, I-50134 Firenze, Italy

M. Freidlin: Department of Mathematics, University of Maryland, College Park, Maryland, USA

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for any $0 \leq T < \infty$ and $\delta > 0$. Statement (1.3) is called Smoluchowski-Kramers approximation of q_t^μ by q_t. This statement justifies the description of the motion of a small particle by the first order equation (1.2) instead of the second order equation (1.1).

Actually, the closeness of q_t^μ and q_t is not restricted to equality (1.3). If $b(q)$ is a potential vector, that is $b(q) = -\nabla U(q)$ for any $q \in R^n$, and if $\sigma(q) = I$ is the unit matrix, then the distribution with the density

$$m_\mu(q,p) = c_\mu \exp\{-(\mu|p|^2 + 2U(q))\}$$

(Boltzman distribution) is invariant for the $2n$-dimensional Markov process $X_t^\mu = (q_t^\mu, p_t^\mu)$, with $p_t^\mu = \dot{q}_t^\mu$, if

$$c_\mu := \left(\int_{R^n \times R^n} m_\mu(q,p)\, dq\, dp\right)^{-1} > 0.$$

To prove this one can check that $m_\mu(q,p)$ satisfies the stationary forward Kolmogorov equation associated with the process X_t^μ.

Then the stationary distribution of q_t^μ is equal to

$$m(q) = \int_{R^n} m_\mu(q,p) dp = \tilde{c} \exp\{-2U(q)\}.$$

On the other hand, $m(q)$ is the stationary density of the process q_t defined by (1.2) with $\sigma = I$. Thus, if $b(q) = -\nabla U(q)$ and $\sigma(q) \equiv I$, the stationary distributions of q_t^μ and q_t coincide for any $\mu > 0$. This means that, under certain conditions providing ergodicity, we can conclude that q_t^μ and q_t are close not just on finite time intervals, but also have similar long time behavior.

If $b(x)$ is not potential, then the invariant measures are not the same. The process $X_t^\mu = (q_t^\mu, p_t^\mu)$ may have no finite invariant measure when q_t has such a measure. For all details and proofs on this finite dimensional case we refer to [7]. We also refer to [16] and [17] for related problems in finite dimension.

In this paper we consider the equation

$$\begin{cases} \mu \frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) - \frac{\partial u}{\partial t}(t,x) + b(x, u(t,x)) \\ \qquad\qquad + \frac{\partial W^Q}{\partial t}(t,x), \quad t > 0, \quad x \in \mathcal{O}, \\ \\ u(0,x) = u_0, \quad \dfrac{\partial u}{\partial t}(0,x) = v_0, \quad u(t,x) = 0, \quad x \in \partial\mathcal{O}. \end{cases} \tag{1.4}$$

where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^d$, with $d \geq 1$. Here $W^Q(t,x)$ is a Gaussian mean zero random field, δ-correlated in time and the operator Q characterizes the correlation in the space variables (see below for detailed assumptions). In particular, in the one-dimensional case $W^Q(t,x)$ can be the Brownian sheet, so that $\frac{\partial^2 W^Q}{\partial t\, \partial x}(t,x)$ in this case is the space-time white noise.

Together with the semi-linear wave equation with the damping term (1.4), consider the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + b(x, u(t,x)) + \dfrac{\partial W^Q}{\partial t}(t,x), \quad t > 0, \ \ x \in \mathcal{O}, \\[10pt] u(0,x) = u_0, \qquad u(t,x) = 0, \quad x \in \partial \mathcal{O}. \end{cases} \tag{1.5}$$

Our first result concerns the convergence of $u^\mu(t,x)$ to $u(t,x)$, as $\mu \downarrow 0$. We prove this convergence in Section 4 under some natural assumptions. The proof follows, in general, the arguments used in the finite-dimensional case. But, of course, in the infinite-dimensional case we have to introduce appropriate functional spaces and obtain certain bounds uniform with respect to $\mu \in (0,1]$, whose proof requires some work. These auxiliary results together with some notations and assumptions are presented in Sections 2 and 3.

The results of these sections allow also to address the questions concerning the invariant measures and the long time behavior for $u^\mu(t,x)$ and $u(t,x)$. First, we give an explicit (in a sense) expression for the Boltzman distribution of the process $(u^\mu(t,x), \frac{\partial u^\mu}{\partial t}(t,x))$. Of course, since there is no universal measure in the functional space similar to the Lebesgue measure, we have to introduce an auxiliary Gaussian measure with respect to which one can write down the density of the Boltzman distribution. This auxiliary Gaussian measure is the stationary measure of the linear wave equation related to problem (1.4). Using the fact that the vector field $\mathcal{B}[u] := \Delta u + b(t,u)$ in the appropriate functional space is of gradient type, we can express the invariant density through the corresponding potential.

The explicit expression for the invariant measure of the process $(u^\mu, \frac{\partial u^\mu}{\partial t})$ allows to prove that $u^\mu(t,x)$ has the same stationary distributions for each $\mu > 0$, which coincide with the invariant measure of the process $u(t,x)$ defined as the unique solution of the heat equation (1.5).

The convergence result of Section 4 will be preserved if we replace the Laplacian $\triangle$ by any second order uniformly elliptic operator with sufficiently smooth coefficients. The results on stationary distributions and invariant measures of Section 5 can be generalized only to self-adjoint non-degenerate second order differential operators with regular coefficients. Actually, if the operator is not self-adjoint, the problem will not be of gradient type.

Now, let $q_t^{\mu,\varepsilon}$ and q_t^ε be the solutions of equations (1.1) and (1.2) with $\sigma(x) = \varepsilon I$, $0 < \varepsilon << 1$. Let a point $x_0 \in R^n$ be an asymptotically stable equilibrium of the field $b(x)$, and let a domain $G \subset R^n$ be attracted to x_0. Then, as it shown in [7] for the case of potential field $b(x)$, the asymptotics in the exit problem from G for the process $q_t^{\mu,\varepsilon}$ and q_t^ε as $\varepsilon \downarrow 0$ is, to some extend, the same. This fact also shows the advantage of the Smoluchowski-Kramers approximation. One can expect that a similar result holds for equations (1.4) and (1.5) with a small noise.

These problems, as well as questions related to ergodic properties of processes $u^\mu(t,x)$ and $u(t,x)$, will be addressed elsewhere.

2. Assumptions and notations

Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$, with $d \geq 1$, and assume that the boundary is of class C^3. In what follows we shall denote by H the Hilbert space $L^2(\mathcal{O})$ and by $\{e_k\}_{k\in\mathbb{N}}$ the complete orthonormal basis of H which diagonalizes the Laplace operator Δ, endowed with Dirichlet boundary conditions in $\mathcal{O}$. Moreover we shall denote by $\{-\alpha_k\}_{k\in\mathbb{N}}$ the corresponding sequence of eigenvalues.

As we are assuming the boundary of $\mathcal{O}$ to be smooth, for any $\delta \in (0,1)$ we have

$$|e_k|_{C^{2+\delta}(\bar{\mathcal{O}})} \leq c\,|\Delta e_k|_{C^\delta(\bar{\mathcal{O}})} = c\,\alpha_k\,|e_k|_{C^\delta(\bar{\mathcal{O}})}$$

(for a proof see [11, Theorem 6.3.2]). Moreover, by interpolation we have

$$|e_k|_{C^\delta(\bar{\mathcal{O}})} \leq c\,|e_k|_\infty^{2/2+\delta}|e_k|_{C^{2+\delta}(\bar{\mathcal{O}})}^{\delta/2+\delta}$$

(for a proof see [11, Lemma 6.3.1]). Then

$$|e_k|_{C^\delta(\bar{\mathcal{O}})} \leq c\,\alpha_k^{\delta/2+\delta}\,|e_k|_\infty^{2/2+\delta}|e_k|_{C^\delta(\bar{\mathcal{O}})}^{\delta/2+\delta},$$

so that

$$|e_k|_{C^\delta(\bar{\mathcal{O}})} \leq c\,\alpha_k^{\delta/2}\,|e_k|_\infty. \tag{2.1}$$

Hypothesis 1. *The bounded linear operator $Q : H \to H$ is diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$. If $\{\lambda_k\}_{k\in\mathbb{N}}$ denotes the corresponding sequence of eigenvalues, there exists a constant $\theta \in (0,1)$ such that*

$$\sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k^{1-\theta}}\,|e_k|_\infty^2 < \infty. \tag{2.2}$$

Hypothesis 2. *The mapping $b : \bar{\mathcal{O}}\times\mathbb{R}\to\mathbb{R}$ is measurable and*

$$\sup_{x\in\bar{\mathcal{O}}}|b(x,\sigma)-b(x,\rho)| \leq L\,|\sigma-\rho|, \qquad \sigma,\ \rho\in\mathbb{R},$$

for some positive constant L. Moreover

$$\sup_{x\in\bar{\mathcal{O}}}|b(x,0)| =: b_0 < \infty.$$

Remark 2.1. 1. In several cases, as for example in the case of space dimension $d = 1$ and in the case of the Laplace operator on the square with Dirichlet boundary conditions, the eigenfunctions e_k are equi-bounded in the sup-norm and then condition (2.2) becomes

$$\sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k^{1-\theta}} < \infty.$$

In general it holds

$$|e_k|_\infty \leq c\,k^\alpha, \qquad k\in\mathbb{N},$$

for some $\alpha \geq 0$. Thus, condition (2.2) is fulfilled if

$$\sum_{k=1}^\infty \frac{\lambda_k^2\,k^{2\alpha}}{\alpha_k^{1-\theta}} < \infty.$$

2. For any reasonable domain one has $\alpha_k \sim k^{2/d}$, for any $k \in \mathbb{N}$. Thus, in dimension $d = 1$ condition (2.2) is fulfilled by the white noise. As soon as one goes to higher dimension, this of course is no more possible. In any case, notice that if the sup-norms of the eigenfunctions e_k are equi-bounded, it is never required to have a noise with Hilbert-Schmidt covariance.

For any $\delta \in \mathbb{R}$ we denote by $H^\delta(\mathcal{O})$ the completion of $C_0^\infty(\mathcal{O})$ with respect to the norm

$$\|h\|^2_{H^\delta(\mathcal{O})} = \sum_{i=1}^\infty \alpha_i^\delta \langle h, e_i\rangle_H^2 = \sum_{i=1}^\infty \alpha_i^\delta h_i^2.$$

Note that here and in what follows for each $h \in H^\delta(\mathcal{O})$ we denote by h_k the k-th Fourier coefficient of h, that is

$$h_k = \langle h, e_k\rangle_H .$$

$H^\delta(\mathcal{O})$ is a Hilbert space, endowed with the scalar product

$$\langle h, k\rangle_{H^\delta(\mathcal{O})} = \sum_{i=1}^\infty \alpha_i^\delta h_i k_i, \quad h, k \in H^\delta(\mathcal{O}).$$

Moreover, for any $\delta \in \mathbb{R}$ we denote by $\mathcal{H}_\delta$ the Hilbert space $H^\delta(\mathcal{O}) \times H^{\delta-1}(\mathcal{O})$, endowed with the natural scalar product and norm inherited from each component.

Next, for any $\mu > 0$ and $\delta \in \mathbb{R}$ we define on $\mathcal{H}_\delta$ the unbounded operator A_μ by setting

$$A_\mu(h, k) = \frac{1}{\mu}\left(\mu k, \Delta h - k\right), \quad (h, k) \in D(A_\mu) := \mathcal{H}_{\delta+1}.$$

Here for the sake of simplicity we have not written the dependence of A_μ on δ, as the operators A_μ defined on different $\mathcal{H}_\delta$ are all consistent. It is known that A_μ is the generator of a group of bounded linear transformations $\{S_\mu(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}_\delta$ which is strongly continuous (for a proof see e.g. [18, section 7.4]).

Note that the adjoint operator to A_μ is given by

$$A_\mu^\star(h, k) = \frac{1}{\mu}\left(-k, -\mu\Delta h - k\right), \quad (h, k) \in D(A_\mu^\star) := \mathcal{H}_{\delta+1}.$$

In what follows we shall denote by $\{S_\mu^\star(t)\}_{\{t\geq 0\}}$ the semigroup generated by $A_\mu^\star$.

Clearly, for any $(u_0, v_0) \in \mathcal{H}_\delta$ and for any $\mu > 0$, $S_\mu(t)(u_0, v_0)$ is the solution of the deterministic linear system

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = v(t, x), \quad \mu\dfrac{\partial v}{\partial t}(t, x) = \Delta u(t, x) - v(t, x), \quad t > 0, \ \ x \in \mathcal{O}, \\ \\ u(0) = u_0, \quad v(0) = v_0, \quad u(t, x) = 0, \quad t \geq 0, \ \ x \in \partial\mathcal{O}, \end{cases}$$

which can be written as the following abstract evolution problem in $\mathcal{H}_\delta$

$$\frac{dz}{dt}(t) = A_\mu z(t), \qquad z(0) = (u_0, v_0),$$

where $z(t) := (u(t), v(t))$.

