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Translation invariant Gibbs states for the Ising model

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Abstract. We prove that all the translation invariant Gibbs states of the Ising model are a linear combination of the pure phases μ_β^+, μ_β^- for any $\beta \neq \beta_c$. This implies that the average magnetization is continuous for $\beta > \beta_c$. Furthermore, combined with previous results on the slab percolation threshold [B2] this shows the validity of Pisztora's coarse graining [Pi] up to the critical temperature.

1. Introduction

The set of Gibbs measures associated to the Ising model is a simplex (see [Ge]) and the complete characterization of the extremal measures at any inverse temperature $\beta = 1/T$ remains an important issue. The most basic states are the two pure phases μ_β^+, μ_β^- which are obtained as the thermodynamic limit of the finite Gibbs measures with boundary conditions uniformly equal to 1 or -1. In the phase transition regime ($\beta > \beta_c$), these two Gibbs states are distinct and translation invariant. An important result by Aizenman and Higuchi [A, H] (see also [GH]) asserts that for the two dimensional nearest neighbor Ising model these are the only two extremal Gibbs measures and that any other Gibbs measure on $\{\pm 1\}^{\mathbb{Z}^2}$ belongs to $[\mu_\beta^+, \mu_\beta^-]$, i.e. is a linear combination of μ_β^+, μ_β^-. In higher dimensions Dobrushin [D] proved the existence of other extremal invariant measures. They arise from well chosen mixed boundary conditions which create a rigid interface separating the system into two regions. Thus, contrary to the previous pure phases, the Dobrushin states are non-translation invariant. We refer the reader to the survey by Dobrushin, Shlosman [DS] for a detailed account on these states.

In this paper we are going to focus on the translation invariant Gibbs states in the phase transition regime in dimension $d \geqslant 3$ and prove that they belong to $[\mu_\beta^+, \mu_\beta^-]$. This problem has a long history and has essentially already been solved, with the exception of one detail which we will now tie up.

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Two strategies have been devised to tackle the problem. The first one, implemented by Gallavotti and Miracle-Solé [GS], is a constructive method based on Peierls estimates. They proved that for any β large enough the set of translation invariant Gibbs states is $[\mu_\beta^+, \mu_\beta^-]$. This result was extended in [BMP] to the Ising model with Kac interactions for any $\beta > 1$ as soon as the interaction range is large enough. More generally, in the framework of Pirogov-Sinai theory, similar results can be obtained for β large enough (see e.g. [Z, M]). A completely different approach relying on ferromagnetic inequalities was introduced by Lebowitz [L1] and generalized to the framework of FK percolation by Grimmett [Gr]. The key argument is to relate the differentiability of the pressure wrt β and the characterization of the translation invariant Gibbs states. As the pressure is a convex function, it is differentiable for all β, except possibly for an at most countable set of inverse temperatures $\mathcal{B} \subset [\beta_c, \infty[$. For the Ising model, $\mathcal{B}$ is conjectured to be empty, although the previous method does not provide any explicit control on $\mathcal{B}$. We stress the fact that the non differentiability of the pressure has other implications, namely that for any inverse temperature in $\mathcal{B}$, the average magnetization would be discontinuous; and that the number of pure phases would be uncountable (see [BL]).

We will show that in dimension $d \geqslant 3$, for any $\beta > \beta_c$ there is a unique infinite volume FK measure. Several consequences can be drawn from this by using previous results in [Gr, L1]: the set of translation invariant Gibbs states is $[\mu_\beta^+, \mu_\beta^-]$, the average magnetization is continuous in $]\beta_c, \infty[$. Finally, combining this statement with the characterization of the slab percolation threshold in [B2], we deduce that Pisztora's coarse graining is valid up to the critical temperature. All these facts are summarized in Subsection 2.3. Our method is restricted to $\beta > \beta_c$. At $\beta = \beta_c$, the magnetization is known to be continuous in dimension $d = 2$ for the nearest neighbor Ising model (see e.g. [O]). The continuity in dimension $d \geqslant 4$ has also been derived by using the random current representation [AF]. In dimension $d = 3$, it is widely believed that the phase transition of the Ising model is of second order and thus similar results should also hold at β_c.

2. Notation and Results

2.1. The Ising model

We consider the Ising model on $\mathbb{Z}^d$ ($d \geqslant 3$) with finite range interactions and spins $\{\sigma_i\}_{i \in \mathbb{Z}^d}$ taking values ± 1. Let $\sigma_\Lambda \in \{\pm 1\}^\Lambda$ be the spin configuration restricted to $\Lambda \subset \mathbb{Z}^d$. The Hamiltonian associated to σ_Λ with boundary conditions σ_{Λ^c} is defined by

$$H(\sigma_\Lambda \mid \sigma_{\Lambda^c}) = -\frac{1}{2} \sum_{i,j \in \Lambda} J(i-j)\sigma_i\sigma_j - \sum_{i \in \Lambda, j \in \Lambda^c} J(i-j)\sigma_i\sigma_j,$$

where the couplings $J(i-j)$ are ferromagnetic and equal to 0 for $\|i-j\| \geqslant R$ (R will be referred to as the range of the interaction). Furthermore, we assume that $J(i-j) > 0$ for any pair of nearest neighbors (i, j) so that the Gibbs measure (defined below) cannot be decomposed as a product of measures on disjoint sub-lattices.

The Gibbs measure in Λ at inverse temperature $\beta > 0$ is defined by

$$\mu_{\beta,\Lambda}^{\sigma_{\Lambda^c}}(\sigma_\Lambda) = \frac{1}{Z_{\beta,\Lambda}^{\sigma_{\Lambda^c}}} \exp\big(- \beta H(\sigma_\Lambda \,|\, \sigma_{\Lambda^c})\big),$$

where the partition function $Z_{\beta,\Lambda}^{\sigma_{\Lambda^c}}$ is the normalizing factor. The boundary conditions act as boundary fields, therefore more general values of the boundary conditions can be used. For any $h > 0$, let us denote by $\mu_{\beta,\Lambda}^h$ the Gibbs measure with boundary magnetic field h, i.e. with Hamiltonian

$$H_h(\sigma_\Lambda) = -\frac{1}{2} \sum_{i,j\in\Lambda} J(i-j)\sigma_i\sigma_j - h \sum_{i\in\Lambda, j\in\Lambda^c} J(i-j)\sigma_i \,.$$

The phase transition is characterized by symmetry breaking for any β larger than the inverse critical temperature β_c defined by

$$\beta_c = \inf\{\beta > 0, \qquad \lim_{N\to\infty} \mu_{\beta,\Lambda_N}^{+}(\sigma_0) > 0\}\,.$$

2.2. The random cluster measure

The random cluster measure was originally introduced by Fortuin and Kasteleyn [FK] (see also [ES, Gr]) and it can be understood as an alternative representation of the Ising model (or more generally of the q-Potts model). This representation will be referred to as the FK representation.

