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Strong solutions of stochastic equations with singular time dependent drift

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Abstract. We prove existence and uniqueness of strong solutions to stochastic equations in domains $G \subset \mathbb{R}^d$ with unit diffusion and singular time dependent drift b up to an explosion time. We only assume local L_q-L_p-integrability of b in $\mathbb{R} \times G$ with $d/p + 2/q < 1$. We also prove strong Feller properties in this case. If b is the gradient in x of a nonnegative function ψ blowing up as $G \ni x \to \partial G$, we prove that the conditions $2D_t\psi \le K\psi$, $2D_t\psi + \Delta\psi \le Ke^{\varepsilon\psi}$, $\varepsilon \in [0, 2)$, imply that the explosion time is infinite and the distributions of the solution have sub Gaussian tails.

1. Introduction

In this paper we prove existence and uniqueness of strong solutions for stochastic equations of type

$$x_t = x + \int_0^t b(s + r, x_r)\,dr + w_t, \ \ t \ge 0, \tag{1.1}$$

in open subsets $Q \subset \mathbb{R} \times \mathbb{R}^d$ for singular drifts b. Here w_t is a standard Wiener process in $\mathbb{R}^d$ and $(s, x) \in \mathbb{R} \times \mathbb{R}^d$ is the initial starting point. Since b is not regular, we emphasize that solutions of (1.1) are supposed to be such that (1.1) makes sense, that is

$$\int_0^T |b(s + r, x_r)|\,dr < \infty \quad \forall T \in [0, \infty) \quad \text{(a.s.)}.$$

Observe that the equation itself expresses w_t as a function of x_r, $r \le t$. However, from the point of view of applications, in particular, in mathematical physics, it is desirable to look for solutions of (1.1) which are functions of the Wiener process w_t, i.e. so-called strong solutions.

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Diffusions with singular drift as in (1.1) have been studied from various points of view and under various assumptions. If b is locally Lipschitz in x and for some reasons the process $(s+t, x_t)$ cannot exit from Q or go to infinity when Q is unbounded, then the existence and uniqueness of strong solutions is a classical fact (see [13]). Even in this situation interesting issues arise when we have many moving particles $x_t^{(1)}, ..., x_t^{(M)}$ in $\mathbb{R}^d$ and assume that their joint dynamics is given by equation (1.1) in $\mathbb{R}^{Md}$ with $x_t = (x_t^{(1)}, ..., x_t^{(M)})$ and some repulsive locally Lipschitz continuous function b which can blow up when some particles become close, so that Q is $\mathbb{R} \times \mathbb{R}^{Md}$ without several hyperplanes. There are quite a few articles dealing with the limit behavior of such systems as $M \to \infty$ (see, for instance, [7] and the references therein). Our emphasis is on the cases in which b can be nowhere continuous let alone Lipschitz and one of the main difficulties to overcome is that there are no explicit Lyapunov functions showing that the process does not exit from Q or go to infinity.

Diffusions with singular drift as in (1.1) have been also studied extensively if b does not depend explicitly on time and is the gradient of a function. In this case the solution to (1.1) is called a distorted Brownian motion. We refer e.g. to [5], [2], [1], [8–10], but there are many more. Moreover, there have been generalizations to infinite dimensions. The reader should consult the references in the recent paper [4] to which we also refer for more historical comments. The latter work and especially its applications to finite particle systems in $\mathbb{R}^d$ (see also [16]) and to diffusions in random environments with very singular interaction has been the starting point of this paper. It should be considered as preparation to further analyze the case of infinite particles where progress has been made a few years ago (cf. [3]). But in all of these papers only weak solutions to (1.1) were constructed. In this paper we improve these results (however, partly under slightly stronger conditions) and obtain strong solutions.

The organization of this paper is as follows. In Section 2 we state our main results precisely. Sections 3-8 are devoted to proofs which are developed step by step with some further extensions of our results presented in Section 8. We only want to emphasize here that our approach is based on the Yamada-Watanabe Theorem. The necessary pathwise uniqueness we show employing a method due to A. Yu. Veretennikov ([27]) though in a substantially modified and more general form. In Section 9 we present two applications, both proposed in [4], i.e. first, diffusions in random media, i.e. their (singular) drifts depend on according to a Ruelle-type Gibbs measure distributed impurities in $\mathbb{R}^d$, given by a locally finite point configuration; second, we consider M-particles in $\mathbb{R}^d$ with a gradient dynamics which becomes singular when particles come very close. We also discuss the relation with earlier works, in particular the recent paper [26]. Finally, we fix some notation used below. As usual $\mathbb{R}^d = \{x = (x^1, ..., x^d) : x^i \in \mathbb{R}\}$, for $p, q \in [1, \infty]$ we denote $L_p = L_p(\mathbb{R}^d)$, $L_q\text{-}L_p = L_q(\mathbb{R}, L_p)$. Also we introduce $C = C([0, \infty), \mathbb{R}^d)$, $\mathcal{N}_t = \sigma\{x_s : x_\cdot \in C, s \leq t\}$. By N with or without indices we denote various *finite* constants. By u_x and u_{xx} we mean the gradient and the matrix of second-order derivatives, respectively, of u with respect to x.

2. Main results

Let $b(t, x)$ be an $\mathbb{R}^d$-valued Borel function defined on an open set $Q \subset \mathbb{R}^{d+1}$. Let Q^n, $n \geq 1$, be bounded open subsets of Q such that $\bar{Q}^n \subset Q^{n+1}$ and $\cup_n Q^n = Q$. Assume that for each n there exist $p = p(n)$, $q = q(n)$ satisfying

$$p \geq 2, \quad q > 2, \quad \frac{d}{p} + \frac{2}{q} < 1 \tag{2.1}$$

and such that $bI_{Q^n} \in L_q\text{-}L_p$. Of course, if $d \geq 2$, then automatically, $p > d \geq 2$.

One of our main results is saying that equation (1.1) is uniquely solvable up to the first exit time of its trajectories from all Q^n. There is a standard and convenient way to deal with processes defined on a random time interval. Add an object $\partial \notin Q$ to Q and define the neighborhoods of ∂ as the complements in Q of closed bounded subsets of Q. Then $Q' = Q \cup \partial$ becomes a compact topological space. By $C([0, \infty), Q')$ we denote the space of continuous Q'-valued functions defined on $[0, \infty)$.

Theorem 2.1. *Let w_t be a d-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$, let $\mathcal{F}_t = \mathcal{F}_t^w$ be the completion of $\sigma(w_s : s \leq t)$, and let $(s, x) \in Q$. Then for each $\omega \in \Omega$ there exists a continuous Q'-valued function $z_t = z_t(\omega)$ defined for $t \in [0, \infty)$ such that, with*

$$\zeta = \inf\{t \geq 0 : z_t \notin Q\},$$

we have

(i) z_t is $\mathcal{F}_t$-adapted;
(ii) $\zeta > 0$ and for $0 \leq t < \zeta$ in coordinate form $z_t \in Q$ can be written as $(s + t, x_t)$ with x_t defined as the space component of z_t;
(iii) for any $t \in [0, \infty)$, $z_t = \partial$ on the set $\{\omega : t \geq \zeta(\omega)\}$ (a.s.);
(iv) for any $t \geq 0$ on $\{\omega : t < \zeta(\omega)\}$ (a.s.) equation (1.1) and

$$\int_0^t |b(s + r, x_r)|^2 \, dr < \infty$$

hold.

Furthermore, the process z_t is unique in the sense that, if on $(\Omega, \mathcal{F}, P)$ we are given a continuous Q'-valued function z'_t, $t \in [0, \infty)$, such that for ζ' and x'_t defined from z'_t

(a) the properties (ii)-(iv) hold with ζ', $x'_\cdot$ in place of ζ, $x_\cdot$, respectively,
(b) for any $h, t \geq 0$ the increment $w_{t+h} - w_t$ is independent of $\{w_r, z'_r : r \leq t\}$,

then (a.s.) $\sup_t |z_t - z'_t| = 0$.

Finally, the distribution of $z_\cdot$ on $C([0, \infty), Q')$ is uniquely determined by the function b, that is weak uniqueness holds for equation (1.1).

Remark 2.2. The following is an equivalent and perhaps a more traditional albeit longer way to state property (b): (b$'$) there is an increasing filtration of σ-fields $\hat{\mathcal{F}}_t \subset \mathcal{F}$, $t \geq 0$, such that $(w_t, \hat{\mathcal{F}}_t)$ is a Wiener process and z'_t is $\hat{\mathcal{F}}_t$-adapted. That (b$'$) implies (b) follows directly from definitions. The converse becomes clear if we take $\hat{\mathcal{F}}_t = \sigma(z'_r, w_r : r \leq t)$.

Remark 2.3. In the one-dimensional case $b(t, x) = -x^{-1}$, $Q = \mathbb{R} \times (0, \infty)$, and $Q^n = (-n, n) \times \{x : 1/n < x < n\}$ satisfy the assumptions under which Theorem 2.1 is stated. Observe that in this case the solutions of (1.1) exit from $(0, \infty)$ in finite time, so that $\zeta < \infty$ (a.s.).

Remark 2.4. Write $\zeta = \zeta(s, x)$ and $x_t = x_t(s, x)$ to reflect the dependence of ζ, x_t on the initial data. By using an observation from [11] one can prove that $\zeta(s, x)$ and $x_t(s, x)$ depend continuously on (s, x) in the sense that if $(s^n, x^n) \to (s, x) \in Q$, then we have in probability $\zeta(s^n, x^n) \to \zeta(s, x)$ and $x_t(s^n, x^n) \to x_t(s, x)$ uniformly on any closed bounded subinterval of $[0, \zeta(s, x))$.

The following theorem provides information on additional properties of solutions corresponding to different (s, x).

Theorem 2.5. *For $z = (s, x) \in Q$, let P_z be the distribution of $z_\cdot = z_\cdot(s, x)$ from Theorem 2.1 on the space $C([0, \infty), Q')$. For $z = \partial$ let P_z be the measure concentrated on the function identically equal to ∂ on $[0, \infty)$. Define $\mathcal{N}_t(Q') = \sigma\{z_r : r \leq t, z_\cdot \in C([0, \infty), Q')\}$. Then*

$$(C([0, \infty), Q'), \mathcal{N}_t(Q'), z_t, P_{s,x}) \tag{2.2}$$

is a strong Markov (time homogeneous) process. Furthermore, this process is strong Feller (not in the sense of time-homogeneous processes but) in the sense that for any Borel bounded f defined on Q' and $T \in \mathbb{R}$, the function $E_{s,x} f(z_{T-s})$ is continuous with respect to (s, x) in $Q \cap \{(s, x) : s < T\}$.

Remark 2.6. Take a Borel bounded f defined on Q' and $t \in \mathbb{R}$. For $(s, x) \in Q \cap \{s < T\}$ define $u(s, x) = E_{s,x} f(z_{T-s})$. It turns out that not only u is continuous in $Q \cap \{s < T\}$ but u_x is Hölder continuous there. Indeed, Theorem 10.3 below and a standard localization procedure (see, for instance, [17]) show that, for any $T' \in (0, \infty)$ and $\eta \in C_0^\infty(Q \cap \{s < T\})$ we have that $(u\eta)(\cdot + T - T', \cdot) \in H_p^{2,q}(T')$ which along with Lemma 10.2 below lead to our conclusion. For the definition of spaces $H_p^{2,q}(T)$ we refer the reader to the end of Section 3.

Our next main result concerns a particular case of equation (1.1) in which we can prove the existence of solutions for all times. Let $\psi(t, x)$ be a continuous function defined on Q.

Assumption 2.1. (i) The function ψ is nonnegative.

(ii) For each n there exist $p = p(n), q = q(n)$ satisfying (2.1) such that $\psi_x I_{Q^n} \in L_q\text{-}L_p$, where ψ_x is understood in the sense of distributions.

(iii) The function ψ blows up near the parabolic boundary of Q, that is for any $(s, x) \in Q$, $\tau \in (0, \infty)$, and continuous bounded $\mathbb{R}^d$-valued function x_t defined on $[0, \tau)$ and such that $(s + t, x_t) \in Q$ for all $t \in [0, \tau)$ and

$$\varliminf_{t \uparrow \tau} \operatorname{dist}((s + t, x_t), \partial Q) = 0,$$

we have

$$\varlimsup_{t \uparrow \tau} \psi(s + t, x_t) = \infty.$$

(iv) For some constants $K_0 \in [0, \infty)$ and $\varepsilon \in [0, 2)$ in the sense of distributions on Q we have

$$2D_t\psi \le K_0\psi, \quad 2D_t\psi + \Delta\psi \le he^{\varepsilon\psi}, \tag{2.3}$$

where h is a continuous nonnegative function on Q satisfying the following condition
(H) for any $\sigma > 0$ and $T \in [0, \infty)$ there is an $r = r(T, \sigma) \in (1, \infty)$ such that

$$\mathcal{H}(T, \sigma, r) := \mathcal{H}_Q(T, \sigma, r) := \int_Q h^r(t, x) I_{(-T,T)}(t) e^{-\sigma|x|^2} \, dt dx < \infty.$$

Observe that $\mathcal{H}(T, \sigma, r) < \infty$ if h is just a constant.

Theorem 2.7. *Let Assumption 2.1 be satisfied and let w_t be a d-dimensional Wiener process defined on a complete probability space. Then for any $(s, x) \in Q$ there exists a continuous $\mathbb{R}^d$-valued and $\mathcal{F}_t^w$-adapted random process x_t, $t \ge 0$, such that almost surely for all $t \ge 0$*

$$(s + t, x_t) \in Q, \quad \int_0^t |\psi_x(s + r, x_r)|^2 \, dr < \infty,$$
$$x_t = x + w_t - \int_0^t \psi_x(s + r, x_r) \, dr. \tag{2.4}$$

Furthermore, for each $T \in (0, \infty)$ and $n \ge 1$ there exists a constant N, depending only on d, $p(n+1)$, $q(n+1)$, ε, T, $\|\psi_x I_{Q^{n+1}}\|_{L_{q(n+1)}\text{-}L_{p(n+1)}}$, $\operatorname{dist}(\partial Q^n, \partial Q^{n+1})$, $\sup\{\psi + h, Q^{n+1}\}$, and the function $\mathcal{H}$, such that for $(s, x) \in Q^n$ we have

$$E \sup_{t \le T} \exp(\mu\psi(s + t, x_t) + \mu\nu|x_t|^2) \le N,$$

where

$$\mu = (\delta/2)e^{-TK_0/(2\delta)}, \quad \delta = 1/2 - \varepsilon/4, \quad \nu = \mu/(12T), \tag{2.5}$$

Remark 2.8. The uniqueness of solutions to (2.4) follows from Theorem 2.1.

Remark 2.9. The known condition for the existence of weak solutions in the time homogeneous case (see [4, Assumptions (H1) and (H2)]) is the following:

$$\exists \delta > 0 : |\psi_x|^{2\vee(d+\delta)} e^{-2\psi} \in L_{1,loc}(\mathbb{R}^d). \tag{2.6}$$

By the way, we write this condition for our equation (2.4) which is slightly different from the one considered in [4] where there is the factor $\sqrt{2}$ in front of w_t.

Let us compare this with our results. Notice at once that of course there is a substantial difference in the number of derivatives of ψ involved. However, as far as the singularities of ψ are concerned our assumptions are pretty reasonable and sometimes are even weaker.