Our aim now is giving an explicit expression both of $S_\mu(t)(u, v)$ and of $S^\star_\mu(t)(u, v)$, for any $t \geq 0$ and $(u, v) \in \mathcal{H}_\delta$.

By writing $S_\mu(t)(u, v)$ in Fourier coefficients, if we set $\Pi_1(u, v) := u$ and $\Pi_2(u, v) := v$ we have that

$$\Pi_1\, S_\mu(t)(u, v) = \sum_{k=1}^\infty f_k^\mu(t) e_k, \qquad \Pi_2\, S_\mu(t)(u, v) = \sum_{k=1}^\infty g_k^\mu(t) e_k,$$

where the pair $(f_k^\mu(t), g_k^\mu(t))$ is for each $k \in \mathbb{N}$ and $\mu > 0$ the solution of the system

$$\begin{cases} f'(t) = g(t), \qquad f(0) = u_k \\ \\ \mu\, g'(t) = -\alpha_k\, f(t) - g(t), \qquad g(0) = v_k. \end{cases} \tag{2.3}$$

In the next proposition we provide an explicit formula for f_k^μ and g_k^μ.

Proposition 2.2. *For any $\mu > 0$ and $k \in \mathbb{N}$, let us define*

$$\gamma_k^\mu := \frac{1}{2\mu}\sqrt{1 - 4\alpha_k \mu}.$$

Then, we have

$$\begin{aligned} f_k^\mu(t) = \frac{1}{2} \exp\left(-\frac{t}{2\mu}\right) \Bigg(\Bigg[\left(1 + \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(\gamma_k^\mu t\right) \\ + \left(1 - \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(-\gamma_k^\mu t\right) \Bigg]\, u_k \\ + \frac{1}{\gamma_k^\mu} \left[\exp\left(\gamma_k^\mu t\right) - \exp\left(-\gamma_k^\mu t\right)\right]\, v_k \Bigg), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} g_k^\mu(t) = \frac{1}{2} \exp\left(-\frac{t}{2\mu}\right) \Bigg(-\frac{\alpha_k}{\mu\gamma_k^\mu} \left[\exp\left(\gamma_k^\mu t\right) - \exp\left(-\gamma_k^\mu t\right)\right] u_k \\ + \left[\left(1 - \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(\gamma_k^\mu t\right) + \left(1 + \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(-\gamma_k^\mu t\right) \right] v_k \Bigg), \end{aligned} \tag{2.5}$$

where, in the case $\gamma_k^\mu = 0$, we have set

$$\frac{1}{\gamma_k^\mu} \left[\exp\left(\gamma_k^\mu t\right) - \exp\left(-\gamma_k^\mu t\right)\right] = 2t.$$

Proof. Differentiating the second equation in system (2.3) we have

$$\mu\,\frac{d^2 g_k^\mu}{dt^2}(t) = -\alpha_k\,\frac{df_k^\mu}{dt}(t) - \frac{dg_k^\mu}{dt}(t) = -\alpha_k\, g_k^\mu(t) - \frac{dg_k^\mu}{dt}(t).$$

Thus, by taking into account the initial conditions, by standard computations we obtain formulas (2.4) and (2.5). □

Next, we show that we can express $S^\star_\mu(t)$ in terms of $S_\mu(t)$.

Proposition 2.3. *For any $\mu > 0$ and $(u, v) \in \mathcal{H}_\delta$ we have*

$$S^\star_\mu(t)(u, v) = \left(\Pi_1\, S_\mu(t)\,(u, -v/\mu)\,, \Pi_2\, S_\mu(t)\,(-\mu u, v)\right), \qquad t \geq 0. \tag{2.6}$$

Proof. If we write $S^\star_\mu(t)(u, v)$ in Fourier coefficients, we have

$$\Pi_1\, S^\star_\mu(t)(u, v) = \sum_{k=1}^\infty \hat{f}_k^\mu(t) e_k, \qquad \Pi_2\, S^\star_\mu(t)(u, v) = \sum_{k=1}^\infty \hat{g}_k^\mu(t) e_k,$$

where the pair $(\hat{f}_k^\mu(t), \hat{g}_k^\mu(t))$ is for each $k \in \mathbb{N}$ and $\mu > 0$ the solution of the system

$$\begin{cases} \mu f'(t) = -g(t), \qquad f(0) = u_k \\ \\ \mu\, g'(t) = \mu\, \alpha_k\, f(t) - g(t), \qquad g(0) = v_k. \end{cases}$$

This means that the pair $(-\mu \hat{f}_k^\mu(t), \hat{g}_k^\mu(t))$ is the solution of system (2.3) with initial conditions $(-\mu u_k, v_k)$, so that for any $t \geq 0$ we have

$$\hat{f}_k^\mu(t) = -\frac{1}{\mu}\left[\Pi_1 S_\mu(t)(-\mu u, v)\right]_k, \qquad \hat{g}_k^\mu(t) = \left[\Pi_2 S_\mu(t)(-\mu u, v)\right]_k.$$

This allows us to conclude, as

$$\begin{aligned} \left[\Pi_1 S^\star_\mu(t)(u, v)\right]_k &= \hat{f}_k^\mu(t) = -\frac{1}{\mu}\left[\Pi_1 S_\mu(t)(-\mu u, v)\right]_k \\ &= \left[\Pi_1 S_\mu(t)\,(u, -v/\mu)\right]_k, \end{aligned}$$

and

$$\left[\Pi_2 S^\star_\mu(t)(u, v)\right]_k = \hat{g}_k^\mu(t) = \left[\Pi_2 S_\mu(t)\,(-\mu u, v)\right]_k.$$

□

Finally, an important consequence of Proposition 2.2 is the following result on the asymptotic behavior of $S_\mu(t)$.

Proposition 2.4. *For any fixed $\mu > 0$ and any $\delta \in \mathbb{R}$, the semigroup $\{S_\mu(t)\}_{t\geq 0}$ is of negative type in $\mathcal{H}_\delta$, that is there exist some $\omega_\mu > 0$ and $M_\mu > 0$ such that*

$$\|S_\mu(t)\|_{\mathcal{L}(\mathcal{H}_\delta)} \leq M_\mu\, e^{-\omega_\mu t}, \qquad t \geq 0. \tag{2.7}$$

Proof. Fix $\mu > 0$. Multiplying the second equation in (2.3) by $g_k^\mu(t)$ we get

$$\mu \frac{d\,|g_k^\mu|^2}{dt}(t) + \alpha_k \frac{d\,|f_k^\mu|^2}{dt}(t) + 2\,|g_k^\mu(t)|^2 = 0,$$

and hence, integrating with respect to $t \geq 0$ and multiplying both sides by $\alpha_k^{\delta-1}$, we get

$$\begin{aligned}&\mu\,\alpha_k^{\delta-1}\,|g_k^\mu(t)|^2 + \alpha_k^\delta\,|f_k^\mu(t)|^2 + 2\,\alpha_k^{\delta-1}\int_0^t |g_k^\mu(s)|^2\,ds\\ &\quad = \mu\,\alpha_k^{\delta-1}\,|v_k|^2 + \alpha_k^\delta\,|u_k|^2.\end{aligned} \tag{2.8}$$

Now, in order to prove (2.7), we note that thanks to Proposition 2.2 for any constant $c > 0$ and any $k \in \mathbb{N}$

$$\lim_{t\to\infty}\sup_{|u_k|+|v_k|\leq c}|f_k^\mu(t)| = \lim_{t\to\infty}\sup_{|u_k|+|v_k|\leq c}|g_k^\mu(t)| = 0.$$

Thus, according to (2.8) we can conclude that for any fixed $\mu > 0$

$$\lim_{t\to\infty}\|S_\mu(t)\|_{\mathcal{H}_\delta} = 0.$$

As a consequence of the Datko theorem (see [1] for a proof) this yields (2.7). □

3. Estimates for the stochastic convolution

For each $\mu > 0$, let us consider the linear problem

$$\begin{cases}\mu\dfrac{\partial^2\eta}{\partial t^2}(t,x) = \Delta\eta(t,x) - \dfrac{\partial\eta}{\partial t}(t,x) + \dfrac{\partial W^Q}{\partial t}(t,x), & t > 0,\ \ x \in \mathcal{O},\\[2ex] \eta(0) = 0, \quad \dfrac{\partial\eta}{\partial t}(0) = 0, \quad \eta(t,x) = 0, & t \geq 0,\ \ x \in \partial\mathcal{O},\end{cases} \tag{3.1}$$

where W^Q is the noise with covariance given by

$$\mathbb{E}\left\langle W^Q(t), h\right\rangle_H \left\langle W^Q(s), k\right\rangle_H = (t \wedge s)\,\langle Qh, k\rangle_H\,.$$

Note that $W^Q(t)$ is formally defined as

$$W^Q(t) = \sum_{k=1}^\infty Qe_k\,\beta_k(t) = \sum_{k=1}^\infty \lambda_k e_k\,\beta_k(t), \qquad t \geq 0,$$

where $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions, all defined on some complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.
It is well known that if for some $\theta \in \mathbb{R}$ condition (2.2) holds, then for any $\mu > 0$ there exists a unique solution η^μ to problem (3.1) such that for any $T > 0$ and $p \geq 1$

$$\eta^\mu \in L^p(\Omega; C([0,T]; H^\theta(\mathcal{O}))), \qquad \frac{\partial\eta^\mu}{\partial t} \in L^p(\Omega; C([0,T]; H^{\theta-1}(\mathcal{O}))) \tag{3.2}$$

(for a proof we refer for example to [4] and [9], see also [3]).

Our aim here is proving that if the constant θ above is strictly positive (as in Hypothesis 1), then for any $\delta < \theta/2$ the process η^μ has a version which is δ-Hölder continuous with respect to $t \geq 0$ and $\xi \in \bar{\mathcal{O}}$ and the momenta of the δ-Hölder norms of η^μ are equi-bounded with respect to $\mu > 0$. Namely we prove the following result.

Proposition 3.1. *Assume that Hypothesis 1 is satisfied. Then for any $\mu > 0$ and $\delta < \theta/2$ the process η^μ has a version (which we still denote by η^μ) which is δ-Hölder continuous with respect to $(t,x) \in [0,T] \times \bar{\mathcal{O}}$, for any $T > 0$.*

Moreover, for any $p \geq 1$

$$\sup_{\mu>0} \mathbb{E}\,|\eta^\mu|^p_{C^\delta([0,T]\times\bar{\mathcal{O}})} =: c_{T,p} < \infty. \tag{3.3}$$

Proof. For all $(t,x) \in [0,\infty) \times \bar{\mathcal{O}}$ we have

$$\eta^\mu(t,x) = \sum_{k=1}^\infty \eta_k^\mu(t) e_k(x), \tag{3.4}$$

where, for each $k \in \mathbb{N}$, $\eta_k^\mu(t)$ is the solution of the one dimensional problem

$$\begin{cases} d\eta_k^\mu(t) = \theta_k^\mu(t)\,dt \\ \\ \mu\, d\theta_k^\mu(t) = -\left(\alpha_k \eta_k^\mu(t) + \theta_k^\mu(t)\right)\,dt + \lambda_k\,d\beta_k(t), \\ \\ \eta_k^\mu(0) = 0, \qquad \theta_k^\mu(0) = 0. \end{cases} \tag{3.5}$$

Then, by the variation of constants formula, it is immediate to check that

$$\eta_k^\mu(t) = \frac{\lambda_k}{\mu}\int_0^t f_k^\mu(t-s)\,d\beta_k(s), \tag{3.6}$$

with f_k^μ defined as the solutions of the system (2.3) with initial conditions $f_k^\mu(0) = 0$ and $g_k^\mu(0) = 1$.

Therefore, since for any $t, s \geq 0$ and $x, y \in \bar{\mathcal{O}}$ the random variable $\eta^\mu(t,x) - \eta^\mu(s,y)$ is Gaussian, the proof of (3.3) is a consequence of the following lemma and of the Garcia-Rademich-Rumsey theorem. □

Lemma 3.2. *Under Hypothesis 1 there exists a constant $c > 0$ such that*

$$\sup_{\mu>0} \mathbb{E}\,|\eta^\mu(t,x) - \eta^\mu(s,y)|^2 \leq c\left(|t-s|^\theta + |x-y|^{2\theta}\right), \tag{3.7}$$

for any $t, s \geq 0$ and $x, y \in \bar{\mathcal{O}}$.

