Probab. Theory Relat. Fields 134, 565–602 (2006)
Digital Object Identifier (DOI) 10.1007/s00440-005-0448-1

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On symmetric random walks with random conductances on $\mathbb{Z}^d$

Received: 14 March 2004 / Revised version: 3 January 2005 /
Published online: 14 July 2005 –

Abstract. We study models of continuous time, symmetric, $\mathbb{Z}^d$-valued random walks in random environments. One of our aims is to derive estimates on the decay of transition probabilities in a case where a uniform ellipticity assumption is absent. We consider the case of independent conductances with a polynomial tail near 0 and obtain precise asymptotics for the annealed return probability and convergence times for the random walk confined to a finite box.

1. Introduction

We study continuous time, irreducible, symmetric, nearest neighbor random walks in random environments on $\mathbb{Z}^d$. Our aim is to derive estimates on the decay of transition probabilities in the absence of a uniform ellipticity assumption.

The paper has four sections (other than this introduction). Sections 2 and 4 deal with the decay of the mean or annealed return probability. In Section 2, we consider quite general reversible random walks in a random environment and we establish a comparison lemma for the annealed return probability. The proof is based on a trace formula (in fact an extension of the trace formula for central probability for random walks on amenable groups, see [9]). In Section 4, we derive sharp bounds on the decay of the annealed return probability from direct investigation of traces and eigenvalues when the rates are i.i.d. random variables chosen from a law with polynomial tail near 0. We then prove that one might get the classical $t^{-d/2}$ decay or a slower decay of the form $t^{-\gamma}$, where $\gamma < d/2$ is related to the tail of the law of the rates near 0. In Section 5 we deal with the quenched decay and obtain a partial result (Theorem 5.1) that nonetheless establishes a difference with respect to the annealed decay for small values of γ.

In Section 3, we discuss finite volume random walks taking their values in a torus. We obtain some quenched estimates on convergence times when the random rates are i.i.d., chosen from a law with polynomial tail near 0. These follow from sharp bounds on the spectral gap. In particular we prove a universal lower bound for the spectral gap of a symmetric random walk on a torus of side length N (Prop-

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osition 3.14 below) which allows to separate the effects of the usual diffusive N^{-2} factor and the contribution of small values of the rates.

The paper is written in such a way to ease independent reading of the different parts at the cost of some repetition. Sections 2 and 3 are self-contained; only the spectral gap lower bound (3.12) from Section 3 is needed to proceed through Section 4.

2. A comparison lemma for the annealed return probability

We study a family of symmetric, irreducible, nearest neighbors Markov chains taking their values in $\mathbb{Z}^d$ and constructed in the following way. Let Ω be the set of functions $\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}_+$ such that $\omega(x, y) > 0$ iff $x \sim y$, and $\omega(x, y) = \omega(y, x)$. ($y \sim x$ means that x and y are nearest neighbors.) We call elements of Ω environments.

Define the Markov generator

$$\mathcal{G}^\omega f(x) = \sum_{y \sim x} \omega(x, y)\, [f(y) - f(x)]. \tag{2.1}$$

As usual, $\{X_t,\ t \in \mathbb{R}_+\}$ will be the coordinate process on path space $(\mathbb{Z}^d)^{\mathbb{R}_+}$ and we use the notation $\mathbb{P}^\omega_x$ to denote the unique probability measure on path space under which $\{X_t,\ t \in \mathbb{R}_+\}$ is the Markov process generated by (2.1) and satisfying $X_0 = x$. Under $\mathbb{P}^\omega_x$, $X_0 = x$; then the process waits for an exponentially distributed random time of parameter $\sum_{y \sim x} \omega(x, y)$ and jumps to point x_1 with probability $\omega(x, x_1)/ \sum_{y \sim x} \omega(x, y)$; this procedure is then iterated choosing independent hopping times. Equivalently, one can define $\mathbb{P}^\omega_x$ using the theory of symmetric Dirichlet forms, see [4]. The reference space is then $L^2(\mathbb{Z}^d)$, equipped with the counting measure. For functions f and g with finite support, let

$$\mathcal{D}^\omega(f, g) = \frac{1}{2} \sum_{x \sim y \in \mathbb{Z}^d} \omega(x, y)\, [f(x) - f(y)]\, [g(x) - g(y)].$$

The bilinear form $\mathcal{D}^\omega$ is closable and its closure is a regular, symmetric Dirichlet form. Thus, there exists a Hunt process associated to $\mathcal{D}^\omega$. Note that points have non zero capacity. Therefore, the measure $\mathbb{P}^\omega_x$ is uniquely determined by $\mathcal{D}^\omega$. It is easy to prove that both constructions yield the same law $\mathbb{P}^\omega_x$.

Since $\omega(x, y) > 0$ for all neighboring pairs (x, y), X_t is irreducible under $\mathbb{P}^\omega_x$ for all x. The counting measure on $\mathbb{Z}^d$ is reversible because we have assumed that $\omega(x, y) = \omega(y, x)$.

We now choose the rates ω at random, according to a translation invariant law $\mathbb{Q}$ on Ω.

In the sequel $\mathbb{Q}.\mathbb{P}^\omega_x$ will be used as a short hand notation for the annealed law defined by $\mathbb{Q}.\mathbb{P}^\omega_x[\,\cdot\,] = \int P^\omega_x[\,\cdot\,]\, d\mathbb{Q}(\omega)$. Note that X_t is Markov under $\mathbb{P}^\omega_x$ for any ω, but is not Markov anymore under $\mathbb{Q}.\mathbb{P}^\omega_x$ for nontrivial $\mathbb{Q}$. Let $\mathbb{P}^\omega = \mathbb{P}^\omega_0$ and $\mathbb{Q}.\mathbb{P}^\omega = \mathbb{Q}.\mathbb{P}^\omega_0$.

We are interested in estimating the decay of the annealed return probability $\mathbb{Q}.\mathbb{P}^\omega[X_t = 0]$, as t tends to $+\infty$.

As a subset of $(\mathbb{R}_+)^{\mathbb{Z}^d\times\mathbb{Z}^d}$, Ω is a partially ordered set. By duality, one can define a partial order on the set of probabilities on Ω in the following way. Given two probabilities, $\mathbb{Q}$ and $\mathbb{Q}'$, we say that $\mathbb{Q}' \geq \mathbb{Q}$ if, for any measurable, bounded, increasing function $f : \Omega \to \mathbb{R}$, we have $\mathbb{Q}'(f) \geq \mathbb{Q}(f)$. ($f$ is increasing if, whenever $\omega, \omega' \in \Omega$ satisfy $\omega'(x, y) \geq \omega(x, y)$ for all x, y, then $f(\omega') \geq f(\omega)$.)

Remark 2.1. The function $\omega \to \mathbb{P}^\omega[X_t = 0]$ is not monotonous in ω. It is clearly not increasing. It is also not very difficult to find subgraphs of $\mathbb{Z}^d$ for which the removal of an edge decreases the value of $\mathbb{P}^\omega[X_t = 0]$ (left as an exercise), which implies that the function $\omega \to \mathbb{P}^\omega[X_t = 0]$ is not decreasing.

Lemma 2.2. *Let $\mathbb{Q}$ and $\mathbb{Q}'$ be two probabilities on Ω such that $\mathbb{Q}' \geq \mathbb{Q}$. Assume that for $\mathbb{Q}' + \mathbb{Q}$-almost all ω, the Markov chain X_t is conservative under $\mathbb{P}^\omega$. Then, for all time t, we have*

$$\mathbb{Q}'.\mathbb{P}^\omega[X_t = 0] \leq \mathbb{Q}.\mathbb{P}^\omega[X_t = 0].$$

Proof. We prove that $\mathbb{Q}.\mathbb{P}^\omega[X_t = 0]$ can be written as a supremum of the $\mathbb{Q}$-expectation of decreasing in ω functions. More precisely, let $B_N = [-N, N]^d$ be the box centered at the origin and of radius N. Let $\mathcal{G}^{\omega,N}$ be the restriction of the operator $\mathcal{G}^\omega$ to B_N with Dirichlet boundary conditions outside B_N (that is, $\mathcal{G}^{\omega,N}$ is the generator of the process which coincides with the one given by $\mathcal{G}^\omega$ until the latter process leaves B_N for the first time, and then it is killed). Then $-\mathcal{G}^{\omega,N}$ is a positive symmetric operator. Let $\{\mu_i^\omega(B_N), i \in [1, \#B_N]\}$ be the set of its eigenvalues labeled in increasing order. We shall prove that

$$\mathbb{Q}.\mathbb{P}^\omega[X_t = 0] = \sup_N \frac{1}{\#B_N}\mathbb{Q}\left[\sum_i e^{-\mu_i^\omega(B_N)t}\right]. \tag{2.2}$$

Let

$$\begin{aligned}\mathcal{E}^{\omega,N}(f, g) = \frac{1}{2}\sum_{\substack{x\sim y\\ x,y\in B_N}} \omega(x, y)\,[f(x) - f(y)]\,[g(x) - g(y)]\\ + \sum_{x\in B_N} f(x)g(x)\sum_{\substack{y\sim x\\ y\notin B_N}} \omega(x, y)\end{aligned}$$

be the Dirichlet form of $-\mathcal{G}^{\omega,N}$. From the min-max caracterization of $\mu_i^\omega(B_N)$, we have

$$\mu_i^\omega(B_N) = \max_{f_1,\ldots,f_{i-1}} \min_f \frac{\mathcal{E}^{\omega,N}(f, f)}{\sum_{x\in B_N} f^2(x)},$$

where the 'max' is computed on choices of $i - 1$ functions defined on B_N and the 'min' is computed on functions f such that, for all $j \in \{1, \ldots, i - 1\}$, $\sum_{x\in B_N} f(x) f_j(y) = 0$. For any function f, $\mathcal{E}^{\omega,N}(f, f)$ is clearly increasing in ω, therefore for given N, and i, $\mu_i^\omega(B_N)$ is an increasing function of ω and $\sum_i e^{-\mu_i^\omega(B_N)t}$ is decreasing in ω. Thus (2.2) implies the lemma. □

Proof of (2.2). Let τ_N be the exit time of X_t outside B_N. Note that $\sum_i e^{-\mu_i^\omega(B_N)t}$ is just the trace of the semi-group of the process X_t killed when leaving the box B_N, i.e.,

$$\sum_i e^{-\mu_i^\omega(B_N)t} = \sum_{x\in B_N} \mathbb{P}_x^\omega[X_t = x; t < \tau_N].$$

We compute $\mathbb{Q}.\mathbb{P}_0^\omega[X_t = 0]$ using that, from the translation invariance of the probability $\mathbb{Q}$, we know that $\mathbb{Q}.\mathbb{P}_x^\omega[X_t = x]$ does not depend on x. Therefore

$$\begin{aligned}\mathbb{Q}.\mathbb{P}_0^\omega[X_t = 0] &= \frac{1}{\#B_N}\sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^\omega[X_t = x] \ge \frac{1}{\#B_N}\sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^\omega[X_t = x; t < \tau_N]\\ &= \frac{1}{\#B_N}\mathbb{Q}\left[\sum_i e^{-\mu_i^\omega(B_N)t}\right]\end{aligned}$$

proves the lower bound.

As far as the upper bound is now concerned, note that

$$\begin{aligned}\mathbb{Q}.\mathbb{P}_0^\omega[X_t = 0] &= \frac{1}{\#B_N}\sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^\omega[X_t = x]\\ &= \frac{1}{\#B_N}\sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^\omega[X_t = x; t < \tau_{N+k}]\\ &\quad+\frac{1}{\#B_N}\sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^\omega[X_t = x; t \ge \tau_{N+k}]\\ &\le \frac{1}{\#B_N}\sum_{x\in B_{N+k}} \mathbb{Q}.\mathbb{P}_x^\omega[X_t = x; t < \tau_{N+k}]\\ &\quad+\frac{1}{\#B_N}\sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^\omega[t \ge \tau_{N+k}].\end{aligned}$$

We have

$$\begin{aligned}\sum_{x\in B_{N+k}} \mathbb{Q}.\mathbb{P}_x^\omega[X_t = x; t < \tau_{N+k}] &= \mathbb{Q}\left[\sum_i e^{-\mu_i^\omega(B_{N+k})t}\right]\\ &\le \#B_{N+k}\, \sup_M \frac{1}{\#B_M}\,\mathbb{Q}\left[\sum_i e^{-\mu_i^\omega(B_M)t}\right].\end{aligned}$$

Let n_t be the number of jumps the process X_t performs by time t. For $x \in B_N$, under $\mathbb{P}_x^\omega$, $t \ge \tau_{N+k}$ implies that $n_t \ge k$. Therefore

$$\begin{aligned}\sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^\omega[t \ge \tau_{N+k}] &\le \sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^\omega[n_t \ge k]\\ &= \#B_N\, \mathbb{Q}.\mathbb{P}^\omega[n_t \ge k],\end{aligned}$$

using the translation invariance in the last equality.

So far, we have obtained the bound

$$\mathbb{Q}.\mathbb{P}_0^{\omega}[X_t = 0] \leq \frac{\#B_{N+k}}{\#B_N} \sup_M \frac{1}{\#B_M} \mathbb{Q}\left[\sum_i e^{-\mu_i^{\omega}(B_M)t}\right] + \mathbb{Q}.\mathbb{P}^{\omega}[n_t \geq k].$$

First let N tend to $+\infty$, then let k tend to $+\infty$ to deduce that

$$\mathbb{Q}.\mathbb{P}_0^{\omega}[X_t = 0] \leq \sup_M \frac{1}{\#B_M} \mathbb{Q}\left[\sum_i e^{-\mu_i^{\omega}(B_M)t}\right] + \mathbb{Q}.\mathbb{P}^{\omega}[n_t = +\infty].$$

Now the conservativeness assumption and the fact that there are no instantaneous points of X_t in $\mathbb{Z}^d$ imply that $\mathbb{P}^{\omega}[n_t = +\infty] = 0$ $\mathbb{Q}$-a.s. □

3. Times of convergence to equilibrium of random walks on the torus

Let S_N be the discrete, d-dimensional torus of side length N. When convenient, we consider S_N as a subset of $\mathbb{Z}^d$. We construct a family of Markov chains taking their values in S_N. Let $\omega : S_N \to \mathbb{R}_+^*$, and define the Markov generator

$$\mathcal{L}^{\omega,N} f(x) = \sum_{y\sim x} [\omega(x) \wedge \omega(y)]\,[f(y) - f(x)], \tag{3.1}$$

where the sum is over sites y which are nearest neighbors to x (relation that is denoted $y \sim x$). Let $\{X_t,\ t \in \mathbb{R}_+\}$ be the process with distribution $\mathbb{P}_x^{\omega,N}$ generated by (3.1) and the condition $X_0 = x$. Since $\omega(x) > 0$ for all x, X_t is ergodic under $\mathbb{P}_x^{\omega,N}$ for all x. The unique invariant probability measure is the uniform law, denoted by η_N. Furthermore, η_N is reversible.

We choose the family $\{\omega(x),\ x \in \mathbb{Z}^d\}$ i.i.d. according to a law $\mathbb{Q}$ on $(\mathbb{R}_+^*)^{\mathbb{Z}^d}$ such that

$$\omega(x) \leq 1 \text{ for all } x; \tag{3.2}$$

$$\mathbb{Q}(\omega(0) \leq a) \sim a^{\gamma} \text{ as } a \downarrow 0, \tag{3.3}$$

where $\gamma > 0$ is a parameter.

Remark 3.1. We note that this generator has the same form as $\mathcal{G}^{\omega}$ in (2.1) by making $\omega(x, y) = \omega(x) \wedge \omega(y)$, but for a process in finite volume. We could have defined ω on edges, instead of points, as in the previous section, with i.i.d. values for different edges, and the same technique would apply, with similar results, and heavier computation.