In one space dimension let $\psi(x) = -\alpha \ln|x|$, $\psi'(x) = -\alpha x^{-1}$ near zero. Then (2.6) is satisfied near zero if and only if $\alpha > 1/2$. For such α condition (2.3) is satisfied with $h = \alpha|x|^{\varepsilon\alpha-2}$, which is summable near zero to some power $r > 1$ if ε is sufficiently close to 2. By the way, this example shows an advantage of allowing h to be a *function* rather than just a constant.

Again for $d = 1$ and $\psi(x) = |x|^{-\delta}$, $\delta > 0$, conditions (2.6) and (2.3) are both satisfied if in the latter one we take h to be an appropriate constant.

On the other hand, for $d = 3$ and $\psi(x) = -\alpha \ln|x^1|$ the assumptions of Theorem 2.7 are satisfied in $Q = \mathbb{R}^4 \setminus \{x^1 = 0\}$ if $\alpha > 1/2$ and (2.6) is satisfied only if $\alpha > 1$.

One more situation when Theorem 2.7 is applicable and the results of [4] are not occurs if $d = 2$ and $\psi(x) = \alpha \ln|\ln|x||$ near the origin with constant $\alpha > 0$. Here condition (2.3) is satisfied near the origin with $K_0 = h = \varepsilon = 0$ just because $\Delta\psi \le 0$. However, $|\psi_x|^{2+\delta}\exp(-2\psi)$ is summable near the origin only if $\delta \le 0$. This example shows, in particular, the advantage of requiring estimates only from above in (2.3).

3. Local weak solutions

Let $b_t(x_\cdot)$ be an $\mathbb{R}^d$-valued function defined on $(0,\infty) \times C$, let p and q be two numbers satisfying (2.1), and let $T, K \in (0,\infty)$ be some constants. Assume that

(i) $b_t(x_\cdot)$ is jointly measurable $\mathcal{N}_t$-adapted and $b_t(x_\cdot) = 0$ for $t > T$ and $x_\cdot \in C$;

(ii) there exists a Borel real-valued function $g(t,x)$ such that $|b_t(x_\cdot)| \le g(t, x_t)$ on $(0,\infty) \times C$ and $\|g\|_{L_q\text{-}L_p} \le K$.

Remark 3.1. For us the most important case when requirement (ii) is satisfied occurs if $b_t(x_\cdot)$ has the form $b(t,x_t)I_{t<\tau(x_\cdot)}$, where $\tau(x_\cdot)$ is a bounded $\mathcal{N}_t$-stopping time. In that case a good candidate for g is $|b(t,x)|$. In the future we deal with increasing sequences of $\mathcal{N}_t$-stopping times and this somewhat justifies the title of the section.

Below we basically reproduce an approach developed by N. Portenko in [23].

Lemma 3.2. *There exist processes w_t, x_t defined on a probability space such that*

(i) w_t, $t \ge 0$, is a d-dimensional Wiener process and x_t, $t \ge 0$, is a continuous d-dimensional process;

(ii) $\{x_s, w_s : s \le t\}$ and $w_{t+h} - w_t$ are independent for each $t, h \ge 0$;

(iii) with probability one

$$\int_0^\infty |b_t(x.)|^2\,dt < \infty;$$

(iv) with probability one for all $t \geq 0$

$$x_t = w_t + \int_0^t b_s(x.)\,ds. \tag{3.1}$$

Proof. Let x_t be a d-dimensional Wiener process defined on a probability space and let f be a Borel nonnegative function on $\mathbb{R}^{d+1}$. Observe that for $t > s \geq 0$ and $x \in \mathbb{R}^d$

$$E\int_s^t f(r, x + x_{r-s})\,dr = \int_s^t (2\pi(r-s))^{-d/2}\int_{\mathbb{R}^d} f(r, x+y)e^{-|y|^2/(2r-2s)}\,dydr.$$

By using Hölder's inequality first with respect to y and then with respect to r we find that for any $p', q' \in [1,\infty]$ satisfying

$$\frac{d}{p'} + \frac{2}{q'} < 2 \tag{3.2}$$

we have

$$E\int_s^t f(r, x + x_{r-s})\,dr \leq N(t-s)^{1-1/q'-d/(2p')}\|f\|_{L_{q'}\text{-}L_{p'}}, \tag{3.3}$$

where N depends only on d, p', q'. Next we notice that $p' := p/2 \geq 1$ ($p \geq 2$!) and $q' = q/2 > 1$ and apply (3.3) to $f = |g|^2$ (with g introduced in (ii) in the beginning of the section) to find that

$$E\int_s^t g^2(r, x + x_{r-s})\,dr \leq N(t-s)^{\varepsilon}, \tag{3.4}$$

where N and $\varepsilon > 0$ depend only on d, p, q, and K. Since this estimate is uniform with respect to t, s, and x, Khasminskii's lemma (see [14] or [23]) implies that for any constant $\kappa > 0$ there is a $\delta > 0$ such that

$$\sup_{s,x} E\exp(\kappa\int_s^{s+\delta} g^2(r, x + x_{r-s})\,dr) \leq N(\kappa, K, T, d, p, q).$$

Then one splits $(0, T)$ (where T is taken from the beginning of the section) into a union of intervals of length $\leq \delta$ and one uses the Markov property of the Wiener process to get that

$$\sup_x E\exp(\kappa\int_0^T g^2(r, x + x_r)\,dr) \leq N, \tag{3.5}$$

where N again depends only on κ, T, d, p, q, and K. We replace here g with $|b|$ and then the integral over $(0, T)$ can be replaced with the one over $(0, \infty)$ since $b_t = 0$ for $t > T$. Thus,

$$E \exp(\kappa \int_0^\infty |b_t(x.)|^2 \, dt) < \infty. \tag{3.6}$$

By standard results about exponential martingales now it follows that

$$E\rho = 1, \tag{3.7}$$

where

$$\rho = \exp(\int_0^\infty b_t(x.) \, dx_t - (1/2) \int_0^\infty |b_t(x.)|^2 \, dt). \tag{3.8}$$

Furthermore, by Girsanov's theorem the process

$$w_t := x_t - \int_0^t b_s(x.) \, ds \tag{3.9}$$

is a Wiener process relative to the new probability measure $\tilde{P}$ defined by $\tilde{P}(d\omega) = \rho(\omega) \, P(d\omega)$. In addition, the increments of w_t are independent of the past values of x_s and w_s or, in somewhat more traditional language, $(w_t, \mathcal{F}_t)$ is a Wiener process, where $\mathcal{F}_t$ is the completion of $\sigma(w_s, x_s : s \leq t)$ or the completion of $\sigma(x_s : s \leq t)$ (the two completions coincide owing to (3.9)). We see that assertions (i), (ii), and (iv) hold under the new probability measure. Assertion (iii) holds with respect to the old probability measure due to (3.6). That it holds with respect to the new measure as well follows from the fact that the new measure is absolutely continuous with respect to the old one. The lemma is proved. □

Lemma 3.3. *Let w_t, x_t be processes for which the assertions (i)-(iv) of Lemma 3.2 hold. Then for any Borel nonnegative function f defined on the space $C([0, \infty), \mathbb{R}^d)$ we have*

$$Ef(x.) = Ef(w.) \exp(\int_0^\infty b_t(w.) \, dw_t - (1/2) \int_0^\infty |b_t(w.)|^2 \, dt), \tag{3.10}$$

$$Ef(w.) = Ef(x.) \exp(- \int_0^\infty b_t(x.) \, dw_t - (1/2) \int_0^\infty |b_t(x.)|^2 \, dt).$$

Furthermore, for any $S \in (0, \infty)$ and $p', q' > 1$ satisfying (3.2) there exists a constant N, depending only on S, p, q, p', q', d, T, and K, such that for any Borel nonnegative function f on $\mathbb{R}^{d+1}$ we have

$$E \int_0^S f(t, x_t) \, dt \leq N \|f\|_{L_{q'}\text{-}L_{p'}}. \tag{3.11}$$

Proof. As we have seen from the proof of Lemma 3.2, property (iii) holds if we take w_t in place of x_t. Therefore the first assertion of the lemma is a direct consequence of the Liptser-Shiryaev theorem about absolutely continuous change of measure (see, for instance, Theorem 7.7 in [22]).

To prove the second assertion we use notation (3.8) with $w.$ in place of $x.$. Then by (3.10) and Hölder's inequality for $\alpha,\ \beta > 1$ satisfying $1/\alpha + 1/\beta = 1$ we find

$$
\begin{aligned}
E\int_0^S f(t,x_t)\,dt &= E\rho\int_0^S f(t,w_t)\,dt \\
&\le (E\rho^\alpha)^{1/\alpha}S^{1/\alpha}(E\int_0^S f^\beta(t,w_t)\,dt)^{1/\beta}
\end{aligned}
$$

Owing to (3.6) all moments of ρ are finite. Indeed, use the notation $\rho(b)$ for the right-hand side of (3.8) with $w.$ in place of $x.$ (remember that $x.$ in (3.8) is a Wiener process). Then, for any b we have $E\rho(b) \le 1$ and our assertion follows from Hölder's inequality and the equation

$$
E\rho^\alpha = E(\rho(2\alpha b))^{1/2}\big(\exp((4\alpha^2-\alpha)\int_0^\infty |b_t(w.)|^2\,dt)\big)^{1/2}
$$

Hence by (3.3)

$$
E\int_0^S f(t,x_t)\,dt \le N\|f^\beta\|^{1/\beta}_{L_{q''}\text{-}L_{p''}} = N\|f\|_{L_{\beta q''}\text{-}L_{\beta p''}}
$$

if $p'', q'' \in [1,\infty]$ are such that $d/p''+2/q'' < 2$. One can certainly choose $\beta > 1$ sufficiently close to 1 so that this condition holds for $p'' = p'/\beta$ and $q'' = q'/\beta$. This yields the desired result and the lemma is proved. □

Corollary 3.4. *In the situation of Lemma 3.3 take a constant $\kappa \ge 0$ and a function $\hat b_t(x.)$ satisfying the conditions in the beginning of the section with $\hat T$, $\hat K$, $\hat g$, $\hat p$, and $\hat q$ in place of T, K, g, p, q. Then*

$$
E\exp(\kappa\int_0^\infty |\hat b_t(x.)|^2\,dt) \le N(d,\hat T,\hat K,\hat p,\hat q,T,K,p,q).
$$

Proof. By (3.10) the expectation equals

$$
\begin{aligned}
&E\rho\exp(\kappa\int_0^\infty |\hat b_t(w.)|^2\,dt) \\
&\le (E\rho^2)^{1/2}\big(E\exp(2\kappa\int_0^\infty |\hat b_t(w.)|^2\,dt)\big)^{1/2},
\end{aligned}
$$

which is finite by (3.6). □

Remark 3.5. Equation (3.10) shows that different solutions of (3.1) have the same distribution on C. In other words, weak uniqueness holds for (3.1).

Lemma 3.6. *Let $b_t^{(i)}(x.)$, $i = 1, 2$, satisfy the assumptions in the beginning of the section and let $|b_t^{(1)}(x.) - b_t^{(2)}(x.)| \le \bar{b}(t, x_t)$, where $\bar{b} \in L_q\text{-}L_p$. Let some couples $(w_t^{(i)}, x_t^{(i)})$, $i = 1, 2$, possess the properties (i)-(iv) of Lemma 3.2 with b replaced by $b^{(i)}$. Then for any bounded Borel functions $f^{(i)}(x.)$, $i = 1, 2$, given on C we have*

$$|Ef^{(1)}(x_\cdot^{(1)}) - Ef^{(2)}(x_\cdot^{(2)})| \le N\big(E|f^{(1)}(w.) - f^{(2)}(w.)|^2\big)^{1/2} + N \sup_C |f^{(1)}| \, \|\bar{b}\|_{L_q\text{-}L_p}, \tag{3.12}$$

where $w.$ is a d-dimensional Wiener process and N is a constant depending only on p, q, d, T, and K.

Proof. The distributions of $x_\cdot^{(i)}$ are mutually absolutely continuous and the corresponding Radon-Nikodym densities are known (see [22] or Lemma 3.3). We have

$$Ef^{(2)}(x_\cdot^{(2)}) = Ef^{(2)}(x_\cdot^{(1)})\bar{\rho},$$

where $\bar{\rho} = \bar{\rho}_\infty$ and for $\Delta b_t = b_t^{(2)} - b_t^{(1)}$

$$\bar{\rho}_t = \exp(\int_0^t \Delta b_s(x_\cdot^{(1)})\, dw_s^{(1)} - (1/2)\int_0^t |\Delta b_s(x_\cdot^{(1)})|^2\, ds).$$

Hence the left-hand side of (3.12) is less than

$$E|f^{(1)} - f^{(2)}|(x_\cdot^{(1)})\bar{\rho} + \sup_C |f^{(1)}|\, E|\bar{\rho} - 1| =: I_1 + I_2 \sup_C |f^{(1)}|.$$

By Corollary 3.4 all moments of the exponential martingale $\bar{\rho}$ are finite, so that $I_1^{3/2} \le NE|f^{(1)} - f^{(2)}|^{3/2}(x_\cdot^{(1)})$ and the latter is estimated through the first term on the right in (3.12) in the same way as Corollary 3.4 is proved. To estimate I_2 we use Itô's formula to get

$$\bar{\rho} = \bar{\rho}_T = 1 + \int_0^T \Delta b_s(x_\cdot^{(1)})\bar{\rho}_s\, dw_s^{(1)}.$$

It follows that for any $\beta > 1$

$$\begin{aligned} I_2^2 \le E|\bar{\rho} - 1|^2 &\le E\int_0^T \bar{b}^2(s, x_s^{(1)})\bar{\rho}_s^2\, ds \\ &\le N\Big(\int_0^T E\bar{\rho}_s^{2\beta/(\beta-1)}\, ds\Big)^{1-1/\beta}\Big(E\int_0^T \bar{b}^{2\beta}(s, x_s^{(1)})\, ds\Big)^{1/\beta}. \end{aligned} \tag{3.13}$$

To estimate the second factor we use (3.11) with $\beta > 1$ so close to 1 that $2\beta/q + \beta d/p < 1$. The first factor is estimated by means of $E\bar{\rho}_T^{2\beta/(\beta-1)}$ since $\bar{\rho}_t$ is a martingale. Then we recall again how all moments of $\bar{\rho}$ are estimated. The lemma is proved. □

Before stating other properties of solutions to (3.1) we introduce some Banach spaces. For $\nu \in \mathbb{R}$ let $H_p^\nu = (1-\Delta)^{-\nu/2} L_p$ be the usual space of Bessel potentials on $\mathbb{R}^d$ and for $0 \le S < T < \infty$ introduce

$$\mathbb{H}_p^{2,q}((S,T)) = L_q((S,T), H_p^2), \quad \mathbb{L}_p^q((S,T)) = L_q((S,T), L_p),$$

$$\mathbb{H}_p^{2,q}(T) = \mathbb{H}_p^{2,q}((0,T)), \quad \mathbb{L}_p^q(T) = \mathbb{L}_p^q((0,T)).$$

One knows that the norm in H_p^2 can be taken to be $\|u_{xx}\|_{L_p} + \|u\|_{L_p}$, where u_{xx} is the matrix of second order derivatives of u. We also introduce the space $H_p^{2,q}(T)$ consisting of functions $u = u(t)$ defined on $[0,T]$ with values in the space of distributions on $\mathbb{R}^d$ such that $u \in \mathbb{H}_p^{2,q}(T)$ and $D_t u \in \mathbb{L}_p^q(T)$. It is worth making precise that by writing $D_t u \in \mathbb{L}_p^q(T)$ we mean that there is an $f \in \mathbb{L}_p^q(T)$ such that for any $s, t \in [0,T]$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$(u(t), \phi) - (u(s), \phi) = \int_s^t (f(r), \phi)\, dr.$$

In that case naturally we write $D_t u = f$.

