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L^p estimates for the uniform norm of solutions of quasilinear SPDE's

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Abstract. In this paper we prove L^p estimates ($p \geqslant 2$) for the uniform norm of the paths of solutions of quasilinear stochastic partial differential equations (SPDE) of parabolic type. Our method is based on a version of Moser's iteration scheme developed by Aronson and Serrin in the context of non-linear parabolic PDE.

1. Introduction

The aim of this paper is to study the following Stochastic Partial Differential Equation :

$$du_t(x)+Au_t(x)dt+f\left(t,x,u_t(x),\nabla u_t(x)\right)dt+\sum_{i=1}^{d}\partial_i g_i\left(t,x,u_t(x),\nabla u_t(x)\right)dt$$

$$=\sum_{j=1}^{d_1} h_j\left(t,x,u_t(x),\nabla u_t(x)\right)\, dB_t^j, \tag{1}$$

where A is a second order symmetric differential operator defined in some domain $\mathcal{O}\subset\mathbb{R}^d$. We are interested in studying the behavior of the weak solution. More precisely, if $H_0^1(\mathcal{O})$ denotes the standard Sobolev space with zero Dirichlet condition, then under suitable Lipschitz hypotheses on the coefficients f, g, h, we get the following estimate

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Theorem 1. *Let u be a $H_0^1(\mathcal{O})$-valued predictable process which is a weak solution of the equation (1) in the variational sense with initial condition $\xi \in L^p(\Omega; L^\infty(\mathcal{O}))$ for some $p \geqslant 2$. Then the following estimate holds*

$$E\,\|u\|_{\infty,\infty;t}^p \le k(t)\,E\left(\|\xi\|_\infty^p + \left\|f^0\right\|_{\theta;t}^{*p} + \left\||g^0|^2\right\|_{\theta;t}^{*p/2} + \left\||h^0|^2\right\|_{\theta;t}^{*p/2}\right).$$

where

$$\|u(\omega)\|_{\infty,\infty,t} = \sup_{0\leqslant s\leqslant t,\, x\in\mathcal{O}} |u(s,\omega,x)|,$$

k is a function which only depends on the structure constants of the SPDE ,

$$f^0(t,\omega,x) = f(t,\omega,x,0,0), \quad g^0(t,\omega,x) = g(t,\omega,x,0,0),$$
$$h^0(t,\omega,x) = h(t,\omega,x,0,0)$$

and $\|\cdot\|_{\theta;t}^*$ is a certain norm which will be defined in the next section.

This result implies in particular that P-almost surely, $u(t,x)$ is uniformly bounded in t and x.

Motivated by numerical problems, Krylov proved in [10] some fundamental results concerning the L^p -estimates of solutions of SPDE's. His approach is based on the theory of Sobolev spaces, in particular, the classical Sobolev embeding theorem ensures L^p -estimates for the uniform norm. His method requires the coefficients to have some smoothness. In the present paper we introduce Moser's iteration technique in the context of SPDE's. This method allows us to obtain L^p estimates for the uniform norm of the paths of solutions under weaker conditions on the coefficients (all the coefficients are only assumed to be measurable). We should also mention that Gyöngy and Rovira [9] derived L^p -estimates by deriving first L^p -estimates for the Green kernel. They assume that the coefficients of the elliptic operator are smooth and, in particular, deduce that the solution is pathwise continuous. However their method does not produce L^p estimates for the uniform norm of the paths.

This paper is divided as follows: in the next section, we recall some facts concerning $L^{p,q}$-spaces and set the hypotheses and notations for the rest of the paper. Next we establish Itô's formula for the spacial integral of solution of the SPDE which permits to obtain L^p-estimates. In the fourth section we prove the desired estimates. Finally, in an appendix we give a technical lemma on $L^{p,q}$-norm.

2. Preliminaries

2.1. The $L^{p,q}$- spaces

Let $\mathcal{O} \subset \mathbb{R}^d$ be an open domain with finite Lebesgue measure in $\mathbb{R}^d$ and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$.

We shall recall some preliminary facts from Aronson Serrin [1] in a slightly modified form. For each $t > 0$ and for all real numbers $p, q \geqslant 1$, we denote by $L^{p,q}([0, t] \times \mathcal{O})$ the space of measurable functions $u : [0, t] \times \mathcal{O} \longrightarrow \mathbb{R}$ such that

$$\|u\|_{p,q;t} := \Big(\int_0^t \Big(\int_{\mathcal{O}} |u(s,x)|^p \, dx \Big)^{q/p} ds \Big)^{1/q}$$

is finite. The limiting cases with p or q taking the value ∞ are also considered with the obvious use of the essential sup norm. In our analysis of solutions of SPDE's we need the following interpolation result between the spaces $L^{p,q}$ which is a consequence of Hölder's inequality (see [1]), as well as the classical Sobolev inequality.

Lemma 2. *If* $u \in L^{p_1,q_1} \bigcap L^{p_2,q_2}$, *then* $u \in L^{p,q}$ *where* $p_i, q_i \in [1, \infty]$, $i = 1, 2$ *and*

$$\frac{1}{p} = \frac{\mu}{p_1} + \frac{\nu}{p_2}, \qquad \frac{1}{q} = \frac{\mu}{q_1} + \frac{\nu}{q_2}, \qquad \mu,\ \nu \geqslant 0 \quad \text{and} \quad \mu + \nu = 1\,.$$

Moreover we have

$$\|u\|_{p,q;t} \leqslant \|u\|^{\mu}_{p_1,q_1;t} \, \|u\|^{\nu}_{p_2,q_2;t} \tag{2}$$

Lemma 3 (Sobolev's inequality). *Assume* $d > 2$. *Let* $u \in H^1_0(\mathcal{O})$, *then* $u \in L^{2^*}(\mathcal{O})$ *where* $2^* = 2d/d - 2$ *and there exists a constant* $c > 0$ *which depends only on the dimension d such that*

$$\|u\|_{2^*} \leqslant c\,\|\nabla u\|_2\,.$$

If $d = 1, 2$, *then* 2^* *may be any finite real greater than 2.*

A consequence of Sobolev's inequality which will be used in our context is the following

$$\||u|^2\|_{\frac{2^*}{2},1;t} \leqslant c\,\||\nabla u|^2\|_{1,1;t}, \qquad \forall\, u \in L^2\big([0, t];\ H^1_0(\mathcal{O})\big)\,. \tag{3}$$

Now we introduce the following sets associated to a fixed $\theta \in [0, 1)$:

$$\Gamma_\theta = \left\{ (p, q) \in [1, \infty]^2 \,,\ \frac{d}{2p} + \frac{1}{q} = \frac{d}{2} + \theta \right\},$$

$$\Gamma^*_\theta = \left\{ (p, q) \in [1, \infty]^2 \,,\ \frac{d}{2p} + \frac{1}{q} = 1 - \theta \right\}.$$

It is easy to check the following properties:

1. The relations $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ imply that the pair (p, q) belongs to Γ_θ if and only if the pair $\big(p', q'\big)$ belongs to Γ^*_θ.

2. If $(p_i, q_i) \in \Gamma_\theta,\ i = 1, 2$ and $\mu, \nu \geq 0,\ \mu + \nu = 1$, then the pair (p, q) defined by
$$\frac{1}{p} = \frac{\mu}{p_1} + \frac{\nu}{p_2},\ \frac{1}{q} = \frac{\mu}{q_1} + \frac{\nu}{q_2}$$
is in Γ_θ too. A similar property is valid for Γ_θ^*.
3. The pair (p, q) belongs to Γ_θ if and only if there exists $\mu, \nu \geq 0,\ \mu + \nu = 1$ such that
$$\frac{1}{p} = \frac{\mu}{\frac{d}{d-2(1-\theta)}} + \frac{\nu}{1},\ \frac{1}{q} = \frac{\mu}{1} + \frac{\nu}{\frac{1}{\theta}}.$$
This means that the pair of inverses $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to the segment in the plane with end points $\left(\frac{1}{\frac{d}{d-2(1-\theta)}}, 1\right)$ and $(1, \theta)$.
4. If $(p, q) \in \Gamma_\theta$ and $u : [0,\ t] \times \mathcal{O} \longrightarrow \mathbb{R}$, one has
$$\|u\|_{p,q;\,t} \leq \|u\|_{\frac{d}{d-2(1-\theta)},1;\,t} \vee \|u\|_{1,\frac{1}{\theta};\,t}\,.$$
Therefore one has
$$\|u\|_{\theta;\,t} := \|u\|_{\frac{d}{d-2(1-\theta)},1;\,t} \vee \|u\|_{1,\frac{1}{\theta};\,t} = \sup_{(p,q)\in\Gamma_\theta} \|u\|_{p,q;\,t}$$
and this represents a norm on the space $L_\theta := L^{\frac{d}{d-2(1-\theta)},1} \cap L^{1,\frac{1}{\theta}}$. For $\theta = 0$ one has $\left(\frac{2^*}{2}, 1\right) = \left(\frac{d}{d-2}, 1\right)$, $(1, \infty) \in \Gamma_0$ and
$$\|u\|_{0;\,t} := \|u\|_{\frac{d}{d-2},1;\,t} \vee \|u\|_{1,\infty;\,t}\,.$$
5. On the space defined as the algebric sum
$$L_\theta^* := \sum_{(p,q)\in\Gamma_\theta^*} L^{p,q},$$
we introduce the norm
$$\|u\|_{\theta;t}^* := \inf\left\{\sum_{i=1}^{n} \|u_i\|_{p_i,q_i;\,t}\ \middle|\ u = \sum_{i=1}^{n} u_i,\, u_i \in L^{p_i,q_i},\, (p_i, q_i) \in \Gamma_\theta^*,\ i = 1, \ldots n;\ n \in \mathbf{N}\right\}$$
This space represents the dual of L_θ in the sense of the following inequality
$$\int_0^t \int_{\mathcal{O}} u\,(s, x)\, v\,(s, x)\, dx ds \leq \|u\|_{\theta;\,t}\, \|v\|_{\theta;\,t}^* \tag{4}$$
which holds for $u \in L_\theta$ and $v \in L_\theta^*$.

