

Alexei Borodin · Grigori Olshanski

Markov processes on partitions

Received: 22 December 2004 / Revised version: 9 May 2005 /
Published online: 17 August 2005 – © Springer-Verlag 2005

Abstract. We introduce and study a family of Markov processes on partitions. The processes preserve the so-called z -measures on partitions previously studied in connection with harmonic analysis on the infinite symmetric group. We show that the dynamical correlation functions of these processes have determinantal structure and we explicitly compute their correlation kernels. We also compute the scaling limits of the kernels in two different regimes. The limit kernels describe the asymptotic behavior of large rows and columns of the corresponding random Young diagrams, and the behavior of the Young diagrams near the diagonal.

Our results show that recently discovered analogy between random partitions arising in representation theory and spectra of random matrices extends to the associated time-dependent models.

Introduction

In a series of papers (see [BO1], [OI2], references therein, and also [BO5]) we have been studying a remarkable family of probability distributions on partitions (equivalently, Young diagrams) called z -measures. These objects have a representation theoretic origin, they arise in harmonic analysis on the infinite symmetric group, see [KOV1], [KOV2]. Surprisingly enough, the z -measures turned out to be related to a number of probabilistic models of random matrix theory, stochastic growth, random tilings, percolation theory, etc. In this paper, we introduce and study a family of Markov processes on partitions which preserve the z -measures. Our main result is the computation of the dynamical correlation functions for these Markov processes. We also compute the scaling limits of the correlation functions corresponding to two different limit regimes as the size of partitions tends to infinity. In the first regime we look at the largest rows and columns of the random Young diagram¹ while in the second one we focus on the boundary of the random Young diagram near the diagonal.

Examples of dynamical models of random matrix type are well known. The sources of dynamics may be very different: in the Gaussian random matrix ensembles one allows the matrix elements to evolve according to the stationary Ornstein–

A. Borodin: Mathematics 253–37, Caltech, Pasadena, CA 91125, USA.
e-mail: borodin@caltech.edu

G. Olshanski: Dobrushin Mathematics Laboratory, Institute for Information Transmission Problems, Bolshoy Karetny 19, 127994 Moscow GSP-4, Russia.
e-mail: olsh@online.ru

¹ This limit regime has a representation theoretic meaning, see [BO2], [OI2].

Uhlenbeck process (Dyson’s Brownian motion [Dy]), in tiling models one reads the two–dimensional picture section by section [Jo3], [Jo6], [Jo7], [OR], in growth models the time parameter is present from the very beginning [PS].

In our setting the construction of dynamics is different; it is based on representation theory. We heavily rely on the fact that the z -measures define characters of the infinite symmetric group and thus possess a special *coherency property*. It reflects the consistency of restrictions of a character of the infinite symmetric group to various finite subgroups. The resulting Markov processes are analogous to those arising in other models, and in degenerations they even coincide with some of those, see [BO7]. It is rather surprising that the similarity of the z -measures to measures of different origin extends to dynamics associated with those models.

One of the elements of our construction is a special family of birth and death processes associated with Meixner orthogonal polynomials. Such birth and death processes, among many others, were extensively studied by Karlin–McGregor [KMG1], [KMG2]. Certain degenerations of our Markov processes admit a natural description in the language of Karlin–McGregor, see §6.5 below.

Let us now describe our results in more detail.

Let \mathbb{Y} denote the set of all Young diagrams. We consider a family $M_{z,z',\xi}$ of probability measures on \mathbb{Y} which depend on two complex parameters z and z' and a real parameter $\xi \in (0, 1)$. The weight of a Young diagram λ with respect to $M_{z,z',\xi}$ is given by

$$M_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'} \xi^{|\lambda|} (z)_\lambda (z')_\lambda \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 .$$

Here

$$(z)_\lambda = \prod_{(i,j) \in \lambda} (z + j - i)$$

(product over the boxes of λ) is the generalized Pochhammer symbol, and $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group of degree $|\lambda|$ associated to λ . In order for $M_{z,z',\xi}(\lambda)$ to be nonnegative for all $\lambda \in \mathbb{Y}$, we need to impose certain restrictions on z and z' , for instance, $z' = \bar{z}$. All possibilities for (z, z') are given before Proposition 1.2 below.

Let \mathbb{Y}_n denote the set of all Young diagrams with n boxes. Restricting $M_{z,z',\xi}$ to $\mathbb{Y}_n \subset \mathbb{Y}$ and renormalizing it, we obtain a probability measure $M_{z,z'}^{(n)}$ on \mathbb{Y}_n , which does not depend on ξ . The measure $M_{z,z',\xi}$ may be viewed as a mixture of the finite level measures $M_{z,z'}^{(n)}$.

The Markov processes that we are about to construct, are jump processes with countable state space \mathbb{Y} and continuous time $t \in \mathbb{R}$. The jumps are of two types: one either adds a box to the random Young diagram, or one removes a box from the diagram.

The event of adding or removing a box is governed by a birth and death process $N_{c,\xi}(t) := |\lambda(t)|$ on \mathbb{Z}_+ . This process depends on ξ and the product $c = zz'$, and its jump rates are given by

$$\begin{aligned} \text{Prob}\{N_{c,\xi}(t+dt) = n+1 \mid N_{c,\xi}(t) = n\} &= \frac{\xi(c+n)dt}{1-\xi}, \\ \text{Prob}\{N_{c,\xi}(t+dt) = n-1 \mid N_{c,\xi}(t) = n\} &= \frac{ndt}{1-\xi}. \end{aligned}$$

This is special case of the birth and death processes considered in [KMG2]. Its invariant distribution, the so-called negative binomial distribution, is the weight function for the Meixner orthogonal polynomials.

Conditioned on the jump $n \rightarrow n+1$, the choice of the box (i, j) to be added to λ is made according to the *transition probabilities*

$$p^\uparrow(n, \lambda; n+1, \nu) = \frac{(z+j-i)(z'+j-i) \dim \nu}{(zz'+n)(n+1) \dim \lambda},$$

and conditioned on the jump $n \rightarrow n-1$, the choice of the box (i, j) to be removed from λ is made according to the *cotransition probabilities*

$$p^\downarrow(n, \lambda; n-1, \mu) = \frac{\dim \mu}{\dim \lambda}$$

(here $\nu = \lambda \cup (i, j)$ and $\mu = \lambda \setminus (i, j)$).

The transition and cotransition probabilities are naturally associated with finite level measures $M_{z,z'}^{(n)}$. These probabilities were introduced in [VK] in the context of general characters of the infinite symmetric group (see also [Ke2]).

The jump rates $\lambda \nearrow \nu$ and $\lambda \searrow \mu$ correctly define a stationary Markov process $\Lambda_{z,z',\xi}(t)$ on \mathbb{Y} . The measure $M_{z,z',\xi}$ is the invariant measure for this process. Moreover, $\Lambda_{z,z',\xi}$ is reversible. In the degenerate case of z or z' being an integer, $\Lambda_{z,z',\xi}$ can be interpreted in terms of finitely many independent birth and death processes subject to a nonintersection condition, see §6.5 below.

One can also construct Markov chains which preserve the finite level measures $M_{z,z'}^{(n)}$. The key idea is that finite level measures are preserved by transition and cotransition probabilities. Thus, adding a random box and removing a random box afterwards leaves $M_{z,z'}^{(n)}$ invariant. Alternatively, one can first remove a box and then add a box. These two procedures yield two different Markov chains. They were suggested by Kerov a long time ago (unpublished). The same idea was independently exploited by Fulman [Fu]. It should be noted that our methods based on determinantal point processes are not directly applicable to such Markov chains. The idea of mixing all finite level measures together² is essential for us, it allows us to obtain explicit formulas for dynamical correlation functions, as we explain below.

It is well known that Young diagrams can be viewed as infinite subsets (point configurations) in a one-dimensional lattice. This parametrization of Young diagrams turns out to be very useful.

Let \mathbb{Z}' be the lattice of (proper) half-integers

$$\mathbb{Z}' = \mathbb{Z} + \frac{1}{2} = \{\dots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}.$$

² which may be viewed as a passage to the *grand canonical ensemble*, cf. [Ve].

For any $\lambda \in \mathbb{Y}$ we set

$$\underline{X}(\lambda) = \{\lambda_i - i + \frac{1}{2} \mid i = 1, 2, \dots\} \subset \mathbb{Z}'.$$

For instance, for the empty diagram $\lambda = \emptyset$, $\underline{X}(\emptyset) = \{\dots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}\}$. Using the correspondence $\lambda \mapsto \underline{X}(\lambda)$ we interpret the measure $M_{z,z',\xi}$ on \mathbb{Y} as a probability measure on $2^{\mathbb{Z}'}$. This makes it possible to speak about the *dynamical correlation functions* of $\Lambda_{z,z',\xi}$ which uniquely determine the process. They are defined by

$$\rho_n(t_1, x_1; t_2, x_2; \dots; t_n, x_n) = \text{Prob} \left\{ \underline{X}(\lambda) \text{ at time } t_i \text{ contains } x_i \text{ for } 1 \leq i \leq n \right\}.$$

Here $n = 1, 2, \dots$, and the n th correlation function ρ_n is a function of n pairwise distinct arguments $(t_1, x_1), \dots, (t_n, x_n) \in \mathbb{R} \times \mathbb{Z}'$.

The notion of the dynamical correlation functions is a hybrid of the finite-dimensional distributions of a stochastic process and standard correlation functions of probability measures on point configurations.

The reason why we are interested in dynamical correlation functions is the same as in the “static” (fixed time) case: As we take scaling limits of our processes, the notion of weight of a point configuration ceases to make any sense because the space of relevant point configurations becomes uncountable. On the other hand, the scaling limits of the correlation functions do exist, and they carry complete information about the asymptotic behavior of our processes.

Theorem A (Part 1). *The dynamical correlation functions of $\Lambda_{z,z',\xi}$ have the determinantal form ($n = 1, 2, \dots$)*

$$\rho_n(t_1, x_1; \dots; x_n, t_n) = \det \left[\underline{K}_{z,z',\xi}(t_i, x_i; t_j, x_j) \right]_{i,j=1}^n,$$

where the correlation kernel $\underline{K}_{z,z',\xi}(s, x; t, y)$ is a function on $(\mathbb{R} \times \mathbb{Z}')^2$ which can be explicitly computed.

One way of writing the kernel is by a double contour integral

$$\begin{aligned} & \underline{K}_{z,z',\xi}(s, x; t, y) \\ &= \frac{e^{\frac{1}{2}(s-t)} \Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2}) (-1)^{x+y+1}}{(\Gamma(-z - x + \frac{1}{2}) \Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}} \\ & \times \frac{1 - \xi}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} (1 - \sqrt{\xi} \omega_1)^{-z'} (1 - \sqrt{\xi} \omega_1^{-1})^z (1 - \sqrt{\xi} \omega_2)^{-z} (1 - \sqrt{\xi} \omega_2^{-1})^z \\ & \times \frac{\omega_1^{-x-\frac{1}{2}} \omega_2^{-y-\frac{1}{2}} d\omega_1 d\omega_2}{e^{s-t} (\omega_1 - \sqrt{\xi}) (\omega_2 - \sqrt{\xi}) - (1 - \sqrt{\xi} \omega_1) (1 - \sqrt{\xi} \omega_2)} \end{aligned}$$

with the contours $\{\omega_1\}$ and $\{\omega_2\}$ of ω_1 and ω_2 satisfying the following conditions:

- $\{\omega_1\}$ and $\{\omega_2\}$ go around 0 in positive direction and pass between $\sqrt{\xi}$ and $1/\sqrt{\xi}$.
- The contours are chosen so that the denominator in the formula above does not vanish. There are two possibilities of doing that; one of them is used for $s \geq t$, and the other one is used for $s < t$, see Theorem 6.1 below for details.

This integral representation is convenient for computing the scaling limits of the correlation functions. However, it does not reveal important structural features of the kernel. Let us now present another way of writing the correlation kernel.

Consider a second order difference operator D on \mathbb{Z}' , depending on parameters (z, z', ξ) and acting on functions $f(\cdot) \in \ell^2(\mathbb{Z}')$ as follows

$$(Df)(x) = \sqrt{\xi(z+x+\frac{1}{2})(z'+x+\frac{1}{2})} f(x+1) \\ + \sqrt{\xi(z+x-\frac{1}{2})(z'+x-\frac{1}{2})} f(x-1) - (x+\xi(z+z'+x)) f(x).$$

This is a self-adjoint operator with discrete simple spectrum $\text{Sp } D = \{(1-\xi)\mathbb{Z}'\}$. Its eigenfunctions ψ_a ,

$$D\psi_a = (1-\xi)a \cdot \psi_a,$$

are explicitly written through the Gauss hypergeometric function, see (5.1) below. We normalize them by the condition $\|\psi_a\| = 1$.

Theorem A (Part 2). *The correlation kernel for the dynamical correlation functions of the Markov process $\Lambda_{z,z',\xi}$ can also be written as*

$$\underline{K}_{z,z',\xi}(s, x; t, y) = \pm \sum_{a=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots} e^{-a|s-t|} \psi_{\pm a}(x) \psi_{\pm a}(y)$$

with “+” taken for $s \geq t$ and “-” taken for $s < t$.

The functions $\{\psi_a\}$ form an orthonormal basis in $\ell^2(\mathbb{Z}')$. Thus, for $s = t$ the kernel $\underline{K}_{z,z',\xi}$ defines a projection operator whose range is the span of the eigenfunctions of D corresponding to the positive part of the spectrum of D . The kernel of this projection operator can be written in a simpler, so-called *integrable form*:

$$\underline{K}_{z,z',\xi}(x, y) = \frac{P(x)Q(y) - Q(x)P(y)}{x - y},$$

where P and Q are expressed through the Gauss hypergeometric function. This kernel is called the *discrete hypergeometric kernel*. It was found in [BO2] and further studied in [BO4], [BO5] (see also [BO8, §3]).

The formula of Theorem A (Part 2) shows that our Markov process is determined by the following data: a state space \mathfrak{X} , a Hilbert space H of functions on \mathfrak{X} , a self-adjoint operator D in H , and two complementary spectral projection operators P_{\pm} for D . In our case, $\mathfrak{X} = \mathbb{Z}'$, $H = \ell^2(\mathbb{Z}')$, D is the difference operator given above, and P_{\pm} are projections on the positive and negative parts of the spectrum of D .

It seems that generating Markov processes with determinantal correlation functions by data $(\mathfrak{X}, H, D, P_{\pm})$ of this type is a rather general phenomenon. Similar structures have appeared earlier in the dynamics arising in polynuclear growth models [PS], [Jo4], in tiling models [Jo6], [Jo7], and in random matrix theory [NF], [Jo5], [TW]. Following the terminology of those papers, we call the kernel of Theorem A the *extended hypergeometric kernel*.

The reader might notice that in our Theorem A as well as in all the papers cited above, the values of the extended (dynamical) kernels are always given by somewhat different expressions depending on the relative order of the time variables. This dichotomy is unavoidable because of a certain discontinuity of the dynamical correlation functions. For example, we must have

$$s \approx t \Rightarrow \rho_2(s, x; t, y) \approx \begin{cases} \rho_2(s, x; s, y), & x \neq y, \\ \rho_1(s, x), & x = y. \end{cases}$$

If we assume the determinantal structure of the dynamical correlation functions with a kernel $K(s, x; t, y)$ then

$$\rho_1(s, x) = K(s, x; s, x), \quad \rho_2(s, x; t, y) = \begin{vmatrix} K(s, x; s, x) & K(s, x; t, y) \\ K(t, y; s, x) & K(t, y; t, y) \end{vmatrix}.$$

If we further assume that the kernel is continuous in s, t subject to the condition $s \geq t$,³ then the above relations imply

$$K(s, x; s, y) = \lim_{\epsilon \rightarrow +0} K(s, x; s - \epsilon, y) = \lim_{\epsilon \rightarrow +0} K(s, x; s + \epsilon, y) + \delta_{xy}.$$

The validity of the last relation for the extended hypergeometric kernel can be immediately observed from Part 2 of Theorem A using the fact that $\{\psi_a\}$ form an orthonormal basis.

Let us now describe our results on scaling limits of the dynamical correlation functions. In our previous works we considered three asymptotic regimes for random Young diagrams without dynamics: one for largest rows and columns, one for rows and columns of intermediate growth, and one for the behavior of the boundary of the Young diagrams near the diagonal, see [BO5] and references therein. In all three limit regimes the parameter ξ tends to 1, which makes the expected number of boxes in the random Young diagram go to infinity.

In this paper we concentrate on the first and the third limit regime, but with the presence of dynamics. Let us start with the behavior of large rows and columns.

In order to catch the largest rows and columns in the limit $\xi \nearrow 1$, we need to scale them by $(1 - \xi)$. This leads to scaling of the state space \mathbb{Z}' by the same factor. That is, \mathbb{Z}' is replaced by $(1 - \xi)\mathbb{Z}'$ which in the limit turns into $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

The parametrization of Young diagrams by point configurations $\underline{X}(\lambda)$ is not suitable for this limit transition. Or, rather to say, the positive part of $\underline{X}(\lambda)$ indeed reflects the behavior of largest rows, while the behavior of the largest columns is captured by the complement of the negative part of $\underline{X}(\lambda)$ in $\{\dots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}\}$. Thus, instead of encoding λ by $\underline{X}(\lambda)$ we use the map

$$\lambda \mapsto X(\lambda) = \left(\underline{X}(\lambda) \cap \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right\} \right) \cup \left(\left\{ \dots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2} \right\} \setminus \underline{X}(\lambda) \right).$$

We refer the reader to [BO2], [OI2] for representation theoretic interpretation of this map and for further details.

³ We could have used $s \leq t$ equally well, this is a question of convention. For instance, transposition of the kernel does not affect the correlation functions, and this operation just turns $s \geq t$ into $s \leq t$.

Theorem B. *The scaling limits, as $\xi \rightarrow 1$, of the dynamical correlation functions of $\Lambda_{z, z', \xi}$ corresponding to the map $\lambda \mapsto X(\lambda)$, under the rescaling of \mathbb{Z}' by $(1 - \xi)$, have determinantal form with the correlation kernel $K_{z, z'}^W(s, u; t, v)$ on $(\mathbb{R} \times \mathbb{R}^*)^2$. This kernel has four blocks according to the choices of signs of u and v .*

The block with $u, v > 0$ has an integral representation

$$K_{z, z'}^W(s, u; t, v) = e^{\pi i(z+z')(u/v)} (u/v)^{\frac{z-z'}{2}} e^{\frac{1}{2}(s-t)} \frac{1}{(2\pi i)^2} \\ \times \oint_{+\infty}^{0-} \oint_{+\infty}^{0-} \zeta_1^{-z'} (1+\zeta_1)^z \zeta_2^{-z} (1+\zeta_2)^{z'} \frac{e^{-u(\zeta_1+\frac{1}{2})-v(\zeta_2+\frac{1}{2})} d\zeta_1 d\zeta_2}{e^{s-t}(1+\zeta_1)(1+\zeta_2)-\zeta_1\zeta_2}$$

with different choices of contours for $s \geq t$ and $s < t$, see Theorem 8.4 below.

The same block has a series representation

$$K_{z, z'}^W(s, u; t, v) = \pm \sum_{a=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots} e^{-a|s-t|} w_{\pm a}(u) w_{\pm a}(v),$$

where “+” is taken for $s \geq t$, “-” is taken for $s < t$, and

$$w_a(u) = \lim_{\xi \rightarrow 1} (1 - \xi)^{\frac{1}{2}} \psi_a([(1 - \xi)^{-1}u])$$

are eigenfunctions of a second order differential operator on \mathbb{R}_+ :

$$u w_a''(u) + w_a'(u) + \left(-\frac{u}{4} + \frac{z+z'}{2} - \frac{(z-z')^2}{4u}\right) w_a(u) = a w_a(u),$$

which are explicitly written through the Whittaker functions, see (8.1) below.

Similar expressions are available for three other blocks of $K_{z, z'}^W(s, u; t, v)$, see Theorems 8.2 and 8.4 below.

We call $K_{z, z'}^W(s, u; t, v)$ the *extended Whittaker kernel*.

In the “static” case $s = t$ the kernel admits a simpler “integrable” form, see [BO1], [BO2], [B1], [Ol2], and (8.4) below.

Let us now proceed to the other limit regime which describes the behavior of the Young diagrams near the diagonal. This just means that we stay on the lattice \mathbb{Z}' . For this asymptotic regime it does not really matter whether we use $\underline{X}(\lambda)$ or $X(\lambda)$ to encode the Young diagrams. We refer to [BO5] for a detailed discussion of this regime.

In the following statement we will use a more detailed notation $w_a(u; z, z')$ for the functions $w_a(u)$ introduced above.

Theorem C. *The limits, as $\xi \rightarrow 1$, of the dynamical correlation functions of $\Lambda_{z, z', \xi}$ corresponding to the map $\lambda \mapsto \underline{X}(\lambda)$, under the rescaling of time by $(1 - \xi)^{-1}$, have determinantal form with the correlation kernel $\underline{K}_{z, z'}^{\text{gamma}}(\sigma, x; \tau, y)$ on $(\mathbb{R} \times \mathbb{Z}')^2$.*

For $\sigma \geq \tau$, the correlation kernel can be written in two different ways: as a double contour integral

$$\begin{aligned} & \underline{K}_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) \\ &= \frac{\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})e^{-\pi i(z+z')}(-1)^{x+y+1}}{\left(\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2})\right)^{\frac{1}{2}}} \\ & \times \frac{1}{(2\pi i)^2} \oint_{+\infty}^{0-} \oint_{+\infty}^{0-} \frac{\zeta_1^{z'+x-\frac{1}{2}}(1+\zeta_1)^{-z-x-\frac{1}{2}}\zeta_2^{z+y-\frac{1}{2}}(1+\zeta_2)^{-z'-y-\frac{1}{2}} d\zeta_1 d\zeta_2}{1 + (\sigma - \tau) + \zeta_1 + \zeta_2} \end{aligned}$$

and as a single integral

$$\underline{K}_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) = \int_0^{+\infty} e^{-u(\sigma-\tau)} w_x(u; -z, -z') w_y(u; -z, -z') du.$$

The values of the kernel for $\sigma < \tau$ are obtained from the above formulas using the symmetry property

$$\underline{K}_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) = (-1)^{x+y+1} \underline{K}_{-z,-z'}^{\text{gamma}}(\tau, -x; \sigma, -y), \quad \sigma \neq \tau.$$

For $\sigma = \tau$ the kernel admits a simpler expression of “integrable” type

$$\frac{P(x)Q(y) - Q(x)P(y)}{x - y}$$

where P and Q are expressed through gamma functions only, see [BO5] and (8.3) below. That kernel was called the gamma kernel, and for this reason we call $\underline{K}_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y)$ the *extended gamma kernel*.

Note that the extended gamma kernel fits into the same abstract scheme as the extended hypergeometric kernel: one takes $\mathfrak{X} = \mathbb{Z}'$, $H = \ell^2(\mathbb{Z}')$, D is the special case of the difference operator given above corresponding to the limit value $\xi = 1$. The spectrum of this operator fills the whole real axis, the eigenfunctions are $x \mapsto w_x(u; -z, -z')$, and the spectral projections P_{\pm} again correspond to the positive and negative parts of the spectrum.

The functions $\psi_a(x) = \psi_a(x; z, z', \xi)$ used in the discussion of the extended hypergeometric kernel have the following symmetry:

$$\psi_a(x; z, z', \xi) = \psi_x(a; -z, -z', \xi).$$

This means, in particular, that $\psi_a(x)$ satisfies second order difference equations both in a and x (the *bispectrality property*, see [Gr]). The two limit transitions considered above (Theorems B and C) correspond to taking continuous limits in x and a , respectively. This explains why we end up with the same functions $w_a(u)$ in Theorems B and C.

Let us make some remarks about our proof of Theorem A. As a matter of fact, we prove the theorem in a greater generality. We introduce certain *time inhomogeneous* Markov processes on partitions. Their fixed time distributions are also the measures $M_{z,z',\xi}$, but now $\xi = \xi(t)$ varies with time t . The construction of these

processes is similar to the stationary ones except that the birth and death process on \mathbb{Z}_+ becomes time inhomogeneous. In particular, we consider pure birth and pure death processes for which the Young diagrams either always gain new boxes or always lose their boxes. These “pure” processes are simpler, their transition probabilities can be evaluated explicitly. They can also be viewed as building blocks of general processes, more exactly, the transition matrix $P(s, t)$ for a general process can be represented as a product $P(s, t) = P^\downarrow(s, u)P^\uparrow(u, t)$ of transition matrices of “pure” processes for a suitable intermediate time moment $u \in (s, t)$.

This product representation of the transition matrix plays an important role in the proof of Theorem A. We first prove the theorem for a degenerate case, when one of the parameters z, z' is an integer, and the process is “finite-dimensional”, that is, it lives on the Young diagrams with bounded number of rows or columns. Then the needed formulas are derived from a version of Eynard–Mehta theorem on spectral correlations of coupled random matrices [EM].⁴ The passage from the degenerate case to the general one is based on analytic continuation in the parameters z and z' . This passage is not trivial since we need to extrapolate from the integer points to a complex domain. The needed analytic properties of the dynamical correlation functions are derived from the product formula for the transition matrix $P(s, t)$ mentioned above. Let us also emphasize that in our approach, the introduction of time inhomogeneous processes is necessary for handling the stationary case.

Let us point out that there exists another way of obtaining the dynamical correlation functions of Theorem A, based on the formalism of infinite wedge Fock space. In [Ok2] Okounkov gave an elegant derivation of static ($s = t$) correlation functions (initially computed in [BO2]) using a representation of $SL(2)$ by the so-called Kerov operators. We can extend Okounkov’s approach to derive the formula of Theorem A. This alternative path bears some similarity to the formalism of *Schur processes* of [OR], [Ok3]. However, the Schur processes seem to be not applicable in our situation. Note also that despite the beauty of Okounkov’s idea, a rigorous realization of this approach would have to overcome certain nontrivial technical difficulties.

One more important subject that we do not touch upon in this paper, is a family of Markov processes on partitions related to Plancherel measures. In the limit $z, z' \rightarrow \infty, \xi \rightarrow 0, zz'\xi \rightarrow \theta > 0$, the measures $M_{z, z', \xi}$ tend to the so-called *poissonized Plancherel measure* on \mathbb{Y} with Poisson parameter θ . This connection was used in [BOO] to study the asymptotics of the Plancherel measures. Using the general scheme presented in this paper, one constructs Markov processes on \mathbb{Y} which preserve the poissonized Plancherel measures. These processes may be viewed as degenerations of the processes considered in this paper. They are equivalent to the droplet model of polynuclear growth. Our results on this other family of Markov processes and their scaling limits are presented in [BO7]. Let us note that the analog of Theorem A for those processes can be obtained either by limit transition from Theorem A or by using the Schur process of [OR].

The present paper is organized as follows. In Section 1 we introduce the z -measures, the associated transition and cotransition probabilities, and other

⁴ Other proofs of this theorem can be found in [NF], [Jo4], [TW], [BR].

notions related to the Young graph. In Section 2 we introduce time homogeneous and inhomogeneous Markov processes on \mathbb{Y} , prove their existence and uniqueness, and compute the transition probabilities for “pure” ascending and descending processes. In Section 3 we evaluate the transition matrices for integral values of parameters. In Section 4 we study the analytic nature of the dependence of the dynamical correlation functions on the parameters. In Section 5 we list necessary facts about the eigenfunctions ψ_a of the second order difference operator D on \mathbb{Z}' (more detailed information including proofs can be found in [BO8, §2]). In Section 6 we prove Theorem A first in the degenerate case using Eynard–Mehta theorem and Meixner polynomials, and then in the general case using analytic continuation. In Section 7 we derive the dynamical correlation functions of $\Lambda_{z,z',\xi}$ corresponding to the map $\lambda \mapsto X(\lambda)$ (as opposed to the map $\lambda \mapsto \underline{X}(\lambda)$ used in Theorem A). In Section 8 we prove Theorem B, and in Section 9 we prove Theorem C.

1. Z-measures

As in Macdonald [Ma] we identify partitions and Young diagrams. By \mathbb{Y}_n we denote the set of partitions of a natural number n , or equivalently, the set of Young diagrams with n boxes. By \mathbb{Y} we denote the set of all Young diagrams, that is, the disjoint union of the finite sets \mathbb{Y}_n , where $n = 0, 1, 2, \dots$ (by convention, \mathbb{Y}_0 consists of a single element, the empty diagram \emptyset). Given $\lambda \in \mathbb{Y}$, let $|\lambda|$ denote the number of boxes of λ (so that $\lambda \in \mathbb{Y}_{|\lambda|}$), let $\ell(\lambda)$ be the number of nonzero rows in λ (the length of the partition), and let λ' denote the transposed diagram.

For two Young diagrams λ and μ we write $\mu \nearrow \lambda$ (equivalently, $\lambda \searrow \mu$) if $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$, or, in other words, μ is obtained from λ by removing one box.

The *Young graph* is the graph whose vertices are the elements of \mathbb{Y} and the edges join all pairs (μ, λ) such that $\mu \nearrow \lambda$. The Young graph will also be denoted by \mathbb{Y} . Clearly, $\mu \nearrow \lambda$ implies $\mu' \nearrow \lambda'$, so that the transposition operation $\lambda \mapsto \lambda'$ induces an involutive automorphism of the Young graph.