It turns out (see Lemma 10.2 below) that if $2/q + d/p < 1$ and $u \in H_p^{2,q}(T)$, then u and u_x are continuous in $[0,T] \times \mathbb{R}^d$ or, to be more rigorous, for each $t \in [0,T]$ the distribution $u(t)$ is realized by a real-valued function $u(t,x)$ and $u(t,x)$ and $u_x(t,x)$ are continuous in $[0,T] \times \mathbb{R}^d$. Therefore, the following statement makes sense. Observe that p', q' in Theorem 3.7 need not coincide with p, q introduced and fixed in the beginning of the section.

Theorem 3.7 (Itô's formula). *Let* $2/q' + d/p' < 1$ *and* $u \in H_{p'}^{2,q'}(T)$. *Let* w_t, x_t *be processes for which the assertions (i)-(iv) of Lemma 3.2 hold. Then with probability one for any* $0 \le s \le t \le T$

$$u(t, x_t) = u(s, x_s) + \int_s^t u_x(r, x_r)\, dw_r$$
$$+ \int_s^t \left[D_t u(r, x_r) + (1/2)\Delta u(r, x_r) + b_r^i(x_\cdot) u_{x^i}(r, x_r) \right] dr.$$

The proof of this theorem is obtained right away by approximating u by smooth functions and by using estimate (3.11) and Lemma 10.2. It is perhaps also worth noting that no matter which versions of $D_t u, \Delta u, u_x$ we take the integral of $(D_t u, \Delta u, u_x)(r, x_r)$ over (s,t) remains the same (a.s.) since owing to (3.11) we have

$$E \int_0^T |f(t, x_t)|\, dt = 0$$

if $f = 0$ (a.e.).

Remark 3.8. Under the same conditions on b one can prove that the assertions of Lemma 3.2 hold true if in (3.1) instead of w_t we have $\int_0^t a(s, x_s)\, dw_s$, where $a(s,x)$ is a uniformly nondegenerate bounded symmetric matrix which is Borel in (s,x) and uniformly continuous in $x \in \mathbb{R}^d$ uniformly with respect to s.

In that case instead of using an explicit formula for the distribution of w_t one can use Remark 10.4 and Lemma 10.2.

4. The strong Markov property and the strong Feller property of weak solutions

In this section we consider a particular case of $b_t(x.)$ from Section 3, namely, we assume that

$$b_t(x.) = b(t, x_t),$$

where $b(t, x)$ is a Borel function defined on $\mathbb{R}^{d+1}$ vanishing for large t and such that $|b| \in L_q\text{-}L_p$ for some p, q satisfying (2.1). By shifting the origin in $\mathbb{R}^{d+1}$ we get that for each $(s, x) \in \mathbb{R}^{d+1}$ there exists a probability space and a d-dimensional Wiener process w_t, $t \geq 0$, and a continuous d-dimensional process $x_t = x_t(s, x)$, $t \geq 0$, defined on that space such that $\{x_s, w_s : s \leq t\}$ and $w_{t+h} - w_t$ are independent for any $t, h \geq 0$ and with probability one

$$\int_0^\infty |b(s+t, x_t)|^2 \, dt < \infty, \quad x_t = x + w_t + \int_0^t b(s+r, x_r)\, dr.$$

By $P_{s,x}$ we denote the distribution of the $\mathbb{R}^{d+1}$-valued process

$$z_t = z_t(s, x) = (s+t, x_t(s, x))$$

on $C([0,\infty), \mathbb{R}^{d+1})$ (see Remark 3.5). In an obvious way one introduces the σ-fields $\mathcal{N}_t(\mathbb{R}^{d+1})$ of subsets of $C([0,\infty), \mathbb{R}^{d+1})$. If Q is a domain in $\mathbb{R}^{d+1}$, we define an $\mathcal{N}_t(\mathbb{R}^{d+1})$-stopping time on $C([0,\infty), \mathbb{R}^{d+1})$ by

$$\tau_Q = \tau_Q(z.) = \inf\{t \geq 0 : z_t \notin Q\}.$$

By $E_{s,x}$ we denote the expectation sign relative to $P_{s,x}$.

The following lemma will be used in the proof of Theorem 2.5.

Lemma 4.1. *(i) Let g be a Borel bounded function on ∂Q. Then the function*

$$u(s, x) := E_{s,x} g(z_{\tau_Q})$$

is a continuous function in Q.

(ii) The term

$$(C([0,\infty), \mathbb{R}^{d+1}), \mathcal{N}_t(\mathbb{R}^{d+1}), z_t, P_{s,x})$$

is a strong Markov process. In particular, if γ is an $\mathcal{N}_t(\mathbb{R}^{d+1})$-stopping time such that $\gamma \leq \tau_Q$ and η is an $\mathcal{N}_\gamma(\mathbb{R}^{d+1})$-measurable bounded function on $C([0,\infty), \mathbb{R}^{d+1})$, then

$$E_{s,x} \eta g(z_{\tau_Q}) = E_{s,x} \eta u(z_\gamma).$$

Proof. Assertion (ii) follows immediately from (3.10), the strong Markov property of the Wiener process, and the formula

$$\rho_{\bar\gamma} = E(\rho_\infty | \mathcal{F}^w_{\bar\gamma}) \quad \text{(a.s.)}, \tag{4.1}$$

where $\bar\gamma = \gamma(\bar z.)$, $\bar z_t = (s+t, x+w_t)$, and

$$\rho_t = \exp\Big(\int_0^t b(s+r, x+w_r)\, dw_r - (1/2)\int_0^t |b(s+r, x+w_r)|^2\, dr\Big).$$

In other words assertion (ii) follows from the fact that the change of measure preserves the strong Markov property.

To prove (i) we first claim that for any bounded *continuous* function $f(z_\cdot)$ given on $C([0,\infty),\mathbb{R}^{d+1})$, the function $E_{s,x}f(z_\cdot)$ is continuous with respect to (s,x). Indeed, this follows easily from Lemma 3.6 and the fact that summable functions are continuous in the mean after observing that $x_t(s,x)-x$ satisfies

$$y_t = w_t + \int_0^t \tilde{b}(r,y_r)\,dr,$$

where $\tilde{b}(r,y) := b(s+r,y+x)$. From thus proved claim it follows by a standard measure-theoretic argument that $E_{s,x}f(z_\cdot)$ is Borel measurable for all Borel nonnegative or bounded $f(z_\cdot)$.

Next, without losing generality we assume that $0\in Q$ and only concentrate on proving the continuity of u at 0. Denote $Q_{\pm t,r} = (-t,t)\times\{x\in\mathbb{R}^d : |x|<r\}$, $r,t>0$, and for $z_\cdot\in C([0,\infty),\mathbb{R}^{d+1})$ define $\gamma_r=\gamma_r(z_\cdot)$ as the first exit time of z_t from $Q_{\pm r,r}$. According to (ii) for $(s,x)\in Q_{\pm r,r}$ and for r small enough so that $Q_{\pm r,r}\subset Q$ we have

$$u(s,x) = E_{s,x}u(z_{\gamma_r}),$$

which owing to (3.10) and (4.1) is rewritten as

$$\begin{aligned} u(s,x) = {} & Eu(s+\tau_r(s,x), x+w_{\tau_r(s,x)}) + Eu(s+\tau_r(s,x), x \\ & + w_{\tau_r(s,x)})(\rho_{\tau_r(s,x)}-1) =: I_1(s,x)+I_2(s,x), \end{aligned}$$

where $\tau_r(s,x)$ is the first exit time of $(s+t,x+w_t)$ from $Q_{\pm r,r}$. As is well known $I_1(s,x)$ is infinitely differentiable and satisfies the heat equation in $Q_{\pm r,r}$. In particular, $I_1(s,x)$ is continuous at 0. Furthermore, similarly to the argument about (3.13) (notice that $\gamma_r\le 2r$)

$$|I_2(s,x)|^2 \le N\big(E\int_0^{2r} |(I_{Q_{\pm r,r}}b)(s+t,x+w_t)|^{2\beta}\,dt\big)^{1/\beta} \le N\|I_{Q_{\pm r,r}}b\|_{L_q\text{-}L_p}.$$

Hence

$$\varlimsup_{(s,x)\to 0} |u(s,x)-u(0)| \le N\|I_{Q_{\pm r,r}}b\|_{L_q\text{-}L_p},$$

for any $r>0$, where N is independent of r. By letting $r\downarrow 0$ we see that the left-hand side is zero. The lemma is proved. □

The following lemma will allow us to prove that in the situation of Theorem 2.1 the solutions do not bounce back deep into the interior of Q from near ∂Q too often on any finite interval of time.

Lemma 4.2. *Let w_t, x_t be processes for which the assertions (i)-(iv) of Lemma 3.2 hold. Let G, Q be bounded open subsets of $\mathbb{R}^{d+1}$, containing the origin and such that $\bar{G}\subset Q$. Define $\nu_0=0$,*

$$\mu_k = \inf\{t\ge\nu_k : (t,x_t)\notin Q\},\quad \nu_{k+1} = \inf\{t\ge\mu_k : (t,x_t)\notin\mathbb{R}^{d+1}\setminus\bar{G}\}.$$

Then for any $S \in (0,\infty)$ there exists a constant N, depending only on d, q, p, S, $\|bI_Q\|_{L_q\text{-}L_p}$, and the diameter of Q, such that

$$\sum_{k=0}^{\infty}\big(E|x_{S\wedge\mu_k}-x_{S\wedge\nu_k}|^2\big)^2\le N,\quad \sum_{k=0}^{\infty}\big(E|S\wedge\mu_k-S\wedge\nu_k|^2\big)^2\le S^4.$$

Proof. We have $E|x_{S\wedge\mu_k}-x_{S\wedge\nu_k}|^2\le 2I_k+2J_k$, where

$$I_k:=E|w_{S\wedge\mu_k}-w_{S\wedge\nu_k}|^2,\quad J_k:=E\Big(\int_{S\wedge\nu_k}^{S\wedge\mu_k}|b(s,x_s)|\,ds\Big)^2.$$

Observe that

$$I_k^2=\big(E|S\wedge\mu_k-S\wedge\nu_k|d\big)^2\le d^2E|S\wedge\mu_k-S\wedge\nu_k|^2=:d^2\bar I_k$$

$$\le d^2SE|S\wedge\mu_k-S\wedge\nu_k|,\quad \sum_{k=0}^{\infty}\big(E|w_{S\wedge\mu_k}-w_{S\wedge\nu_k}|^2\big)^2\le d^2S^2,$$

$$\sum_{k=0}^{\infty}(\bar I_k)^2\le\Big(\sum_{k=0}^{\infty}\bar I_k\Big)^2\le S^4.$$

Furthermore,

$$J_k\le E|S\wedge\mu_k-S\wedge\nu_k|\int_{S\wedge\nu_k}^{S\wedge\mu_k}|b(s,x_s)|^2\,ds,\quad J_k^2\le\bar I_k\bar J_k,$$

where

$$\bar J_k:=E\Big(\int_{S\wedge\nu_k}^{S\wedge\mu_k}|b(s,x_s)|^2\,ds\Big)^2.$$

It only remains to estimate $\bar J_k$ by a constant N, depending only on d, q, p, S, $\|bI_Q\|_{L_q\text{-}L_p}$, and the diameter of Q.

Observe that on the set $\{S\wedge\nu_k<S\wedge\mu_k\}$ we have $S\wedge\nu_k=\nu_k$ and $(\nu_k,x_{\nu_k})\in\bar G\subset Q$. Furthermore, $(t,x_t)\in Q$ for $S\wedge\nu_k<t<S\wedge\mu_k$. Now from the strong Markov property of z_t it follows that

$$\bar J_k\le\sup_{(s,x)\in Q}E_{s,x}\Big(\int_0^{S\wedge\tau_Q}|b(s+t,x_t)|^2\,dt\Big)^2.$$

By using (3.10) we easily see that the latter expression will not change if we change arbitrarily b outside of Q only preserving the property that the new b belongs to $L_q\text{-}L_p$. We choose to let b to be zero outside of Q and then get the desired estimate from Corollary 3.4 (after shifting the origin to (s,x)). The lemma is proved. □

Corollary 4.3. *Naturally, we say that on the time interval* $[\nu_k, \mu_k]$ *the trajectory* (t, x_t) *makes a run from* $\bar{G}$ *to* Q^c *provided that* $\mu_k < \infty$. *Denote by* $\nu(S)$ *the number of runs which* (t, x_t) *makes from* $\bar{G}$ *to* Q *before time* S. *Then for any* $\alpha \in [0, 1/2)$, $E\nu^\alpha(S)$ *is dominated by a constant* N, *which depends only on* α, d, q, p, S, $\|bI_Q\|_{L_q\text{-}L_p}$, *the diameter of* Q, *and the distance between the boundaries of* G *and* Q.

Proof. Observe that for any integer $k \geq 1$

$$kP^2(\mu_{k-1} \leq S) \leq P^2(\mu_0 \leq S) + ... + P^2(\mu_{k-1} \leq S) + ... \tag{4.2}$$

Since

$$E\{|x_{S\wedge\mu_k} - x_{S\wedge\nu_k}|^2 + |S \wedge \mu_k - S \wedge \nu_k|^2\}$$
$$\geq E\{|x_{\mu_k} - x_{\nu_k}|^2 + |\mu_k - \nu_k|^2\}I_{\mu_k \leq S} \geq \operatorname{dist}^2(\partial G, \partial Q)P(\mu_k \leq S),$$

by Lemma 4.2 we see that the series in (4.2) converges and its sum is bounded by a constant with proper dependence on the data. After that it only remains to note that $P(\nu(S) \geq k) = P(\mu_{k-1} \leq S)$. □

5. Pathwise uniqueness and strong solutions

In this section we consider a particular case of $b_t(x.)$ from Section 3, namely, we assume that

$$b_t(x.) = b(t, x_t)I_{t<\tau(x.)},$$

for $(t, x.) \in (0, \infty) \times C$, where $\tau(x.)$ is a bounded $\mathcal{N}_t$-stopping time defined on C and $b(t, x)$ is a Borel $\mathbb{R}^d$-valued function defined on $\mathbb{R}^{d+1}$ such that $|b(\cdot, \cdot)| \in L_q\text{-}L_p$ for some p, q satisfying (2.1).

Theorem 5.1. *Let* w_t, y_t, *and* z_t *be* $\mathbb{R}^d$*-valued processes defined on a complete probability space for* $t \geq 0$. *Assume that* w_t *is a Wiener process and the properties (ii)-(iv) of Lemma 3.2 hold true if we take* (y_t, w_t) *and* (z_t, w_t) *in place of* (x_t, w_t). *Furthermore, let* $\{y_s, z_s, w_s : s \leq t\}$ *be independent of* $w_{t+h} - w_t$ *for each* $t, h \geq 0$. *Then*

$$P(\sup_{t\geq 0} |y_t - z_t| > 0) = 0.$$

The Yamada-Watanabe principle (see [28] or [13]) immediately allows us to deduce from Theorem 5.1 and Lemma 3.2 the following result about existence and uniqueness of so-called strong solutions.