6. Setting $\sigma = 1 + \frac{2\theta}{d}$, it is easy to see that $(\sigma p, \sigma q) \in \Gamma_0$ if and only if $(p, q) \in \Gamma_\theta$. Therefore one has

$$\left\| |u|^\sigma \right\|_{\theta;\, t}^{\frac{1}{\sigma}} = \sup_{(p,q)\in\Gamma_\theta} \left\| |u|^\sigma \right\|_{p,q;\, t}^{\frac{1}{\sigma}} = \sup_{(p,q)\in\Gamma_\theta} \|u\|_{\sigma p, \sigma q;\, t} \leqslant \|u\|_{0;\, t} \,.$$

In order to understand the main point of the above facts related to the $L^{p,q}$-spaces one should have in mind the following remark. In the theory of parabolic PDEs the norms of the type

$$\|u^2\|_{1,\infty;t} + c_1 \int_0^t \int_{\mathcal{O}} |\nabla u_s|^2 dx ds$$

with some positive constant c_1 appear naturally. From the Sobolev inequality (3) we have

$$c \int_0^t \int_{\mathcal{O}} |\nabla u_s|^2 dx ds \geqslant \| |u|^2 \|_{\frac{d}{d-2},1;t} \,.$$

Therefore one has

$$\| |u|^2 \|_{1,\infty;t} + c \int_0^t \int_{\mathcal{O}} |\nabla u_s|^2 dx dt s \geqslant \| |u|^{2\sigma} \|_{\theta;t}^{\frac{1}{\sigma}}$$

for $\sigma = 1 + \frac{2\theta}{d}$. This is the main point which allows one to apply Moser's iteration scheme.

2.2. Hypotheses and definitions

Let $\{B_t := (B_t^j)_{j\in\{1,\cdots,d_1\}}\}_{t\,\geqslant\,0}$ be a d_1-dimensional Brownian motion defined on a standard filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\,\geqslant\,0}, P\big)$.

Let A be a symmetric second order differential operator expressed by

$$A := -L = -\sum_{i,j=1}^{d} \partial_i (a^{i,j}\, \partial_j)$$

with zero Dirichlet boundary conditions. We assume that a is a measurable and symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda |\xi|^2 \leqslant \sum_{i,j} a^{i,j}(x) \xi^i \xi^j \leqslant \Lambda |\xi|^2, \quad \forall x \in \mathcal{O},\ \xi \in \mathbb{R}^d \,, \tag{5}$$

where λ and Λ are positive constants.

Let $(F, \mathcal{E})$ be the associated Dirichlet form given by $F := \mathcal{D}(A^{1/2}) = H_0^1(\mathcal{O})$ which implies

$$\mathcal{E}(u, v) := \big(A^{1/2}u,\ A^{1/2}v\big) \quad \text{and} \quad \mathcal{E}(u, u) = \|\sqrt{A}u\|^2,\ \forall u, v \in F$$

where $(\cdot,\cdot)$ and $\|\,.\,\|$ are respectively the inner product and the norm on $L^2(\mathcal{O})$. $H_0^1(\mathcal{O})$ is the first order Sobolev space of functions vanishing at the boundary . For the notion of Dirichlet form we refer to [8] or [4].

We consider the quasilinear stochastic partial differential equation (in short SPDE) for the real-valued random field $u_t(x)$

$$du_t(x)+Au_t(x)dt+f(t,x,u_t(x),\nabla u_t(x))dt+\sum_{i=1}^{d}\partial_i g_i(t,x,u_t(x),\nabla u_t(x))dt$$
$$=\sum_{j=1}^{d_1} h_j(t,x,u_t(x),\nabla u_t(x))\,dB_t^j, \tag{6}$$

with initial condition $u(0,.)=:\xi(.)$ and Dirichlet boundary condition

$$u_t(x)=0, \quad \text{for all } (t,x)\in\ (0,+\infty)\times\partial\mathcal{O}\,.$$

We assume that we have predictable random functions

$$\begin{aligned} f\ &:\ \mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}\,,\\ g=(g_1,...,g_d)\ &:\ \mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d\\ h\ &:\ \mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^{d_1}\ , \end{aligned}$$

which satisfy the following Lipschitz conditions with respect to the last two variables

$$\begin{aligned} |f(t,\omega,x,y,z)-f(t,\omega,x,y',z')| &\leqslant C\big(|y-y'|+|z-z'|\big)\,,\\ \Big(\sum_{i=1}^{d}|g_i(t,\omega,x,y,z)-g_i(t,\omega,x,y',z')|^2\Big)^{\frac12} &\leqslant C\,|y-y'|+\alpha\,|z-z'|\,,\\ \Big(\sum_{j=1}^{d_1}|h_j(t,\omega,x,y,z)-h_j(t,\omega,x,y',z')|^2\Big)^{\frac12} &\leqslant C\,|y-y'|+\beta\,|z-z'|\,, \end{aligned} \tag{7}$$

where C,α,β are non negative constants.

For the existence of solutions we will make use of the result from [7] and, in order to ensure the validity of the result, we assume that the constants α and β satisfy the condition

$$\alpha+\frac{\beta^2}{2}<\lambda\,. \tag{8}$$

This last condition means that the size of the second order perturbation and the first order perturbation associated with the Brownian motion, should be small. Moreover we define

$$\begin{aligned} f(\cdot,\cdot,\cdot,0,0) &:= f^0\\ h(\cdot,\cdot,\cdot,0,0) &:= h^0=(h_1^0,...,h_{d_1}^0)\\ g(\cdot,\cdot,\cdot,0,0) &:= g^0=(g_1^0,...,g_d^0). \end{aligned}$$

We also assume that ξ is a $\mathcal{F}_0$-measurable, $L^\infty(\mathcal{O})$-valued random variable. The coefficient f^0, $|g^0|^2$, $|h^0|^2$ are L^*_θ -valued random variable, for a certain $\theta \in [0,1)$, and for each $t \geqslant 0$ one has

$$\|\xi\|_\infty , \quad \left\| f^0 \right\|^*_{\theta;t} , \quad \left(\left\| |g^0|^2 \right\|^*_{\theta;t} \right)^{\frac{1}{2}}, \quad \left(\left\| |h^0|^2 \right\|^*_{\theta;t} \right)^{\frac{1}{2}} \quad \in \quad L^p\left(\Omega, P\right) . \tag{9}$$

for a certain number $p \geqslant 2$.

2.3. Weak solutions

Let $L^2_{loc}\left(\mathbb{R}_+; H^1_0(\mathcal{O})\right)$ be the space of all measurable functions $u : \mathbb{R}_+ \to H^1_0(\mathcal{O})$ such that

$$\left(\int_0^T \left(\|u_t\|^2 + \mathcal{E}\left(u_t\right) \right) dt \right)^{1/2} < \infty, \qquad \text{for any } \ T > 0.$$

$\mathcal{H}$ is the space of $H^1_0(\mathcal{O})$-valued predictable processes $(u_t)_{t \geqslant 0}$ such that

$$\left(E \int_0^T \|u_t\|^2 \, dt + \int_0^T E\,\mathcal{E}\left(u_t\right) dt \right)^{1/2} < \infty , \quad \text{for each } T > 0 .$$

Of special interest is the subspace $\widehat{F} \subset L^2_{loc}\left(\mathbf{R}_+; H^1_0\left(\mathcal{O}\right)\right)$ consisting of functions $u \in L^2_{loc}\left(\mathbf{R}_+; H^1_0\left(\mathcal{O}\right)\right)$,which admit a continuous version in $L^2\left(\mathcal{O}\right)$. On this space we have the natural seminorms expressed by

$$\|u\|_T = \left(\sup_{s \leq T} \|u_s\|^2 + \int_0^T \mathcal{E}\left(u_s\right) ds \right)^{\frac{1}{2}} , T > 0.$$

The space of test functions in our study will be $\mathcal{D} = \mathcal{C}^\infty_c\left([0, \ \infty)\right) \otimes \mathcal{C}^2_c(\mathcal{O})$, where $\mathcal{C}^\infty_c\left([0, \infty)\right)$ denotes the space of functions with compact support definde on $[0, \infty)$ which admit an extention as an infinity differentiable function on $(-\infty, \infty)$ and $\mathcal{C}^2_c(\mathcal{O})$ the set of C^2-functions with compact support on $\mathcal{O}$. Since $\mathcal{C}^\infty_c\left([0, \ \infty)\right)$ is dense in $L^2_{loc}\left(\mathbb{R}_+\right)$ and $\mathcal{C}^2_c(\mathcal{O})$ is dense in $H^1_0(\mathcal{O})$ and in $L^2(\mathcal{O})$ respectively, it follows that $\mathcal{D}$ is bothly dense in $L^2_{loc}\left(R_+; H^1_0(\mathcal{O})\right)$ and in $L^2_{loc}\left(\mathbb{R}_+; L^2(\mathcal{O})\right)$.

Definition 4. *We say that $u \in \mathcal{H}$ is a weak solution of equation* (6) *with initial condition $\xi \in L^2(\Omega \times \mathcal{O})$, if the following relation holds almost surely, for each $\varphi \in \mathcal{D}$,*

$$\begin{aligned} \int_0^\infty & [(u_s, \partial_s\varphi) - \mathcal{E}\left(u_s, \varphi_s\right) - \left(f\left(s, u_s, \nabla u_s\right), \varphi_s\right) \\ & + \sum_{i=1}^d \left(g_i\left(s, u_s, \nabla u_s\right), \partial_i\varphi_s\right)]ds \\ & + \sum_{j=1}^{d_1} \int_0^\infty \left(h_j\left(s, u_s, \nabla u_s\right), \varphi_s\right) dB^j_s + (\xi, \varphi_0) = 0. \end{aligned} \tag{10}$$

3. L^p-estimates of the solution and Itô's formula

3.1. A stronger Hypothesis

Concerning existence of solutions we shall rely on the result from [7]. Since the condition (9) does not ensure applicability of the existence theorem (Theorem 8 from [7]), in the first stage we will strengthen it and assume that

$$\xi \in L^{\infty}(\Omega \times \mathcal{O}), \qquad f^0,\ g^0,\ h^0 \in L^{\infty}(\mathbb{R}_+ \times \Omega \times \mathcal{O}) \tag{11}$$

We point out that we always denote by $c > 0$ a constant whose value may change from line to line and that, for any $\epsilon > 0$, we denote by c_ϵ a constant which depends on ϵ like the one appearing in the following typical inequality

$$ab \leqslant \epsilon a^2 + c_\epsilon b^2, \ \ a, b \in \mathbb{R}\,.$$

The proof of the next theorem may be found in [7].