For any $\lambda \in \mathbb{Y}_n$, standard Young tableaux of shape λ can be viewed as paths

$$\emptyset \nearrow \lambda^{(1)} \nearrow \dots \nearrow \lambda^{(n)} = \lambda$$

in \mathbb{Y} . Let $\dim \lambda$ be the number of all such paths. A convenient explicit formula for $\dim \lambda$ is

$$\dim \lambda = \frac{n!}{\prod_{i=1}^N (\lambda_i + N - i)!} \prod_{1 \leq i < j \leq N} (\lambda_i - i - \lambda_j + j), \quad \lambda \in \mathbb{Y}_n,$$

where N is an arbitrary integer $\geq \ell(\lambda)$ (the above expression is stable in N). For $\lambda \in \mathbb{Y}_n, \mu \in \mathbb{Y}_{n-1}$ set

$$p^\downarrow(n, \lambda; n - 1, \mu) = \begin{cases} \frac{\dim \mu}{\dim \lambda}, & \mu \nearrow \lambda, \\ 0, & \text{otherwise} \end{cases} \quad (1.1)$$

and note that

$$\sum_{\mu \in \mathbb{Y}_{n-1}} p^\downarrow(n, \lambda; n-1, \mu) = 1.$$

The numbers $p^\downarrow(n, \lambda; n-1, \mu)$ are called the *cotransition probabilities* of the Young graph. They admit a nice interpretation in terms of the “hook walk” algorithm of Greene, Nijenhuis, and Wilf [GNW1].

A family $\{M^{(n)}\}$ of probability measures $M^{(n)}$ on \mathbb{Y}_n , $n = 0, 1, 2, \dots$, is called a *coherent system on \mathbb{Y}* if the measures are consistent with the cotransition probabilities in the following sense:

$$M^{(n-1)}(\mu) = \sum_{\lambda \in \mathbb{Y}_n} M^{(n)}(\lambda) p^\downarrow(n, \lambda; n-1, \mu), \quad \mu \in \mathbb{Y}_{n-1}, \quad n=1, 2, \dots \quad (1.2)$$

This concept has an important representation theoretic meaning. Namely, there is a 1–1 correspondence between coherent systems on \mathbb{Y} and normalized positive definite class functions on the infinite symmetric group, see [VK], [Ke2], [OI2].

Note that the cotransition probabilities are invariant under the involution $\lambda \mapsto \lambda'$ of the Young graph. Consequently, the push-forward of a coherent system under this involution is again a coherent system.

Example 1.1. The *Plancherel measures* defined by

$$M_{Plancherel}^{(n)}(\lambda) = \frac{(\dim \lambda)^2}{n!}$$

form a coherent family of probability measures, see [VK].

Let \mathcal{T} be the set of all infinite paths in \mathbb{Y} of the form

$$\emptyset \nearrow \lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \dots \nearrow \lambda^{(n)} \nearrow \dots, \quad \lambda^{(n)} \in \mathbb{Y}_n.$$

This is a compact topological space (a closed subset of the product space $\prod_{n \geq 0} \mathbb{Y}_n$).

A probability measure \mathcal{M} on \mathcal{T} is called *central* if for any $n = 1, 2, \dots$ and any $\lambda \in \mathbb{Y}_n$, the mass of each cylinder set consisting of all paths with fixed $\lambda^{(1)}, \dots, \lambda^{(n)} = \lambda$ depends on λ only (and does not depend on $\lambda^{(1)}, \dots, \lambda^{(n-1)}$).

Any coherent system $\{M^{(n)}\}$ generates a central measure \mathcal{M} on \mathcal{T} . By definition, the mass of the cylinder set mentioned above equals $M^{(n)}(\lambda) / \dim \lambda$. The relation (1.2) ensures that \mathcal{M} is correctly defined. This defines a one-to-one correspondence between coherent systems $\{M^{(n)}\}$ on \mathbb{Y} and central measure \mathcal{M} on \mathcal{T} , see [VK], [Ke2], [OI2].

For any central measure \mathcal{M} ,

$$p^\downarrow(n, \lambda; n-1, \mu) = \text{Prob}\{\lambda^{(n-1)} = \mu \mid \lambda^{(n)} = \lambda\},$$

which is a justification of the term “cotransition probability”.

Assuming $M^{(|\lambda|)}(\lambda) > 0$ for all $\lambda \in \mathbb{Y}$, set

$$p^\uparrow(n, \lambda; n+1, \nu) = \text{Prob}\{\lambda^{(n+1)} = \nu \mid \lambda^{(n)} = \lambda\}, \quad n = |\lambda|.$$

In contrast to $p^\downarrow(n, \lambda; n - 1, \mu)$, these numbers depend on \mathcal{M} . We call them the *transition probabilities* of the central measure \mathcal{M} (or of the corresponding coherent system $\{M^{(n)}\}$). The transition probabilities define \mathcal{M} and $\{M^{(n)}\}$ uniquely.

Note an important relation between the transition and cotransition probabilities:

$$M^{(n)}(\lambda)p^\uparrow(n, \lambda; n + 1, \nu) = p^\downarrow(n + 1, \nu; n, \lambda)M^{(n+1)}(\nu). \quad (1.3)$$

It implies, in particular, that

$$p^\uparrow(n, \lambda; n + 1, \nu) = \begin{cases} \frac{M^{(n+1)}(\nu) \dim \lambda}{M^{(n)}(\lambda) \dim \nu}, & \lambda \nearrow \nu, \\ 0, & \text{otherwise.} \end{cases} \quad (1.4)$$

If $M^{(\lambda)}(\lambda)$ vanishes for some $\lambda \in \mathbb{Y}$ then the definition has to be slightly modified. Namely, let $\text{supp } \mathcal{M}$ be the set of those $\lambda \in \mathbb{Y}$ for which $M^{(\lambda)}(\lambda) > 0$. Equivalently, $\lambda \in \text{supp } \mathcal{M}$ if the set of paths passing through λ has positive mass with respect to \mathcal{M} . Note that $\lambda \in \text{supp } \mathcal{M}$ implies $\mu \in \text{supp } \mathcal{M}$ for all $\mu \nearrow \lambda$. The set $\text{supp } \mathcal{M}$ spans a subgraph of \mathbb{Y} (which may be called the *support of \mathcal{M}*), and the transition probabilities are correctly defined on this subgraph by the same formula (1.3). Again, the initial central measure \mathcal{M} is uniquely determined by its support and the transition probabilities.

Note two useful equations

$$M^{(n-1)}(\mu) = \sum_{\lambda} M^{(n)}(\lambda)p^\downarrow(n, \lambda; n - 1, \mu), \quad (1.5)$$

$$M^{(n+1)}(\nu) = \sum_{\lambda} M^{(n)}(\lambda)p^\uparrow(n, \lambda; n + 1, \nu). \quad (1.6)$$

We shall need the *generalized Pochhammer symbol* $(z)_\lambda$:

$$(z)_\lambda = \prod_{i=1}^{\ell(\lambda)} (z - i + 1)_{\lambda_i}, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{Y},$$

where

$$(x)_k = x(x + 1) \dots (x + k - 1) = \frac{\Gamma(x + k)}{\Gamma(x)}$$

is the conventional Pochhammer symbol. Note that

$$(z)_\lambda = \prod_{(i,j) \in \lambda} (z + j - i)$$

(product over the boxes of λ), which implies at once the symmetry relation

$$(z)_\lambda = (-1)^{|\lambda|} (-z)_{\lambda'}.$$

For two complex parameters z, z' set

$$M_{z,z'}^{(n)}(\lambda) = \frac{(z)_\lambda (z')_\lambda}{(zz')_n} \frac{(\dim \lambda)^2}{n!}, \quad n = 0, 1, \dots, \quad \lambda \in \mathbb{Y}_n, \quad (1.7)$$

where $\dim \lambda$ was defined in the beginning of the section. The expression (1.7) makes sense if $(zz')_n$ does not vanish, i.e., if $zz' \notin \{0, -1, -2, \dots\}$. Obviously, (1.7) is symmetric in z, z' .

Note that (see Example 1.1)

$$\lim_{z, z' \rightarrow \infty} M_{z, z'}^{(n)}(\lambda) = M_{\text{Plancherel}}^{(n)}(\lambda). \quad (1.8)$$

Let us say that two nonzero complex numbers z, z' form an *admissible* pair of parameters if one of the following three conditions holds:

- The numbers z, z' are not real and are conjugate to each other.
- Both z, z' are real and are contained in the same open interval of the form $(m, m+1)$, where $m \in \mathbb{Z}$.
- One of the numbers z, z' (say, z) is a nonzero integer while z' has the same sign and, moreover, $|z'| > |z| - 1$.

Proposition 1.2. *If (z, z') is an admissible pair of parameters then $\{M_{z, z'}^{(n)}\}$ is a coherent family of probability measures.*

Proof. It is readily checked that if (and only if) one of the conditions above holds then $(z)_\lambda (z')_\lambda \geq 0$ for all λ , see [BO5, Proposition 1.8]. Moreover, $(zz')_n > 0$ for all n . Hence (1.7) is nonnegative. The fact that each $M_{z, z'}^{(n)}$ is a probability measure and the coherency property can be proved in several ways. See, e.g., [O11], [BO3]. \square

We call the measures $M_{z, z'}^{(n)}$ the z -measures on the floors \mathbb{Y}_n of the Young graph. Depending on which of the three conditions of Proposition 1.1 holds we will speak about the *principal*, *complementary* or *degenerate* series of z -measures, respectively. By virtue of (1.8), the z -measures may be viewed as a deformation of the Plancherel measure (for any fixed n). The principal series of z -measures first appeared in [KOV1], see also [KOV2]. For more information about the z -measures and their generalizations, see [BO2], [BO3], [BO4], [BO5], [BO6], [Ke1].

Note that the involution $\lambda \mapsto \lambda'$ of the Young graph takes $M_{z, z'}^{(n)}$ to $M_{-z, -z'}^{(n)}$.

Let $\mathcal{M}_{z, z'}$ be the central measure corresponding to the coherent family $\{M_{z, z'}^{(n)}\}_{n=0, 1, \dots}$. In the case of the principal or complementary series the support of $\mathcal{M}_{z, z'}$ is the whole \mathbb{Y} . For the degenerate series it is a proper subset of \mathbb{Y} : if $z = k = 1, 2, \dots$ and $z' > k - 1$ then $\text{supp } \mathcal{M}_{z, z'}$ consists of diagrams with at most k rows, and if $z = -k = -1, -2, \dots$ and $z' < -(k - 1)$ then $\text{supp } \mathcal{M}_{z, z'}$ consists of diagrams with at most k columns.

The transition probabilities of the z -measures are given by

$$p_{z, z'}^\uparrow(n, \lambda; n+1, \nu) = \frac{(z + c(\nu/\lambda))(z' + c(\nu/\lambda)) \dim \nu}{(zz' + n)(n+1) \dim \lambda}, \quad \lambda \nearrow \nu, \quad (1.9)$$

where $c(\nu/\lambda)$ denotes the *content* of the box $(i, j) = \nu/\lambda$, that is, $c = j - i$. Indeed, (1.9) follows immediately from (1.4) and (1.7). Note that if λ is in $\text{supp } \mathcal{M}_{z, z'}$ while ν is not (which may happen for the degenerate series) then (1.9) vanishes due to vanishing of one of the factors $z + c(\nu/\lambda), z' + c(\nu/\lambda)$.

For the Plancherel measure, the transition probabilities are

$$p_{Plancherel}^\uparrow(n, \lambda; n + 1, \nu) = \frac{\dim \nu}{(n + 1) \dim \lambda}, \quad \lambda \nearrow \nu,$$

see [VK].

The Plancherel transition probabilities admit an interpretation in terms of the “dual” hook walk algorithm described in [GNW2]. A similar interpretation for the z -measures is unknown.

Consider a special case of the negative binomial distribution on \mathbb{Z}_+ depending on two parameters $a > 0$ and $\xi \in (0, 1)$:

$$\pi_{a,\xi}(n) = (1 - \xi)^a \frac{(a)_n \xi^n}{n!}, \quad n = 0, 1, 2, \dots \tag{1.10}$$

The next formula defines a probability measure on \mathbb{Y} which is the mixture of all z -measures $M_{z'}^{(n)}$ with given fixed parameters z, z' and varying n by means of the distribution (1.10) on n 's, with parameters $a = zz'$ and ξ :

$$M_{z,z',\xi}(\lambda) = M_{z,z'}^{(|\lambda|)}(\lambda) \pi_{z,z',\xi}(|\lambda|) = (1 - \xi)^{zz'} \xi^{|\lambda|} (z)_\lambda (z')_\lambda \left(\frac{\dim \lambda}{|\lambda|!} \right)^2. \tag{1.11}$$

We call (1.11) the *mixed z -measure*. An interpretation of formula (1.11) is given in [BO5, Definition 1.4].

Likewise, consider a mixture of the Plancherel measures, depending on a parameter $\theta > 0$:

$$M_{Plancherel,\theta}(\lambda) = M_{Plancherel}^{(|\lambda|)}(\lambda) e^{-\theta} \frac{\theta^{|\lambda|}}{|\lambda|!} = e^{-\theta} \theta^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2. \tag{1.12}$$

We call (1.12) the *poissonized Plancherel measure*. Note that it can be obtained as a limit case of the mixed z -measures:

$$\lim_{\substack{z, z' \rightarrow \infty \\ \xi \rightarrow 0 \\ zz'\xi \rightarrow \theta}} M_{z,z',\xi}(\lambda) = M_{Plancherel,\theta}(\lambda).$$

The main objects of this paper are the z -measures and related Markov processes. One can also develop a parallel theory associated with the Plancherel measure. We do not pursue this goal in the present paper. An interested reader can find the statements of the main results related to the Plancherel measure in our paper [BO7].

2. Construction of Markov processes

In this section we introduce the continuous time Markov processes on partitions which will be studied in the paper. Their fixed time distributions are the z -measures introduced in section 1.

It is fairly easy to give the jump rates for these processes. We then prove that the jump rates define the processes uniquely and compute the transition probabilities for underlying birth–death processes.

2.1. Preliminaries on Markov processes

Let us recall some basic facts about continuous time Markov processes with discrete state space, and introduce the notation.

The time parameter t always ranges over an open interval (t_{\min}, t_{\max}) where $t_{\min} \in \mathbb{R} \cup \{-\infty\}$ and $t_{\max} \in \mathbb{R} \cup \{+\infty\}$. Let us denote the state space by \mathbb{A} , it is assumed to be either finite or countable.

We also denote by $P(s, t)$, $s \leq t$, the matrix of transition probabilities of a Markov process. This is a matrix with rows and columns marked by elements of \mathbb{A} , its elements will be denoted by $P_{ab}(s, t)$, $a, b \in \mathbb{A}$. By definition, $P_{ab}(s, t)$ is the probability that the process will be in the state b at the time moment t conditioned that it is in the state a at time s . Thus, $P(s, t)$ is a (row) stochastic matrix: its elements are nonnegative, and their sum along any row is equal to 1. The transition matrices $P(s, t)$ satisfy the *Chapman-Kolmogorov equation*

$$P(s, u)P(u, t) = P(s, t), \quad s \leq u \leq t. \quad (2.1)$$

We assume that there exist $\mathbb{A} \times \mathbb{A}$ matrices $Q(t)$ with continuously depending on t entries, such that

$$P_{ab}(s, t) = \delta_{ab} + Q_{ab}(t)(t - s) + o(|t - s|), \quad |t - s| \rightarrow 0. \quad (2.2)$$

This relation implies that $Q_{ab}(t) \geq 0$ for $a \neq b$ and $Q_{aa}(t) \leq 0$. Further, we assume that

$$\sum_{b \neq a} Q_{ab}(t) = -Q_{aa}(t), \quad \text{for any } a \in \mathbb{A}.$$

This is the infinitesimal analog of the condition $\sum_{b \in \mathbb{A}} P_{ab}(s, t) = 1$.

It is well known that (2.1) then implies that $P(s, t)$ satisfies *Kolmogorov's backward equation*

$$-\frac{\partial}{\partial s} P(s, t) = Q(s)P(s, t), \quad s \leq t, \quad (2.3)$$

with the initial condition

$$P(t, t) = \text{Id}. \quad (2.4)$$

Under certain additional conditions, $P(s, t)$ will also satisfy Kolmogorov's forward equation

$$\frac{\partial}{\partial t} P(s, t) = P(s, t)Q(t). \quad (2.5)$$

In our concrete situation we would like to *define* a Markov process by specifying the *jump rates* $Q(t)$. As is well known, it may happen that the jump rates do not specify the process uniquely (then the backward equation has many solutions $P(s, t)$). Uniqueness always holds if \mathbb{A} is finite or, more generally, if \mathbb{A} is infinite but the functions $|Q_{aa}(t)|$ are bounded on any closed time interval (see, e.g.

[Fe1]). However, these conditions are not satisfied in our case. There exist other, more involved uniqueness conditions for time homogeneous (stationary) Markov processes. However, in our approach, even if we restrict our attention to stationary processes, we still need to handle some non stationary processes as auxiliary objects. For these reasons we had to find some more special uniqueness condition.

Let us write $Q(t)$ in the form $Q(t) = -R(t) + \tilde{Q}(t)$, where $-R(t)$ is the diagonal part of $Q(t)$ and $\tilde{Q}(t)$ is the off-diagonal part of $Q(t)$. In other words,

$$R_{ab}(t) = -\delta_{ab}Q_{aa}(t), \quad \tilde{Q}_{ab}(t) = \begin{cases} Q_{ab}(t), & a \neq b, \\ 0, & a = b. \end{cases}$$

For $s \leq t$ set

$$F(s, t) = \exp\left(-\int_s^t R(\tau)d\tau\right), \quad G(s, t) = F(s, t)\tilde{Q}(t).$$

Define $P^{[n]}(s, t)$ recursively by

$$P^{[0]}(s, t) = F(s, t), \quad P^{[n]}(s, t) = \int_s^t G(s, \tau)P^{[n-1]}(\tau, t)d\tau, \quad n \geq 1,$$

and set

$$\bar{P}(s, t) = \sum_{n=0}^{\infty} P^{[n]}(s, t), \quad s \leq t.$$

Theorem 2.1. [Fe1]

- (i) *The matrix $\bar{P}(s, t)$ is substochastic (i.e., its elements are nonnegative and $\sum_b P_{ab}(s, t) \leq 1$). Its elements are absolutely continuous and almost everywhere differentiable with respect to both s and t , and it provides a solution of Kolmogorov's backward and forward equations (2.3), (2.5) with the initial condition (2.4).*
- (ii) *$\bar{P}(s, t)$ also satisfies the Chapman-Kolmogorov equation (2.1).*
- (iii) *$\bar{P}(s, t)$ is the minimal solution of (2.3) (or (2.5)) in the sense that for any other solution $P(s, t)$ of (2.3) (or (2.5)) with the initial condition (2.4) in the class of substochastic matrices, one has $P_{ab}(s, t) \geq \bar{P}_{ab}(s, t)$ for any $a, b \in \mathbb{A}$.*

Corollary 2.2. *If the minimal solution $\bar{P}(s, t)$ is stochastic (the sums of matrix elements along the rows are all equal to 1) then it is the unique solution of (2.3) (or (2.5)) with the initial condition (2.4) in the class of substochastic matrices.*

Let us note that the construction of $\bar{P}(s, t)$ is very natural: the summands $P_{ab}^{[n]}(s, t)$ are the probabilities to go from a to b in n jumps. The condition of $\bar{P}(s, t)$ being stochastic exactly means that we cannot make infinitely many jumps in a finite amount of time.

Our next goal is to provide a convenient sufficient condition for $\bar{P}(s, t)$ to be stochastic.

Fix $s \in (t_{\min}, t_{\max})$ and $a \in \mathbb{A}$. For any finite X , $X \subset \mathbb{A}$, $a \in X$, we denote by $T_{s,a,X}$ the time of the first exit from X under the condition that the process is in a at time s . Formally, we can modify \mathbb{A} and $Q(t)$ by contracting all the states $b \in \mathbb{A} \setminus X$ into one absorbing state \tilde{b} with $Q_{\tilde{b},c} \equiv 0$ for any $c \in X \cup \{\tilde{b}\}$. We obtain a process with a finite number of states for which the solution $\tilde{P}(s, t)$ of the backward equation is unique. Then $T_{s,a,X}$ is a random variable with values in $(s, +\infty]$ defined by

$$\text{Prob}\{T_{s,a,X} \leq t\} = \tilde{P}_{a\tilde{b}}(s, t).$$

Proposition 2.3. *Assume that for any $a \in \mathbb{A}$ and any $s < t$, $\varepsilon > 0$, there exists a finite set $X(\varepsilon) \subset \mathbb{A}$ such that*

$$\text{Prob}\{T_{s,a,X(\varepsilon)} \leq t\} \leq \varepsilon.$$

Then the minimal solution $\bar{P}(s, t)$ provided by Theorem 2.1 is stochastic.

Proof. Consider the modified process on the finite state space $X(\varepsilon) \cup \{\tilde{b}\}$ described above. Since its transition matrix $\tilde{P}(s, t)$ is stochastic,

$$\sum_{b \in X(\varepsilon)} \tilde{P}_{ab}(s, t) = 1 - \tilde{P}_{a\tilde{b}}(s, t) \geq 1 - \varepsilon.$$

The construction of the minimal solution as the sum of $P^{[n]}$'s, see above, immediately implies that $P_{ab}(s, t) \geq \tilde{P}_{ab}(s, t)$. Thus, $\sum_b P_{ab}(s, t) \geq 1 - \varepsilon$ for any $\varepsilon > 0$. \square

2.2. An application to birth-death processes

A *birth-death process* is a continuous time Markov process on $\mathbb{A} = \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ such that the rates $Q_{mn}(t)$ vanish if $|n - m| > 1$. In other words, the process can make jumps only of size 1. Our assumption (2.2) means that

$$-Q_{nn}(t) = R_{nn}(t) = Q_{n,n+1}(t) + Q_{n,n-1}(t).$$

Proposition 2.4. *Assume that for any closed segment $[t', t''] \subset (t_{\min}, t_{\max})$ there exists a sequence $\{\gamma_n\}_{n=0}^\infty$ of positive real numbers such that $Q_{n,n+1}(t) \leq \gamma_n$ for any $t \in [t', t'']$, and $\sum_n \gamma_n^{-1} = \infty$. Then the minimal solution $\bar{P}(s, t)$ is stochastic.*

Proof. We will apply Proposition 2.3. Let us fix $a \in \mathbb{A} = \mathbb{Z}_+$. As $X = X(\varepsilon)$ we will take a set of the form $\{0, 1, \dots, N - 1\}$ for a suitable fixed N . Then $T_{s,a,X}$ is the moment of the first arrival at N given that we start at a at the time moment s . To simplify the notation, set $T = T_{s,a,\{0,1,\dots,N-1\}}$.

In order to estimate T we will compare our inhomogeneous birth-death process to the pure birth homogeneous process with jump rates $\hat{Q}_{n,n+1} = \gamma_n$, $\hat{Q}_{n+1,n} = 0$, for all $n \in \mathbb{Z}_+$. Let \hat{T} denote the time of reaching N given that we start at a at time s . Note that since this is a pure birth process, once the process leaves $\{0, 1, \dots, N - 1\}$ it never comes back.

It is known that for the pure birth process with rates γ_n the minimal solution is stochastic if and only if $\sum_n \gamma_n^{-1} = \infty$, see [Fe1], [Fe3, ch. XIV, §8]. By our hypothesis, $\sum_n \gamma_n^{-1} = \infty$, and we may denote by $\widehat{P}(s, t)$ the unique stochastic solution of the backward and forward equations. Clearly,

$$\text{Prob}\{\widehat{T} \leq t\} = \sum_{b=N}^{\infty} \widehat{P}_{ab}(s, t),$$

which tends to zero as $N \rightarrow \infty$. Thus, by virtue of Proposition 2.3 it suffices to show that $\text{Prob}\{T \leq t\} \leq \text{Prob}\{\widehat{T} \leq t\}$ for any $t > s$.

To prove this inequality we use a simple coupling argument: we construct a process in the cone $\{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid m \leq n\}$ such that the projections onto the first and the second components are, respectively, our initial process with jump rates Q and the pure birth process with jump rates \widehat{Q} .

Outside the diagonal $m = n$ (that is, in the domain $m < n$), the process behaves as if its m - and n -coordinates were independent. Namely, the jump rates are defined by

$$(m, n) \rightarrow \begin{cases} (m - 1, n) : & Q_{m,m-1}, \\ (m + 1, n) : & Q_{m,m+1}, \\ (m, n + 1) : & \gamma_n, \\ (m, n) : & -\gamma_n - Q_{m,m-1} - Q_{m,m+1}. \end{cases}$$

On the diagonal $m = n$, the jump rates are slightly modified:

$$(n, n) \rightarrow \begin{cases} (n - 1, n) : & Q_{n,n-1} \\ (n, n + 1) : & \gamma_n - Q_{n,n+1} \\ (n + 1, n + 1) : & Q_{n,n+1} \\ (n, n) : & -\gamma_n - Q_{n,n-1}. \end{cases}$$

Here the hypothesis $Q_{n,n+1} \leq \gamma_n$ has been used to ensure nonnegativity of the transition rate $(n, n) \rightarrow (n, n + 1)$.

It is directly verified that the sum of the jump rates $(m, n) \rightarrow (m', n')$ with fixed m, m', n and varying n' gives $Q_{m,m'}$ (independently of n), while the sum with fixed m, n, n' and varying m' gives $Q_{n,n'}$ (independently of m). This just means that the projections have the required form.

Since the process does not leave the cone $m \leq n$, the desired inequality follows.

□

2.3. Birth–death process $N_{c,\xi(\cdot)}$

From now on we restrict our attention to birth–death processes with

$$Q_{n,n+1} = \alpha(t)(c + n), \quad Q_{n,n-1} = \beta(t)n, \quad (2.6)$$

where $\alpha(t) \geq 0, \beta(t) \geq 0$ are continuous functions on (t_{\min}, t_{\max}) and $c > 0$ is a constant. Proposition 2.4 implies that for any process of this kind there exists a

unique stochastic solution $P(s, t)$ of Kolmogorov's backward and forward equations. By breaking the time interval into finitely many subintervals, we may as well assume that $\alpha(t)$ and $\beta(t)$ are piecewise continuous functions with finitely many points of discontinuity at which they have finite left and right limits.

The *negative binomial distribution* $\pi_{c,\xi}$ on \mathbb{Z}_+ with parameters $c > 0$ and $\xi \in (0, 1)$ is defined by

$$\pi_{c,\xi}(n) = (1 - \xi)^c \frac{(c)_n}{n!} \xi^n, \quad n = 0, 1, 2, \dots$$

It will be convenient to interpret $\pi_{c,\xi}$ as an infinite row-vector.

Proposition 2.5. *Let $\xi(t)$ be a continuous, piecewise continuously differentiable function in t with values in $(0, 1)$. Assume that $\xi(t)$ solves the differential equation*

$$\frac{\dot{\xi}(t)}{\xi(t)(1 - \xi(t))} = \frac{\alpha(t)}{\xi(t)} - \beta(t), \quad t \in (t_{\min}, t_{\max}). \quad (2.7)$$

Then the row vector $\pi_{c,\xi(t)}$ solves $\pi_{c,\xi(s)}P(s, t) = \pi_{c,\xi(t)}$ for any $s \leq t$.

Proof. Let us differentiate $\pi_{c,\xi(s)}P(s, t)$ with respect to s and use Kolmogorov's backward equation. Collecting the coefficients of $P_{xy}(s, t)$ in the y th coordinate gives

$$\begin{aligned} \pi_{c,\xi(s)}(x)P_{xy}(s, t) & \left(-\frac{c\dot{\xi}(s)}{1 - \xi(s)} + \frac{x\dot{\xi}(s)}{\xi(s)} + \alpha(s)(c + x) + \beta(s)x \right. \\ & \left. - \alpha(s)(c + x - 1)\frac{\pi_{c,\xi(s)}(x - 1)}{\pi_{c,\xi(s)}(x)} - \beta(s)(x + 1)\frac{\pi_{c,\xi(s)}(x + 1)}{\pi_{c,\xi(s)}(x)} \right). \end{aligned}$$

Simplifications show that this expression is zero for all x, y if (2.7) holds. The initial condition $\pi_{c,\xi(s)}P(s, t)|_{s=t} = \pi_{c,\xi(t)}$ is obviously satisfied. \square

Once we have a family of distributions $\pi_{c,\xi(t)}$ satisfying $\pi_{c,\xi(s)}P(s, t) = \pi_{c,\xi(t)}$, we can define a birth-death process by the matrix of transition probabilities $P(s, t)$ (which is uniquely determined by the jump rates) and one-dimensional distributions $\pi_{c,\xi(t)}$.

It is not *a priori* clear what is a convenient way to parametrize these processes. In particular, multiplying both $\alpha(t)$ and $\beta(t)$ by the same function of t leads only to a reparametrization of time in our process. In order to eliminate this freedom, we will always use one specific choice of time in our processes which we call *interior* or *canonical* time of the corresponding process. The convenience of this choice will soon become clear.

The interior time is uniquely determined by the condition that $\alpha(t)$ and $\beta(t)$ are expressed through $\xi(t)$ by

$$\alpha(t) = \left(1 + \frac{\dot{\xi}(t)}{2\xi(t)}\right) \frac{\xi(t)}{1 - \xi(t)}, \quad \beta(t) = \left(1 - \frac{\dot{\xi}(t)}{2\xi(t)}\right) \frac{1}{1 - \xi(t)}. \quad (2.8)$$

Evidently, these formulas imply (2.7). Moreover, for any $(\alpha(t), \beta(t), \xi(t))$ satisfying (2.7), if $\alpha(t)$ and $\beta(t)$ do not vanish simultaneously, we can choose a new time variable $\tau(t)$ with

$$\dot{\tau} = \frac{1}{2} \left(\frac{\alpha}{\xi} + \beta \right) (1 - \xi)$$

so that $(\dot{\tau}^{-1}\alpha(t(\tau)), \dot{\tau}^{-1}\beta(t(\tau)), \xi(t(\tau)))$ satisfy both (2.7) and (2.8) as functions in τ .