Let $\mathbb{E}$ be the set of bonds, i.e. of pairs (i, j) in $\mathbb{Z}^d$ such that $J(i-j) > 0$. For any subset Λ of $\mathbb{Z}^d$ we consider two sets of bonds

$$\begin{cases} \mathbb{E}_\Lambda^{\mathrm{w}} = \{(i,j)\in\mathbb{E}, \quad i\in\Lambda, j\in\mathbb{Z}^d\}\,, \\ \mathbb{E}_\Lambda^{\mathrm{f}} = \{(i,j)\in\mathbb{E}, \quad i,j\in\Lambda\}\,. \end{cases} \tag{2.1}$$

The set $\Omega = \{0,1\}^{\mathbb{E}}$ is the state space for the dependent percolation measures. Given $\omega \in \Omega$ and a bond $b = (i, j) \in \mathbb{E}$, we say that b is open if $\omega_b = 1$. Two sites of $\mathbb{Z}^d$ are said to be connected if one can be reached from another via a chain of open bonds. Thus, each $\omega \in \Omega$ splits $\mathbb{Z}^d$ into the disjoint union of maximal connected components, which are called the open clusters of Ω. Given a finite subset $B \subset \mathbb{Z}^d$ we use $c_B(\omega)$ to denote the number of different open finite clusters of ω which have a non-empty intersection with B.

For any $\Lambda \subset \mathbb{Z}^d$ we define the random cluster measure on the bond configurations $\omega \in \Omega_\Lambda = \{0,1\}^{\mathbb{E}_\Lambda^{\mathrm{f}}}$. The boundary conditions are specified by a frozen percolation configuration $\pi \in \Omega_\Lambda^c = \Omega \setminus \Omega_\Lambda$. Using the shortcut $c_\Lambda^\pi(\omega) = c_\Lambda(\omega \vee \pi)$ for the joint configuration $\omega \vee \pi \in \mathbb{E}$, we define the finite volume random cluster measure $\Phi_{\beta,\Lambda}^\pi$ on Ω_Λ with the boundary conditions π as:

$$\Phi_{\beta,\Lambda}^\pi(\omega) \;=\; \frac{1}{Z_\Lambda^{\beta,\pi}} \left(\prod_{b\in\mathbb{E}_\Lambda^{\mathrm{f}}} \big(1-p_b\big)^{1-\omega_b}\; p_b^{\omega_b} \right) 2^{c_\Lambda^\pi(\omega)}\,, \tag{2.2}$$

where the bond intensities are such that $p_{(i,j)} = 1 - \exp(-2\beta J(i-j))$. We will sometimes use the same notation for the FK measure on $\mathbb{E}_\Lambda^{\mathrm{w}}$, in which case we will state it explicitly.

The measures $\Phi_{\beta,\Lambda}^{\pi}$ are FKG partially ordered with respect to the lexicographical order of the boundary condition π. Thus, the extremal ones correspond to the free ($\pi \equiv 0$) and wired ($\pi \equiv 1$) boundary conditions and are denoted as $\Phi_{\beta,\Lambda}^{\mathrm{f}}$ and $\Phi_{\beta,\Lambda}^{\mathrm{w}}$ respectively. The corresponding infinite volume limits Φ_β^{f} and Φ_β^{w} always exist.

The phase transition of the random cluster model is characterized by the occurrence of percolation

$$\forall \beta > \beta_c, \qquad \lim_{N\to\infty} \Phi_{\beta,\Lambda_N}^{\mathrm{w}}\left(0 \leftrightarrow \Lambda_N^c\right) = \Phi_\beta^{\mathrm{w}}(0 \leftrightarrow \infty) > 0. \tag{2.3}$$

2.3. Results and consequences

Our main result applies in dimension $d \geqslant 3$

Theorem 2.1. *We consider the Ising model in dimension $d \geqslant 3$. Then for any $\beta \neq \beta_c$*

$$\Phi_\beta^{\mathrm{f}}\big(\{0 \leftrightarrow \infty\}\big) = \Phi_\beta^{\mathrm{w}}\big(\{0 \leftrightarrow \infty\}\big). \tag{2.4}$$

The proof is postponed to Subsection 3.5 and we first draw some consequences from this Theorem.

• **Continuity of the average magnetization.**

Grimmett proved in [Gr] (Theorem 5.2) that the function $\beta \to \Phi_\beta^{\mathrm{w}}\big(\{0 \leftrightarrow \infty\}\big)$ is right continuous in $[0,1]$ and $\beta \to \Phi_\beta^{\mathrm{f}}\big(\{0 \leftrightarrow \infty\}\big)$ is left continuous in $[0,\infty[\setminus\{\beta_c\}$. Therefore Theorem 2.1 implies that the average magnetization

$$\mu_\beta^+(\sigma_0) = \Phi_\beta^{\mathrm{w}}\big(\{0 \leftrightarrow \infty\}\big) \tag{2.5}$$

is a continuous function of β except possibly at β_c.

• **Translation invariant states.**

According to Theorem 5.3 (b) in [Gr], equality (2.4) implies that there exists only one random cluster measure. This means that $\Phi_\beta^{\mathrm{w}} = \Phi_\beta^{\mathrm{f}}$ for $\beta \neq \beta_c$.

Alternatively for the spin counterpart, Lebowitz proved in [L1] (Theorem 3 and remark (iii) page 472) that the continuity of the average magnetization implies the existence of only two extremal invariant states, i.e. that for $\beta > \beta_c$ all the translation invariant Gibbs states are of the form $\lambda\mu_\beta^+ + (1-\lambda)\mu_\beta^-$ for some $\lambda \in [0,1]$.

• **Continuity of the energy density.**

Using Lebowitz's result [L1] (Theorem 3 and remark (iii) page 472), one obtains the continuity of the energy density wrt β, i.e. of $\mu_\beta^+(\sigma_i\sigma_j)$ for any pair of sites. More generally, for any finite set A, the function $\beta \to \mu_\beta^+(\sigma_A)$ is continuous where $\sigma_A = \prod_{i\in A}\sigma_i$.

• **Pisztora's coarse graining.**

A description of the Ising model close to the critical temperature requires a renormalization procedure in order to deal with the diverging correlation length.

A crucial tool for implementing this is the Pisztora's coarse graining [Pi] which provides an accurate description of the typical configurations of the Ising model (and more generally of the q-Potts model) in terms of the FK representation. This renormalization scheme is at the core of many works on the Ising model and in particular it was essential for the analysis of phase coexistence (see [C, CP, B1, BIV]).