Theorem 5. *Under hypotheses (7), (8) and (11), the SPDE (6) admits a unique solution $u \in \mathcal{H}$ and this solution has $L^2(\mathcal{O})$-continuous trajectories.*

3.2. Itô's formula for the L^p-norm

We will denote by $u := \mathcal{U}(\xi, f, g, h)$ the solution of the equation (6) with initial condition ξ and coefficients $f,\ g,\ h$. In order to prove an Itô type formula with respect to the p -integral over $\mathcal{O}$ we first study solutions of the equation (6) with ξ, f, g, h of a particular type. In the next lemma we consider the linear case, that is we assume that f, g, h do not depend on the last two variables.

Lemma 6. *1) If $f, h_1, \cdots, h_{d_1}$ belong to $\mathcal{C}_c^{\infty}([0,\infty)) \otimes L^2(\Omega) \otimes \mathcal{D}(A)$, $g_1, \cdots, g_d$ belong to $\mathcal{C}_c^{\infty}([0,\infty)) \otimes L^2(\Omega) \otimes \mathcal{D}(A^{3/2})$ and if ξ belongs to $L^2(\Omega) \otimes \mathcal{D}(A)$ then $u := \mathcal{U}(\xi, f, g, h)$ is an $L^2(\mathcal{O})$-valued square integrable semimartingale.*

2) If $f,\ h_1, \cdots, h_{d_1},\ g_1, \cdots, g_d$ belong to $L^2_{loc}\big(\mathbb{R}^+; L^2(\Omega \times \mathcal{O})\big)$ and $\xi \in L^2\big(\Omega \times \mathcal{O}\big)$, then there exists a sequence $(u^k)_{k\in\mathbb{N}}$ of $L^2(\mathcal{O})$-valued square integrable semimartingales which approximates $u := \mathcal{U}(\xi, f, g, h)$ in the sense that $\lim_{k\to\infty} E\,\|u^k - u\|_T^2 = 0$ for all $T > 0$.

Proof. 1) The fact that u is a semi-martingale is a consequence of Lemma 3, Lemma 5 and Proposition 6 in [7]. Namely one has the decomposition

$$\forall t \geqslant 0,\ u_t = \xi + \int_0^t Lu_s ds - \int_0^t f(s) - \sum_{i=1}^{d} \int_0^t \partial_i g_i(s)\, ds$$
$$+ \sum_{j=1}^{d_1} \int_0^t h_j(s)\, dB_s^j\, ds,$$

where each term makes sense because in this case $u \in L^2([0,T] \times \Omega; \mathcal{D}(A))$ for all $T > 0$.

2) For the second part of the Lemma, by Theorem 9 in [7], there exists a constant $K > 0$ which does not depend on f, h, ξ and such that, for each $T > 0$,

$$E(\| u \|_T^2) \leqslant K E\big(\|\xi\|^2 + \int_0^T \|f(s)\|^2 + \sum_{i=1}^d \|g_i(s)\|^2 + \sum_{j=1}^{d_1} \|h_j(s)\|^2 \, ds\big).$$

Consider now sequences $(f^k)_{k\in\mathbb{N}^*}$, $(h_j^k)_{k\in\mathbb{N}^*}$, $1 \leqslant j \leqslant d_1$ in $\mathcal{C}_c^\infty([0, \infty)) \otimes L^2(\Omega) \otimes \mathcal{D}(A)$ which converge in $L^2_{loc}\big([0, \infty) \times \Omega \times \mathcal{O}\big)$ to f and (h_j), $1 \leqslant j \leqslant d_1$ respectively; for each $i \in \{1 \cdots d\}$ a sequence $(g_i^k)_{k\in\mathbb{N}^*}$ in $\mathcal{C}_c^\infty \otimes L^2(\Omega) \otimes \mathcal{D}(A^{3/2})$ which converges to g_i in $L^2_{loc}\big(\mathbb{R}_+; L^2(\Omega \times \mathcal{O})\big)$ and a sequence $(\xi^k)_{k\in\mathbb{N}^*}$ in $L^2(\Omega) \otimes \mathcal{D}(A)$ which converges to ξ in $L^2\big(\Omega \times \mathcal{O}\big)$. We set

$$\forall k \in \mathbb{N}^*,\ u^k := \mathcal{U}(\xi^k, f^k, g^k, h^k),$$

and then, thanks to the first part of this proposition and the inequality we have just recalled, it is easy to conclude. □

Lemma 7. *Assume that $f, h_1 \cdots h_{d_1}, g_1, \cdots, g_d$ belong to $L^2_{loc}\big(\mathbb{R}_+; L^2(\Omega \times \mathcal{O})\big)$ and $\xi \in L^2\big(\Omega \times \mathcal{O}\big)$ and consider $u := \mathcal{U}(\xi, f, g, h)$. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice differentiable function with continuous and bounded second order derivative. Then P-a.s. for all $t \in [0, T]$*

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi(u_t(x))\, dx + \int_0^t \mathcal{E}\big(\varphi'(u_s), u_s\big)\, ds \\
&= \int_{\mathcal{O}} \varphi(\xi)\, dx - \int_0^t \big(\varphi'(u_s), f_s\big)\, ds \\
&\quad + \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi''(u_s(x))\partial_i u_s(x)\, g_i(s, x)\, dx\, ds \\
&\quad + \sum_{j=1}^{d_1} \int_0^t \big(\varphi'(u_s), h_j(s)\big)\, dB_s^j \\
&\quad + \frac{1}{2} \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi''(u_s(x)) h_j^2(s, x)\, dx\, ds,
\end{aligned} \tag{12}$$

where the term $t \to \sum_{j=1}^{d_1} \int_0^t \big(\varphi'(u_s), h_j(s)\big)\, dB_s^j$ is a well-defined martingale hence integrable.

Proof. Assume first that $f, h_1, \cdots, h_{d_1} \in \mathcal{C}_c^\infty \otimes L^2(\Omega) \otimes \mathcal{D}(A)$, $g_1, \cdots, g_d$ belong to $\mathcal{C}_c^\infty \otimes L^2(\Omega) \otimes \mathcal{D}(A^{3/2})$ and $\xi \in L^2(\Omega) \otimes \mathcal{D}(A)$, then u is a semimartingale and one has

$$\forall t \geqslant 0,\ u_t = \xi + \int_0^t Lu_s ds - \int_0^t f(s)\, ds - \sum_{i=1}^{d} \int_0^t \partial_i g_i(s)\, ds + \sum_{j=1}^{d_1} \int_0^t h_j(s)\, dB_s^j\,.$$

Ito's formula for Hilbert-valued semimartingales (see [5] for example) yields

$$\begin{aligned}\int_{\mathcal{O}} \varphi(u_t(x))\, dx = &\int_{\mathcal{O}} \varphi(\xi(x))\, dx + \int_0^t \left(\varphi'(u_s), Lu_s\right) ds - \int_0^t \left(\varphi'(u_s), f_s\right) ds \\ &- \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \varphi'(u_s(x))\partial_i\, g_i(s,x)\, dx\, ds \\ &+ \sum_{j=1}^{d_1} \int_0^t \left(\varphi'(u_s),\ h_j(s)\right) dB_s^j \\ &+ \frac{1}{2}\sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi''(u_s(x)) h_j(s,x)^2\, dx\, ds\,. \qquad (13)\end{aligned}$$

Then, as

$$\left(\varphi'(u_s), Lu_s\right) = -\mathcal{E}\left(\varphi'(u_s), u_s\right),\ \ \forall s \geqslant 0,$$

and

$$\sum_{i=1}^{d} \int_{\mathcal{O}} \varphi'(u_s(x))\partial_i\, g_i(s,x)\, dx = -\sum_{j=1}^{d_1} \int_{\mathcal{O}} \varphi''(u_s(x))\partial_i u_s(x)\, g_i(s,x)\, dx,$$

we get the desired equality.

For the martingale part, let us remark that the square root of its brackets is dominated as follows

$$\begin{aligned}\left(\sum_{j=1}^{d_1} \int_0^T \left(\varphi'(u_s),\ h_j(s)\right)^2 ds\right)^{\frac{1}{2}} &\leqslant \left(\sum_{j=1}^{d_1} \int_0^T \parallel \varphi'(u_s) \parallel^2 \parallel h_j(s) \parallel^2 ds\right)^{\frac{1}{2}} \\ &\leqslant \sup_{s\in[0,T]} \parallel \varphi'(u_s) \parallel^2 + \sum_{j=1}^{d_1} \int_0^T \parallel h_j(s) \parallel^2 ds.\end{aligned}$$

Since φ'' is bounded, the first derivative φ' has at most linear growth, and since $u \in \mathcal{H}$ it follows that $\sup_{s\in[0,T]} \left\|\varphi'(u_s)\right\|^2$ belongs to $L^1(\Omega)$. Therefore the square root of the bracket belongs $L^1(\Omega)$, so that, by the Burkholder-Davis-Gundy inequality we deduce that $t \to \sum_{j=1}^{d_1} \int_0^t \left(\varphi'(u_s),\ h_j(s)\right) dB_s^j$ is a martingale.

The general case is obtained by approximation thanks to the previous lemma. □

Lemma 8. *Assume that hypothesis of section 2.2 and (11) hold. We denote by*

$$K = \|\xi\|_{L^\infty(\Omega\times\mathcal{O})} \vee \|f^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O})} \\ \vee\|h^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O})} \vee \|g^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O};\mathbb{R}^d)} \cdot$$

Then the solution u of the equation (6) belongs to $\bigcap_{p\geqslant 2} L^p\big([0,\ T]\times\mathcal{O}\times\Omega\big)$, for each $T>0$. Moreover there exist constants $c,\ c'>0$ which only depend on $K, C,\ \alpha$ and β such that, for all real $l\geqslant 2$, one has

$$E\int_{\mathcal{O}}|u_t(x)|^l\,dx\ \leqslant\ cK^2 l(l-1)e^{c\,l(l-1)t} \tag{14}$$

and

$$E\int_0^t\int_{\mathcal{O}}|u_t(x)|^{l-2}|\nabla u_t(x)|^2\,dx\ \leqslant\ c'K^2 l(l-1)e^{c\,l(l-1)t}\,. \tag{15}$$

Proof. Notice first that if u is a solution of the equation (6), then

$$f\,(u,\nabla u)\,,\,g_i\,(u,\nabla u)\,,\,h_i\,(u,\nabla u)\in L^2_{loc}\Big(R_+;\,L^2\,(\Omega\times\mathcal{O})\Big)$$

and consequently we may apply Lemma 7 to u.