Remark 3.2. If $\omega(0)$ were a Bernoulli random variable, then we would have a random walk on a (independent, site) percolation cluster (provided we started in an occupied site). See [6].

Our main results refer to the following convergence time. For $\epsilon \in (0, 1)$, let

$$T_1^{\omega,N} = \inf\{t \geq 0 : \sup_{x\in S_N} \sup_{|f|\leq 1} |\mathbb{E}_x^{\omega,N}[f(X_t)] - \eta_N(f)| \leq \epsilon\} \tag{3.4}$$

where $\mathbb{E}_x^{\omega,N}$ is the expectation with respect to $\mathbb{P}_x^{\omega,N}$ and $\mathbb{E}_{\eta_N}^{\omega,N}(\cdot) = \int \mathbb{E}_x^{\omega,N}(\cdot)\, d\eta_N(x)$.

Theorem 3.3. *For all* $\gamma > 0$ *and* $\epsilon \in (0,1)$*, we have* $\mathbb{Q}$*-a.s.*

$$\lim_{N\to\infty} \frac{\log T_1^{\omega,N}}{\log N} = 2 \vee \frac{d}{\gamma}. \tag{3.5}$$

Remark 3.4. If $\omega(0)$ were bounded away from zero, that is, if $\omega(0) > c_1$ $\mathbb{Q}$-a.s. for some constant $c_1 > 0$, then $\limsup_{N\to\infty} N^{-2} T_1^{\omega,N} \le c_2$ $\mathbb{Q}$-a.s. for some constant $c_2 > 0$.

From now on, we shall drop the 'N' in some of our notation. For example, we use the short hand notation $S = S_N$.

One has to establish both lower and upper bounds:

$$\limsup_{N\to\infty} \frac{\log T_1^{\omega,N}}{\log N} \le 2 \vee \frac{d}{\gamma}, \tag{3.6}$$

$$\liminf_{N\to\infty} \frac{\log T_1^{\omega,N}}{\log N} \ge 2 \vee \frac{d}{\gamma}. \tag{3.7}$$

The lower bound (3.7) will be discussed in part 3.6. The proof of (3.6) is given below. The main step is a lower bound on the spectral gap of the random walk. We first start with some preliminary lemmata.

3.1. Preliminaries

In the next lemmas, for given $0 < \xi < 1$, we choose G as the largest connected component of the set $\{x : \omega(x) \ge \xi\}$ (following a deterministic order in case of ties) and we set $B = S \setminus G$.

Lemma 3.5. *For* $\xi > 0$*, there exists a number* $c_2(\xi)$ *such that* $c_2(\xi) \to 0$ *as* $\xi \to 0$ *and* $\mathbb{Q}$*-a.s.*

$$\limsup_{N\to\infty} \eta_N(B) \le c_2(\xi).$$

Lemma 3.6. *There exists a finite number* c_3 *depending only on* d *such that* $\mathbb{Q}$*-a.s., for all* N *large enough*

$$\inf_{x\in S} \omega(x) \ge N^{-c_3}.$$

In the proof below, we will see that c_3 can be taken as $\frac{d}{\gamma} + \epsilon$ for arbitrary $\epsilon > 0$.

Proof of Lemma 3.5. Consider the site percolation model on $\mathbb{Z}^d$ where a site x is occupied if $\omega(x) \ge \xi$. Let ξ_0 be positive and satisfy $\mathbb{Q}(\omega(x) \ge \xi_0) > p_c$, the critical density for the a.s. appearance of an infinite connected component C. Then, if $\xi < \xi_0$, C exists a.s. Let $\tilde{C}_N = C \cap \tilde{S}_N$, where $\tilde{S}_N$ is S_N viewed as a subset of $\mathbb{Z}^d$ (that is, without the *boundary identification*), say, $\tilde{S}_N = (-N/2, N/2]^d \cap \mathbb{Z}^d$. Let

C_N be $\tilde{C}_N$ viewed as a subset of the torus S_N (that is, with the boundary identification). Then, it follows by standard ergodicity arguments that $\lim_{N\to\infty} \eta_N(\tilde{C}_N) = \theta(\xi) := \mathbb{Q}(0 \in C)$ $\mathbb{Q}$-a.s. Since $\theta(\xi) \to 1$ as $\xi \to 0$ (a well known result [5]), the result would follow if C_N were connected, which it is not necessarily.

Consider then $\hat{C}_N := \tilde{C}_{N-\lfloor\sqrt{N}\rfloor}$. We claim that $\hat{C}_N$ is connected in $\tilde{S}_N$, and thus also in S_N, for all large enough N $\mathbb{Q}$-a.s. Indeed, in the event that $\hat{C}_N$ is not connected in $\tilde{S}_N$, there exist two sites at the boundary of $\tilde{S}_{N-\lfloor\sqrt{N}\rfloor}$ that are connected to the boundary of $\tilde{S}_N$ but are not connected to one another. This implies that there exists a site $\tilde{x}$ at the boundary of $\tilde{S}_N$ whose (occupied) cluster (in $\tilde{S}_N$) has a boundary (of vacant sites) of size at least $\lfloor\sqrt{N}\rfloor$. Now, the (bond) boundary of any finite cluster of a site in $\tilde{S}_N$ can be identified with a surface of *plaquettes* around the given site, each plaquette crossing orthogonally a boundary bond. For each such plaquette, there corresponds thus an inner occupied site and an outer vacant one. For a given such surface of plaquettes of size (total number of plaquettes) n, there is at least $n/(2d)$ distinct outer vacant sites (since a vacant site can not be adjacent to more than $2d$ sites[1]. In the case of $\tilde{x}$, the surface of plaquettes will intersect the boundary of $\tilde{S}_N$ in a closed curve. It will also have to cross the region between the boundaries of $\tilde{S}_N$ and $\tilde{S}_{N-\lfloor\sqrt{N}\rfloor}$. For this reason it will contain at least $\lfloor\sqrt{N}\rfloor$ plaquettes.

From the arguments in the latter paragraph, we get the following estimate.

$$\begin{aligned}&\mathbb{Q}(\hat{C}_N \text{ is not connected in } \tilde{S}_N)\\ &\quad\le \sum_{\tilde{x}\in\partial\tilde{S}_N}\ \sum_{\substack{\Gamma \text{ around } x:\\ \#\Gamma\ge\lfloor\sqrt{N}\rfloor}} \mathbb{Q}(\text{all the sites at the outer boundary of } \Gamma \text{ are vacant}), \qquad (3.8)\end{aligned}$$

where the latter sum above is over surfaces of plaquettes Γ around $\tilde{x}$. The number of distinct such surfaces which have size n can be estimated to be exponential in n [10]. Proceeding with the estimation we get that the right hand side of (3.8) equals

$$\begin{aligned}&\sum_{\tilde{x}\in\partial\tilde{S}_N}\ \sum_{n\ge\lfloor\sqrt{N}\rfloor}\ \sum_{\substack{\Gamma \text{ around } x:\\ |\Gamma|=n}} \mathbb{Q}(\text{all the sites at the outer boundary of } \Gamma \text{ are vacant})\\ &\quad\le N^d \sum_{n\ge\lfloor\sqrt{N}\rfloor} e^{\nu n}[\mathbb{Q}(\omega(0)<\xi)]^{n/(2d)},\end{aligned}$$

where ν depends only on d. Thus, by taking $0<\xi<\xi_0$ small enough, the probability in the left hand side of (3.8) can be made summable and the claim at the beginning of the previous paragraph follows by Borel-Cantelli. The lemma then follows. □

Proof of Lemma 3.6. We will prove that $\mathbb{Q}$-a.s.

$$\lim_{N\to\infty} \frac{\log \inf_x \omega(x)}{\log N} = -\frac{d}{\gamma}.$$

[1] Actually, $n/(2d-1)$ is a better bound.

For that, let $c < d/\gamma$. Then

$$\mathbb{Q}(\inf_x \omega(x) \geq N^{-c}) = [\mathbb{Q}(\omega(x) \geq N^{-c})]^{N^d} \leq (1 - c_1 N^{-c\gamma})^{N^d} \leq e^{-c_1 N^{d-c\gamma}} \quad (3.9)$$

for N large enough and some constant c_1. Thus the Borel-Cantelli lemma implies the upper bound in (3.9).

Now, let $c > d/\gamma$. For $e^k \leq N \leq e^{k+1}$, we have

$$\inf_{x \in S_N} \omega(x) \geq \inf_{x \in S_{e^k}} \omega(x) \wedge \inf_{x \in S_{e^{k+1}} \setminus S_{e^k}} \omega(x).$$

Therefore,

$$\begin{aligned}
&\mathbb{Q}\left(\exists N \in [e^k, e^{k+1}) : \inf_{x \in S_N} \omega(x) \leq N^{-c}\right) \\
&\leq \mathbb{Q}\left(\inf_{x \in S_{e^k}} \omega(x) \leq e^{-ck}\right) + \mathbb{Q}\left(\inf_{x \in S_{e^k} \setminus S_{e^k}} \omega(x) \leq e^{-ck}\right) \\
&= (1 - (1 - c_1 e^{-c\gamma k})^{e^{dk}}) + (1 - (1 - c_1 e^{-c\gamma k})^{e^{d(k+1)} - e^{dk}}) \\
&\leq c_2\, e^{-(c\gamma - d)k} \qquad (3.10)
\end{aligned}$$

and the result follows from Borel-Cantelli and the summability of the probabilities on the left hand sides of (3.9) and (3.10), implied by their right hand sides. □

3.2. *Proof of (3.6): Spectral gap estimates*

Let $\mathcal{B}$ denote the set of nearest neighbor bonds of S, i.e., $\mathcal{B} = \{(x, y) : x, y \in S, x \sim y\}$. For $x, y \in S$, define $r^\omega(b) = N^{-d}(\omega(x) \wedge \omega(y))$, if $b \in \mathcal{B}$, and $r^\omega(b) = 0$, otherwise. The Dirichlet form of $\mathcal{L}^{\omega,N}$ on $L_2(S, \eta_N)$ can be written as

$$\mathcal{E}^{\omega,N}(f, f) = \frac{1}{2} \sum_{b \in \mathcal{B}} (d_b f)^2 r^\omega(b),$$

where $d_b f = f(x) - f(y)$ and the sum ranges over $b = (x, y)$, $x, y \in S$.

Let

$$\tau^{\omega,N} = \sup_{f \not\equiv 0, \eta_N(f) = 0} \frac{\eta_N(f^2)}{\mathcal{E}^{\omega,N}(f, f)}$$

be the inverse of the spectral gap. From general facts [12], we have

$$|\mathbb{E}_x^{\omega,N}[f(X_t)] - \eta_N(f)| \leq \eta_N(x)^{-1/2} e^{-t/\tau^{\omega,N}},$$

whenever f is any function uniformly bounded by 1. Thus

$$\limsup_{N \to \infty} \frac{\log T_1^{\omega,N}}{\log N} \leq \limsup_{N \to \infty} \frac{\log \tau^{\omega,N}}{\log N}.$$

Using a formula of Saloff-Coste (see Theorem 3.2.3 in [12]), we get

$$\begin{aligned}\tau^{\omega,N} &\le N^{-d} \max_{b\in\mathcal{B}} \frac{W(b)}{\omega(b)} \sum_{\substack{(x,y):\\ \pi_{x,y}\ni b}} |\pi_{x,y}|_W \\ &= N^{-d} \max_{b\in\mathcal{B}} \frac{W(b)}{\omega(b)} \sum_{b'\in\mathcal{B}} \frac{1}{W(b')} \mathcal{N}(b,b'), \end{aligned} \tag{3.11}$$

where $\omega(b') = \omega(x') \wedge \omega(y')$ for $b' = (x', y') \in \mathcal{B}$, $W : \mathcal{B} \to (0,\infty)$ is an arbitrary weight function, $\{\pi_{x,y} : (x, y) \in S \times S\}$ is an arbitrary complete set of paths (where $\pi_{x,y}$ is a path with end points x and y), for an arbitrary path π in S, $|\pi|_W = \sum_{b\in\pi} 1/W(b)$, and $\mathcal{N}(b, b') := \#\{(x, y) \in S \times S : b, b' \in \pi_{x,y}\}$.

We need to estimate the right hand side of (3.11). The key point here is the choices of the weight function and the complete set of paths. Roughly speaking, the latter will be taken in such a way that no path in it has *interior sites* with low values of ω; and the former will give low weight to bonds with low values of ω. We are precise in the definitions below.

Remark 3.7. If the rates ω were bounded away from 0, then by taking $\{\eta_{x,y},\ x, y \in S_N\}$ as the complete set of paths, where $\eta_{x,y}$ is defined below (in Definition 3.11), and making $W(\cdot) \equiv 1$, then it is a straightforward matter to verify an upper bound of const N^2 for $\tau^{\omega,N}$ from (3.11).

To control the ω's which are close to 0 in our case, we will consider a modification of the above set of paths and weight function below. For that we start with the following definitions.

Definition 3.8. *Given $\epsilon > 0$, a site $x \in S_N$ will be called ϵ-*good *if $\omega(x) > N^{-\epsilon}$. Otherwise, it will be called ϵ-*bad*. A bond $b = (x, y) \in \mathcal{B}$ will be ϵ-*good *if x and y are ϵ-*good*. Otherwise, it will be called ϵ-*bad.

Definition 3.9. *Given $L > 0$ and a path $\pi \in S$ connecting given sites x, y, a site z in π will be called an L-*interior site *of π if $||z - x||_\infty, ||z - y||_\infty > L$.*

Definition 3.10. *Given $L, \epsilon > 0$ and Γ, a set of paths of S, Γ will be called (L, ϵ)-good if all the paths of Γ have all their L-interior sites, if any, ϵ-good.*

We now construct for every N a complete set of paths for S_N which will turn out to be almost surely (L, ϵ)-good for all large enough N and which will have other properties leading to the validity of (3.6).

We start with an auxiliary set of paths.

Definition 3.11. *For $x, y \in S$, let $\eta_{x,y}$ be the path given by moving sequentially in the 1-st, 2-nd,..., d-th coordinate direction one step at a time, along the longest segment (and according to an arbitrary predetermined order in case of a tie), from x to y, until the coordinates are successively matched.*

For example, if $d = 3$, $N = 100$, $S_N = \{1, 2, \dots, 100\}^3$ (with the boundaries appropriately identified), $x = (1, 1, 1)$ and $y = (2, 20, 80)$, then $\eta_{x,y} = \gamma_1 \cup \gamma_2 \cup \gamma_3$

is the union of the segments

$$\gamma_1 = \{(1, 1, 1) \equiv (100, 1, 1), (99, 1, 1), \dots , (3, 1, 1), (2, 1, 1)\}$$
$$\gamma_2 = \{(2, 1, 1) \equiv (2, 100, 1), (2, 99, 1), \dots , (2, 21, 1), (2, 20, 1)\}$$
$$\gamma_3 = \{(2, 20, 1), (2, 20, 2), \dots , (2, 20, 79), (2, 20, 80)\}.$$

Now for $L > 0$ we define the *L-sausage* $\mathcal{S}_L = \mathcal{S}_L(x, y)$ with base $\eta_{x,y}$ and width L as follows. We suppose $N > 3L$. Let $i_1, i_2, \dots , i_k$, $1 \le k \le d$ be the coordinates where x differs from y in increasing order, so that $\eta_{x,y}$ is the union of the segments $\gamma_1, \dots , \gamma_k$, each of length at least $N/2$, with γ_i parallel to the coordinate direction i. If $k < d$, then let $i^* = \min\{i : 1 \le i \le d \text{ and } i \ne i_k, 1 \le k \le d\}$ and

$$\mathcal{S}_L = \{(z_1, \dots , z_{i^*-1}, w_{i^*}, z_{i^*+1}, \dots , z_d) : \\ z_{i^*} \le w_{i^*} \le z_{i^*} + L - 1, (z_1, \dots , z_{i^*-1}, z_{i^*}, z_{i^*+1}, \dots , z_d) \in \eta_{x,y}\}.$$

If $k = d$, then let

$$\mathcal{S}'_L = \{(w_1, z_2 \dots , z_d) : z_1 \le w_1 \le z_1 + L - 1, (z_1, \dots , z_d) \in \cup_{j=2}^k \gamma_j\},$$
$$\mathcal{S}''_L = \{(w_1, z_2 \dots , z_d) : z_1 - L + 1 \le w_1 \le z_1, (z_1, \dots , z_d) \in \cup_{j=2}^k \gamma_j\}.$$

Now let R_1 be the uniquely defined rectangle with base γ_1 and width L such that either $R_1 \cap \mathcal{S}'_L$ or $R_1 \cap \mathcal{S}''_L$ is a $L \times L$ square (one and only one of these possibilities occurs). In the latter case, $\mathcal{S}_L = R_1 \cup \mathcal{S}''_L$; in the former one, $\mathcal{S}_L = R_1 \cup \mathcal{S}'_L$. See Figure 1 below.