The main features of the coarse graining will be recalled in Subsection 3.1. Nevertheless, we stress that its implementation is based upon two hypothesis:

(1) The inverse temperature β should be above the slab percolation threshold (see [Pi]).
(2) The uniqueness of the FK measure, i.e. $\Phi_\beta^{\mathrm{f}} = \Phi_\beta^{\mathrm{w}}$.

The first assumption was proved to hold for the Ising model as soon as $\beta > \beta_c$ [B2] and as a consequence of Theorem 2.1, the second is also valid for $\beta > \beta_c$. Thus for the Ising model, Pisztora's coarse graining applies in the whole of the phase transition regime and from [CP] the Wulff construction in dimension $d \geqslant 3$ is valid up to the critical temperature.

3. Proof of Theorem 2.1

Let us briefly comment on the structure of the proof. It is well known that the wired measure Φ_β^{w} dominates the free measure Φ_β^{f} in the FKG sense thus the core of the proof is to prove the reverse inequality. The first step is to show that Φ_β^{f} dominates the FK counterpart of the finite volume Gibbs measure $\mu_{\beta,\Lambda}^h$ for some value of $h > 0$ and independently of Λ. This is achieved by introducing intermediate random variables Z (Subsection 3.2) and $\widehat{Z}$ (Subsection 3.3) which can be compared thanks to a coupling (Subsection 3.4). We then rely on a result by Lebowitz [L2] and Messager, Miracle Sole, Pfister [MMP] which ensures that $\mu_{\beta,\Lambda}^h$ converges to μ_β^+ in the thermodynamic limit (see (3.12)). From this, we deduce that Φ_β^{f} dominates Φ_β^{w} in the FKG sense (Subsection 3.5).

Finally, we stress the fact that the proof is restricted to the Ising model because it relies on two specific features which are not known for more general q-Potts models

(1) The occurrence of percolation in a slab up to the critical temperature [B2].
(2) The convergence of $\mu_{\beta,\Lambda}^h$ to μ_β^+ in the thermodynamic limit for any $h > 0$ [L2, MMP].

Both assertions should be valid in the q-Potts model for $q > 1$ and any $\beta > \beta_c$.

The second statement has been studied for a broad class of models on trees by Pemantle, Steif [PS] and was referred to as a *robust phase transition*. Van Enter proved in [vE] that (2) fails at β_c for the q-Potts model with large q. This does not contradict the fact that assertion (2) should be correct for any $\beta > \beta_c$, nevertheless it indicates that the proof implemented in [L2, MMP] for the Ising model cannot be easily generalized as it is based on ferromagnetic inequalities which are valid for all β.

3.1. Renormalization

We recall the salient features of Pisztora's coarse graining and refer to the original paper [Pi] for the details. The reference scale for the coarse graining is an integer K which will be chosen large enough. The space $\mathbb{Z}^d$ is partitioned into blocks of side length K

$$\forall x \in K\,\mathbb{Z}^d, \qquad \mathbb{B}_K(x) = x + \left\{ -\frac{K}{2} + 1, \dots, \frac{K}{2} \right\}^d .$$

First of all we shall set up the notion of *good* block on the K-scale which characterizes a local equilibrium in a pure phase.

Definition 3.1. *A block $\mathbb{B}_K(x)$ is said to be good with respect to the bond configuration $\omega \in \Omega$ if the following events are satisfied*

(1) *There exists a crossing cluster $\mathbf{C}^*$ in $\mathbb{B}_K(x)$ connected to all the faces of the inner vertex boundary of $\mathbb{B}_K(x)$.*
(2) *Any FK-connected cluster in $\mathbb{B}_K(x)$ of diameter larger than $\sqrt{K}/10$ is contained in $\mathbf{C}^*$.*
(3) *There are crossing clusters in each block $\left(\mathbb{B}_{\sqrt{K}}(x \pm \frac{K}{2}\vec{e}_i)\right)_{1 \leqslant i \leqslant d}$, where $\left(\vec{e}_i\right)_{1 \leqslant i \leqslant d}$ are the unit vectors (see (4.2) in [Pi]).*
(4) *There is at least a closed bond in $\mathbb{B}_{K^{1/2d}}(x)$.*

The important fact which can be deduced from (1,2,3) is that the crossing clusters in two neighboring good blocks are connected. Thus a connected cluster of good blocks at scale K induces also the occurrence of a connected cluster at the microscopic level.

To each block $\mathbb{B}_K(x)$, we associate a coarse grained variable $u_K(x)$ equal to 1 if this is a good block or 0 otherwise. Fundamental techniques developed by Pizstora (see (4.15) in [Pi]) imply that a block is good with high probability conditionally to the states of its neighboring blocks. For any $\beta > \beta_c$, there is K_0 large enough such that for all scales $K \geqslant K_0$ one can find a constant $C > 0$ (depending on K, β) such that

$$\Phi^{\mathrm{f}}_\beta \left(u_K(x) = 0 \;\middle|\; u_K(y) = \eta_y, \quad y \neq x \right) \leqslant \exp(-C)\,, \tag{3.1}$$

this bound holds uniformly over the values $\eta_j \in \{0, 1\}$ of the neighboring blocks. Furthermore, the constant C diverges as K tends to infinity. The previous estimate was originally derived beyond the slab percolation threshold. The latter has been proved to coincide with the critical temperature in the case of the Ising model [B2].

A last feature of Pisztora's coarse graining is a control of the density of the crossing cluster in each good block. Under the assumption that (2.4) holds, one can prove that with high probability, the density of the crossing cluster in each block is close to the one of the infinite cluster. Thus, one of the goals of this paper is to prove that the complete renormalization scheme is valid up to the critical temperature. Throughout the paper, we will use only the estimate (3.1) and not the full Pisztora's coarse graining which includes as well the control on the density.

For $N = n\frac{K}{2}$, we define

$$\Lambda_N = \{-N+1, \dots, N\}^d, \qquad \partial\Lambda_N = \Lambda_{N+R} \setminus \Lambda_N\,, \tag{3.2}$$

where R is the interaction range. $\partial\Lambda_N$ will be partitioned into $(d-1)$-dimensional slabs of side length $L = \ell K$ (for some appropriate choice of n and ℓ). We define the slab

$$T_L = \{0, \dots, R\} \times \{-L/2+1, \dots, L/2\}^{d-1}$$

and $\Xi_{N,L}$ a subset of $\partial\Lambda_N$ such that $\partial\Lambda_N$ can be covered by non intersecting slabs with centers in $\Xi_{N,L}$

$$\partial\Lambda_N = \bigcup_{x\in\Xi_{N,L}} T_L(x)\,, \tag{3.3}$$

where $T_L(x)$ denotes the slab centered at site x and deduced from T_L by rotation and translation (see figure 1). More precisely, if $F^{(i)}$ is the face of Λ_N with outward normal the unit vector $\vec{e}_i$ we write

$$F^{(i)} = \{-N, N\}^{i-1} \times \{N+1\} \times \{-N, N\}^{d-i}\,.$$

The sites of $\Xi_{N,L}$ associated to the face $F^{(i)}$ are

$$\Xi^{(i)}_{N,L} = \left\{ \bigcup_{\substack{j\in\mathbb{Z}^{i-1}\\ j'\in\mathbb{Z}^{d-i}}} \{Lj\} \times \{N+1\} \times \{Lj'\} \right\} \bigcap F^{(i)}\,.$$

For any x in $\Xi^{(i)}_{N,L}$ the corresponding slab is

$$T_L(x) = x + \{-L/2+1, \dots, L/2\}^{i-1} \times \{0, \dots, R\} \times \{-L/2+1, \dots, L/2\}^{d-i}\,.$$

The rest of the set $\Xi_{N,L}$ is obtained in the same way by symmetry.