Proof. *First step.* There exists $c_1 > 0$ such that for any $t \geq 0$ and $x, y \in \bar{\mathcal{O}}$

$$\sup_{\mu>0} \mathbb{E}\,|\eta^\mu(t,x) - \eta^\mu(t,y)|^2 \leq c_1\,|x-y|^{2\theta}. \tag{3.8}$$

Due to (3.4) and (3.6), for any $t \geq 0$ and $x, y \in \bar{\mathcal{O}}$ we have

$$\eta^\mu(t,x) - \eta^\mu(t,y) = \sum_{k=1}^\infty \frac{\lambda_k}{\mu} \int_0^t f_k^\mu(t-s)\, d\beta_k(s)\, [e_k(x) - e_k(y)],$$

so that

$$\mathbb{E}\, |\eta^\mu(t,x) - \eta^\mu(t,y)|^2 = \sum_{k=1}^\infty \frac{\lambda_k^2}{\mu^2} \int_0^t |f_k^\mu(s)|^2\, ds\, |e_k(x) - e_k(y)|^2 .$$

Hence, due to (2.1)

$$\mathbb{E}\, |\eta^\mu(t,x) - \eta^\mu(t,y)|^2 \leq c \sum_{k=1}^\infty \frac{\lambda_k^2 \alpha_k^\theta}{\mu^2}\, |e_k|_\infty^2 \int_0^t |f_k^\mu(s)|^2\, ds\, |x-y|^{2\theta}. \quad (3.9)$$

Now, in order to estimate the series above we can assume $\mu \neq 1/4\alpha_k$, for any $k \in \mathbb{N}$. Actually, if we can prove the upper bound

$$\sup_{\mu \neq 1/4\alpha_k} \frac{1}{\mu^2} \int_0^t |f_k^\mu(s)|^2\, ds =: c_k < \infty,$$

since

$$\lim_{\mu \to 1/4\alpha_k} f_k^\mu(t) = f_k^{1/4\alpha_k}(t), \quad t \geq 0, \tag{3.10}$$

due to the Fatou lemma we have the same upper bound for any $\mu > 0$.

As we are assuming $\mu \neq 1/4\alpha_k$, with a change of variable we have

$$\begin{aligned}
\mathbb{E}\, |\eta_k^\mu(t)|^2 &= \frac{\lambda_k^2}{\mu^2} \int_0^t \left|f_k^\mu(s)\right|^2\, ds \\
&= \frac{\lambda_k^2}{|2\mu\, \gamma_k^\mu|^2} \int_0^t \exp\left(-\frac{s}{\mu}\right) \left|\exp(\gamma_k^\mu s) - \exp(-\gamma_k^\mu s)\right|^2\, ds \\
&= \frac{\lambda_k^2\, \mu}{|1 - 4\alpha_k \mu|} \int_0^{\frac{t}{\mu}} \exp\left(-\left(1 - \sqrt{(1-4\alpha_k\mu)^+}\right) s\right) \\
&\quad \times \left|1 - \exp(-2\mu\, \gamma_k^\mu s)\right|^2\, ds.
\end{aligned}$$

If $0 < \sqrt{(1-4\alpha_k\mu)^+} \leq 1/2$, we have

$$\exp\left(-\left(1 - \sqrt{(1-4\alpha_k\mu)^+}\right) s\right) \frac{\left|1 - \exp(-2\mu\, \gamma_k^\mu s)\right|^2}{|1 - 4\alpha_k\mu|} \leq c\, \exp\left(-\frac{s}{2}\right) s^2,$$

so that

$$\mathbb{E}\, |\eta_k^\mu(t)|^2 \leq c\, \lambda_k^2\, \mu \int_0^\infty \exp\left(-\frac{s}{2}\right) s^2\, ds = c\, \lambda_k^2\, \mu.$$

Since

$$\frac{\left|1-\exp(-2\mu\,\gamma_k^\mu s)\right|^2}{|1-4\alpha_k\mu|}\leq c,$$

if $\sqrt{(1-4\alpha_k\mu)^+}>1/2$ we have

$$\begin{aligned}\mathbb{E}\,|\eta_k^\mu(t)|^2&\leq c\,\lambda_k^2\,\mu\int_0^\infty \exp\left(-\left(1-\sqrt{(1-4\alpha_k\mu)^+}\right)s\right)\,ds\\&=\frac{c\,\lambda_k^2}{\alpha_k}\,\left(1+\sqrt{(1-4\alpha_k\mu)^+}\right).\end{aligned}$$

Finally, if $\sqrt{(1-4\alpha_k\mu)^+}=0$ we have

$$\begin{aligned}&\exp\left(-\left(1-\sqrt{(1-4\alpha_k\mu)^+}\right)s\right)\frac{\left|1-\exp(-2\mu\,\gamma_k^\mu s)\right|^2}{|1-4\alpha_k\mu|}\\&=2\,\exp(-s)\,\frac{1-\cos\sqrt{(1-4\alpha_k\mu)^-}s}{(1-4\alpha_k\mu)^-}.\end{aligned}$$

Thus, since for any $\delta\in\,[0,2]$ there exists $c_\delta>0$ such that

$$1-\cos\beta\leq c_\delta\frac{\beta^\delta}{\beta^\delta\vee 1},\qquad \beta>0, \tag{3.11}$$

for $\delta=2$ we have

$$\begin{aligned}\mathbb{E}\,|\eta_k^\mu(t)|^2&\leq\frac{c\lambda_k^2}{\alpha_k}\,\int_0^{\frac{t}{\mu}}\exp\left(-s\right)\,\frac{4\alpha_k\mu\,s^2}{(1-4\alpha_k\mu)^-s^2\vee 1}ds\\&\leq\frac{c\lambda_k^2}{\alpha_k}\,\int_0^\infty\exp\left(-s\right)\left(1+s^2\right)\,ds.\end{aligned}$$

Therefore, in all these three cases we obtain

$$\mathbb{E}\,|\eta_k^\mu(t)|^2\leq\frac{c\lambda_k^2}{\alpha_k}, \tag{3.12}$$

and hence, according to (3.9), we obtain (3.8).

Second step. There exists a constant $c_2>0$ such that for any $t,s\geq 0$ and $x\in\overline{\mathcal{O}}$

$$\sup_{\mu>0}\mathbb{E}\,|\eta^\mu(t,x)-\eta^\mu(s,x)|^2\leq c_2|t-s|^\theta. \tag{3.13}$$

We can assume $t>s$. As a consequence of (3.6), for any $x\in\bar{\mathcal{O}}$ we have

$$\begin{aligned}\eta^\mu(t,x)-\eta^\mu(s,x)=\sum_{k=1}^\infty\frac{\lambda_k}{\mu}&\left(\int_0^t f_k^\mu(t-r)\,d\beta_k(r)\right.\\&\left.-\int_0^s f_k^\mu(s-r)\,d\beta_k(r)\right)e_k(x)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{\lambda_k}{\mu} \left(\int_s^t f_k^\mu(t-r)\, d\beta_k(r) \right.\\
&\quad \left. + \int_0^s \left(f_k^\mu(t-r) - f_k^\mu(s-r) \right) \, d\beta_k(r) \right) e_k(x),
\end{aligned}
$$

and hence we have

$$
\begin{aligned}
&\mathbb{E} \left| \eta^\mu(t,x) - \eta^\mu(s,x) \right|^2 \\
&= \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\mu^2} \int_s^t |f_k^\mu(t-r)|^2 \, dr \, |e_k|_\infty^2 \\
&\quad + \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\mu^2} \int_0^s \left| f_k^\mu(t-r) - f_k^\mu(s-r) \right|^2 \, dr \, |e_k|_\infty^2 \\
&=: \sum_{k=1}^{\infty} I_k^\mu \, |e_k|_\infty^2 + \sum_{k=1}^{\infty} J_k^\mu \, |e_k|_\infty^2. \qquad (3.14)
\end{aligned}
$$

Concerning the terms I_k^μ, since for any $\delta \in [0,1]$ there exists $c_\delta > 0$ such that

$$
1 - \exp(-\beta) \le c_\delta \beta^\delta, \qquad \beta > 0, \qquad (3.15)
$$

due to (3.12) we have

$$
\begin{aligned}
I_k^\mu &= \frac{\lambda_k^2}{\mu^2} \int_0^{t-s} |f_k^\mu(r)|^2 \, dr \\
&\le \frac{c\lambda_k^2}{\alpha_k} \left[1 - \exp\left(-\left(\frac{1 - \sqrt{(1-4\alpha_k\mu)^+}}{2\mu} \right)(t-s) \right) \right] \\
&\le \frac{c\lambda_k^2}{\alpha_k} \left(\frac{1 - (1-4\alpha_k\mu)^+}{2\mu(1+\sqrt{(1-4\alpha_k\mu)^+})} \right)^\theta (t-s)^\theta \le \frac{c\lambda_k^2}{\alpha_k^{1-\theta}} (t-s)^\theta. \qquad (3.16)
\end{aligned}
$$

Now we go to the estimate of the terms J_k^μ, which is more delicate. As in the first step, due to (3.10) we can assume that $4\alpha_k\mu \neq 1$. We have

$$
\begin{aligned}
&2\gamma_k^\mu \left(f_k^\mu(t-r) - f_k^\mu(s-r) \right) \\
&= \exp\left(-\frac{s-r}{2\mu} \right) \left[\exp(\gamma_k^\mu(s-r)) \right.\\
&\quad \left. - \exp(-\gamma_k^\mu(s-r)) \right] \left[\exp\left(-\frac{(1-2\gamma_k^\mu\mu)}{2\mu}(t-s) \right) - 1 \right] \\
&\quad + \exp\left(-\frac{t-s}{2\mu} \right) \left[\exp(\gamma_k^\mu(t-s)) \right.\\
&\quad \left. - \exp(-\gamma_k^\mu(t-s)) \right] \exp\left(-\frac{(1+2\gamma_k^\mu\mu)}{2\mu}(s-r) \right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
J_k^\mu \leq & \frac{c\lambda_k^2}{|2\gamma_k^\mu \mu|^2} \int_0^s \exp\left(-\frac{s-r}{\mu}\right) \left|\exp(\gamma_k^\mu(s-r)) - \exp(-\gamma_k^\mu(s-r))\right|^2 \, dr \\
& \times \left|\exp\left(-\frac{(1-2\gamma_k^\mu \mu)}{2\mu}(t-s)\right) - 1\right|^2 \\
& + \frac{c\lambda_k^2}{|2\gamma_k^\mu \mu|^2} \int_0^s \exp\left|-\frac{(1+2\gamma_k^\mu \mu)}{\mu}(s-r)\right| \, dr \\
& \times \exp\left(-\frac{t-s}{\mu}\right) \left|\exp(\gamma_k^\mu(t-s)) - \exp(-\gamma_k^\mu(t-s))\right|^2 =: (J_k^\mu)_1 + (J_k^\mu)_2.
\end{aligned}
$$

We estimate separately the terms $(J_k^\mu)_1$ and $(J_k^\mu)_2$. Since

$$
\frac{\lambda_k^2}{|2\gamma_k^\mu \mu|^2} \int_0^s \exp\left(-\frac{r}{\mu}\right) \left|\exp(\gamma_k^\mu r) - \exp(-\gamma_k^\mu r)\right|^2 \, dr = \mathbb{E}\, |\eta_k^\mu(s)|^2,
$$

according to (3.12) we have

$$
(J_k^\mu)_1 \leq \frac{c\lambda_k^2}{\alpha_k} \left|1 - \exp\left(-\frac{(1-2\gamma_k^\mu \mu)}{2\mu}(t-s)\right)\right|^2 .
$$

It is not difficult to check that if $z \in \mathbb{C}$

$$
|1-\exp(-z)|^2 = (1-\exp(-\operatorname{Re} z))^2 + 2\, \exp(-\operatorname{Re} z)\, (1-\cos \operatorname{Im} z)\, . \tag{3.17}
$$

Then we have

$$
\begin{aligned}
& \left|1 - \exp\left(-\frac{(1-2\gamma_k^\mu \mu)}{2\mu}(t-s)\right)\right|^2 \\
& \quad \leq \left[1 - \exp\left(-\left(\frac{1-\sqrt{(1-4\alpha_k\mu)^+}}{2\mu}\right)(t-s)\right)\right]^2 \\
& \quad\quad + 2\left(1 - \cos \frac{\sqrt{(1-4\alpha_k\mu)^-}}{2\mu}(t-s)\right).
\end{aligned}
$$

Then, from (3.15) and (3.11) it follows

$$
\begin{aligned}
(J_k^\mu)_1 & \leq \frac{c\lambda_k^2}{\alpha_k} \left[\left(\frac{1-\sqrt{(1-4\alpha_k\mu)^+}}{2\mu}\right)^\theta + \left(\frac{\sqrt{(1-4\alpha_k\mu)^-}}{2\mu}\right)^\theta\right] (t-s)^\theta \\
& \leq \frac{c\lambda_k^2}{\alpha_k^{1-\theta}}\, (t-s)^\theta.
\end{aligned} \tag{3.18}
$$