Theorem 5.2. *(i) For each* $t \in [0, \infty)$ *on* C *there exists an* $\mathcal{N}_t$*-measurable* $\mathbb{R}^d$*-valued function* $F_t(y.)$ *such that if*

(a) w_t *and* x_t *are* $\mathbb{R}^d$*-valued continuous processes defined on a probability space for* $t \geq 0$,
(b) w_t *is a Wiener process and the properties (ii)-(iv) of Lemma 3.2 hold true for* x_t, w_t,

then for each $t \in [0, \infty)$ *we have* $x_t = F_t(w.)$ *(a.s.).*

(ii) Let w_t be an $\mathbb{R}^d$-valued Wiener process defined on a complete probability space for $t \geq 0$. Then on the same probability space there exists a continuous process x_t, $t \geq 0$, such that the properties (ii)-(iv) of Lemma 3.2 hold. Furthermore, by assertion (i) this process is $\mathcal{F}_t^w$-adapted and unique.

Remark 5.3. The function $F_t(y.)$ does not change if we change w_t or x_t or the underlying probability space. Therefore, for each $b_t(y.)$ under consideration we may and we do choose and fix an appropriate function $F_t(y.)$.

To prove Theorem 5.1 we need two lemmas before which we introduce $\Sigma_t = \sigma\{y_s, z_s, w_s : s \leq t\}$ and let $\bar{\Sigma}_t$ be the completion of Σ_t and $\mathcal{F}_t = \bigcap_{s>t} \bar{\Sigma}_s$. It is easy to see that w_t is a Wiener process relative to $\mathcal{F}_t$.

Lemma 5.4. *Let $u \in H_p^{2,q}(T)$. Then there exists a continuous $\mathcal{F}_t$-adapted increasing process A_t such that $A_0 = 0$, $EA_T < \infty$, and for $t \in [0, T]$*

$$\int_0^t |u_x(s, y_s) - u_x(s, z_s)|^2 \, ds = \int_0^t |y_s - z_s|^2 \, dA_s.$$

Proof. Generally the process A_t we are looking for is not unique and the smallest one is given by

$$A_t = \int_0^t I_{y_s \neq z_s} \frac{|u_x(s, y_s) - u_x(s, z_s)|^2}{|y_s - z_s|^2} \, ds$$

provided that the right hand side is finite. To prove that this is indeed the case and also to prove all other assertions of the lemma, we prove that

$$E \int_0^T I_{y_s \neq z_s} \frac{|u_x(s, y_s) - u_x(s, z_s)|^2}{|y_s - z_s|^2} \, ds \leq N \|u\|_{H_p^{2,q}(T)}, \tag{5.1}$$

where the constant N is independent of u.

By Lemma 10.2 below if $u^n \in H_p^{2,q}(T)$, $n = 1, 2, \ldots$, and $u^n \to u$ in $H_p^{2,q}(T)$, then $u_x^n \to u_x$ uniformly in $[0, T] \times \mathbb{R}^d$. Bearing in mind Fatou's lemma we conclude that it suffices to prove (5.1) for $u \in C_0^\infty([0, T] \times \mathbb{R}^d)$.

In that case by Hadamard's formula

$$u_x(s, y_s) - u_x(s, z_s) = (y_s^j - z_s^j) \int_0^1 u_{xx^j}(s, ry_s + (1 - r)z_s) \, dr.$$

Therefore the left-hand side of (5.1) is less than a constant times

$$\int_0^1 E \int_0^T |u_{xx}(t, ry_t + (1 - r)z_t)|^2 \, dt dr.$$

Here

$$ry_t + (1 - r)z_t = w_t + \int_0^t [rb_s(y.) + (1 - r)b_s(z.)] \, ds.$$

Furthermore, for any $\kappa \geq 0$ by convexity and by Corollary 3.4

$$E \exp(\kappa \int_0^T |rb_s(y.) + (1-r)b_s(z.)|^2 \, ds) \tag{5.2}$$

$$\leq rE \exp(\kappa \int_0^T |b_s(y.)|^2 \, ds) + (1-r)E \exp(\kappa \int_0^T |b_s(z.)|^2 \, ds) < \infty.$$

Now for fixed $r \in [0, 1]$ denote $\bar{b}_t = rb_t(y.) + (1-r)b_t(z.)$ and

$$\rho = \exp(-\int_0^T \bar{b}_t \, dw_t - (1/2) \int_0^T |\bar{b}_t|^2 \, dt).$$

Then (5.2) implies that all (positive and negative) moments of ρ are finite and $E\rho = 1$. Hence by Hölder's inequality and Girsanov's theorem for any $\alpha > 1$

$$\begin{aligned} &E \int_0^T |u_{xx}(t, ry_t + (1-r)z_t)|^2 \, dt \\ &= E\rho^{-\alpha}\rho^{\alpha} \int_0^T |u_{xx}(t, ry_t + (1-r)z_t)|^2 \, dt \\ &\leq N\big(E\rho \int_0^T |u_{xx}(t, ry_t + (1-r)z_t)|^{2\alpha} \, dt\big)^{1/\alpha} \\ &= N\big(E \int_0^T |u_{xx}(t, w_t)|^{2\alpha} \, dt\big)^{1/\alpha} =: NI. \end{aligned}$$

Thus, the left-hand side of (5.1) is less than a constant times I and now (5.1) follows from (3.11) if $\alpha > 1$ is so close to 1 that $|u_{xx}|^{2\alpha} \in L_{q'}\text{-}L_{p'}$ with $d/p' + 2/q' < 2$. The lemma is proved. □

The proof of the second lemma is rather long. Therefore, to understand better its idea, borrowed from [27], we advice the reader to take $\Gamma = \Omega$, assume formally that our construction is valid for $B_1 = B_2 = \mathbb{R}^d$, and drop the first inf in the definition of $\bar{\nu}$.

Lemma 5.5. *Let γ be a finite $\mathcal{F}_t$-stopping time. Assume that $y_t = z_t$ for $t \leq \gamma$ and let a $\Gamma \in \mathcal{F}_\gamma$ be such that $P(\Gamma) > 0$. Then there exists a finite $\mathcal{F}_t$-stopping time $\sigma \geq \gamma$ such that $y_t = z_t$ for $\gamma \leq t \leq \sigma$ and $P(\Gamma, \sigma > \gamma) > 0$.*

Proof. Define $\bar{\tau} = \tau(y.) \wedge \tau(z.)$ and split the proof according as $P(\Gamma, \gamma \geq \bar{\tau})$ is > 0 or $= 0$.

Case 1: $P(\Gamma, \gamma \geq \bar{\tau}) > 0$. Introduce

$$\sigma = \gamma + I_{\Gamma, \gamma \geq \bar{\tau}}.$$

The random variable σ is a stopping time, which is seen from the following formula

$$\begin{aligned} \{\sigma \leq t\} = \big([\Gamma \cap \{\gamma \geq \bar{\tau}\}] \cap \{\gamma \leq t-1\}\big) \\ \cup\big([\Gamma \cap \{\gamma \geq \bar{\tau}\}]^c \cap \{\gamma \leq t\}\big), \end{aligned}$$

where $\Gamma \cap \{\gamma \geq \bar{\tau}\} \in \mathcal{F}_\gamma$.

Next we observe that, due to the fact that $\tau(x.)$ is an $\mathcal{N}_t$-stopping time, if we are given $\alpha., \beta. \in C$ and $\alpha_t = \beta_t$ for $t \in [0, \tau(\alpha.))$, then $\tau(\alpha.) = \tau(\beta.)$. It follows that on the set where $\gamma \geq \bar{\tau}$ we have $\bar{\tau} = \tau(y.) = \tau(z.) \leq \gamma$ and for all $t \geq \gamma$

$$y_t = y_\gamma + w_t - w_\gamma = z_\gamma + w_t - w_\gamma = z_t.$$

In particular, $y_t = z_t$ for $\gamma \leq t \leq \sigma$. Finally, the assumption $P(\Gamma, \gamma \geq \bar{\tau}) > 0$ implies that $P(\Gamma, \sigma > \gamma) > 0$.

Case 2: $\gamma < \bar{\tau}$ (a.s.) on Γ. Obviously for any $\varepsilon > 0$ there exists $T \in (0, \infty)$ and a unit ball $B_1 \subset \mathbb{R}^d$ such that

$$P(T - \varepsilon < \gamma < T, y_\gamma = z_\gamma \in B_1, \Gamma) > 0. \tag{5.3}$$

Let B_2 be the ball of radius 2 with center x_0 being that of B_1. Take ε from Lemma 10.6 below, find a T which suits (5.3), and take u^i, $i = 1, ..., d$, from Lemma 10.6. By Itô's formula for $t \in [S, T] := [(T - \varepsilon)_+, T]$ we have

$$u^i(t \wedge \bar{\tau}, y_{t\wedge\bar{\tau}}) = u^i(S \wedge \bar{\tau}, y_{S\wedge\bar{\tau}}) + \int_{S\wedge\bar{\tau}}^{t\wedge\bar{\tau}} u^i_x(r, y_r)\, dw_r.$$

We multiply this equation by the indicator of the set

$$\Pi = \{\gamma > S, y_\gamma = z_\gamma \in B_1, \Gamma\}$$

and notice that on this set $S \wedge \bar{\tau} = S$ and $y_S = z_S$. We also do the same for the process z_t. Then we obtain

$$\begin{aligned}
[u^i(t \wedge \bar{\tau}, y_{t\wedge\bar{\tau}}) - u^i(t \wedge \bar{\tau}, z_{t\wedge\bar{\tau}})]I_\Pi &= I_\Pi \int_S^{t\wedge\bar{\tau}} [u^i_x(r, y_r) - u^i_x(r, z_r)]\, dw_r \\
&= I_\Pi \int_S^{t\wedge\bar{\tau}} [u^i_x(r, y_r) - u^i_x(r, z_r)] I_{\gamma\leq r}\, dw_r \\
&= \int_S^{t\wedge\bar{\tau}} [u^i_x(r, y_r) - u^i_x(r, z_r)] I_{\Pi,\gamma\leq r}\, dw_r.
\end{aligned}$$

We square the extreme terms of this equation, sum up the results with respect to i, then take sup's over t in the range $[S, \nu]$ where ν is a stopping time with values in $[S, T]$. After that we take expectations and use Doob's inequality to obtain

$$\begin{aligned}
&EI_\Pi \sup_{t\in[S,\nu]} \sum_{i=1}^d [u^i(t \wedge \bar{\tau}, y_{t\wedge\bar{\tau}}) - u^i(t \wedge \bar{\tau}, z_{t\wedge\bar{\tau}})]^2 \\
&\leq 4E \int_S^{\nu\wedge\bar{\tau}} I_{\Pi,\gamma\leq r} \sum_{i=1}^d |u^i_x(r, y_r) - u^i_x(r, z_r)|^2\, dr.
\end{aligned} \tag{5.4}$$

By Lemma 5.4 the right-hand side of (5.4) equals

$$E \int_S^{\nu\wedge\bar{\tau}} I_{\Pi,\gamma\leq r} |y_r - z_r|^2\, dA_r. \tag{5.5}$$

We now take $\nu := S \vee \bar{\nu} \wedge T$, where

$$\bar{\nu} := \inf\{t \geq \gamma : |y_t - x_0| \vee |z_t - x_0| \geq 2\} \wedge \inf\{t \geq \gamma : A_t - A_\gamma \geq 1/4\}.$$

Observe that on Π if $t \in [S, \nu]$ and

$$\sum_{i=1}^{d} [u^i(t \wedge \bar{\tau}, y_{t\wedge\bar{\tau}}) - u^i(t \wedge \bar{\tau}, z_{t\wedge\bar{\tau}})]^2 \neq 0, \tag{5.6}$$

then $t > \gamma > S$, $\nu > S$, and $\nu = \bar{\nu} \wedge T$. In that case $\gamma < t \leq \bar{\nu}$. Also $\gamma < \bar{\tau}$ (a.s.) by assumption and hence, a.e. on Π, if $t \in [S, \nu]$ and (5.6) holds, then $\gamma < t \wedge \bar{\tau} \leq \bar{\nu}$ and therefore $|y_{t\wedge\bar{\tau}} - x_0| \leq 2$. A similar statement holds for $z_\cdot$. By Lemma 10.6 we conclude that the left-hand side of (5.4) is greater than

$$(1/2) E I_\Pi \sup_{t\in[S,\nu]} |y_{t\wedge\bar{\tau}} - z_{t\wedge\bar{\tau}}|^2. \tag{5.7}$$

On the other hand, the right-hand side of (5.4), which equals (5.5), is less than

$$E \int_S^\nu I_{\Pi,\gamma\leq r} |y_{r\wedge\bar{\tau}} - z_{r\wedge\bar{\tau}}|^2 \, dA_r$$

$$\leq E I_\Pi \sup_{t\in[S,\nu]} |y_{t\wedge\bar{\tau}} - z_{t\wedge\bar{\tau}}|^2 \int_S^{\bar{\nu}} I_{\gamma\leq r} \, dA_r \leq (1/4) E I_\Pi \sup_{t\in[S,\nu]} |y_{t\wedge\bar{\tau}} - z_{t\wedge\bar{\tau}}|^2.$$

We see that expression (5.7) is less than its half, which implies that it is zero. Introduce

$$\Theta := \{ \sup_{t\in[S,\nu]} |y_{t\wedge\bar{\tau}} = z_{t\wedge\bar{\tau}}| = 0, \gamma < \bar{\tau}\} \cap \Pi.$$

Then $\Theta \subset \Pi$ and by the above $P(\Theta) = P(\Pi)$.

Our last step is to prove that

$$\sigma := [(\bar{\nu} \wedge T \wedge \bar{\tau}) \vee \gamma] I_\Theta + \gamma I_{\Theta^c}$$

possesses the required properties. First, by definition $\bar{\nu} > \gamma$. In addition, the event $\Theta \cap \{\gamma < T \wedge \bar{\tau}\}$ ($= \Pi \cap \{\gamma < T \wedge \bar{\tau}\}$ (a.s.)) happens with strictly positive probability due to (5.3) and the condition of the case under consideration. When this event happens, we have $\bar{\nu} \wedge T \wedge \bar{\tau} > \gamma$ and $\sigma = \bar{\nu} \wedge T \wedge \bar{\tau} > \gamma$. Also Γ happens. Therefore $P(\Gamma, \sigma > \gamma) \geq P(\Theta \cap \{\gamma < T \wedge \bar{\tau}\}) > 0$. Furthermore, obviously $\sigma(\omega) \geq \gamma(\omega)$ for all ω.

Next, we have

$$\{\sigma > \gamma\} \subset \{\sigma = \bar{\nu} \wedge T \wedge \bar{\tau}\} \cap \Theta, \quad \Theta \subset \Pi \subset \{\gamma > S\},$$

$$\{\gamma > S\} \subset \{\bar{\nu} \wedge T > S\} \subset \{\nu = \bar{\nu} \wedge T\}$$

implying that

$$\{\sigma > \gamma\} \subset \{\sigma = \bar{\nu} \wedge T \wedge \bar{\tau}\} \cap \{\gamma > S\}$$

$$\subset \{\sigma = \bar{\nu} \wedge T \wedge \bar{\tau}\} \cap \{\nu = \bar{\nu} \wedge T\} \subset \{\sigma = \nu \wedge \bar{\tau}\}.$$

Therefore, if $\sigma > \gamma$, then

$$\sup_{t\in[\gamma,\sigma]} |y_t - z_t| \le \sup_{t\in[S,\nu\wedge\bar\tau]} |y_t - z_t| = 0.$$

Of course, if $\sigma = \gamma$, then the equality between the extreme terms is given by the assumption on γ.