3.2. Free boundary conditions

We define new random variables indexed by the set $\Xi_{N,L}$ introduced in (3.3).

Definition 3.2. *The collection $(Z_x)_{x\in\Xi_{N,L}}$ depends on the bond configurations in $\mathbb{E} \setminus \mathbb{E}^{\mathrm{f}}_{\Lambda_N}$. For any x in $\Xi_{N,L}$, we declare that $Z_x = 1$ if the three following events are satisfied (see figure 2)*

(1) *All the bonds in $\mathbb{E}^{\mathrm{w}}_{\Lambda_N} \setminus \mathbb{E}^{\mathrm{f}}_{\Lambda_N}$ intersecting $T_L(x)$ are open, as well as those in $\mathbb{E}^{\mathrm{f}}_{T_L(x)}$.*
(2) *If $\vec{n}$ denotes the outward normal to Λ_{N+1} at x then the $3K/4$ edges $\big\{\big(x + i\vec{n}, x + (i+1)\vec{n}\big)\big\}_{0\leqslant i\leqslant 3K/4}$ are open. Let y be the site $x + K\vec{n}$. Then $\mathbb{B}_K(y)$ is a good block, i.e. $u_K(y) = 1$.*

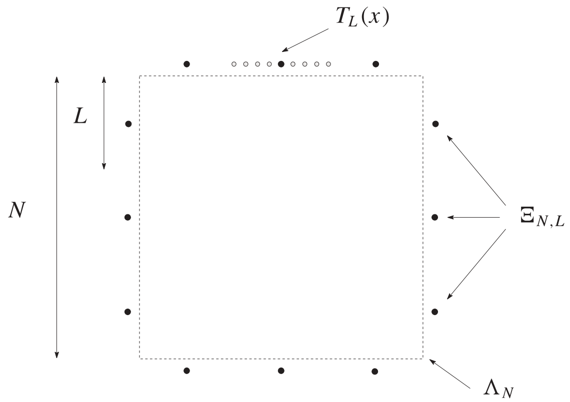

Fig. 1. The figure corresponds to the nearest neighbor Ising model. The scales are not accurate and one should imagine $1 \ll K \ll L \ll N$. The set Λ_N is depicted in dashed lines. The subset $\Xi_{N,L}$ is the union of the black dots which all belong to $\partial \Lambda_N$. Only one set $T_L(x)$ has been depicted at the top.

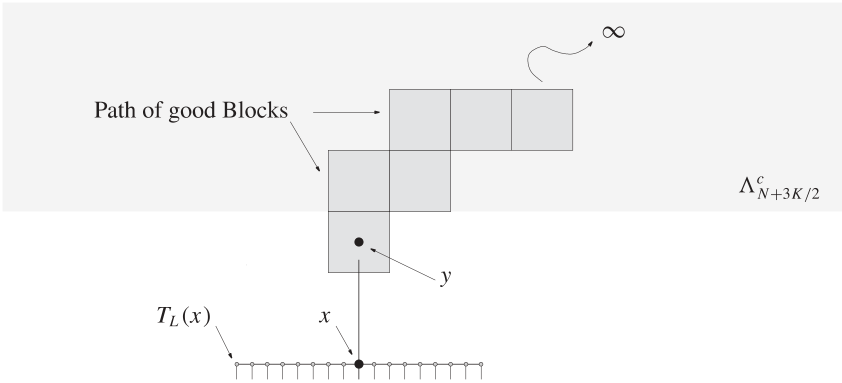

Fig. 2. The event $Z_x = 1$ is depicted (the scales are not accurate). The black lines are the open bonds attached to $T_L(x)$. The block $\mathbb{B}_K(y)$ is good and connected to infinity by a path of good blocks included in $\Lambda^c_{N+3K/2}$ (represented by the light gray region).

(3) *The block $\mathbb{B}_K(y)$ is connected to infinity by an open path of good blocks included in $\Lambda^c_{N+3K/2}$.*

If one of the events is not satisfied, then $Z_x = 0$.

Let $\mathbb{Q}$ be the image measure on $\{0,1\}^{\Xi_{N,L}}$ of Φ^{f}_β by the application $\omega \to \{Z_x(\omega)\}_{x \in \Xi_{N,L}}$.

It is convenient to order the sites of $\Xi_{N,L}$ wrt the lexicographic order and to index the random variables by $\{Z_k\}_{k \leqslant M}$, where M is the cardinality of $\Xi_{N,L}$. The k^{th} element x_k of $\Xi_{N,L}$ is associated to $Z_k = Z_{x_k}$.

We will associate to a given sequence $\{Z_k\}_{k \leqslant M}$ a random cluster measure in $\mathbb{E}^{\mathrm{f}}_{\Lambda_N}$ with boundary conditions which will be wired in the regions where $Z_k = 1$ and free otherwise. More precisely, $\partial\Lambda_N$ is split into two regions

$$\partial^{\mathrm{f}}\Lambda_N = \bigcup_{k \text{ such that } Z_k=0} T_L(x_k), \qquad \partial^{\mathrm{w}}\Lambda_N = \bigcup_{k \text{ such that } Z_k=1} T_L(x_k)\,.$$

We set

$$\forall (i,j) \in \mathbb{E}^{\mathrm{w}}_{\Lambda_N} \setminus \mathbb{E}^{\mathrm{f}}_{\Lambda_N}, \qquad \pi^Z_{(i,j)} = \begin{cases} 0, & \text{if } i \in \partial^{\mathrm{f}}\Lambda_N,\ j \in \Lambda_N\,, \\ 1, & \text{if } i \in \partial^{\mathrm{w}}\Lambda_N,\ j \in \Lambda_N\,. \end{cases} \tag{3.4}$$

Outside $\mathbb{E}^{\mathrm{w}}_{\Lambda_N}$ the boundary conditions will be wired and we set $\pi^Z_b = 1$ for b in $\mathbb{E} \setminus \mathbb{E}^{\mathrm{w}}_{\Lambda_N}$. Finally, let us introduce for the FK measure in $\mathbb{E}^{\mathrm{f}}_{\Lambda_N}$ with boundary conditions π^Z

$$\forall Z \in \{0,1\}^{\Xi_{N,L}}, \qquad \Psi(Z) = \Phi^{\pi^Z}_{\beta,\Lambda_N}\big(0 \leftrightarrow \partial^{\mathrm{w}}\Lambda_N\big)\,. \tag{3.5}$$

If $\partial^{\mathrm{w}}\Lambda_N$ is empty then $\Psi(Z) = 0$.