Remark 3.12. Notice that $\mathcal{S}_L$ can be seen as either a single bidimensional[2] strip of length at least $N/2$ and at most dN and width L, when $k < d$, or the union of two such strips (one of which is the rectangle R_1), when $k = d$.

Given $\epsilon > 0$ and a strip $\mathcal{S}$ of length at least $N/2$ and at most dN and width L, we consider the site percolation model in $\mathcal{S}$ in which a site is open if and only if it is ϵ-good and define the event $A_{\mathcal{S}} = A_{\mathcal{S}}(L)$ that there exists an open path connecting the two smaller sides of $\mathcal{S}$ (within $\mathcal{S}$). Then one argues as usually that $A^c_{\mathcal{S}}$ is the event that there exists a $*$-closed path connecting the two larger sides of $\mathcal{S}$ (within $\mathcal{S}$). (Here we adopt the usual concept of $*$-connectedness from site percolation in $\mathbb{Z}^2$.) It is clear that $A_{\mathcal{S}}(L) \subseteq A_{\mathcal{S}}(L')$ if $L \le L'$.

Now consider the event $A_N = A_N(L)$ that $A_{\mathcal{S}}$ occurs for all the strips involved in the sausages $\mathcal{S}_L(x, y)$ for all $x, y \in S$. Clearly, $A_N(L) \subseteq A_N(L')$ if $L \le L'$.

Definition 3.13. *Let*

$$\ell_\epsilon = \inf\{L : 3L < N \textit{ and } A_N(L) \textit{ occurs}\},$$

with the convention that $\inf \emptyset = \infty$.

The following result will be proven below.

[2] Even if living in k-dimensional space.

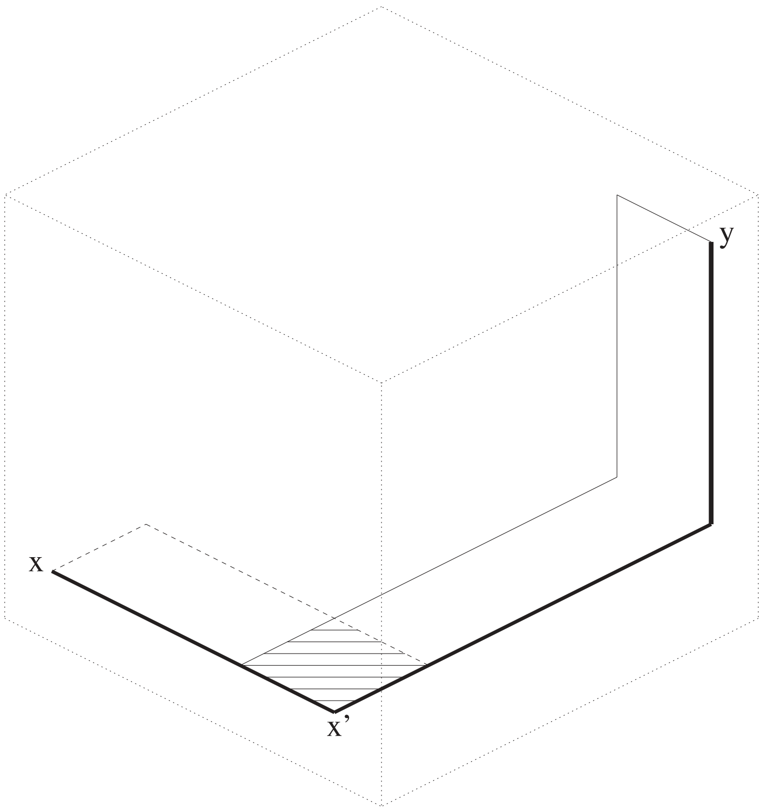

Fig. 1. Dotted cube represents the torus S_N ($d = 3$); x, x', y are sites of S_N; thick polygonal is $\eta_{x,y}$; rectangle delimited by dashed lines is R_1; strip delimited by thin and thick lines is $\mathcal{S}''_L$ (it is also $\mathcal{S}_L(x', y)$); shaded region is the $L \times L$ square determining R_1

Proposition 3.14.

$$\tau^{\omega,N} \leq C(\ell_\epsilon + 1)^{2d} \left(N^{2+\epsilon} + \max_{x \in S_N} \frac{1}{\omega(x)} \right), \tag{3.12}$$

where the positive finite C *depends only on* d.

End of the proof of (3.6). (3.12) is a deterministic statement. Together with Lemma 3.15, a probabilistic result, it readily yields (3.6), after one uses Lemma 3.6 and Borel-Cantelli. Its proof uses (3.11) with a choice of the weight function W assigning small values to the ϵ-bad bonds (see (3.14) below), and a choice of complete set of paths Γ with all paths $(\ell_\epsilon, \epsilon)$-good. For each $(x, y) \in S_N$, the path in Γ connecting them will be contained in $\mathcal{S}_{\ell_\epsilon}(x, y)$ (see (3.15) below).

Lemma 3.15. *For all large enough* N

$$\mathbb{P}\left(\ell_\epsilon > \left\lceil 4 \frac{d+1}{\gamma\epsilon} \right\rceil \right) \leq \frac{c}{N^{1+\delta}}, \tag{3.13}$$

where c depends only on d and $\delta > 0$ *is independent of* N.

Proof of Lemma 3.15. (3.13) follows by a standard path counting argument, which we now outline. For $L > 0$ fixed, we have that

$$\mathbb{P}(A^c_{\mathcal{S}}(L)) \leq dN \max_{x\in\bar{\mathcal{S}}} \mathbb{P}(x \text{ is connected within } \mathcal{S} \text{ by a } *\text{-closed path to } \underline{\mathcal{S}}),$$

where $\bar{\mathcal{S}}$ and $\underline{\mathcal{S}}$ are the two larger sides of $\mathcal{S}$. Now the latter probability can be bounded above in a standard way by

$$\sum_{l\geq L} \lambda_l N^{-\gamma\epsilon l},$$

where λ_l is the number of distinct $*$-paths of length l within $\mathcal{S}$ and starting at x. This is bounded above in a standard way by 7^l and thus

$$\mathbb{P}(A^c_{\mathcal{S}}(L)) \leq dN \sum_{l\geq L} (7N^{-\gamma\epsilon})^l \leq cN^{1-\gamma\epsilon L/2},$$

for some constant c and all large enough N.

Then

$$\mathbb{P}(A^c_N(L)) \leq cN^{2d+1-\gamma\epsilon L/2}.$$

The result now follows from the observation that $\{\ell_\epsilon > L\} \subset A^c_N(L)$. $\square$

3.3. Proof of Proposition 3.14

We assume $\ell_\epsilon < \infty$; otherwise, the bound is obvious. We choose the weight function W. For $b \in \mathcal{B}$, we make

$$W(b) = \begin{cases} 1, & \text{if } b \text{ is } \epsilon\text{-good}, \\ \frac{1}{N}, & \text{if } b \text{ is } \epsilon\text{-bad}. \end{cases} \tag{3.14}$$

We now choose a complete set of paths for S, Γ. Since $\ell_\epsilon < \infty$, we have that for all $x, y \in S$, there will be a $(\ell_\epsilon, \epsilon)$-good path within $\mathcal{S}_{\ell_\epsilon}(x, y)$ connecting x and y, so we choose one of them (according to some arbitrary predetermined order), call it $\pi_{x,y}$, and make

$$\Gamma = \{\pi_{x,y}; x, y \in S\}. \tag{3.15}$$

We now use the above W and Γ in (3.11). Let $\mathcal{B}_1 = \{b \in \mathcal{B} : b \text{ is } \epsilon\text{-good}\}$ and $\mathcal{B}_2 = \{b \in \mathcal{B} : b \text{ is } \epsilon\text{-bad}\} = \mathcal{B} \setminus \mathcal{B}_1$. Then

$$\tau^{\omega,N} \leq \tau^\omega_{11} + \tau^\omega_{12} + \tau^\omega_{21} + \tau^\omega_{22}, \tag{3.16}$$

where, for $i, j = 1, 2$,

$$\tau^\omega_{ij} = N^{-d} \max_{b\in\mathcal{B}_i} \frac{W(b)}{\omega(b)} \sum_{b'\in\mathcal{B}_j} \frac{1}{W(b')} \mathcal{N}(b, b').$$

For $x, y \in S$, let Q_x, resp. Q_y, denote the $\ell_\epsilon \times \ell_\epsilon$ square contained in $\mathcal{S}_{\ell_\epsilon}(x, y)$ with x, resp. y, as one of its corners.

Remark 3.16. Notice that for every $x, y \in S$, all bonds of $\pi_{x,y} \setminus (Q_x \cup Q_y)$ are ϵ-good.

Given $b, b' \in \mathcal{B}$, let $\mathcal{M}(b, b') = \#\{(x, y) \in S \times S : b, b' \in \eta_{x,y}\}$; see Definition 3.11.

Estimation of τ_{11}^{ω}.

$$\tau_{11}^{\omega} \le N^{\epsilon-d} \max_{b\in\mathcal{B}} \sum_{b'\in\mathcal{B}} \mathcal{N}(b, b'). \tag{3.17}$$

Now for every $b, b' \in \mathcal{B}$

$$\begin{aligned}\mathcal{N}(b, b') &\le \#\{(x, y) \in S \times S : b, b' \in \mathcal{S}_L(x, y)\} \\ &\le \#\{(x, y) \in S \times S : a, a' \in \eta_{x,y} \text{ for some } a, a' \in \mathcal{B} : \\ &\qquad \operatorname{dist}(a, b) \vee \operatorname{dist}(a', b') \le \ell_\epsilon\} \\ &\le \sum_{a,a' : \operatorname{dist}(a,b)\vee\operatorname{dist}(a',b')\le\ell_\epsilon} \mathcal{M}(a, a').\end{aligned}$$

where dist is the usual Hausdorff distance between sets. Thus

$$\tau_{11}^{\omega} \le \ell_\epsilon^{2d} N^{\epsilon} M_N \tag{3.18}$$

where $M_N := N^{-d} \max_{a\in\mathcal{B}} \sum_{a'\in\mathcal{B}} \mathcal{M}(a, a')$.

To estimate M_N, we start with the observation that since our paths are described in an oriented way, we must specify which of a or a' is traversed first and in which direction. Given $a = (w, z)$, we have

$$\sum_{a'=(w',z')\in\mathcal{B}} \mathcal{M}(a, a') = \sum_{a'\in\mathcal{B}} \#\{(x, y) : a, a' \in \eta_{x,y} \text{ in the order } w, z, w', z'\} \tag{3.19}$$

$$+ \sum_{a'\in\mathcal{B}} \#\{(x, y) : a, a' \in \eta_{x,y} \text{ in the order } z, w, w', z'\} \tag{3.20}$$

$$+ \sum_{a'\in\mathcal{B}} \#\{(x, y) : a, a' \in \eta_{x,y} \text{ in the order } w', z', w, z\} \tag{3.21}$$

$$+ \sum_{a'\in\mathcal{B}} \#\{(x, y) : a, a' \in \eta_{x,y} \text{ in the order } w', z,' z, w\}. \tag{3.22}$$

We estimate the sum in (3.19). The estimation for the ones in (3.20-3.22) is similar. Let j be the coordinate where w, z differ, that is $z_i = w_i$ if $i \ne j$ and $z_j = w_j \pm 1$. Then the ordering imposes that $z'_i = w'_i = w_i$ if $i < j$. The sum in (3.19) can then be decomposed as follows.

$$\sum_{k=j}^{d} \sum_{a'\in\Lambda_k} \mathcal{M}'(a, a'), \tag{3.23}$$

where $\mathcal{M}'(a, a') = \#\{(x, y) : a, a' \in \eta_{x,y} \text{ in the order } w, z, w', z'\}$ and

$$\begin{aligned}\Lambda_k = \{(w', z') \in \mathcal{B} : z'_i = w'_i = w_i,\ \text{if } i < j;\ z'_i \ne w_i,\ \text{if } j \le i \le k;\ z'_i = w_i, \\ \text{if } k < i \le d\}.\end{aligned}$$

It is clear that $|\Lambda_k| \leq N^{k-j+1}$. Now, for $a' \in \Lambda_k$

$$\begin{aligned}\mathcal{M}'(a,a') &\leq \#\{x \in S : x_i = w_i \text{ for } i > j\} \times \#\{y \in S : y_i = z_i' \text{ for } i < k\} \\ &\leq N^j N^{d-k+1}.\end{aligned}$$

Thus (3.23) and (3.19) are bounded above by dN^{2+d}. After a similar reasoning for (3.20-3.22), with the same bounds, we finally get from (3.18) that

$$\tau_{11}^{\omega} \leq 4d\ell_\epsilon^{2d} N^{2+\epsilon}. \tag{3.24}$$

Remark 3.17. This estimation is similar to the one for the case of ω's bounded away from 0, if we take $\{\eta_{x,y},\ x, y \in S_N\}$ as complete set of paths, and $W(\cdot) \equiv 1$.