Thus, from now on we will parametrize our processes by continuous, piecewise continuously differentiable functions $\xi(t)$ taking values in $(0, 1)$ such that $|\dot{\xi}(t)/\xi(t)| \leq 2$ (this condition is necessary to guarantee the nonnegativity of α and β). Such curves $\xi(t)$ will be called *admissible*. Then the corresponding birth–death process is determined by jump rates given by (2.6), (2.8) and one–dimensional distributions $\pi_{c,\xi(t)}$. We will denote this process by $N_{c,\xi(\cdot)}$.

In other words, if we set $A(t) = -\frac{1}{2} \ln \xi(t)$ then $A(t)$ has to satisfy three conditions: $A(t) \geq 0$ for all t ; $|\dot{A}(t)| \leq 1$ for all t ; and $A(t)$ is continuous and piecewise continuously differentiable.

In terms of $A(t)$ it is convenient to single out important special cases: $A(t) \equiv \text{const}$ corresponds to the homogeneous birth–death process; $A(t) = t + \text{const}$ corresponds to pure death processes; and $A(t) = -t + \text{const}$ corresponds to pure birth processes.

Note that the pure birth process is the well–known Yule process with immigration. The pure death processes is also well known: it describes a collection of particles which disappear with constant rate independently of each other.

Further, note that in case of a pure birth process $A(t)$ will hit zero in finite time which means that in terms of the canonical time parametrization, the process reaches infinity in a finite amount of time.

2.4. The transition matrix of $N_{c,\xi(\cdot)}$

Here we derive an expression of $P(s, t)$, the transition matrix of $N_{c,\xi(\cdot)}$, in terms of the classical Meixner polynomials.

Recall that the Meixner polynomials are the orthogonal polynomials with respect to the following weight function on \mathbb{Z}_+ :

$$W_{c,\xi}(x) = \frac{(c)_x \xi^x}{x!} = \frac{\Gamma(c+x)\xi^x}{\Gamma(c)x!}, \quad x \in \mathbb{Z}_+.$$

Here $c > 0$ and $\xi \in (0, 1)$ are parameters. Our notation for the Meixner polynomials is $\mathfrak{M}_n(x; c, \xi)$, where $n = 0, 1, \dots$ is the degree. We use the same normalization of the polynomials as in the handbook [KS] (note that in [KS], our parameter c is denoted as β , and our ξ is denoted as c).

Set

$$\tilde{\mathfrak{M}}_n(x; c, \xi) = (-1)^n \frac{\mathfrak{M}_n(x; c, \xi)}{\|\mathfrak{M}_n(\cdot; c, \xi)\|} \sqrt{W_{c,\xi}(x)}, \quad x \in \mathbb{Z}_+,$$

where

$$\|\mathfrak{M}_n(\cdot; c, \xi)\|^2 = \sum_{x=0}^{\infty} \mathfrak{M}_n^2(x; c, \xi) W_{c, \xi}(x)$$

and the factor $(-1)^n$ is introduced for convenience (see (5.9) below). These functions form an orthonormal basis in $\ell^2(\mathbb{Z}_+)$.

Proposition 2.6. *The matrix $P(s, t)$ has the form*

$$P_{xy}(s, t) = \left(\frac{\pi_{c, \xi(t)}(y)}{\pi_{c, \xi(s)}(x)} \right)^{\frac{1}{2}} \sum_{n=0}^{\infty} e^{n(s-t)} \tilde{\mathfrak{M}}_n(x; c, \xi(s)) \tilde{\mathfrak{M}}_n(y; c, \xi(t)), \quad (2.9)$$

where $s \leq t$ and $x, y \in \mathbb{Z}_+$.

Comments. 1. In the stationary case $\xi(t) \equiv \text{const}$ this formula was derived by Karlin and McGregor [KMG2] as part of a much more general formalism, see also [KMG1].

2. The formula implies that $P(s, t)$ depends on the initial value $\xi(s)$, final value $\xi(t)$ and the length $t - s$ of the time interval. However, $P(s, t)$ does *not* depend on the behavior of the curve $\xi(\cdot)$ inside this time interval, as one might expect.
3. The simplicity of the factor $e^{n(s-t)}$ is a consequence of our choice of the interior time of the process.
4. Since $\tilde{\mathfrak{M}}_0(x; c, \xi) = (\pi_{c, \xi}(x))^{\frac{1}{2}}$, the prefactor may be rewritten as

$$\left(\frac{\pi_{c, \xi(t)}(y)}{\pi_{c, \xi(s)}(x)} \right)^{\frac{1}{2}} = \frac{\tilde{\mathfrak{M}}_0(y; c, \xi(t))}{\tilde{\mathfrak{M}}_0(x; c, \xi(s))}.$$

5. The formula implies that the following kernel on $\mathbb{Z}_+ \times \mathbb{Z}_+$

$$(x, y) \mapsto \sum_{n=0}^{\infty} q^n \tilde{\mathfrak{M}}_n(x; c, \zeta) \tilde{\mathfrak{M}}_n(y; c, \eta), \quad (2.10)$$

where $\zeta, \eta \in (0, 1)$ and $0 < q \leq \min \left\{ \sqrt{\zeta/\eta}, \sqrt{\eta/\zeta} \right\}$,

takes nonnegative values. This result is generalized in Corollary 3.3 below. Note that the bound on q follows from the inequality $|\dot{\xi}/\xi| \leq 2$.

Our proof of Proposition 2.6 consists of few steps. Let us denote the right-hand side of (2.9) by $\widehat{P}_{xy}(s, t)$.

First, we show that $\widehat{P}(s, t)$ satisfies Kolmogorov’s backward equation. Since we know that there exists only one stochastic solution, it remains to prove that $\widehat{P}(s, t)$ is a stochastic matrix.

The fact that the sum of the matrix elements along any row is equal to 1 is obvious (only $n = 0$ term gives a nonzero contribution due to orthogonality of nonconstant Meixner polynomials to constants). The fact that $\widehat{P}(s, t)$ is always nonnegative is not so obvious. In order to prove that we explicitly evaluate $\widehat{P}(s, t)$ in the cases of pure birth and pure death processes, and then show that in the general case, $\widehat{P}(s, t)$ is always a product of a “pure death” and a “pure birth” transition matrices.

Lemma 2.7. *The following relations hold*

$$\begin{aligned} & \sqrt{\xi(x+1)(x+c)} \tilde{\mathfrak{M}}_n(x+1; c, \xi) + \sqrt{\xi x(x+c-1)} \tilde{\mathfrak{M}}_n(x-1; c, \xi) \\ & - (x(1+\xi) + c\xi) \tilde{\mathfrak{M}}_n(x; c, \xi) = -n(1-\xi) \tilde{\mathfrak{M}}_n(x; c, \xi), \end{aligned} \quad (2.11)$$

$$\begin{aligned} & 2\xi(1-\xi) \frac{\partial}{\partial \xi} \tilde{\mathfrak{M}}_n(x; c, \xi) + \sqrt{\xi(x+1)(x+c)} \tilde{\mathfrak{M}}_n(x+1; c, \xi) \\ & - \sqrt{\xi x(x+c-1)} \tilde{\mathfrak{M}}_n(x-1; c, \xi) = 0. \end{aligned} \quad (2.12)$$

Proof. Straightforward computation using

$$\xi^2 \frac{\partial}{\partial \xi} \mathfrak{M}_n(x; c, \xi) = \frac{nx}{c} \mathfrak{M}_{n-1}(x-1, c+1, \xi)$$

and [KS, 1.9.5, 1.9.6, 1.9.8]. □

Proof of Proposition 2.6. First of all, we need to verify that $\widehat{P}(s, t)$ (the right-hand side of (2.9)) satisfies the backward equation. This is the equality

$$\begin{aligned} -\frac{\partial}{\partial s} \widehat{P}_{xy}(s, t) &= Q_{xx}(s) \widehat{P}_{xy}(s, t) \\ &+ Q_{x,x+1}(s) \widehat{P}_{x+1,y}(s, t) + Q_{x,x-1}(s) \widehat{P}_{x-1,y}(s, t) \end{aligned} \quad (2.13)$$

with $Q_{xx}(s) = -Q_{x,x+1}(s) - Q_{x,x-1}(s)$ and

$$\begin{aligned} Q_{x,x+1}(s) &= (c+x) \left(1 + \frac{\dot{\xi}(s)}{2\xi(s)} \right) \frac{\xi(s)}{1-\xi(s)}, \\ Q_{x,x-1}(s) &= x \left(1 - \frac{\dot{\xi}(s)}{2\xi(s)} \right) \frac{1}{1-\xi(s)}. \end{aligned}$$

The computation proceeds as follows. One substitutes the sum in the right-hand side of (2.9) into the needed equality (2.13) and collects the coefficients of $\tilde{\mathfrak{M}}_n(y; c, \xi(t))$ using the relation $\partial/\partial s = \dot{\xi}(s) \partial/\partial(\xi(s))$. Each such coefficient has two parts: one of them does not involve $\dot{\xi}(s)$ while the other one is equal to $\dot{\xi}(s)$ times an expression not involving $\dot{\xi}(s)$. It turns out that each of these parts vanishes, for the first part the needed relation is (2.11), and for the second part one uses (2.12). The details are tedious but straightforward, and we omit them.

As was mentioned before, it remains to prove that $\widehat{P}_{xy}(s, t)$ is always nonnegative.

Let us use the notation $P^\uparrow(s, t)$, $P^\downarrow(s, t)$ for $P(s, t)$ when we consider a pure birth or a pure death process (that is, $\xi(\tau) = e^{\text{const}+2\tau}$ or $\xi(\tau) = e^{\text{const}-2\tau}$, respectively).

Lemma 2.8. *$P^\downarrow(s, t)$ is the unique solution of the backward equation for the pure death process with the initial condition $P(t, t) \equiv \text{Id}$, and $P^\uparrow(s, t)$ is the unique*

solution of the forward equation for the pure birth process with the same initial condition. Furthermore, with the notation $\zeta = \xi(s)$, $\eta = \xi(t)$, we have

$$P_{xy}^\downarrow(s, t) = \begin{cases} \left(\frac{(1-\zeta)\eta}{(1-\eta)\zeta} \right)^x \left(\frac{\zeta-\eta}{(1-\zeta)\eta} \right)^{x-y} \frac{x!}{(x-y)!y!}, & x \geq y, \\ 0, & x < y, \end{cases} \quad (2.14)$$

$$P_{xy}^\uparrow(s, t) = \begin{cases} \left(\frac{1-\eta}{1-\zeta} \right)^{c+x} \left(\frac{\eta-\zeta}{1-\zeta} \right)^{y-x} \frac{(c+x)_{y-x}}{(y-x)!}, & x \leq y, \\ 0, & x > y. \end{cases} \quad (2.15)$$

Proof. Consider $P^\downarrow(s, t)$ first. Since $Q_{x,x+1} \equiv 0$, Kolmogorov's backward equation takes the form

$$\begin{aligned} -\frac{\partial}{\partial s} P_{xy}^\downarrow(s, t) &= -Q_{x,x-1}(s) P_{xy}^\downarrow(s, t) + Q_{x,x-1}(s) P_{x-1,y}^\downarrow(s, t) \\ &= \frac{2x}{1-\xi(s)} \left(-P_{xy}^\downarrow(s, t) + P_{x-1,y}^\downarrow(s, t) \right). \end{aligned} \quad (2.16)$$

If we fix y then these differential equations can be solved recursively: we subsequently find $P_{0,y}^\downarrow, P_{1,y}^\downarrow, P_{2,y}^\downarrow, \dots$, using the initial conditions $P_{x,y}^\downarrow(t, t) = \delta_{xy}$. This shows that the backward equation for the pure death process has a unique solution. A straightforward calculation shows that the expression in the right-hand side of (2.14) satisfies this equation (with $\xi(s) = e^{\text{const}-2s}$).

The case of the pure birth process is completely analogous. \square

Since for the pure death process the backward equation has a unique solution, we have just shown that $\widehat{P}(s, t) = P^\downarrow(s, t)$ is the corresponding transition matrix.

In order to make a similar conclusion for the pure birth process, we need to know that $\widehat{P}(s, t)$ satisfies the forward equation. This fact can be proved directly using Lemma 2.7. It can also be reduced to the case of the backward equation as follows.

Note that $\pi_{c,\xi(s)}(x) \widehat{P}_{xy}(s, t)$ remains invariant under the changes

$$s \mapsto -t, \quad t \mapsto -s, \quad \xi(\tau) \mapsto \widetilde{\xi}(\tau) := \xi(-\tau), \quad x \leftrightarrow y. \quad (2.17)$$

Thus, instead of computing $\frac{\partial}{\partial t} \widehat{P}_{xy}(s, t)$ we may compute

$$-\frac{\partial}{\partial u} \left(\pi_{c,\widetilde{\xi}(u)}(y) \widehat{P}_{yx}(u, v) (\pi_{c,\widetilde{\xi}(v)}(x))^{-1} \right) \Big|_{u=-t, v=-s}$$

with the new $\widetilde{\xi}(\tau)$ obtained from $\xi(\tau)$ by the time inversion. Since we already know that $\widehat{P}_{xy}(s, t)$ solves the backward equation, for $\xi(\tau) = e^{\text{const}+2\tau}$ we obtain,⁵ cf. (2.16),

$$\begin{aligned} \frac{\partial}{\partial u} \frac{\pi_{c,\widetilde{\xi}(u)}(y) \widehat{P}_{yx}(u, v)}{\pi_{c,\widetilde{\xi}(v)}(x)} &= \left(-\frac{2c \widetilde{\xi}(u)}{1-\widetilde{\xi}(u)} + 2y \right) \frac{\pi_{c,\widetilde{\xi}(u)}(y) \widehat{P}_{yx}(u, v)}{\pi_{c,\widetilde{\xi}(v)}(x)} \\ &\quad + \frac{2y}{1-\widetilde{\xi}(u)} \frac{\pi_{c,\widetilde{\xi}(u)}(y)}{\pi_{c,\widetilde{\xi}(v)}(x)} \left(-\widehat{P}_{yx}(u, v) + \widehat{P}_{y-1,x}(u, v) \right). \end{aligned}$$

⁵ The argument goes through for any admissible curve $\xi(\cdot)$, it just becomes more tedious.

Using $\pi_{c,\tilde{\xi}(u)}(y) = \pi_{c,\xi(u)}(y-1) \cdot (c+y-1)\tilde{\xi}(u)/y$ and substituting $u = -t$, $v = -s$, we obtain the needed forward equation

$$\frac{\partial}{\partial t} \widehat{P}_{xy}(s, t) = -\frac{2(c+y)\xi(t)}{1-\xi(t)} \widehat{P}_{xy}(s, t) + \frac{2(c+y-1)\xi(t)}{1-\xi(t)} \widehat{P}_{x,y-1}(s, t).$$

The conclusion is that in case of the pure birth process, $\widehat{P}(s, t)$ satisfies the forward equation, and by Lemma 2.8 we have $\widehat{P}(s, t) = P^\uparrow(s, t)$.

Explicit expressions (2.14) and (2.15) show that $\widehat{P}_{xy}(s, t) \geq 0$ for the pure death and the pure birth processes. The nonnegativity of $\widehat{P}_{xy}(s, t)$ for arbitrary admissible curves $\xi(\cdot)$ follows from

Lemma 2.9. *Let $\xi(\cdot)$ be an admissible curve and $N_{c,\xi(\cdot)}$ be the corresponding birth-death process. Then for any $s < t$, $\widehat{P}(s, t)$ is a product of $P^\downarrow(s, u)$ with $\xi(\tau) = e^{-2(\tau-s)+\ln \xi(s)}$ and $P^\uparrow(u, t)$ with $\xi(\tau) = e^{2(\tau-t)+\ln \xi(t)}$ for a certain choice of u . Specifically, u is determined from the continuity condition:*

$$e^{-2(u-s)+\ln \xi(s)} = e^{2(u-t)+\ln \xi(t)} \iff u = \frac{s+t}{2} + \frac{\ln \xi(s) - \ln \xi(t)}{4}. \quad (2.18)$$

Proof. This statement follows from the Chapman–Kolmogorov equation (2.1), which $\widehat{P}(s, t)$ obviously satisfies due to the orthogonality of Meixner polynomials, and from the fact that $\widehat{P}(s, t)$ does not depend on the specific form of the curve $\xi(\cdot)$, see Comment 2 after the statement of Proposition 2.6. Thus, we may just replace $\xi(\tau)$ by a continuous combination of $e^{-2\tau+\text{const}}$ and $e^{2\tau+\text{const}}$ and preserve $\xi(s)$, $\xi(t)$, and $t - s$. Note that the fact that u given by the formula above is between s and t follows from the inequality $|\dot{\xi}/\xi| \leq 2$. \square

Lemma 2.9 implies that $\widehat{P}_{xy}(s, t)$ is always nonnegative, and this completes the proof of Proposition 2.6. \square

Corollary 2.10. *The process obtained from $N_{c,\xi(\cdot)}$ by the time reversion is also of the form $N_{c,\tilde{\xi}(\cdot)}$ with $\tilde{\xi}(\tau) = \xi(-\tau)$.*

Proof. $N_{c,\xi(\cdot)}$ is characterized by the fact that it is a Markov process with two-dimensional distributions

$$\text{Prob}\{N_{c,\xi(s)} = x, N_{c,\xi(t)} = y\} = \pi_{c,\xi(s)}(x) P_{xy}(s, t).$$

As was already mentioned above, the right-hand side of (2.9) multiplied by $\pi_{c,\xi(s)}(x)$ is invariant with respect to (2.17). This implies the statement. \square

Note that, in particular, time inversion turns our pure birth process into the pure death process and *vice versa* (essentially, we gave a proof of this fact before Lemma 2.9), and the stationary process $N_{c,\xi}$ with $\xi \equiv \text{const}$ is reversible. This is well known; any stationary birth-death process with an invariant measure is reversible with respect to this measure.

2.5. Markov processes on partitions

Our next goal is to extend birth–death processes $N_{c,\xi(\cdot)}$ to partitions in the following sense. We construct continuous time Markov processes on the state space \mathbb{Y} (the set of all Young diagrams, see §1) parametrized by admissible pairs (z, z') , see §1, and admissible curves $\xi(\cdot)$. The projection of such a process on \mathbb{Z}_+ obtained by looking at the number of boxes of the random Young diagrams, coincides with $N_{zz',\xi(\cdot)}$.

Let us fix a pair (z, z') of admissible parameters and set $c = zz' > 0$. Given an admissible curve $\xi(\cdot)$, we define the matrix Q of jump rates of our future Markov process $\Lambda_{z,z',\xi}$ on \mathbb{Y} by (set $n = |\lambda|$)

$$\begin{aligned}
 & Q_{\lambda\mu}(s) \\
 &= \begin{cases} (c+n) \left(1 + \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{\xi(s)}{1-\xi(s)} \cdot p_{zz'}^\uparrow(n, \lambda; n+1, \mu), & \lambda \nearrow \mu, \\ n \left(1 - \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{1}{1-\xi(s)} \cdot p^\downarrow(n, \lambda; n-1, \mu), & \lambda \searrow \mu, \\ -(c+n) \left(1 + \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{\xi(s)}{1-\xi(s)} - n \left(1 - \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{1}{1-\xi(s)}, & \mu = \lambda, \end{cases} \\
 & \hspace{15em} (2.19)
 \end{aligned}$$

and $Q_{\lambda\mu}(s) \equiv 0$ in all other cases. Here $p_{zz'}^\uparrow$ and p^\downarrow are transition and cotransition probabilities from §1, see (1.9) and (1.1), and the expressions involving ξ come from (2.8). Note that under the projection $\mathbb{Y} \rightarrow \mathbb{Z}_+, \lambda \mapsto |\lambda|$, this matrix Q turns into the matrix of jump rates for $N_{c,\xi(\cdot)}$.

Proposition 2.11. *The minimal solution $P(s, t)$ of Kolmogorov’s backward equation with the matrix Q defined above is stochastic.*

Proof. We apply Propositions 2.3, 2.4. In the proof of Proposition 2.4 it was shown that for any $a \in \mathbb{Z}_+$ there exists a set of the form $X = \{0, 1, \dots, n-1\}$ such that the probability of exiting X during the time period from s to t with the initial state a is smaller than any given positive number ε . This means that if we start at time s from $\lambda \in \mathbb{Y}$ with $|\lambda| = a$ then the probability of exiting $\mathbb{Y}_0 \cup \mathbb{Y}_1 \cup \dots \cup \mathbb{Y}_{n-1}$ before time t is just the same as for the birth–death process and, hence, is less than ε . Proposition 2.3 concludes the proof. \square

Proposition 2.12 (cf. Proposition 2.5). *For any $s < t$*

$$M_{z,z',\xi(s)} P(s, t) = M_{z,z',\xi(t)}, \tag{2.20}$$

where $P(s, t)$ is the transition matrix of Proposition 2.11, and $M_{z,z',\xi}$ is the mixed z -measure (1.11) viewed as a row-vector with coordinates indexed by elements of \mathbb{Y} .

Proof. Since the formula obviously holds for $s = t$, it suffices to show that the derivative with respect to s of the left–hand side of (2.20) vanishes. Thus, it suffices to show that

$$-\frac{\partial}{\partial s} M_{z,z',\xi(s)}(\mu) + \sum_{\lambda \in \mathbb{Y}} M_{z,z',\xi(s)}(\lambda) Q_{\lambda\mu}(s) = 0 \tag{2.21}$$

for any $\mu \in \mathbb{Y}$.

Recall that

$$M_{z,z',\xi}(\lambda) = M_{z,z'}^{(n)}(\lambda)\pi_{c,\xi}(n) \quad \text{with } c = zz', \quad n = |\lambda|.$$

Substituting this relation into (2.21) we notice that we can perform the summation over λ using (1.5) and (1.6). Factoring out $M_{z,z'}^{(|\mu|)}(\mu)$ leads to the formula which states that the derivative of $\pi_{c,\xi(s)}P(s, t)$ with $\hat{P}(s, t)$ being the transition matrix for the birth-death process $N_{c,\xi}$, with respect to s vanishes. But this has already been proved in Proposition 2.5. \square

We conclude that given an admissible pair (z, z') and an admissible curve $\xi(\tau)$, there exists a unique continuous time Markov process on \mathbb{Y} with jump rates Q defined above and with one-dimensional distributions $M_{z,z',\xi(\tau)}$. This Markov process will be denoted by $\Lambda_{z,z',\xi(\cdot)}$.

As for the birth-death processes, we single out three important special cases: the stationary process $\xi(\tau) \equiv \text{const}$, the *ascending process* $\xi(\tau) = e^{2\tau + \text{const}}$ and the *descending process* $\xi(\tau) = e^{-2\tau + \text{const}}$. The projections of these processes on \mathbb{Z}_+ are the stationary birth-death process, the pure birth and the pure death processes, respectively.

As in §1, for $\lambda \in \mathbb{Y}$ we denote by $\dim \lambda$ the number of ascending paths in the Young graph leading from \emptyset to λ . More generally, we denote by $\dim(\mu, \lambda)$ the number of ascending paths in \mathbb{Y} leading from μ to λ ; if there are no such paths we set $\dim(\mu, \lambda) = 0$. Also, for $\mu, \lambda \in \mathbb{Y}$ such that $\mu \subset \lambda$ we set

$$(x)_{\lambda \setminus \mu} = \prod_{(i,j) \in \lambda \setminus \mu} (x + j - i), \quad x \in \mathbb{C},$$

where the product is taken over all boxes in $\lambda \setminus \mu$.

Proposition 2.13 (cf. Lemma 2.8). *The transition matrix of the descending process $\Lambda_{z,z',\xi(\cdot)}$ has the form*

$$P_{\lambda\mu}^\downarrow(s, t) = \left(\frac{(1-\zeta)\eta}{(1-\eta)\zeta} \right)^x \left(\frac{\zeta-\eta}{(1-\zeta)\eta} \right)^{x-y} \frac{x!}{(x-y)! y!} \frac{\dim \mu \dim(\mu, \lambda)}{\dim \lambda} \quad (2.22)$$

and the transition matrix of the ascending process $\Lambda_{z,z',\xi(\cdot)}$ has the form

$$P_{\lambda\mu}^\uparrow(s, t) = \left(\frac{1-\eta}{1-\zeta} \right)^{zz'+x} \left(\frac{\eta-\zeta}{1-\zeta} \right)^{y-x} \frac{x!}{(y-x)! y!} \frac{\dim \mu \dim(\lambda, \mu)}{\dim \lambda} \times (z)_{\mu \setminus \lambda} (z')_{\mu \setminus \lambda} \quad (2.23)$$

where $\zeta = \xi(s)$, $\eta = \xi(t)$, $x = |\lambda|$, $y = |\mu|$.

Proof. Let us consider the descending process first. It is immediate to check that the matrix $P_{\lambda\mu}^\downarrow(s, t)$ obtained from the transition matrix $P_{xy}^\downarrow(s, t)$ of the pure death process by

$$P_{\lambda\mu}^\downarrow(s, t) = P_{xy}^\downarrow(s, t) \sum p^\downarrow(x, \lambda; x-1, \mu^{(x-y-1)}) \times p^\downarrow(x-1, \mu^{(x-y-1)}; x-2, \mu^{(x-y-2)}) \cdots p^\downarrow(y+1, \mu^{(1)}; y, \mu)$$

where the sum is taken over all paths $\mu = \mu^{(0)} \nearrow \mu^{(1)} \nearrow \dots \nearrow \mu^{(x-y)} = \lambda$ from μ to λ , satisfies the backward equation. All terms in the above sum are equal to $\dim \mu / \dim \lambda$, and the number of terms is equal to $\dim(\mu, \lambda)$. Together with (2.14) this implies (2.22).

Similarly, for the ascending process one has

$$P_{\lambda\mu}^\uparrow(s, t) = P_{xy}^\uparrow(s, t) \sum p_{zz'}^\uparrow(x, \lambda; x + 1, \lambda^{(1)}) \times p_{zz'}^\uparrow(x + 1, \lambda^{(1)}; x + 2, \lambda^{(2)}) \dots p_{zz'}^\uparrow(y - 1, \lambda^{(y-x-1)}; y, \mu)$$

where the sum is taken over all paths $\lambda = \lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \dots \nearrow \lambda^{(x-y)} = \mu$ from λ to μ . Again, the product of transition probabilities does not depend on the path and it is equal to

$$\frac{1}{(c + x)_{y-x}} \frac{x! \dim \mu}{y! \dim \lambda} \cdot (z)_{\mu \setminus \lambda} (z')_{\mu \setminus \lambda}$$

while the number of paths is equal to $\dim(\lambda, \mu)$. Together with (2.15) this gives (2.23). □

3. Transition matrix for integral values of z

Our main goal in this section is to obtain a formula for the transition matrix of the process $\Lambda_{z,z',\xi(\cdot)}$ in the case when z is a nonnegative integer. For $z = 1$, the process $\Lambda_{z,z',\xi(\cdot)}$ coincides with the birth–death process $N_{c,\xi(\cdot)}$, where $c = zz' = z'$ (because $\Lambda_{z,z',\xi(\cdot)}$ lives on the Young diagrams with only one row), and our formula is reduced to (2.9).

Fix $z = N \in \{1, 2, \dots\}$. In order for (z, z') to be an admissible pair, we must have $z' \in \mathbb{R}$ and $z' > N - 1$. We will use the notation $z' = N + \alpha$, $\alpha > -1$.

According to the notation of §2.5 we set

$$c = zz' = N(N + \alpha).$$

Note, however, that in the present situation, we take as the first parameter of the Meixner polynomials not c but $\alpha + 1$ (these quantities coincide for $N = 1$ only).

As was mentioned in §1, the support of $M_{N,N+\alpha,\xi}$ consists of the Young diagrams with no more than N rows. It is convenient to parameterize such diagrams λ by sequences of N strictly decreasing nonnegative integers (x_1, \dots, x_N) ,

$$x_i = \lambda_i + N - i, \quad i = 1, \dots, N.$$

Given an admissible curve $\xi(\cdot)$, we set for $s \leq t$

$$v_{s,t}(x, y) = \sum_{k=0}^\infty e^{k(s-t)} \tilde{\mathfrak{M}}_k(x; \alpha + 1, \xi(s)) \tilde{\mathfrak{M}}_k(y; \alpha + 1, \xi(t)), \quad x, y \in \mathbb{Z}_+. \tag{3.1}$$

Recall that the functions $\tilde{\mathfrak{M}}_k$ were defined in §2.4.

Theorem 3.1. *Let λ, μ denote Young diagrams with no more than N rows, and let $(x_1, \dots, x_N), (y_1, \dots, y_N)$ be the corresponding collections of decreasing non-negative integers. For any admissible curve $\xi(\cdot)$ the transition matrix of the Markov process $\Lambda_{N, N+\alpha, \xi(\cdot)}$ has the form*

$$P_{\lambda, \mu}(s, t) = e^{-\frac{(t-s)N(N-1)}{2}} \left(\frac{M_{N, N+\alpha, \xi(t)}(\mu)}{M_{N, N+\alpha, \xi(s)}(\lambda)} \right)^{\frac{1}{2}} \det[v_{s,t}(x_i, y_j)]_{i,j=1}^N. \tag{3.2}$$

We will use the term *Karlin–McGregor representation* for this formula. Theorem 3.1 and Proposition 2.6 show that the process $\Lambda_{N, N+\alpha, \xi}$ is a Doob h -transform of N independent birth–death processes $N_{\alpha+1, \xi}$. In our situation this transform can also be realized through a certain conditioning procedure, see §6.5. Similar constructions have appeared in the literature earlier, see [Bi1], [Bi2] [Jo3], [KOR], [O’C].

Proof. The arguments follow the same pattern as in the proof of Proposition 2.6 (which is a special case of this theorem). The first step is to show that the right-hand side of (3.2) satisfies Kolmogorov’s backward equation. After that we prove that this solution is stochastic.