We fix a real $l\geqslant 2$, $T>0$ and introduce the sequence $(\varphi_n)_{n\in\mathbb{N}^*}$ of functions such that for all $n\in\mathbb{N}^*$:

$$\forall x\in\mathbb{R},\ \varphi_n(x)=\begin{cases}|\ x\ |^l & \text{if } |\ x\ |\ \leqslant n\\ n^{l-2}\left[\frac{l(l-1)}{2}(|x|-n)^2+l\,n(|x|-n)+n^2\right] & \text{if } |\ x\ |> n\end{cases}$$

One can easily verify that for fixed n, φ_n is twice differentiable with bounded second derivative, $\varphi_n''(x)\geqslant 0$, and as $n\to\infty$ one has $\varphi_n(x)\longrightarrow|x|^l$, $\varphi_n'(x)\longrightarrow l sgn(x)|x|^{l-1}$, $\varphi_n''(x)\longrightarrow l(l-1)|x|^{l-2}$. Moreover, the following relations hold, for all $x\in\mathbb{R}$ and $n\geqslant l$:

1. $|\ x\varphi_n'(x)\ |\ \leqslant l\varphi_n(x)$.
2. $|\ \varphi_n'(x)\ |\ \leqslant\ |\ x\varphi_n''(x)\ |$.
3. $|\ x^2\varphi_n''(x)\ |\ \leqslant l(l-1)\varphi_n(x)$.
4. $|\varphi_n'(x)|\leqslant l(\varphi_n(x)+1)$.
5. $|\varphi_n''(x)|\leqslant l(l-1)(\varphi_n(x)+1)$.

From Lemma 7 we have P-a.s. for all $t \in [0,T]$

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi_n(u_t(x))\,dx \;+\; \int_0^t \mathcal{E}\big(\varphi_n'(u_s),\; u_s\big)\,ds \\
&\quad = \int_{\mathcal{O}} \varphi_n(\xi)\,dx - \int_0^t \int_{\mathcal{O}} \varphi_n'(u_s) f(s,x,u_s,\nabla u_s)\,dxds \\
&\qquad + \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi_n''(u_s(x))\partial_i u_s(x)\, g_i(s,x,u_s,\nabla u_s)\,dx\,ds \\
&\qquad + \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi_n'(u_s)\, h_j(s,x,u_s,\nabla u_s)\,dxdB_s^j \\
&\qquad + \frac{1}{2} \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi_n''(u_s(x)) h_j^2(s,x,u_s,\nabla u_s)\,dx\,ds\,. \qquad (16)
\end{aligned}$$

By the uniform ellipticity of the operator A we get

$$\mathcal{E}\big(\varphi_n'(u_s),\; u_s\big) \;\geqslant\; \lambda \int_{\mathcal{O}} \varphi_n''(u_s)|\nabla u_s|^2\,dx.$$

Let $\epsilon > 0$ be fixed. Using the Lipschitz condition on f and the properties of the functions $(\varphi_n)_n$ we get

$$\begin{aligned}
&|\varphi_n'(u_s)|\,|f(s,x,u_s,\nabla u_s)| \\
&\quad \leqslant |\varphi_n'(u_s)|\,\big(|f^0(s,x)| + C\,(|u_s| + |\nabla u_s|)\,\big) \\
&\quad \leqslant\; |\varphi_n'(u_s)||f^0(s,x)| + |u_s||\varphi_n''(u_s)|\,(C|u_s| + C|\nabla u_s|)\,) \\
&\quad \leqslant l(\varphi_n(u_s)+1)\,|f^0(s,x)| + C|u_s|^2|\varphi_n''(u_s)| + C|u_s||\nabla u_s||\varphi_n''(u_s)| \\
&\quad \leqslant l(\varphi_n(u_s)+1)\,|f^0(s,x)| + (C+c_\epsilon)\,|u_s|^2\varphi_n''(u_s) +\; \epsilon\varphi_n''(u_s)|\nabla u_s|^2.
\end{aligned}$$

Now using Cauchy-Schwarz inequality and the Lipschitz condition on g we get

$$\begin{aligned}
&\sum_{i=1}^d \varphi_n''(u_s)\partial_i u_s g_i(s,x,u_s,\nabla u_s) \\
&\quad \leqslant \varphi_n''(u_s)|\nabla u_s|\,\big(|g^0(s,x)| + C|u_s| + \alpha|\nabla u_s|\,\big) \\
&\quad \leqslant \epsilon\varphi_n''(u_s)|\nabla u_s|^2 + 2c_\epsilon\varphi_n''(u_s)\big(K^2 + C^2|u_s|^2\big) + \alpha\varphi_n''(u_s)|\nabla u_s|^2 \\
&\quad \leqslant l(l-1)c_\epsilon K^2 + 2c_\epsilon(K^2+C^2)l(l-1)|\varphi_n(u_s)| + (\alpha+\epsilon)\varphi_n''(u_s)|\nabla u_s|^2
\end{aligned}$$

In the same way as before

$$\begin{aligned}
&\sum_{j=1}^{d_1} \varphi_n''(u_s)h_j^2(s,u_s,\nabla u_s) \\
&\quad \leqslant \varphi_n''(u_s)\big(c_\epsilon'(|h^0(s,x)| + C|u_s|)^2 + (1+\epsilon)\beta^2|\nabla u_s|^2\big) \\
&\quad \leqslant \varphi_n''(u_s)\big(2c_\epsilon' K^2 + 2c_\epsilon' C^2|u_s|^2 + (1+\epsilon)\beta^2|\nabla u_s|^2\big) \\
&\quad \leqslant 2c_\epsilon' l(l-1)K^2 + 2c_\epsilon'(K^2+C^2)l(l-1)\varphi_n(u_s) + (1+\epsilon)\beta^2\varphi_n''(u_s)|\nabla u_s|^2
\end{aligned}$$

Thus taking the expectation (one has to remember that, as a consequence of the previous lemma, the expectation of the martingale part is null), we deduce

$$
\begin{aligned}
E\int_{\mathcal{O}} \varphi_n(u_t(x))dx + (\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha+2\epsilon))E\int_0^t\int_{\mathcal{O}} \varphi_n''(u_s(x))|\nabla u_s|^2 dxds \\
\leqslant l(l-1)c_\epsilon'' K^2 + c_\epsilon'' l(l-1)\big(K^2 + C^2 + C + c_\epsilon\big)E\int_0^t\int_{\mathcal{O}} \varphi_n(u_s(x))dxds
\end{aligned}
\tag{17}
$$

On account of the condition (8), one can choose $\epsilon > 0$ small enough such that

$$
\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha + 2\epsilon) > 0
$$

and then

$$
E\int_{\mathcal{O}} \varphi_n(u_t(x))\,dx \leqslant cK^2 l(l-1) + cl(l-1)E\int_0^t\int_{\mathcal{O}} \varphi_n(u_s(x))\,dx\,ds\,.
$$

We obtain by Gronwall's Lemma, that

$$
E\int_{\mathcal{O}} \varphi_n(u_t(x))\,dx \leqslant c\,K^2 l(l-1)\,\exp\big(c\,l(l-1)\,t\big)
$$

and so it is now easy from (17) to get

$$
E\int_0^t\int_{\mathcal{O}} \varphi_n''(u_s(x))\,|\nabla u_s|^2\,dx\,ds \leqslant c'\,K^2 l\,(l-1)\,\exp\big(cl(l-1)\,t\big)
$$

Finally, letting $n \to \infty$ by Fatou's lemma we deduce (14) and (15). □

First we use the above estimates to get uniqueness of solutions under the general conditions (9) without (11).

Corollary 9. *Assume all the hypotheses of section 2.2. Then the uniqueness of the weak solution in $\mathcal{H}$ of SPDE in the sense (10) holds.*

Proof. Let u and u' be two weak solutions in $\mathcal{H}$ of the SPDE (10) associated to (ξ, f, g, h). Let us define

$$
\bar{u} = u - u', \quad \bar{f}(t,\omega,x,y,z) = f(t,\omega,x,y+u_t',z+\nabla u_t') - f(t,\omega,x,u_t',\nabla u_t')
$$

in the same way we define $\bar{g}$ and $\bar{h}$. Then $\bar{u}$ is a weak solution in $\mathcal{H}$ of the SPDE (10) associated to $(0, \bar{f}, \bar{g}, \bar{h})$ with $\bar{f}^0 = \bar{g}^0 = \bar{h}^0 = 0$. Then by (14) we have that $\bar{u} = 0$, a.e. which gives our uniqueness result. □

Proposition 10. *Assume the hypotheses of the previous lemma. Let $u = u(t,x)$ be the solution of the SPDE (6). Then for $l \geqslant 2$, we get the following Itô's formula, P-almost surely, for all $t \geqslant 0$*

$$
\begin{aligned}
&\int_{\mathcal{O}} |u_t(x)|^l \, dx + \int_0^t \mathcal{E}\left(l\,(u_s)^{l-1}\, sgn(u_s),\, u_s\right) ds \\
&= \int_{\mathcal{O}} |\xi(x)|^l \, dx - \; l \int_0^t \int_{\mathcal{O}} sgn(u_s)|u_s(x)|^{l-1} f(s,x,u_s,\nabla u_s)\, dx ds \\
&\quad + l(l-1) \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} |u_s(x)|^{l-2} \partial_i u_s(x)\, g_i(s,x,u_s,\nabla u_s)\, dx\, ds \\
&\quad + l \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} sgn(u_s)|u_t(x)|^{l-1} h_j(s,x,u_s,\nabla u_s)\, dx dB_s^j \\
&\quad + \frac{l(l-1)}{2} \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} |u_t(x)|^{l-2} h_j^2(s,x,u_s,\nabla u_s)\, dx\, ds\,. \qquad (18)
\end{aligned}
$$

where

$$
\mathcal{E}\left(l\,(u_s)^{l-1}\, sgn(u_s),\, u_s\right) = l(l-1) \sum_{i,j=1}^{d} \int_{\mathcal{O}} |u_s(x)|^{l-2} a^{ij}(x)\, \partial_i u_s(x)\, \partial_j u_s(x)\, dx.
$$