Estimation of τ_{12}^{ω}.

$$\tau_{12}^{\omega} \leq N^{\epsilon-d+1} \max_{b\in\mathcal{B}} \sum_{b'\in\mathcal{B}_2} \mathcal{N}(b,b'). \tag{3.25}$$

By Remark 3.16, if $b' \in \mathcal{B}_2$ is in $\pi_{x,y} \in \Gamma$, then b' must be either in Q_x or in Q_y (see definition right above Remark 3.16). Thus

$$\begin{aligned}\mathcal{N}(b,b') \leq\ & \#\{(x,y) \in S \times S :\ a \in \eta_{x,y} \text{ for some } a \in \mathcal{B} \text{ and} \\ &\quad \operatorname{dist}(a,b) \vee \operatorname{dist}(x,b') \leq \ell_\epsilon\} \\ &+ \#\{(x,y) \in S \times S :\ a \in \eta_{x,y} \text{ for some } a \in \mathcal{B} \text{ and} \\ &\quad \operatorname{dist}(a,b) \vee \operatorname{dist}(y,b') \leq \ell_\epsilon\} \\ \leq & \sum_{a\in\mathcal{B}, z\in S:\, \operatorname{dist}(a,b)\vee\operatorname{dist}(z,b')\leq\ell_\epsilon} [\mathcal{J}(a,z) + \tilde{\mathcal{J}}(a,z)],\end{aligned}$$

where

$$\mathcal{J}(a,z) = \#\{x \in S :\ a \in \eta_{x,z}\}, \quad \tilde{\mathcal{J}}(a,z) = \#\{y \in S :\ a \in \eta_{z,y}\}.$$

We conclude that

$$\begin{aligned}\tau_{12}^{\omega} &\leq \ell_\epsilon^d N^{\epsilon-d+1} \left[\max_{a\in\mathcal{B}} \sum_{z\in S} \mathcal{J}(a,z)\mathcal{W}(z) + \max_{a\in\mathcal{B}} \sum_{z\in S} \tilde{\mathcal{J}}(a,z)\mathcal{W}(z)\right] \\ &\leq \text{const } \ell_\epsilon^{2d} N^{\epsilon-d+1} \left[\max_{a\in\mathcal{B}} \sum_{z\in S} \mathcal{J}(a,z) + \max_{a\in\mathcal{B}} \sum_{z\in S} \tilde{\mathcal{J}}(a,z)\right],\end{aligned} \tag{3.26}$$

since

$$\mathcal{W}(z) := \#\{b' \in \mathcal{B} : \operatorname{dist}(z,b') \leq \ell_\epsilon\} \leq \text{const } \ell_\epsilon^d.$$

We estimate the first max term in (3.26). The other one is treated similarly, with the same bound. Let $a = (u,v)$. We decompose $\mathcal{J}(a,z)$ in $\mathcal{J}'(a,z)$ and $\mathcal{J}''(a,z)$, where

$$\begin{aligned}\mathcal{J}'(a,z) &= \#\{x \in S :\ a \in \eta_{x,z},\ \text{with } u \text{ traversed before } v\}, \\ \mathcal{J}''(a,z) &= \#\{x \in S :\ a \in \eta_{x,z},\ \text{with } v \text{ traversed before } u\}.\end{aligned}$$

We estimate $\max_{a\in\mathcal{B}}\sum_{z\in S}\mathcal{J}'(a,z)$. The expression involving $\mathcal{J}''(a,z)$ is treated similarly, with the same bound. Let j be the coordinate where u and v differ. Then z must satisfy $z_i = u_i$, if $1\le i\le j-1$. We conclude that there are at most N^{d-j+1} such z's. For each one, if $a\in\eta_{x,z}$, then x must satisfy $x_i = u_i$, if $j+1\le i\le d$. We conclude that there are at most N^j such x's. Thus,

$$\max_{a\in\mathcal{B}}\sum_{z\in S}\mathcal{J}'(a,z)\le\max_{1\le j\le d}N^j\,N^{d-j+1}=N^{d+1}.$$

We conclude that

$$\tau_{12}^{\omega}\le\text{const}\;\ell_\epsilon^{2d}N^{2+\epsilon}. \tag{3.27}$$

Remark 3.18. Note that there is an extra N in the factor before the sum in (3.25) as compared to the one in (3.17). This is compensated by the estimate of the latter sum having an extra N as compared to the estimate for the former sum; that arises from the fact that the restriction that a path passes through a bad bond imposes that that bond be at the begginning or the end of the path.

Estimation of τ_{21}^{ω}.

$$\tau_{21}^{\omega}\le\left(\max_{x\in S}\frac{1}{\omega(x)}\right)N^{-d-1}\max_{b\in\mathcal{B}_2}\sum_{b'\in\mathcal{B}}\mathcal{N}(b,b'). \tag{3.28}$$

We now estimate the max of the sum above, in much the same way as we estimated $\max_{b\in\mathcal{B}}\sum_{b'\in\mathcal{B}_2}\mathcal{N}(b,b')$ above. By Remark 3.16, if $b\in\mathcal{B}_2$ is in $\pi_{x,y}\in\Gamma$, then b must be either in Q_x or in Q_y. Thus

$$\begin{aligned}\mathcal{N}(b,b')\le{}&\#\{(x,y)\in S\times S:\ a\in\eta_{x,y}\text{ for some }a\in\mathcal{B}\text{ and}\\&\quad\operatorname{dist}(a,b')\vee\operatorname{dist}(x,b)\le\ell_\epsilon\}\\&+\#\{(x,y)\in S\times S:\ a\in\eta_{x,y}\text{ for some }a\in\mathcal{B}\text{ and}\\&\quad\operatorname{dist}(a,b')\vee\operatorname{dist}(y,b)\le\ell_\epsilon\}\\\le{}&\sum_{a\in\mathcal{B},z\in S:\,\operatorname{dist}(a,b')\vee\operatorname{dist}(z,b)\le\ell_\epsilon}[\mathcal{J}(a,z)+\tilde{\mathcal{J}}(a,z)].\end{aligned}$$

Thus,

$$\begin{aligned}\max_{b\in\mathcal{B}_2}\sum_{b'\in\mathcal{B}}\mathcal{N}(b,b')&\le\text{const}\;\ell_\epsilon^{d}\left[\max_{z\in S}\sum_{a\in\mathcal{B}}\mathcal{J}(a,z)\bar{\mathcal{W}}(a)+\max_{z\in S}\sum_{a\in\mathcal{B}}\tilde{\mathcal{J}}(a,z)\bar{\mathcal{W}}(a)\right]\\&\le\text{const}\;\ell_\epsilon^{2d}\left[\max_{z\in S}\sum_{a\in\mathcal{B}}\mathcal{J}(a,z)+\max_{z\in S}\sum_{a\in\mathcal{B}}\tilde{\mathcal{J}}(a,z)\right],\end{aligned} \tag{3.29}$$

where

$$\bar{\mathcal{W}}(a):=\#\{b'\in\mathcal{B}:\operatorname{dist}(a,b')\le\ell_\epsilon\}\le\text{const}\;\ell_\epsilon^{d}.$$

We estimate the first summand within square brackets in (3.29). The second one can be similarly estimated with the same resulting bound.

$$\max_{z\in S}\sum_{a\in\mathcal{B}}\mathcal{J}(a,z)\le\max_{z\in S}\sum_{a\in\mathcal{B}}\mathcal{J}'(a,z)+\max_{z\in S}\sum_{a\in\mathcal{B}}\tilde{\mathcal{J}}''(a,z) \tag{3.30}$$

and we estimate the first summand within square brackets in (3.30) only. The second one can be similarly treated with the same bound. Let $z\in S$ be fixed and j be the coordinate where u and v differ, where $(u,v)=a$. Then u must satisfy $u_i=z_i$, if $1\le i\le j-1$. We conclude that there are at most N^{d-j+1} such u's. For each one, if $a\in\eta_{x,z}$, then x must satisfy $x_i=u_i$, if $j+1\le i\le d$. We conclude that there are at most N^j such x's. We then conclude that

$$\max_{z\in S}\sum_{a\in\mathcal{B}}\mathcal{J}'(a,z)\le\max_{z\in S}\sum_{j=1}^{d}\sum_{\substack{a=(u,v)\in\mathcal{B}\\ u\text{ and }v\text{ differ in }j}}\mathcal{J}'(a,z)\le d\,N^{d-j+1}\,N^j=d\,N^{d+1},$$

which eventually yields

$$\tau_{21}^{\omega}\le\text{const}\,\ell_\epsilon^{2d}\left(\max_{x\in S}\frac{1}{\omega(x)}\right). \tag{3.31}$$

Remark 3.19. Here, the missing factor of N in the estimate of the sum in (3.28) as compared to the one for the sum in (3.17) is explained in the same way as the analogous issue discussed in Remark 3.18.

Estimation of τ_{22}^{ω}.

$$\tau_{22}^{\omega}\le\left(\max_{x\in S}\frac{1}{\omega(x)}\right)N^{-d}\max_{b\in\mathcal{B}_2}\sum_{b'\in\mathcal{B}_2}\mathcal{N}(b,b'). \tag{3.32}$$

By Remark 3.16, if $b,b'\in\mathcal{B}_2$ is in $\pi_{x,y}\in\Gamma$, then b' must be either in Q_x or in Q_y (see definition right above Remark 3.16). Thus for $b\in\mathcal{B}_2$, we have

$$\begin{aligned}\sum_{b'\in\mathcal{B}_2}\mathcal{N}(b,b')\le\sum_{x,y\in S}\sum_{b'\in\mathcal{B}_2}&[1\{b,b'\in Q_x\}+1\{b,b'\in Q_y\}\\&+1\{b\in Q_x,b'\in Q_y\}+1\{b\in Q_y,b'\in Q_x\}].\end{aligned}$$

Now

$$\sum_{x,y\in S}\sum_{b'\in\mathcal{B}_2}1\{b,b'\in Q_x\}\le\sum_{y\in S}\sum_{x\in S}1\{b\in Q_x\}\sum_{b'\in\mathcal{B}_2}1\{b'\in Q_x\}. \tag{3.33}$$

The two inner summands in the left hand side of (3.33) are uniformly bounded by const ℓ_ϵ^d, so the left hand side of (3.33) is bounded by const $\ell_\epsilon^{2d}N^d$. For similar reasons, the same bound holds for $\sum_{x,y\in S}\sum_{b'\in\mathcal{B}_2}1\{b,b'\in Q_y\}$, $\sum_{x,y\in S}\sum_{b'\in\mathcal{B}_2}1\{b\in Q_x,b'\in Q_y\}$ and $\sum_{x,y\in S}\sum_{b'\in\mathcal{B}_2}1\{b\in Q_y,b'\in Q_x\}$, and thus, from (3.32)

$$\tau_{22}^{\omega}\le\text{const}\,\ell_\epsilon^{2d}\left(\max_{x\in S}\frac{1}{\omega(x)}\right). \tag{3.34}$$

Remark 3.20. The missing factor of N^2 in the estimate of the sum in (3.32) as compared to the one for the sum in (3.17) comes about from the fact that, since b, b' are both bad, they must both be at the extremes of the path.

The result of Proposition 3.14 now follows from (3.24), (3.27), (3.31), (3.34) and (3.16). □

3.4. Averaged convergence times and generalized Poincaré inequalities

We now introduce a second 'convergence time':

$$T_2^{\omega,N} = \inf\{t \geq 0 : \sup_{|f|\leq 1} \sup_{|g|\leq 1} |\mathbb{E}_{\eta_N}^{\omega,N}[f(X_0)g(X_t)] - \eta_N(f)\eta_N(g)| \leq \epsilon\}. \tag{3.35}$$

Remark 3.21. The first convergence time $T_1^{\omega,N}$ is a worst-case one, that is, it is the longest convergence time among all initial conditions. $T_2^{\omega,N}$ is an average convergence time among all initial conditions (under uniform weighting).

Remark 3.22. Clearly, $T_2^{\omega,N} \leq T_1^{\omega,N}$ for all ω.

Theorem 3.23. *For all $\gamma > 0$ and $\epsilon \in (0, 1/4)$, we have $\mathbb{Q}$-a.s.*

$$\lim_{N\to\infty} \frac{\log T_2^{\omega,N}}{\log N} = 2. \tag{3.36}$$

In fact, for all $\epsilon \in (0, 1/4)$, there exists a constant $c > 0$ such that for all ω

$$\liminf_{N\to\infty} N^{-2} T_2^{\omega,N} \geq c. \tag{3.37}$$

Remark 3.24. Theorems 3.3 and 3.23 establish that $\mathbb{Q}$-a.s.

$$\lim_{N\to\infty} \frac{\log T_1^{\omega,N}}{\log N} = 2 \vee \frac{d}{\gamma} \text{ and } \lim_{N\to\infty} \frac{\log T_2^{\omega,N}}{\log N} = 2. \tag{3.38}$$

We thus have distinct asymptotic behaviors of $T_1^{\omega,N}$ and $T_2^{\omega,N}$ when $d/\gamma > 2$. A heuristic argument to justify that follows. When $d/\gamma > 2$, $T_1^{\omega,N}$, as a worst case convergence time, is greater than or equal to the convergence time starting at a site with minimal ω, whose order is clearly smaller than or equal to $N^{-d/\gamma}$. On the other hand, choosing a site uniformly at random as a starting point will miss the low ω sites and, starting at high ω, the walk will get to equilibrium faster than it will get to any low ω site. It will be as if there were no low ω sites, and that means $T_2^{\omega,N}$ is of order N^2 (see Remark 3.4).

3.5. Proof of the upper bound

We first prove that

$$\limsup_{N\to\infty} \frac{\log T_2^{\omega,N}}{\log N} \le 2. \tag{3.39}$$

In a similar way as uniform convergence times of the form $T_1^{\omega,N}$ are, for general reversible Markov chains on finite sets, estimated by the inverse spectral gap, it turns out that averaged convergence times of the form $T_2^{\omega,N}$ can be bounded in terms of a family of suitable modifications of the classical Poincaré inequalities, the so-called *generalized Poincaré inequalities* [7], which we recall now. Keep in mind the notation $\mathcal{E}^{\omega,N}(f, f)$ for the Dirichlet form of $\mathcal{L}^{\omega,N}$, see part (3.2).

For $p \in (0, 2)$, let q be such that $1 + 1/q = 2/p$ and

$$\tau^{\omega,N}(p) = \sup_{f\not\equiv 0,\eta_N(f)=0} \frac{\eta_N(f^2)^{2/p}}{\mathcal{E}^{\omega,N}(f, f)||f||_\infty^{2/q}}. \tag{3.40}$$

As a particular case of the general results of [7], we then have

$$T_2^{\omega,N} \le q\epsilon^{-1/q}\tau^{\omega,N}(p) \tag{3.41}$$

for all $p \in (0, 2)$.

Remark 3.25. In the notation of [7], $\tau^{\omega,N}(p)$, as defined in (3.40), equals $1/\mathcal{K}^\omega(p)$.

For all $x, y \in S$, let $\pi_{x,y}$ be a nearest neighbor path from x to y and let $\ell^* = \sup_{x,y} |\pi_{x,y}|$ be the length of the longest path.

As in part (3.1), consider now a partitioning of $S = B \cup G$ where, for given $0 < \xi < 1$, we choose G as the largest connected component of the set $\{x : \omega(x) \ge \xi\}$ (following a deterministic order in case of ties). Let

$$\tau_G^{\omega,N} = \sup_{f\not\equiv 0} \frac{\sum_{b=(x,y)\in G\times G}(d_b f)^2\eta_N(x)\eta_N(y)}{\sum_{b\in G\times G}(d_b f)^2 r^\omega(b)}.$$

Lemma 3.26.

$$\tau^{\omega,N}(p) \le 2^{2/q}3^{2/p}\eta_N(B)^{2/p}\ell^* \sup_{b=(x,y):x\sim y} \frac{1}{r^\omega(b)} + 2^{2/q}\tau_G^{\omega,N}$$

Lemma 3.27. *For $\xi > 0$ small enough, there exists a positive number c_1 that depends only on d such that $\mathbb{Q}$-a.s.*

$$\liminf_{N\to\infty} N^2/\tau_G^{\omega,N} \ge \xi c_1.$$

Proof of (3.39). With ω and $\xi > 0$ fixed, we choose N big enough so that the conclusions of Lemmas 3.26, 3.27, 3.5 and 3.6 hold. Then, using also (3.41),

$$\begin{aligned} T_2^{\omega,N} &\le q\epsilon^{-1/q}\tau^{\omega,N}(p) \\ &\le \frac{q\epsilon}{4}\{[12\epsilon^{-1}c_2(\xi)]^{2/p}N^{1+d+c_3} + (c_1\xi)^{-1}(4\epsilon^{-1})^{2/p}N^2]\}. \end{aligned} \tag{3.42}$$

Assuming that ξ is small enough, let p satisfy

$$\frac{2}{p} = \frac{(d-1+c_3)\log N + \log(c_1\xi)}{\log c_2(\xi)^{-1} - \log 3}.$$

With this choice, the two summands in the expression within braces in (3.42) are equal and thus (3.42) equals

$$\begin{aligned} &\frac{q\epsilon}{2}(c_1\xi)^{-1}\exp\frac{\log(c_1\xi)}{\log(c_2(\xi))^{-1} - \log 3} \\ &\quad\times\exp\left\{\left[2 + \frac{(d-1+c_3)\log(4\epsilon^{-1})}{\log(c_2(\xi))^{-1} - \log 3}\right]\log N\right\}. \end{aligned} \tag{3.43}$$