Concerning $(J_k^\mu)_2$ we have

$$
\begin{aligned}
(J_k^\mu)_2 \leq\ & \frac{c\lambda_k^2}{|2\gamma_k^\mu \mu|^2}\,\frac{\mu}{1+\operatorname{Re}2\gamma_k^\mu\mu}\exp\left(-\frac{t-s}{\mu}\right)\left|\exp(\gamma_k^\mu(t-s)) -\right. \\
& \qquad \left.\exp(-\gamma_k^\mu(t-s))\right|^2 \\
=\ & \frac{c\lambda_k^2\mu}{|1-4\alpha_k\mu|(1+\sqrt{(1-4\alpha_k\mu)^+})} \\
& \qquad \exp\left(-\left(\frac{1-\sqrt{(1-4\alpha_k\mu)^+}}{\mu}\right)(t-s)\right) \\
& \quad \times\left|1-\exp\left(-2\gamma_k^\mu(t-s)\right)\right|^2.
\end{aligned}
$$

Then, due to (3.17), (3.15) and (3.11) for any $\delta \in [0,2]$ we have

$$
\begin{aligned}
(J_k^\mu)_2 \leq\ & \frac{c_\delta\,\lambda_k^2\mu^{1-\delta}}{1+\sqrt{(1-4\alpha_k\mu)^+}}\exp\left(-\left(\frac{1-\sqrt{(1-4\alpha_k\mu)^+}}{\mu}\right)(t-s)\right)(t-s)^\delta \\
& \times\left[\frac{\left((1-4\alpha_k\mu)^+\right)^{\delta/2}}{|1-4\alpha_k\mu|}\right. \\
& \quad \left. +\frac{\left((1-4\alpha_k\mu)^-\right)^{\delta/2}}{|1-4\alpha_k\mu|}\left(\left(\frac{(1-4\alpha_k\mu)^-}{\mu^2}\right)^{\delta/2}(t-s)^\delta\vee 1\right)^{-1}\right].
\end{aligned}
\tag{3.19}
$$

If we take $\delta=\theta$ in (3.19) and assume $4\alpha_k\mu \in (0,1/2]$, we have

$$
(J_k^\mu)_2 \leq \frac{c_\theta\,\lambda_k^2\mu^{1-\theta}}{(1-4\alpha_k\mu)^{1-\theta/2}}(t-s)^\theta \leq \frac{c\,\lambda_k^2}{\alpha_k^{1-\theta}}(t-s)^\theta. \tag{3.20}
$$

If we take $\delta=2$ in (3.19) and assume $4\alpha_k\mu \in (1/2,1)$, we have

$$
(J_k^\mu)_2 \leq \frac{c\lambda_k^2}{\mu(1+\sqrt{1-4\alpha_k\mu})}(t-s)^2\exp\left(-\frac{(1-\sqrt{1-4\alpha_k\mu})}{\mu}(t-s)\right).
$$

Now, we remark that for any $\beta>0$ there exists a constant $c_\beta>0$ such that

$$
s^\beta e^{-s} \leq c_\beta, \qquad s\geq 0, \tag{3.21}
$$

so that

$$
\exp\left(-\frac{(1-\sqrt{1-4\alpha_k\mu})}{\mu}(t-s)\right)(t-s)^{2-\theta} \leq c\left(\frac{1+\sqrt{1-4\alpha_k\mu}}{4\alpha_k\mu}\right)^{2-\theta}\mu^{2-\theta}.
$$

As $4\alpha_k\mu > 1/2$ this yields

$$
\begin{aligned}
(J_k^\mu)_2 &\leq \frac{c\lambda_k^2}{\mu(1+\sqrt{1-4\alpha_k\mu})}(t-s)^\theta\left(\frac{1+\sqrt{1-4\alpha_k\mu}}{4\alpha_k\mu}\right)^{2-\theta}\mu^{2-\theta} \\
&\leq c\lambda_k^2\mu^{1-\theta}(t-s)^\theta,
\end{aligned}
$$

and hence, as $4\alpha_k\mu<1$, (3.20) follows.

Next, if we take $\delta = 2$ in (3.19) and assume $4\alpha_k\mu \in (1, 2)$, due to (3.21) we get

$$(J_k^\mu)_2 \leq \frac{c\lambda_k^2}{\mu} \exp\left(-\frac{t-s}{\mu}\right)(t-s)^2 \leq \frac{c\lambda_k^2}{\alpha_k^{1-\theta}}(t-s)^\theta.$$

Finally, if we assume $4\alpha_k\mu \geq 2$, by taking again $\delta = 2$ in (3.19) we have

$$\begin{aligned}(J_k^\mu)_2 &\leq \frac{c\lambda_k^2}{\mu} \exp\left(-\frac{t-s}{\mu}\right)(t-s)^2 \left(\frac{(4\alpha_k\mu - 1)}{\mu^2}(t-s)^2 \vee 1\right)^{-1} \\ &\leq \frac{c\lambda_k^2\mu^{1-\theta}}{(\alpha_k\mu)^{1-\theta/2}}(t-s)^\theta \leq \frac{c\lambda_k^2}{\alpha_k^{1-\theta}}(t-s)^\theta,\end{aligned}$$

so that (3.20) holds.

According to (3.14), thanks to (3.18) and (3.20) we obtain (3.13).

Conclusion. Estimate (3.7) follows combining together (3.8) and (3.13). □

4. The convergence result

In this section we are concerned with the stochastic semi-linear damped wave equation

$$\begin{cases} \mu\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) - \frac{\partial u}{\partial t}(t,x) + b(x,u(t,x)) \\ \qquad\qquad + \frac{\partial W^Q}{\partial t}(t,x), \quad t > 0, \ x \in \mathcal{O}, \\ u(0) = u_0, \quad \dfrac{\partial u}{\partial t}(0) = v_0, \quad u(t,x) = 0, \quad t \geq 0, \ x \in \partial\mathcal{O}. \end{cases} \tag{4.1}$$

Our aim is proving that the solution $u^\mu(t)$ converges to the solution of the stochastic semi-linear heat equation

$$\begin{cases} \frac{\partial z}{\partial t}(t,x) = \Delta z(t,x) + b(x,z(t,x)) \\ \qquad\qquad + \frac{\partial W^Q}{\partial t}(t,x), \quad t > 0, \ x \in \mathcal{O}, \\ z(0) = u_0, \quad z(t,x) = 0, \quad t \geq 0, \ x \in \partial\mathcal{O}, \end{cases} \tag{4.2}$$

as the parameter μ converges to zero.

For any $\mu > 0$ and $\delta \in [0, 1]$ we define the operators

$$B_\mu(h,k)(x) := \frac{1}{\mu}(0, b(x,h(x))), \quad (h,k) \in \mathcal{H}_\delta, \quad x \in \mathcal{O}, \tag{4.3}$$

and

$$Q_\mu h = \frac{1}{\mu}(0, Qh), \quad h \in H. \tag{4.4}$$

Note that, since $\delta \in [0, 1]$, for any $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2) \in \mathcal{H}_\delta$

$$|B_\mu(z_1) - B_\mu(z_2)|_{\mathcal{H}_\delta} = \frac{1}{\mu}|b(\cdot,u_1) - b(\cdot,u_2)|_{H^{\delta-1}(\mathcal{O})} \leq \frac{c}{\mu}|b(\cdot,u_1) - b(\cdot,u_2)|_H,$$

and then, thanks to Hypothesis 2

$$|B_\mu(z_1) - B_\mu(z_2)|_{\mathcal{H}_\delta} \leq \frac{c\,L}{\mu}|u_1 - u_2|_H \leq \frac{c\,L}{\mu}|z_1 - z_2|_{\mathcal{H}_\delta}. \tag{4.5}$$

Definition 4.1. *Let $\delta \in [0, 1]$. A process u^μ is a* mild solution *of problem (4.1) if*

$$u^\mu \in L^2(\Omega; C([0, T]; H^\delta(\mathcal{O}))), \quad v^\mu := \frac{\partial u^\mu}{\partial t} \in L^2(\Omega; C([0, T]; H^{\delta-1}(\mathcal{O}))),$$

for any $T > 0$, and

$$z^\mu(t) = S_\mu(t)(u_0, v_0) + \int_0^t S_\mu(t-s)B_\mu(z^\mu(s))\,ds + \int_0^t S_\mu(t-s)\,dW^{Q_\mu}(s),$$

where $z^\mu(t) := (u^\mu(t), v^\mu(t))$.

Note that with these notations, the weak solution η^μ of problem (3.1) introduced in Section 3 can be written as

$$\eta^\mu(t) = \Pi_1 \int_0^t S_\mu(t-s)\,dW^{Q_\mu}(s), \quad t \geq 0,$$

and hence if u^μ is a mild solution of problem (4.1) we have

$$\begin{aligned} u^\mu(t) = {} & \Pi_1 S_\mu(t)(u_0, v_0) \\ & + \Pi_1 \int_0^t S_\mu(t-s)B_\mu(u^\mu(s), v^\mu(s))\,ds + \eta^\mu(t), \quad t \geq 0. \end{aligned} \tag{4.6}$$

Now we can establish the existence of a unique mild solution to problem (4.1), for any $\mu > 0$. This result is known in the literature (for a proof see e.g. [5, Theorem 5.3.1]), but here, for the safe of completeness, we give a self-contained proof.

Proposition 4.2. *Assume Hypotheses 1 and 2. Then for any $\mu > 0$ and for any initial data $u_0 \in H^\theta(\mathcal{O})$ and $v_0 \in H^{\theta-1}(\mathcal{O})$ there exists a unique mild solution $z^\mu(t)$ to problem (4.1). Moreover, for any $T > 0$ and $p \geq 1$ there exists $c_{\mu,p}(T) > 0$ such that*

$$\mathbb{E} \sup_{t\in[0,T]} |z^\mu(t)|^p_{\mathcal{H}_\theta} \leq c_{\mu,p}(T)\left(1 + |(u_0, v_0)|^p_{\mathcal{H}_\theta}\right). \tag{4.7}$$

Proof. For any $z = (u, v) \in L^2(\Omega; C([0, T]; H^\theta(\mathcal{O}))) \times L^2(\Omega; C([0, T]; H^{\theta-1}(\mathcal{O}))) =: \mathcal{H}_\theta(T)$ we define

$$\mathcal{F}_\mu(z)(t) := S_\mu(t)(u_0, v_0) + \int_0^t S_\mu(t-s)B_\mu(z(s))\,ds + \int_0^t S_\mu(t-s)\,dW^{Q_\mu}(s).$$

If we show that for some $T_0 > 0$ small enough $\mathcal{F}_\mu$ is a contraction on $\mathcal{H}_\theta(T_0)$, then we have a unique mild solution to problem (4.1) in the interval $[0, T_0]$.

Due to (2.7), we have

$$S_\mu(t)(u_0, v_0) \in \mathcal{H}_\theta(T), \quad T > 0.$$

Moreover, as seen in (4.5) B_μ maps $\mathcal{H}_\theta$ into itself, so that, by using again (2.7) we have

$$t \mapsto \int_0^t S_\mu(t-s) B_\mu(z(s))\, ds \in \mathcal{H}_\theta(T).$$

Thanks to (3.2) we can conclude that $\mathcal{F}_\mu(z) \in \mathcal{H}_\theta(T)$, for any $z \in \mathcal{H}_\theta(T)$.

Now, thanks to (4.5) $\mathcal{F}_\mu$ is a contraction on $\mathcal{H}_\theta(T_0)$, provided T_0 is sufficiently small. This means that $\mathcal{F}_\mu$ admits a unique fixed point in $\mathcal{H}_\theta(T_0)$, which is the unique mild solution defined in the time interval $[0, T_0]$. As the same arguments can be repeated in the intervals $[T_0, 2T_0]$, $[2T_0, 3T_0]$ and so on, we obtain a unique global solution in the time interval $[0, T]$.

We skip here the proof of estimate (4.7). A proof can be found for example in [5, Theorem 5.3.1]. □

Next step is proving that the family of probability measures $\{\mathcal{L}(u^\mu)\}_{\mu \in (0,1]}$ is tight on $C([0, T]; H)$, for any $T > 0$.