It only remains to check that σ is a stopping time. To do that observe that for any $r \ge 0$ we have $\Theta^c \cap \{\sigma \le r\} = \Theta^c \cap \{\gamma \le r\} = \Pi^c \cap \{\gamma \le r\}$ (a.s.) and $\Pi \in \mathcal{F}_\gamma$ so that

$$\Theta^c \cap \{\sigma \le r\} \in \mathcal{F}_r.$$

Furthermore, $\Theta \cap \{\sigma \le r\}$ is empty if $r \le S$ since $\sigma \ge \gamma > S$ on Θ. If $r \ge S$, then (a.s.)

$$\Theta \cap \{\sigma \le r\} = \{\bar\nu \wedge T \wedge \bar\tau \le r\} \cap \{\gamma \le r\} \cap \Pi,$$

Here $\{\gamma \le r\} \cap \Pi \in \mathcal{F}_r$ since $\Pi \in \mathcal{F}_\gamma$ and $\{\bar\nu \wedge T \wedge \bar\tau \le r\} \in \mathcal{F}_r$ because $\bar\nu$, T, and $\bar\tau$ are stopping times. Hence $\Theta \cap \{\sigma \le r\}$ belongs to $\mathcal{F}_r$ along with $\Theta^c \cap \{\sigma \le r\}$, σ is indeed a stopping time, and the lemma is proved. □

Proof of Theorem 5.1. Almost everything, actually, has been done in the proof of Lemma 5.5. Indeed, define

$$\mu = \inf\{t \ge 0 : |y_t - z_t| > 0\}.$$

Our goal is to prove that $\mu = \infty$ (a.s.) or that for any constant $S \in (0, \infty)$ we have $\mu \wedge S = S$ (a.s.). Take a constant S and introduce $\gamma = \mu \wedge S$ and $\Gamma = \{\omega : \gamma(\omega) < S\}$. We claim that $P(\Gamma) = 0$.

Indeed, if $P(\Gamma) > 0$, then according to Lemma 5.5 we can find a stopping time $\sigma \ge \gamma$, such that $\sigma > \gamma$ on a subset of Γ of positive probability and $y_t = z_t$ for $t \le \sigma$. We therefore have at least one ω at which $\mu = \gamma < \sigma$, and $y_t = z_t$ for $t \le \sigma$. But this contradicts the very definition of μ, which requires $|y_t - z_t|$ to be strictly bigger than zero for points arbitrarily close to μ from the right. This completes the proof of the theorem. □

6. Proofs of Theorems 2.1 and 2.5

Proof of Theorem 2.1. Without losing generality we may and will assume that in (1.1) we have $s = 0$ and $x = 0$. We split the proof into two parts.

Existence. For any $z_\cdot \in C$ and $n \ge 1$ define $\mathcal{N}_t$-stopping times

$$\tau^n(z_\cdot) = \inf\{t \ge 0 : (t, z_t) \notin Q^n\}.$$

Notice that since Q^n are bounded, $\tau^n(z_\cdot)$ are bounded stopping times.

Let $b^n = bI_{Q^n}$ and consider the equation

$$x_t^n = w_t + \int_0^t I_{s<\tau^n(x_\cdot^n)} b^n(s, x_s^n)\,ds \quad \left(= w_t + \int_0^t I_{s<\tau^n(x_\cdot^n)} b(s, x_s^n)\,ds\right). \quad (6.1)$$

By Theorem 5.2 equation (6.1) has an $\mathcal{F}_t$-adapted solution x^n_t. As is easy to see the process

$$\bar{x}^n_t := x^{n+1}_t I_{t<\tau^n(x^{n+1}_\cdot)} + (w_t - w_{\tau^n(x^{n+1}_\cdot)} + x^{n+1}_{\tau^n(x^{n+1}_\cdot)}) I_{t\geq\tau^n(x^{n+1}_\cdot)}$$

also satisfies (6.1). By Theorem 5.1 we have $x^n_t = \bar{x}^n_t$ for all $t \geq 0$ (a.s.). In particular, these processes coincide before $\tau^n(x^n_\cdot) = \tau^n(\bar{x}^n_\cdot) = \tau^n(x^{n+1}_\cdot)$ and $\tau^n(x^n_\cdot) < \tau^{n+1}(x^{n+1}_\cdot)$ (a.s.). Therefore, the definitions

$$\zeta = \lim_{n\to\infty} \tau^n(x^n_\cdot), \quad x_t = \lim_{n\to\infty} x^n_{t\wedge\tau^n(x^n_\cdot)}, \quad t < \zeta,$$

$$z_t = (t, x_t), \quad t < \zeta, \quad z_t = \partial, \quad \zeta \leq t < \infty$$

make sense almost surely. We may throw away the set of ω where the above definitions do not make sense and work only on the remaining part of Ω. Certainly, $\tau^n(x^n_\cdot)$ and ζ are $\mathcal{F}_t$-stopping times. Also, $x^n_{t\wedge\tau^n(x^n_\cdot)}$ and $x_t I_{t<\zeta}$ are $\mathcal{F}_t$-adapted. It follows that z_t is $\mathcal{F}_t$-adapted. Next, if $t < \zeta$, then there exists an n such that $t < \tau^n(x^n_\cdot) < \zeta$. Since $x_s = x^n_s$ for $s \leq \tau^n(x^n_\cdot)$, this easily implies assertion (iv) of the theorem and it only remains to prove that z_t is left continuous at ζ (a.s.).

By using the terminology of Corollary 4.3 we denote by $\nu_k(S)$ the number of runs of (t, x_t) from $\bar{Q}^k$ to $(Q^{k+1})^c$ before time $S \wedge \zeta$. For $n > k+1$ obviously, $\nu_k(S \wedge \tau^n(x^n_\cdot))$ is also the number of runs that (t, x^n_t) makes from $\bar{Q}^k$ to $(Q^{k+1})^c$ before time $S \wedge \tau^n(x^n_\cdot)$, which increases if we increase the time interval to S. It follows by Corollary 4.3 that $E\nu_k^{1/4}(S \wedge \tau^n(x^n_\cdot))$ is bounded by a constant independent of n. By Fatou's theorem $E\nu_k^{1/4}(S \wedge \zeta)$ is finite. In particular, on the set $\{\omega : \zeta(\omega) < \infty\}$ (a.s.) we have $\nu_k(\zeta) < \infty$. The latter also holds on the set $\{\omega : \zeta(\omega) = \infty\}$ because (t, x_t) is continuous on $[0, \zeta)$ and Q^k is bounded.

Thus, $\nu_k(\zeta) < \infty$ (a.s.) for any k. In addition, $\tau^n(x^n_\cdot) < \zeta$ and $\tau^n(x^n_\cdot) \uparrow \zeta$. Since $(\tau^n(x^n_\cdot), x^n_{\tau^n(x^n_\cdot)}) \in \partial Q^n$ we conclude that (a.s.) there can exist only finitely many n such that (t, x_t) visits $\bar{Q}^k$ after exiting from Q^n. This is the same as to say that $(t, x_t) \to \partial$ as $t \uparrow \zeta$ (a.s.). "Cleaning" again the probability space finishes the proof of the existence part of the theorem.

Uniqueness. The process z'_t is continuous and Q'-valued. Furthermore, for $t < \zeta'$ it is Q-valued. It follows that for any $n \geq 1$

$$\tau^n(x'_\cdot) = \inf\{t \geq 0 : (t, x'_t) \notin Q^n\} < \zeta'. \tag{6.2}$$

Also observe that (a.s.)

$$\bar{\zeta} := \lim_{n\to\infty} \tau^n(x'_\cdot) = \zeta'. \tag{6.3}$$

Indeed, that $\bar{\zeta} \leq \zeta'$ follows from (6.2). On the other hand, on the set where $\bar{\zeta} < \zeta'$ we have that, on the one hand, $z'_{\bar{\zeta}} \in Q$ since $\bar{\zeta} < \zeta'$, but on the other hand, $z'_{\bar{\zeta}} = \partial$ since $z'_{\bar{\zeta}}$ is the limit of points getting outside of any Q^n.

Next, introduce

$$x^n_t = x'_{t\wedge\tau^n(x'_\cdot)} + w_{t\vee\tau^n(x'_\cdot)} - w_{\tau^n(x'_\cdot)}.$$

Then $x_t^n = x_t'$ for $t \le \tau^n(x_\cdot')$, so that $\tau^n(x_\cdot') = \tau^n(x_\cdot^n)$ and it is easy to see that x_t^n satisfies (6.1). It follows by Theorem 5.2 that for each $t \in [0,\infty)$ there exists an $\mathcal{N}_t$-measurable $\mathbb{R}^d$-valued function $F_t^n(y_\cdot)$ determined uniquely by b and Q^n such that $x_t^n = F_t^n(w_\cdot)$ (a.s.). Hence, the formulas

$$\{\omega : \tau^n(x_\cdot') > t\} = \{\omega : \tau^n(x_\cdot^n) > t\}$$

$$= \{\omega : \inf_{r \in \rho_t} \operatorname{dist}\big((r, F_r^n(w_\cdot)), \partial Q^n\big) > 0\}, \quad \text{(a.s.)},$$

where ρ_t is the set of rationals on $[0,t]$, and

$$\tau^n(x_\cdot') = \int_0^\infty I_{\tau^n(x_\cdot')>t}\,dt = \lim_{m\to\infty} m^{-1}\sum_{i=0}^\infty I_{\tau^n(x_\cdot')>i/m},$$

show that $\tau^n(x_\cdot') = G^n(w_\cdot)$ (a.s.), where G^n is a Borel function on C uniquely determined by b and Q^n.

Equation (6.3) now shows that ζ' is expressed (a.s.) as a Borel function of $w_\cdot$ uniquely determined by b and Q^n, $n = 1, 2, \ldots$. In addition,

$$x_t' I_{t<\zeta'} = \lim_{n\to\infty} x'_{t\wedge\tau^n(x_\cdot')} I_{t<\tau^n(x_\cdot')} = \lim_{n\to\infty} x^n_{t\wedge\tau^n(x_\cdot')} I_{t<\tau^n(x_\cdot')}$$

$$= \lim_{n\to\infty} x_t^n I_{t<\tau^n(x_\cdot')} = \lim_{n\to\infty} F_t^n(w_\cdot) I_{t<G^n(w_\cdot)} \quad \text{(a.s.)}.$$

It follows that z_t' is expressed (a.s.) as a Borel function of $w_\cdot$ and this function is uniquely determined by b and Q^n, $n \ge 1$. Obviously this implies both statements about uniqueness in the theorem which is thereby proved. □

Proof of Theorem 2.5. Strong Feller property. For $z_\cdot \in C([0,\infty), Q')$ let

$$\zeta(z_\cdot) = \inf\{t \ge 0 : z_t = \partial\}, \quad \tau^n(z_\cdot) = \inf\{t \ge 0 : z_t \notin Q^n\}.$$

On $\{z_\cdot : t < \zeta\}$ we have $z_t \in Q$ so that the projection of z_t on $\mathbb{R}^d$, which we denote by $x_t = x_t(z_\cdot)$ is well defined. Observe that by definition of $P_{s,x}$ we have $z_t = (s+t, x_t)$ ($P_{s,x}$-a.s.). Also observe that

$$E_{s,x} f(z_{T-s}) = f(\partial) + E_{s,x}[f(T, x_{T-s}) - f(\partial)] I_{\zeta > T-s}.$$

Hence, it suffices to prove that for any $T \in \mathbb{R}$ and Borel bounded f given on $\mathbb{R}^d$, the function

$$u(s,x) := E_{s,x} f(x_{T-s}) I_{\zeta > T-s}$$

is continuous with respect to (s,x) on $Q(T) := Q \cap \{(s,x) : s < T\}$.

By examining the way $P_{s,x}$ is defined, it is easy to see that on $Q(T)$ we have

$$\begin{aligned} u(s,x) &= \lim_{n\to\infty} E_{s,x} f(x_{T-s}) I_{\tau^n \ge T-s} \\ &= \lim_{n\to\infty} E^n_{s,x} f(x_{T-s}) I_{\tau^n \ge T-s} =: \lim_{n\to\infty} u^n(s,x), \end{aligned} \tag{6.4}$$

where $E^n_{s,x}$ is the symbol of integrating against $P^n_{s,x}$, which is the distribution on $C([0,\infty),\mathbb{R}^{d+1})$ of $(s+t,x^n_t)$, where x^n_t is the unique solution to

$$x^n_t = x + w_t + \int_0^t b_n(s+r,x^n_r)\,dr$$

with $b_n := bI_{Q_n}$. Observe that in the notation introduced before Lemma 4.1

$$u^n(s,x) = E^n_{s,x} g(z_{\tau_{Q^n(T)}}),$$

where $Q^n(T) = Q^n \cap \{t < T\}$, $g(T,y) = f(y)$ if $(T,y) \in \partial Q^n(T)$, and $g = 0$ on the remaining part of $\partial Q^n(T)$. By Lemma 4.1 (ii) for $n \geq m \geq 1$

$$u^n(s,x) = E^n_{s,x} u^n(z_{\tau_{Q^m(T)}}) = E^m_{s,x} u^n(z_{\tau_{Q^m(T)}}).$$

Upon letting $n \to \infty$ and using (6.4) and the dominated convergence theorem we conclude that for any $m \geq 1$

$$u(s,x) = E^m_{s,x} u(z_{\tau_{Q^m(T)}}),$$

which implies the continuity of u in $Q^m(T)$ by Lemma 4.1 (i). Since m is arbitrary, u is continuous in $Q(T)$.

Strong Markov property. Define for $n \geq 1$

$$z^n_t := (s+t, x^n_t), \quad t \geq 0.$$

Then for $(s,x) \in Q$ and any Borel bounded f on $\mathbb{R}^d$ and any $\mathcal{N}_t(Q')$-stopping time τ

$$E_{s,x} f(z_{\tau+t}) = f(\partial) + E_{s,x}(f(z_{\tau+t}) - f(\partial)) I_{\zeta > \tau+t}. \tag{6.5}$$

But

$$\begin{aligned} E_{s,x} f(z_{\tau+t}) I_{\zeta>\tau+t} &= \lim_{n\to\infty} E_{s,x} f(z_{\tau+t}) I_{\tau_n \geq \tau+t} \\ &= \lim_{n\to\infty} E^n_{s,x} f(z^n_{\tau+t}) I_{\tau_n\geq\tau+t} I_{\tau_n\geq\tau} \end{aligned}$$

which, since $\{\tau_n \geq \tau\} \in \mathcal{N}_\tau$, by Lemma 4.1 equals

$$\begin{aligned} \lim_{n\to\infty} E^n_{s,x} I_{\tau_n\geq\tau} E^n_{z^n_\tau} f(z^n_t) I_{\tau_n\geq t} &= \lim_{n\to\infty} E_{s,x} I_{\tau_n\geq\tau} E_{z_\tau} f(z_t) I_{\tau_n\geq t} \\ &= E_{s,x} I_{\zeta>\tau} E_{z_\tau} f(z_t) I_{\zeta>t}. \end{aligned}$$

By (6.5) it follows that

$$E_{s,x} f(z_{\tau+t}) = E_{s,x} E_{z_\tau} f(z_t).$$

Since this equality obviously also holds with (s,x) replaced by ∂, the strong Markov property follows by a well-known result from Markov process theory (see e.g. [6, Proposition IV. 6.3]). □

7. Some auxiliary estimates

In order to be able to show that under certain conditions our solutions stay in Q for all times, we need certain estimates which we collect in this section. We fix a $T \in (0, \infty)$ and a real-valued function $\phi(t, x)$ which is infinitely differentiable on $[0, \infty) \times \mathbb{R}^d$. Let w_t, $t \geq 0$, be a d-dimensional Wiener process given on a complete probability space.