Combining (3.42-3.43), we get

$$\limsup_{N\to\infty}\frac{\log T_2^{\omega,N}}{\log N} \le 2 + \frac{(d-1+c_3)\log(4\epsilon^{-1})}{\log(c_2(\xi))^{-1} - \log 3}.$$

Since this holds for all $\xi > 0$ sufficiently small and $c_2(\xi) \to 0$ as $\xi \to 0$, the result follows. $\square$

Proof of Lemma 3.26. This is very similar to the results of part III in [8]. We estimate the three terms in the decomposition

$$\begin{aligned} \eta_N(f^2) &= \left(\frac{1}{2}\sum_{x,y\in G} + \sum_{x\in G,y\in B} + \frac{1}{2}\sum_{x,y\in B}\right)(f(x)-f(y))^2\eta_N(x)\eta_N(y) \\ &=: I + II + III \end{aligned} \tag{3.44}$$

in turn.

$$\begin{aligned} I &\le \frac{1}{2}(2\|f\|_\infty)^{2-p}\sum_{x,y\in G}(f(x)-f(y))^p\eta_N(x)\eta_N(y) \\ &\overset{\text{Hölder}}{\le} 2^{1-p}\|f\|_\infty^{2-p}\left(\sum_{x,y\in G}(f(x)-f(y))\eta_N(x)\eta_N(y)\right)^{p/2} \\ &\le 2^{1-p}\|f\|_\infty^{2-p}\left(2\mathcal{E}^{\omega,N}(f,f)/\tau_G^{\omega,N}\right)^{p/2}. \end{aligned} \tag{3.45}$$

$$
\begin{aligned}
II &\le (2||f||_\infty)^{2-p} \sum_{x\in G, y\in B} (f(x)-f(y))^p \eta_N(x)\eta_N(y)\\
&\overset{\text{Hölder}}{\le} (2||f||_\infty)^{2-p} \left(\sum_{x\in G, y\in B} (f(x)-f(y))\eta_N(x)\eta_N(y)\right)^p (\eta_N(G)\eta_N(B))^{1-p}\\
&\le (2||f||_\infty)^{2-p} \left(\sum_{x\in G, y\in B} \mid \sum_{b\in\pi_{x,y}} d_b f \mid \eta_N(x)\eta_N(y)\right)^p \eta_N(B)^{1-p}\\
&\le (2||f||_\infty)^{2-p} \left(\sum_b |d_b f| \sum_{\substack{x\in G, y\in B:\\ \pi_{x,y}\ni b}} \eta_N(x)\eta_N(y)\right)^p \eta_N(B)^{1-p}\\
&\overset{\text{Hölder}}{\le} (2||f||_\infty)^{2-p} \left(\sum_b |d_b f|^2 r^\omega(b)\right)^{p/2}\\
&\quad\times \left(\sum_b (r^\omega(b))^{-1} \left(\sum_{\substack{x\in G, y\in B:\\ \pi_{x,y}\ni b}} \eta_N(x)\eta_N(y)\right)^2\right)^{p/2} \eta_N(B)^{1-p}\\
&\le (2||f||_\infty)^{2-p} \left(2\mathcal{E}^{\omega,N}(f,f)\right)^{p/2} \left(\sup_b (r^\omega(b))^{-1}\right)^{p/2} \eta_N(B)^{p/2}\\
&\quad\times \left(\sum_b \sum_{\substack{x\in G, y\in B:\\ \pi_{x,y}\ni b}} \eta_N(x)\eta_N(y)\right)^{p/2} \eta_N(B)^{1-p}\\
&\le 2^{1-p/2}||f||_\infty^{2-p} \left(\mathcal{E}^{\omega,N}(f,f) \sup_b (r^\omega(b))^{-1}\right)^{p/2} \left(\ell^*\eta_N(B)\right)^{p/2} \eta_N(B)^{1-p/2}\\
&= 2^{1-p/2}||f||_\infty^{2-p} \eta_N(B) \left(\mathcal{E}^{\omega,N}(f,f) \sup_b (r^\omega(b))^{-1}\ell^*\right)^{p/2}, \qquad (3.46)
\end{aligned}
$$

where the last inequality follows from

$$
\sum_b \sum_{\substack{x\in G, y\in B:\\ \pi_{x,y}\ni b}} \eta_N(x)\eta_N(y) = \sum_{x\in G, y\in B} |\pi_{x,y}|\eta_N(x)\eta_N(y) \le \ell^*\eta_N(B).
$$

Similarly,

$$
III \le 2^{1-p/2}||f||_\infty^{2-p} \eta_N(B) \left(\mathcal{E}^{\omega,N}(f,f) \sup_b (r^\omega(b))^{-1}\ell^*\right)^{p/2}. \qquad (3.47)
$$

We conclude from (3.44), (3.45), (3.46) and (3.47) that

$$
\begin{aligned}
\eta_N(f^2) \le &\left\{3\eta_N(B)\left(\sup_b (r^\omega(b))^{-1}\ell^*\right)^{p/2} + \left(\tau_G^{\omega,N}\right)^{p/2}\right\}\\
&\times 2^{1-p/2}||f||_\infty^{2-p} (\mathcal{E}^{\omega,N}(f,f))^{p/2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\tau^{\omega,N}(p) &\leq \left\{6\,2^{-p/2}\eta_N(B)\left(\sup_b (r^{\omega}(b))^{-1}\ell^*\right)^{p/2} + 2^{1-p/2}\left(\tau_G^{\omega,N}\right)^{p/2}\right\}^{2/p} \\
&\leq 2^{\frac{4}{p}-1}\left\{(3\eta_N(B))^{2/p}\sup_b (r^{\omega}(b))^{-1}\ell^* + \tau_G^{\omega,N}\right\} \\
&= 2^{\frac{2}{q}+1}\left\{(3\eta_N(B))^{2/p}\sup_b (r^{\omega}(b))^{-1}\ell^* + \tau_G^{\omega,N}\right\}. \qquad \square
\end{aligned}
$$

Proof of Lemma 3.27. Since $\omega(\cdot) \geq \xi$ on G, we have

$$
\tau_G^{\omega,N} \leq \frac{\#G}{N^d}\xi^{-1}\tau_G^{\mathbf{1}} \leq \xi^{-1}\tau_G^{\mathbf{1}},
$$

where

$$
\tau_G^{\mathbf{1}} := \sup_{f\not\equiv 0} \frac{\sum_{b=(x,y)\in G\times G}(d_b f)^2(\#G)^{-2}}{\sum_{b\in G\times G}(d_b f)^2(\#G)^{-1}}
$$

is the inverse of the spectral gap for the ordinary rate 1 random walk on G. From Cheeger's inequality, we get that

$$
\tau_G^{\mathbf{1}} \leq 8\,\Xi_G^2,
$$

and therefore

$$
\tau_G^{\omega,N} \leq 8\xi^{-1}\,\Xi_G^2, \tag{3.48}
$$

where the isoperimetric constant Ξ_G is defined by:

$$
\Xi_G := \sup_{A\subset G} \frac{\#A\,\#G\setminus A}{\#G\,\#\partial_G A},
$$

where $\partial_G A = \{(x, y) : x \sim y, x \in A, y \in G \setminus A\}$ is the bond boundary of A with respect to G. The statement of the Lemma will thus follow if we can prove that $N\,\Xi_G^{-1}$ is bounded from below for large N by some constant that only depends on the dimension. We shall rather show that

$$
\sum_N \mathbb{Q}(\Xi_G \geq \alpha N) < \infty, \tag{3.49}
$$

for some α. One then uses the Borel-Cantelli Lemma to deduce from (3.49) that, $\mathbb{Q}$.a.s., for large N, we have $\Xi_G \leq \alpha N$ and therefore, as follows from (3.48), $\tau_G^{\omega,N} \leq 8\xi^{-1}\alpha^2N^2$.

Following [6], Subsection 3.1, we note that we can restrict ourselves to connected A's such that $G \setminus A$ is connected.

Since $\#\partial_G A \geq 1$, we have $\frac{\#A\,\#G\setminus A}{\#G\,\#\partial_G A} \leq \frac{\alpha}{2}N$ as soon as $\#A \leq \frac{\alpha}{2}N$ or $\#G \setminus A \leq \frac{\alpha}{2}N$. Thus we may also assume that $\#A \geq \frac{\alpha}{2}N$ and $\#G \setminus A \geq \frac{\alpha}{2}N$.

The same argument as in [6], Subsection 3.1, based on the classical isoperimetric inequality on S, shows that (3.49) follows from

$$\sum_N \mathbb{Q}\left(\sup_{F\subset\mathcal{B}} \frac{\#F}{\#\{(x,y)\in F;\omega(x)\geq\xi,\omega(y)\geq\xi\}}\geq\alpha\right)<\infty. \tag{3.50}$$

In (3.50), $\mathcal{B}=\{(x,y):x,y\in S,x\sim y\}$ denotes the set of nearest neighbor bonds of S. The sup is computed on $*$-connected sets $F\subset\mathcal{B}$ such that $\#F\geq\alpha_1 N^{\frac{d-1}{d}}$, for some constant α_1 that depends on α and the dimension.

Given such an F, choose a subset, say $\widetilde{F}$, such that $b=(x,y)\neq b'=(x',y')\in\widetilde{F}\Rightarrow x\neq x'$ and $y\neq y'$. Since any point has at most $2d$ neighbors and $\#F\geq\alpha_1 N^{\frac{d-1}{d}}$, we may assume that $\#\widetilde{F}\geq\alpha_2\#F$, for some positive α_2.

Now, for all $\lambda>0$

$$\begin{aligned}
&\mathbb{Q}(\#\{(x,y)\in F;\omega(x)\geq\xi,\omega(y)\geq\xi\}\leq\#F/\alpha)\\
&\quad\leq\mathbb{Q}(\#\{(x,y)\in\widetilde{F};\omega(x)\geq\xi,\omega(y)\geq\xi\}\leq\#F/\alpha)\\
&\quad=\mathbb{Q}\left(\sum_{(x,y)\in\widetilde{F}}\mathbf{1}_{\omega(x)\geq\xi}\mathbf{1}_{\omega(y)\geq\xi}\leq\#F/\alpha\right)\\
&\quad\leq e^{\frac{\lambda}{\alpha}\#F}(1-\pi^2+e^{-\lambda}\pi^2)^{\#\widetilde{F}}\\
&\quad\leq e^{\frac{\lambda}{\alpha}\#F}(1-\pi^2+e^{-\lambda}\pi^2)^{\alpha_2\#F},
\end{aligned}$$

where $\pi=\mathbb{Q}(\omega(x)\geq\xi)$.

By the above inequality, and the fact that the number of distinct $*$-connected subsets F with $\#F=n$ is bounded above by $N^d e^{\alpha_3 n}$ for some α_3 [10], we get

$$\begin{aligned}
&\mathbb{Q}\left(\sup_F\frac{\#F}{\#\{(x,y)\in F;\omega(x)\geq\xi,\omega(y)\geq\xi\}}\geq\alpha\right)\\
&\quad\leq N^d\sum_{n\geq\alpha_1 N^{\frac{d-1}{d}}} e^{[\alpha_3+\lambda\alpha^{-1}+\alpha_2\log(1-\pi^2+e^{-\lambda}\pi^2)]n}\\
&\quad=N^d\sum_{n\geq\alpha_1 N^{\frac{d-1}{d}}} e^{-\alpha_4 n},
\end{aligned}$$

where $\alpha_4:=-[\alpha_3+\lambda\alpha^{-1}+\alpha_2\log(1-\pi^2+e^{-\lambda}\pi^2)]>0$, provided we choose λ and α such that $\alpha_3+\lambda/\alpha<\lambda\alpha_2$ and $\xi\leq\xi_0$, for ξ_0 close enough to 0, depending on $\alpha,\lambda,\alpha_2,\alpha_3$ and γ only. □

3.6. Lower bounds for $T_1^{\omega,N}$ and $T_2^{\omega,N}$

Proof of (3.37). Let $A=\{x=(x_1,\dots,x_d)\in S:x_1\in[0,N/2]\}$, $T_A=\inf\{t\geq 0:X_t\in A\}$ and, for $\lambda\geq 0$, $h_x^{\omega}(\lambda)=\mathbb{E}_x^{\omega,N}(e^{-\lambda T_A})$. Choosing $f=1_A$ and

$g = 1_{A^c}$, we have

$$\sup_{|f|,|g|\le 1} |\mathbb{E}^{\omega,N}_{\eta_N}[f(X_0)g(X_t)] - \eta_N(f)\eta_N(g)| \ge \eta_N(A)\eta_N(A^c)$$
$$-\mathbb{P}^{\omega,N}_{\eta_N}(X_0 \notin A, X_t \in A)$$
$$\ge \eta_N(A)\eta_N(A^c) - \mathbb{P}^{\omega,N}_{\eta_N}(X_0 \notin A, T_A \le t) \ge \eta_N(A)\eta_N(A^c)$$
$$- \inf_{\lambda>0} \eta_N(1_{A^c}h^\omega(\lambda))e^{\lambda t}. \tag{3.51}$$

We now estimate $\eta_N(1_{A^c}h^\omega(\lambda))$. We will compare with the case $\omega \equiv 1$, which corresponds to the usual random walk on S_N. The Dirichlet form of X_t is given by

$$\mathcal{E}^{\omega,N}(f,f) = \frac{1}{2N^d} \sum_{x\sim y\in S} (\omega(x) \wedge \omega(y))(f(x) - f(y))^2. \tag{3.52}$$

It is clear that $\mathcal{E}^{\omega,N}(f,f)$ is nondecreasing in (the natural partial ordering of) ω. We have also that, for $\lambda \ge 0$,

$$\lambda\eta_N(h^\omega(\lambda)) = \inf_{f|_A=1} \mathcal{E}^{\omega,N}(f,f) + \lambda\eta_N(f^2). \tag{3.53}$$

Since $\mathcal{E}^{\omega,N}(f,f) \le \mathcal{E}^{\mathbf{1}}(f,f)$, where $\mathbf{1}$ is the identically 1 vector indexed by S, we have that

$$\eta_N(h^\omega(\lambda)) \le \eta_N(h^{\mathbf{1}}(\lambda)). \tag{3.54}$$

Since T_A is a hitting time for an ordinary rate 1 random walk on $\mathbb{Z}$ under $\mathbb{P}^{\mathbf{1}}_{\cdot}$, the invariance principle yields that for all $\lambda > 0$

$$\eta_N(h^{\mathbf{1}}(N^{-2}\lambda)) \to \frac{1}{2} + \phi(\lambda) \tag{3.55}$$

as $N \to \infty$, where $\phi(\lambda) \to 0$ as $\lambda \to \infty$. We also have that $\eta_N(h^\omega(\lambda)) = \eta_N(A) + \eta_N(1_{A^c}h^\omega(\lambda))$ and $\eta_N(A) \to 1/2$ when $N \to \infty$. Thus, from (3.53), (3.54) and (3.55),

$$\limsup_{N\to\infty} \eta_N(h^\omega(N^{-2}\lambda)) = \frac{1}{2} + \limsup_{N\to\infty} \eta_N(1_{A^c}h^\omega(N^{-2}\lambda)) \le \frac{1}{2} + \phi(\lambda) \tag{3.56}$$

and it follows that

$$\eta_N(1_{A^c}h^\omega(N^{-2}\lambda)) \le \phi(\lambda). \tag{3.57}$$

We conclude that

$$\liminf_{N\to\infty} \sup_{|f|,|g|\le 1} |\mathbb{E}^{\omega,N}_{\eta_N}[f(X_0)g(X_{cN^2})] - \eta_N(f)\eta_N(g)| \ge \frac{1}{4} - e\phi(1/c). \tag{3.58}$$

Since $\phi(1/c) \to 0$ as $c \to 0$, we get that for all $\epsilon < 1/4$, $\liminf_{N\to\infty} N^{-2}T_2^{\omega,N} \ge c^*$, where c^* is any positive constant satisfying $\phi(1/c^*) < (1/4 - \epsilon)/e$. $\square$

Proof of (3.7). From $T_1^{\omega,N} \geq T_2^{\omega,N}$ and $\liminf_{N\to\infty} N^{-2} T_2^{\omega,N} > c$ $\mathbb{Q}$-a.s., we deduce that $\liminf_{N\to\infty} N^{-2} T_1^{\omega,N} > c$ $\mathbb{Q}$-a.s. and, thus, $\liminf_{N\to\infty} \log T_1^{\omega,N}/\log N \geq 2$ $\mathbb{Q}$-a.s.