Proposition 4.3. *Assume that $u_0 \in H^1(\mathcal{O})$ and $v_0 \in H$. Then, under Hypotheses 1 and 2 the family of probability measures $\{\mathcal{L}(u^\mu)\}_{\mu \in (0,1]}$ is tight on $C([0, T]; H)$, for any $T > 0$.*

Proof. If η^μ is the solution of the stochastic linear damped wave equation (3.1) and if we define

$$\rho^\mu(t) := u^\mu(t) - \eta^\mu(t), \qquad t \geq 0,$$

then the process $\rho^\mu(t)$ solves the deterministic equation

$$\begin{cases} \mu \frac{\partial^2 \rho^\mu}{\partial t^2}(t,x) = \Delta \rho^\mu(t,x) - \frac{\partial \rho^\mu}{\partial t}(t,x) + b(x, \rho^\mu(t,x) \\ \qquad\qquad\qquad +\eta^\mu(t,x)), \quad t > 0, \;\; x \in \mathcal{O}, \\ \rho^\mu(0) = u_0, \quad \frac{\partial \rho^\mu}{\partial t}(0) = v_0, \qquad \rho^\mu(t,x) = 0, \qquad t \geq 0, \;\; x \in \partial\mathcal{O}. \end{cases}$$

If we multiply both sides of the first equation above by $(-\Delta)^{\theta-1} \partial\rho^\mu/\partial t$ and integrate with respect to $x \in \mathcal{O}$, we easily get

$$\begin{aligned} &\mu \frac{d}{dt} \left| \frac{\partial \rho^\mu}{\partial t}(t) \right|^2_{H^{\theta-1}(\mathcal{O})} + \frac{d}{dt} \left| \rho^\mu(t) \right|^2_{H^\theta(\mathcal{O})} + 2 \left| \frac{\partial \rho^\mu}{\partial t}(t) \right|^2_{H^{\theta-1}(\mathcal{O})} \\ &= 2 \left\langle b(\cdot, \rho^\mu(t) + \eta^\mu(t)), (-\Delta)^{\theta-1} \frac{\partial \rho^\mu}{\partial t}(t) \right\rangle_H \\ &\quad \leq |b(\cdot, \rho^\mu(t) + \eta^\mu(t))|^2_H + \left| \frac{\partial \rho^\mu}{\partial t}(t) \right|^2_{H^{\theta-1}(\mathcal{O})}. \end{aligned}$$

Hence, integrating with respect to $t \geq 0$ it follows

$$\begin{aligned} &\mu \left| \frac{\partial \rho^\mu}{\partial t}(t) \right|^2_{H^{\theta-1}(\mathcal{O})} + \left| \rho^\mu(t) \right|^2_{H^\theta(\mathcal{O})} + \int_0^t \left| \frac{\partial \rho^\mu}{\partial t}(s) \right|^2_{H^{\theta-1}(\mathcal{O})} ds \\ &\quad \leq \mu \, |v_0|^2_{H^{\theta-1}(\mathcal{O})} + |u_0|^2_{H^\theta(\mathcal{O})} + \int_0^t |b(\cdot, \rho^\mu(s) + \eta^\mu(s))|^2_H \, ds. \end{aligned}$$

Now, due to Hypothesis 2, for any $s \in [0, T]$ we have

$$|b(\cdot, \rho^\mu(s) + \eta^\mu(s))|_H^2 \leq c \left(1 + |\rho^\mu(s)|_{L^2(\mathcal{O})}^2 + \sup_{t \in [0,T]} |\eta^\mu(t)|_H^2\right),$$

and then

$$\begin{aligned}
&\mu \left|\frac{\partial \rho^\mu}{\partial t}(t)\right|_{H^{\theta-1}(\mathcal{O})}^2 + \left|\rho^\mu(t)\right|_{H^\theta(\mathcal{O})}^2 + \int_0^t \left|\frac{\partial \rho^\mu}{\partial t}(s)\right|_{H^{\theta-1}(\mathcal{O})}^2 ds \\
&\quad \leq \mu\, |v_0|_{H^{\theta-1}(\mathcal{O})}^2 + |u_0|_{H^\theta(\mathcal{O})}^2 \\
&\qquad + c_T \left(1 + \sup_{t \in [0,T]} |\eta^\mu(t)|_H^2\right) + c \int_0^t |\rho^\mu(s)|_{H^\theta(\mathcal{O})}^2\, ds.
\end{aligned}$$

Thanks to the Gronwall lemma, for any $\mu \in (0, 1]$ and $T > 0$ this yields

$$\begin{aligned}
&\sup_{t \in [0,T]} \left|\rho^\mu(t)\right|_{H^\theta(\mathcal{O})}^2 + \int_0^T \left|\frac{\partial \rho^\mu}{\partial t}(s)\right|_{H^{\theta-1}(\mathcal{O})}^2 ds \\
&\quad \leq c_T \left(|v_0|_{H^{\theta-1}(\mathcal{O})}^2 + |u_0|_{H^\theta(\mathcal{O})}^2 + \sup_{t \in [0,T]} |\eta^\mu(t)|_H^2 + 1\right).
\end{aligned} \tag{4.8}$$

According to (3.3), this means that we can find some constant c_T independent of $\mu \in (0, 1]$ such that

$$\mathbb{E} \sup_{t \in [0,T]} \left|\rho^\mu(t)\right|_{H^\theta(\mathcal{O})}^2 + \mathbb{E} \int_0^T \left|\frac{\partial \rho^\mu}{\partial t}(s)\right|_{H^{\theta-1}(\mathcal{O})}^2 ds \leq c_T.$$

In particular $\rho^\mu \in L^2(\Omega; C([0, T]; H^\theta(\mathcal{O})))$ and, since $u^\mu = \rho^\mu + \eta^\mu$, due to (3.3) we have that $u^\mu \in L^2(\Omega; C([0, T]; H))$ and

$$\sup_{\mu \in (0,1]} \mathbb{E}\, |u^\mu|_{C([0,T];H)}^2 \leq c_T. \tag{4.9}$$

Next, by using once more (3.3), for any $\epsilon > 0$ we can find $\lambda > 0$ such that

$$\mathbb{P}\left(\eta^\mu \in K_{\lambda,1}\right) \geq 1 - \epsilon, \quad \mu > 0, \tag{4.10}$$

where, by the Ascoli-Arzelà theorem, $K_{\lambda,1}$ is the compact subset of $C([0, T]; H)$ defined by

$$K_{\lambda,1} := \left\{ f : [0, T] \times \bar{\mathcal{O}} \to \mathbb{R},\ :\ |f|_{C^\delta([0,T]\times\bar{\mathcal{O}})} \leq \lambda \right\},$$

with $\delta < \theta/2$. Now, as we are assuming that $u_0 \in H^1(\mathcal{O})$ and $v_0 \in H$, due to (4.8) we have

$$\eta^\mu \in K_{\lambda,1} \Longrightarrow \sup_{t \in [0,T]} \left|\rho^\mu(t)\right|_{H^1(\mathcal{O})}^2 + \sup_{t \in [0,T]} \int_0^t \left|\frac{\partial \rho^\mu}{\partial t}(s)\right|_H^2 ds \leq c_{T,\lambda},$$

so that

$$\left\{\eta^\mu \in K_{\lambda,1}\right\} \subset \left\{u^\mu = \rho^\mu + \eta^\mu \in K_{\lambda,2} + K_{\lambda,1}\right\},$$

where, again by the Ascoli-Arzelà theorem, $K_{\lambda,2}$ is the compact subset of $C([0, T]; H)$ defined by

$$K_{\lambda,2} := \left\{ f \,:\, \sup_{t\in[0,T]} |f(t)|_{H^1(\mathcal{O})} \leq c_{T,\lambda}, \quad \sup_{\substack{t,s\in[0,T]\\ t\neq s}} \frac{|f(t)-f(s)|_H}{|t-s|^{1/2}} \leq c_{T,\lambda}^{1/2} \right\}.$$

Hence, in view of (4.10) we have

$$\mathbb{P}\left(u^\mu \in K_{\lambda,1} + K_{\lambda,2}\right) \geq 1 - \epsilon,$$

and this concludes the proof of tightness. □

Next, we prove an integration by parts formula.

Lemma 4.4. *Assume Hypotheses 1 and 2 and fix $u_0 \in H^\theta(\mathcal{O})$ and $v_0 \in H^{\theta-1}(\mathcal{O})$. Then for any $\mu > 0$ and for any $\varphi \in C^2([0, T] \times \bar{\mathcal{O}})$, such that $\varphi \equiv 0$ on $\partial\mathcal{O}$, we have*

$$\begin{aligned}\int_{\mathcal{O}} u^\mu(t,x)\varphi(t,x)\,dx = &\int_{\mathcal{O}} u_0(x)\varphi(0,x)\,dx\\ &+\int_0^t\int_{\mathcal{O}} u^\mu(s,x)\left[\frac{\partial\varphi}{\partial t}(s,x)+\Delta\varphi(s,x)\right]ds\,dx\\ &+\int_0^t\int_{\mathcal{O}} b(x,u^\mu(s,x))\varphi(s,x)\,ds\,dx\\ &+\int_0^t\int_{\mathcal{O}}\varphi(s,x)\,W^Q(ds,dx)+R_\mu(t),\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}R_\mu(t) := &\ \mu\left(1-e^{-\frac{t}{\mu}}\right)\int_{\mathcal{O}} v_0(x)\varphi(0,x)\,dx - \int_0^t e^{-\frac{t-s}{\mu}}\,M_\mu(s)\,ds\\ &-\int_0^t e^{-\frac{t-s}{\mu}}\left[\int_{\mathcal{O}}\left(u_0(x)\frac{\partial\varphi}{\partial t}(0,x)-u^\mu(s,x)\frac{\partial\varphi}{\partial t}(s,x)\right.\right.\\ &\left.\left.+\int_0^s u^\mu(r,x)\frac{\partial^2\varphi}{\partial t^2}(r,x)\,dr\right)dx\right]ds\\ &-\int_0^t\int_{\mathcal{O}} e^{-\frac{t-s}{\mu}}\varphi(s,x)W^Q(ds,dx),\end{aligned} \tag{4.12}$$

and

$$M_\mu(t) := \int_{\mathcal{O}}\left(u^\mu(t,x)\Delta\varphi(t,x)+b(x,u^\mu(t,x))\varphi(t,x)\right)dx. \tag{4.13}$$

Proof. Since we are assuming $u_0 \in H^\theta(\mathcal{O})$ and $v_0 \in H^{\theta-1}(\mathcal{O})$, due to Proposition 4.2 we only have

$$u^\mu \in L^2(\Omega; C([0,T]; H^\theta(\mathcal{O}))), \quad \frac{\partial u^\mu}{\partial t} \in L^2(\Omega; C([0,T]; H^{\theta-1}(\mathcal{O}))).$$

Thus, in order to have enough regularity, in our computations we replace u^μ with its finite dimensional Galerkin approximations which belong to $C^{2,2}([0,T] \times \bar{\mathcal{O}})$. Once we get formula (4.11) for the Galerkin approximations, we pass easily to the limit and we get formula (4.11) for u^μ.

Note that for simplicity of notations we still denote the Galerkin approximations by u^μ. If we set $v^\mu := \partial u^\mu/\partial t$, multiplying both sides of the first equation in problem (4.1) by some $\varphi \in C^2([0,T] \times \bar{\mathcal{O}})$ and integrating with respect to $t \geq 0$ and $x \in \mathcal{O}$, we get

$$\begin{aligned}\int_0^t \int_{\mathcal{O}} \frac{\partial v^\mu}{\partial t}(s,x)\varphi(s,x)\, ds\, dx &= \frac{1}{\mu}\int_0^t \int_{\mathcal{O}} \varphi(s,x)\, W^Q(ds,dx) \\ &+\frac{1}{\mu}\int_0^t \int_{\mathcal{O}} \left[\Delta u^\mu(s,x) - v^\mu(s,x) + b(x, u^\mu(s,x))\right]\varphi(s,x)\, ds\, dx.\end{aligned}$$

Now, integrating by parts we have

$$\begin{aligned}\int_0^t \int_{\mathcal{O}} \frac{\partial v^\mu}{\partial t}(s,x)\varphi(s,x)\, ds\, dx &= \left\langle v^\mu(t), \varphi(t)\right\rangle_H - \langle v_0, \varphi(0)\rangle_H \\ &- \int_0^t \left\langle v^\mu(s), \frac{\partial \varphi}{\partial t}(s)\right\rangle_H ds,\end{aligned}$$

and if φ vanishes at the boundary of $\mathcal{O}$

$$\int_0^t \int_{\mathcal{O}} \Delta u^\mu(s,x)\varphi(s,x)\, ds\, dx = \int_0^t \left\langle u^\mu(s), \Delta\varphi(s)\right\rangle_H ds.$$

Thus, if we define

$$\begin{aligned}H(t) :&= \int_0^t \left\langle v^\mu(s), \varphi(s)\right\rangle_H ds = \left\langle u^\mu(t), \varphi(t)\right\rangle_H \\ &- \langle u_0, \varphi(0)\rangle_H - \int_0^t \left\langle u^\mu(s), \frac{\partial \varphi}{\partial t}(s)\right\rangle_H ds,\end{aligned} \tag{4.14}$$

and if $M_\mu(t)$ is defined as in (4.13), we obtain

$$\begin{aligned}H'(t) &= -\frac{1}{\mu}H(t) + \langle v_0, \varphi(0)\rangle_H + \frac{1}{\mu}\int_0^t \left\langle \varphi(s), dW^Q(s)\right\rangle_H \\ &+\frac{1}{\mu}\int_0^t \left[M_\mu(s) + \mu\left\langle v^\mu(s), \frac{\partial \varphi}{\partial t}(s)\right\rangle_H\right] ds.\end{aligned}$$