Lemma 7.1. *For any $x \in \mathbb{R}^d$ and $B_t := w_{T-t} - w_T$ we have that (a.s.)*

$$\phi(0, x) - \phi(T, x + w_T) - \int_0^T \phi_x(T - s, y + B_s)\, dB_s|_{y=x+w_T}$$
$$= \phi(T, x + w_T) - \phi(0, x) - \int_0^T \phi_x(s, x + w_s)\, dw_s$$
$$-2\int_0^T D_t\phi(s, x + w_s)\, ds. \tag{7.1}$$

Proof. Observe that B_s is a Wiener process on $[0, T]$, so that the stochastic integral in (7.1) with respect to B_s is well defined. Furthermore, by Itô's formula

$$\int_0^T \phi_x(T - s, y + B_s)\, dB_s = \phi(0, y + B_T) - \phi(T, y)$$
$$+ \int_0^T (D_t\phi - (1/2)\Delta\phi)(T - s, y + B_s)\, ds.$$

This shows that the left-hand side in this equation is a well defined continuous function of y so that formula (7.1) makes perfect sense. This also shows that the left-hand side of (7.1) equals

$$-\int_0^T (D_t\phi - (1/2)\Delta\phi)(T - s, x + w_{T-s})\, ds$$
$$= \int_0^T ((1/2)\Delta\phi - D_t\phi)(s, x + w_s)\, ds.$$

That the latter expression coincides with the right-hand side of (7.1) follows again by Itô's formula. The lemma is proved. □

Corollary 7.2. *Take a nonnegative Borel function $f(x)$ and for $t \in [0, T]$ introduce*

$$\beta_T(t, x) = \exp(-\int_0^{T-t} \phi_x(t + s, x + w_s)\, dw_s$$
$$-(1/2)\int_0^{T-t} |\phi_x(t + s, x + w_s)|^2\, ds$$
$$-2\int_0^{T-t} D_t\phi(t + s, x + w_s)\, ds),$$

$$v_T(t, x) = v(t, x) = E\beta_T(t, x)f(x + w_{T-t}), \quad c(t) = \int_{\mathbb{R}^d} e^{-2\phi(t,x)}v(t, x)\, dx.$$

Then $c(t)$ is an increasing function. In particular, $c(0) \le c(T)$, that is

$$\int_{\mathbb{R}^d} e^{-2\phi(0,x)} v(0,x)\,dx \le \int_{\mathbb{R}^d} e^{-2\phi(T,x)} f(x)\,dx. \tag{7.2}$$

Furthermore, one can replace $\le$ in (7.2) with $=$ if $\phi_x \in L_q\text{-}L_p$ for some p, q satisfying (2.1) and in that case $c(t)$ is constant on $[0, T]$.

Proof. First observe that by the Markov property of the Wiener process, for $0 \le t \le r \le T$, we have

$$v(t,x) = E\beta_r(t,x)v(r, x + w_{r-t}).$$

Furthermore, by applying (7.1) to $r - t, \phi(t + \cdot, \cdot)$ in place of T, ϕ, respectively, we obtain

$$\beta_r(t,x)e^{-2\phi(t,x)} = \xi(x + w_{r-t})e^{-2\phi(r,x+w_{r-t})},$$

where

$$\xi(y) = \exp(-\int_0^{r-t} \phi_x(r-s, y+B_s)\,dB_s - (1/2)\int_0^{r-t} |\phi_x(r-s, y+B_s)|^2\,ds)$$

and $B_s = w_{r-t-s} - w_{r-t}$. Therefore,

$$\begin{aligned} c(t) &= E\int_{\mathbb{R}^d} \xi(x + w_{r-t})e^{-2\phi(r,x+w_{r-t})} v(r, x + w_{r-t})\,dx \\ &= E\int_{\mathbb{R}^d} \xi(x)e^{-2\phi(r,x)} v(r,x)\,dx = \int_{\mathbb{R}^d} e^{-2\phi(r,x)} v(r,x)E\xi(x)\,dx, \end{aligned}$$

and it only remains to use (3.7) and remember that generally the expectation of an exponential martingale is less than 1. $\square$

Remark 7.3. Since the above corollary plays a very important role in what follows, it is probably worth giving it at least an outline of a different analytic proof. Under mild conditions $v(t,x)$ satisfies the corresponding Kolmogorov equation, that is

$$D_t v(t,x) + (1/2)\Delta v(t,x) - \phi_{x^i}(t,x)v_{x^i}(t,x) - 2v(t,x)D_t\phi(t,x) = 0$$

which is rewritten in an equivalent form as

$$D_t(e^{-2\phi}v) + (1/2)\big(e^{-2\phi}v_{x^i}\big)_{x^i} = 0.$$

We integrate through the equation with respect to x and very naturally, however maybe not quite rigorously, arrive at

$$D_t\int_{\mathbb{R}^d} e^{-2\phi(t,x)} v(t,x)\,dx = 0,$$

which says that $c(t)$ is constant.

Below we use the notation

$$B_r = \{x \in \mathbb{R}^d : |x| < r\}, \quad Q_{t,r} = [0,t) \times B_r, \quad Q_t = [0,t) \times \mathbb{R}^d.$$

Lemma 7.4. *On an extension of the probability space there is a stopping time γ such that the distribution of (γ, w_γ) has a bounded density concentrated on $Q_{1,1}$.*

Proof. Denote $n = d + 2$. On an extension of our probability space there exists a random variable ρ with values in $[0, 1]$ and density nr^{n-1} such that ρ is independent of all $\mathcal{F}^w_t$. Let $\hat{\mathcal{F}}_t = \mathcal{F}^w_t \vee \sigma(\rho)$, and define γ as the first exit time of (t, w_t) from $Q_{\rho^2,\rho}$. We claim that γ is a random variable of the type we are looking for.

That γ is an $\hat{\mathcal{F}}_t$-stopping time is obvious. As is well known the exit distribution of (t, w_t) from $Q_{1,1}$ has a bounded density with respect to the surface measure on

$$S := ((0, 1) \times \partial B_1) \cup (\{1\} \times B_1\}.$$

Denote this density by $\pi(t, x)$. Since by a straightforward computation

$$\big(\tau_{Q_{r^2,r}}(z_\cdot), w_{\tau_{Q_{r^2,r}}(z_\cdot)}\big) = \big(r^2\tau_{Q_{1,1}}(\bar{z}_\cdot), r\bar{w}_{\tau_{Q_{1,1}}(\bar{z}_\cdot)}\big),$$

where $z_t := (t, w_t)$, $\bar{w}_t := \frac{1}{r}w_{r^2 t}$, $\bar{z}_t := (t, \bar{w}_t)$, $t \geq 0$, it follows by the self-similarity of w_t that for any Borel nonnegative $f(t, x)$ we have

$$\begin{aligned}
Ef(\gamma, w_\gamma) &= n\int_0^1 r^{n-1}\int_S f(r^2 t, rx)\pi(t, x)\,dSdr \\
&\leq N\int_0^1 r^{n-1}\int_S f(r^2 t, rx)\,dSdr \\
&= N\int_0^1 r^{n-1}\int_0^1\int_{\partial B_1} f(r^2 t, rx)\,d(\partial B_1)dtdr \\
&\quad +N\int_0^1 r^{n-1}\int_{B_1} f(r^2, rx)\,dxdr =: NI_1 + NI_2.
\end{aligned}$$

Here

$$\begin{aligned}
I_1 &= \int_0^1 r^{n-3}\int_0^{r^2}\int_{\partial B_1} f(t, rx)\,d(\partial B_1)dtdr \\
&\leq \int_0^1 r^{d-1}\int_0^1\int_{\partial B_1} f(t, rx)\,d(\partial B_1)dtdr = \int_{Q_{1,1}} f(t, x)\,dxdt, \\
I_2 &= \int_0^1 r^{n-1-d}\int_{B_r} f(r^2, x)\,dxdr \leq (1/2)\int_{Q_{1,1}} f(t, x)\,dxdt.
\end{aligned}$$

Hence

$$Ef(\gamma, w_\gamma) \leq N\int_{Q_{1,1}} f(t, x)\,dxdt,$$

where N is independent of f and the lemma is proved. □

In the following lemma we use the notation

$$\mathbb{R}^{d+1}_+ = \{(t, x) : t \in (0, \infty), x = (x^1, ..., x^d) \in \mathbb{R}^d\}.$$

Lemma 7.5. *Let $K \in [0, \infty)$ be a constant. Assume that for some p, q satisfying (2.1) we have*

$$\phi I_{Q_{1,1}} \le K, \quad \|\phi_x I_{Q_{1,1}}\|_{L_q\text{-}L_p} \le K.$$

Take an $r \in (1, \infty)$ and a Borel nonnegative function $f = f(t, x)$ on $\mathbb{R}^{d+1}_+$, such that $f(t, x) = 0$ for $t > T$. For $0 \le s \le t \le T$ and $x \in \mathbb{R}^d$ introduce

$$\begin{aligned}\rho_t(s, x) &= \exp(-\int_0^{t-s} \phi_x(s + r, x + w_r)\, dw_r \\ &\qquad -(1/2)\int_0^{t-s} |\phi_x(s + r, x + w_r)|^2\, dr), \\ \alpha_t(s, x) &= \exp(-2\int_0^{t-s} (D_t\phi)_+(s + r, x + w_r)\, dr), \\ u_t(s, x) &= E\rho_t(s, x)\alpha_t(s, x) f(t, x + w_{t-s}).\end{aligned}$$

Then there is a constant N, depending only on r, p, q, K, and T, such that

$$\int_0^T u_t(0, 0)\, dt \le N\big(\int_{\mathbb{R}^{d+1}_+} f^r e^{-2\phi}\, dtdx\big)^{1/r} + N\big(\int_{Q_{1,1}} f^{d+3}\, dtdx\big)^{1/(d+3)}. \tag{7.3}$$

Proof. By the strong Markov property of the Wiener process for any stopping time τ we have

$$E I_{\tau\le t}\rho_t(0, 0)\alpha_t(0, 0) f(0, w_t) = E I_{\tau\le t}\rho_\tau(0, 0)\alpha_\tau(0, 0)u_t(\tau, w_\tau).$$

Therefore, upon assuming without losing generality that $T \ge 1$, for γ from Lemma 7.4

$$\begin{aligned}\int_0^T u_t(0, 0)\, dt &= E\int_0^\gamma \rho_t(0, 0)\alpha_t(0, 0) f(t, w_t)\, dt \\ &\qquad + E\rho_\gamma(0, 0)\alpha_\gamma(0, 0)\int_\gamma^T u_t(\gamma, w_\gamma)\, dt =: I_1 + I_2.\end{aligned}$$

Observe that $\alpha_t \le 1$ and for $t \le \gamma$ we have $(t, w_t) \in Q_{1,1}$ so that, in particular, in the formula defining $\rho_t(0, 0)$ we can replace ϕ_x with $\phi_x I_{Q_{1,1}}$ and hence all moments of $\rho_t(0, 0)I_{t\le\gamma}$ and $\rho_\gamma(0, 0)$ are finite and uniformly bounded in t. It follows by (3.3) that for any $\nu \in (1, \infty)$

$$I_1 \le N\big(E\int_0^T f^\nu(t, w_t)I_{Q_{1,1}}(t, w_t)\, dt\big)^\nu \le N\|f^\nu I_{Q_{1,1}}\|^{1/\nu}_{L_{d+5/2}(\mathbb{R}^{d+1})}.$$

We choose ν so that $\nu(d + 5/2) = d + 3$ and get that I_1 is less than the second term on the right in (7.3).

In what concerns I_2 we again use $\alpha_\gamma(0, 0) \le 1$ and the finiteness of all moments of $\rho_\gamma(0, 0)$. Then we find

$$I_2 \le N\big(\int_0^1\int_s^T \big(\int_{B_1} u^r_t(s, x)\, dx\big)\, dtds\big)^{1/r}.$$

To estimate the interior integral with respect to x we insert there $\exp(-2\phi(s,x))$ and again use Hölder's inequality and the fact that $E\rho_t(s,x) \le 1$. This yields

$$I_2(s,t) := \int_{B_1} u_t^r(s,x)\,dx \le e^{2K} \int_{\mathbb{R}^d} e^{-2\phi(s,x)} \hat{v}_t(s,x)\,dx,$$

where

$$\hat{v}_t(s,x) = E\rho_t(s,x)\alpha_t(s,x) f^r(t, x + w_{t-s}) \le E\beta_t(s,x) f^r(t, x + w_{t-s}).$$

Hence by Corollary 7.2

$$I_2(s,t) \le e^{2K} \int_{\mathbb{R}^d} e^{-2\phi(t,x)} f^r(t,x)\,dx,$$

which shows that I_2 is less than the first term on the right in (7.3). The lemma is proved. □

In the following lemma by $Q_{\pm T}$ we mean $(-T,T)\times\mathbb{R}^d$ and use the notation

$$\tau_Q(x.) = \inf\{t \ge 0 : (t, x_t) \notin Q\}, \quad x. \in C \tag{7.4}$$

instead of $\tau_Q(z.)$, where $z_t = (t, x_t)$.