We argue now for the inequality $\liminf_{N\to\infty} \log T_1^{\omega,N}/\log N \geq d/\gamma$ $\mathbb{Q}$-a.s. Let $x \in S$. During an exponential time of parameter $\sum_{y:y\sim x} \omega(y) \wedge \omega(x)$, the process X starting at x stays still. Therefore,

$$\begin{aligned}\sup_{|f|\leq 1} |\mathbb{E}_x^{\omega,N} f(X_t) - \eta_N(f)| &\geq \mathbb{P}_x^{\omega,N}(X_t = x) - N^{-d} \\ &\geq e^{-t\sum_{y\sim x}\omega(y)\wedge\omega(x)} - N^{-d} \geq e^{-2d\omega(x)t} - N^{-d},\end{aligned}$$

i.e.,

$$T_1^{\omega,N} \geq \frac{1}{2d} \sup_x \omega(x)^{-1} \log(\epsilon + N^{-d})^{-1}.$$

Therefore,

$$\frac{\log T_1^{\omega,N}}{\log N} \geq \frac{\log \sup_x \omega(x)^{-1}}{\log N} + o(1).$$

Now, let $0 < \delta < 1$ be arbitrary.

$$\begin{aligned}\mathbb{Q}\left(\log \sup_x \omega(x)^{-1} \leq (1-\delta)\frac{d}{\gamma}\log N\right) &= [\mathbb{Q}(\omega(x) \geq N^{-(1-\delta)d/\gamma})]^{N^d} \\ &\leq [1 - N^{-(1-\delta')d}]^{N^d},\end{aligned}$$

for any $1 > \delta' > \delta$, provided N is large enough. Thus, the above probability is summable in N for any $\delta > 0$, and the result follows by Borel-Cantelli. □

4. Decay of the annealed return probability for random walks on $\mathbb{Z}^d$

We go back to the study of Markov chains taking their values in $\mathbb{Z}^d$. Let $\omega : \mathbb{Z}^d \to \mathbb{R}_+^*$, and define the Markov generator

$$\mathcal{L}^\omega f(x) = \sum_{y\sim x} [\omega(x) \wedge \omega(y)]\,[f(y) - f(x)], \tag{4.1}$$

where the sum is over sites y which are nearest neighbors to x.

As in Section 2, $\{X_t,\ t \in \mathbb{R}_+\}$ will be the coordinate process on path space $(\mathbb{Z}^d)^{\mathbb{R}_+}$ and we use the notation $\mathbb{P}_x^\omega$ to denote the unique probability measure on path space under which $\{X_t,\ t \in \mathbb{R}_+\}$ is the Markov process generated by (4.1) and satisfying $X_0 = x$.

As in Section 3, we choose the family $\{\omega(x),\ x \in \mathbb{Z}^d\}$ at random, according to a law $\mathbb{Q}$ on $(\mathbb{R}_+^*)^{\mathbb{Z}^d}$ such that

$$\begin{gathered}\text{the random variables } \{\omega(x),\ x \in \mathbb{Z}^d\} \text{ are i.i.d. ;} \\ \omega(x) \leq 1 \text{ for all } x; \\ \mathbb{Q}(\omega(0) \leq a) \sim a^\gamma \text{ as } a \downarrow 0,\end{gathered} \tag{4.2}$$

where $\gamma > 0$ is a parameter.

Remark 4.1. We note that this generator has the same form as $\mathcal{G}^\omega$ in (2.1) by making $\omega(x, y) = \omega(x) \wedge \omega(y)$, and also the same form as $\mathcal{L}^{\omega,N}$ in (3.1), but in infinite volume. There would also be similar results for ω defined on edges, instead of points, with i.i.d. values for different edges, and the same technique would apply.

Remark 4.2. If $\omega(0)$ were a Bernoulli random variable, then we would have a random walk on a (independent, site) percolation cluster, see [6].

In the sequel $\mathbb{Q}.\mathbb{P}^\omega_x$ will be used as a short hand notation for the annealed law defined by $\mathbb{Q}.\mathbb{P}^\omega_x[\,\cdot\,] = \int P^\omega_x[\,\cdot\,]d\mathbb{Q}(\omega)$. We are interested in estimating the decay of the return probability under $\mathbb{Q}.\mathbb{P}^\omega$, $\mathbb{Q}.\mathbb{P}^\omega[X_t = 0]$, as t tends to $+\infty$. It is actually quite easy to derive lower bounds for $\mathbb{Q}.\mathbb{P}^\omega[X_t = 0]$. Indeed, on one hand, one can use the comparison lemma 2.2 with the usual nearest neighbor random walk on $\mathbb{Z}^d$ to prove that

$$\mathbb{Q}.\mathbb{P}^\omega[X_t = 0] \geq ct^{-d/2}, \tag{4.3}$$

for some contant c that depends on the dimension d. There is another way to prove (4.3), as follows. It is known [3] that, under $\mathbb{Q}.\mathbb{P}^\omega$, X_t satisfies the central limit theorem. Together with the reversibility and the translation invariance of the law $\mathbb{Q}$, the C.L.T. implies (4.3) (See Appendix D, in [6]).

On the other hand, for any realization of ω, the first jump of X_t follows an exponential law of parameter $\sum_{y\sim 0} \omega(0) \wedge \omega(y) \leq 2d\omega(0)$. Therefore

$$\mathbb{P}^\omega_0[X_t = 0] \geq \mathbb{P}^\omega_0[X_s = 0, \forall s \leq t] = e^{-t\sum_{y\sim 0}\omega(0)\wedge\omega(y)} \geq e^{-2d\omega(0)t}.$$

Taking expectation w.r.t. $\mathbb{Q}$ and using the condition (4.2) on the law of $\omega(0)$, a simple computation leads to a lower bound of the form

$$\mathbb{Q}.\mathbb{P}^\omega[X_t = 0] \geq ct^{-\gamma}. \tag{4.4}$$

As is indicated in the next statement, these lower bounds turn out to be of the correct logarithmic order.

Theorem 4.3.

$$\lim_{t\to+\infty} \frac{\log \mathbb{Q}.\mathbb{P}^\omega[X_t = 0]}{\log t} = -\left(\frac{d}{2} \wedge \gamma\right). \tag{4.5}$$

Remark 4.4. From the point of view of statistical mechanics — here the statistical mechanics of a disordered system — one might consider Theorem 4.3 as an example of a dynamical phase transition.

Remark 4.5. Such tools as Sobolev embeddings, isoperimetric or Nash inequalities of constant use for estimating transition probabilities of Markov chains, see [2], cannot be directly applied here because of the lack of ellipticity of the transition rates ω. Thus (4.5) is also an example of exotic 'heat kernel decay' for a non uniformly elliptic generator.

Remark 4.6. A fruitful technique to handle r.w.r.e. is to isolate the effect of the fluctuations of the environment ω in a given scale. See for instance random walks in Poisson environments [11] where one single eigenvalue dominates the rest of the spectrum. There does not seem to exist such a separating scale in our model.

In view of (4.3) and (4.4), only the upper bound is missing in the proof of (4.5).

We use spectral theory. We rely on a trace formula similar to the one obtained in Section 2 and on our spectral gap estimates from Proposition 3.14.

4.1. *Trace formula*

We express the annealed return probability as a trace. The argument is the same as in Section 2, except that we restrict ourselves to computing the trace on cubes whose radius can be chosen as a function of time. This is possible because rates are assumed to be uniformly bounded.

Let $\xi > 0$. In the sequel, we shall use the notation $N = t^{(1+\xi)/2}$. (In fact, N should be defined as the integer part of $t^{(1+\xi)/2}$, but, for notational ease, we will omit integer parts.)

Let $B_N = [-N, N]^d$, be the box centered at the origin and of radius N. Let $\mathcal{L}^{\omega,N}$ be the restriction of the operator $\mathcal{L}^{\omega}$ to B_N. Thus $\mathcal{L}^{\omega,N}$ is defined by

$$\mathcal{L}^{\omega,N} f(x) = \sum_{y\sim x} [\omega(x) \wedge \omega(y)]\,[f(y) - f(x)], \tag{4.6}$$

where the sum is now restricted to neighboring points x and y in B_N and we impose periodic boundary conditions. $-\mathcal{L}^{\omega,N}$ is then a symmetric operator. We denote by $\{\lambda_i^{\omega}(B_N), i \in [1, \#B_N]\}$ the set of its eigenvalues in increasing order.

Let τ_N be the exit time of X_t outside B_N.

We compute $\mathbb{Q}.\mathbb{P}_0^{\omega}[X_t = 0]$ using the translation invariance of the probability $\mathbb{Q}$. Since $\mathbb{Q}.\mathbb{P}_x^{\omega}[X_t = x]$ does not depend on x, we have

$$\begin{aligned}
\mathbb{Q}.\mathbb{P}_0^{\omega}[X_t = 0] &= \frac{1}{\#B_N} \sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^{\omega}[X_t = x] \\
&= \frac{1}{\#B_N} \sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^{\omega}[X_t = x; t < \tau_{2N}] \\
&\quad + \frac{1}{\#B_N} \sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^{\omega}[X_t = x; t \geq \tau_{2N}] \\
&\leq \frac{1}{\#B_N} \sum_{x\in B_{2N}} \mathbb{Q}.\mathbb{P}_x^{\omega}[X_t = x; t < \tau_{2N}] \\
&\quad + \frac{1}{\#B_N} \sum_{x\in B_N} \mathbb{Q}.\mathbb{P}_x^{\omega}[t \geq \tau_{2N}].
\end{aligned}$$

If under $\mathbb{P}_x^{\omega}$, $x \in B_N$, we have $t \geq \tau_{2N}$, then the process must have left the ball $x + B_N$ before time t. Since the probability $\mathbb{Q}.\mathbb{P}_x^{\omega}[\exists s \leq t \text{ s.t. } X_s \notin x + B_N]$ does not depend on x, we have that $\mathbb{Q}.\mathbb{P}_x^{\omega}[t \geq \tau_{2N}] \leq \mathbb{Q}.\mathbb{P}_0^{\omega}[t \geq \tau_N]$.

We note that $\sum_{x\in B_{2N}} \mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}]$ is the trace of the semi-group of the process X_t killed when leaving the box B_{2N}, i.e., with Dirichlet boundary conditions outside B_{2N}. It is therefore dominated by the trace of $\exp(t\mathcal{L}^{\omega,2N})$, that is

$$\sum_{x\in B_{2N}} \mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] \le \sum_i e^{-\lambda_i^\omega(B_{2N})t}.$$

Thus, we have proved that

$$\mathbb{Q}.\mathbb{P}_0^\omega[X_t = 0] \le \frac{1}{\# B_N}\sum_i \mathbb{Q}[e^{-\lambda_i^\omega(B_{2N})t}] + \mathbb{Q}.\mathbb{P}^\omega[t \ge \tau_N].$$

From the Carne-Varopoulos inequality, it follows that

$$\mathbb{P}^\omega[t \ge \tau_N] \le 2tN^{d-1}e^{-\frac{N^2}{4t}} + e^{-ct}, \tag{4.7}$$

where c is a numerical constant, see Appendix C in [6]. With our choice of $N = t^{(1+\xi)/2}$, we get that $\mathbb{P}^\omega[t \ge \tau_N]$ decays faster than any polynomial as t tends to $+\infty$.

Thus Theorem 4.3 will be proved if we can check that

$$\lim_{\xi\to 0}\limsup_{t\to+\infty} \frac{\log \mathbb{Q}[\sum_i e^{-\lambda_i^\omega(B_N)t}]}{\log t} \le 0 \vee \left(\frac{d}{2} - \gamma\right). \tag{4.8}$$

4.2. Min-Max

C is a constant that depends only on d and $\mathbb{Q}$. For constants depending on other parameters, we indicate it.

Let us first recall the lower bound on the first non trivial eigenvalue of an operator of the form $\mathcal{L}^{\omega,N}$. In Section 3, we proved that

$$\frac{1}{\lambda_2^\omega(B_N)} \le C\left(N^{2+\varepsilon} + \sup_{x\in B_N}\frac{1}{\omega(x)}\right)d_\varepsilon^N. \tag{4.9}$$

In (4.9), ε is any positive number; C is a constant depending on the dimension only; d_ε^N is a measure of the set $\{x \in B_N : \omega(x) \le N^{-\varepsilon}\}$.

With the notation of Section 3, Proposition 3.14, $d_\varepsilon^N = (\ell_\varepsilon + 1)^{2d}$. (But note that ℓ_ε depends on N.) Thus d_ε^N is a random variable, i.e., depends on ε, N and also ω.

Using the properties of $\mathbb{Q}$, we get that, for some constant c, that depends on $\mathbb{Q}$ only, we have

$$\mathbb{Q}(d_\varepsilon^N \ge A) \le cN^{-\frac{\varepsilon\gamma A}{2}}, \tag{4.10}$$

where A can be chosen such that $A \ge \frac{4d}{\varepsilon\gamma}$ and N is supposed to be large enough. (How large depends on the dimension only.) A proof of (4.10) can be found in the proof of Lemma 3.15.

From the min-max characterization of the eigenvalues of symmetric operators, we have

$$\lambda^{\omega}_{i+1}(B_N) = \max_{f_1,\dots,f_i} \min_{f} \frac{1}{2} \frac{\sum_{x\sim y\in B_N}[\omega(x)\wedge\omega(y)]\,[f(x)-f(y)]^2}{\sum_{x\in B_N} f^2(x)},$$

where the 'max' is computed on choices of i functions defined on B_N and the 'min' is computed on functions f such that, for all $j\in[1,i]$, $\sum_{x\in B_N} f(x)f_j(x)=0$.

Thus, in the computation of $\lambda^{\omega}_{i+1}(B_N)$, we may impose at most i different linear constraints on the test function f. We consider two kind of conditions.

Let $k\in\mathbb{N}^*$. We chop $\mathbb{Z}^d$ into a disjoint union of boxes of radius k, say $\mathbb{Z}^d=\cup_{z\in\mathbb{Z}^d}\mathbf{B}_z$, where $\mathbf{B}_z=(2k+1)z+B_k$. We now choose for some of the function f_j's, the indicator function of the boxes $\mathbf{B}_z$ that intersect B_N, i.e., we require that

$$\sum_{x\in B_N\cap\mathbf{B}_z} f(x)=0,$$

for all $z\in\mathbb{Z}^d$ such that $B_N\cap\mathbf{B}_z\neq\emptyset$. The number of such z's is at most

$$n_2=\left(\frac{2N+1+2k+1}{2k+1}\right)^d.$$

Clearly,

$$\sum_{x\in B_N} f^2(x)=\sum_z\sum_{x\in B_N\cap\mathbf{B}_z} f^2(x),$$

and

$$\sum_{x\sim y\in B_N}\omega(x)\wedge\omega(y)(f(x)-f(y))^2\geq\sum_z\sum_{x\sim y\in B_N\cap\mathbf{B}_z}[\omega(x)\wedge\omega(y)]\,[f(x)-f(y)]^2.$$

Therefore

$$\begin{aligned}&\frac{\sum_{x\sim y\in B_N}[\omega(x)\wedge\omega(y)]\,[f(x)-f(y)]^2}{\sum_{x\in B_N} f^2(x)}\\ &\quad\geq\min_z\frac{\sum_{x\sim y\in B_N\cap\mathbf{B}_z}[\omega(x)\wedge\omega(y)]\,[f(x)-f(y)]^2}{\sum_{x\in B_N\cap\mathbf{B}_z} f^2(x)},\end{aligned}$$

where, for each $z\in\mathbb{Z}^d$, $\sum_{x\in B_N\cap\mathbf{B}_z} f(x)=0$.