As $H(0)=0$, this yields

$$H(t)=\mu\left(1-e^{-\frac{t}{\mu}}\right)\langle v_0,\varphi(0)\rangle_H+\frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\int_0^s \left\langle \varphi(r),dW^Q(r)\right\rangle_H\,ds$$
$$+\frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\int_0^s\left[M_\mu(r)+\mu\left\langle v^\mu(r),\frac{\partial\varphi}{\partial t}(r)\right\rangle_H\right]dr\,ds.$$

Hence, due to (4.14) we get

$$\begin{aligned}\left\langle u^\mu(t),\varphi(t)\right\rangle_H=&\ \langle u_0,\varphi(0)\rangle_H+\int_0^t\left\langle u^\mu(s),\frac{\partial\varphi}{\partial t}(s)\right\rangle_H ds\\&+\mu\left(1-e^{-\frac{t}{\mu}}\right)\langle v_0,\varphi(0)\rangle_H\\&+\frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\int_0^s\left\langle\varphi(r),dW^Q(r)\right\rangle_H ds\\&+\frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\int_0^s\left[M_\mu(r)+\mu\left\langle v^\mu(r),\frac{\partial\varphi}{\partial t}(r)\right\rangle_H\right]dr\,ds.\end{aligned}\tag{4.15}$$

Now, integrating by parts it is immediate to check that for any function $f:[0,T]\to\mathbb{R}$

$$\frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\int_0^s f(r)\,dr\,ds=\int_0^t f(s)\,ds-\int_0^t e^{-\frac{t-s}{\mu}}f(s)\,ds.\tag{4.16}$$

Similarly, for the stochastic integral we have

$$\begin{aligned}&\frac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\int_0^s\left\langle\varphi(r),dW^Q(r)\right\rangle_H ds\\&=\int_0^t\left\langle\varphi(s),dW^Q(s)\right\rangle_H-\int_0^t e^{-\frac{t-s}{\mu}}\left\langle\varphi(s),dW^Q(ds)\right\rangle_H.\end{aligned}\tag{4.17}$$

Therefore, recalling how $M_\mu(t)$ is defined, if we plug (4.16) and (4.17) into (4.15) we obtain

$$\begin{aligned}\left\langle u^\mu(t),\varphi(t)\right\rangle_H=&\ \langle u_0,\varphi(0)\rangle_H+\int_0^t\left\langle u^\mu(s),\frac{\partial\varphi}{\partial t}(s)\right\rangle_H ds\\&+\mu\left(1-e^{-\frac{t}{\mu}}\right)\langle v_0,\varphi(0)\rangle_H\\&+\int_0^t\left\langle\varphi(s),dW^Q(s)\right\rangle_H+\int_0^t\left(\left\langle u^\mu(s),\Delta\varphi(s)\right\rangle_H\right.\\&\left.+\left\langle b(\cdot,u^\mu(s)),\varphi(s)\right\rangle_H\right)ds\\&-\int_0^t e^{-\frac{t-s}{\mu}}\left\langle\varphi(s),dW^Q(s)\right\rangle_H-\int_0^t e^{-\frac{t-s}{\mu}}M_\mu(s)\,ds\\&+\int_0^t e^{-\frac{t-s}{\mu}}\int_0^s\left\langle v^\mu(r),\frac{\partial\varphi}{\partial t}(r)\right\rangle_H dr\,ds.\end{aligned}$$

This concludes the proof of the lemma, as

$$\int_0^t e^{-\frac{t-s}{\mu}} \int_0^s \left\langle v^\mu(r), \frac{\partial \varphi}{\partial t}(r)\right\rangle_H dr\, ds = \int_0^t e^{-\frac{t-s}{\mu}} \left(\left\langle u^\mu(s), \frac{\partial \varphi}{\partial t}(s)\right\rangle_H - \left\langle u_0, \frac{\partial \varphi}{\partial t}(0)\right\rangle_H - \int_0^s \left\langle u^\mu(r), \frac{\partial^2 \varphi}{\partial t^2}(r)\right\rangle_H dr\right) ds.$$

□

Concerning the remainder term $R_\mu(t)$ defined in (4.12), we have the following limiting result.

Lemma 4.5. *Under the same hypotheses of Lemma 4.4 we have*

$$\lim_{\mu \to 0} \mathbb{E}\, |R_\mu(t)|^2 = 0, \qquad t \geq 0.$$

Proof. We have

$$\begin{aligned}|R_\mu(t)|^2 \leq 3\,\mu^2 \left|\langle v_0, \varphi(0)\rangle_H\right|^2 + 3\left|\int_0^t e^{-\frac{t-s}{\mu}} \left\langle \varphi(s), dW^Q(s)\right\rangle_H\right|^2 \\ +3\left|\int_0^t e^{-\frac{t-s}{\mu}} \left[M_\mu(s) + \left\langle u_0, \frac{\partial \varphi}{\partial t}(0)\right\rangle_H - \left\langle u^\mu(s), \frac{\partial \varphi}{\partial t}(s)\right\rangle_H \right.\right. \\ \left.\left. + \int_0^s \left\langle u^\mu(r), \frac{\partial^2 \varphi}{\partial t^2}(r)\right\rangle_H dr\right] ds\right|^2 =: I^1_\mu(t) + I^2_\mu(t) + I^3_\mu(t).\end{aligned}$$

Now, with a change of variable we have

$$\begin{aligned}\mathbb{E}\, I^2_\mu(t) &= 3\int_0^t e^{-\frac{2(t-s)}{\mu}} |\varphi(s)|^2_H\, ds = 3\,\mu \int_0^{\frac{t}{\mu}} e^{-2s}\, |\varphi(t-\mu s)|^2_H\, ds \\ &\leq \frac{3}{2}\,\mu \sup_{s \in [0,T]} |\varphi(s)|^2_H\end{aligned}$$

Moreover, recalling how $M_\mu(t)$ is defined in (4.13), with a change of variables we easily get

$$I^3_\mu(t) \leq \mu\, c\, t \int_0^\infty e^{-2s}\, ds\, (1+T^2) |u^\mu|^2_{C([0,T];H)} |\varphi|^2_{C^2([0,T]\times \bar{\mathcal{O}})},$$

and hence, due to (4.9) we conclude

$$\mathbb{E}\, I^3_\mu(t) \leq \mu\, c_T\, \left(1 + \mathbb{E}\,|u^\mu|^2_{C([0,T];H)}\right) |\varphi|^2_{C^2([0,T]\times \bar{\mathcal{O}})} \leq \mu\, c_T\, |\varphi|^2_{C^2([0,T]\times \bar{\mathcal{O}})}.$$

This implies that

$$\mathbb{E}\, |R_\mu(t)|^2 \leq I^1_\mu(t) + \mathbb{E}\, I^2_\mu(t) + \mathbb{E}\, I^3_\mu(t) \leq \mu\, c_T,$$

for some constant c_T depending only on T, u_0 and φ, and the lemma is proved. □

Now we can prove the main result of this section.

Theorem 4.6. *Assume Hypotheses 1 and 2 and fix $u_0 \in H^1(\mathcal{O})$ and $v_0 \in H$. Then, if u^μ is the solution of the stochastic semi-linear damped wave equation (4.1) and z is the solution of the stochastic semi-linear heat equation (4.2), for any $T > 0$ and for any $\epsilon > 0$ we have*

$$\lim_{\mu \to 0} \mathbb{P}\left(|u^\mu - z|_{C([0,T];H)} > \epsilon\right) = 0.$$

Proof. Due to the tightness in $C([0,T];H)$ of the sequence $\{\mathcal{L}(u^\mu)\}_{\mu \in (0,1]}$, the Skorokhod theorem assures that for any two sequences $\{\mu_n\}_n$ and $\{\mu_m\}_m$ converging to zero there exist subsequences $\{\mu_{n(k)}\}_{k \in \mathbb{N}}$ and $\{\mu_{m(k)}\}_{k \in \mathbb{N}}$ and a sequence of random elements

$$\{\rho_k\}_{k \in \mathbb{N}} := \left\{(u_1^k, u_2^k, \hat{W}_k{}^Q)\right\}_{k \in \mathbb{N}},$$

in $\mathcal{C} := C([0,T];H)^2 \times C([0,T];\mathcal{D}'(\mathcal{O}))$, defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, such that the law of ρ_k coincides with the law of $(u^{\mu_{n(k)}}, u^{\mu_{m(k)}}, W^Q)$, for each $k \in \mathbb{N}$, and ρ_k converges $\hat{\mathbb{P}}$-a.s. to some random element $\rho := (u_1, u_2, \hat{W}^Q) \in \mathcal{C}$.

Now, if we show that $u_1 = u_2$, we have that there exists some $z \in C([0,T];H)$ such that u^μ converges to z in probability. Actually, as observed by Gyöngy and Krylov in [8], if E is any Polish space equipped with the Borel σ-algebra, a sequence $\{\rho_n\}$ of E-valued random variables converges in probability if and only if for every pair of subsequences $\{\rho_m\}$ and $\{\rho_l\}$ there exists an E^2-valued subsequence $w_k := (\rho_{m(k)}, \rho_{l(k)})$ converging weakly to a random variable w supported on the diagonal $\{(h,k) \in E^2 \,:\, h = k\}$.

Note that both u_1^k and u_2^k solve equation (4.1) with W^Q replaced by $\hat{W}_k{}^Q$. Then they both verify formula (4.11), with R_1^k and R_2^k obtained replacing u^μ respectively with u_1^k and u_2^k and W^Q with $\hat{W}_k^Q$. According to Lemma 4.5 we have that both R_1^k and R_2^k converge to zero in $L^2(\hat{\Omega})$, as $\mu_{n(k)}$ and $\mu_{m(k)}$ go to zero, and then, possibly for a subsequence, they converge $\hat{\mathbb{P}}$-a.s. to zero. Due to formula (4.11) this implies

$$\begin{aligned}\int_{\mathcal{O}} u_i(t,x)\varphi(t,x)\,dx = {} & \int_{\mathcal{O}} u_0(x)\varphi(0,x)\,dx \\ & + \int_0^t \int_{\mathcal{O}} u_i(t,x)\left[\frac{\partial \varphi}{\partial t}(s,x) + \Delta\varphi(s,x)\right] ds\,dx \\ & + \int_0^t \int_{\mathcal{O}} b(x, u_i(s,x))\varphi(s,x)\,ds\,dx \\ & + \int_0^t \int_{\mathcal{O}} \varphi(s,x)\,\hat{W}^Q(ds,dx), \qquad i = 1, 2,\end{aligned}$$

and then both u_1 and u_2 coincide with the solution of the semi-linear heat equation perturbed by the noise $\hat{W}^Q$, which is unique.

As we have recalled above, thanks to the remark by Gyöngy-Krylov in [8] this implies that u^μ converges in probability to some random variable $z \in C([0,T];H)$.

But, by using again formula (4.11) and Lemma 4.5 we have that z solves the heat equation (4.2). This completes the proof of the theorem. □

5. Stationary distributions

In this section we study the relation between the stationary distributions of the processes $u^\mu(t)$ and $z(t)$, defined respectively as the solution of the semi-linear stochastic damped wave equation (4.1) and as the solution of the semi-linear stochastic heat equation (4.2).

If we set

$$z^\mu(t) := (u^\mu(t), v^\mu(t)), \quad t \geq 0, \ \ \mu > 0,$$

with the notations introduced in Section 2 and Section 4 we can write equation (4.1) as the abstract evolution equation on the Hilbert space $\mathcal{H}_0 = H \times H^{-1}(\mathcal{O})$

$$dz^\mu(t) = \left[A_\mu z^\mu(t) + B_\mu(z^\mu(t))\right] dt + dW^{\mathcal{Q}_\mu}, \qquad z^\mu(0) = (u_0, v_0), \tag{5.1}$$

where B_μ and $\mathcal{Q}_\mu$ are the operators already defined in (4.3) and (4.4), respectively.