Proof. From Lemma 7 with the same notations, we have P-almost surely, and for all $t \geqslant 0$ and $n \in \mathbb{N}$

$$
\begin{aligned}
&\int_{\mathcal{O}} \varphi_n(u_t(x))\, dx \;+\; \int_0^t \mathcal{E}\big(\varphi_n'(u_s),\, u_s\big)\, ds \\
&= \int_{\mathcal{O}} \varphi_n(\xi(x))\, dx - \int_0^t \int_{\mathcal{O}} \varphi_n'(u_s(x)) f(s,x,u_s,\nabla u_s)\, dx ds \\
&\quad + \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \varphi_n''(u_s(x)) \partial_i u_s(x)\, g_i(s,x,u_s,\nabla u_s)\, dx\, ds \\
&\quad + \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi_n'(u_s(x))\, h_j(s,x,u_s,\nabla u_s)\, dx dB_s^j \\
&\quad + \frac{1}{2} \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi_n''(u_s(x)) h_j^2(s,x,u_s,\nabla u_s)\, dx\, ds\,.
\end{aligned}
$$

Passing to the limit as $n \to \infty$, thanks to the Lemma 8 and the the dominated convergence theorem, we obtain the desired result. □

4. The main result

We will need the following stronger hypothesis :

$$\alpha + \frac{1}{2}\beta^2 + 72\beta^2 < \lambda\,. \tag{19}$$

Then we choose a constant $\epsilon \in (0,\ 1)$ such that

$$\lambda > \alpha + \frac{\epsilon l}{l-1} + \frac{1+\epsilon}{2}\beta^2 + \frac{36\,(1+\epsilon)}{(1-6\epsilon)^2}\frac{l}{l-1}\beta^2,$$

for all $l \geqslant 2$.

The constants λ, C, α, β, θ, p and $|\mathcal{O}|$, the volume of the open set $\mathcal{O}$, represent the structure parameters of our SPDE and are considered fixed from now on. The constant ϵ, chosen above, will be considered fixed too. The estimates of solutions will be expressed only in terms of this constants and the norms of the random variables ξ, f^0, g^0, h^0. Our main theorem is the following:

Theorem 11. *Assume that hypotheses of section 2.2 and (19) hold. Then the equation (6) admits a unique solution u in $\mathcal{H}$. Moreover, there exists a function k_2 : $\mathbb{R}_+ \to \mathbb{R}_+$, which involves only the structure constants, such that the following estimate holds for all $p \geqslant 2$*

$$E\,\|u\|^p_{\infty,\infty;t} \leq k_2\,(t)\,E\left(\|\xi\|^p_\infty + \left\|f^0\right\|^{*p}_{\theta;t} + \left\||g^0|^2\right\|^{*p/2}_{\theta;t} + \left\||h^0|^2\right\|^{*p/2}_{\theta;t}\right). \tag{20}$$

The proof of this result is rather lengthy so we will go on in several steps.

First we will assume that the hypothesis (11) of section 3.1 is fulfilled and in the next subsection prove several lemmas. We have in mind to get estimates in which the constants depend on C, $\|\xi\|_{L^\infty(\Omega\times\mathcal{O})}$, $\|f^0\|^*_{\theta;t}$, $\||g^0|^2\|^*_{\theta;t}$, $\||h^0|^2\|^*_{\theta;t}$, α and β which will allow us to consider the general case by an approximation argument.

4.1. Preliminary estimates

For each $l \geqslant 2$, we define the processes v and v' by

$$v_t := \sup_{s\leqslant t}\left(\int_{\mathcal{O}} |u_s|^l\,dx + \gamma l\,(l-1)\int_0^s\int_{\mathcal{O}} |u_r|^{l-2}\,|\nabla u_r|^2\,dx\,dr\right)$$

$$v'_t := \int_{\mathcal{O}} |\xi|^l\,dx + l^2 c_1 \left\||u|^l\right\|_{1,1;t} + l\left\|f^0\right\|^*_{\theta;t}\left\||u|^{l-1}\right\|_{\theta;t}$$
$$+\,l^2\left(c_2\left\||g^0|^2\right\|^*_{\theta;t} + c_3\left\||h^0|^2\right\|^*_{\theta;t}\right)\left\||u|^{l-2}\right\|_{\theta;t},$$

where the constants are given by

$$\gamma = \lambda - \alpha - \frac{\epsilon l}{l-1} - \frac{1+\epsilon}{2}\beta^2$$

$$c_1 = \frac{C}{2}\left(1 + \frac{C}{4\epsilon}\right) + \frac{3+2\epsilon}{2\epsilon}C^2 + 3\frac{1+\epsilon}{\epsilon^2}C^2$$

$$c_2 = \frac{1}{2\epsilon} \quad \text{and} \quad c_3 = \frac{(3+\epsilon)\,(1+\epsilon)}{\epsilon} \tag{21}$$

We start with an estimate of the bracket of the local martingale appearing in (18), expressed by

$$M_t := l \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} sgn(u_s)|u_s(x)|^{l-1} h_j(s,x,u_s,\nabla u_s)\, dx dB_s^j$$

Lemma 12. *For arbitrary* $\varepsilon > 0$, *one has*

$$\langle M \rangle_t^{\frac{1}{2}} \le \varepsilon v_t + \frac{l^2}{2\varepsilon}\left(\frac{1+\varepsilon}{\varepsilon} \left\| |h^0|^2 \right\|_{\theta;t}^* \left\| |u|^{l-2} \right\|_{\theta;t} + \frac{1+\varepsilon}{\varepsilon} C^2 \left\| |u|^l \right\|_{1,1;t} \right) + \sqrt{1+\varepsilon} \sqrt{\frac{l}{l-1}} \frac{\beta}{\sqrt{\gamma}} v_t \,. \tag{22}$$

Proof. The Lipschitz conditions (7) imply

$$|h(s,x,u_s,\nabla u_s)|^2 \le 2\left(1+\frac{1}{\varepsilon}\right)\left|h_s^0\right|^2 + 2\left(1+\frac{1}{\varepsilon}\right) C^2 |u_s|^2 + (1+\varepsilon)\beta^2 |\nabla u_s|^2 \,.$$

First the bracket of M is estimated by

$$\begin{aligned}\langle M \rangle_t &= l^2 \sum_j^{d_1} \int_0^t \left(\int_{\mathcal{O}} |u_s|^{l-1} sign(u_s) h_j(s,x,u_s,\nabla u_s)\, dx \right)^2 ds \\ &\le l^2 \int_0^t \left(\int_{\mathcal{O}} |u_s|^l dx \right) \left(\int_{\mathcal{O}} |u_s|^{l-2} |h(s,x,u_s,\nabla u_s)|^2 dx \right) ds \\ &\le l^2 \left(\sup_{s\le t} \int_{\mathcal{O}} |u_s|^l dx \right) \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} |h(s,x,u_s,\nabla u_s)|^2 dx ds\end{aligned}$$

Now using the Hölder's inequality (4)

$$\begin{aligned}\langle M \rangle_t \leqslant\ & l^2 \left(\sup_{s\le t} \int_{\mathcal{O}} |u_s|^l dx \right) \left(2\frac{1+\varepsilon}{\varepsilon} \left\| |h^0|^2 \right\|_{\theta;t}^* \left\| |u|^{l-2} \right\|_{\theta;t} \right) \\ & + l^2 \left(\sup_{s\le t} \int_{\mathcal{O}} |u_s|^l dx \right) \left(2\frac{1+\varepsilon}{\varepsilon} C^2 \left\| |u|^l \right\|_{1,1;t} \right. \\ & \left. + (1+\varepsilon)\beta^2 \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} |\nabla u_s|^2 ds \right).\end{aligned}$$

Thus we may write the estimate of the bracket as follows

$$\begin{aligned}\langle M \rangle_t \le l^2 v_t & \left(2\frac{1+\varepsilon}{\varepsilon} \left\| |h^0|^2 \right\|_{\theta;t}^* \left\| |u|^{l-2} \right\|_{\theta;t} \right. \\ & \left. + 2\frac{1+\varepsilon}{\varepsilon} C^2 \left\| |u|^l \right\|_{1,1;t} + \frac{(1+\varepsilon)\beta^2}{l(l-1)\gamma} v_t \right),\end{aligned}$$

which yields the result thanks to the trivial inequalities:

$$\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b} \text{ and } \sqrt{ab} \leqslant \varepsilon a + \frac{b}{4\varepsilon},\ a,b > 0.$$

□

In what follows we will use the notion of domination as a technical tool. We recall the definition from Revuz and Yor [18].

Definition 13. *A non-negative, adapted right continuous process X is dominated by an increasing process A, if*

$$E\left[X_\tau\right] \leqslant E\left[A_\tau\right]$$

for any bounded stopping time, τ.

One important result related to this notion is the following domination inequality (see Proposition IV.4.7 in Revuz-Yor, p. 163), for any $k \in]0,\ 1[$,

$$E\left[(X_\infty^*)^k\right] \leqslant C_k\, E\left[(A_\infty)^k\right] \tag{23}$$

where C_k is a positive constant and $X_t^* := \sup_{s \leqslant t} |X_s|$.

We will also use the fact that if A, A' are increasing processes, then the domination of a process X by A is equivalent to the domination of $X + A'$ by $A + A'$.

Lemma 14. *The Process τv is dominated by the process v' where*

$$\tau = 1 - 6\epsilon - 6\sqrt{1+\epsilon}\sqrt{\frac{l}{l-1}}\frac{\beta}{\sqrt{\gamma}}.$$

In other words, we have

$$\begin{aligned}
&\tau\, E \sup_{0 \leqslant s \leqslant t} \left(\int_{\mathcal{O}} |u_s|^l\, dx + \gamma l\,(l-1) \int_0^s \int_{\mathcal{O}} |u_r|^{l-2}\, |\nabla u_r|^2\, dx dr \right) \\
&\quad \leqslant E \int_{\mathcal{O}} |\xi|^l\, dx \,+\, l^2 c_1 E\, \left\| |u|^l \right\|_{1,1;t} + l E\, \left\| f^0 \right\|_{\theta;t}^* \left\| |u|^{l-1} \right\|_{\theta;t} \\
&\quad + l^2 E\, \left(c_2 \left\| |g^0|^2 \right\|_{\theta;t}^* + c_3 \left\| |h^0|^2 \right\|_{\theta;t}^* \right) \left\| |u|^{l-2} \right\|_{\theta;t},
\end{aligned} \tag{24}$$

where γ, c_1, c_2 and c_3 are the constants given above.