We will use the notation

$$\begin{aligned} x &= (x_1, \dots, x_N), & y &= (y_1, \dots, y_N), & \varepsilon_r &= (\overbrace{0, \dots, 0}^{r-1}, \overbrace{1, 0, \dots, 0}^{N-r}), \\ & & & & & 1 \leq r \leq N, \\ \zeta &= \xi(s), & \eta &= \xi(t), & n = |\lambda| &= \sum_{i=1}^N x_i - \frac{N(N-1)}{2}, \\ f_k(\cdot) &= \tilde{\mathfrak{M}}_k(\cdot; \alpha + 1, \xi(s)), & g_k(\cdot) &= \tilde{\mathfrak{M}}_k(\cdot; \alpha + 1, \xi(t)). \end{aligned}$$

Also, denote the right-hand side of (3.2) by $\hat{P}_{xy}(s, t)$.

The formulas of §1 imply

$$\left(\frac{M_{N, N+\alpha, \xi(t)}(\mu)}{M_{N, N+\alpha, \xi(s)}(\lambda)} \right)^{\frac{1}{2}} = \frac{\eta^{\frac{|\mu|}{2}} (1 - \eta)^{\frac{\zeta}{2}}}{\zeta^{\frac{|\lambda|}{2}} (1 - \zeta)^{\frac{\zeta}{2}}} \left(\prod_{j=1}^N \frac{\Gamma(y_j + \alpha + 1) \Gamma(x_j + 1)}{\Gamma(x_j + \alpha + 1) \Gamma(y_j + 1)} \right)^{\frac{1}{2}} \frac{V(y)}{V(x)} \tag{3.3}$$

where $V(u) = \prod_{1 \leq i < j \leq N} (u_i - u_j)$ stands for the Vandermonde determinant.

Note that adding to λ one box or removing from λ one box is equivalent to adding ε_r to x or subtracting ε_r from x , where r is the row number of the box. We have (see (1.9), (1.1), and the dimension formula just before (1.1))

$$\begin{aligned} p_{N, N+\alpha}^{\uparrow}(n, x; n + 1, x + \varepsilon_r) &= \frac{x_r + \alpha + 1}{c + n} \frac{V(x + \varepsilon_r)}{V(x)}, \\ p^{\downarrow}(n, x; n - 1, x - \varepsilon_r) &= \frac{x_r}{n} \frac{V(x - \varepsilon_r)}{V(x)}. \end{aligned}$$

The right-hand sides of these relations conveniently vanish exactly when $x \pm \varepsilon_r$ does not represent a Young diagram (two of the coordinates of $x \pm \varepsilon_r$ are equal). Using (2.19) we can now write down the needed backward equation:

$$\begin{aligned} & -\frac{\partial}{\partial s} \widehat{P}_{xy}(s, t) \\ &= -\left(\left(1 + \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{\xi(s)(c+n)}{1-\xi(s)} + \left(1 - \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{n}{1-\xi(s)} \right) \widehat{P}_{xy}(s, t) \\ &+ \left(1 + \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{\xi(s)}{1-\xi(s)} \sum_{r=1}^N \frac{(x_r + \alpha + 1)V(x + \varepsilon_r)}{V(x)} \cdot \widehat{P}_{x+\varepsilon_r, y}(s, t) \\ &+ \left(1 - \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{1}{1-\xi(s)} \sum_{r=1}^N \frac{x_r V(x - \varepsilon_r)}{V(x)} \cdot \widehat{P}_{x-\varepsilon_r, y}(s, t). \end{aligned}$$

It is time to use the definition of $\widehat{P}_{xy}(s, t)$. The Cauchy–Binet identity (see, e.g., [Ga, ch. I, §2]) implies (in all determinants below the indices run from 1 to N)

$$\begin{aligned} \det[v_{s,t}(x_i, y_j)] &= \det \left[\sum_{k=0}^{\infty} e^{(s-t)k} f_k(x_i) g_k(y_j) \right] \\ &= \sum_{k_1 > k_2 > \dots > k_N \geq 0} e^{(s-t)(k_1 + \dots + k_N)} \det[f_{k_i}(x_j)] \det[g_{k_i}(y_j)]. \quad (3.4) \end{aligned}$$

Let us use this relation and (3.2) for $\widehat{P}_{xy}(s, t)$, $\widehat{P}_{x \pm \varepsilon_r, y}(s, t)$ in the backward equation and collect the coefficients of $\det[g_{k_i}(y_j)]$. Factoring out

$$\left(\frac{M_{N, N+\alpha, \xi(t)}(\mu)}{M_{N, N+\alpha, \xi(s)}(\lambda)} \right)^{\frac{1}{2}} e^{(s-t)(k_1 + \dots + k_N - \frac{N(N-1)}{2})}$$

we obtain (Vandermonde determinants cancel out quite conveniently)

$$\begin{aligned} & \left(\frac{N(N-1)}{2} - (k_1 + \dots + k_N) + \frac{\dot{\xi}(s)}{2} \left(\frac{n}{\zeta} - \frac{c}{1-\zeta} \right) \right) \det[f_{k_i}(x_j)] \\ & - \frac{\partial}{\partial s} \det[f_{k_i}(x_j)] = - \left(\left(1 + \frac{\dot{\xi}(s)}{2\zeta}\right) \frac{\zeta(c+n)}{1-\zeta} + \left(1 - \frac{\dot{\xi}(s)}{2\zeta}\right) \frac{n}{1-\zeta} \right) \det[f_{k_i}(x_j)] \\ & + \left(1 + \frac{\dot{\xi}(s)}{2\zeta}\right) \frac{1}{1-\zeta} \sum_{r=1}^N \sqrt{(x_r + 1)(x_r + \alpha + 1)\zeta} \det[f_{k_i}((x + \varepsilon_r)_j)] \\ & + \left(1 - \frac{\dot{\xi}(s)}{2\zeta}\right) \frac{1}{1-\zeta} \sum_{r=1}^N \sqrt{x_r(x_r + \alpha)\zeta} \det[f_{k_i}((x - \varepsilon_r)_j)]. \quad (3.5) \end{aligned}$$

We claim that the part of the left-hand side of this relation that does not involve derivatives $\dot{\xi}(s)$ and $-\frac{\partial}{\partial s}$ equals the part of the right-hand side without $\dot{\xi}(s)$, and the part of the left-hand side with derivatives equals that of the right-hand side with $\dot{\xi}(s)$.

The part without derivatives gives

$$\begin{aligned} & \left(\zeta(c+n) + n + \frac{N(N-1)}{2}(1-\zeta) - (k_1 + \dots + k_N)(1-\xi) \right) \det[f_{k_i}(x_j)] \\ &= \sum_{r=1}^N \left(\sqrt{(x_r+1)(x_r+\alpha+1)\zeta} \det[f_{k_i}((x+\epsilon_r)_j)] \right. \\ & \quad \left. + \sqrt{x_r(x_r+\alpha)\zeta} \det[f_{k_i}((x-\epsilon_r)_j)] \right). \end{aligned} \tag{3.6}$$

The right-hand side of this equality can be rewritten as

$$\begin{aligned} & \sum_{\sigma \in S_N} \sum_{r=1}^N \operatorname{sgn} \sigma f_{k_{\sigma(1)}}(x_1) \cdots f_{k_{\sigma(N)}}(x_N) \\ & \quad \times \frac{\sqrt{(x_r+1)(x_r+\alpha+1)\zeta} f_{\sigma(r)}(x_r+1) + \sqrt{x_r(x_r+\alpha)\zeta} f_{\sigma(r)}(x_r-1)}{f_{\sigma(r)}(x_r)}. \end{aligned}$$

By (2.11) the last ratio equals $x_r(1+\zeta) + (\alpha+1)\zeta - k_{\sigma(r)}(1-\zeta)$, and the whole expression equals

$$\left((x_1 + \dots + x_N)(1+\zeta) + N(\alpha+1)\zeta - (k_1 + \dots + k_N)(1-\zeta) \right) \det[f_{k_i}(x_j)],$$

which is exactly the left-hand side of (3.6) with $c = N(N+\alpha)$ and $n = \sum x_i - N(N-1)/2$.

The part of (3.5) with derivatives gives

$$\begin{aligned} -\frac{2\zeta(1-\zeta)}{\dot{\xi}(s)} \frac{\partial}{\partial s} \det[f_{k_i}(x_j)] &= \sum_{r=1}^N \left(\sqrt{(x_r+1)(x_r+\alpha+1)\zeta} \det[f_{k_i}((x+\epsilon_r)_j)] \right. \\ & \quad \left. - \sqrt{x_r(x_r+\alpha)\zeta} \det[f_{k_i}((x-\epsilon_r)_j)] \right). \end{aligned}$$

The same operation with determinants and (2.12) show that this equality holds.

This concludes the proof of the fact that $\widehat{P}_{\lambda,\mu}(s, t)$ (the right-hand side of (3.2)) satisfies the backward equation for $\Lambda_{z,z',\xi(\cdot)}$. It remains to verify that the $\mathbb{Y} \times \mathbb{Y}$ matrix $\widehat{P}_{\lambda,\mu}(s, t)$ is stochastic.

Let us check that $\sum_{\mu} \widehat{P}_{\lambda,\mu}(s, t) = 1$ for any $\lambda \in \mathbb{Y}$ and $s < t$.

Using elementary row operations on the Vandermonde matrix, we obtain

$$\begin{aligned} V(u) &= \prod_{1 \leq i < j \leq N} (u_i - u_j) = \det[u_j^{N-i}] \\ &= (-1)^{\frac{N(N-1)}{2}} \det[\mathfrak{M}_{N-i}(u_j; \alpha+1, \xi)] \end{aligned} \tag{3.7}$$

where the sign $(-1)^{\frac{N(N-1)}{2}}$ appears because the highest coefficient of $\mathfrak{M}_k(u; \alpha+1, \xi)$ is $(-1)^k$. Then (3.2) and (3.3) imply

$$\widehat{P}_{\lambda,\mu}(s, t) = e^{\frac{(t-s)N(N-1)}{2}} \frac{\det[g_{N-i}(y_j)]}{\det[f_{N-i}(x_j)]} \det[v_{s,t}(x_i, y_j)].$$

Apply (3.4) and sum the result over all $y_1 > \dots > y_N \geq 0$. The Cauchy–Binet identity gives

$$\sum_{y_1 > \dots > y_N \geq 0} \det[g_{k_i}(y_j)] \det[g_{N-i}(y_j)] = \det \left[\sum_{u=0}^{\infty} g_{k_i}(u) g_{N-j}(u) \right]. \quad (3.8)$$

Orthogonality of Meixner polynomials means that $\sum_{u \geq 0} g_k(u) g_l(u) = \delta_{kl}$. Hence, the last determinant equals 1 if $k_i = N - i$ for all $i = 1, \dots, N$ and vanishes otherwise. This gives the desired result.

Finally, the nonnegativity of $\widehat{P}_{\lambda, \mu}(s, t)$ follows from Proposition 2.13 and

Lemma 3.2 (cf. Lemma 2.9). *For any admissible curve $\xi(\cdot)$, the matrix $\widehat{P}(s, t)$ given by the right–hand side of (3.2) is the product of the transition matrix $P^\downarrow(s, u)$ for the descending process with $\xi(\tau) = e^{-2(\tau-s)+\ln \xi(s)}$ and the transition matrix $P^\uparrow(u, t)$ for the ascending process with $\xi(\tau) = e^{2(\tau-t)+\ln \xi(t)}$ with u given by (2.18):*

$$u = \frac{s+t}{2} + \frac{\ln \xi(s) - \ln \xi(t)}{4}.$$

The proof of this lemma is very similar to that of Lemma 2.9. The Chapman–Kolmogorov equation for $\widehat{P}(s, t)$ is easily verified by means of (3.4), the orthogonality of Meixner polynomials, and the trick with the Cauchy–Binet formula used above.

Since the transition matrices of the descending and ascending processes have nonnegative matrix elements (2.22), (2.23), the matrix elements of $\widehat{P}(s, t)$ are also nonnegative, and the proof of Theorem 3.1 is complete. \square

Corollary 3.3. *The kernel (2.10) is totally positive.*

Proof. Indeed, Theorem 3.1 shows that the minors of this kernel are, up to positive factors, transition probabilities. \square

In the stationary case ($\xi(\cdot) \equiv \text{const}$) this also follows from old results of Karlin and McGregor (see [KMG2], [KMG3]).

4. Analytic continuation

This section contains two claims related to analytic continuation in the parameters (z, z') . In Proposition 4.1 we show that the factorization of the transition matrix for $\Lambda_{z, z', \xi(\cdot)}$ into the product of transition matrices for descending and ascending processes is carried over from the case of integral z (Lemma 3.2 above) to the general case. Proposition 4.2 is a technical result about the analytic dependence of the finite–dimensional distributions of $\Lambda_{z, z', \xi(\cdot)}$ on the parameters; it will be used in §6 to compute the dynamical correlation functions of $\Lambda_{z, z', \xi(\cdot)}$.

Proposition 4.1. *The statement of Lemma 3.2 holds for arbitrary admissible pair of parameters (z, z') . That is, the transition matrix $P(s, t)$ for $\Lambda_{z, z', \xi(\cdot)}$ is equal to the product $P^\downarrow(s, u)P^\uparrow(u, t)$ with the descending and ascending processes and the time moment u specified as in Lemma 3.2.*

Comments. 1. This statement together with Proposition 2.13 allows to write $P(s, t)$ down rather explicitly. Namely, we obtain

$$P_{\lambda\nu}(s, t) = \frac{(\zeta - \theta)^l (\eta - \theta)^n (1 - \eta)^{zz'}}{\zeta^l (1 - \theta)^{l+n+zz'}} \frac{l! \dim \nu}{n! \dim \lambda} (z)_\nu (z')_\nu \\ \times \sum_{\mu \in \mathbb{Y}} \frac{(1 - \zeta)^m (1 - \theta)^m \eta^m \dim(\mu, \lambda) \dim(\mu, \nu)}{(l - m)! (n - m)!} \frac{1}{(z)_\mu (z')_\mu} \quad (4.1)$$

with the notation

$$\zeta = \xi(s), \quad \eta = \xi(t), \quad \theta = e^{s-t} \sqrt{\zeta \eta}, \quad l = |\lambda|, \quad m = |\mu|, \quad n = |\nu|.$$

The sum above is actually finite; μ ranges over all Young diagrams which are smaller than both λ and ν .

2. The formula for $P_{\lambda\mu}(s, t)$ above after the multiplication by

$$M_{z, z', \zeta}(\lambda) = (1 - \zeta)^{zz'} \zeta^l (z)_\lambda (z')_\lambda \frac{\dim^2 \lambda}{l!}$$

becomes symmetric with respect to $(\lambda, \zeta) \longleftrightarrow (\nu, \eta)$. This shows that the time reversion of $\Lambda_{z, z', \xi(\cdot)}$ is again a process of this form with new $\tilde{\xi}(\tau) = \xi(-\tau)$, cf. Corollary 2.10.

Proof. Set $\widehat{P}(s, t) = P^\downarrow(s, u) P^\uparrow(u, t)$. Clearly, this is a stochastic matrix that satisfies the initial condition $\widehat{P}(t, t) = \text{Id}$. Thus, it suffices to verify that it satisfies Kolmogorov’s backward equation

$$-\frac{\partial}{\partial s} \widehat{P}_{\lambda\nu}(s, t) = Q_{\lambda\lambda}(s) \widehat{P}_{\lambda\lambda}(s, t) + \sum_{\mu: \lambda \nearrow \mu} Q_{\lambda\mu}(s) \widehat{P}_{\mu\nu}(s, t) \\ + \sum_{\mu: \lambda \searrow \mu} Q_{\lambda\mu}(s) \widehat{P}_{\mu\nu}(s, t)$$

with $Q(s)$ given by (2.19). As we substitute the expression in the right-hand side of (4.1) for $\widehat{P}(s, t)$, we see that both sides of the equality above, as function in z and z' , have the form

$$\left(\frac{1 - \eta}{1 - \theta} \right)^{zz'} \times \{ \text{a polynomial in } z, z' \}.$$

Since the equality has already been established for $(z, z') = (N, N + \alpha)$ with $N = 1, 2, \dots$ and $\alpha > -1$, it must hold for arbitrary (z, z') . \square

We proceed to the second claim, which concerns finite-dimensional distributions. Take an admissible pair of parameters (z, z') and an admissible curve $\xi(\cdot)$, and consider the Markov process $\Lambda_{z, z', \xi(\cdot)}$. Let $t_1 < t_2 < \dots < t_n$ be arbitrary time moments. Set

$$\xi_i = \xi(t_i), \quad \eta_{i, i+1} = e^{t_i - t_{i+1}} \sqrt{\xi_i \xi_{i+1}}. \quad (4.2)$$

Proposition 4.1 (or formula (4.1)) and the fact that the process is Markovian imply that the finite-dimensional distributions

$$\text{Prob}\{\Lambda_{z,z',\xi(\cdot)}(t_i) = \lambda(i); \quad i = 1, \dots, n\}$$

with given Young diagrams $\lambda(1), \dots, \lambda(n)$, depend on parameters ξ_1, \dots, ξ_n and $\eta_{12}, \eta_{23}, \dots, \eta_{n-1,n}$ but they do not depend on the behavior of $\xi(t)$ inside the intervals (t_i, t_{i+1}) . Thus, in order to compute these finite-dimensional distributions we may replace our process by a sequence of alternating descending and ascending processes: We start off at the time moment t_1 and go down till time

$$s_{12} = \frac{t_1 + t_2}{2} + \frac{\ln \xi_1 - \ln \xi_2}{4},$$

the value of ξ at this moment is exactly η_{12} . Then we go up till ξ_2 and then again down, etc. The time moments when we change directions are

$$t_1 \leq s_{12} \leq t_2 \leq s_{23} \leq \dots \leq s_{n,n+1} \leq t_n$$

with $s_{i,i+1} = (t_i + t_{i+1})/2 + (\ln \xi_i - \ln \xi_{i+1})/4$, and the values of ξ at these points are $\eta_{i,i+1}$. At t_i 's we switch from going up to going down, and at $s_{i,i+1}$'s we switch from going down to going up.

Fix arbitrary subsets $\mathcal{D}_1, \dots, \mathcal{D}_n$ of \mathbb{Y} . We want to compute the probability that our new descending–ascending process hits $\mathcal{D}_1, \dots, \mathcal{D}_n$ at the time moments t_1, \dots, t_n , respectively. It is equal to the sum

$$\begin{aligned} &F(\xi, \eta; \mathcal{D}_1, \dots, \mathcal{D}_n) \\ &:= \sum_{\substack{\lambda(i) \in \mathcal{D}_i, i=1, \dots, n \\ \mu(j, j+1) \in \mathbb{Y}, j=1, \dots, n-1}} M_{z, z', \xi_1}(\lambda(1)) P_{\lambda(1), \mu(1,2)}^\downarrow(t_1, s_{12}) P_{\mu(1,2), \lambda(2)}^\uparrow(s_{12}, t_2) \cdots \\ &\quad \cdots P_{\lambda(n-1), \mu(n-1,n)}^\downarrow(t_{n-1}, s_{n-1,n}) P_{\mu(n-1,n), \lambda(n)}^\uparrow(s_{n-1,n}, t_n), \end{aligned} \tag{4.3}$$

where ξ_i and $\eta_{i,i+1}$ are given by (4.2). This definition makes sense because the right-hand side is actually a function in ξ, η . Note that the new variables ξ_i and $\eta_{i,i+1}$ may take arbitrary values in $(0, 1)$ subject to the inequalities

$$\xi_1 \geq \eta_{12} \leq \xi_2 \geq \eta_{23} \leq \xi_3 \geq \dots \leq \xi_{n-1} \geq \eta_{n,n-1} \leq \xi_n. \tag{4.4}$$

A “static” version of the next result was proved in [BO8, Lemma 3.11].

Proposition 4.2. *Let $F(\xi, \eta; \mathcal{D}_1, \dots, \mathcal{D}_n)$ be the function defined above, where variables ξ_i and $\eta_{i,i+1}$ are in $(0, 1)$ and satisfy inequalities (4.4).*

- (i) $F(\xi, \eta; \mathcal{D}_1, \dots, \mathcal{D}_n)$ is a real-analytic function in ξ and η .
- (ii) For fixed ξ and η , the function $\varepsilon \mapsto F(\varepsilon\xi, \varepsilon\eta; \mathcal{D}_1, \dots, \mathcal{D}_n)$, which is initially defined for real $\varepsilon \in (0, 1]$, can be analytically continued to a small neighborhood of 0 in \mathbb{C} . The coefficients of the Taylor decomposition of this function at $\varepsilon = 0$ are polynomial functions in z, z' .

Comment. The proposition implies that the function $F(\xi, \eta; \mathcal{D}_1, \dots, \mathcal{D}_n)$ viewed also as a function in z, z' , is uniquely determined by its values on the arguments (ξ, η, z, z') such that ξ, η satisfy inequalities $0 < \xi_i, \eta_{j, j+1} < 1$ and (4.4), and $(z, z') = (N, N + \alpha)$ with $N = 1, 2, \dots$ and $\alpha > -1$. This uniqueness property will be used in the next section to extend certain formulas derived in the case of integral z to the general case.

Proof. In order to prove (i), by Weierstrass' uniform convergence theorem it suffices to check that the series (4.2) with $M_{z, z', \xi_1}(\lambda(1))$ and all P^\downarrow, P^\uparrow replaced by their expressions given by (1.11), (2.22), (2.23), converges absolutely and uniformly in $\xi \in \mathbb{C}^n$ and $\eta \in \mathbb{C}^{n-1}$ varying in small discs around their initial (real) values. It is more convenient to work with the case when all \mathcal{D}_i coincide with \mathbb{Y} ; clearly, the needed convergence of the restricted sum follows from that of the unrestricted sum.

Set $l_i = |\lambda(i)|$, $m_{i, i+1} = |\mu(i, i + 1)|$. As seen from the proof of Proposition 2.13, the matrix elements $P_{\lambda\mu}^\downarrow$ and $P_{\lambda\mu}^\uparrow$ split into products of transition probabilities for pure death and pure birth processes of Lemma 2.8 and (co)transition probabilities on the Young graph. By (1.11), $M_{z, z', \xi}(\lambda)$ is also a product of the negative binomial distribution $\pi_{c, \xi}(l)$ on \mathbb{Z}_+ and the probability distributions $M_{z, z'}^{(l)}(\lambda)$ on \mathbb{Y}_l 's. The probabilities related to the Young graph do not depend on ξ and η . Thus, we can represent the sum in (4.2) (remember that all \mathcal{D}_i are equal to \mathbb{Y}) as a double sum: the outer sum is taken over all nonnegative integers l_1, \dots, l_n and $m_{12}, \dots, m_{n-1, n}$ satisfying

$$l_1 \geq m_{12} \leq l_2 \geq m_{23} \leq \dots \geq m_{n-1, n} \leq l_n, \tag{4.5}$$

and the inner sum is taken over all Young diagrams $\lambda(i) \in \mathbb{Y}_{l_i}, \mu(i, i + 1) \in \mathbb{Y}_{m_{i, i+1}}$.

Since all factors related to the Young graph are nonnegative we can write (using the relations (1.5), (1.6) applied to the z -measures)

$$\begin{aligned} & \sum_{\substack{\lambda(i) \in \mathbb{Y}, i=1, \dots, n \\ \mu(j, j+1) \in \mathbb{Y}, j=1, \dots, n-1}} \left| M_{z, z', \xi_1}(\lambda(1)) P_{\lambda(1), \mu(1,2)}^\downarrow(t_1, s_{12}) P_{\mu(1,2), \lambda(2)}^\uparrow(s_{12}, t_2) \dots \right. \\ & \quad \left. \dots P_{\lambda(n-1), \mu(n-1,n)}^\downarrow(t_{n-1}, s_{n-1,n}) P_{\mu(n-1,n), \lambda(n)}^\uparrow(s_{n-1,n}, t_n) \right| \\ & = \sum_{\substack{l_1, \dots, l_n \\ m_{1,2}, \dots, m_{n-1,n}}} \left| \pi_{z, z', \xi_1}(l_1) P_{l_1, m_{12}}^\downarrow(t_1, s_{12}) P_{m_{12}, l_2}^\uparrow(s_{12}, t_2) \dots \right. \\ & \quad \left. \dots P_{l_{n-1}, m_{n-1,n}}^\downarrow(t_{n-1}, s_{n-1,n}) P_{m_{n-1,n}, l_n}^\uparrow(s_{n-1,n}, t_n) \right| \tag{4.6} \end{aligned}$$

where, in the right-hand side, the quantities P^\downarrow and P^\uparrow are given by Lemma 2.8, and the summation is taken over indices subject to (4.5). Thus, we need to verify the uniform convergence of the right-hand side under small complex variations of ξ and η .

It is convenient to set

$$u_i = \frac{\xi_i - \eta_{i-1,i}}{1 - \eta_{i-1,i}}, \quad i = 1, \dots, n, \quad \text{with } \eta_{0,1} := 0, \quad (4.7)$$

$$v_{i,i+1} = \frac{(1 - \xi_i) \eta_{i,i+1}}{(1 - \eta_{i,i+1}) \xi_i}, \quad i = 1, \dots, n - 1. \quad (4.8)$$

The inequalities (4.4) imply that $u_i \in [0, 1)$ and $v_{i,i+1} \in (0, 1]$ for all i . Clearly, small (complex) variations of ξ and η lead to small (complex) variations of u_i 's and $v_{i,i+1}$'s. What is important for us here is that if the variations are small enough then u_i 's are bounded away from 1 and $v_{i,i+1}$'s are bounded away from 0.

The probabilities entering the right-hand side of (4.6) can be expressed through variables (4.7) and (4.8), as follows (here we use (1.10), (2.14) and (2.15))

$$\pi_{zz', \xi_1}(l_1) = (1 - u_1)^c u_1^{l_1} \frac{(c)_{l_1}}{l_1!}, \quad (4.9)$$

$$P_{m_{i-1,i}, l_i}^\uparrow(s_{i-1,i}, t_i) = (1 - u_i)^{c+m_{i-1,i}} u_i^{l_i - m_{i-1,i}} \frac{(c + m_{i-1,i})_{l_i - m_{i-1,i}}}{(l_i - m_{i-1,i})!} \quad (i = 2, \dots, n), \quad (4.10)$$

$$P_{l_i, m_{i,i+1}}^\downarrow(t_i, s_{i,i+1}) = v_{i,i+1}^{l_i} \left(\frac{1}{v_{i,i+1}} - 1 \right)^{l_i - m_{i,i+1}} \frac{l_i!}{(l_i - m_{i,i+1})! m_{i,i+1}!} \quad (i = 1, \dots, n - 1), \quad (4.11)$$

where, as before, $c = zz'$. Note that (4.9) is a special case of (4.10) with $m_{0,1} = 0$.

We proceed to estimating the right-hand side of (4.6). Start with summation over l_n . The only factor in the right-hand of (4.6) that depends on l_n is $\left| P_{m_{n-1,n}, l_n}^\uparrow(s_{n-1,n}, t_n) \right|$. Using (4.11) we obtain

$$\sum_{l_n \geq m_{n-1,n}} \left| P_{m_{n-1,n}, l_n}^\uparrow(s_{n-1,n}, t_n) \right| = |1 - u_n|^{c+m_{n-1,n}} \times \sum_{l_n \geq m_{n-1,n}} |u_n|^{l_n - m_{n-1,n}} \frac{(c+m_{n-1,n})_{l_n - m_{n-1,n}}}{(l_n - m_{n-1,n})!} = \left(\frac{|1 - u_n|}{1 - |u_n|} \right)^{c+m_{n-1,n}}. \quad (4.12)$$

We conclude that this expression can be estimated, as a function of $m_{n-1,n}$, by a constant times a geometric progression of the form $r^{m_{n-1,n}}$ with a suitable $r > 0$. If the variation of u_n is small enough, u_n is close to the real axis. Hence, by decreasing the variations we can take r arbitrarily close to 1.

Next, consider the double sum over $m_{n,n-1} \leq l_{n-1}$ and $l_n \geq m_{n-1,n}$, where we have two relevant factors. Applying the above estimate and using (4.10) we obtain

$$\begin{aligned} & \sum_{m_{n-1,n} \leq l_{n-1}} \sum_{l_n \geq m_{n-1,n}} \left| P_{l_{n-1}, m_{n-1,n}}^\downarrow(t_{n-1}, s_{n-1,n}) P_{m_{n-1,n}, l_n}^\uparrow(s_{n-1,n}, t_n) \right| \\ & \leq \text{const } |v_{n-1,n}|^{l_{n-1}} \sum_{m_{n-1,n} \leq l_{n-1}} \left| \frac{1}{v_{n-1,n}} - 1 \right|^{l_{n-1}-m_{n-1,n}} r^{m_{n-1,n}} \\ & \times \frac{l_{n-1}!}{(l_{n-1} - m_{n-1,n})! m_{n-1,n}!} = \text{const} \left(\left| \frac{1}{v_{n-1,n}} - 1 \right| + r \right)^{l_{n-1}} |v_{n-1,n}|^{l_{n-1}}. \end{aligned}$$

Again, this is bounded by a constant times a geometric progression $\tilde{r}^{l_{n-1}}$ where \tilde{r} can be made arbitrarily close to 1 by considering small enough variations of u_n and $v_{n-1,n}$.

The next step, the triple summation, is performed similarly to (4.12). The only difference is in the presence of the additional geometric progression $\tilde{r}^{l_{n-1}}$. The summation yields

$$\text{const} \left(\frac{|1 - u_{n-1}|}{1 - \tilde{r} |u_{n-1}|} \right)^{c+m_{n-2,n-1}}.$$

Once again, for small variations of u_{n-1} , $v_{n-1,n}$, u_n , this is bounded by a constant times a geometric progression with exponent $m_{n-2,n-1}$ and a ratio that is close to 1.