By construction, to any bond configuration ω outside $\mathbb{E}^{\mathrm{f}}_{\Lambda_N}$, one can associate a collection $\{Z_k(\omega)\}$ and a bond configuration $\pi^{Z(\omega)}$. Almost surely wrt Φ^{f}_β, the infinite cluster is unique for any $\beta > \beta_c$ [BK] and all the sites x_k such that $Z_k = 1$ belong to the same cluster. Thus the following FKG domination holds

$$\Phi^{\omega}_{\beta,\Lambda_N} \succ \Phi^{\pi^{Z(\omega)}}_{\beta,\Lambda_N}, \qquad \omega - a.s \text{ wrt } \Phi^{\mathrm{f}}_\beta.$$

As the event $\{0 \leftrightarrow \infty\}$ is increasing, we get

$$\Phi^{\mathrm{f}}_\beta(0 \leftrightarrow \infty) \geqslant \mathbb{Q}\left(\Psi(Z)\right)\,. \tag{3.6}$$

We claim that for an appropriate choice of the parameters K, L the collection of variables $\{Z_k\}$ dominates a product measure

Proposition 3.1. *There exists K, L, N_0 and $\alpha > 0$ such that for $N \geqslant N_0$*

$$\forall k \leqslant M, \qquad \mathbb{Q}\left(Z_k = 1 \big| \, Z_j = \eta_j, \quad j \leqslant k-1\right) \geqslant \alpha\,,$$

for any collection of variables $\{\eta_j\}_{j \leqslant M}$ taking values in $\{0,1\}^M$.

The proof is postponed to Section 4.

3.3. Wired boundary conditions

Following the previous Subsection, we are going to define another type of random variables which are related to the wired FK measure. The FK counterpart of the Gibbs measure μ^h_{β,Λ_N} with boundary magnetic field $h > 0$ is denoted by $\Phi^{s,\mathrm{w}}_{\beta,\Lambda_N}$ and is defined as the wired FK measure in $\mathbb{E}^{\mathrm{w}}_{\Lambda_N}$ for which a bond (i, j) in $\mathbb{E}^{\mathrm{w}}_{\Lambda_N} \setminus \mathbb{E}^{\mathrm{f}}_{\Lambda_N}$ has intensity $s_{(i,j)} = 1 - \exp(-2hJ(i-j))$ instead of $p_{(i,j)}$. The intensities of the bonds in $\mathbb{E}^{\mathrm{f}}_{\Lambda_N}$ remain as defined in Subsection 2.2.

Using the notation of Definition 3.2, we introduce new random variables indexed by the set $\Xi_{N,L}$.

Definition 3.3. *For any x in $\Xi_{N,L}$, we declare that $\widehat{Z}_x = 1$ if there exists at least one open bond in $\mathbb{E}^{\mathrm{w}}_{\Lambda_N} \setminus \mathbb{E}^{\mathrm{f}}_{\Lambda_N}$ joining $T_L(x)$ to Λ_N. Otherwise we set $\widehat{Z}_x = 0$.*

Let $\widehat{\mathbb{Q}}$ be the image measure on $\{0, 1\}^{\Xi_{N,L}}$ of $\Phi^{s,\mathrm{w}}_{\beta,\Lambda_N}$ by the application $\omega \to \{\widehat{Z}_x(\omega)\}$.

As in the previous Subsection, the random variables $\{\widehat{Z}_k = \widehat{Z}(x_k)\}_{k \leqslant M}$ are ordered wrt the lexicographic order in $\Xi_{N,L}$.

To any bond configuration ω in $\mathbb{E}^{\mathrm{w}}_{\Lambda_N} \setminus \mathbb{E}^{\mathrm{f}}_{\Lambda_N}$, one associates two types of boundary conditions: $\pi^{\widehat{Z}(\omega)}$ which is defined as in (3.4) and

$$\forall b \notin \mathbb{E}^{\mathrm{f}}_{\Lambda_N}, \qquad \pi^{\omega}_b = \begin{cases} \omega_b, & \text{if } b \in \mathbb{E}^{\mathrm{w}}_{\Lambda_N} \setminus \mathbb{E}^{\mathrm{f}}_{\Lambda_N}\,, \\ 1, & \text{otherwise}\,. \end{cases} \tag{3.7}$$

Thus the following FKG domination holds $\pi^{\widehat{Z}(\omega)} \succ \pi^{\omega}$ and conditionally to the bond configuration outside $\mathbb{E}^{\mathrm{f}}_{\Lambda_N}$

$$\Psi(\widehat{Z}(\omega)) \geqslant \Phi^{\pi^{\omega}}_{\beta,\Lambda_N}\big(0 \leftrightarrow \partial\Lambda_N\big)\,,$$

where Ψ was introduced in (3.5). This leads to

$$\widehat{\mathbb{Q}}\big(\Psi(\widehat{Z})\big) \geqslant \Phi^{s,\mathrm{w}}_{\beta,\Lambda_N}\big(0 \leftrightarrow \partial\Lambda_N\big)\,. \tag{3.8}$$

Finally, we check that uniformly in N the variables $\{\widehat{Z}_k\}$ satisfy

Proposition 3.2. *For any collection of variables $\{\eta_j\}_{j \leqslant M}$ taking values in $\{0, 1\}^M$*

$$\forall k \leqslant M, \qquad \widehat{\mathbb{Q}}\left(\widehat{Z}_k = 1 \middle|\, \widehat{Z}_j = \eta_j, \quad j \leqslant k-1\right) \leqslant C_R L^{d-1} s_h\,,$$

where $s_h = \max s_{(i,j)}$ and C_R is a constant depending only on the interaction range.