Proof. One starts with the relation (18) :

$$\begin{aligned}
&\int_{\mathcal{O}} |u_t|^l\, dx + l\,(l-1) \int_0^t \left(\int_{\mathcal{O}} |u_s|^{l-2} \sum_{i,j} a^{ij}\, \partial_i u_s \partial_j u_s\, dx \right) ds \\
&\quad = \int_{\mathcal{O}} |\xi|^l\, dx - l \int_0^t \left(\int_{\mathcal{O}} |u_s|^{l-1}\, sign\,(u_s)\, f\,(s, x, u_s, \nabla u_s)\, dx \right) ds \\
&\quad + l\,(l-1) \int_0^t \left(\int_{\mathcal{O}} |u_s|^{l-2} \sum_i \partial_i u_s g_i\,(s, x, u_s, \nabla u_s)\, dx \right) ds \\
&\quad + \frac{l(l-1)}{2} \int_0^t \left(\int_{\mathcal{O}} |u_s|^{l-2} \sum_j^{d_1} h_j^2\,(s, x, u_s, \nabla u_s)\; dx \right) ds \\
&\quad + l \sum_{j=1}^{d_1} \int_0^t \left(\int_{\mathcal{O}} |u_s|^{l-1}\, sign\,(u_s)\, h_j\,(s, x, u_s, \nabla u_s)\; dx \right) dB_s^j.
\end{aligned}$$

Using the uniform ellipticity condition (5) one has

$$\sum_{i,j} a^{ij} \partial_i u_s \partial_j u_s \geq \lambda \left|\nabla u_s\right|^2$$

and the Lipschitz conditions (7) imply

$$\begin{aligned}
&\left|f\left(s,x,u_s,\nabla u_s\right)\right| \leq \left|f_s^0\right| + C\left|u_s\right| + C\left|\nabla u_s\right| \\
&\left|g\left(s,x,u_s,\nabla u_s\right)\right| \leq \left|g_s^0\right| + C\left|u_s\right| + \alpha\left|\nabla u_s\right| \\
&\left|h\left(s,x,u_s,\nabla u_s\right)\right|^2 \leq 2\big(1+\frac{1}{\varepsilon}\big)\left|h_s^0\right|^2 + 2\big(1+\frac{1}{\varepsilon}\big)C^2\left|u_s\right|^2 + \big(1+\varepsilon\big)\beta^2\left|\nabla u_s\right|^2,
\end{aligned}$$

for all $\varepsilon > 0$. This leads to

$$\begin{aligned}
&\int_{\mathcal{O}} |u_t|^l\, dx + l\,(l-1)\,\lambda \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} \left|\nabla u_s\right|^2 dxds \\
&\quad\leq \int_{\mathcal{O}} |\xi|^l\; dx + l \int_0^t \int_{\mathcal{O}} |u_s|^{l-1} \left|f_s^0\right| dxds \\
&\qquad + l\left(C + \frac{C^2}{4\varepsilon}\right) \int_0^t \int_{\mathcal{O}} |u_s|^l\, dxds + l\varepsilon \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} \left|\nabla u_s\right|^2 dxds \\
&\qquad + l\,(l-1) \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} \left(\frac{1}{2\varepsilon}\left|g_s^0\right|^2 + \frac{1}{2\varepsilon} C^2 \left|u_s\right|^2 + (\alpha+\varepsilon)\left|\nabla u_s\right|^2\right) dxds \\
&\qquad + l\,(l-1) \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} \left(\left(1+\frac{1}{\varepsilon}\right)\left|h_s^0\right|^2 + \left(1+\frac{1}{\varepsilon}\right) C^2 \left|u_s\right|^2\right. \\
&\qquad \left. + \frac{1}{2}\,(1+\varepsilon)\,\beta^2 \left|\nabla u_s\right|^2\right) dxds + M_t\,.
\end{aligned}$$

This further leads to

$$\begin{aligned}
&\int_{\mathcal{O}} |u_t|^l\, dx + l\,(l-1)\,\gamma \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} \left|\nabla u_s\right|^2 dxds \\
&\quad\leq \int_{\mathcal{O}} |\xi|^l\, dx + l\,(l-1)\,c_4 \int_0^t \int_{\mathcal{O}} |u_s|^l\, dxds \\
&\qquad + l \int_0^t \int_{\mathcal{O}} |u_s|^{l-1} \left|f_s^0\right| dxds + \frac{l\,(l-1)}{2\varepsilon} \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} \left|g_s^0\right|^2 dxds \\
&\qquad + \frac{l\,(l-1)\,(1+\varepsilon)}{\varepsilon} \int_0^t \int_{\mathcal{O}} |u_s|^{l-2} \left|h_s^0\right|^2 dxds + M_t \\
&\quad\leq \int_{\mathcal{O}} |\xi|^l\, dx + l\,(l-1)\,c_4 \left\| |u|^l \right\|_{1,1;t} + +l \left\| |u|^{l-1} \right\|_{\theta;t} \left\| f^0 \right\|_{\theta;t}^* \\
&\qquad + \frac{l\,(l-1)}{2\varepsilon} \left\| |u|^{l-2} \right\|_{\theta;t} \left\| |g^0|^2 \right\|_{\theta;t}^* \\
&\qquad + \frac{l\,(l-1)\,(1+\varepsilon)}{\varepsilon} \left\| |u|^{l-2} \right\|_{\theta;t} \left\| |h^0|^2 \right\|_{\theta;t}^* + M_t
\end{aligned}$$

where

$$c_4 := \frac{C}{(l-1)}\left(1+\frac{C}{4\varepsilon}\right)+\frac{3+2\varepsilon}{2\varepsilon}C^2.$$

Taking the supremum one deduces

$$v_t \le \int_{\mathcal{O}} |\xi|^l\, dx + l\,(l-1)\,c_4 \left\| |u|^l \right\|_{1,1;t} + l \left\| f^0 \right\|^*_{\theta;t} \left\| |u|^{l-1} \right\|_{\theta;t} + \left(\frac{l\,(l-1)}{2\varepsilon} \left\| |g^0|^2 \right\|^*_{\theta;t} + \frac{l\,(l-1)\,(1+\varepsilon)}{\varepsilon} \left\| |h^0|^2 \right\|^*_{\theta;t} \right) \left\| |u|^{l-2} \right\|_{\theta;t} + M^*_t.$$

Since the process M^*_t is dominated by $6\,\langle M\rangle_t^{\frac{1}{2}}$ we deduce that the process v is dominated by

$$\int_{\mathcal{O}} |\xi|^l\, dx + l\,(l-1)\,c_4 \left\| |u|^l \right\|_{1,1;t} + l \left\| f^0 \right\|^*_{\theta;t} \left\| |u|^{l-1} \right\|_{\theta;t} + \left(\frac{l\,(l-1)}{2\varepsilon} \left\| |g^0|^2 \right\|^*_{\theta;t} + \frac{l\,(l-1)\,(1+\varepsilon)}{\varepsilon} \left\| |h^0|^2 \right\|^*_{\theta;t} \right) \left\| |u|^{l-2} \right\|_{\theta;t} + 6\,\langle M\rangle_t^{\frac{1}{2}}.$$

We may finally use the estimate (22) to get the result of the lemma with $\varepsilon = \epsilon$ choosen as it is required in the statement. □

Lemma 15. *The process v satisfies the estimate*

$$v_t \ge \delta \left\| |u|^l \right\|_{0;t}$$

with $\delta = 1 \wedge \left(2c^{-1}\gamma\right)$, where c is the constant in the Sobolev inequality of Lemma 3.

Proof. Denote by $y := |u|^{l/2}$, then $\partial_i y = (l/2)\partial_i u\, sign(u)\, |u|^{(l-2)/2}$ and so $|\nabla y|^2 = (l/2)^2 |u|^{l-2} |\nabla u|^2$. Making the change of variables in the second term of the left hand side of the last equation and using the Sobolev's inequality (3) we get

$$\begin{aligned} l(l-1)\gamma \int_0^t \int_{\mathcal{O}} |u_s|^{l-2}\, |\nabla u_s|^2\, dxds &= \frac{4\gamma(l-1)}{l} \int_0^t \int_{\mathcal{O}} |\nabla y_s|^2\, dxds \\ &\geqslant \frac{4\gamma(l-1)}{cl} \int_0^t \left(\int_{\mathcal{O}} |y_s|^{2^*}\, dx \right)^{2/2^*} ds \\ &= \frac{4\gamma(l-1)}{cl} \, \| |u| \|_{\frac{2^*}{2},1;t} \\ &\geqslant \delta \, \| |u| \|_{\frac{2^*}{2},1;t} \end{aligned}$$

where $2^* := \dfrac{2d}{d-2}$ for $d > 2$ and $2^* = 4$ for $d = 1,\ 2$. It is now easy to conclude our lemma. □

The next lemma is a key technical ingredient for our purposes.

Lemma 16. *The process*

$$w_t := \left[\left\| |u|^{\sigma l} \right\|_{\theta;t}^{\frac{1}{\sigma}} \vee \|\xi\|_\infty^l \vee \left\| f^0 \right\|_{\theta;t}^l \vee \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right]$$

is dominated by the process

$$w'_t := 6k\,(t)\, l^2 \left[\left\| |u|^l \right\|_{\theta;t} \vee \|\xi\|_\infty^l \vee \left\| f^0 \right\|_{\theta;t}^l \vee \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right],$$

where $\sigma = \frac{d+2\theta}{d}$ *and* $k : \mathbb{R}_+ \to \mathbb{R}_+$ *is a function independent of* l*, depending only on the structure constants.*

Proof. The main problem is to dominate $\| |u|^{\sigma l} \|_{\theta;t}^{\frac{1}{\sigma}}$, because the other factors participating in the expression of w are repeated in the expression of w'. Thanks to the property 6. from the preliminaries and the preceding lemma, we deduce that $\delta \| |u|^{\sigma l} \|_{\theta;t}^{\frac{1}{\sigma}} \le v_t$. Then we apply Lemma 14 and deduce that the process $t \to \delta\tau \| |u|^{\sigma l} \|_{\theta;t}^{\frac{1}{\sigma}}$ is dominated by v'. So the problem that remains is the estimation of the process v'.