Note that the adjoint of the operator $\mathcal{Q}_\mu : H \to \mathcal{H}_0$ is the operator $\mathcal{Q}_\mu^\star : \mathcal{H}_0 \to H$ defined by

$$\mathcal{Q}_\mu^\star(u, v) = \frac{1}{\mu}(-\Delta)^{-1} Q v.$$

In particular we have that $\mathcal{Q}_\mu \mathcal{Q}_\mu^\star : \mathcal{H}_0 \to \mathcal{H}_0$ is given by

$$\mathcal{Q}_\mu \mathcal{Q}_\mu^\star \,(u, v) = \frac{1}{\mu^2}\,(0, (-\Delta)^{-1} Q^2 v), \qquad (u, v) \in \mathcal{H}_0. \tag{5.2}$$

Next, for any $\mu > 0$ we introduce the operator $C_\mu \in \mathcal{L}^+(\mathcal{H}_0)$ by setting

$$C_\mu := \int_0^\infty S_\mu(s)\, \mathcal{Q}_\mu \mathcal{Q}_\mu^\star\, S_\mu^\star(s)\, ds,$$

where $\{S_\mu^\star(t)\}_{t \geq 0}$ is the semigroup generated by $A_\mu^\star$, the adjoint to the operator A_μ.

Proposition 5.1. *Under Hypothesis 1, with $\theta = 0$, we have*

$$C_\mu(u, v) = \frac{1}{2}\left((-\Delta)^{-1} Q^2 u, \frac{1}{\mu}(-\Delta)^{-1} Q^2 v\right), \qquad (u, v) \in \mathcal{H}_0. \tag{5.3}$$

In particular C_μ is a trace-class operator with

$$\operatorname{Tr} C_\mu = \frac{1}{2}\left(1 + \frac{1}{\mu}\right) \sum_{k=1}^\infty \frac{\lambda_k^2}{\alpha_k}.$$

Proof. Due to (2.6) and (5.2), for any $(u, v) \in \mathcal{H}_0$ and $\mu > 0$ we have

$$\begin{aligned} S_\mu(t)\, \mathcal{Q}_\mu \mathcal{Q}_\mu^\star\, S_\mu^\star(t)(u, v) &= \frac{1}{\mu^2}\, S_\mu(t)(0, (-\Delta)^{-1} Q^2\, \Pi_2 S_\mu^\star(t)(u, v)) \\ &= \frac{1}{\mu^2}\, S_\mu(t)(0, (-\Delta)^{-1} Q^2\, \Pi_2 S_\mu(t)(-\mu u, v)). \end{aligned}$$

Thus, from (2.4) and (2.5), for any $k \in \mathbb{N}$ we easily have

$$\begin{aligned} \left[\Pi_1\, S_\mu(t)\, \mathcal{Q}_\mu \mathcal{Q}_\mu^\star\, S_\mu^\star(t)(u, v)\right]_k = \frac{\lambda_k^2}{4\alpha_k\, \mu^2 \gamma_k^\mu}\, \exp\left(-\frac{t}{\mu}\right) \Bigg(\frac{\alpha_k}{\gamma_k^\mu} \big[\exp\left(\gamma_k^\mu t\right) \\ - \exp\left(-\gamma_k^\mu t\right)\big]^2\, u_k + \big[\exp\left(\gamma_k^\mu t\right) \\ - \exp\left(-\gamma_k^\mu t\right)\big] \left[\left(1 - \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(\gamma_k^\mu t\right) \right. \\ \left. + \left(1 + \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(-\gamma_k^\mu t\right)\right] v_k \Bigg), \end{aligned}$$

with the usual assumption that if $\gamma_k^\mu = 0$ then for any $t \geq 0$

$$\frac{1}{\gamma_k^\mu}\, \left[\exp\left(\gamma_k^\mu t\right) - \exp\left(-\gamma_k^\mu t\right)\right] = 2t.$$

Therefore, by some computations for any $k \in \mathbb{N}$ we obtain

$$\left[\Pi_1\, C_\mu(u, v)\right]_k = \int_0^\infty \left[\Pi_1\, S_\mu(t)\, \mathcal{Q}_\mu \mathcal{Q}_\mu^\star\, S_\mu^\star(t)(u, v)\right]_k\, dt = \frac{\lambda_k^2}{2\alpha_k}\, u_k.$$

Concerning the second component, due to (2.5) we have

$$\begin{aligned} &\left[\Pi_2\, S_\mu(t)\, \mathcal{Q}_\mu \mathcal{Q}_\mu^\star\, S_\mu^\star(t)(u, v)\right]_k = \frac{\lambda_k^2}{4\alpha_k\, \mu^2}\, \exp\left(-\frac{t}{\mu}\right) \\ &\left[\left(1 - \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(\gamma_k^\mu t\right) + \left(1 + \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(-\gamma_k^\mu t\right)\right] \\ &\Bigg(\frac{\alpha_k}{\gamma_k^\mu} \left[\exp\left(\gamma_k^\mu t\right) - \exp\left(-\gamma_k^\mu t\right)\right] u_k \\ &\quad + \left[\left(1 - \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(\gamma_k^\mu t\right) + \left(1 + \frac{1}{2\mu\gamma_k^\mu}\right) \exp\left(-\gamma_k^\mu t\right)\right] v_k \Bigg), \end{aligned}$$

and, as for the first component, by some computations this implies that

$$\left[\Pi_2\, C_\mu(u, v)\right]_k = \int_0^\infty \left[\Pi_2\, S_\mu(t)\, \mathcal{Q}_\mu \mathcal{Q}_\mu^\star\, S_\mu^\star(t)(u, v)\right]_k\, dt = \frac{\lambda_k^2}{2\mu\alpha_k}\, v_k.$$

This allows to obtain (5.3) and hence to conclude the proof of the proposition. $\square$

5.1. The linear case

Our aim here is studying the invariant measure of the system

$$dz(t) = A_\mu z(t)\,dt + d\,W^{Q_\mu}(t), \qquad z(0) = (u_0, v_0) \in \mathcal{H}_0, \tag{5.4}$$

and showing that the stationary distribution for the solution of the linear stochastic damped wave equation

$$\begin{aligned}\mu\frac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) - \frac{\partial u}{\partial t}(t,x) + \frac{\partial W^Q}{\partial t}(t,x),\\ u(t,x) &= 0, \quad x \in \partial\mathcal{O},\end{aligned} \tag{5.5}$$

coincides for all $\mu > 0$ with the unique invariant measure of the linear stochastic heat equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \frac{\partial W^Q}{\partial t}(t,x), \qquad u(0,x) = 0, \quad x \in \partial\mathcal{O}. \tag{5.6}$$

Theorem 5.2. *Under Hypothesis 1, with $\theta = 0$, the Gaussian measure $\mathcal{N}(0, C_\mu)$ is the unique invariant measure of system (5.4), for each $\mu > 0$, and for any $\varphi \in C_b(\mathcal{H}_0)$ and $z_0 \in \mathcal{H}_0$*

$$\lim_{t\to\infty} \mathbb{E}^{z_0}\,\varphi(z^\mu(t)) = \int_{\mathcal{H}_0} \varphi(z)\,\mathcal{N}(0, C_\mu)(dz), \tag{5.7}$$

so that $\mathcal{N}(0, C_\mu)$ is ergodic and strongly mixing.

Moreover the Gaussian measure $\nu = \mathcal{N}(0, (-\Delta)^{-1}/2)$ is the stationary distribution of (5.5). In particular, ν does not depend on $\mu > 0$ and coincides with the unique invariant measure of the stochastic heat equation (5.6).

Proof. According to Proposition 5.1, the operator C_μ is non-negative, symmetric and of trace-class on $\mathcal{H}_0$. Thus problem (5.4) admits an invariant measure of the form

$$\nu \star \mathcal{N}(0, C_\mu),$$

where ν is an invariant measure for the semigroup $S_\mu(t)$ and $\mathcal{N}(0, C_\mu)$ is the Gaussian measure, with zero mean and covariance operator C_μ (for a proof see e.g. [4, Theorem 11.7]). Moreover, as the semigroup $\{S_\mu(t)\}_{t\geq 0}$ is of negative type (see Proposition 2.4), due to [4, Theorem 11.11] $\mathcal{N}(0, C_\mu)$ is the unique invariant measure for (5.1) and (5.7) holds. As well known this implies that $\mathcal{N}(0, C_\mu)$ is ergodic and strongly mixing.

Next, due to (5.3) the measure $\mathcal{N}(0, C_\mu)$ defined on $\mathcal{H}_0$ is the product of two Gaussian measures, defined respectively on H and $H^{-1}(\mathcal{O})$. Namely

$$\mathcal{N}(0, C_\mu) = \mathcal{N}\left(0, (-\Delta)^{-1}/2\right) \times \mathcal{N}\left(0, (-\Delta)^{-1}/2\mu\right).$$

In particular the marginal measure $\nu_\mu := \Pi_1\mathcal{N}(0, C_\mu)$ equals $\mathcal{N}(0, (-\Delta)^{-1}/2)$, so that it does not depend on $\mu > 0$ and coincides with the unique invariant measure ν of the Ornstein-Uhlenbeck process solving problem (5.6).

This allows us to conclude the proof of the theorem, as the process $\bar{u}^{\mu}(t) = \Pi_1 \bar{z}^{\mu}(t)$, with

$$\bar{z}^{\mu}(t) = (\bar{u}^{\mu}(t), \bar{v}^{\mu}(t)) := \int_0^t S_{\mu}(t-s)dW^{Q_{\mu}}(s),$$

is the stationary solution to problem (5.5) and its distribution is $\Pi_1 \mathcal{N}(0, C_{\mu})$. □

5.2. The semi-linear case

We show that an analogous result holds also in the non-linear case, if Q is the identity operator (and in particular if $d = 1$).

Our aim first is proving that system (5.1) is of gradient type and admits an invariant measure of the following type

$$\nu_{\mu}(dz) = c_{\mu}\, e^{2U(z)}\, \mathcal{N}(0, C_{\mu})(dz),$$

for some mapping $U : \mathcal{H}_0 \to \mathbb{R}$ which does not depend on $\mu > 0$ and is a function of $u \in H$ only.

To this purpose we introduce some notations. For any $n \in \mathbb{N}$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ we define

$$T_n \xi := \sum_{k \leq n} \xi_k e_k.$$

Clearly the mapping T_n is well defined from $\mathbb{R}^n$ into $H^{\delta}(\mathcal{O})$ and the mapping $\bar{T}_n(\xi, \eta) := (T_n\xi, T_n\eta)$ is well defined from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathcal{H}_{\delta}$, for any $\delta \in \mathbb{R}$. Moreover, if we define

$$R_n u := (\langle u, e_1 \rangle_H, \dots, \langle u, e_n \rangle_H),$$

we have that R_n maps $H^{\delta}(\mathcal{O})$ into $\mathbb{R}^n$, for any $\delta \in \mathbb{R}$, and $R_n T_n = \mathrm{Id}_{\mathbb{R}^n}$. Furthermore, if we set $P_n := T_n R_n$, we have that P_n is the projection of $H^{\delta}(\mathcal{O})$ onto the finite dimensional space generated by $\{e_1, \dots, e_n\}$ and for any fixed $u \in H^{\delta}(\mathcal{O})$ we have that $P_n u$ converges to u in H^{δ}, as n goes to infinity. In particular, setting

$$\bar{P}_n(z) := (P_n u, P_n v), \qquad z = (u, v) \in \mathcal{H}_{\delta},$$

we have

$$\lim_{n\to\infty} \bar{P}_n z = z, \qquad \text{in} \quad \mathcal{H}_{\delta} \tag{5.8}$$

In what follows, for any Banach space X we denote by $B_b(X)$ the Banach space of Borel and bounded functions from X into $\mathbb{R}$, endowed with the sup-norm, and we denote by $C_b(X)$ the subspace of uniformly continuous functions.

We recall that the transition semigroup $\{P^{\mu}(t)\}_{t\geq 0}$ associated with system (5.1) in $\mathcal{H}_0$ is defined for any $t \geq 0$ and $\varphi \in B_b(\mathcal{H}_0)$ by

$$P^{\mu}(t)\varphi(z) = \mathbb{E}\,\varphi(u^{\mu}(t), v^{\mu}(t)), \qquad z = (u, v) \in \mathcal{H}_0,$$

where $(u^{\mu}(t), v^{\mu}(t))$ is the solution to (5.1) with initial datum $z = (u, v)$.