Proof. For a given $k \leqslant M$, the variable $\widehat{Z}_k$ is an increasing function supported only by the set of bonds joining $T_L(x_k)$ to Λ_N which we denote by $\mathcal{T}_k$.

$$\begin{aligned} &\widehat{\mathbb{Q}}\left(\widehat{Z}_k = 1 \middle|\, \widehat{Z}_j = \eta_j \quad j \leqslant k-1\right) \\ &\quad = \Phi^{s,\mathrm{w}}_{\beta,\Lambda_N}\left(\exists \text{ an open bond in } \mathcal{T}_k \middle|\, \widehat{Z}_j = \eta_j \quad j \leqslant k-1\right) \\ &\quad \leqslant \sum_{b \in \mathcal{T}_k} \Phi^{s,\mathrm{w}}_{\beta,\Lambda_N}\left(\omega_b \middle|\, \widehat{Z}_j = \eta_j \quad j \leqslant k-1\right)\,. \end{aligned}$$

By construction the support $\mathcal{T}_k$ of $\widehat{Z}_k$ is disjoint from any other $\mathcal{T}_{k'}$, so that each bond b in $\mathcal{T}_k$ is open with intensity at most s_h uniformly wrt the bond configurations in the rest of the graph $\mathbb{E}^{\mathrm{w}}_{\Lambda_N}$ (see e.g. equation (3.10) in [Gr]). As the total number of bonds in $\mathcal{T}_k$ is less than $C_R L^{d-1}$, the Proposition follows. □

3.4. The coupling measure

We are going to define a joint measure $\mathbb{P}$ for the variables $\{Z_k, \widehat{Z}_k\}_{k \leqslant M}$. The coupling will be such that

$$\mathbb{P}\ a.s. \quad \{Z_k\} \succ \{\widehat{Z}_k\}, \quad \text{i.e.} \quad \mathbb{P}\left(\left\{Z_k \geqslant \widehat{Z}_k, \quad \forall k \leqslant M\right\}\right) = 1\,, \tag{3.9}$$

and the marginals coincide with $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$, i.e. for any function ϕ in $\{0,1\}^{\Xi_{N,L}}$

$$\mathbb{P}\big(\phi(Z)\big) = \mathbb{Q}\big(\phi(Z)\big) \qquad \text{and} \qquad \mathbb{P}\big(\phi(\widehat{Z})\big) = \widehat{\mathbb{Q}}\big(\phi(\widehat{Z})\big)\,. \tag{3.10}$$

Proposition 3.3. *There exists K, L and $h > 0$ such that for any N large enough, one can find a coupling $\mathbb{P}$ satisfying the conditions* (3.9) *and* (3.10).

Proof. The existence of the coupling is standard and follows from Propositions 3.1 and 3.2. First choose K, L large enough such that Proposition 3.1 holds and then fix h such that $\alpha > C_R L^{d-1} s_h$. The coupling $\mathbb{P}$ is defined recursively. Suppose that the first $k \leqslant M-1$ variables $\mathcal{Z}_k = \{Z_i\}_{i \leqslant k}$, $\widehat{\mathcal{Z}}_k = \{\widehat{Z}_i\}_{i \leqslant k}$ are fixed such that

$$\forall i \leqslant k, \qquad Z_i \geqslant \widehat{Z}_i\,.$$

We define

$$\begin{cases} \mathbb{P}\big(Z_{k+1} = 1, \widehat{Z}_{k+1} = 0\,\big|\mathcal{Z}_k, \widehat{\mathcal{Z}}_k\big) = \mathbb{Q}\left(Z_{k+1} = 1\,\big|\mathcal{Z}_k\right) - \widehat{\mathbb{Q}}\big(\widehat{Z}_{k+1} = 1\,\big|\widehat{\mathcal{Z}}_k\big)\,, \\ \mathbb{P}\big(Z_{k+1} = 1, \widehat{Z}_{k+1} = 1\,\big|\mathcal{Z}_k, \widehat{\mathcal{Z}}_k\big) = \widehat{\mathbb{Q}}\big(\widehat{Z}_{k+1} = 1\,\big|\widehat{\mathcal{Z}}_k\big)\,, \\ \mathbb{P}\big(Z_{k+1} = 0, \widehat{Z}_{k+1} = 0\,\big|\mathcal{Z}_k, \widehat{\mathcal{Z}}_k\big) = \mathbb{Q}\big(Z_{k+1} = 0\,\big|\widehat{\mathcal{Z}}_k\big)\,. \end{cases}$$

Thanks to Propositions 3.1 and 3.2 the measure is well defined and one can check that the conditions (3.9) and (3.10) are fulfilled. □

3.5. Conclusion

For $\beta < \beta_c$ Theorem 2.1 holds (see Theorem 5.3 (a) in [Gr]), thus we focus on the case $\beta > \beta_c$. As the wired FK measure dominates the free FK measure in the FKG sense, it is enough to prove

$$\Phi^{\mathrm{f}}_\beta\big(\{0 \leftrightarrow \infty\}\big) \geqslant \Phi^{\mathrm{w}}_\beta\big(\{0 \leftrightarrow \infty\}\big)\,. \tag{3.11}$$

Let us first fix K, L, h such that Proposition 3.3 holds. From (3.6) and (3.10)

$$\Phi^{\mathrm{f}}_\beta\big(\{0 \leftrightarrow \infty\}\big) \geqslant \mathbb{Q}\left(\Psi(Z)\right) = \mathbb{P}\left(\Psi(Z)\right)\,.$$

As Ψ is an increasing function, we get from (3.9)

$$\mathbb{P}\left(\Psi(Z)\right) \geqslant \mathbb{P}\left(\Psi(\widehat{Z})\right)\,.$$

Finally from (3.10) and (3.8) we conclude that

$$\mathbb{P}\left(\Psi(\widehat{Z})\right) = \widehat{\mathbb{Q}}\left(\Psi(\widehat{Z})\right) \geqslant \Phi^{s,\mathrm{w}}_{\beta,\Lambda_N}\left(0 \leftrightarrow \partial\Lambda_N\right).$$

Thus the previous inequalities imply that for any N large enough

$$\Phi^{\mathrm{f}}_{\beta}\left(\{0 \leftrightarrow \infty\}\right) \geqslant \Phi^{s,\mathrm{w}}_{\beta,\Lambda_N}\left(0 \leftrightarrow \partial\Lambda_N\right) = \mu^{h}_{\beta,\Lambda_N}(\sigma_0)\,,$$

where μ^h_{β,Λ_N} denotes the Gibbs measure with boundary magnetic field $h = -\frac{1}{2}\log(1-s)$. It was proven by Lebowitz [L2] and Messager, Miracle Sole, Pfister [MMP] that for any $h > 0$

$$\lim_{N\to\infty} \mu^h_{\beta,\Lambda_N}(\sigma_0) = \mu^+_\beta(\sigma_0)\,. \tag{3.12}$$

Therefore the correspondence between the Ising model and the FK representation (2.5) completes the derivation of inequality (3.11).