To estimate v' we shall use the following consequence of Hölder's inequality

$$\left\| u^\kappa \right\|_{p,q;t} \le |\mathcal{O}|^{\frac{1-\kappa}{p}} \, |t|^{\frac{1-\kappa}{q}} \, \|u\|_{p,q;t}^\kappa ,$$

where $\kappa \in (0,1)$. So, we first use the definition of $L_{\theta,t}$ -norm and apply this inequality with $\kappa = \frac{l-1}{l}$ and then Young's inequality, obtaining

$$\begin{aligned} \left\| f^0 \right\|_{\theta;t}^* \left\| |u|^{l-1} \right\|_{\theta;t} &\le \left(|\mathcal{O}| \vee |\mathcal{O}|^{\frac{d-2(1-\theta)}{d}} \right)^{\frac{1}{l}} \left(t \vee t^\theta \right)^{\frac{1}{l}} \left\| f^0 \right\|_{\theta;t}^* \left\| |u|^l \right\|_{\theta;t}^{\frac{l-1}{l}} \\ &\le \frac{1}{l} \left(|\mathcal{O}| \vee |\mathcal{O}|^{\frac{d-2(1-\theta)}{d}} \right) \left(t \vee t^\theta \right) \left\| f^0 \right\|_{\theta;t}^{*l} + \frac{l-1}{l} \left\| |u|^l \right\|_{\theta;t} . \end{aligned}$$

In the same way, by taking $\kappa = \frac{l-2}{l}$, one has

$$\left\| |g^0|^2 \right\|_{\theta;t}^* \left\| |u|^{l-2} \right\|_{\theta;t} \le \frac{2}{l} \left(|\mathcal{O}| \vee |\mathcal{O}|^{\frac{d-2(1-\theta)}{d}} \right) \left(t \vee t^\theta \right) \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + \frac{l-2}{l} \left\| |u|^l \right\|_{\theta;t} ,$$

$$\left\| |h^0|^2 \right\|_{\theta;t}^* \left\| |u|^{l-2} \right\|_{\theta;t} \le \frac{2}{l} \left(|\mathcal{O}| \vee |\mathcal{O}|^{\frac{d-2(1-\theta)}{d}} \right) \left(t \vee t^\theta \right) \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + \frac{l-2}{l} \left\| |u|^l \right\|_{\theta;t} .$$

Further, still from the definition of the $L_{\theta,t}$ -norm and Hölder's inequality, one gets

$$\left\| |u|^l \right\|_{1,1;t} \leqslant \left(|\mathcal{O}|^{\frac{2}{d}} \vee t \right)^{1-\theta} \left\| |u|^l \right\|_{\theta;t} .$$

Now we may treat each term in the expresion of v' and the preceding estimates lead to the following one

$$v_t' \leqslant |\mathcal{O}| \,\|\xi\|_\infty^l + \left[l^2 c_1 \left(|\mathcal{O}|^{\frac{2}{d}} \vee t\right)^{1-\theta} + l - 1 + l\,(l-2)\,(c_2 + c_3)\right] \left\| |u|^l \right\|_{\theta;t}$$
$$+ \left(|\mathcal{O}| \vee |\mathcal{O}|^{\frac{d-2(1-\theta)}{d}}\right)\left(t \vee t^\theta\right) \left(\left\| f^0 \right\|_{\theta;t}^{*l} + 2lc_2 \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right.$$
$$\left. + 2lc_3 \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right). \qquad (25)$$

Then we write

$$w_t \leq \left[\left\| |u|^{\sigma l} \right\|_{\theta;t}^{\frac{1}{\sigma}} + \|\xi\|_\infty^l + \left\| f^0 \right\|_{\theta;t}^{*l} + \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right]$$

and conclude that w is dominated by

$$k\,(t)\, l^2 \left[\left\| |u|^l \right\|_{\theta;t} + \|\xi\|_\infty^l + \left\| f^0 \right\|_{\theta;t}^{*l} + \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right] \leq w_t'$$

where the function k may be choosen to have the expression

$$k\,(t) = (\delta\tau)^{-1} \left(1 + |\mathcal{O}| \vee |\mathcal{O}|^{\frac{d-2(1-\theta)}{d}} \vee |\mathcal{O}|^{\frac{2(1-\theta)}{d}}\right)\left(1 + t \vee t^\theta \vee t^{1-\theta}\right)$$
$$\times (1 + c_1 + c_2 + c_3)\,.$$

The proof is complete. □

Lemma 17. *There exists a function $k_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which involves only the structure constants of our SPDE and such that the following estimate holds*

$$E v_t \leq k_1\,(l,t)\, E \left(\int_{\mathcal{O}} |\xi|^l \, dx + \left\| f^0 \right\|_{\theta;t}^{*l} + \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right).$$

Proof. We start with the calculation done in the proof of the preceding lemma. Namely, the same procedure that has been employed to establish the relation (25) leads to

$$v_t' \leq S_t + l^2 c_1 \left\| |u|^l \right\|_{1,1;t} + b\,(l) \left\| |u|^l \right\|_{\theta;t}, \qquad (26)$$

where

$$S_t := \int_{\mathcal{O}} |\xi|^l \, dx + a\,(t) \left(\left\| f^0 \right\|_{\theta;t}^{*l} + 2lc_2 \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + 2lc_3 \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right),$$

and $a\,(t) = \left(|\mathcal{O}| \vee |\mathcal{O}|^{\frac{d-2(1-\theta)}{d}}\right)\left(t \vee t^\theta\right)$, $b\,(l) = l - 1 + l\,(l-2)\,(c_2 + c_3)\,.$

On the other hand, by Lemma 15 we know that $\tau v_t \geq \frac{\tau}{2} v_t + \frac{\tau\delta}{2} \left\| |u|^l \right\|_{0;t}$, and by Lemma 18 in the appendix, one further deduces

$$\tau v_t \geq \frac{\tau}{2} v_t + \frac{\tau\delta}{2\varepsilon} \left(\left\| |u|^l \right\|_{\theta;t} - C\,(\theta, \varepsilon, t) \left\| |u|^l \right\|_{1,1;t} \right).$$

Taking $\varepsilon = \frac{\tau\delta}{2b(l)}$, with any $\varepsilon > 0$ in this inequality and using it in conjunction with Lemma 14 and the above relation (26), we obtain that the process

$$S_t + \left(l^2 c_1 + b\,(l)\, C\,(\theta, \varepsilon, t)\right) \left\| |u|^l \right\|_{1,1;t}$$

dominates the process $\frac{\tau}{2}v$. In particular, setting $a_1\,(l, t) = l^2 c_1 + b\,(l)\, C\,(\theta, \varepsilon, t)\,,$ we have

$$\frac{\tau}{2} E \int_{\mathcal{O}} |u_t|^l \, dx \leq ES_t + a_1\,(l, t) \int_0^t E \int_{\mathcal{O}} |u_s|^l \, dx\, ds.$$

Moreover, it is clear that the functions $t \to a_1\,(l, t)$ and $t \to ES_t$ are increasing. So, for all $t' \in [0, t]$,

$$\frac{\tau}{2} E \int_{\mathcal{O}} |u_{t'}|^l \, dx \leq ES_t + a_1\,(l, t) \int_0^{t'} E \int_{\mathcal{O}} |u_s|^l \, dx\, ds.$$

Thanks to the Gronwall's lemma, we get

$$E \int_{\mathcal{O}} |u_t|^l \leq \frac{2}{\tau} ES_t \exp \frac{2}{\tau} t a_1\,(l, t)\,.$$

Then we deduce

$$\left\| |u|^l \right\|_{1,1;t} \leq \frac{2}{\tau} ES_t \int_0^t \exp \frac{2}{\tau} s a_1\,(l, s)\, ds.$$

Returning back to the domination obtained for $\frac{\tau}{2}v$, we use it again and get

$$\frac{\tau}{2} Ev_t \leq ES_t \left(1 + \frac{2}{\tau} a_1\,(l, t) \int_0^t \exp \frac{2}{\tau} s a_1\,(l, s)\, ds\right).$$

The estimate of the lemma is obtained with

$$k_1\,(l, t) = \frac{4la\,(l)\, c_3}{\tau} \left(1 + \frac{2}{\tau} a_1\,(l, t) \int_0^t \exp \frac{2}{\tau} s a_1\,(l, s)\, ds\right).$$

□

4.2. *Proof of Theorem 11*

The uniqueness of the solution is ensured by Corollary 9. Next we will prove the existence and the asserted estimate. This will be done in two steps.

Step 1 : We first assume that hypotheses of section 2.2, (11) and (19) hold.

We set $l = p\sigma^n$, with some $n \in \mathbf{N}$. By Lemma 16 and the domination inequality (23) we deduce, for $n \geq 1$,

$$E\left(\left\| |u|^{\sigma l} \right\|_{\theta;t}^{\frac{1}{\sigma}} \vee \|\xi\|_\infty^l \vee \left\| f^0 \right\|_{\theta;t}^{*l} \vee \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}}\right)^{\frac{1}{\sigma^n}}$$
$$\leqslant C_{\sigma^{-n}} \left(6k\,(t)\, l^2\right)^{\frac{1}{\sigma^n}} E\left(\left\| |u|^l \right\|_{\theta;t} \vee \|\xi\|_\infty^l \vee \left\| f^0 \right\|_{\theta;t}^{*l}\right.$$
$$\left. \vee \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}}\right)^{\frac{1}{\sigma^n}},$$

where $C_{\sigma^{-n}}$ is the constant in the domination inequality. This constant is estimated by

$$C_{\sigma^{-n}} \leq \sigma^{\frac{n}{\sigma^n}} \left(1 - \frac{1}{\sigma^n}\right)^{-1}.$$

(See the exercise IV.4.30 in Revuz -Yor, p. 171). So let us denote by

$$a_n := \left\| |u|^{p\sigma^n} \right\|_{\theta;t}^{\frac{1}{\sigma^n}} \vee \|\xi\|_\infty^p \vee \left\| f^0 \right\|_{\theta;t}^{*p} \vee \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \vee \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}}$$

and deduce from the above inequality the following one

$$Ea_{n+1} \leq \sigma^{\frac{n}{\sigma^n}} \left(1 - \frac{1}{\sigma^n}\right)^{-1} \left(6k(t)\left(p\sigma^n\right)^2\right)^{\frac{1}{\sigma^n}} Ea_n.$$