Iterating this procedure we finally obtain that the whole sum in the right-hand side of (4.6) is bounded by

$$\text{const } |1 - u_1|^c \sum_{l_1 \geq 0} |u_1|^{l_1} r^{l_1} \frac{(c)_{l_1}}{l_1!}$$

with a certain $r > 0$ close to 1. This sum is finite, which completes the argument.

Let us prove (ii). Fix ξ and η as in the formulation of the proposition. For a small complex ε , let $u_i^{(\varepsilon)}$ and $v_{i,i+1}^{(\varepsilon)}$ be defined according to (4.7) and (4.8) with ξ and η replaced by $\varepsilon\xi$ and $\varepsilon\eta$, respectively. Remark that $u_i^{(\varepsilon)}$ is a small variation of 0 while $v_{i,i+1}^{(\varepsilon)}$ is a small variation of $\eta_{i,i+1}/\xi_i$ which a real number in $(0, 1]$. Recall now that the only assumption on the variables u_i and $v_{i,i+1}$ that we exploited in the proof of (i) is that u_i and $v_{i,i+1}$ are small complex variations of certain real numbers in $[0, 1)$ and $(0, 1]$, respectively. Therefore, exactly the same argument as in (i) shows that $\varepsilon \mapsto F(\varepsilon\xi, \varepsilon\eta; \mathcal{D}_1, \dots, \mathcal{D}_n)$ is well defined as an analytic function for small complex ε .

It remains to check that the Taylor coefficients of this function are polynomials in z, z' . To do this we return to the initial expression (4.3). Using formulas (2.22) and (2.23) (which we adapt to our present notation) we can write (4.3) in the following form

$$F(\xi, \eta; \mathcal{D}_1, \dots, \mathcal{D}_n) = \sum_{\substack{\lambda(i) \in \mathcal{D}_i, i=1, \dots, n \\ \mu(j, j+1) \in \mathbb{Y}, j=1, \dots, n-1}} C(\vec{\lambda}, \vec{\mu}),$$

where $\vec{\lambda}, \vec{\mu} = (\lambda(1), \mu(1, 2), \lambda(2), \dots, \lambda(n-1), \mu(n-1, n), \lambda(n))$,

$$\begin{aligned} C(\vec{\lambda}, \vec{\mu}) &= M_{z, z', \xi_1}(\lambda(1)) \mathcal{P}_{\lambda(1), \mu(1,2)}^\downarrow(\xi_1, \eta_{12}) \mathcal{P}_{\mu(1,2), \lambda(2)}^\uparrow(\eta_{12}, \xi_2) \cdots \\ &\cdots \mathcal{P}_{\lambda(n-1), \mu(n-1,n)}^\downarrow(\xi_{n-1}, \eta_{n-1,n}) \mathcal{P}_{\mu(n-1,n), \lambda(n)}^\uparrow(\eta_{n-1,n}, \xi_n) \end{aligned}$$

with

$$M_{z,z',\xi_1}(\lambda(1)) = (1 - \xi_1)^{zz'} \xi_1^{l_1} (z)_{\lambda(1)}(z')_{\lambda(1)} \left(\frac{\dim \lambda(1)}{l_1!} \right)^2$$

(see (1.11)),

$$\begin{aligned} \mathcal{P}_{\lambda_i, \mu_{i,i+1}}^\downarrow(\xi_i, \eta_{i,i+1}) &= \left(\frac{(1 - \xi_i)\eta_{i,i+1}}{(1 - \eta_{i,i+1})\xi_i} \right)^{l_i} \left(\frac{\xi_i - \eta_{i,i+1}}{(1 - \xi_i)\eta_{i,i+1}} \right)^{l_i - m_{i,i+1}} \\ &\times \frac{l_i!}{(l_i - m_{i,i+1})! m_{i,i+1}!} \frac{\dim \mu(i, i+1) \dim(\mu(i, i+1), \lambda(i))}{\dim \lambda(i)} \end{aligned}$$

and

$$\begin{aligned} P_{\mu(i-1,i), \lambda(i)}^\uparrow(\eta_{i-1,i}, \xi_i) &= \left(\frac{1 - \xi_i}{1 - \eta_{i-1,i}} \right)^{zz' + m_{i-1,i}} \left(\frac{\xi_i - \eta_{i-1,i}}{1 - \eta_{i-1,i}} \right)^{l_i - m_{i-1,i}} \\ &\times \frac{m_{i-1,i}!}{(l_i - m_{i-1,i})! l_i!} \frac{\dim \lambda(i) \dim(\mu(i-1, i), \lambda(i))}{\dim \mu(i-1, i)} \\ &\times (z)_{\lambda(i) \setminus \mu(i-1,i)}(z')_{\lambda(i) \setminus \mu(i-1,i)}. \end{aligned}$$

Remark that each term $C(\lambda, \mu)$ with ξ, η replaced by $\varepsilon\xi, \varepsilon\eta$, involves the factor

$$\varepsilon^{l_1 + \dots + l_n - m_{12} - \dots - m_{n-1,n}}.$$

Indeed, it comes from the factors $(\varepsilon\xi_i - \varepsilon\eta_{i-1,i})^{l_i - m_{i-1,i}}$ in P^\uparrow 's and from $(\varepsilon\xi_1)^{l_1}$ in $M_{z,z',\varepsilon\xi_1}(\lambda(1))$.

Next, remark that inequalities (4.5) imply

$$l_1 + \dots + l_n - m_{12} - \dots - m_{n-1,n} \geq \max(l_1, \dots, l_n).$$

Indeed, let i be such that $l_i = \max(l_1, \dots, l_n)$. Then we have

$$\lambda_j \geq \mu_{j,j+1} \quad (j = 1, \dots, i-1), \quad \lambda_{j+1} \geq \mu_{j,j+1} \quad (j = i, \dots, n-1),$$

whence $l_1 + \dots + l_n - m_{12} - \dots - m_{n-1,n} \geq l_i$.

Thus, only finitely many terms $C(\lambda, \mu)$ contribute to a fixed Taylor coefficient. Each of these terms involves polynomial expressions in z, z' and expressions of the form $(1 - \varepsilon\xi_i)^{zz'}$, $(1 - \varepsilon\eta_{i,i+1})^{-zz'}$, and their Taylor coefficients at $\varepsilon = 0$ are also polynomials in z, z' . \square

5. A basis in the ℓ^2 space on the lattice and the Meixner polynomials

This short section contains a summary of formulas which will be used in what follows. For proofs and more details we refer to [BO8, §2].

Throughout the section we will assume (unless otherwise stated) that (z, z') is in the principal series or in the complementary series but not in the degenerate series. In particular, z, z' are not integers.

Consider the lattice of (proper) half-integers

$$\mathbb{Z}' = \mathbb{Z} + \frac{1}{2} = \{ \dots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \}.$$

We introduce a family of functions on \mathbb{Z}' depending on a parameter $a \in \mathbb{Z}'$ and also on our parameters z, z', ξ :

$$\begin{aligned} \psi_a(x; z, z', \xi) &= \left(\frac{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})}{\Gamma(z-a+\frac{1}{2})\Gamma(z'-a+\frac{1}{2})} \right)^{\frac{1}{2}} \xi^{\frac{1}{2}(x+a)} (1-\xi)^{\frac{1}{2}(z+z')-a} \\ &\times \frac{F(-z+a+\frac{1}{2}, -z'+a+\frac{1}{2}; x+a+1; \frac{\xi}{\xi-1})}{\Gamma(x+a+1)}, \quad x \in \mathbb{Z}', \end{aligned} \tag{5.1}$$

where $F(A, B; C; w)$ is the Gauss hypergeometric function. Due to the Γ -factor in the denominator, these functions are well defined on the whole lattice. Note that they are real-valued.

Further, we introduce a second order difference operator $D(z, z', \xi)$ on the lattice \mathbb{Z}' , depending on parameters z, z', ξ and acting on functions $f(x)$ (where x ranges over \mathbb{Z}') as follows

$$\begin{aligned} D(z, z', \xi) f(x) &= \sqrt{\xi(z+x+\frac{1}{2})(z'+x+\frac{1}{2})} f(x+1) \\ &+ \sqrt{\xi(z+x-\frac{1}{2})(z'+x-\frac{1}{2})} f(x-1) - (x+\xi(z+z'+x)) f(x). \end{aligned}$$

Note that $D(z, z', \xi)$ is a symmetric operator in $\ell^2(\mathbb{Z}')$.

The functions $\psi_a(x; z, z', \xi)$ are eigenfunctions of the operator $D(z, z', \xi)$:

$$D(z, z', \xi) \psi_a(x; z, z', \xi) = a(1-\xi) \psi_a(x; z, z', \xi), \quad a \in \mathbb{Z}'. \tag{5.2}$$

For any fixed triple (z, z', ξ) , they form an orthonormal basis in $\ell^2(\mathbb{Z}')$.

We have the following integral representation

$$\begin{aligned} \psi_a(x; z, z', \xi) &= \left(\frac{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})}{\Gamma(z-a+\frac{1}{2})\Gamma(z'-a+\frac{1}{2})} \right)^{\frac{1}{2}} \frac{\Gamma(z'-a+\frac{1}{2})}{\Gamma(z'+x+\frac{1}{2})} (1-\xi)^{\frac{z'-z+1}{2}} \\ &\times \frac{1}{2\pi i} \oint_{\{\omega\}} \left(1-\sqrt{\xi}\omega\right)^{-z'+a-\frac{1}{2}} \left(1-\frac{\sqrt{\xi}}{\omega}\right)^{z-a-\frac{1}{2}} \omega^{-x-a} \frac{d\omega}{\omega} \end{aligned} \tag{5.3}$$

Here $\{\omega\}$ is an arbitrary simple contour which goes around the points 0 and $\sqrt{\xi}$ in the positive direction leaving $1/\sqrt{\xi}$ outside. We also assume that the argument of $1-\sqrt{\xi}\omega$ and $1-\sqrt{\xi}/\omega$ equals 0 for real negative values of ω .

A related useful formula is

$$\begin{aligned} & \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi) \\ &= \varphi_{z, z'}(x, y) \frac{1 - \xi}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} \left(1 - \sqrt{\xi} \omega_1\right)^{-z'+a-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{z-a-\frac{1}{2}} \\ & \quad \times \left(1 - \sqrt{\xi} \omega_2\right)^{-z+a-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z'-a-\frac{1}{2}} \\ & \quad \times \omega_1^{-x-a} \omega_2^{-y-a} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2}, \end{aligned} \tag{5.4}$$

where

$$\varphi_{z, z'}(x, y) = \frac{\sqrt{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})\Gamma(y+z'+\frac{1}{2})}}{\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})}. \tag{5.5}$$

It is worth noting that (5.4), in contrast to (5.3), does not contain Γ -prefactors depending on index a . This is achieved due to switching $z \leftrightarrow z'$ in the second integral (for more detail, see the proof of (5.4) in [BO8, Proposition 2.3]).

The following symmetry relations hold

$$\psi_a(x; z, z', \xi) = \psi_x(a; -z, -z', \xi) \tag{5.6}$$

$$\psi_a(x; z, z', \xi) = (-1)^{x+a} \psi_{-a}(-x; -z, -z', \xi). \tag{5.7}$$

Let us explain how the functions ψ_a are related to the Meixner polynomials. Below elements of \mathbb{Z}_+ will be denoted by symbols \tilde{x}, \tilde{y} (we reserve the notation x and y for points of the lattice \mathbb{Z}'). Let us drop the assumption that (z, z') is not in the degenerate series, and assume, just on the contrary, that $z = N$ and $z' = N + \alpha$, where $N = 1, 2, \dots$ and $\alpha > -1$. Then expression (5.1) for the functions $\psi_a(x; z, z', \xi)$ still makes sense provided that

$$\tilde{x} := x + N - \frac{1}{2}, \quad n := N - a - \frac{1}{2} \tag{5.8}$$

are in \mathbb{Z}_+ , and in this notation we have

$$\psi_a(x; z, z', \xi) = \tilde{\mathfrak{M}}_n(\tilde{x}; \alpha + 1, \xi), \tag{5.9}$$

where the functions $\tilde{\mathfrak{M}}_n$ on \mathbb{Z}_+ were defined in §2.4.

The following explanation might help the reader to follow the analytic continuation arguments in §6. Our functions ψ_a can be obtained from the Meixner polynomials by the following procedure:

- We replace the initial polynomials \mathfrak{M}_n by the functions $\tilde{\mathfrak{M}}_n$. This step is quite clear: as a result we get functions which form an orthonormal basis in the ℓ^2 space on \mathbb{Z}_+ with respect to the weight function 1.

- Next, we make a change of the argument. Namely, we introduce an additional parameter $N = 1, 2, \dots$ and we set $x = \tilde{x} - N + \frac{1}{2}$. Then we get orthogonal functions on the subset

$$\{-N + \frac{1}{2}, -N + \frac{3}{2}, -N + \frac{5}{2}, \dots\} \subset \mathbb{Z}'$$

which exhausts the whole \mathbb{Z}' in the limit as N goes to infinity.

- Then we also need a change of the index. Namely, instead of n we have to take $a = N - n - \frac{1}{2}$. We cannot give a conceptual explanation of this transformation, it is dictated by the formulas. Again, the range of the possible values for a becomes larger together with N , and in the limit as $N \rightarrow +\infty$ we get the whole lattice \mathbb{Z}' .
- Finally, we make a (formal) analytic continuation in parameters N and α , using an appropriate analytic expression for the Meixner polynomials (see [BO8, §3]). Note that the difference equation (5.2) precisely corresponds to a similar relation for the Meixner polynomials.

Of course, instead of the lattice \mathbb{Z}' we could equally well deal with the lattice \mathbb{Z} , and then numerous “ $\frac{1}{2}$ ” would disappear. However, dealing with the lattice \mathbb{Z}' makes main formulas more symmetric.

6. Dynamical correlation functions

6.1. Definitions

As in §5, we are dealing with the lattice $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ of (proper) half-integers. For an arbitrary $\lambda \in \mathbb{Y}$ we set

$$\underline{X}(\lambda) = \{\lambda_i - i + \frac{1}{2} \mid i = 1, 2, \dots\} \subset \mathbb{Z}'.$$

For instance, $\underline{X}(\emptyset) = \{-\frac{1}{2}, -\frac{3}{2}, \dots\}$. The correspondence $\lambda \mapsto \underline{X}(\lambda)$ is an embedding of \mathbb{Y} into the set of point configurations in \mathbb{Z}' (here “point configuration” means simply “subset”).

Assume we are given a continuous time stochastic process $\Lambda(t)$ with the state space \mathbb{Y} . Using the above correspondence we can convert $\Lambda(t)$ into a process with values in point configurations in \mathbb{Z}' ; let us denote the latter process by $\underline{X}(t)$.

For any $n = 1, 2, \dots$ define the n th dynamical correlation function of \underline{X} pairwise distinct arguments $(t_1, x_1), \dots, (t_n, x_n) \in (t_{\min}, t_{\max}) \times \mathbb{Z}'$ by

$$\rho_n(t_1, x_1; t_2, x_2; \dots; t_n, x_n) = \text{Prob} \{ \underline{X}(t_i) \text{ contains } x_i \text{ for all } i = 1, \dots, n \}.$$

In other words, the dynamical correlation functions describe the probabilities of events of the following type: for given time moments $s_1 < \dots < s_m$ and given finite sets Y_1, \dots, Y_m , the random point configurations $\underline{X}(s_1), \dots, \underline{X}(s_m)$ contain Y_1, \dots, Y_m , respectively. Thus, the notion of the dynamical correlation functions is a hybrid of the finite-dimensional distributions of a stochastic process and standard correlation functions of probability measures on point configurations.

Clearly, the dynamical correlation functions uniquely determine the finite-dimensional distributions of the process and, thus, the process itself. The reason why we are interested in these quantities is the same as in the “static” (fixed time) case: As we take scaling limits of our processes, for the limiting object the notion of the weight of a point configuration does not make sense anymore. Thus, the probabilities of the form $\text{Prob} \{ \underline{X}(s_1) = \underline{X}_1, \dots, \underline{X}(s_m) = \underline{X}_m \}$ do not have any meaning in the limit while the scaling limits of the correlation functions are well-defined and, moreover, carry a lot of useful information about the limit process.

We say that the process $\underline{X}(t)$ is *determinantal* if there exists a kernel

$$K : ((t_{\min}, t_{\max}) \times \mathbb{Z}^l) \times ((t_{\min}, t_{\max}) \times \mathbb{Z}^l) \rightarrow \mathbb{C}$$

such that for any $n = 1, 2, \dots$

$$\rho_n(t_1, x_1; \dots; x_n, t_n) = \det [K(t_i, x_i; t_j, x_j)]_{i,j=1}^n.$$

If such a kernel exists then it is not unique. In particular, transformations of the form

$$K(s, x; t, y) \longrightarrow \frac{f(s, x)}{f(t, y)} K(s, x; t, y) \tag{6.1}$$

do not change the correlation functions.

6.2. Main results

Theorem 6.1. *Let (z, z') be a pair of admissible parameters and $\xi(\cdot)$ be an admissible curve. Consider the Markov process $\Lambda_{z,z',\xi(\cdot)}$ defined in §2, and denote by $\underline{X}_{z,z',\xi(\cdot)}$ the corresponding process with values in the space of point configurations in \mathbb{Z}^l . Then the process $\underline{X}_{z,z',\xi(\cdot)}$ is determinantal.*

Recall that in §5 we introduced the functions $\psi_a(x; z, z', \xi)$ which form, for any $\xi \in (0, 1)$, an orthonormal basis in $\ell^2(\mathbb{Z}^l)$. These functions were defined under the condition that (z, z') belong to either principal or complementary series.

Theorem 6.2. *Assume that (z, z') is either in principal or complementary series. Then the kernel*

$$\underline{K}_{z,z',\xi(\cdot)}(s, x; t, y) = \pm \sum_{a \in \mathbb{Z}^l_+} e^{\pm a(t-s)} \psi_{\pm a}(x; z, z', \xi(s)) \psi_{\pm a}(y; z, z', \xi(t)) \tag{6.2}$$

with “+” taken for $s \geq t$ and “−” taken for $s < t$, is a correlation kernel of the process $\underline{X}_{z,z',\xi(\cdot)}$.

Theorem 6.3. *The correlation kernel (6.2) can also be written in the form*

$$\underline{K}_{z,z',\xi(\cdot)}(s, x; t, y) = e^{\frac{1}{2}(s-t)} \varphi_{z,z'}(x, y) \widehat{K}_{z,z',\xi(\cdot)}(s, x; t, y), \tag{6.3}$$

where, as in (5.5),

$$\varphi_{z,z'}(x, y) = \frac{\sqrt{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})\Gamma(y+z'+\frac{1}{2})}}{\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})}$$

and the kernel $\widehat{K}_{z,z',\xi(\cdot)}(s, x; t, y)$ can be written as a double contour integral (set $\zeta = \sqrt{\xi(s)}$, $\eta = \sqrt{\xi(t)}$)

$$\begin{aligned} \widehat{K}_{z,z',\xi(\cdot)}(s, x; t, y) &= \frac{\sqrt{(1-\zeta)(1-\eta)}}{(2\pi i)^2} \\ &\times \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} (1-\zeta\omega_1)^{-z'} (1-\zeta\omega_1^{-1})^z (1-\eta\omega_2)^{-z} (1-\eta\omega_2^{-1})^{z'} \quad (6.4) \\ &\times \frac{\omega_1^{-x-\frac{1}{2}}\omega_2^{-y-\frac{1}{2}}}{e^{s-t}(\omega_1-\zeta)(\omega_2-\eta)-(1-\zeta\omega_1)(1-\eta\omega_2)} d\omega_1 d\omega_2 \end{aligned}$$

with the contours $\{\omega_1\}$ and $\{\omega_2\}$ of ω_1 and ω_2 satisfying the following conditions:

- $\{\omega_1\}$ goes around 0 in positive direction and passes between ζ and ζ^{-1} ;
- $\{\omega_2\}$ goes around 0 in positive direction and passes between η and η^{-1} ;
- if $s \geq t$ then the image of $\{\omega_1\}$ under the fractional-linear map

$$\omega \mapsto \mathcal{A}(\omega) = \frac{\omega(e^{s-t}\eta - \zeta) + 1 - e^{s-t}\zeta\eta}{\omega(e^{s-t} - \zeta\eta) + \eta - e^{s-t}\zeta} \quad (6.5)$$

is contained inside $\{\omega_2\}$;

- if $s < t$ then the domain bounded by $\{\omega_2\}$ does not intersect the image of $\{\omega_1\}$ under the map above.

The kernels $\underline{K}_{z,z',\xi(\cdot)}(s, x; t, y)$ and $\widehat{K}_{z,z',\xi(\cdot)}(s, x; t, y)$ are equivalent. Namely, they are related by a “gauge transformation” (6.1),

$$\widehat{K}_{z,z',\xi(\cdot)}(s, x; t, y) = \frac{f_{z,z'}(s, x)}{f_{z,z'}(t, y)} \underline{K}_{z,z',\xi(\cdot)}(x, y), \quad x, y \in \mathbb{Z}',$$

where

$$f_{z,z'}(s, x) = \frac{e^{-\frac{1}{2}s} \Gamma(x+z'+\frac{1}{2})}{\sqrt{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})}} \quad (6.6)$$

The kernel $\widehat{K}_{z,z,\xi(\cdot)}(s, x; t, y)$ can serve as a correlation kernel for all admissible values of parameters (z, z') , including the degenerate series.

Comments. 1. The fractional-linear transformation (6.5) arises from the condition that the denominator in the integral representation (6.4) has to be nonzero. Indeed, the equation “denominator=0” is equivalent to $\omega_2 = \mathcal{A}(\omega_1)$.

2. It is not *a priori* clear why the needed contours $\{\omega_1\}$ and $\{\omega_2\}$ exist. Let us show that it is indeed so. Set $q = e^{s-t}$ and note that $\mathcal{A}(\zeta) = \eta^{-1}$ and $\mathcal{A}(\zeta^{-1}) = \eta$.

Consider the case $q \geq 1$ (i.e. $s \geq t$) first. Let us take a circle with center at the origin and radius r slightly smaller than ζ^{-1} as $\{\omega_1\}$. Then its image under \mathcal{A} is again a circle which is symmetric with respect to the real axis and which passes through $\mathcal{A}(r)$ and $\mathcal{A}(-r)$. Remark that $\mathcal{A}(r)$ is close to $\mathcal{A}(\zeta^{-1}) = \eta$, and $\mathcal{A}(-r)$ is close to

$$\mathcal{A}(-\zeta^{-1}) = \frac{-q\eta(\zeta + \zeta^{-1}) + 2}{-q(\zeta + \zeta^{-1}) + 2\eta}.$$

Since ζ and η are strictly between 0 and 1, we immediately see that the denominator is negative, and the whole expression is $< \eta$. Thus, the image of $\{\omega_1\}$ is a finite circle that lies to the left of $\eta + \varepsilon$ with a small $\varepsilon > 0$. Clearly, there exists a contour $\{\omega_2\}$ that passes between η and η^{-1} and encircles both 0 and the image of $\{\omega_1\}$.

Let us consider the case $q < 1$ now. As $\{\omega_1\}$ we again take a circle with center at the origin but with radius slightly greater than ζ . Then its image $\mathcal{A}(\{\omega_1\})$ is a circle which is symmetric with respect to the real axis and which passes near $\mathcal{A}(\zeta) = \eta^{-1}$ and

$$\mathcal{A}(-\zeta) = \frac{\zeta + \zeta^{-1} - 2q\eta}{\eta(\zeta + \zeta^{-1}) - 2q}.$$

If the denominator of this ratio is negative then the whole expression is negative, and there exists a contour $\{\omega_2\}$ inside this circle that passes between η and η^{-1} and goes around the origin. If the denominator is positive then the whole expression is $> \eta^{-1}$, and $\{\omega_2\}$ can be a circle of radius between η and η^{-1} with center at the origin. The same choice is possible in the limit case when the denominator is 0 and $\mathcal{A}(\{\omega_1\})$ degenerates to the vertical line passing through η^{-1} . \square

6.3. Proof of Theorems 6.1, 6.2, and 6.3

Although our argument is self-contained, it might be helpful to the reader to look first at [BO8, §3], where similar claims are proved in the static case $t_1 = \dots = t_n$.

The first step is to examine the case $z = N \in \{1, 2, \dots\}$, $z' = N + \alpha$, $\alpha > -1$. As was already mentioned in §1, the measure $M_{N, N+\alpha, \xi}$ is concentrated on a proper subset of \mathbb{Y} — the set of diagrams with at most N rows; let us denote it by $\mathbb{Y}(N)$. Consider the embedding of $\mathbb{Y}(N)$ into the set of N -point configurations of \mathbb{Z}_+ given by

$$\lambda \in \mathbb{Y}(N) \mapsto \tilde{X}(\lambda) = (\tilde{x}_1, \dots, \tilde{x}_N), \quad \tilde{x}_i = \lambda_i + N - i, \quad i = 1, \dots, N.$$

Our starting point is the following observation from [BO2, Proposition 4.1]:

$$M_{N, N+\alpha, \xi}(\lambda) = \text{const} \cdot \prod_{1 \leq i < j \leq N} (\tilde{x}_i - \tilde{x}_j)^2 \prod_{i=1}^n W_{\alpha+1, \xi}(\tilde{x}_i), \quad \lambda \in \mathbb{Y}(N),$$

where $W_{\alpha+1, \xi}$ is the Meixner weight function on \mathbb{Z}_+ with parameters $\alpha + 1$ and ξ (see §2.4). This means that the pushforward of $M_{N, N+\alpha, \xi}$ under the above embedding is the N -point Meixner orthogonal polynomial ensemble on \mathbb{Z}_+ .⁶

Denote by $\tilde{X}_{N, N+\alpha, \xi(\cdot)}$ the stochastic process on the space of N -point configurations in \mathbb{Z}_+ corresponding to $\Lambda_{N, N+\alpha, \xi(\cdot)}$ under the embedding $\lambda \mapsto \tilde{X}(\lambda)$. (The definition is correct because $\Lambda_{N, N+\alpha, \xi(\cdot)}$ lives on the subset $\mathbb{Y}(N) \subset \mathbb{Y}$.) Thus, $\Lambda_{N, N+\alpha, \xi(\cdot)}$ gives rise to two processes on point configurations: $\underline{X}_{N, N+\alpha, \xi(\cdot)}$ and $\tilde{X}_{N, N+\alpha, \xi(\cdot)}$. Obviously, these two processes are essentially equivalent: if $\lambda \in \mathbb{Y}(N)$ then $\underline{X}(\lambda) = (x_1, x_2, \dots)$ and $\tilde{X}(\lambda) = (\tilde{x}_1, \dots, \tilde{x}_N)$ are related by

$$x_i = \begin{cases} \tilde{x}_i - N + \frac{1}{2}, & i = 1, \dots, N, \\ -i + \frac{1}{2}, & i \geq N + 1. \end{cases}$$

This implies that the dynamical correlation functions ρ_l of $\underline{X}_{N, N+\alpha, \xi(\cdot)}$ and $\tilde{\rho}_l$ of $\tilde{X}_{N, N+\alpha, \xi(\cdot)}$ are related by

$$\rho_l(\tau_1, x_1; \dots; \tau_l, x_l) = \tilde{\rho}_l(\tau_1, \tilde{x}_1; \dots; \tau_l, \tilde{x}_l)$$

with $\tilde{x}_i = x_i + N - \frac{1}{2} \in \mathbb{Z}_+$ for $i = 1, \dots, l$.

Define the extended Meixner kernel $K_{N, \alpha, \xi(\cdot)}^{\text{Meixner}}$ by

$$K_{N, \alpha, \xi(\cdot)}^{\text{Meixner}}(s, \tilde{x}; t, \tilde{y}) = \begin{cases} K_{N, \alpha, \xi(\cdot)}^{\text{Meixner}, +}(s, \tilde{x}; t, \tilde{y}), & s \geq t, \\ K_{N, \alpha, \xi(\cdot)}^{\text{Meixner}, -}(s, \tilde{x}; t, \tilde{y}), & s < t, \end{cases}$$

with $\tilde{x}, \tilde{y} \in \mathbb{Z}_+$ and

$$K_{N, \alpha, \xi(\cdot)}^{\text{Meixner}, +}(s, \tilde{x}; t, \tilde{y}) = \sum_{m=0}^{N-1} e^{m(s-t)} \tilde{\mathfrak{M}}_m(\tilde{x}; \alpha + 1, \xi(s)) \tilde{\mathfrak{M}}_m(\tilde{y}; \alpha + 1, \xi(t)), \tag{6.7}$$

$$K_{N, \alpha, \xi(\cdot)}^{\text{Meixner}, -}(s, \tilde{x}; t, \tilde{y}) = - \sum_{m=N}^{\infty} e^{m(s-t)} \tilde{\mathfrak{M}}_m(\tilde{x}; \alpha + 1, \xi(s)) \tilde{\mathfrak{M}}_m(\tilde{y}; \alpha + 1, \xi(t)). \tag{6.8}$$

Lemma 6.4. *The process $\tilde{X}_{N, N+\alpha, \xi(\cdot)}$ is determinantal. Its correlation functions have the form*

$$\tilde{\rho}_l(\tau_1, \tilde{x}_1; \dots; \tau_l, \tilde{x}_l) = \det \left[K_{N, \alpha, \xi(\cdot)}^{\text{Meixner}}(\tau_i, \tilde{x}_i; \tau_j, \tilde{x}_j) \right]_{i, j=1}^l, \quad l = 1, 2, \dots$$

We postpone the proof of this lemma till §6.4. Note that in the particular case $s = t$ the extended Meixner kernel turns into the conventional Meixner kernel associated with the Meixner ensemble,

$$K_{N, \alpha, \xi}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \sum_{m=0}^{N-1} \tilde{\mathfrak{M}}_m(\tilde{x}; \alpha + 1, \xi) \tilde{\mathfrak{M}}_m(\tilde{y}; \alpha + 1, \xi),$$

⁶ The Meixner ensemble was also exploited in Johansson's papers [Jo1], [Jo2].

and the statement of the lemma reduces to a well-known general fact about orthogonal polynomial ensembles.