Next, we denote by $z_n^\mu(t)$ the solution to the finite dimensional problem

$$dz(t) = \left[A_\mu z(t) + \bar{P}_n B_\mu(\bar{P}_n z(t))\right] dt + dW^{\mathcal{I}_{\mu,n}}(t), \qquad z(0) = \bar{P}_n z, \quad (5.9)$$

where $\mathcal{I}_{\mu,n} = \bar{P}_n \mathcal{I}_\mu$. Due to (5.8), to the fact that B_μ is Lipschitz continuous (see (4.5)) and to estimate (4.7), it is possible to prove the following approximation result

$$\lim_{n\to\infty} \mathbb{E}\,|z_n^\mu(t) - z^\mu(t)|^2_{\mathcal{H}_0} = 0, \qquad t \geq 0.$$

An important consequence of this fact is that the semigroup $P^\mu(t)$ can be approximated by the semigroup $P_\mu^n(t)$ associated with $z_n^\mu(t)$. Namely, for any $\varphi \in C_b(\mathcal{H}_0)$ and $t \geq 0$ it holds

$$\lim_{n\to\infty} P_n^\mu(t)\varphi(z) = \lim_{n\to\infty} \mathbb{E}\,\varphi(z_n^\mu(t)) = P^\mu(t)\varphi(z), \qquad z \in \mathcal{H}_0. \quad (5.10)$$

Now, for any $u \in H$ we set

$$U(u) := \int_{\mathcal{O}} \int_0^{u(x)} b(x,\sigma)\, d\sigma\, dx. \quad (5.11)$$

Since $b(x,\cdot) : \mathbb{R} \to \mathbb{R}$ has linear growth, uniformly with respect to $x \in \mathcal{O}$ (see Hypothesis 2), it is not difficult to check that

$$|U(u)| \leq c\left(1 + |u|_H^2\right), \quad |U(u) - U(v)| \leq c\,|u - v|_H\,(1 + |u|_H + |v|_H)\,,$$

so that $U : H \to \mathbb{R}$ is well defined and locally Lipschitz continuous. Moreover it is differentiable and $DU(u) = b(\cdot, u)$, for any $u \in H$.

Hypothesis 3. *The mapping $U : H \to \mathbb{R}$ defined in (5.11) is bounded from above, that is*

$$\sup_{u\in H} U(u) < \infty.$$

Remark 5.3. 1. The assumption of boundedness from above for U implies that

$$Z := \int_H e^{2U(u)}\,\mathcal{N}(0, (-\Delta)^{-1}/2)(du) < \infty,$$

and

$$Z_n := \int_H e^{2U(P_n u)}\,\mathcal{N}(0, (-\Delta)^{-1}/2)(du) < \infty, \qquad n \in \mathbb{N}.$$

2. Hypothesis 3 above is satisfied if for example

$$b(x,\sigma) = -c_1(x)\,\sigma + c_2(x), \qquad (x,\sigma) \in \bar{\mathcal{O}} \times \mathbb{R},$$

for some continuous mappings $c_1, c_2 : \bar{\mathcal{O}} \to \mathbb{R}$, with $\min_{x\in\bar{\mathcal{O}}} c_1(x) > 0$.

3. From the proof of Theorem 5.4 one sees that it is sufficient to assume a weaker condition than Hypothesis 3. Namely, what is needed is that both Z and Z_n are finite and

$$\lim_{n\to\infty}\int_{\mathcal{H}_0}\varphi_n(z)e^{2U(P_n u)}\mathcal{N}(0,C_\mu)\,dz=\int_{\mathcal{H}_0}\varphi(z)e^{2U(u)}\mathcal{N}(0,C_\mu)\,dz,$$

for any sequence $\{\varphi_n\}\subset C_b(\mathcal{H}_0)$ uniformly bounded and pointwise convergent to some $\varphi\in C_b(\mathcal{H}_0)$.

Theorem 5.4. *Assume that Hypotheses 1, 2 and 3 hold and take $Q=I$. Then the probability measure*

$$\nu_\mu(dz):=\frac{1}{Z}\,e^{2U(u)}\,\mathcal{N}(0,C_\mu)(dz)$$

is invariant for system (5.1).

Moreover the distribution

$$\nu(du):=\frac{1}{Z}\,e^{2U(u)}\,\mathcal{N}(0,(-\Delta)^{-1}/2)(du)$$

is stationary for equation (4.1), for any $\mu>0$, and coincides with the unique invariant measure for the stochastic semi-linear heat equation (4.2).

Proof. For any $\mu>0$, $n\in\mathbb{N}$ and $(q,p)\in\mathbb{R}^n\times\mathbb{R}^n$, we denote by $\zeta_n^\mu(t):=(q_n^\mu(t),p_n^\mu(t))$ the solution of the system in $\mathbb{R}^n$

$$\begin{cases}\dot q_n^{\mu}(t)=p_n^\mu(t),\quad q_n^\mu(0)=q\\[2ex] \mu\,\dot p_n^{\mu}(t)=R_n\Delta T_n q_n^\mu(t)+R_n b(\cdot,T_n q_n^\mu(t))-p_n^\mu(t)+\dot W_n(t),\quad p_n^\mu(0)=p,\end{cases}\tag{5.12}$$

where $W_n(t)=(\beta_1(t),\dots,\beta_n(t))$, for any $t\geq 0$. The transition semigroup associated with system (5.12) is defined for any $\varphi\in C_b(\mathbb{R}^{2n})$ by

$$\hat P_n^\mu(t)\varphi(q,p)=\mathbb{E}\,\varphi\left(\zeta_n^\mu(t)\right),\quad t\geq 0.$$

Note that if we define $U_n(q):=U(T_n q)$, we have

$$DU_n(q)=R_n\,b(\cdot,T_n q),\quad q\in\mathbb{R}^n,$$

and

$$\frac{1}{2}\,D\,\langle R_n\Delta T_n q,q\rangle_{\mathbb{R}^n}=R_n\Delta T_n q\quad q\in\mathbb{R}^n.$$

Moreover, since

$$\begin{aligned}&\int_{\mathbb{R}^n}\exp\left(\langle R_n\Delta T_n q,q\rangle_{\mathbb{R}^n}+2\,U(T_n q)\right)\,dq\\&\qquad=c_n\int_{\mathbb{R}^n}e^{2\,U(T_n q)}\,\mathcal{N}(0,R_n(-\Delta)^{-1}T_n/2)\,dq,\end{aligned}$$

for the obvious normalizing constant c_n, by a change of variable from Hypothesis 3 we have

$$\int_{\mathbb{R}^n} \exp\left(\langle R_n \Delta T_n q, q\rangle_{\mathbb{R}^n} + 2\,U(T_n q)\right) dq = c_n \int_H e^{2\,U(P_n u)}\,\mathcal{N}(0, (-\Delta)^{-1}/2)\,du < \infty.$$

As a well-known fact, the Boltzmann distribution

$$\begin{aligned}\hat{\nu}_{\mu,n}(dq, dp) &= c_{\mu,n} \exp\left(\langle R_n \Delta T_n q, q\rangle_{\mathbb{R}^n} + 2\,U_n(q)\right) \exp\left(-\mu |p|^2_{\mathbb{R}^n}\right)(dq, dp)\\ &= \frac{1}{Z_n}\, e^{2\,U_n(q)} \mathcal{N}(0, R_n(-\Delta)^{-1} T_n/2)(dq) \times \mathcal{N}(0, I_{\mathbb{R}^n}/2\mu)(dp)\end{aligned}$$

is invariant for system (5.12), so that for any $\hat{\varphi} \in C_b(\mathbb{R}^n)$ and $t \geq 0$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \hat{P}^\mu_n(t)\hat{\varphi}(q, p)\,\hat{\nu}_{\mu,n}(dq, dp) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \hat{\varphi}(q, p)\,\hat{\nu}_{\mu,n}(dq, dp). \tag{5.13}$$

Now, it is immediate to check that the $\mathcal{H}_0$-valued process $\bar{T}_n \zeta^\mu_n(t)$ coincides with the solution $z^\mu_n(t)$ of the approximating system (5.9) with initial datum $\bar{T}_n(q, p)$. For any $\varphi \in C_b(\mathcal{H}_0)$ this yields

$$P^\mu_n(t)\varphi(\bar{T}_n(q, p)) = \hat{P}^\mu_n(t)(\varphi \circ \bar{T}_n)(q, p), \qquad (q, p) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and hence from (5.13) for any $\varphi \in C_b(\mathcal{H}_0)$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n \times \mathbb{R}^n} P^\mu_n(t)\varphi(\bar{T}_n(q, p))\,\hat{\nu}_{\mu,n}(dq, dp)\\ &\quad= \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi(\bar{T}_n(q, p))\,\hat{\nu}_{\mu,n}(dq, dp).\end{aligned} \tag{5.14}$$

If T_n is considered as a mapping from $\mathbb{R}^n$ into H by reasoning as above we have

$$\begin{aligned}&\left[e^{2\,U_n(q)} \mathcal{N}(0, R_n(-\Delta)^{-1} T_n/2)\right] \circ T_n^{-1}(du)\\ &\quad= e^{2\,U(u)} \mathcal{N}(0, (-\Delta)^{-1} P_n/2)(du).\end{aligned} \tag{5.15}$$

Moreover, if T_n is considered as a mapping from $\mathbb{R}^n$ into $H^{-1}(\mathcal{O})$ we have

$$\mathcal{N}(0, I_{\mathbb{R}^n}/2\mu) \circ T_n^{-1} = \mathcal{N}(0, (-\Delta)^{-1} P_n/2\mu). \tag{5.16}$$

Actually, for any $\lambda \in H^{-1}(\mathcal{O})$ we have

$$\begin{aligned}&\int_{H^{-1}(\mathcal{O})} \exp\left(i\,\langle \lambda, v\rangle_{H^{-1}(\mathcal{O})}\right) \left[\mathcal{N}(0, I_{\mathbb{R}^n}/2\mu) \circ T_n^{-1}\right] dv\\ &= \int_{\mathbb{R}^n} \exp\left(i\,\left\langle (-\Delta)^{-1}\lambda, T_n p\right\rangle_H\right) \mathcal{N}(0, I_{\mathbb{R}^n}/2\mu)\,dp\\ &= \exp\left(-\frac{1}{4\mu}\left\langle R_n(-\Delta)^{-1}\lambda, R_n(-\Delta)^{-1}\lambda\right\rangle_{\mathbb{R}^n}\right)\\ &= \exp\left(-\frac{1}{4\mu}\left\langle (-\Delta)^{-1} P_n\lambda, P_n\lambda\right\rangle_{H^{-1}(\mathcal{O})}\right)\\ &= \int_{H^{-1}(\mathcal{O})} \exp\left(i\,\langle \lambda, v\rangle_{H^{-1}(\mathcal{O})}\right) \mathcal{N}(0, (-\Delta)^{-1} P_n/2\mu)\,dv,\end{aligned}$$

and by uniqueness of the Fourier transform we obtain (5.16).

Therefore, from (5.15) and (5.16) we have

$$\begin{aligned}\hat{\nu}_{\mu,n} \circ \bar{T}_n^{-1}(dz) &= \frac{1}{Z_n}\, e^{2\,U(u)} \mathcal{N}(0, (-\Delta)^{-1} P_n/2) \times \mathcal{N}(0, (-\Delta)^{-1} P_n/2\mu)\,(dz)\\ &= \frac{1}{Z_n}\, e^{2\,U(u)} \left[\mathcal{N}(0, C_\mu) \circ \bar{P}_n^{-1}\right](dz),\end{aligned}$$

and hence, since

$$P_n^\mu(t)\varphi(z) = P_n^\mu(t)\varphi(\bar{P}_n z), \qquad z \in \mathcal{H}_0,$$

from (5.14) it follows

$$\begin{aligned}&\frac{1}{Z_n} \int_{\mathcal{H}_0} P_n^\mu(t)\varphi(z)\, e^{2\,U(P_n u)} \mathcal{N}(0, C_\mu)\, dz\\ &\quad = \frac{1}{Z_n} \int_{\mathcal{H}_0} \varphi(\bar{P}_n z)\, e^{2\,U(P_n u)} \mathcal{N}(0, C_\mu)\, dz. \end{aligned} \tag{5.17}$$

Now, due to (5.8) and (5.10) we have

$$\lim_{n\to\infty} P_n^\mu(t)\varphi(z)\, e^{2\,U(P_n u)} = P^\mu(t)\varphi(z)\, e^{2\,U(u)}.$$

Then, thanks to Hypothesis 3, by the dominated convergence theorem we can take the limit as n goes to infinity in both sides of (5.17) and we get

$$\frac{1}{Z} \int_{\mathcal{H}_0} P^\mu(t)\varphi(z)\, e^{2\,U(u)} \mathcal{N}(0, C_\mu)\, dz = \frac{1}{Z} \int_{\mathcal{H}_0} \varphi(z)\, e^{2\,U(u)} \mathcal{N}(0, C_\mu)\, dz,$$

for any $\varphi \in C_b(\mathcal{H}_0)$. By a monotone class argument the same identity follows for arbitrary $\varphi \in B_b(\mathcal{H}_0)$. This in particular implies that the measure

$$\nu_\mu = \frac{1}{Z}\, e^{2\,U(u)} \mathcal{N}(0, C_\mu)(dz)$$

is invariant for $P^\mu(t)$.

Finally, we obtain the second part of the theorem, as we have

$$\Pi_1\, [\frac{1}{Z}\, e^{2\,U(u)} \mathcal{N}(0, C_\mu)(dz)] = \frac{1}{Z}\, e^{2\,U(u)} \mathcal{N}(0, (-\Delta)^{-1}/2)(dz).$$

□

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