Lemma 7.6. *Let Q be a domain such that $Q \subset Q_{\pm T}$. Let the assumptions of Lemma 7.5 be satisfied and let $\varepsilon \in [0,2)$ be a constant and h a nonnegative Borel function on Q such that on Q*

$$2D_t\phi + \Delta\phi \le h e^{\varepsilon\phi}. \tag{7.5}$$

Then for any $\delta \in [0, 2-\varepsilon)$, $r \in (1, 2/(\delta+\varepsilon)]$, there exists a constant N, depending only on $T, p, q, K, \varepsilon, \delta$, and r (but not Q), such that, for any stopping time $\tau \le \tau_Q(w.)$ we have

$$E\Phi_\tau \le N + N\Big(\int_Q h^r e^{-(2-r\theta)\phi}\,dtdx\Big)^{1/r} + N \sup_{Q_{1,1}} h, \tag{7.6}$$

where $\theta = \delta + \varepsilon$, so that $r\theta \le 2$, and

$$\begin{aligned}\Phi_t := \exp(-\int_0^t \phi_x(s,w_s)\,dw_s - (1/2)\int_0^t |\phi_x(s,w_s)|^2\,ds \\ -2\int_0^t (D_t\phi)_+(s,w_s)\,ds + \delta\phi(t,w_t)).\end{aligned}$$

Proof. By Itô's formula

$$\begin{aligned}\Phi_\tau = \Phi_0 + m_\tau + \int_0^\tau \Phi_t[\delta D_t\phi \\ +(\delta/2)\Delta\phi - 2(D_t\phi)_+ + (1/2)(|\delta-1|^2 - 1)|\phi_x|^2](t,w_t)\,dt,\end{aligned}$$

where m_t is a local martingale starting at zero. By using (7.5) and the inequality $|\delta-1|\leq 1$ we obtain

$$\Phi_\tau \leq \Phi_0+\delta\int_0^\tau \Phi_t h(t,w_t)\exp(\varepsilon\phi(t,w_t))\,dt+m_\tau. \tag{7.7}$$

Since $\Phi_t\geq 0$ we take the expectations of both sides and drop Em_τ. More precisely, we introduce $\sigma_n:=\inf\{t\geq 0:|m_t|\geq n\}$ and substitute $\tau\wedge\sigma_n$ in place of τ in (7.7). After that we take expectations, use the fact that $Em_{t\wedge\sigma_n}=0$, let $n\to\infty$, and finally use Fatou's lemma along with the monotone convergence theorem. Furthermore, we denote $f=I_Q h\exp(\theta\phi)$ and notice that $\tau\leq T$. Then in the notation of Lemma 7.5 we find that

$$\begin{aligned}E\Phi_\tau &\leq N+NE\int_0^\tau \rho_t(0,0)\alpha_t(0,0)f(t,w_t)\,dt\\ &\leq N+N\int_0^T E\rho_t(0,0)\alpha_t(0,0)f(t,w_t)\,dt=N+N\int_0^T u_t(0,0)\,dt.\end{aligned}$$

It only remains to note that the first term in the right-hand side of (7.3) is just the second one on the right in (7.6) and the second integral on the right in (7.3) is less than $\operatorname{vol}Q_{1,1}\sup_{Q_{1,1}}h\exp[\theta K(d+3)]$. The lemma is proved. □

Theorem 7.7. *Let $K,K_0\in[0,\infty)$ and $\varepsilon\in[0,2)$ be some constants and let Q be a subdomain of $Q_{\pm T}$ and h be a nonnegative Borel function on Q. Assume that for some p,q satisfying* (2.1) *we have*

$$hI_{Q_{1,1}}\leq K,\quad \phi I_{Q_{1,1}}\leq K,\quad \|\phi_x I_{Q_{1,1}}\|_{L_q\text{-}L_p}\leq K.$$

Also assume that on Q

$$\phi\geq 0,\quad 2D_t\phi\leq K_0\phi,\quad 2D_t\phi+\Delta\phi\leq he^{\varepsilon\phi}.$$

Finally, let ϕ_x satisfy the linear growth condition:

$$\sup_{t\in[0,T],x\in\mathbb{R}^d}|\phi_x|/(1+|x|)<\infty.$$

and denote by x_t, $t\in[0,T]$, the solution of

$$x_t=w_t-\int_0^t\phi_x(s,x_s)\,ds.$$

Then for any $r\in(1,4/(2+\varepsilon)]$ there exists a constant N, depending only on r,d,T,p,q,K, and ε, such that

$$E\sup_{t\leq\tau_Q(x.)}\exp[\mu(\phi(t,x_t)+\nu|x_t|^2)]\leq N+N\mathcal{H}_Q(T,\sigma,r), \tag{7.8}$$

where $\tau_Q(x.)$ is introduced in (7.4)*, $\mathcal{H}_Q$ is introduced before Theorem 2.7, $\sigma=(2-r\theta)\nu$, $\theta=2\delta+\varepsilon$, μ, ν, and δ are taken from* (2.5).

Proof. Define $\bar\phi = \phi + \nu|x|^2$,

$$M_t = \exp(\delta\bar\phi(t, x_t) - (K_0/2)\int_0^t \bar\phi(s, x_s)\,ds), \quad M_* = \sup_{t\le\tau_Q(x.)} M_t.$$

Then for $t \le \tau_Q(x.)$

$$\bar\phi(t, x_t) \le \ln M_*^{1/\delta} + (K_0/(2\delta))\int_0^t \bar\phi(s, x_s)\,ds$$

and hence by Gronwall's inequality

$$\bar\phi(t, x_t) \le e^{tK_0/(2\delta)} \ln M_*^{1/\delta} \le e^{TK_0/(2\delta)} \ln M_*^{1/\delta}.$$

Therefore, to prove (7.8), it suffices to prove that $E\sqrt{M_*} \le N$. In turn by a well-known result on transformations of stochastic inequalities (see, for instance, Lemma 3.2 in [12]), if $EM_\tau \le N_1$ for all stopping times $\tau \le \tau_Q(x.)$, then $E\sqrt{M_*} \le 3N_1$. Thus, it suffices to estimate EM_τ.

On a probability space carrying a d-dimensional Wiener process $\bar w_t$ introduce $\bar x_t$ as the solution of the equation

$$\bar x_t = \bar w_t - \int_0^t I_{s<\tau_Q(\bar x.)}\bar\phi_x(s, \bar x_s)\,ds. \tag{7.9}$$

Also set

$$\bar M_t = \exp(2\delta\bar\phi(t, \bar x_t) - 2\int_0^t (D_t\bar\phi)_+(s, \bar x_s)\,ds).$$

Write $\bar E$ for the expectation sign on the new probability space and observe that on Q

$$2D_t\bar\phi + \Delta\bar\phi = 2D_t\phi + \Delta\phi + 2\nu d \le (h + 2\nu d)e^{\varepsilon\bar\phi}.$$

Then after an obvious change of measure (cf. (3.10)) equation (7.6) with 2δ, $\bar E$, $\bar\phi$, and $\bar w_t$ in place of δ, E, ϕ, and w_t, respectively, $\theta = 2\delta + \varepsilon$, and $r \in (1, 2/(2\delta+\varepsilon)]$ is rewritten as

$$\bar E\bar M_\tau \le N + N\Big(\int_Q h^r I_{(0,T)} e^{-(2-r\theta)\bar\phi}\,dtdx\Big)^{1/r}$$

and since $\bar\phi \ge \nu|x|^2$ on Q, we obtain

$$\bar E\bar M_\tau \le N + N\mathcal{H}_Q^{1/r}(T, (2-r\theta)\nu, r) =: N_0$$

for all stopping times $\tau \le \tau_Q(\bar x.)$. Combining this with the inequality

$$\exp(2\delta\bar\phi(t, \bar x_t) - K_0\int_0^t \bar\phi(s, \bar x_s)\,ds) \le \bar M_t, \quad t \le \tau_Q(\bar x.),$$

the left-hand side of which is quite similar to M_t but with $2\bar\phi$ in place of $\bar\phi$, by the above argument we get

$$\bar E \sup_{t\le\tau_Q(\bar x.)} \exp(2\mu\nu|\bar x_t|^2) \le \bar E \sup_{t\le\tau_Q(\bar x.)} \exp(2\mu\bar\phi(t, \bar x_t)) \le NN_0. \tag{7.10}$$

We now estimate EM_τ through $\bar{E}\bar{M}_\tau$ by using Girsanov's theorem and Hölder's inequality. We use a certain freedom in choosing $\bar{x}_t$ and $\bar{w}_t$ and on the probability space where $w.$ and $x.$ are given we introduce a new measure by the formula

$$\bar{P}(d\omega) = \exp(-2\nu \int_0^\infty x_t I_{t<\tau_Q(x.)}\, dw_t - 2\nu^2 \int_0^\infty |x_t|^2 I_{t<\tau_Q(x.)}\, dt)\; P(d\omega).$$

The linear growth condition guarantees that $\bar{P}$ is a probability measure. Furthermore, as is easy to see

$$\bar{x}_t := x_t I_{t<\tau_Q(x.)} + (w_t - w_{\tau_Q(x.)} + x_{\tau_Q(x.)}) I_{t\ge\tau_Q(x.)}$$

coincides with x_t for $t \le \tau_Q(x.)$ and satisfies (7.9) with

$$\bar{w}_t = w_t + 2\nu \int_0^t x_s I_{s<\tau_Q(x.)}\, ds,$$

which is a Wiener process with respect to $\bar{P}$. In this notation for $\tau \le \tau_Q(x.) = \tau_Q(\bar{x}.)$

$$\begin{aligned} EM_\tau &\le \bar{E}\bar{M}_\tau^{1/2} \exp(2\nu \int_0^\infty \bar{x}_t I_{t<\tau_Q(\bar{x}.)}\, d\bar{w}_t - 2\nu^2 \int_0^\infty |\bar{x}_t|^2 I_{t<\tau_Q(\bar{x}.)}\, dt), \\ &\le (\bar{E}\bar{M}_\tau)^{1/2} \big(\bar{E}\rho^{1/2} \exp\big(12\nu^2 \int_0^\infty |\bar{x}_t|^2 I_{t<\tau_Q(\bar{x}.)}\, dt\big)\big)^{1/2}, \end{aligned}$$

where

$$\rho = \exp(8\nu \int_0^\infty \bar{x}_t I_{t<\tau_Q(\bar{x}.)}\, d\bar{w}_t - 32\nu^2 \int_0^\infty |\bar{x}_t|^2 I_{t<\tau_Q(\bar{x}.)}\, dt).$$

Observe that $\bar{E}\rho = 1$ and $\bar{E}\bar{M}_\tau \le N_0$. Therefore,

$$EM_\tau \le N_0^{1/2} \big(\bar{E} \exp\big(24\nu^2 \int_0^{\tau_Q(\bar{x}.)} |\bar{x}_t|^2\, dt\big)\big)^{1/4}.$$

It only remains to refer to (7.10) after noticing that

$$24\nu^2 \int_0^{\tau_Q(\bar{x}.)} |\bar{x}_t|^2\, dt \le 24\nu^2 T \sup_{t\le\tau_Q(\bar{x}.)} |\bar{x}_t|^2 = 2\mu\nu \sup_{t\le\tau_Q(\bar{x}.)} |\bar{x}_t|^2$$

and use the inequality $a^\alpha \le 1 + a$ if $a \ge 0$, $0 \le \alpha \le 1$. The theorem is proved. □

8. Proof of Theorem 2.7 and concluding Remarks

By Theorem 2.1 the solution x_t is defined at least until the time ζ when $(s+t, x_t)$ exits from all Q^n. We claim that to prove that $\zeta = \infty$ (a.s.) and also to prove the second assertion of the theorem, it suffices to prove that for each $T \in (0, \infty)$ and $m \geq 1$ there exists a constant N, depending only on d, $p(m+1)$, $q(m+1)$, ε, T, K, $\|\psi_x I_{Q^{m+1}}\|_{L_{q(m+1)}\text{-}L_{p(m+1)}}$, dist $(\partial Q^m, \partial Q^{m+1})$, $\sup\{\psi + h, Q^{m+1}\}$, and the function $\mathcal{H}$, such that for $(s, x) \in Q^m$ we have

$$E \sup_{t<\zeta\wedge T} \exp(\mu\psi(s+t, x_t) + \mu\nu|x_t|^2) \leq N. \tag{8.1}$$

To prove the claim notice that (8.1) implies that

$$\sup_{t<\zeta\wedge T} (\psi(s+t, x_t) + |x_t|^2) < \infty \quad \text{(a.s.)}. \tag{8.2}$$

It follows that (a.s.) there exists an $n \geq 1$ such that up to time $\zeta \wedge T$ the trajectory $z_t = (s+t, x_t)$ lies in Q^n. Indeed, on the set of all ω where this is wrong, for the first exit time τ^n of z_t from Q^n we have $\tau^n < T$ for all n. However owing to (8.2), the sequence x_{τ^n} is bounded and therefore the sequence z_{τ^n} has limit points on ∂Q. By the assumptions before Theorem 2.7

$$\overline{\lim_{n\to\infty}}\ \psi(z_{\tau^n}) = \infty,$$

which only happens with probability zero again due to (8.2). Hence, (a.s.) there is an $n \geq 1$ such that $T \leq \tau^n$. Since this happens for any T and $\tau^n < \zeta$ we conclude $\zeta = \infty$ (a.s.), which proves our intermediate claim.

Since $\text{dist}(\partial Q^m, \partial Q^{m+1}) > 0$ we can find $\kappa \in (0, 1]$ sufficiently small so that $(s, x) + Q_{\kappa^2,\kappa} \subset Q^{m+1}$ for all $(s, x) \in Q^m$. Therefore, by translation and dilation, without losing generality, we may assume that $s = 0$, $x = 0$, and that $Q_{1,1} \subset Q^{m+1}$.

Next we notice that obviously, to prove (8.1) it suffices to prove that with N of the same kind as in (8.1) for any $n \geq m+2$

$$E \sup_{t<\tau^n\wedge T} \exp(\mu\psi(t, x_t) + \mu\nu|x_t|^2) \leq N. \tag{8.3}$$

Fix an $n \geq m+2$. By virtue of Theorem 5.1 the left-hand side of (8.3) will not change if we change ψ_x outside of Q^n. Therefore we may replace ψ with $\psi\eta$, where η is an infinitely differentiable function equal 1 on a neighborhood of $\bar{Q}^n$ and 0 outside of Q^{n+1}. To simplify the notation we just assume that ψ itself vanishes outside of Q^{n+1} and (2.3) holds in a neighborhood of $\bar{Q}^n$. This is harmless as long as we prove that N depends appropriately on the data.

Now we mollify ψ by convolving it with a δ-like smooth functions $\xi^\gamma(t, x) = \gamma^{-d-1}\xi(t/\gamma, x/\gamma)$ with compact support. Denote by $\psi^{(\gamma)}$ the result of the convolution and use an analogous notation for the convolution of $\xi^\gamma(t, x)$ with other functions. Also denote by x_t^γ the solution of (2.4) with $s = 0$, $x = 0$, and $\psi^{(\gamma)}$ in place of ψ. Consider the bounded function f on C given by the formula

$$f(y.) = \sup_{t<\tau^n(y.)\wedge T} \exp(\mu\psi(t, y_t) + \mu\nu|y_t|^2)$$

with an obvious meaning of $\tau^n(y.)$ and let f^γ be defined by the same formula with $\psi^{(\gamma)}$ in place of ψ. By using Lemma 3.6 we conclude that the left-hand side of (8.3) is equal to the limit as $\gamma \downarrow 0$ of

$$E \sup_{t<\tau^n(x^\gamma_\cdot)\wedge T} \exp(\mu\psi^{(\gamma)}(t, x_t^\gamma) + \mu\nu|x_t^\gamma|^2).$$

In the light of the fact that (2.3) holds in a neighborhood of $\bar{Q}^n$ we have that on Q^n for sufficiently small γ

$$2D_t\psi^{(\gamma)} + \Delta\psi^{(\gamma)} = (2D_t\psi + \Delta\psi)^{(\gamma)} \le (he^{\varepsilon\psi})^{(\gamma)} = h^\gamma e^{\varepsilon\psi^{(\gamma)}},$$

where $h^\gamma := (he^{\varepsilon\psi})^{(\gamma)}e^{-\varepsilon\psi^{(\gamma)}} \to h$ uniformly on Q^n since h is continuous. The functions $\mathcal{H}_{Q^n}$ corresponding to h^γ converge to the original $\mathcal{H}_{Q^n}$ as $\gamma \downarrow 0$ since Q^n is a bounded subset of Q. Furthermore, the condition $2D_t\psi^{(\gamma)} \le K_0\psi^{(\gamma)}$ also holds in a neighborhood of $\bar{Q}^n$ for sufficiently small γ.