The next step is to connect the extended Meixner kernel K^{Meixner} and the kernel \underline{K} of Theorem 6.2.

Lemma 6.5. *We have*

$$K_{N,\alpha,\xi(\cdot)}^{\text{Meixner}}(s, \tilde{x}; t, \tilde{y}) = e^{(N-\frac{1}{2})(s-t)} \underline{K}_{N,N+\alpha,\xi(\cdot)}(s, x; t, y) \tag{6.9}$$

with $\tilde{x} = x + N - \frac{1}{2} \in \mathbb{Z}_+$, $\tilde{y} = y + N - \frac{1}{2} \in \mathbb{Z}_+$.

Proof. For $s \geq t$ take (6.7) and change the summation index: $m = N - a - \frac{1}{2}$. Then, see the end of §5,

$$e^{m(s-t)} = e^{(N-\frac{1}{2})(s-t)} \cdot e^{a(t-s)}, \quad \tilde{\mathfrak{M}}_m(\tilde{x}; \alpha + 1, \xi) = \psi_a(x; N, N + \alpha, \xi)$$

with $\tilde{x} = x + N - \frac{1}{2}$. Furthermore, $\psi_a(x; N, N + \alpha, \xi) \equiv 0$ if $a = N + \frac{1}{2}, N + \frac{3}{2}, \dots$ because of the factor $\Gamma(z - a + \frac{1}{2})$ in (5.1). This yields (6.9). For $s < t$ the argument is similar; it uses (6.8), the summation index change $m = N - a - \frac{1}{2}$ and

$$e^{m(s-t)} = e^{(N-\frac{1}{2})(s-t)} \cdot e^{-a(s-t)}, \quad \tilde{\mathfrak{M}}_m(\tilde{x}; \alpha + 1, \xi) = \psi_{-a}(x; N, N + \alpha, \xi). \tag{6.10}$$

□

Lemma 6.4 and Lemma 6.5 imply that the correlation functions ρ_l of $X_{N,N+\alpha,\xi(\cdot)}$ can be written as

$$\rho_l(\tau_1, x_1; \dots; \tau_l, x_l) = \det \left[\underline{K}_{N,N+\alpha,\xi(\cdot)}(\tau_i, x_i; \tau_j, x_j) \right]_{i,j=1}^l, \quad l = 1, 2, \dots,$$

provided that $x_i \geq -N + \frac{1}{2}$ for all $i = 1, \dots, l$ (indeed, the factor $e^{(N-\frac{1}{2})(s-t)}$ in (6.9) does not affect the correlation functions).

Lemma 6.6. *Assume that*

- either (z, z') is not in the degenerate series and $x, y \in \mathbb{Z}'$ are arbitrary
- or $z = N = 1, 2, \dots, z' > N - 1$, and both x, y are in $\mathbb{Z}_+ - N + \frac{1}{2}$.

Then the kernel $\widehat{K}_{z,z',\xi(\cdot)}(s, x; t, y)$ of Theorem 6.3 is related to the kernel $\underline{K}_{z,z',\xi(\cdot)}(s, x; t, y)$ of Theorem 6.2 by equality (6.3). Or, equivalently, by the “gauge transformation”

$$\widehat{K}_{z,z',\xi(\cdot)}(s, x; t, y) = \frac{f_{z,z'}(s, x)}{f_{z,z'}(t, y)} \underline{K}_{z,z',\xi(\cdot)}(s, x; t, y),$$

where $f_{z,z'}$ is defined in (6.6).

Proof. Applying formula (5.3) to ψ_a and setting $a = k + \frac{1}{2}$ we obtain

$$\begin{aligned} &\psi_{k+\frac{1}{2}}(x; z, z', \xi(s)) \\ &= \left(\frac{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})}{\Gamma(z-k)\Gamma(z'-k)} \right)^{\frac{1}{2}} \frac{\Gamma(z'-k)(1-\xi(s))^{\frac{z'-z+1}{2}}}{\Gamma(x+z'+\frac{1}{2})} \\ &\quad \times \oint \left(1-\sqrt{\xi(s)}\omega_1\right)^{-z'+k} \left(1-\sqrt{\xi(s)}\omega_1^{-1}\right)^{z-k-1} \omega_1^{-x-k-\frac{1}{2}} \frac{d\omega_1}{\omega_1}, \\ &\psi_{k+\frac{1}{2}}(y; z, z', \xi(t)) \\ &= \left(\frac{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})}{\Gamma(z-k)\Gamma(z'-k)} \right)^{\frac{1}{2}} \frac{\Gamma(z-k)(1-\xi(t))^{\frac{z-z'+1}{2}}}{\Gamma(x+z+\frac{1}{2})} \\ &\quad \times \oint \left(1-\sqrt{\xi(t)}\omega_2\right)^{-z+k} \left(1-\sqrt{\xi(t)}\omega_2^{-1}\right)^{z'-k-1} \omega_2^{-y-k-\frac{1}{2}} \frac{d\omega_2}{\omega_2}. \end{aligned}$$

(Here we used the symmetry $\psi_a(x; z, z', \xi) = \psi_a(x; z', z, \xi)$ to get the second relation.)

Note that both sides of (6.3) are real-analytic functions of $q = e^{s-t}$ on the ray $q > 1$ and on the interval $0 < q < 1$ with all other parameters $(z, z', \xi(s), \xi(t), x, y)$ being fixed. Thus, it suffices to prove (6.3) for q large enough in the case of $q \geq 1$, and for q small enough in the case $q < 1$.

Let us consider the case $q \geq 1$. Substituting the integral representations above in (6.2), we observe that the summation index $k = a - \frac{1}{2} \in \mathbb{Z}_+$ enters the resulting expression only as the exponent in

$$\left(\frac{(1-\sqrt{\xi(s)}\omega_1)(1-\sqrt{\xi(t)}\omega_2)}{q(\omega_1-\sqrt{\xi(s)})(\omega_2-\sqrt{\xi(t)})} \right)^k.$$

For large enough q the geometric progression $\sum_{k \geq 0}$ with this ratio converges uniformly on any fixed contours $\{\omega_1\}, \{\omega_2\}$. Computing the sum yields (6.3), (6.4).

Similarly, in the case $q < 1$ the computation reduces to summing the geometric progression with the ratio

$$\frac{q(\omega_1-\sqrt{\xi(s)})(\omega_2-\sqrt{\xi(t)})}{(1-\sqrt{\xi(s)}\omega_1)(1-\sqrt{\xi(t)}\omega_2)}$$

which always makes sense for small enough q .

For large q the image of any finite contour under (6.5) is concentrated near $\eta = \sqrt{\xi(t)}$, and for small q such image is concentrated near η^{-1} . These points are always inside/outside of any contour $\{\omega_2\}$ used in the integral representation of §5. The conditions on contours in the statement of Theorem 6.3 reflect the deformations of the contours of §5. Note that the denominator of (6.4) must stay away from zero while deforming the contours, which means that the image of $\{\omega_1\}$ under (6.5) is not allowed to intersect ω_2 . □

The relations (6.10) and (6.3) yield the following formula for the correlation functions of $\underline{X}_{N, N+\alpha, \xi(\cdot)}$:

$$\rho_l(\tau_1, x_1; \dots; \tau_l, x_l) = \det \left[\widehat{K}_{N, N+\alpha, \xi(\cdot)}(\tau_i, x_i; \tau_j, x_j) \right]_{i, j=1}^l, \quad l = 1, 2, \dots, \tag{6.11}$$

where $x_i + N - \frac{1}{2} \in \mathbb{Z}_+$ for all $i = 1, \dots, l$. In order to prove Theorem 6.3, we need to extend this formula to arbitrary admissible (z, z') instead of $(N, N + \alpha)$, and to arbitrary $x_i \in \mathbb{Z}'$. We will do that by means of Proposition 4.2.

Up till now the time moments τ_1, \dots, τ_l were arbitrary, they were not ordered and some of them were allowed to coincide. Let us denote by $t_1, \dots, t_n, n \leq l$, the same numbers but ordered and without repetitions ($t_1 < \dots < t_n$). Thus, each τ_j is equal to one and only one t_i . As in §4, we set

$$\xi_i = \xi(t_i), \quad 1 \leq i \leq n; \quad \eta_{i, i+1} = e^{t_i - t_{i+1}} \sqrt{\xi_i \xi_{i+1}}, \quad 1 \leq i \leq n - 1.$$

Then in the notation of §4, see (4.2) and below, $\rho_l(\tau_1, x_1; \dots; \tau_l, x_l)$ is equal to $F(\xi, \eta; \mathcal{D}_1, \dots, \mathcal{D}_n)$ with a suitable choice of the sets $\mathcal{D}_1, \dots, \mathcal{D}_n$. Namely, the set \mathcal{D}_i is determined according to the following recipe: take all numbers $j \in \{1, \dots, l\}$ such that $\tau_j = t_i$, then the corresponding points $x_j \in \mathbb{Z}'$ must be pairwise distinct. Then

$$\mathcal{D}_i = \{ \lambda \in \mathbb{Y} \mid \underline{X}(\lambda) \text{ contains all } x_j \text{ such that } \tau_j = t_i \}.$$

Proposition 4.2 says that $\rho_l(\tau_1, x_1; \dots; \tau_l, x_l)$ is a real analytic function in ξ, η , and after the substitution $(\xi, \eta) \mapsto (\varepsilon \xi, \varepsilon \eta)$ the corresponding function in ε can be analytically continued in a neighborhood of $\varepsilon = 0$. Moreover, its Taylor coefficients at this point are polynomials in z and z' .

Now let us look at the right-hand side of (6.11) with $N, N + \alpha$ replaced by z, z' . The values $\widehat{K}_{z, z', \xi(\cdot)}(\tau_i, x_i; \tau_j, x_j)$ of the kernel are given by (6.4). This formula involves $\sqrt{\xi(\tau_i)}, \sqrt{\xi(\tau_j)}$, and $e^{\tau_i - \tau_j}$, which is expressible through $\sqrt{\xi_i}$'s and $\eta_{i, i+1}$'s. Moreover, $e^{\tau_i - \tau_j}$ do not change if we scale ξ and η by ε . Indeed, this follows from the fact that

$$e^{t_{i+1} - t_i} = \frac{\sqrt{\xi_i \xi_{i+1}}}{\eta_{i, i+1}}, \quad i = 1, \dots, n - 1.$$

The integral representation (6.4) implies that $\widehat{K}_{z, z', \xi(\cdot)}(\tau_i, x_i; \tau_j, x_j)$ are real analytic functions in $\sqrt{\xi_i}$'s and $\eta_{i, i+1}$'s (the η variables enters only the factors $e^{t_i - t_j}$). Further, if we scale ξ and η by ε , then (6.4) viewed as a function in $\delta = \varepsilon^{\frac{1}{2}}$, extends to an analytic function in a small enough disc $\{ \delta : |\delta| < \text{const} \}$.⁷ Moreover, its Taylor coefficients at $\delta = 0$ are polynomials in z and z' because the Taylor coefficients of $(1 - u)^\kappa$ at $u = 0$ are polynomials in κ .

⁷ It is worth noting that for small ε we can choose the contours of integration in (6.4) which would be independent of ε ; it suffices to consider suitable circles centered at the origin.

We conclude that both sides of (6.11) are uniquely determined by their values for $(z, z') = (N, N + \alpha)$ with any large enough natural N and any $\alpha > -1$. This completes the proof of Theorem 6.3.

Theorem 6.2 is a direct corollary of Theorem 6.3 and Lemma 6.6. Theorem 6.1 follows from Theorem 6.3. \square

6.4. Eynard–Mehta theorem and the proof of Lemma 6.4

Here we state the Eynard–Mehta theorem [EM] in a form which is convenient for our purposes and then we show that Lemma 6.4 is a corollary of this theorem.

Let m be a fixed natural number and let the index k range over $\{1, \dots, m\}$. Consider the Hilbert space $\ell^2(\mathbb{Z}_+)$ taken with respect to the counting measure on the set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Assume that for each k we are given an orthonormal basis $\{\phi_{k,n}\}_{n=0,1,\dots}$ of real-valued functions in $\ell^2(\mathbb{Z}_+)$. Next, assume that for each $k = 1, \dots, m - 1$ and each $n = 0, 1, \dots$ we are given a number $c_{k,k+1;n} > 0$. As $n \rightarrow \infty$, these numbers have to decay fast enough to make convergent certain infinite sums specified below. Finally, we will impose on these data certain positivity conditions, see below.

We aim to construct a probability measure on collections (X_1, \dots, X_m) , where each X_k is an arbitrary N -point subset in \mathbb{Z}_+ and N is a fixed natural number. This measure can be regarded as a Markov process with “time” $k = 1, \dots, m$, the state space being the set of N -point subsets in \mathbb{Z}_+ .

The construction goes as follows. For an arbitrary N -point subset $X = (x_1 < \dots < x_N) \subset \mathbb{Z}_+$ we introduce the $N \times N$ matrix $\phi_k(X)$ with the entries $\phi_{k,i}(x_j)$, where the row index i takes values in $\{0, \dots, N - 1\}$, and the column index j takes values in $\{1, \dots, N\}$.

As X ranges over all N -point subsets of \mathbb{Z}_+ , one has

$$\sum_X (\det \phi_k(X))^2 = 1.$$

The proof follows from the Cauchy–Binet identity and orthonormality of $\phi_{k,n}$ ’s, cf. (3.8).

Thus, for each $k = 1, \dots, m$ we have a probability measure σ_k on N -point subsets in \mathbb{Z}_+ which assigns to a subset X its weight $(\det \phi_k(X))^2$. The measures σ_k are the 1-dimensional distributions for our future Markov process.

For each $k = 1, \dots, m - 1$ we set

$$v_{k,k+1}(x, y) = \sum_{n=0}^{\infty} c_{k,k+1;n} \phi_{k,n}(x) \phi_{k+1,n}(y), \quad x, y \in \mathbb{Z}_+,$$

where the sum is assumed to be convergent. Since $\{\phi_{k,n}\}$ is an orthonormal basis for each fixed k , we have

$$\begin{aligned} \sum_{x \in \mathbb{Z}_+} \phi_{k,n}(x) v_{k,k+1}(x, y) &= c_{k,k+1;n} \phi_{k+1,n}(y), \\ \sum_{y \in \mathbb{Z}_+} v_{k,k+1}(x, y) \phi_{k+1,n}(y) &= c_{k,k+1;n} \phi_{k,n}(x). \end{aligned} \tag{6.12}$$

For arbitrary subsets $X = (x_1 < \dots < x_N)$ and $Y = (y_1 < \dots < y_N)$ we form an $N \times N$ matrix $v_{k,k+1}(X, Y)$ with entries $v_{k,k+1}(x_i, y_j)$, and we set

$$\sigma_{k,k+1}(X, Y) = \frac{\det \phi_k(X) \det v_{k,k+1}(X, Y) \det \phi_{k+1}(Y)}{\prod_{n=0}^{N-1} c_{k,k+1;n}}$$

Once again, using the Cauchy-Binet identity and (6.12), it is not hard to show that for any $k = 1, \dots, m - 1$ one has

$$\sum_Y \sigma_{k,k+1}(X, Y) = \sigma_k(X), \quad \sum_X \sigma_{k,k+1}(X, Y) = \sigma_{k+1}(Y).$$

Assume that $\sigma_{k,k+1}(X, Y) \geq 0$ for all X and Y . Then we may regard $\sigma_{k,k+1}$ as a probability measure on couples (X, Y) with marginal measures σ_k and σ_{k+1} . This is the “one step” 2-dimensional distribution of our Markov process. We also assume that $\det \phi_k(X)$ does not vanish. Then we define the “one step” transition probability function as follows

$$\mathbf{P}_{X,Y}(k, k + 1) = \frac{\sigma_{k,k+1}(X, Y)}{\sigma_k(X)} = \frac{\det v_{k,k+1}(X, Y) \det \phi_{k+1}(Y)}{\det \phi_k(X) \prod_{n=0}^{N-1} c_{k,k+1;n}}. \quad (6.13)$$

We regard this as a matrix $\mathbf{P}(k, k + 1)$ whose rows and columns are labelled by N -point subsets.

Finally, we define a Markov process $X(k)$, where the “time” k takes values from 1 to m and $X(k)$ is an N -point subset of \mathbb{Z}_+ , using the initial distribution σ_1 and the “one step” transition probabilities (6.13):

$$\text{Prob}(X(1), \dots, X(m)) = \sigma_1(X_1) \mathbf{P}_{X(1),X(2)}(1, 2) \dots \mathbf{P}_{X(m-1),X(m)}(m - 1, m).$$

For arbitrary indices k, l such that $1 \leq k < l \leq m$ we set

$$c_{k,l;n} = c_{k,k+1;n} c_{k+1,k+2;n} \dots c_{l-1,l;n}, \quad n = 0, 1, \dots,$$

$$v_{k,l}(x, y) = \sum_{n=0}^{\infty} c_{k,l;n} \phi_{k,n}(x) \phi_{l,n}(y), \quad x, y \in \mathbb{Z}_+.$$

Theorem 6.7 (Eynard–Mehta [EM]). *Under the above assumptions, let us regard the Markov process $X(k)$ as a probability measure on mN -point configurations $X = (X(1), \dots, X(m))$ in the space $\{1, \dots, m\} \times \mathbb{Z}_+$. Then this measure is determinantal, and its correlation kernel has the form*

$$K(k, x; l, y) = \sum_{i=0}^{N-1} \frac{1}{c_{l,k;i}} \phi_{k,i}(x) \phi_{l,i}(y), \quad k \geq l,$$

(where we agree that $c_{k,k;i} = 1$) and

$$\begin{aligned}
 K(k, x; l, y) &= \sum_{i=0}^{N-1} c_{k,l;i} \phi_{k,i}(x) \phi_{l,i}(y) - v_{k,l}(x, y) \\
 &= - \sum_{i=N}^{\infty} c_{k,l;i} \phi_{k,i}(x) \phi_{l,i}(y), \quad k < l.
 \end{aligned}$$

In other words, for any $n = 1, 2, \dots$ we have

$$\begin{aligned}
 \rho_n(k_1, x_1; \dots; k_n, x_n) &:= \text{Prob}\{X(k_i) \ni x_i \text{ for each } i = 1, \dots, n\} \\
 &= \det [K(k_i, x_i; k_j, x_j)]_{i,j=1}^n,
 \end{aligned}$$

where $(k_i, x_i) \in \{1, \dots, m\} \times \mathbb{Z}$ and $(k_i, x_i) \neq (k_j, x_j)$ for $i \neq j$.

Proof. See [EM], [NF], [Jo4], [TW], [BR]. □

Proof of Lemma 6.4. The process $\Lambda_{N,N+\alpha,\xi(\cdot)}$ restricted to any finite sequence of time moments $t_1 < \dots < t_m$ fits into this formalism perfectly. Indeed, we take

$$\phi_{k,n}(x) = \tilde{\mathfrak{M}}_n(x; \alpha + 1, \xi(t_k)), \quad c_{k,k+1;n} = e^{n(t_k - t_{k+1})}.$$

Recall the connection between measure $M_{N,N+\alpha,\xi}$ and the N -point Meixner orthogonal ensemble (see the beginning of §6.3). It readily follows that (under the correspondence $\lambda \mapsto \tilde{X}(\lambda)$) $M_{N,N+\alpha,\xi(t_k)}$ is exactly σ_k , and the transition matrix (3.2) coincides with (6.13). Lemma 6.4 is thus a direct corollary of Theorem 6.7. □

6.5. An interpretation via nonintersecting paths

In this section we are dealing with the process $\Lambda_{N,N+\alpha,\xi(\cdot)}$ with $\xi(\cdot) \equiv \xi$, where $\xi \in (0, 1)$ is a fixed number. We denote this stationary process as $\Lambda_{N,N+\alpha,\xi}$, and we also write ξ instead of $\xi(\cdot)$ for some related processes. Our aim is to interpret $\Lambda_{N,N+\alpha,\xi}$ in terms of N nonintersecting trajectories of independent birth–death processes. This is done using formulas of Karlin–McGregor [KMG3] and an idea of Johansson [Jo3] (see also [Bi1], [Bi2], [KOR], [O’C]).

Instead of dealing with $\Lambda_{N,N+\alpha,\xi}$ we will use the associated process $\tilde{X}_{N,N+\alpha,\xi}$ introduced in the beginning of §6.3. Recall that its state space consists of N -point configurations in \mathbb{Z}_+ .

In the special case $N = 1$ the process $\tilde{X}_{1,1+\alpha,\xi}$ is just the birth–death process $N_{1+\alpha,\xi}$. We aim to construct $\tilde{X}_{N,N+\alpha,\xi}$ directly in terms of $N_{1+\alpha,\xi}$.

Let us take a large $T > 0$ and consider a new process $Y_{N,\alpha,\xi,T}$ introduced as follows. This process is defined on the time interval $[-T, T]$. Let us take N independent birth–death processes which start at the moment $-T$ at points $a_1 < \dots < a_N$ and end up at the moment T at points $b_1 < \dots < b_N$ conditioned on the event that the trajectories $x_i(t)$ do not intersect on $[-T, T]$:

$$x_1(t) < x_2(t) < \dots < x_N(t), \quad -T \leq t \leq T.$$

The boundary conditions $\{a_i\}$ and $\{b_i\}$ are arbitrary but fixed while the parameter T will vary.

Theorem 6.8. *In the above notation, as $T \rightarrow \infty$ the processes $Y_{N,\alpha,\xi,T}$ converge to $\tilde{X}_{N,N+\alpha,\xi}$ in the sense of convergence of the finite dimensional distributions.*

Proof. Consider the orthonormal basis

$$\phi_n(x) = \tilde{\mathfrak{M}}_n(x; \alpha + 1, \xi), \quad n = 0, 1, \dots, \quad x \in \mathbb{Z}_+$$

in $\ell^2(\mathbb{Z}_+)$. According to (3.1) set

$$v_{s,t}(x, y) = \sum_{n=0}^{\infty} e^{(s-t)n} \phi_n(x) \phi_n(y), \quad s \leq t, \quad x, y \in \mathbb{Z}_+.$$

Let us fix arbitrary time moments $t_1 < \dots < t_k$ inside $(-T, T)$. Then by [KMG3] and Theorem 3.1, the corresponding finite-dimensional distribution of $Y_{N,\alpha,\xi,T}$ has the form

$$\begin{aligned} & \text{Prob}\{Y_{N,\alpha,\xi,T}(t_i) = (y_1^{(i)} < \dots < y_N^{(i)}) \text{ for all } i = 1, \dots, k\} \\ & \det[v_{-T,t_1}(a_i, y_j^{(1)})] \cdot \prod_{r=1}^{k-1} \det[v_{t_r,t_{r+1}}(y_i^{(r)}, y_j^{(r+1)})] \cdot \det[v_{t_k,T}(y_i^{(k)}, b_j)] \\ & = \frac{\det[v_{-T,t_1}(a_i, y_j^{(1)})] \cdot \prod_{r=1}^{k-1} \det[v_{t_r,t_{r+1}}(y_i^{(r)}, y_j^{(r+1)})] \cdot \det[v_{t_k,T}(y_i^{(k)}, b_j)]}{\det[v_{-T,T}(a_i, b_j)]}. \end{aligned} \tag{6.14}$$

On the other hand, the finite-dimensional distribution of $\tilde{X}_{N,N+\alpha,\xi}$ is given by

$$\begin{aligned} & \text{Prob}\{\tilde{X}_{N,N+\alpha,\xi}(t_i) = (y_1^{(i)} < \dots < y_N^{(i)}) \text{ for all } i = 1, \dots, k\} \\ & = e^{\frac{(t_k-t_1)N(N-1)}{2}} \det[\phi_{i-1}(y_j^{(1)})] \det[\phi_{i-1}(y_j^{(k)})] \prod_{r=1}^{k-1} \det[v_{t_r,t_{r+1}}(y_i^{(r)}, y_j^{(r+1)})]. \end{aligned} \tag{6.15}$$

Indeed, by Theorem 3.1,

$$\begin{aligned} & \text{Prob}\{\tilde{X}_{N,N+\alpha,\xi}(t_i) = (y_1^{(i)} < \dots < y_N^{(i)}) \text{ for all } i = 1, \dots, k\} \\ & = e^{\frac{(t_k-t_1)N(N-1)}{2}} (M_{N,N+\alpha,\xi}(\lambda(1)) M_{N,N+\alpha,\xi}(\lambda(k)))^{1/2} \\ & \quad \times \prod_{r=1}^{k-1} \det[v_{t_r,t_{r+1}}(y_i^{(r)}, y_j^{(r+1)})], \end{aligned}$$

where $\lambda(1)$ and $\lambda(k)$ are the Young diagrams such that

$$\tilde{X}(\lambda(1)) = (y_1^{(1)}, \dots, y_N^{(1)}), \quad \tilde{X}(\lambda(k)) = (y_1^{(k)}, \dots, y_N^{(k)}).$$

Using the fact that the measure $M_{N,N+\alpha,\xi}$ corresponds to the Meixner ensemble (see the beginning of §6.3) one can readily see that

$$(M_{N,N+\alpha,\xi}(\lambda(1)) M_{N,N+\alpha,\xi}(\lambda(k)))^{1/2} = \det[\phi_{i-1}(y_j^{(1)})] \det[\phi_{i-1}(y_j^{(k)})],$$

which proves (6.15).

Comparing (6.14) and (6.15) we see that it remains to prove that

$$\begin{aligned} \lim_{T \rightarrow +\infty} \frac{\det[v_{-T,t_1}(a_i, y_j^{(1)})] \det[v_{t_k,T}(y_i^{(k)}, b_j)]}{\det[v_{-T,T}(a_i, b_j)]} \\ = e^{\frac{(t_k-t_1)N(N-1)}{2}} \det[\phi_{i-1}(y_j^{(1)})] \det[\phi_{i-1}(y_j^{(k)})]. \end{aligned} \tag{6.16}$$

Note that we have the following asymptotic relation: for arbitrary $x'_1, x''_1, \dots, x'_N, x''_N \in \mathbb{Z}_+$

$$\begin{aligned} \det \left[\sum_{n=0}^{\infty} \epsilon^n \phi_n(x'_i) \phi_n(x''_j) \right] = \epsilon^{\frac{N(N-1)}{2}} \det[\phi_{i-1}(x'_j)] \det[\phi_{i-1}(x''_j)] \\ + O \left(\epsilon^{\frac{N(N-1)}{2} + 1} \right) \end{aligned}$$

as $\epsilon \rightarrow 0$, cf. (3.4). Applying this asymptotic relation to $v_{-T,t_1}, v_{t_k,T}, v_{-T,T}$, we obtain

$$\begin{aligned} \det[v_{-T,t_1}(a_i, y_j^{(1)})] &\sim e^{-\frac{(t_1+T)N(N-1)}{2}} \det[\phi_{i-1}(a_j)] \det[\phi_{i-1}(y_j^{(1)})], \\ \det[v_{t_1,T}(y_i^{(k)}, b_j)] &\sim e^{-\frac{(T-t_k)N(N-1)}{2}} \det[\phi_{i-1}(y_j^{(k)})] \det[\phi_{i-1}(b_j)], \\ \det[v_{-T,T}(a_i, b_j)] &\sim e^{-TN(N-1)} \det[\phi_{i-1}(a_j)] \det[\phi_{i-1}(b_j)], \end{aligned}$$

as $T \rightarrow +\infty$. This implies (6.16) and completes the proof. □

6.6. Remark on the Charlier limit

Keeping N fixed, let α go to ∞ and ξ go to 0 in such a way that $\alpha\xi$ tends to a finite value $\theta > 0$. Then the measure $M_{N,N+\alpha,\xi}$ on $\mathbb{Y}(N) \subset \mathbb{Y}$ converges to a probability measure with parameters N and θ . Under the embedding $\lambda \mapsto \tilde{X}(\lambda)$ the latter measure turns into a discrete orthogonal polynomial ensemble on \mathbb{Z}_+ — the N -point *Charlier ensemble* associated with the Charlier polynomials (cf. [BO4, §6]). The Charlier ensemble also possesses a dynamical version $\tilde{X}_{N,\theta(\cdot)}$, which is a limit of the process $\tilde{X}_{N,N+\alpha,\xi(\cdot)}$. Everything said in §6.3–§6.5 about the process $\tilde{X}_{N,N+\alpha,\xi(\cdot)}$ has a counterpart for the process $\tilde{X}_{N,\theta(\cdot)}$. In the pure birth case ($\theta(t) = e^{2t+\text{const}}$), the process $\tilde{X}_{N,\theta(\cdot)}$ and its discrete time analog (which may be viewed as a random walk on $\mathbb{Y}(N)$) have been previously studied in [Bi1], [Bi2], [KOR], [O’C].

7. Particle–hole involution

For any set \mathfrak{X} and its subset \mathfrak{Y} one can define an involution on point configurations $X \subset \mathfrak{X}$ by $X \mapsto X \Delta \mathfrak{Y}$ (here Δ is the symbol of symmetric difference). This map leaves intact the “particles” of X outside of \mathfrak{Y} , and inside \mathfrak{Y} it picks the “holes” (points of \mathfrak{Y} free of particles). This involution is called the *particle-hole involution* on \mathfrak{Y} .

The goal of this section is to give a different description of the z-measures using a new identification of Young diagrams and point configurations on \mathbb{Z}' . Instead of using the configurations

$$\underline{X}(\lambda) = \{\lambda_i - i + \frac{1}{2} \mid i = 1, 2, \dots\}$$

we will use the configurations

$$X(\lambda) = (\underline{X}(\lambda) \cap \mathbb{Z}'_+) \cup (\mathbb{Z}'_- \setminus \underline{X}(\lambda)) = \underline{X}(\lambda) \Delta \mathbb{Z}'_-$$

which are obtained from $\underline{X}(\lambda)$ by applying the particle-hole involution on \mathbb{Z}'_- .