Next, let us choose n_1 points in B_N, say $\delta_1,\dots,\delta_{n_1}$. We choose for some of the f_j's, the indicator function of the points δ_j and their neighbors in B_N, i.e., we specify that $f(x)=0$, for $x\in\{\delta_1,\dots,\delta_{n_1}\}$ or $x\sim\delta_j$, for some j. This recipe leads to, at most, $(2d+1)n_1$ different conditions. We note that, for such a function f, the value of the Dirichlet form

$$\sum_{x\sim y\in B_N}[\omega(x)\wedge\omega(y)]\,[f(x)-f(y)]^2$$

does not depend on the value of $\omega(\delta_j)$ anymore. Therefore, we may assume that $\omega(\delta_j) = 1$, for $j \in [1, n_1]$.

Thus we see that, if $i \geq n_2 + (2d+1)n_1$, then

$$\lambda_{i+1}^{\omega}(B_N) \geq \min_z \lambda_2^{\tilde{\omega}}(B_N \cap \mathbf{B}_z),$$

where $\tilde{\omega}$ is a new environment obtained by modifying the value of ω to 1 on all points δ_i and z ranges through those points in $\mathbb{Z}^d$ such that $\mathbf{B}_z$ intersects B_N.

We now choose for δ_j the points in B_N where ω achieves its lowest values. Let us use (4.9) to estimate each eigenvalue $\lambda_2^{\tilde{\omega}}(B_N \cap \mathbf{B}_z)$:

$$\frac{1}{\lambda_{i+1}^{\omega}(B_N)} \leq C \left(k^{2+\varepsilon} + \sup_{x \in B_N}^{n_1} \frac{1}{\omega(x)} \right) d_\varepsilon^N. \tag{4.11}$$

We used d_ε^N as a uniform upper bound for the minimal side length of strips for which the event $A_N(L)$ in Definition 3.13 occurs.

$\sup_{x \in B_N}^{n_1} 1/\omega(x)$ denotes the maximal value of $1/\tilde{\omega}(x)$, i.e.,

$$\sup_{x \in B_N}^{n_1} \frac{1}{\omega(x)} = \max\{h : \#\{x \in B_N : \omega(x) = 1/h\} \geq n_1 + 1\}.$$

4.3. *Proof of Theorem 4.3*

Remember that we have already chosen some parameter $\xi > 0$ (that we want to choose close to 0 and which is related to t by $N = t^{(1+\xi)/2}$), and another parameter $\varepsilon > 0$ which is arbitrarily close to 0. We need a third parameter $a \in (0, 1)$. The constant A in (4.10) is at our disposal. We also still have to choose n_1 and n_2, depending on i and such that $i \geq n_2 + (2d+1)n_1$.

Write

$$\begin{aligned}
&\mathbb{Q}\left[\sum_i e^{-\lambda_i^{\omega}(B_N)t}\right] \\
&\leq \mathbb{Q}\left[\sum_i e^{-\lambda_i^{\omega}(B_N)t}; d_\varepsilon^N \geq A\right] + \sum_i \mathbb{Q}\left[e^{-\lambda_i^{\omega}(B_N)t}; \lambda_i^{\omega}(B_N) \geq N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2\right] \\
&\quad + \sum_i \mathbb{Q}\left[e^{-\lambda_i^{\omega}(B_N)t}; d_\varepsilon^N \leq A \text{ and } \lambda_i^{\omega}(B_N) \leq N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2\right] \\
&\leq (2N+1)^d \mathbb{Q}[d_\varepsilon^N \geq A] + \sum_i e^{-N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2 t} \\
&\quad + \sum_i \mathbb{Q}\left[d_\varepsilon^N \leq A \text{ and } \lambda_i^{\omega}(B_N) \leq N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2\right].
\end{aligned} \tag{4.12}$$

Using (4.10), we see that we can choose A in such a way that

$$\limsup_{t\to+\infty} \frac{\log[(2N+1)^d \mathbb{Q}(d_\varepsilon^N \geq A)]}{\log t} \leq 0 \vee \left(\frac{d}{2} - \gamma\right). \tag{4.13}$$

An easy computation shows that

$$\limsup_{t\to+\infty} \frac{\log \sum_i e^{-N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2 t}}{\log t} \leq \frac{d}{2a}\left(\xi + \frac{\varepsilon}{2} + \frac{\varepsilon\xi}{2}\right). \tag{4.14}$$

Let us now bound the last term in (4.12). Assume that $d_\varepsilon^N \leq A$ and $\lambda_i^\omega(B_N) \leq N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2$. From (4.11), we must have

$$N^{\varepsilon+2} i^{-\frac{2a}{d}} \leq C\left(k^{2+\varepsilon} + \sup_{x\in B_N}^{n_1} \frac{1}{\omega(x)}\right) A.$$

We choose $n_2 = i^a$ and assume that i is large enough, how large depending on the dimension, a and γ only, which we may do. Then $k^{2+\varepsilon} \leq N^{2+\varepsilon} i^{-\frac{a(2+\varepsilon)}{d}}$. Therefore, we must have

$$N^{\varepsilon+2} i^{-\frac{2a}{d}} \leq C \sup_{x\in B_N}^{n_1} \frac{1}{\omega(x)},$$

with a possibly different value for C.

From now on, we deal separately with the cases of large or small values of γ.

Case $\gamma \geq \frac{d}{2}$. We then choose $\varepsilon < 2\frac{d}{\gamma}$ and $a = 1 - \frac{d}{2\gamma} + \frac{\varepsilon}{4}$.

The computation goes as follows (the value of C changes from line to line)

$$\begin{aligned}
&\sum_i \mathbb{Q}\left[d_\varepsilon^N \leq A \text{ and } \lambda_i^\omega(B_N) \leq N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2\right] \\
&\leq \sum_i \mathbb{Q}\left[C \sup_{x\in B_N}^{n_1} \frac{1}{\omega(x)} \geq N^{\varepsilon+2} i^{-\frac{2a}{d}}\right] \\
&\leq \sum_i \mathbb{Q}\left[C \sup_{x\in B_N}^{n_1} \frac{1}{\omega(x)} \geq N^{\varepsilon+2-2a}\right] \\
&= \mathbb{Q}\left[\#\left\{i : C \sup_{x\in B_N}^{n_1} \frac{1}{\omega(x)} \geq N^{\varepsilon+2-2a}\right\}\right] \\
&\leq C\mathbb{Q}\left[\#\left\{i : C \sup_{x\in B_N}^{i} \frac{1}{\omega(x)} \geq N^{\varepsilon+2-2a}\right\}\right] \\
&= C(2N+1)^d \mathbb{Q}\left[\frac{1}{\omega(x)} \geq N^{\varepsilon+2-2a}\right] \\
&\leq CN^d N^{-\gamma(\varepsilon+2-2a)} = CN^{-\frac{3\varepsilon\gamma}{4}},
\end{aligned} \tag{4.15}$$

where the second inequality follows because $i \leq (2N+1)^d$, and the third one because $n_1 = (i - i^a)/(2d+1)$ and $a < 1$. Thus we deduce from (4.15), (4.12), (4.13) and (4.14) that

$$\limsup_{t\to+\infty} \frac{\log \mathbb{Q}\left[\sum_i e^{-\lambda_i^\omega(B_N)t}\right]}{\log t} \leq \frac{d}{2a}\left(\xi + \frac{\varepsilon}{2} + \frac{\varepsilon\xi}{2}\right).$$

Let ε tend to 0 and then ξ tend to 0 to deduce (4.8). This ends the proof of Theorem 4.3 in the case $\gamma \geq \frac{d}{2}$.

Case $\gamma < \frac{d}{2}$. Let $\delta \in (0, \gamma)$, to be chosen later. We have

$$\begin{aligned}
&\sum_i \mathbb{Q}\left[d_\varepsilon^N \leq A \text{ and } \lambda_i^\omega(B_N) \leq N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2\right] \\
&\quad\leq \sum_i \mathbb{Q}\left[C \overset{n_1}{\sup_{x\in B_N}} \frac{1}{\omega(x)} \geq N^{\varepsilon+2} i^{-\frac{2a}{d}}\right] \\
&\quad\leq N^{(2a-\varepsilon-2)\delta} \sum_i \mathbb{Q}\left[\left(\overset{n_1}{\sup_{x\in B_N}} \frac{1}{\omega(x)}\right)^\delta\right],
\end{aligned}$$

since $i \leq (2N+1)^d$. Remember that $n_2 = i^a$ is much smaller than $n_1 = (i - i^a)/(2d+1)$ for large values of i, say $i/(4d+2) \leq n_1 \leq i/(2d+1)$. Let $x_0, \ldots x_j, \ldots x_{(2N+1)^d - 1}$ be an enumeration of the points in B_N such that the sequence $\omega(x_j)$ is increasing. Thus

$$\overset{n_1}{\sup_{x\in B_N}} \frac{1}{\omega(x)} = \frac{1}{\omega(x_{n_1})} \leq \frac{1}{\omega\left(x_{i/(4d+2)}\right)}.$$

Therefore

$$\sum_i \mathbb{Q}\left[\left(\overset{n_1}{\sup_{x\in B_N}} \frac{1}{\omega(x)}\right)^\delta\right] \leq (4d+2)\sum_{x\in B_N} \mathbb{Q}\left[\left(\frac{1}{\omega(x)}\right)^\delta\right] = c_\delta(4d+2)(2N+1)^d,$$

where $c_\delta = \mathbb{Q}\left[\left(\frac{1}{\omega(x)}\right)^\delta\right]$. Note that $\mathbb{Q}\left[\left(\frac{1}{\omega(x)}\right)^\delta\right]$ is finite and does not depend on x. Therefore

$$\sum_i \mathbb{Q}\left[d_\varepsilon^N \leq A \text{ and } \lambda_i^\omega(B_N) \leq N^{-\varepsilon}\left(\frac{i^{a/d}}{N}\right)^2\right] \leq c_\delta N^{(2a-\varepsilon-2)\delta}(2N+1)^d.$$

Gathering this last inequality with (4.13) and (4.14), we get that

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{\log \mathbb{Q}\left[\sum_i e^{-\lambda_i^\omega(B_N)t}\right]}{\log t} \\
&\quad\leq \max\left\{\frac{d}{2} - \gamma; \frac{d}{2a}\left(\xi + \frac{\varepsilon}{2} + \frac{\varepsilon\xi}{2}\right); \frac{1+\xi}{2}[d + (2a - \varepsilon - 2)\delta]\right\},
\end{aligned} \tag{4.16}$$

with $a \in (0,1)$ and $\delta \in (0,\gamma)$. First replace δ by γ. Then let ε tend to 0 and choose $a = \frac{d\xi}{d-2\gamma}$. The upper bound in (4.16) becomes $\max\left[\frac{d}{2}-\gamma; \frac{1+\xi}{2}(d-2\gamma)\right.$ $\left.+\frac{(1+\xi)\xi d\gamma}{d-2\gamma}\right]$. Finally let ξ tend to 0 and conclude that

$$\lim_{\xi\to 0}\limsup_{t\to+\infty}\frac{\log \mathbb{Q}\left[\sum_i e^{-\lambda_i^\omega(B_N)t}\right]}{\log t} \leq \frac{d}{2}-\gamma,$$

and Theorem 4.3 is now proved in the case $\gamma < \frac{d}{2}$. □

5. Quenched decay of the return probability

In this section, we investigate the quenched decay of the return probability when $\gamma < \frac{d}{2}$. Model and notation are the same as in Section 4: a random walk among i.i.d. random conductancies with a power law with an exponent γ. Now we are rather interested in the asymptotics of the return probability $\mathbb{P}_0^\omega[X_t = 0]$ in $\mathbb{Q}$ probability. Let us set α_c to be the best exponent α such that

$$\mathbb{Q}[\mathbb{P}_0^\omega[X_t = 0] \leq t^{-\alpha}] \to 1 \text{ as } t \to \infty\,. \tag{5.1}$$

From Theorem 4.3, it is clear that $\alpha_c \geq \frac{d}{2} \wedge \gamma$. We can do better in the case $\gamma < \frac{d}{2}$:

Theorem 5.1. *For any $\gamma < \frac{d}{2}$ then $\alpha_c > \gamma$.*

Remark 5.2. Although rather unsatisfactory — because it does not give the true value of α_c — Theorem 5.1 shows that the typical decay of the return probability is strictly faster than the averaged decay. Such a situation is sometimes called in the literature a ‘high disorder regime’.

Remark 5.3. The proof of Theorem 5.1 actually yields the lower bound

$$\alpha_c \geq \frac{d}{2}\frac{1+\gamma}{1+d/2}. \tag{5.2}$$

There is no reason to believe that this bound is sharp for a given value of γ. Notice however that, in the regime $\gamma \to \frac{d}{2}$, we get the inequality $\alpha_c \geq \frac{d}{2}$, which seems to be sharp.

Let us sketch the proof: we use the fact that, with large $\mathbb{Q}$ probability, the origin lies in an infinite percolation cluster, say $\mathcal{C}$, of ‘good’ sites, where ω is bounded from below. Estimates on the return probability for random walks on percolation clusters have been proved in [6] (See also [1]). One strategy would then be to try to couple the random walk in the environment ω with the random walk on $\mathcal{C}$: we have no idea on how to do that. We rather rely on spectral theory to compare the behaviours of the eigenvectors for the two random walks. Note that from the results of [6] follow precise estimates on the eigenvalues of the discrete Laplace operator on $\mathcal{C}$. The core of the proof is to show that eigenvectors of the generator of the random walk in the environment ω, when they correspond to small enough eigenvalues, are concentrated outside $\mathcal{C}$, and therefore do not contribute too much to the asymptotics of the return probability as soon as the random walk starts outside $\mathcal{C}$.

5.1. Proof of Theorem 5.1. Step 1

Let $\alpha < \frac{d}{2}\frac{1+\gamma}{1+d/2}$. Choose two parameters $\varepsilon > 0$ and $\xi > 0$. We shall use the notation $N = t^{(1+\xi)/2}$. (In fact, N should be defined as the integer part of $t^{(1+\xi)/2}$, but, for notational ease, we will omit integer parts.) All the limits to be taken are to be understood as $t \to \infty$ or, equivalently $N \to \infty$.

Let $\mathcal{C}^\omega$ be the largest connected component of the set $\{x \in \mathbb{Z}^d : \omega(x) \geq N^{-\varepsilon}\}$. We assume that N is large enough so that $\mathbb{Q}[\omega(x) \geq N^{-\varepsilon}]$ becomes larger than the critical percolation probability on $\mathbb{Z}^d$. Then $\mathcal{C}^\omega$ is the unique infinite connected component of the set $\{x \in \mathbb{Z}^d : \omega(x) \geq N^{-\varepsilon}\}$, see [5]. We denote by $\mathcal{C}^\omega_N$ the largest connected component of the intersection $\mathcal{C}^\omega \cap B_N$, where $B_N = [-N, N]^d$.

In the next step of the proof, we will define a set of environments, denoted Ω_N, such that $\mathbb{Q}[\Omega_N] \to 1$. We further have the property $\mathbb{Q}[\frac{\#\mathcal{C}^\omega_N}{\#B_N}] \to 1$.