4. Proof of Proposition 3.1

For any k, we write $Z_k = Z_{x_k} = X_k Y_k$, where the random variables X_k and Y_k are defined as follows

- $X_k = 1$ if and only if the conditions (1) and (2) of Definition 3.2 are both satisfied. Otherwise $X_k = 0$.
- $Y_k = 1$ if and only if the condition (3) of Definition 3.2 is satisfied. Otherwise $Y_k = 0$.

For any collection of variables $\{\eta_j\}_{j\,\leqslant\, M}$ taking values in $\{0, 1\}^M$, we set

$$\mathcal{C} = \left\{Z_j = \eta_j, \quad j \leqslant k-1\right\}\,.$$

We are going to prove that for K, L large enough there exists $c_1, c_2 \in [0, 1[$ (depending on K, L) such that

$$\mathbb{Q}\left(X_k = 0\middle|\, \mathcal{C}\right) \leqslant c_1\,, \tag{4.1}$$

$$\mathbb{Q}\left(X_k = 1, Y_k = 0\middle|\, \mathcal{C}\right) \leqslant c_2\mathbb{Q}\left(X_k = 1\middle|\, \mathcal{C}\right)\,. \tag{4.2}$$

Proposition 3.1 is a direct consequence of the previous inequalities. First we write

$$\mathbb{Q}\left(Z_k = 0\middle|\, \mathcal{C}\right) = \mathbb{Q}\left(X_k = 0\middle|\, \mathcal{C}\right) + \mathbb{Q}\left(X_k = 1, Y_k = 0\middle|\, \mathcal{C}\right)\,.$$

Using (4.2) and (4.1)

$$\mathbb{Q}\left(Z_k = 0\middle|\, \mathcal{C}\right) \leqslant 1 - (1-c_2)\mathbb{Q}\left(X_k = 1\middle|\, \mathcal{C}\right) \leqslant 1 - (1-c_2)(1-c_1)\,.$$

Thus for K, L large enough there is $\alpha > 0$ such that

$$\mathbb{Q}\left(Z_k = 1\middle|\, \mathcal{C}\right) \geqslant \alpha\,.$$

Proof of (4.1). The counterpart for x_k of the site y in Definition 3.2 is denoted by y_k. The event $X_k = 1$ requires first of all that

- All the bonds in $\mathbb{E}^{\mathrm{w}}_{\Lambda_N} \setminus \mathbb{E}^{\mathrm{f}}_{\Lambda_N}$ intersecting $T_L(x)$ are open, as well as those in $\mathbb{E}^{\mathrm{f}}_{T_L(x)}$.
- The $3K/4$ edges $\{(x_k + i\vec{n}, x_k + (i+1)\vec{n})\}_{0 \leqslant i \leqslant 3K/4}$ are open, where $\vec{n}$ denotes the outward normal to Λ_{N+1} at x_k.

Let $\mathcal{A}$ be the intersection of both events. The support of $\mathcal{A}$ is disjoint from the support of $\mathcal{C}$, so that $\mathcal{A}$ can be satisfied with a positive probability depending on K and L but not on $\mathcal{C}$ or N.

It remains to check that conditionally to $\mathcal{A} \cap \mathcal{C}$, the block $\mathbb{B}_K(y)$ is good with a positive probability depending on K. Notice that this statement is not a direct consequence of (3.1) because $\mathcal{A}$ cannot be expressed in terms of the coarse grained variables. Nevertheless $\mathcal{A}$ is an increasing event and Pisztora proved in Proposition 4.1 (page 452) of [Pi] that (3.1) remains true also after conditioning wrt some increasing event.

Combining the previous statements, we deduce that (4.1) holds with a constant $c_1 < 1$. □

Proof of (4.2). Let y_k be the counterpart of the site y in Definition 3.2. If $Y_k = 0$ then there exists Γ a contour of bad blocks in $\Lambda^c_{N+3K/2}$ disconnecting y_k from infinity (see (3) of Definition 3.2). More precisely, we define the contour Γ as follows. Let $\mathfrak{C}$ be the maximal connected component of good blocks in $\Lambda^c_{N+3K/2}$ connected to $\mathbb{B}_K(y_k)$. If $Y_k = 0$, $\mathfrak{C}$ is finite and γ is defined as the support of the maximal $\star$-connected component of bad blocks in $\Lambda^c_{N+3K/2}$ which intersects the external boundary of $\mathfrak{C}$ or simply intersects the block connected to $\mathbb{B}_K(y_k)$ if $\mathfrak{C}$ is empty. By construction the boundary of γ, denoted by $\partial\gamma$, contains only good blocks. The contour Γ is defined as the intersection of the events Γ_0 and Γ_1, where the configurations in Γ_0 contain only bad blocks in γ and those in Γ_1 contain only good blocks in $\partial\gamma$ (see figure 3).

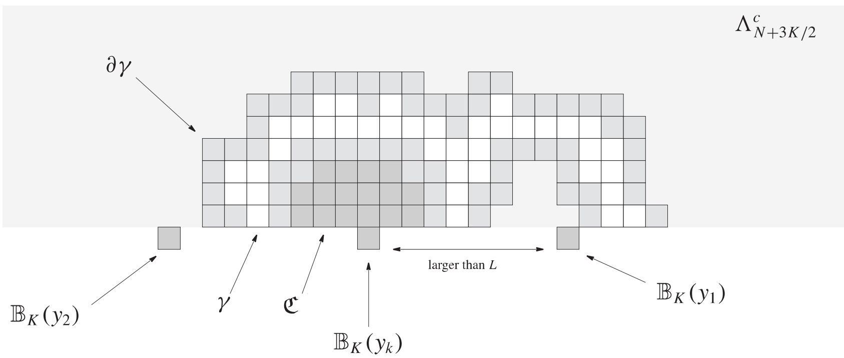

Fig. 3. The support of the contour Γ is $\gamma \cup \partial\gamma$ and is included in $\Lambda^c_{N+3K/2}$ (the light gray region). The blocks $\mathbb{B}_K(y_k)$ and $\mathbb{B}_K(y_1)$ are disconnected from infinity by Γ. The event $Y_2 = 1$ associated to the block $\mathbb{B}_K(y_2)$ is not determined by Γ.

We write

$$\mathbb{Q}\left(\{X_k = 1\} \cap \{Y_k = 0\} \cap \mathcal{C}\right) = \sum_{\Gamma} \Phi_\beta^{\mathrm{f}}\left(\Gamma \cap \{X_k = 1\} \cap \mathcal{C}\right) , \tag{4.3}$$

where the sum is over the contours in $\Lambda_{N+3K/2}^c$ surrounding y_k.