The parametrization of Young diagrams λ by configurations $X(\lambda)$ corresponds to considering the *Frobenius coordinates* of λ , see [BOO, §1.2] for details. The reason for passing to $X(\lambda)$ is very simple: in the continuous limit $\xi \nearrow 1$ which will be considered below in §8, the point process generated by $\underline{X}(\lambda)$ does not survive, while the process corresponding to $X(\lambda)$ has a well defined limit. This fact has a representation theoretic explanation, see e.g. [OI2].

Observe that $X(\lambda') = -X(\lambda)$ for any $\lambda \in \mathbb{Y}$.

Given an arbitrary kernel $K(x, y)$ on $\mathfrak{X} \times \mathfrak{X}$, and a subset \mathfrak{Y} of \mathfrak{X} , we assign to it another kernel,

$$K^\circ(x, y) = \begin{cases} K(x, y), & x \notin \mathfrak{Y}, \\ \delta_{xy} - K(x, y), & x \in \mathfrak{Y}, \end{cases}$$

where δ_{xy} is the Kronecker symbol. Slightly more generally, given an arbitrary map $\varepsilon : \mathfrak{X} \rightarrow \mathbb{R}^*$, we set

$$K^{\circ, \varepsilon}(x, y) = \varepsilon(x)K^\circ(x, y)\varepsilon(y)^{-1}.$$

Proposition 7.1. *Let P be a probability measure in the space of point configurations on a discrete space \mathfrak{X} and let P° be the image of P under the particle-hole involution on $\mathfrak{Y} \subset \mathfrak{X}$. Assume that the correlation functions of P have determinantal form with a certain kernel $K(x, y)$,*

$$\rho_m(x_1, \dots, x_m \mid P) = \det_{1 \leq i, j \leq m} [K(x_i, x_j)], \quad m = 1, 2, \dots$$

Then the correlation functions of the measure P° also have a similar determinantal form, with the kernel $K^\circ(x, y)$ as defined above or, equally well, with the kernel $K^{\circ, \varepsilon}(x, y)$, where the map $\varepsilon : \mathfrak{X} \rightarrow \mathbb{R}^$ may be chosen arbitrarily,*

$$\rho_m(x_1, \dots, x_m \mid P^\circ) = \det_{1 \leq i, j \leq m} [K^\circ(x_i, x_j)] = \det_{1 \leq i, j \leq m} [K^{\circ, \varepsilon}(x_i, x_j)],$$

$$m = 1, 2, \dots$$

Proof. The factor $\varepsilon(\cdot)$ does not affect the values of determinants in the right-hand side of the above formula, so that we may take $\varepsilon(\cdot) \equiv 1$. Then the result is obtained by applying the inclusion/exclusion principle, see Proposition A.8 in [BOO]. \square

Later on we choose the function $\varepsilon(\cdot)$ in a specific way (see (7.3) below) which is appropriate for the limit transition of §8.

The main result of this section is a determinantal formula for the dynamical correlation functions of $\Lambda_{z,z',\xi}$ computed in terms of $X(\lambda)$. For any $n = 1, 2, \dots$ define the *n*th dynamical correlation function of *n* pairwise distinct arguments $(t_1, x_1), \dots, (t_n, x_n) \in (t_{\min}, t_{\max}) \times \mathbb{Y}$ by

$$\begin{aligned} \rho_n(t_1, x_1; t_2, x_2; \dots; t_n, x_n) \\ = \text{Prob} \{X(\lambda) \text{ at the moment } t_i \text{ contains } x_i \text{ for all } i = 1, \dots, n\}. \end{aligned}$$

Here and in what follows we denote by $\Lambda_{z,z',\xi}$ the stationary Markov process corresponding to the constant curve $\xi(t) \equiv \xi$, where $\xi \in (0, 1)$ is fixed.

Theorem 7.2. *Let (z, z') be a pair of admissible parameters not from the degenerate series. Consider the Markov process $\Lambda_{z,z',\xi}$, and denote by $X_{z,z',\xi}$ the process with values in the space of point configurations in \mathbb{Z}' , obtained from $\Lambda_{z,z',\xi}$ via $\lambda \mapsto X(\lambda)$.*

Then the process $X_{z,z',\xi}$ is determinantal and its correlation kernel can be written as follows (in all the formulas below x and y are positive, in symbols \pm and \mp the upper sign corresponds to the case $s \geq t$, and the lower sign corresponds to $s < t$)

$$\begin{aligned} K_{z,z',\xi}(s, x; t, y) &= \pm \sum_{a \in \mathbb{Z}'_+} e^{-a|s-t|} \psi_{\pm a}(x; z, z', \xi) \psi_{\pm a}(y; z, z', \xi), \\ K_{z,z',\xi}(s, x; t, -y) &= \pm \sum_{a \in \mathbb{Z}'_+} (-1)^{\pm a - \frac{1}{2}} e^{-a|s-t|} \\ &\quad \times \psi_{\pm a}(x; z, z', \xi) \psi_{\mp a}(y; -z, -z', \xi), \\ K_{z,z',\xi}(s, -x; t, y) &= \mp \sum_{a \in \mathbb{Z}'_+} (-1)^{\pm a - \frac{1}{2}} e^{-a|s-t|} \\ &\quad \times \psi_{\mp a}(x; -z, -z', \xi) \psi_{\pm a}(y; z, z', \xi), \\ K_{z,z',\xi}(s, -x; t, -y) &= \mp \sum_{a \in \mathbb{Z}'_+} e^{-a|s-t|} \\ &\quad \times \psi_{\mp a}(x; -z, -z', \xi) \psi_{\mp a}(y; -z, -z', \xi), \end{aligned} \tag{7.1}$$

where the fourth formula is valid for $s \neq t$, and for $s = t$ we have

$$K_{z,z',\xi}(s, -x; s, -y) = \sum_{a \in \mathbb{Z}'_+} \psi_a(x; -z, -z', \xi) \psi_a(y; -z, -z', \xi). \tag{7.2}$$

Comments. 1. For $s = t$ this kernel coincides with the hypergeometric kernel derived in [BO2] (see also [BO4], [B2], [BO5]). In those papers the kernel was written in another, so-called “integrable form”.

2. The kernel $K_{z,z',\xi}$ has the following symmetries ($x, y \in \mathbb{Z}'$):

$$K_{z,z',\xi}(s, x; t, y) = (-1)^{\text{sgn } x \cdot \text{sgn } y} K_{z,z',\xi}(s, y; t, x),$$

$$K_{z,z',\xi}(s, x; t, y) = \begin{cases} K_{-z,-z',\xi}(t, -x; s, -y), & s \neq t, \\ (-1)^{\text{sgn } x \cdot \text{sgn } y} K_{-z,-z',\xi}(t, -x; s, -y), & s = t. \end{cases}$$

Proof. We use Proposition 7.1. As the initial kernel we take the expression for $\underline{K}_{z,z',\xi(\cdot)}$ given in Theorem 6.2, the set \mathfrak{X} is the union of finitely many copies of \mathbb{Z}' which correspond to times at which we evaluate the dynamical correlation function, and \mathfrak{Y} is the union of the same number of copies of \mathbb{Z}'_- . On each copy of \mathbb{Z}' the function $\varepsilon(\cdot)$ is chosen in the following way:

$$\varepsilon(x) = \begin{cases} 1, & x > 0, \\ (-1)^{-x-\frac{1}{2}}, & x < 0. \end{cases} \tag{7.3}$$

The statement then follows from (5.7). The last formula (for $s = t$) arises from the relation

$$\begin{aligned} \delta_{x,y} - \sum_{a \in \mathbb{Z}'_+} \psi_{-a}(x; -z, -z', \xi) \psi_{-a}(y; -z, -z', \xi) \\ = \sum_{a \in \mathbb{Z}'_+} \psi_a(x; -z, -z', \xi) \psi_a(y; -z, -z', \xi), \end{aligned}$$

which follows from the fact that ψ_a form an orthonormal basis. □

Remark 7.3. Denote by $K_{z,z',\xi}(x, y)$ the specialization of the kernel of Theorem 7.2 at $s = t$. This is a correlation kernel for the z -measure $M_{z,z',\xi}$, corresponding to the map $\lambda \mapsto X(\lambda)$. Let us abbreviate

$$\psi_{\pm\frac{1}{2}}(x) = \psi_{\pm\frac{1}{2}}(x; z, z', \xi), \quad \tilde{\psi}_{\pm\frac{1}{2}}(x) = \psi_{\pm\frac{1}{2}}(x; -z, -z', \xi).$$

One can check that for $x, y \in \mathbb{Z}'_+$

$$\begin{aligned} K_{z,z',\xi}(x, y) &= \frac{\sqrt{zz'\xi}}{1-\xi} \frac{\psi_{-\frac{1}{2}}(x)\psi_{\frac{1}{2}}(y) - \psi_{\frac{1}{2}}(x)\psi_{-\frac{1}{2}}(y)}{x-y} \\ K_{z,z',\xi}(x, -y) &= \frac{\sqrt{zz'\xi}}{1-\xi} \frac{\psi_{-\frac{1}{2}}(x)\tilde{\psi}_{-\frac{1}{2}}(y) + \psi_{\frac{1}{2}}(x)\tilde{\psi}_{\frac{1}{2}}(y)}{x+y} \\ K_{z,z',\xi}(-x, y) &= -\frac{\sqrt{zz'\xi}}{1-\xi} \frac{\tilde{\psi}_{\frac{1}{2}}(x)\psi_{\frac{1}{2}}(y) + \tilde{\psi}_{-\frac{1}{2}}(x)\psi_{-\frac{1}{2}}(y)}{x+y} \\ K_{z,z',\xi}(-x, -y) &= \frac{\sqrt{zz'\xi}}{1-\xi} \frac{\tilde{\psi}_{-\frac{1}{2}}(x)\tilde{\psi}_{\frac{1}{2}}(y) - \tilde{\psi}_{\frac{1}{2}}(x)\tilde{\psi}_{-\frac{1}{2}}(y)}{x-y} \end{aligned} \tag{7.4}$$

These four formulas coincide with the expressions obtained in [BO2, Theorem 3.3].

8. Limit transition to the Whittaker kernel

In this section we compute the scaling limit of the kernel $K_{z,z',\xi}$ of Theorem 7.2 as $\xi \nearrow 1$ and the arguments x and y are scaled by $(1 - \xi)$. In this limit the lattice \mathbb{Z}' turns into the punctured real line $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Let us introduce the continuous analogs of the functions ψ_a . These new functions $w_a(u; z, z')$ are indexed by $a \in \mathbb{Z}'$ and their argument u varies in $\mathbb{R}_{>0}$. They are expressed through the classical Whittaker functions $W_{\kappa,\mu}(u)$, see [Er, ch. 6] for the definition, as follows:

$$w_a(u; z, z') = \left(\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2})\right)^{-\frac{1}{2}} u^{-\frac{1}{2}} W_{\frac{z+z'}{2}-a, \frac{z-z'}{2}}(u). \tag{8.1}$$

Since $W_{\kappa,\mu}(u) = W_{\kappa,-\mu}(u)$, this expression is symmetric with respect to $z \leftrightarrow z'$.

It will be convenient for us to use the following integral representation of w_a :

$$w_a(u; z, z') = \frac{\Gamma(z' - a + \frac{1}{2})e^{\pi i(z'-a)}u^{\frac{z-z'}{2}}}{2\pi \left(\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2})\right)^{\frac{1}{2}}} \times \int_{+\infty}^{0-} \zeta^{-z'+a-\frac{1}{2}}(1 + \zeta)^{z-a-\frac{1}{2}}e^{-u(\zeta+\frac{1}{2})}d\zeta. \tag{8.2}$$

The (standard) notation for the contour of integration means that we start at $+\infty$, go along the real axis, then around the origin in the clockwise direction, and back to $+\infty$ along the real axis. On the last part of the contour we choose the principal branch of $\zeta^{-z'+a-\frac{1}{2}}$, which uniquely determines the values of this function on the whole contour.

This formula is easily seen to be equivalent to one of the classical integral representations for the confluent hypergeometric function Ψ , see [Er, 6.11.2(9)].

Proposition 8.1. *If $\xi \nearrow 1$ and $x \in \mathbb{Z}'_+$ goes to $+\infty$ so that $(1 - \xi)x \rightarrow u > 0$, then*

$$\psi_a(x; z, z', \xi) \sim (1 - \xi)^{\frac{1}{2}} w_a(u; z, z').$$

Proof. This statement can be proved in a number of ways, see e.g. [Er, 6.8(1)]. We will give an argument which uses the integral representations of ψ_a and w_a . A similar argument will also be employed in the proof of Theorem 8.2 below.

We start with the integral representation (5.3) for ψ_a . Let us choose as $\{\omega\}$ the following contour $C(R, r, \xi)$, where $r > 0$ is small enough (smaller than the distance between $\sqrt{\xi}$ and $1/\sqrt{\xi}$) and $R > 0$ is big enough (greater than $1/\sqrt{\xi} + r$): The contour starts at the point $\omega = R$, goes along the full circle $|\omega| = R$ in the positive direction, then along the real line until the point $\omega = 1/\sqrt{\xi} + r$, further along the full circle $|\omega - 1/\sqrt{\xi}| = r$ in the negative direction, and back along the real line to $\omega = R$. Thus, $C(R, r, \xi)$ consists of a “big circle” of radius R , a “small circle” of radius r , and a “bridge” between them.

We now fix R , pick r of order $(1 - \xi)$, and take the limit $\xi \nearrow 1$ of the integral. The integration over the “big circle” $|\omega| = R$ converges to zero exponentially in

$(1 - \xi)^{-1}$ thanks to the factor ω^{-x-a} . To take care of the rest of the integral, we make the change of the integration variable

$$\omega = 1/\sqrt{\xi} + (1 - \xi)\zeta.$$

Then we have

$$\begin{aligned} 1 - \sqrt{\xi}\omega &= -(1 - \xi)\zeta \cdot \sqrt{\xi}, \\ 1 - \sqrt{\xi}/\omega &= (1 - \xi)(1 + \zeta) \cdot \frac{(1/\sqrt{\xi} - \sqrt{\xi})(1 - \xi)^{-1} + \zeta}{(1 + \zeta)(1/\sqrt{\xi} + (1 - \xi)\zeta)}. \end{aligned}$$

Note that the second factors in these formulas,

$$\sqrt{\xi} \quad \text{and} \quad \frac{(1/\sqrt{\xi} - \sqrt{\xi})(1 - \xi)^{-1} + \zeta}{(1 + \zeta)(1/\sqrt{\xi} + (1 - \xi)\zeta)},$$

are asymptotically equal to 1 for ξ close to 1 and ζ bounded, and are uniformly bounded away from 0 and ∞ for ξ close to 1 and ζ corresponding to arbitrary ω on the contour. Hence, the rest of the integral is asymptotically equal to the absolutely convergent integral

$$\frac{(1 - \xi)^{z-z'}}{2\pi i} \int_{+\infty}^{0-} (-\zeta)^{-z'+a-\frac{1}{2}} (1 + \zeta)^{z-a-\frac{1}{2}} e^{-u(\zeta+\frac{1}{2})} d\zeta.$$

Taking into account the convention about the arguments stated after (5.3) one can check that $\arg(-\zeta) = -\pi i$ on the last part of the contour. Therefore, changing $(-\zeta)^{-z'+a-\frac{1}{2}}$ to $\zeta^{-z'+a-\frac{1}{2}}$ produces the factor

$$e^{\pi i(z'-a+\frac{1}{2})} = i e^{\pi i(z'-a)}.$$

Finally, the prefactor in (5.3) asymptotically equals

$$(1 - \xi)^{z'-z+\frac{1}{2}} \frac{\Gamma(z' - a + \frac{1}{2}) u^{\frac{z-z'}{2}}}{(\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2}))^{\frac{1}{2}}}.$$

Thus, (5.3) asymptotically equals

$$\begin{aligned} (1 - \xi)^{\frac{1}{2}} &\frac{\Gamma(z' - a + \frac{1}{2}) e^{\pi i(z'-a)} u^{\frac{z-z'}{2}}}{2\pi (\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2}))^{\frac{1}{2}}} \\ &\times \int_{+\infty}^{0-} \zeta^{-z'+a-\frac{1}{2}} (1 + \zeta)^{z-a-\frac{1}{2}} e^{-u(\zeta+\frac{1}{2})} d\zeta. \end{aligned}$$

□

Theorem 8.2. Consider the kernel $K_{z,z',\xi}(s, x; t, y)$ as described in Theorem 7.2. Let $\xi \nearrow 1$ and assume that $x, y \rightarrow \infty$ inside \mathbb{Z}' so that $(1-\xi)x \rightarrow u, (1-\xi)y \rightarrow v$, where $u, v \in \mathbb{R}^*$.

Then there exists a limit kernel $K_{z,z'}^W(s, u; t, v)$ on $\mathbb{R}^* \times \mathbb{R}^*$

$$\lim_{\xi \nearrow 1} (1 - \xi)^{-1} K_{z,z',\xi}(s, x; t, y) = K_{z,z'}^W(s, u; t, v). \tag{8.3}$$

For $s \neq t$ the formulas for the limit kernel are obtained from formulas (7.1) for the kernel $K_{z,z',\xi}$ by replacing ψ_a 's with w_a 's and setting $\xi = 1$. Namely, for $u, v > 0$

$$\begin{aligned} K_{z,z'}^W(s, u; t, v) &= \pm \sum_{a \in \mathbb{Z}'_+} e^{-a|s-t|} w_{\pm a}(u; z, z') w_{\pm a}(v; z, z'), \\ K_{z,z'}^W(s, u; t, -v) &= \pm \sum_{a \in \mathbb{Z}'_+} (-1)^{\pm a - \frac{1}{2}} e^{-a|s-t|} w_{\pm a}(u; z, z') w_{\mp a}(v; -z, -z'), \\ K_{z,z'}^W(s, -u; t, v) &= \mp \sum_{a \in \mathbb{Z}'_+} (-1)^{\pm a - \frac{1}{2}} e^{-a|s-t|} w_{\mp a}(u; -z, -z') w_{\pm a}(v; z, z'), \\ K_{z,z'}^W(s, -u; t, -v) &= \mp \sum_{a \in \mathbb{Z}'_+} e^{-a|s-t|} w_{\mp a}(u; -z, -z') w_{\mp a}(v; -z, -z'). \end{aligned}$$

Comments. 1. The prefactor $(1 - \xi)^{-1}$ in (8.3) is due to rescaling of the state space \mathbb{Z}' by $(1 - \xi)$.
 2. The reason of the restriction $s \neq t$ in above formulas is the divergence of the series for $K_{z,z'}^W(s, u; s, -v)$ and $K_{z,z'}^W(s, -u; s, v)$. The series for $K_{z,z'}^W(s, u; s, v)$ and $K_{z,z'}^W(s, -u; s, -v)$ do converge and give the correct answer. For $s = t$ there exist analogs of formulas (7.4):

$$\begin{aligned} K_{z,z'}^W(u, v) &= \sqrt{zz'} \frac{w_{-\frac{1}{2}}(u)w_{\frac{1}{2}}(v) - w_{\frac{1}{2}}(u)w_{-\frac{1}{2}}(v)}{u - v}, \\ K_{z,z'}^W(u, -v) &= \sqrt{zz'} \frac{w_{-\frac{1}{2}}(u)\tilde{w}_{-\frac{1}{2}}(v) + w_{\frac{1}{2}}(u)\tilde{w}_{\frac{1}{2}}(v)}{u + v}, \\ K_{z,z'}^W(-u, v) &= -\sqrt{zz'} \frac{\tilde{w}_{\frac{1}{2}}(u)w_{\frac{1}{2}}(v) + \tilde{w}_{-\frac{1}{2}}(u)w_{-\frac{1}{2}}(v)}{u + v}, \\ K_{z,z'}^W(-u, -v) &= \sqrt{zz'} \frac{\tilde{w}_{-\frac{1}{2}}(u)\tilde{w}_{\frac{1}{2}}(v) - \tilde{w}_{\frac{1}{2}}(u)\tilde{w}_{-\frac{1}{2}}(v)}{u - v}. \end{aligned} \tag{8.4}$$

Here we abbreviate

$$w_{\pm \frac{1}{2}}(u) = w_{\pm \frac{1}{2}}(u; z, z'), \quad \tilde{w}_{\pm \frac{1}{2}}(u) = w_{\pm \frac{1}{2}}(u; -z, -z').$$

Formulas (8.4) can be derived from (7.4) using Proposition 8.1. They were previously obtained in [B1], [BO2, §5].

3. In accordance with the terminology of these papers (where the kernel (8.4) was called the Whittaker kernel) we call the limit kernel $K_{z,z'}^W(s, u; t, v)$ the *extended Whittaker kernel*.

Proof. We use Proposition 8.1. In order to prove the theorem, we need to justify the interchange of the summation and the limit transition in (7.1). To do this it suffices to show that the series converge uniformly in ξ .

We will prove that each of the four expressions

$$|\psi_{\pm a}(x; z, z', \xi)\psi_{\pm a}(y; z, z', \xi)|, \quad |\psi_{\mp a}(x; -z, -z', \xi)\psi_{\mp a}(y; -z, -z', \xi)|, \tag{8.5}$$

$$|\psi_{\pm a}(x; z, z', \xi)\psi_{\mp a}(y; -z, -z', \xi)|, \quad |\psi_{\mp a}(x; -z, -z', \xi)\psi_{\pm a}(y; z, z', \xi)| \tag{8.6}$$

is estimated from above by $\text{const}(u, v)(1 - \xi) \cdot q^{|a|}$, where $q > 1$ can be chosen arbitrarily close to 1, and $\text{const}(u, v)$ does not depend on a and ξ . Together with the factors $e^{-a|s-t|}$ in (7.1) this ensures the needed uniform convergence.

Both expressions (8.5) are estimated in the same way, let us handle the first one. We apply formula (5.4) and we get a double contour integral, in which we single out the terms involving a ; we observe that all together they can be written in the form $(F(\omega_1, \omega_2; \xi))^k$, where $k := a - \frac{1}{2}$ ranges over \mathbb{Z}_+ , and ω_1 and ω_2 are the variables of integration. Let us write down precisely the whole expression separately for the upper and the lower choice of sign in the subscript $\pm a$:

$$\begin{aligned} &\psi_a(x; z, z', \xi)\psi_a(y; z, z', \xi) \\ &= \frac{(\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})\Gamma(y+z'+\frac{1}{2}))^{\frac{1}{2}}}{\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})} (1-\xi) \\ &\quad \times \frac{1}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} (F_{++}(\omega_1, \omega_2; \xi))^k \left(1 - \sqrt{\xi}\omega_1\right)^{-z'} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{z-1} \\ &\quad \times \left(1 - \sqrt{\xi}\omega_2\right)^{-z} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z'-1} \omega_1^{-x-\frac{1}{2}} \omega_2^{-y-\frac{1}{2}} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2} \end{aligned} \tag{8.7}$$

with

$$F_{++}(\omega_1, \omega_2; \xi) = F_+(\omega_1; \xi)F_+(\omega_2, \xi),$$

where

$$F_+(\omega, \xi) = \frac{1 - \sqrt{\xi}\omega}{\omega - \sqrt{\xi}}. \tag{8.8}$$

For $\psi_{-a}(x; z, z', \xi)\psi_{-a}(y; z, z', \xi)$ we obtain a similar expression:

$$\begin{aligned} &\psi_{-a}(x; z, z', \xi)\psi_{-a}(y; z, z', \xi) \\ &= \frac{(\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})\Gamma(y+z'+\frac{1}{2}))^{\frac{1}{2}}}{\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})} (1-\xi) \\ &\quad \times \frac{1}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} (F_{--}(\omega_1, \omega_2; \xi))^k \left(1 - \sqrt{\xi}\omega_1\right)^{-z'-1} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^z \\ &\quad \times \left(1 - \sqrt{\xi}\omega_2\right)^{-z-1} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z'} \omega_1^{-x+\frac{1}{2}} \omega_2^{-y+\frac{1}{2}} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2} \end{aligned} \tag{8.9}$$

with

$$F_{--}(\omega_1, \omega_2; \xi) = F_{-}(\omega_1; \xi)F_{-}(\omega_2, \xi),$$

where

$$F_{-}(\omega, \xi) = (F_{+}(\omega, \xi))^{-1} = \frac{\omega - \sqrt{\xi}}{1 - \sqrt{\xi}\omega}. \tag{8.10}$$

Now we need a lemma.

Lemma 8.3. *Let $F_{\pm}(\omega, \xi)$ be defined by (8.8) and (8.10). For any $q > 1$ there exists a contour $C_{\pm}(\xi, q)$ which is of the same kind as in the proof of Proposition 8.1, and such that*

$$|F_{\pm}(\omega, \xi)| \leq q \quad \forall \omega \in C_{\pm}(\xi, q). \tag{8.11}$$

Proof of Lemma 8.3. Recall that in the proof of Proposition 8.1 we used a specific family $\{C(R, r, \xi)\}$ of contours. We will show that it is possible to take $C_{\pm}(\xi, q) = C(R, r, \xi)$ with an appropriate choice of parameters R and r (the radii of the “big circle” and the “small circle” in $C(R, r, \xi)$).

Consider first the case of F_{+} . Fix $q > 1$ and let $\tilde{\omega}$ be related to ω by the equivalent relations

$$\tilde{\omega} = F_{+}(\omega, \xi)/q = \frac{1 - \sqrt{\xi}\omega}{q\omega - q\sqrt{\xi}}, \quad \omega = \frac{1 + q\sqrt{\xi}\tilde{\omega}}{q\tilde{\omega} + \sqrt{\xi}}.$$

To fulfill inequality (8.11) the contour $C_{+}(\xi, q)$ must be contained in the image of the unit disk $|\tilde{\omega}| \leq 1$ under the conformal map $\tilde{\omega} \mapsto \omega$. Let $S_{+}(\xi, q)$ denote the image of the circle $|\tilde{\omega}| = 1$; $S_{+}(\xi, q)$ is the circle that is symmetric with respect to the real axis and passes through the real points

$$\frac{q\sqrt{\xi} - 1}{q - \sqrt{\xi}}, \quad \frac{q\sqrt{\xi} + 1}{q + \sqrt{\xi}}$$

(these are the images of -1 and 1 , respectively). Since we are interested in the limit transition as $\xi \nearrow 1$ we may assume that ξ is so close to 1 that $q > 1/\sqrt{\xi}$. Then we have

$$\frac{q\sqrt{\xi} - 1}{q - \sqrt{\xi}} < \sqrt{\xi} < \frac{q\sqrt{\xi} + 1}{q + \sqrt{\xi}} < \frac{1}{\sqrt{\xi}}. \tag{8.12}$$

Observe that the image of the disk $|\tilde{\omega}| \leq 1$ is the exterior of $S_{+}(\xi, q)$ (for instance, this follows from the fact that the image of 0 is the point $1/\sqrt{\xi}$ which is outside $S_{+}(\xi, q)$). Now we take $C_{+}(\xi, q) = C(R, r, \xi)$, where R and r are chosen so that both the “big circle” and the “small circle” in $C(R, r, \xi)$ are in the exterior of the circle $S_{+}(\xi, q)$: the “big circle” surrounds $S_{+}(\xi, q)$ while the “small circle” lies to the right of $S_{+}(\xi, q)$. This is possible due to inequalities (8.12) and the fact that the distance between the points $\frac{q\sqrt{\xi}+1}{q+\sqrt{\xi}}$ and $\frac{1}{\sqrt{\xi}}$ is of order $1 - \xi$.

The case of F_- is handled analogously. We define $\tilde{\omega}$ by the equivalent relations

$$\tilde{\omega} = F_-(\omega, \xi)/q = \frac{\omega - \sqrt{\xi}}{q - q\sqrt{\xi}\omega}, \quad \omega = \frac{q\tilde{\omega} + \sqrt{\xi}}{q\sqrt{\xi}\tilde{\omega} + 1}.$$

Instead of the circle $S_+(\xi, q)$ we have another circle, denoted by $S_-(\xi, q)$, which is symmetric with respect to the real axis and passes through the points

$$\frac{q + \sqrt{\xi}}{q\sqrt{\xi} + 1}, \quad \frac{q - \sqrt{\xi}}{q\sqrt{\xi} - 1}.$$

We note that

$$\sqrt{\xi} < \frac{q + \sqrt{\xi}}{q\sqrt{\xi} + 1} < \frac{1}{\sqrt{\xi}} < \frac{q - \sqrt{\xi}}{q\sqrt{\xi} - 1}.$$

The contour $C_-(\xi, q)$ must lie in the exterior of $S_-(\xi, q)$. We again can take $C_-(\xi, q) = C(R, r, \xi)$ with appropriate R and r . But, in contrast to the case of F_+ , now the ‘‘small circle’’ in $C(R, r, \xi)$ must surround $S_-(\xi, q)$ (because $1/\sqrt{\xi}$ is inside $S_-(\xi, q)$, see the inequalities above). This requirement can be satisfied because the diameter of $S_-(\xi, q)$ (the distance between the points $\frac{q + \sqrt{\xi}}{q\sqrt{\xi} + 1}$ and $\frac{q - \sqrt{\xi}}{q\sqrt{\xi} - 1}$) is of order $(1 - \xi)$.