Calling $\{\lambda_i^\omega(B_N), i \in [1, \#B_N]\}$ the eigenvalues of $-\mathcal{L}^{\omega,N}$ in increasing order, and $\{\psi_i^\omega, i \in [1, \#B_N]\}$ the corresponding eigenvectors with due normalization in $L^2(B_N)$, a very similar computation as in Subsection 4.1 leads to the following series of inequalities.

We first use the invariance by translation of $\mathbb{Q}$.

$$\mathbb{Q}[\mathbb{P}_0^\omega[X_t = 0] \geq t^{-\alpha}] = \mathbb{Q}[\mathbb{P}_x^\omega[X_t = x] \geq t^{-\alpha}]$$

holds for any $x \in B_N$. Therefore

$$\mathbb{Q}[\mathbb{P}_0^\omega[X_t = 0] \geq t^{-\alpha}] = \frac{1}{\#B_N} \sum_{x \in B_N} \mathbb{Q}[\mathbb{P}_x^\omega[X_t = x] \geq t^{-\alpha}].$$

Note that

$$\begin{aligned}\mathbb{Q}[\mathbb{P}_x^\omega[X_t = x] \geq t^{-\alpha}] \leq\ & \mathbb{Q}[\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] \geq t^{-\alpha}/2] \\ & + \mathbb{Q}[\mathbb{P}_x^\omega[t \geq \tau_{2N}] \geq t^{-\alpha}/2],\end{aligned}$$

where τ_{2N} is the exit time of B_{2N}. Since $\sup_\omega \sup_x \mathbb{P}_x^\omega[t \geq \tau_{2N}]$ decays faster than any polynomial, see (4.7), we have

$$\begin{aligned}&\limsup \mathbb{Q}[\mathbb{P}_0^\omega[X_t = 0] \geq t^{-\alpha}] \\ &\quad \leq \limsup \frac{1}{\#B_N} \sum_{x \in B_N} \mathbb{Q}[\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] \geq t^{-\alpha}/2].\end{aligned}$$

We now restrict our attention to those environments belonging to Ω_N and to the points $x \in \mathcal{C}^\omega_N$:

$$\begin{aligned}&\mathbb{Q}[\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] \geq t^{-\alpha}/2] \\ &\quad \leq \mathbb{Q}[\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] \geq t^{-\alpha}/2; x \in \mathcal{C}^\omega_N; \Omega_N] + \mathbb{Q}[\Omega_N^c] + \mathbb{Q}[x \notin \mathcal{C}^\omega_N].\end{aligned}$$

Since $\mathbb{Q}[\Omega_N^c] \to 0$, we therefore get that

$$\begin{aligned}
&\limsup \mathbb{Q}[\mathbb{P}_0^\omega[X_t = 0] \geq t^{-\alpha}] \\
&\quad \leq \limsup \frac{1}{\#B_N} \sum_{x \in B_N} \mathbb{Q}[\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] \geq t^{-\alpha}/2; x \in \mathcal{C}_N^\omega; \Omega_N] \\
&\qquad + \limsup \mathbb{Q}\left[\frac{\#(\mathcal{C}_N^\omega)^c}{\#B_N}\right].
\end{aligned}$$

But since $\mathbb{Q}[\frac{\#\mathcal{C}_N^\omega}{\#B_N}] \to 1$ (see step 2 below), we have

$$\begin{aligned}
&\limsup \mathbb{Q}[\mathbb{P}_0^\omega[X_t = 0] \geq t^{-\alpha}] \\
&\quad \leq \limsup \frac{1}{\#B_N} \sum_{x \in B_N} \mathbb{Q}[\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] \geq t^{-\alpha}/2; x \in \mathcal{C}_N^\omega; \Omega_N].
\end{aligned}$$

From the Markov inequality, we deduce that

$$\begin{aligned}
&\mathbb{Q}[\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] \geq t^{-\alpha}/2; x \in \mathcal{C}_N^\omega; \Omega_N] \\
&\quad \leq 2t^\alpha \mathbb{Q}[\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}]; x \in \mathcal{C}_N^\omega; \Omega_N],
\end{aligned}$$

and thus

$$\begin{aligned}
&\limsup \mathbb{Q}[\mathbb{P}_0^\omega[X_t = 0] \geq t^{-\alpha}] \\
&\quad \leq 2 \limsup \frac{t^\alpha}{\#B_N} \mathbb{Q}[\sum_{x \in \mathcal{C}_N^\omega} \mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}]; \Omega_N].
\end{aligned}$$

Finally we express the probability $\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}]$ in the spectral decomposition as

$$\mathbb{P}_x^\omega[X_t = x; t < \tau_{2N}] = \frac{1}{\#B_N} \sum_i e^{-\lambda_i^\omega(B_{2N})\, t} (\psi_i^\omega(x))^2,$$

and get that

$$\begin{aligned}
&\limsup \mathbb{Q}[\mathbb{P}_0^\omega[X_t = 0] \geq t^{-\alpha}] \\
&\quad \leq 2 \limsup \frac{t^\alpha}{\#B_N} \mathbb{Q}\left[\sum_i e^{-\lambda_i^\omega(B_{2N})\, t} \frac{1}{\#B_N} \sum_{x \in \mathcal{C}_N^\omega} (\psi_i^\omega(x))^2; \Omega_N\right]. \qquad (5.3)
\end{aligned}$$

Let us pause a little to look at (5.3). It is true that $\frac{1}{\#B_N} \sum_{x \in \mathcal{C}_N^\omega} (\psi_i^\omega(x))^2 \leq 1$; but if we would use this upper bound, we would be left with $\mathbb{Q}[\sum_i e^{-\lambda_i^\omega(B_{2N})\, t}]$, and the best value for α would then be γ, as the results of Section 4 show. We have to find a better way. Note that terms corresponding to large values of i, and thus large values of $\lambda_i^\omega(B_{2N})$, can be easily controlled. Thus the main point is to show that $\frac{1}{\#B_N} \sum_{x \in \mathcal{C}_N^\omega} (\psi_i^\omega(x))^2$ is small enough for small i, i.e. we have to prove that eigenvectors corresponding to small eigenvalues are concentrated outside $\mathcal{C}_N^\omega$. And in fact one would expect this to be true since small eigenvalues arise because of small values of ω, and these precisely sit outside $\mathcal{C}_N^\omega$.

5.2. Step 2. Definition of Ω_N

The set Ω_N is defined by two requirements: we ask that for any $\omega \in \Omega_N$ we have

$$\text{(i) } 0 \in \mathcal{C}_N^\omega.$$

The second requirement deals with the behaviour of the random walk on $\mathcal{C}_N^\omega$: let $(\mu_i, i \in [1, \#\mathcal{C}_N^\omega])$ be the eigenvalues of the discrete Laplace operator on $\mathcal{C}_N^\omega$ as defined in [6]. We will also use the notation $(\phi_i, i \in [1, \#\mathcal{C}_N^\omega])$ for the corresponding eigenvectors. We assume that the eigenvalues are in increasing order and the eigenvectors are normalized in $L^2(\mathcal{C}_N^\omega)$ for the counting measure. Of course the μ_is and ϕ_is depend on ω and N.

Let

$$\eta = \frac{1+\gamma}{\frac{1}{2}+\frac{1}{d}} \text{ and } j = N^{d-\eta}.$$

Note that since $\gamma < d/2$, then $\eta < d$. We then require that, on Ω_N,

$$\text{(ii) } \mu_j \geq \frac{j^{2/d}}{N^2(\log N)^{8(d-\eta)/d}}.$$

The definition of Ω_N is now complete and all that remains to be done is to check that $\mathbb{Q}(\Omega_N) \to 1$.

That $\mathbb{Q}(\text{(i) holds}) = \mathbb{Q}(0 \in \mathcal{C}_N^\omega) \to 1$ is obvious.

As for condition (ii), we rely on the results of [6]. Calling $P_x^\omega[X_s^N = y]$ the transition probabilities for the random walk on $\mathcal{C}_N^\omega$, we quote from formula (6) of [6]: $\mathbb{Q}$-a.s. on the set where $\mathcal{C}^\omega$ is infinite

$$\sup_{x,y\in\mathcal{C}_N^\omega} \left| \frac{1}{\#\mathcal{C}_N^\omega} - P_x^\omega[X_s^N = y] \right| \leq C \frac{(\log N)^{2d}}{s^{\frac{d}{2}+d\frac{\log\log N}{\log N}}},$$

where C is a dimension dependent constant, s is arbitrary, and $N \geq N_0(\omega)$ is large enough. (In [6], formula (6) is deduced from the isoperimetric inequality (4), (4) is a consequence of (21), and (21) is proved for both site and bond percolation models with parameter p close enough to 1, which is our case here. Besides, we replaced $\varepsilon(N)$ by its value $\varepsilon(N) = d + 2d\frac{\log\log N}{\log N}$, noticing that $(4\varepsilon(N)/\beta^2)^{\varepsilon(N)/2}$ then behaves like a constant.)

We then choose $x = y$, sum over $x \in \mathcal{C}_N^\omega$, and express the result as a trace to get that

$$\sum_i e^{-\mu_i s} \leq 1 + C\#\mathcal{C}_N^\omega \frac{(\log N)^{2d}}{s^{\frac{d}{2}+d\frac{\log\log N}{\log N}}}.$$

Therefore

$$j e^{-\mu_j s} \leq 1 + C\#\mathcal{C}_N^\omega \frac{(\log N)^{2d}}{s^{\frac{d}{2}+d\frac{\log\log N}{\log N}}}.$$

Take now $s = \frac{N^2}{j^{2/d}}(\log N)^{8(d-\eta)/d}$. Then

$$\frac{1}{j} C \# \mathcal{C}_N^\omega \frac{(\log N)^{2d}}{s^{\frac{d}{2}+d\frac{\log\log N}{\log N}}} \to 0,$$

so that $e^{-\mu_j s} \to 0$, and we have proved that $\mathbb{Q}$-a.s. on the set where $\mathcal{C}^\omega$ is infinite, for large enough N, condition (ii) is fullfilled.

Finally we already used the fact that $\mathbb{Q}[\frac{\#\mathcal{C}_N^\omega}{\#B_N}] \to 1$ that should be justified: from the result of Appendix B of [6], we know that the expected density in B_N of the component of $\mathcal{C}^\omega \cap B_N$ that contains the origin goes to 1 as $N \to \infty$, and Lemma 3.5 implies that the expected density in B_N of the largest component of $\mathcal{C}^\omega \cap B_N$ goes to 1 as $N \to \infty$. Thus the component of $\mathcal{C}^\omega \cap B_N$ that contains the origin and $\mathcal{C}_N^\omega$ coincide for large N and its density tends to 1.

5.3. Step 3. Spectral analysis

Assume that $\omega \in \Omega_N$.

We bound the term $\sum_{x\in B_N}(\psi_i^\omega(x))^2$ in (5.3) in two steps by writing that

$$\begin{aligned}\frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}(\psi_i^\omega(x))^2 &= \frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}(\psi_i^\omega(x) - P^j\psi_i^\omega(x))^2\\ &\quad+\frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}(P^j\psi_i^\omega(x))^2,\end{aligned}$$

where P^j is the projection on the subspace of $L^2(\mathcal{C}_N^\omega)$ spanned by the eigenvectors $(\phi_i, i \in [1, j])$.

On one hand, since $\frac{1}{\#B_N}\sum_{x\in B_N}(\phi_i(x))^2 \le 1$, then

$$\begin{aligned}\sum_i e^{-\lambda_i^\omega(B_{2N})t}\frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}(P^j\psi_i^\omega(x))^2 &\le \sum_i \frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}(P^j\psi_i^\omega(x))^2\\ &= \sum_i\sum_{k\le j}\frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}\phi_k(x)\psi_i^\omega(x)\\ &= \sum_{k\le j}\frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}(\phi_k(x))^2\\ &\le j = N^{d-\eta}.\end{aligned}$$

On the other hand, for any function f on $\mathcal{C}_N^\omega$, we have

$$\frac{1}{\#\mathcal{C}_N^\omega}\sum_{x\in\mathcal{C}_N^\omega}(f(x) - P^j f(x))^2 \le \frac{1}{\mu_j}\frac{1}{2\#\mathcal{C}_N^\omega}\sum_{x\sim y\in\mathcal{C}_N^\omega}(f(x)-f(y))^2,$$

this last expression being the Dirichlet form of the random walk on $\mathcal{C}_N^\omega$. Since $\omega(x) \ge N^{-\varepsilon}$ on $\mathcal{C}_N^\omega$, we get

$$\sum_{x\sim y\in\mathcal{C}_N^\omega} (f(x)-f(y))^2 \le N^\varepsilon \sum_{x\sim y\in B_{2N}} (\omega(x)\wedge\omega(y))(f(x)-f(y))^2,$$

this last expression being now the Dirichlet form of the random walk on B_{2N}. Since ψ_i^ω is an eigenvector,

$$\frac{1}{2\#B_{2N}} \sum_{x\sim y\in B_{2N}} (\omega(x)\wedge\omega(y))(\psi_i^\omega(x)-\psi_i^\omega(y))^2 = \lambda_i^\omega(B_{2N}).$$

So

$$\frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}(\psi_i^\omega(x)-P^j\psi_i^\omega(x))^2 \le 2^d N^\varepsilon \frac{\lambda_i^\omega(B_{2N})}{\mu_j}.$$

From these two estimates, we deduce that

$$\sum_i e^{-\lambda_i^\omega(B_{2N})\,t}\frac{1}{\#B_N}\sum_{x\in\mathcal{C}_N^\omega}(\psi_i^\omega(x))^2 \le N^{d-\eta}+2^d\frac{N^\varepsilon}{\mu_j}\sum_i \lambda_i^\omega(B_{2N})e^{-\lambda_i^\omega(B_{2N})\,t}. \tag{5.4}$$

5.4. Step 4

Combining (5.3) and (5.4), we see that Theorem (5.1) will be proved once we have checked that $\frac{t^\alpha}{\#B_N}N^{d-\eta}\to 0$ and that

$$\frac{t^\alpha}{\#B_N}N^\varepsilon\mathbb{Q}\left[\frac{1}{\mu_j}\sum_i\lambda_i^\omega(B_{2N})e^{-\lambda_i^\omega(B_{2N})\,t};\,\Omega_N\right]\to 0. \tag{5.5}$$

We recall that $\alpha < \frac{d}{2}\frac{1+\gamma}{1+d/2}$, $N = t^{(1+\xi)/2}$, $\eta = \frac{1+\gamma}{\frac{1}{2}+\frac{1}{d}}$, and $\mu_j \ge (\frac{j^{1/d}}{N})^2$ $(\log N)^{-8(d-\eta)/d}$ on Ω_N. It is then immediate to see that $\frac{t^\alpha}{\#B_N}N^{d-\eta}\to 0$. Besides, (5.5) will hold for any $\alpha < \frac{d}{2}\frac{1+\gamma}{1+d/2}$ and some $\varepsilon > 0$ if

$$\lim_{\xi\to 0}\limsup\frac{\mathbb{Q}[\sum_i\lambda_i^\omega(B_{2N})e^{-\lambda_i^\omega(B_{2N})\,t}]}{\log t}\le\frac{d}{2}-1-\gamma. \tag{5.6}$$

But using the inequality $\lambda_i e^{-\lambda_i t}\le\frac{1}{t}e^{-\frac{1}{2}\lambda_i t}$, we get

$$\begin{aligned}&\lim_{\xi\to 0}\limsup\frac{\mathbb{Q}[\sum_i\lambda_i^\omega(B_{2N})e^{-\lambda_i^\omega(B_{2N})\,t}]}{\log t}\\ &\quad\le -1+\lim_{\xi\to 0}\limsup\frac{\mathbb{Q}[\sum_i e^{-\lambda_i^\omega(B_{2N})\,t/2}]}{\log t}\\ &\quad\le -1+\frac{d}{2}-\gamma,\end{aligned}$$

by (4.5). □

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