Iterating this relation n times we get

$$Ea_{n+1} \leq \sigma^{3\sum_{m=1}^n \frac{m}{\sigma^m}} \prod_{m=1}^n \left(1 - \frac{1}{\sigma^m}\right)^{-1} \left(6k(t)\, p^2\right)^{\sum_{m=1}^n \frac{1}{\sigma^m}} Ea_1.$$

Now we shall let n tend to infinity in this relation. Since in general one has

$$\lim_{q,q' \to \infty} \|F\|_{q,q';t} = \|F\|_{\infty,\infty;t},$$

for any function $F : \mathbb{R}_+ \times \mathcal{O} \to \mathbb{R}$, it is easy to see that $\lim_{n\to\infty} \left\| |u|^{p\sigma^n} \right\|_{\theta;t}^{\frac{1}{\sigma^n}} = \|u\|_{\infty,\infty;t}^p$.
Therefore we have

$$\lim_{n\to\infty} a_n = \|u\|_{\infty,\infty;t}^p \vee \|\xi\|_\infty^p \vee \left\| f^0 \right\|_{\theta;t}^{*p} \vee \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \vee \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}},$$

which implies

$$E\|u\|_{\infty,\infty;t}^p \leq \rho(t)\, Ea_1,$$

with

$$\rho(t) = \sigma^{3\sum_{m=1}^\infty \frac{m}{\sigma^m}} \prod_{m=1}^\infty \left(1 - \frac{1}{\sigma^m}\right)^{-1} \left(5k(t)\, p^2\right)^{\sum_{m=1}^\infty \frac{1}{\sigma^m}}.$$

Now we estimate Ea_1 by using the fact that $\delta \left\| |u|^{p\sigma} \right\|_{\theta;t}^{\frac{1}{\sigma}} \leq v_t$, with p replacing l in the expression of v. So we have

$$\begin{aligned} Ea_1 &= E\left(\left\| |u|^{p\sigma} \right\|_{\theta;t}^{\frac{1}{\sigma}} \vee \|\xi\|_\infty^p \vee \left\| f^0 \right\|_{\theta;t}^{*p} \vee \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \vee \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \right) \\ &\leqslant E\left(\delta^{-1} v_t + \|\xi\|_\infty^p + \left\| f^0 \right\|_{\theta;t}^{*p} + \left\| |g^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \vee \left\| |h^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \right). \end{aligned}$$

Finally one deduces our estimate (20) by applying Lemma 17 with $l = p$.
Step 2 : The general case

We now assume that ξ and the coefficients f, g, h satisfy the hypothesis (9). We are going to prove Theorem 11 by using an approximation argument. For this, for all $n \in \mathbb{N}^*$, $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant d_1$ and all (t, w, x, y, z) in $\mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d$, we set

$$\begin{aligned} f_n(t,w,x,y,z) &= f(t,w,x,y,z) - f^0(t,w,x) + f^0(t,w,x)\cdot \mathbf{1}_{\{|f^0(t,w,x)|\leqslant n\}} \\ g_{i,n}(t,w,x,y,z) &= g_i(t,w,x,y,z) - g_i^0(t,w,x) + g_i^0(t,w,x)\cdot \mathbf{1}_{\{|g_i^0(t,w,x)|\leqslant n\}} \\ h_{j,n}(t,w,x,y,z) &= h_j(t,w,x,y,z) - h_j^0(t,w,x) + h_j^0(t,w,x)\cdot \mathbf{1}_{\{|h_j^0(t,w,x)|\leqslant n\}} \\ \xi_n(w,x) &= \xi(w,x)\cdot \mathbf{1}_{\{|\xi(\omega,x)|\leqslant n\}} \end{aligned}$$

Define now for each $n \in \mathbb{N}^*$, u^n, the solution of the following s.p.d.e:

$$du_t^n(x) + Au_t^n(x) + f_n(t,x,u_t^n(x),\nabla u_t^n(x)) + \sum_{i=1}^d \partial_i g_{i,n}(t,x,u_t^n(x),\nabla u_t^n(x)) \\ = \sum_{j=1}^{d_1} h_{j,n}(t,x,u_t^n(x),\nabla u_t^n(x))\, dB_t^j,$$

with initial condition $u_0^n = \xi_n$.

Let us remark that u_n is well-defined and satisfies all the estimates of the two preceding sub-sections because the coefficients of the previous s.p.d.e. satisfy the hypotheses we first made at the beginning of subsection 3.1.

We are now going to prove that u_n converges to a solution of the equation (6).

Let us fix $n \leqslant m$ in $\mathbb{N}^*$ and put $u^{n,m} := u^n - u^m$.

We first note that $u^{n,m}$ satisfies the equation

$$\begin{aligned} du_t^{n,m}(x) + Au_t^{n,m}(x)\,dt + f_{n,m}\big(t,x,u_t^{n,m}(x),\nabla u_t^{n,m}(x)\big)\,dt \\ + \sum_{i=1}^d \partial_i g_{i,n,m}\big(t,x,u_t^{n,m}(x),\nabla u_t^{n,m}(x)\big)\,dt \\ = \sum_{j=1}^{d_1} h_{j,n,m}\big(t,x,u_t^{n,m}(x),\nabla u_t^{n,m}(x)\big)\,dB_t^j \end{aligned}$$

where

$$\begin{aligned} f_{n,m}(t,w,x,y,z) = {} & f\big(t,w,x,y+u_t^m(x),z+\nabla u_t^m(x)\big) \\ & - f\big(t,w,x,u_t^m(x),\nabla u_t^m(x)\big) \\ & + f_n^0(t,w,x) - f_m^0(t,w,x) \end{aligned}$$

and $g_{i,n,m}$, $h_{j,n,m}$ have similar expressions. Clearly one has

$$f_{n,m}(t,w,x,0,0) = f_n^0(t,w,x) - f_m^0(t,w,x) := f_{n,m}^0(t,w,x)$$

and some similar relations for $g_{i,n,m}(t, w, x, 0, 0)$ and $h_{j,n,m}(t, w, x, 0, 0)$. On the other hand, one easily verifies that

$$E\|\xi_n - \xi\|_\infty^p \longrightarrow 0, \qquad E\left\|f_n^0 - f^0\right\|_{\theta;T}^{*p} \longrightarrow 0,$$
$$E\left\|g_n^0 - g^0\right\|_{\theta;T}^{*p} \longrightarrow 0, \quad E\left\|h_n^0 - h^0\right\|_{\theta;T}^{*p} \longrightarrow 0.$$

By Lemma 17 with $l = 2$ we deduce that

$$E\left\|u^n - u^m\right\|_T^2 \longrightarrow 0, \quad \forall\, T > 0,$$

as $n, m \to \infty$. Therefore u^n has a limit u and it is easy to check that it is a solution of the equation (6).

On the other hand, the first step of the proof ensures the validity of the relation (20) for each u^n and $u^{n,m}$ with $n \le m$, $n, m \in \mathbb{N}^*$. As a consequence, one has

$$E\left\|u^n - u^m\right\|_{\infty,\infty;T}^p \longrightarrow 0, \quad \forall\, T > 0,$$

as $n, m \to \infty$. In the limit one obtains the relation (20) for u. □

5. Appendix

Lemma 18. *Assume that $\theta \in (0, 1)$, then for each $\varepsilon > 0$, there exists a constant $C(\theta, \varepsilon, t)$ such that the following inequality holds for any function $u \in L_0$,*

$$\|u\|_{\theta;t} \le \varepsilon \|u\|_{0;t} + C(\theta, \varepsilon, t)\|u\|_{1,1;t}\,. \tag{27}$$

Proof. First we need the following interpolation inequality : Let $(U, \mathcal{T}, \mu)$ be a measure space and $p \ge 1, \sigma > 1, \varepsilon > 0$, then there exists a constant C such that

$$\|X\|_p \le \varepsilon \|X\|_{\sigma p} + C\|X\|_1 \tag{28}$$

for each $X \in L^{\sigma p}(\mu) \cap L^1(\mu)$.

In order to check this, one starts with Hölder's inequality and then applies the Young's one

$$\|X\|_p \le \|X\|_{\sigma p}^\nu \|X\|_1^{1-\nu} \le \nu\delta^{\frac{1}{\nu}}\|X\|_{\sigma p} + (1-\nu)\,\delta^{-\frac{1}{1-\nu}}\|X\|_1$$

with $\nu = \frac{\sigma(p-1)}{\sigma p - 1}$ and $\delta > 0$. Then one sets $\varepsilon = \nu\delta^{\frac{1}{\nu}}$ and gets the constant $C = (1-\nu)\,\nu^{\frac{\nu}{1-\nu}}\varepsilon^{-\frac{\nu}{1-\nu}}$, which gives the inequality (28).

We now prove the inequality of the lemma. We first apply the inequality (28) with respect to the space variable with $p = \frac{d}{d-2(1-\theta)}$, $\sigma = 1 + \frac{2\theta}{d}$, and then integrate with respect to the time variable obtaining

$$\|u\|_{\frac{d}{d-2(1-\theta)},1;t} \le \varepsilon \|u\|_{\frac{\sigma d}{d-2(1-\theta)},1;t} + C_1\|u\|_{1,1;t}\,,$$

which, on account of Hölder's inequality, is further dominated by

$$\le \varepsilon t^{\frac{\sigma-1}{\sigma}}\|u\|_{\frac{\sigma d}{d-2(1-\theta)},\sigma;t} + C_1\|u\|_{1,1;t}\,.$$

Then we apply the inequality (28) for the measure space $((0,t), dt)$ with $p = \frac{1}{\theta}$ and integrate with respect to variable space x obtaining

$$\|u\|_{1,\frac{1}{\theta};t} \leq \varepsilon \|u\|_{1,\frac{\sigma}{\theta};t} + C_2 \|u\|_{1,1;t} \leq \varepsilon |\mathcal{O}|^{\frac{\sigma-1}{\sigma}} \|u\|_{\sigma,\frac{\sigma}{\theta};t} + C_2 \|u\|_{1,1;t} .$$

The inequality asserted by the lemma is then deduced taking into account the fact that $\|u\|_{\theta;t} = \|u\|_{\frac{d}{d-2(1-\theta)},1;t} \vee \|u\|_{1,\frac{1}{\theta};t}$ and that $\left(\frac{\sigma d}{d-2(1-\theta)}, \sigma\right)$, $\left(\sigma, \frac{\sigma}{\theta}\right)$ belong to Γ_0. □

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