For a given Γ, we are going to prove

$$\Phi_\beta^{\mathrm{f}}\left(\Gamma \cap \{X_k = 1\} \cap \mathcal{C}\right) \leqslant \exp\left(-\frac{C}{2}|\Gamma|\right) \Phi_\beta^{\mathrm{f}}\left(\{X_k = 1\} \cap \mathcal{C}\right) , \tag{4.4}$$

where $C = C(K, \beta)$ was introduced in (3.1) and $|\Gamma|$ stands for the number of blocks in γ. For K large enough, the constant C can be chosen arbitrarily large so that the combinatorial factor arising by summing over the contours Γ in (4.3) remains under control. This implies that there exists $c_2 \in]0, 1[$ such that

$$\mathbb{Q}\left(\{X_k = 1\} \cap \{Y_k = 0\} \cap \mathcal{C}\right) \leqslant c_2 \, \Phi_\beta^{\mathrm{f}}\left(\{X_k = 1\} \cap \mathcal{C}\right) .$$

Thus the inequality (4.2) follows.

In order to prove (4.4), we specify the set $\mathcal{C}$ and for notational simplicity assume that it is of the form $\mathcal{C} = \mathcal{C}_0 \cap \mathcal{C}_1$ with

$$\mathcal{C}_0 = \left\{Z_j = 0, \quad j \leqslant k_0\right\}, \qquad \mathcal{C}_1 = \left\{Z_j = 1, \quad k_0 + 1 \leqslant j \leqslant k - 1\right\} .$$

The difficulty to derive (4.4) is that Γ may contribute to the event $\mathcal{C}_0$ so that a Peierls argument cannot be applied directly. For this reason we decompose $\mathcal{C}_0$ into 2^{k_0} disjoint sets for which the state of the first k_0 variables is prescribed such that either $\{X_j = 1, Y_j = 0\}$ or $\{X_j = 0\}$. Once again for simplicity we will only consider the subset $\mathcal{D} = \mathcal{D}_0 \cap \mathcal{D}_1$ of $\mathcal{C}_0$ such that

$$\mathcal{D}_0 = \left\{X_j = 1, Y_j = 0, \quad j \leqslant k_1\right\}, \qquad \mathcal{D}_1 = \left\{X_j = 0, \quad k_1 + 1 \leqslant j \leqslant k_0\right\} .$$

The derivation of (4.4) boils down to proving the estimate below

$$\Phi_\beta^{\mathrm{f}}\left(\Gamma \cap \{X_k = 1\} \cap \mathcal{D} \cap \mathcal{C}_1\right) \leqslant \exp\left(-\frac{C}{2}|\Gamma|\right) \Phi_\beta^{\mathrm{f}}\left(\{X_k = 1\} \cap \mathcal{D} \cap \mathcal{C}_1\right) . \tag{4.5}$$

Finally, we suppose that $\mathcal{D}_0$ is such that the first k_2 sites $\{y_j\}_{j \leqslant k_2}$ are disconnected from infinity by Γ and the others $k_1 - k_2$ are not surrounded by Γ (see figure 3). Notice that erasing the contour Γ may affect the state of the first k_2 sites, but not of the other $k_1 - k_2$. By construction, if $\mathcal{E} = \left\{X_j = 1, Y_j = 0, \quad k_2 + 1 \leqslant j \leqslant k_1\right\}$, then

$$\begin{aligned} &\Phi_\beta^{\mathrm{f}}\left(\Gamma \cap \{X_k = 1\} \cap \mathcal{D} \cap \mathcal{C}_1\right) \\ &= \Phi_\beta^{\mathrm{f}}\left(\Gamma \cap \{X_k = 1\} \cap \left\{X_j = 1, \; j \leqslant k_2\right\} \cap \mathcal{E} \cap \mathcal{D}_1 \cap \mathcal{C}_1\right) . \end{aligned}$$

Conditionally to Γ_1, all the events in the RHS are independent of Γ_0 so that by conditioning wrt the configurations in $\partial\gamma$, one can apply the Peierls bound (3.1)

$$\begin{aligned}&\Phi^{\mathrm{f}}_{\beta}\left(\Gamma\cap\{X_k=1\}\cap\mathcal{D}\cap\mathcal{C}_1\right)\\ &\leqslant\exp\left(-C|\Gamma|\right)\ \Phi^{\mathrm{f}}_{\beta}\left(\Gamma_1\cap\{X_k=1\}\cap\left\{X_j=1,\quad j\leqslant k_2\right\}\cap\mathcal{E}\cap\mathcal{D}_1\cap\mathcal{C}_1\right).\end{aligned}$$

By modifying the bonds around each block $\mathbb{B}_K(y_j)$ one can recreate the events $\{Y_j=0\}_{j\leqslant k_2}$ and thus $\mathcal{D}$. First of all notice that Γ_1 screens the blocks $\mathbb{B}_K(y_j)$ from the other events in the RHS. Thus one can turn the blocks in $\Lambda^c_{N+3K/2}$ connected to each site $\{y_j\}_{j\leqslant k_2}$ into bad blocks without affecting the event below

$$\{X_k=1\}\cap\left\{X_j=1,\quad j\leqslant k_2\right\}\cap\mathcal{E}\cap\mathcal{D}_1\cap\mathcal{C}_1.$$

For each block, this has a cost α_K depending only on K (and β)

$$\Phi^{\mathrm{f}}_{\beta}\left(\Gamma\cap\{X_k=1\}\cap\mathcal{D}\cap\mathcal{C}_1\right)\leqslant\exp\left(-C|\Gamma|\right)\left(\alpha_K\right)^{k_2}\Phi^{\mathrm{f}}_{\beta}\left(\{X_k=1\}\cap\mathcal{D}\cap\mathcal{C}_1\right).$$

By construction, the distance between each site $\{y_j\}_{j\leqslant k_2}$ is at least $L=\ell K$. The contour Γ surrounds k_2 sites in $\Xi_{N,L}$ so that $|\Gamma|$ must be larger than ℓk_2 (see figure 3). Therefore for ℓ large enough, the Peierls bound compensates the cost $\left(\alpha_K\right)^{k_2}$

$$\Phi^{\mathrm{f}}_{\beta}\left(\Gamma\cap\{X_k=1\}\cap\mathcal{D}\cap\mathcal{C}_1\right)\leqslant\exp\left(-\frac{C}{2}|\Gamma|\right)\Phi^{\mathrm{f}}_{\beta}\left(\{X_k=1\}\cap\mathcal{D}\cap\mathcal{C}_1\right).$$

This completes (4.5). Similar results would be valid for any decomposition of the set $\mathcal{C}$. In particular $\mathcal{C}_0$ can be represented as the disjoint union of the type $\mathcal{C}_0=\bigvee_{\mathcal{D}_0,\mathcal{D}_1}\mathcal{D}_0\cap\mathcal{D}_1$, thus summing over the sets $\mathcal{D}$, we derive (4.4). □

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