This completes the proof of Lemma 8.3. □

We return to the proof of Theorem 8.2. Let us estimate (8.7). The product of the prefactors is asymptotically

$$(1 - \xi)u^{\frac{1}{2}(z-z')}v^{\frac{1}{2}(z'-z)}.$$

To estimate the integral we take as contours $\{\omega_1\}$ and $\{\omega_2\}$ the contour $C_+(\xi, q^{\frac{1}{2}})$ as described in Lemma 8.3. According to Lemma 8.3, on the product of these contours, $|F_{++}(\omega_1, \omega_2; \xi)| \leq q$, whence we get

$$\begin{aligned} & \oint_{\{\omega_1 \in C_+(\xi, q^{\frac{1}{2}})\}} \oint_{\{\omega_2 \in C_+(\xi, q^{\frac{1}{2}})\}} \left| F_{++}(\omega_1, \omega_2; \xi)^k \left(1 - \sqrt{\xi}\omega_1\right)^{-z'} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{z-1} \right. \\ & \times \left. \left(1 - \sqrt{\xi}\omega_2\right)^{-z} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z-1} \omega_1^{-x-\frac{1}{2}} \omega_2^{-y-\frac{1}{2}} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2} \right| \\ & \leq q^k \oint_{\{\omega_1 \in C_+(\xi, q^{\frac{1}{2}})\}} \oint_{\{\omega_2 \in C_+(\xi, q^{\frac{1}{2}})\}} \left| \left(1 - \sqrt{\xi}\omega_1\right)^{-z'} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{z-1} \right. \\ & \times \left. \left(1 - \sqrt{\xi}\omega_2\right)^{-z} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z-1} \omega_1^{-x-\frac{1}{2}} \omega_2^{-y-\frac{1}{2}} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2} \right| \end{aligned}$$

Arguing as in the proof of Proposition 8.1 we check that the last integral is uniformly bounded as $\xi \nearrow 1$. This yields for (8.7) the required estimate of the form

$\text{const}(1 - \xi)q^k$ with arbitrary $q > 1$. The estimate for (8.9) is obtained in exactly the same way, by using the contours $\{\omega_1\} = \{\omega_2\} = C_-(\xi, q^{\frac{1}{2}})$.

The quantities in (8.6) are estimated similarly. We leave the details to the reader, and only point out two minor differences. First, while writing the double integral representation, we do not need to switch parameters z and z' (as in the proof of (5.4), see [BO8, Proposition 2.3]) to get rid of the gamma factors containing a . Second, we have to use as $\{\omega_1\}$ and $\{\omega_2\}$ two distinct contours: either $(C_+(\xi, q^{\frac{1}{2}})$ and $C_-(\xi, q^{\frac{1}{2}})$) or $(C_-(\xi, q^{\frac{1}{2}})$ and $C_+(\xi, q^{\frac{1}{2}})$). \square

Our next goal is to give an integral representation for the extended Whittaker kernel computed in Theorem 8.2. Let us introduce contours $C_{\pm}(q)$ for $q > 1$. They can be viewed as limits of the images of the contours $C_{\pm}(\xi, q)$ as $\xi \nearrow 1$ in the new variable ζ where

$$\omega = 1/\sqrt{\xi} + (1 - \xi)\zeta.$$

The contour $C_+(q)$ starts at $+\infty$, goes along the real axis, circles around 0 in the negative direction, and returns to $+\infty$ along the real axis. It has to leave on its left the point -1 together with the circle which is symmetric with respect to the real axis and passes through the points $-q/(q - 1)$ and $-q/(q + 1)$ (this circle contains -1).

The contour $C_-(q)$ also starts at $+\infty$, goes along the real axis, circles around 0 in the negative direction, and returns to $+\infty$ along the real axis. It has to leave on its left the point -1 , and it has to surround the circle which is symmetric with respect to the real axis and passes through the points $-1/(q + 1)$ and $1/(q - 1)$ (this circle contains 0).

Note that if $\zeta \in C_+(q)$ then $|\zeta/(1 + \zeta)| < q$, and if $\zeta \in C_-(q)$ then $|(1 + \zeta)/\zeta| < q$.

Theorem 8.4. *The extended Whittaker kernel $K_{z,z'}^W(s, u; t, v)$ of Theorem 8.2 for $s \neq t$ has the following integral representation ($u > 0, v > 0$):*

$$\begin{aligned} K_{z,z'}^W(s, u; t, v) &= e^{\pi i(z+z')} (u/v)^{\frac{z-z'}{2}} e^{\frac{1}{2}(s-t)} \\ &\times \frac{1}{(2\pi i)^2} \oint_{+\infty}^{0-} \oint_{+\infty}^{0-} \zeta_1^{-z'} (1 + \zeta_1)^z \zeta_2^{-z} (1 + \zeta_2)^z \\ &\times \frac{e^{-u(\zeta_1 + \frac{1}{2}) - v(\zeta_2 + \frac{1}{2})} d\zeta_1 d\zeta_2}{e^{s-t}(1 + \zeta_1)(1 + \zeta_2) - \zeta_1 \zeta_2} \end{aligned}$$

where for $s > t$ both contours $\{\zeta_1\}$ and $\{\zeta_2\}$ are of the form $C_+(e^{\frac{1}{2}(s-t)})$, and for $s < t$ both contours are of the form $C_-(e^{\frac{1}{2}(t-s)})$;

$$\begin{aligned} K_{z,z'}^W(s, u; t, -v) &= \frac{(\sin(\pi z) \sin(\pi z'))^{\frac{1}{2}}}{\sin(\pi z')} (u/v)^{\frac{z-z'}{2}} e^{\frac{1}{2}(s-t)} \\ &\times \frac{1}{(2\pi)^2} \oint_{+\infty}^{0-} \oint_{+\infty}^{0-} \zeta_1^{-z'} (1 + \zeta_1)^z \zeta_2^{-z'} (1 + \zeta_2)^z \frac{e^{-u(\zeta_1 + \frac{1}{2}) - v(\zeta_2 + \frac{1}{2})} d\zeta_1 d\zeta_2}{e^{s-t}(1 + \zeta_1)\zeta_2 - \zeta_1(1 + \zeta_2)} \end{aligned}$$

where for $s > t$, the contour $\{\zeta_1\}$ is of the form $C_+(e^{\frac{1}{2}(s-t)})$ and the contour $\{\zeta_2\}$ is of the form $C_-(e^{\frac{1}{2}(s-t)})$, while for $s < t$ the contour $\{\zeta_1\}$ is of the form $C_-(e^{\frac{1}{2}(t-s)})$ and the contour $\{\zeta_2\}$ is of the form $C_+(e^{\frac{1}{2}(t-s)})$;

$$K_{z,z'}^W(s, -u; t, v) = -K_{z,z'}^W(s, v; t, -u);$$

$$K_{z,z'}^W(s, -u; t, -v) = K_{-z,-z'}^W(t, u; s, v).$$

The contours may be chosen differently by deforming the contours above so that the denominators of the integrands do not vanish.

Proof. We take the integral representation (8.2) for the functions w_a and plug it in into the series of Theorem 8.2. Computing the sum of geometric progression under the integral yields the formulas above. The contours are chosen in such a way that the absolute values of the ratios of geometric progressions involved are less than one.

As in the proof of (5.4) (see [BO8, Proposition 2.3]), in the derivation of the first formula we switch z and z' in the integral representation of the second factor, which cancels the gamma factors involving the summation index. In the derivation of the second formula we do not need to do that, the gamma factors disappear thanks to the relation $\Gamma(x)\Gamma(1-x) = \pi / \sin(\pi x)$. \square

9. Limit transition to the gamma kernel

In this section we compute the limit of the extended hypergeometric kernel $K_{z,z',\xi}(s, x; t, y)$ as $\xi \nearrow 1$ and the time variables are scaled $(1 - \xi) : s = (1 - \xi)\sigma, t = (1 - \xi)\tau$ with finite $\sigma, \tau \in \mathbb{R}$.

Theorem 9.1. *There exists a limit of the extended hypergeometric kernel*

$$\underline{K}_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) = \lim_{\xi \rightarrow 1} \underline{K}_{z,z',\xi}((1 - \xi)\sigma, x; (1 - \xi)\tau, y)$$

where $x, y \in \mathbb{Z}'$, $\sigma, \tau \in \mathbb{R}$.

For $\sigma \geq \tau$, the correlation kernel can be written in two different ways: as a double contour integral

$$\begin{aligned} & \underline{K}_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) \\ &= \frac{\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})e^{-\pi i(z+z')}(-1)^{x+y+1}}{(\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}} \\ & \times \frac{1}{(2\pi i)^2} \oint_{+\infty}^{0-} \oint_{+\infty}^{0-} \frac{\zeta_1^{z'+x-\frac{1}{2}}(1+\zeta_1)^{-z-x-\frac{1}{2}}\zeta_2^{z+y-\frac{1}{2}}(1+\zeta_2)^{-z'-y-\frac{1}{2}} d\zeta_1 d\zeta_2}{1 + (\sigma - \tau) + \zeta_1 + \zeta_2} \end{aligned} \tag{9.1}$$

and as a single integral

$$\underline{K}_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) = \int_0^{+\infty} e^{-u(\sigma-\tau)} w_x(u; -z, -z') w_y(u; -z, -z') du. \tag{9.2}$$

The values of the kernel for $\sigma < \tau$ are obtained from the above formulas using the symmetry property

$$\underline{K}_{z,z'}^{gamma}(\sigma, x; \tau, y) = (-1)^{x+y+1} \underline{K}_{-z,-z'}^{gamma}(\tau, -x; \sigma, -y), \quad \sigma \neq \tau.$$

Comment. For $\sigma = \tau$ the kernel $\underline{K}_{z,z'}^{gamma}$ has a simpler “integrable” expression

$$\begin{aligned} \underline{K}_{z,z'}^{gamma}(x; y) &= \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z - z'))} \\ &\times \left\{ \Gamma(z + x + \frac{1}{2}) \Gamma(z' + x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) \right\}^{-1/2} \\ &\times \frac{\Gamma(z + x + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) - \Gamma(z' + x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2})}{x - y}, \end{aligned} \tag{9.3}$$

see [BO5, Theorem 2.3]. We called this kernel the *gamma kernel*. The more general kernel $\underline{K}_{z,z'}^{gamma}(\sigma, x; \tau, y)$ of Theorem 9.1 will be called the *extended gamma kernel*.

Proof. We start with the series representation of the extended hypergeometric kernel, Theorem 6.2. Using the symmetry relations (5.6) and (5.7), we rewrite the kernel in the following form

$$\begin{aligned} &\underline{K}_{z,z',\xi}(s, x; t, y) \\ &= \begin{cases} \sum_{a \in \mathbb{Z}'_+} e^{-a|s-t|} \psi_x(a; -z, -z', \xi) \psi_y(a; -z, -z', \xi), & s \geq t, \\ (-1)^{x+y+1} \sum_{a \in \mathbb{Z}'_+} e^{-a|s-t|} \psi_{-x}(a; z, z', \xi) \psi_{-y}(a; z, z', \xi), & s < t. \end{cases} \end{aligned}$$

This formula implies that if we prove the statement for $\sigma \geq \tau$, then the case $\sigma < \tau$ will follow by symmetry. Thus, we continue with the assumption $s \geq t$ and therefore $\sigma \geq \tau$.

Formula (9.2) is the limit variant of the formula for $\underline{K}_{z,z',\xi}(s, x; t, y)$ above. Indeed, by virtue of Proposition 8.1,

$$\psi_x(a; -z, -z', \xi) \psi_y(a; -z, -z', \xi) \sim (1 - \xi) w_x(u; -z, -z') w_y(u; -z, -z')$$

provided that $a \sim (1 - \xi)^{-1} u$. The factor $(1 - \xi)$ is responsible for turning the sum into an integral over u ; this sum is just an approximation to the integral.

This empirical argument needs a rigorous justification. It is simpler to turn the series representation for $\underline{K}_{z,z',\xi}(s, x; t, y)$ into a double contour integral and then pass to the limit in the integral. The limit integral will be identified with the right-hand side of (9.2).

Using formula (5.4) with appropriately changed parameters, we obtain

$$\begin{aligned} &\underline{K}_{z,z',\xi}(s, x; t, y) \\ &= \frac{\Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2})}{\left(\Gamma(-z - x + \frac{1}{2}) \Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2}) \right)^{\frac{1}{2}}} \\ &\times \frac{1 - \xi}{(2\pi i)^2} \sum_{a \in \mathbb{Z}'_+} \oint \oint e^{-a(s-t)} \left(1 - \sqrt{\xi} \omega_1 \right)^{z'+x-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1} \right)^{-z-x-\frac{1}{2}} \\ &\times \left(1 - \sqrt{\xi} \omega_2 \right)^{z+y-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_2} \right)^{-z'-y-\frac{1}{2}} \omega_1^{-x-a} \omega_2^{-y-a} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2} \end{aligned}$$

with contours $\{\omega_1\}$ and $\{\omega_2\}$ chosen as in (5.3). Now we want to sum the geometric progression inside the integrals. The ratio of the geometric progression is $e^{t-s}\omega_1^{-1}\omega_2^{-1}$. In order to justify the interchange of summation and integration we need to ensure that the absolute value of this ratio, as a function in ω_1, ω_2 , is bounded from above by a constant strictly less than one. This is easy to arrange by requiring, for example, that both contours contain the unit circle.

Performing the summation, we obtain

$$\begin{aligned} & \underline{K}_{z,z',\xi}(s, x; t, y) \\ &= \frac{e^{\frac{1}{2}(s-t)}\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})}{(\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}} \\ & \times \frac{1 - \xi}{(2\pi i)^2} \oint \oint \left(1 - \sqrt{\xi}\omega_1\right)^{z'+x-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{-z-x-\frac{1}{2}} \\ & \times \left(1 - \sqrt{\xi}\omega_2\right)^{z+y-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{-z'-y-\frac{1}{2}} \frac{\omega_1^{-x-\frac{1}{2}}\omega_2^{-y-\frac{1}{2}}}{e^{s-t}\omega_1\omega_2 - 1} d\omega_1 d\omega_2. \end{aligned}$$

Recall the notation $C(R, r, \xi)$ for certain type of contours introduced in the proof of Proposition 8.1. We assume that both integration variables range over such a contour with R being a fixed number greater than 1, and r being of order $1 - \xi$, and such that $1/\sqrt{\xi} - r > 1$.

Let us split each of the contours into two parts: the first one is the “big” circle $|\omega| = R$, and the second part is the rest. If both ω_1 and ω_2 range over their big circles then the integrand is uniformly bounded, and the prefactor $(1 - \xi)$ sends the whole expression to zero as $\xi \rightarrow 1$.

If one of the variables, say, ω_2 ranges over its big circle and ω_1 ranges over the second part of its contour, we observe that all the factors of the integrand involving ω_2 are uniformly bounded. The absolute value of the remaining part of the integrand

$$\left(1 - \sqrt{\xi}\omega_1\right)^{z'+x-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{-z-x-\frac{1}{2}}$$

is uniformly bounded by

$$\text{const} \cdot \left| (1 - \xi)^{z'-z-1} \zeta_1^{z'+x-\frac{1}{2}} (1 + \zeta_1)^{-z-x-\frac{1}{2}} \right|$$

with $\omega_1 = 1/\sqrt{\xi} + (1 - \xi)\zeta_1$, where we used the same argument as in the proof of Proposition 8.1. Thus, our double integral is bounded in absolute value by the following one-dimensional integral in ζ_1 :

$$\text{const} \cdot (1 - \xi)^{\Re(z'-z)} \left| \oint_{\tilde{R}}^{0-} \zeta_1^{z'+x-\frac{1}{2}} (1 + \zeta_1)^{-z-x-\frac{1}{2}} d\zeta_1 \right|$$

with

$$\tilde{R} = 1/\sqrt{\xi} + R(1 - \xi)^{-1} \sim R(1 - \xi)^{-1}.$$

Hence, our expression is bounded by

$$\text{const} \cdot (1 - \xi)^{\Re(z'-z)} \int_1^{\tilde{R}} \zeta_1^{\Re(z'-z)-1} d\zeta_1$$

which is bounded either by a constant (if $\Re(z - z') \neq 0$) or by $|\ln(1 - \xi)|$ (if $\Re(z - z') = 0$). In both cases, the prefactor $(1 - \xi)$ in the integral representation for $\underline{K}_{z,z',\xi}(s, x; t, y)$ sends the whole expression to zero.

The only asymptotically significant part of the integral comes from the case when both ω_1 and ω_2 vary over the second parts of their contours. Making the change of variables

$$\omega_1 = 1/\sqrt{\xi} + (1 - \xi)\zeta_1, \quad \omega_2 = 1/\sqrt{\xi} + (1 - \xi)\zeta_2,$$

and arguing as in the proof of Proposition 8.1, we conclude, using the asymptotic relation

$$\frac{1 - \xi}{e^{s-t}\omega_1\omega_2 - 1} \sim \frac{1}{1 + (\sigma - \tau) + \zeta_1 + \zeta_2},$$

that the limit value of the kernel is given by the right-hand side of (9.1). Note that the integral in (9.1) is absolutely convergent. To see this we use the estimate

$$|1 + (\sigma - \tau) + \zeta_1 + \zeta_2| \geq \text{const} \cdot |\zeta_1|^\nu |\zeta_2|^{1-\nu}$$

which holds for any ζ_1, ζ_2 on our contours and any $\nu \in (0, 1)$. We apply this inequality with $\nu = \frac{1}{2} + \Re(z' - z)/2$. The fact that $\nu \in (0, 1)$ follows from our basic assumptions on z, z' , see §1.

Thus, we have proved the integral representation (9.1). To see the equivalence of (10.1) and (10.2) we substitute the integral representation (8.2) into (10.2) and integrate explicitly over u . □

Acknowledgements. We are deeply grateful to Philippe Biane for a number of remarks and suggestions. This research was partially conducted during the period the first author (A. B.) served as a Clay Mathematics Institute Research Fellow. He was also partially supported by the NSF grant DMS-0402047. The second author (G. O.) was supported by the CRDF grant RM1-2543-MO-03.

References

[Bi1] Biane, P.: Quantum random walk on the dual of $SU(n)$. *Probab. Theory Rel. Fields.* **89**, 117–129 (1991)
 [Bi2] Biane, P.: Minuscule weights and random walks on lattices. *Quant. Probab. Rel. Topics* **VII**, 51–65 (1992)
 [B1] Borodin, A.: Harmonic analysis on the infinite symmetric group and the Whittaker kernel. *St. Petersburg Math. J.* **12** (5), 733–759 (2001)

- [B2] Borodin, A.: Riemann–Hilbert problem and the discrete Bessel kernel. *Intern. Math. Research Notices* (9) 467–494 [arXiv:math.CO/9912093](https://arxiv.org/abs/math.CO/9912093), 2000
- [BOK] Borodin, A., Okounkov, A.: A Fredholm determinant formula for Toeplitz determinants. *Integral Equations Oper. Theory* **37**, 386–396 [arXiv:math.CA/9907165](https://arxiv.org/abs/math.CA/9907165), 2000
- [BOO] Borodin, A., Okounkov, A., Olshanski, G.: Asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.* **13**, 481–515 [arXiv:math.CO/9905032](https://arxiv.org/abs/math.CO/9905032), 2000
- [BO1] Borodin, A., Olshanski, G.: Point processes and the infinite symmetric group. *Math. Research Lett.* **5**, 799–816 [arXiv:math.RT/9810015](https://arxiv.org/abs/math.RT/9810015), 1998
- [BO2] Borodin, A., Olshanski, G.: Distributions on partitions, point processes and the hypergeometric kernel. *Comm. Math. Phys.* **211**, 335–358; [arXiv:math.RT/9904010](https://arxiv.org/abs/math.RT/9904010), 2000
- [BO3] Borodin, A., Olshanski, G. Harmonic functions on multiplicative graphs and interpolation polynomials. *Electronic J. Comb.* **7**, #R28 [math/9912124](https://arxiv.org/abs/math/9912124), 2000
- [BO4] Borodin, A., Olshanski, G.: Z–Measures on partitions, Robinson–Schensted–Knuth correspondence, and $\beta = 2$ random matrix ensembles. In: *Random matrix models and their applications* (P. Bleher and A. Its, eds). Cambridge University Press. *Mathematical Sciences Research Institute Publications* **40**, 71–94; [arXiv:math.CO/9905189](https://arxiv.org/abs/math.CO/9905189), 2001
- [BO5] Borodin, A., Olshanski, G.: Random partitions and the Gamma kernel. *Adv. Math.* **194** (1), 141–202; [arXiv:math-ph/0305043](https://arxiv.org/abs/math-ph/0305043), 2005
- [BO6] Borodin, A., Olshanski, G.: Z-measures on partitions and their scaling limits. *European J. Combin.* **26** (6), 795–834 [arXiv:math-ph/0210048](https://arxiv.org/abs/math-ph/0210048), 2005
- [BO7] Borodin, A., Olshanski, G.: Stochastic dynamics related to Plancherel measure on partitions. To appear in a collection dedicated to A. M. Vershik; [arXiv:math-ph/0402064](https://arxiv.org/abs/math-ph/0402064)
- [BO8] Borodin, A., Olshanski, G.: Markov processes on partitions. [arXiv:math-ph/0409075](https://arxiv.org/abs/math-ph/0409075), version 1
- [BR] Borodin, A., Rains, E.: Eynard–Mehta theorem, Schur process, and their pfaffian analogs. Preprint, [arXiv:math-ph/0409059](https://arxiv.org/abs/math-ph/0409059), 2004
- [De1] Deift, P.: Integrable operators. In: *Differential operators and spectral theory: M. Sh. Birman’s 70th anniversary collection* (V. Buslaev, M. Solomyak, D. Yafaev, eds.) American Mathematical Society Translations, ser. 2, v. **189**, Providence, R.I.: AMS 69–84 (1999)
- [De2] Deift, P.: *Orthogonal polynomials and random matrices: a Riemann–Hilbert approach*. Reprint of the 1998 original American Mathematical Society, Providence, RI 2000
- [Dy] Dyson, F. J.: A Brownian motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3**, 1191–1198 (1962)
- [Er] Erdelyi, A.: (ed.) *Higher transcendental functions*. Bateman Manuscript Project, vol. **I**, McGraw-Hill, New York, 1953
- [EM] Eynard, B., Mehta, M. L.: Matrices coupled in a chain. I. Eigenvalue correlations. *J. Phys. A: Math. Gen.* **31**, 4449–4456 [arXiv:cond-mat/9710230](https://arxiv.org/abs/math-ph/9710230), 1998
- [Fe1] Feller, W.: On the integro–differential equations of purely discontinuous Markoff processes. *Trans. Amer. Math. Soc.* **48**, 488–515 (1940); erratum: **58**, 474 (1945)
- [Fe2] Feller, W.: *An introduction to probability theory and its applications*. Vol. **I**, Wiley, New York, 1970
- [Fe3] Feller, W.: *An introduction to probability theory and its applications*. Vol. **II**, Wiley, New York, 1971

- [Fu] Fulman, J.: Stein's method and Plancherel measure of the symmetric group. *Trans. Amer. Math. Soc.* **357** (2), 555–570 [arXiv:math.RT/0305423](#), 2005
- [Ga] Gantmacher, F. R.: *The theory of matrices*. Vol. **1**, Transl. from the Russian by K. A. Hirsch. Reprint of the 1959 translation. AMS Chelsea Publishing Providence, RI 1998
- [GNW1] Green, C., Nijenhuis, A., Wilf, H.: A probabilistic proof of a formula for the number of Young tableaux of a given shape. *Adv. Math.* **31**, 104–109 (1979)
- [GNW2] Green, C., Nijenhuis, A., Wilf, H.: Another probabilistic method in the theory of Young tableaux *J. Comb. Theory, Ser. A* **37**, 127–135 (1984)
- [Gr] Grünbaum, F. A.: The bispectral problem: an overview. In: *Special functions 2000: current perspective and future directions* (J. Bustoz et al., eds). NATO Sci. Ser. II Math. Phys. Chem. **30**, 129–140 Kluwer Acad. Publ. Dordrecht, 2001
- [IIKS] Its, A. R., Izergin, A. G., Korepin, V. E., Slavnov, N. A.: Differential equations for quantum correlation functions. *Intern. J. Mod. Phys.* **B4**, 10037–1037 (1990)
- [Jo1] Johansson, K.: Shape fluctuations and random matrices. *Comm. Math. Phys.* **209**, 437–476 [arXiv:math.CO/9903134](#), 2000
- [Jo2] Johansson, K.: Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math. (2)* **153** (1), 259–296 [arXiv:math.CO/9906120](#), 2001
- [Jo3] Johansson, K.: Non-intersecting paths, random tilings and random matrices. *Probab. Theory Related Fields* **123** (2), 225–280 [arXiv:math.PR/0011250](#), 2002
- [Jo4] Johansson, K.: Discrete polynuclear growth and determinantal processes. *Comm. Math. Phys.* **242**, 277–329 [math.PR/0206208](#), 2003
- [Jo5] Johansson, K.: Random growth and determinantal processes . MSRI lecture, Sept. 2002 available from www.msri.org/publications/ln/msri/2002/rmt/johansson/1/index.html
- [Jo6] Johansson, K.: The Arctic circle boundary and the Airy process . *Ann. Prob.* **33**, 1–30 [arXiv:math.PR/0306216](#), 2005
- [Jo7] Johansson, K.: Non-intersecting, simple, symmetric random walks and the extended Hahn kernel. Preprint, [arXiv:math.PR/0409013](#), 2004
- [KMG1] Karlin, S., McGregor, J.: The classification of birth and death processes . *Trans. Amer. Math. Soc.* **86**, 366–400 (1957)
- [KMG2] Karlin, S., McGregor, J.: Linear growth, birth and death processes . *J. Math. Mech.* **7**, 643–662, (1958)
- [KMG3] Karlin, S., McGregor, J.: Coincidence probabilities . *Pacific J. Math.* **9**, 1141–1164 (1959)
- [Ke1] Kerov, S. V.: Anisotropic Young diagrams and Jack symmetric functions. *Funktional. Anal. i Prilozhen.* **34** (1), 51–64 (2000); English translation: *Funct. Anal. Appl.* **34**, 41–51 (2000)
- [Ke2] Kerov, S. V.: Asymptotic representation theory of the symmetric group and its applications in analysis. *Amer. Math. Soc.*, Providence, RI, pp. 201 2003
- [KOV1] Kerov, S., Olshanski, G., Vershik, A.: Harmonic analysis on the infinite symmetric group. A deformation of the regular representation . *Comptes Rend. Acad. Sci. Paris, Sér. I* **316**, 773–778, 1993
- [KOV2] Kerov, S., Olshanski, G., Vershik, A.: Harmonic analysis on the infinite symmetric group . *Invent. Math.* **158**, 551–642 [arXiv:math.RT/0312270](#), 2004
- [KS] Koekoek, R., Swarttouw, R. F.: The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. Delft University of Technology, Faculty

- of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98–17 available via <http://aw.twi.tud-e1ft.nl/~koekoek/askey.html>, 1998
- [KOR] König, W., Connell, N. O', Roch, S.: Non-colliding random walks, tandem queues, and discrete orthogonal polynomial ensembles. *Electronic J. Prob.* **7**, 1–24 (2002)
- [NF] T. Nagao and P. J. Forrester .: Multilevel dynamical correlation function for Dyson's Brownian motion model of random matrices . *Phys. Lett.* **A247**, 42–46 (1998)
- [Ma] Macdonald, I. G.: Symmetric functions and Hall polynomials. 2nd edition, Oxford University Press, 1995
- [O'C] Connell, N. O': Conditioned random walks and the RSK correspondence . *J. Phys. A: Math. Gen.* **36**, 3049–3066 (2003)
- [Ok1] Okounkov, A.: Infinite wedge and measures on partitions . *Selecta Math.* **7**, 1–25 math.RT/9907127, 2001
- [Ok2] Okounkov, A.: $SL(2)$ and z -measures In: Random matrix models and their applications (P. M. Bleher and A. R. Its, eds). *Mathematical Sciences Research Institute Publications* **40**, Cambridge Univ. Press, 407–420 math.RT/0002136, 2001
- [Ok3] Okounkov, A.: Symmetric functions and random partitions In: Symmetric functions 2001: Surveys of developments and perspectives (S. Fomin, ed). *Proceedings of the NATO Advanced Study Institute (Cambridge, UK, June 25–July 6, 2001)*. Dordrecht: Kluwer Academic Publishers. *NATO Sci. Ser. II, Math. Phys. Chem.* **74**, 223–252 arXiv:math.CO/0309074, (2002)
- [OkR] Okounkov, A., Reshetikhin, N.: Correlation functions of Schur process with applications to local geometry of a random 3-dimensional Young diagram. *J. Amer. Math. Soc.* **16**, 581–603 arXiv:math.CO/0107056, 2003
- [OI1] Olshanski, G.: Point processes related to the infinite symmetric group. In: *The orbit method in geometry and physics: in honor of A. A. Kirillov* (Ch. Duval et al., eds.), *Progress in Mathematics* **213**, Birkhäuser, 349–393 arXiv:math.RT/9804086, 2003
- [OI2] Olshanski, G.: An introduction to harmonic analysis on the infinite symmetric group. In: *Asymptotic combinatorics with applications to mathematical physics A. M. Vershik A European mathematical summer school held at the Euler Institute, St. Petersburg, Russia, July 9–20, 2001* Springer Lect. Notes Math. **1815**, 127–160 arXiv:math.RT/0311369, 2003
- [OI3] Olshanski, G.: The problem of harmonic analysis on the infinite-dimensional unitary group . *J. Funct. Anal.* **205** (2), 464–524 arXiv:math.RT/0109193, 2003
- [PS] Prähofer, M., Spohn, H.: Scale invariance of the PNG droplet and the Airy process . *J. Stat. Phys.* **108**, 1071–1106; arXiv:math.PR/0105240, 2002
- [TW] Tracy, C. A., Widom, H.: Differential equations for Dyson processes. Preprint, arXiv:math.PR/0309082, 2003
- [Ve] Vershik, A. M.: Statistical mechanics of combinatorial partitions, and their limit shapes. *Funct. Anal. Appl.* **30**, 90–105 (1996)
- [VK] Vershik, A. M., Kerov, S. V.: Asymptotic theory of characters of the symmetric group . *Funct. Anal. Appl.* **15**, 246–255 (1981)