We now apply Theorem 7.7 for $Q^n \cap Q_{\pm T}$ in place of Q to conclude that

$$\begin{aligned}
&E \sup_{t<\tau^n\wedge T} \exp(\mu\psi(t, x_t) + \mu\nu|x_t|^2) \\
&\quad = \lim_{\gamma\downarrow 0} E \sup_{t<\tau^n(x^\gamma_\cdot)\wedge T} \exp(\mu\psi^{(\gamma)}(t, x_t^\gamma) + \mu\nu|x_t^\gamma|^2) \\
&\quad \le N + N\mathcal{H}_{Q^n}(T, (2 - r\theta)\nu, r) \le N + N\mathcal{H}_Q(T, (2 - r\theta)\nu, r),
\end{aligned}$$

where the values of all the parameters are specified in Theorem 7.7 and the constants N depend only on $r, d, p(m+1), q(m+1), \varepsilon, T, K, \|\psi_x I_{Q^{m+1}}\|_{L_{q(m+1)}\text{-}L_{p(m+1)}}$, and $\sup\{\psi + h, Q^{m+1}\}$.

We finally use condition (H) from Assumption 2.1. Fix any $r_0 \in (1, 2/(2\delta+\varepsilon))$, set $\sigma = (2 - r_0\theta)\nu$ (> 0) and take $r = r(T, \sigma)$ from condition (H). Hölder's inequality shows that if condition (H) is satisfied with $r = r'$ where $r' > 1$, then it is also satisfied with any $r \in (1, r']$. Hence without losing generality we may assume that $r = r(T, \sigma) \in (1, r_0]$. Then $(2-r\theta)\nu \ge \sigma$ and $\mathcal{H}_Q(T, (2-r\theta)\nu, r) \le \mathcal{H}_Q(T, \sigma, r(T, \sigma)) < \infty$. Thus, Theorem 7.7 yields (8.3). The theorem is proved.

Remark 8.1. In Theorem 2.7 additionally assume that $(0, 0) \in Q$ and take the solution of (2.4) corresponding to $s = 0$ and $x = 0$. Introduce $\tau = \inf\{t \ge 0 : (t, w_t) \notin Q\}$, $\tau^n(y.) = \inf\{t \ge 0 : (t, y_t) \notin Q^n\}$,

$$\rho_t(n) = \exp(-\int_0^{t\wedge\tau^n(w.)} \psi_x(s, w_s)\, dw_s - (1/2)\int_0^{t\wedge\tau^n(w.)} |\psi_x(s, w_s)|^2\, ds),$$

$$\rho_t = \varliminf_{n\to\infty} \rho_t(n).$$

In the definition of $\rho_t(n)$ we can replace ψ_x with $I_{Q^n}\psi_x$ and therefore $\rho_t(n)$ are martingales. Furthermore, for any m, on the set $\{t < \tau^m(w.)\}$ we obviously have $\rho_t(n) = \rho_t$ for any $n \geq m$. Hence, by Girsanov's theorem for any $T \in (0, \infty)$ and $\mathcal{N}_T$-measurable nonnegative $f = f(y.)$

$$Ef(x.) = \lim_{n\to\infty} Ef(x.)I_{\tau^n(x.)>T} = \lim_{n\to\infty} Ef(w.)\rho_T(n)I_{\tau^n(w.)>T}$$
$$= \lim_{n\to\infty} Ef(w.)\rho_T I_{\tau^n(w.)>T} = Ef(w.)\rho_T I_{\tau>T}.$$

In particular, $E\rho_T I_{\tau>T} = 1$ and since, by Fatou's lemma $E\rho_{T\wedge\tau} \leq 1$, we get that

$$1 \geq E\rho_{T\wedge\tau} = E\rho_\tau I_{\tau\leq T} + E\rho_T I_{\tau>T}, \quad E\rho_\tau I_{\tau\leq T} = 0, \quad E\rho_\tau I_{\tau<\infty} = 0,$$

which is only possible if

$$\int_0^\tau |\psi_x(s, w_s)|^2\, ds = \lim_{n\to\infty} \int_0^{\tau^n(w.)} |\psi_x(s, w_s)|^2\, ds = \infty$$

(a.s.) on the set where $\{\tau < \infty\}$.

Another consequence of this argument is that

$$Ef(x.) = Ef(w.)\rho_{T\wedge\tau},$$

which in turn for f being $\mathcal{F}_S$-measurable with $S \leq T$ implies that $\rho_{t\wedge\tau}$ is a martingale.

Remark 8.2. Under Assumption 2.1 take a Borel locally bounded $\mathbb{R}^d$-valued function $b(t, x)$ defined on $\mathbb{R}^{d+1}$ satisfying the condition $|b(t, x)| \leq K(1 + |x|)$, where K is a finite constant. Then it turns out that the first assertion of Theorem 2.7 still holds with the equation

$$x_t = x + w_t - \int_0^t \psi_x(s + r, x_r)\, dw_r + \int_0^t b(s + r, x_r)\, dr \tag{8.4}$$

in place of (2.4).

To prove this we take the process z_t from Theorem 2.1 corresponding to the drift term $b - \psi_x$ and prove that $\zeta = \infty$ (a.s.). Denote by $\bar{x}_t$ the process from Theorem 2.7 solving equation (2.4) and for $T \in [0, \infty)$ introduce

$$\rho_T(s, x) = \exp\Big(\int_0^T b(s + r, \bar{x}_r)\, dw_r - (1/2)\int_0^T |b(s + r, \bar{x}_r)|^2\, dr\Big).$$

First we claim that

$$E\rho_T(s, x) = 1, \quad \forall (s, x) \in Q, T \in [0, \infty). \tag{8.5}$$

To prove (8.5) fix $(s, x) \in Q$ and observe that by the Markov property of $\bar{x}_t$ for constant $\alpha \in [0, \infty)$ we have

$$E\rho_{T+\alpha}(s, x) = E\rho_T\big[E_{t,y}\rho_\alpha(t, y)\big]\big|_{t=s+T, y=\bar{x}_{s+T}},$$

where we use the notation from Theorem 2.5. Furthermore, for $(t, y) \in Q$

$$E_{t,y} \exp \int_0^\alpha |b(t+r, \bar{x}_r)|^2 \, dr \leq E_{t,y} \exp K\alpha(1 + \sup_{r \leq \alpha} |\bar{x}_r)|^2) \tag{8.6}$$

which is finite due to Theorem 2.7 for $K\alpha \leq \mu^2/(12\alpha)$, where μ is taken from (2.5) with α in place of T. By analyzing the condition $K\alpha \leq \mu^2/(12\alpha)$ we easily conclude that there exists an $\alpha_0 = \alpha_0(\varepsilon, K_0) > 0$ such that the right-hand side of (8.6) is finite for $\alpha = \alpha_0$. It follows (see, for instance, [22]) that $E_{t,y}\rho_{\alpha_0}(t, y) = 1$ and hence for any integer $n \geq 0$

$$E\rho_{(n+1)\alpha_0}(s, x) = E\rho_{n\alpha_0}(s, x) = 1.$$

Since $\rho_T(s, x)$ is a supermartingale, we have proved (8.5).

Now we fix $T \in (0, \infty)$, introduce a new probability measure by $\bar{P}(d\omega) = \rho_T(s, x)\, P(d\omega)$, and by Girsanov's theorem conclude that $\bar{x}_t$ satisfies (8.4) for $t \in [0, T]$ (a.s.) with a certain process $\bar{w}_t$ in place of w_t and $\bar{w}_t$ is a Wiener process relative to the new probability measure. Hence there exists a probability space on which (8.4) has solutions defined at least up to T. By the weak uniqueness in Theorem 2.1 applied to $Q \cap Q_{\pm(s+T)}$, we have $\zeta > T$ (a.s.) and since T is arbitrary, $\zeta = \infty$ (a.s.) indeed.

One of important features of (8.4) is that the added drift coefficient may or may not be the gradient of a function.

Remark 8.3. By using Remark 8.2 and observing that equation (2.4) is equivalent to the following

$$x_t = x + w_t - \int_0^t [K(1 + |x|^2) + \psi]_x(s + r, x_r)\, dr + \int_0^t 2Kx_r \, dr$$

we conclude that (2.4) has a unique solution defined for all times if $(s, x) \in Q$ provided that $\psi + K(1 + |x|^2)$ rather than ψ satisfies Assumption 2.1. This carries our result about existence and uniqueness of strong solutions over to the cases in which ψ is not necessarily nonnegative but $\psi \geq -K(1 + |x|^2)$.

Remark 8.4. Before $\tau^n(x_\cdot)$ the process $x_\cdot$ satisfies (2.4) with $I_{Q^n}\psi_x$ in place of ψ_x. Hence by Lemma 3.3

$$E \int_0^{\tau^n(x_\cdot)} |f(t, x_t)| \, dt \leq N \|f I_{Q^n}\|_{L_q\text{-}L_p},$$

with N independent of f if $d/p + 2/q < 2$. This estimate and Girsanov's theorem allow adding into equation (2.4) a new drift b which vanishes outside of some Q^n and is such that $\|bI_{Q^n}\|_{L_q\text{-}L_p} < \infty$. Again b need not be a gradient.

Remark 8.5. One can obviously combine the observations from Remarks 8.2 and 8.4.

9. Applications

9.1. Diffusions in random media

In this subsection we would like to apply our results to a particle which performs a random motion in $\mathbb{R}^d$, $d \geq 2$, interacting with impurities which are randomly distributed according to a Gibbs measure of Ruelle type. So, the impurities form a locally finite subset $\gamma = \{x_k \mid k \in \mathbb{N}\} \subset \mathbb{R}^d$. The interaction is given by a pair potential V to be specified below defined on $\{x \in \mathbb{R}^d : |x| > \rho\}$, where $\rho \geq 0$ is a given constant. The stochastic dynamics of the particle is then determined by a stochastic equation of type (2.4) as in Theorem 2.7 above with

$$Q := \mathbb{R} \times (\mathbb{R}^d \setminus \gamma^\rho), \;\; \psi(t, x) := \sum_{y \in \gamma} V(x - y), \;\; (t, x) \in Q, \tag{9.1}$$

where γ^ρ is the closed ρ-neighborhood of the set γ, i.e., the random path x_t of the particle should be the unique strong solution of

$$x_t = x + w_t - \int_0^t \sum_{y \in \gamma} V_x(x_s - y)\, ds, \;\; t \geq 0. \tag{9.2}$$

Below we shall give conditions on the pair potential V which imply that this is indeed the case, i.e. that Theorem 2.7 above applies, for all γ outside a set of measure zero for the Gibbs measure. Let us define the set of admissible impurities γ we can treat, namely (cf. [16])

$$\begin{aligned} \Gamma_{ad} := \{\gamma \subset \mathbb{R}^d \mid \forall r > 0\, \exists c(\gamma, r) > 0 \\ : \; |\gamma \cap B_r(x)| \leq c(\gamma, r) \log(1 + |x|) \text{ for all } x \in \mathbb{R}^d\}, \end{aligned} \tag{9.3}$$

where $B_r(x)$ denotes the open ball with center x and radius r and where $|A|$ denotes the cardinality of a set A. We emphasize that for essentially all classes of Gibbs measures in equilibrium statistical mechanics of interacting infinite particle systems in $\mathbb{R}^d$ the set Γ_{ad} has measure one (cf. [15]). In particular, this is true for Ruelle measures (see [20]). Since this is the only fact we use about such Gibbs measures, we do not recall the precise and quite involved definition here, but refer e.g. to [3].

In what follows we fix a $\gamma \in \Gamma_{ad}$. Here are the conditions we need on the pair potential V. Notice that the typical case when $\rho = 0$ is not excluded.
(V1) The function V is once continuously differentiable in $\mathbb{R}^d \cap \{|x| > \rho\}$) and $\lim_{|x| \downarrow \rho} V(x) = \infty$.
(V2) There exist finite constants $\alpha > d/2, K \geq 0, \varepsilon \in [1, 2)$ such that with $U(x) := K(1 + |x|^2)^{-\alpha}$ we have

$$|V(x)| + |V_x(x)| \leq U(x) \quad \text{for} \quad |x| \geq \rho + 1, \tag{9.4}$$

$$\Delta V(x) \leq K(e^{\varepsilon(V(x)+U(x))} - 1) \quad \text{for} \quad |x| > \rho \tag{9.5}$$

in the sense of distributions on $\{x \in \mathbb{R}^d : |x| > \rho\}$.

We emphasize that the above conditions are fulfilled for essentially all potentials of interest in statistical physics.

Introduce $\bar{V}(x) = V(x) + 2U(x)$, $|x| > \rho$, and let

$$\bar{\psi}(t, x) := \sum_{y \in \gamma} \bar{V}(x - y), \quad b(t, x) := 2 \sum_{y \in \gamma} U_x(x - y), \quad (t, x) \in Q.$$

It is easy to see (cf. [16, Lemma 3.1]) that owing to (9.4) and the fact that $\gamma \in \Gamma_{ad}$ with Γ_{ad} described in (9.3), the function ψ is continuously differentiable in Q and $|b(t, x)| \le N \log(2 + |x|)$, where N is independent of (t, x). Also notice that for appropriate constants N we have $2\Delta U \le NU \le N(e^{\varepsilon U} - 1)$. Upon combining this with the inequalities $V + U \ge 0$ and $\sum(\exp a_k - 1) \le \exp \sum a_k - 1$, $a_k \ge 0$, and the fact that one can always differentiate series converging in the sense of distributions, we find

$$\Delta \bar{V} \le K(e^{\varepsilon(V+U)} - 1) + N(e^{\varepsilon U} - 1) \le N(e^{\varepsilon \bar{V}} - 1),$$

$$\Delta \bar{\psi}(t, x) = \sum_{y \in \gamma} \Delta \bar{V}(x - y) \le N \sum_{y \in \gamma} (e^{\varepsilon \bar{V}(x-y)} - 1)$$

$$\le N \exp\Big(\sum_{y \in \gamma} \varepsilon \bar{V}(x - y)\Big) = N e^{\varepsilon \bar{\psi}(t,x)}. \tag{9.6}$$

It follows that all conditions on $\bar{\psi}$ in Theorem 2.7 are fulfilled and therefore by Remark 8.2 the equation

$$x_t = x + w_t - \int_0^t \bar{\psi}_x(r, x_r)\, dw_r + \int_0^t b(r, x_r)\, dr \tag{9.7}$$

has a unique strong solution defined for all times if $x \in \mathbb{R}^d \setminus \gamma^\rho$. Obviously, equation (9.7) coincides with (9.2).

Remark 9.1. For $\rho = 0$ in [4] (which is based in part on the analytic results in [16]) the existence of merely a weak solution to (9.2) was proved. The assumptions in [4] and [16] are different, in [4] they are much stronger than ours and in [16] they are basically weaker (yet see Remark 2.9).

9.2. M- particle systems with gradient dynamics

In this subsection we consider a model of M particles in $\mathbb{R}^d$ interacting via a pair potential V, similar to the one from the previous subsection but satisfying the following weaker conditions:

(V3) The function V is once continuously differentiable in $\mathbb{R}^d \setminus \{0\}$, $\lim_{|x| \downarrow 0} V(x) = \infty$, $V \ge -U$, where $U(x) := K(1 + |x|^2)$ and K is a constant.

(V4) There exists a constant $\varepsilon \in [1, 2)$ such that

$$\Delta V(x) \le K e^{\varepsilon(V(x)+U(x))} \quad \text{for} \quad x \ne 0$$

in the sense of distributions on $\mathbb{R}^d \setminus \{0\}$.

Introduce $\bar{V} := V + 2U$,

$$Q := \mathbb{R} \times \left(\mathbb{R}^{Md} \setminus \bigcup_{1 \le k < j \le M} \{x = (x^{(1)}, \dots, x^{(M)}) \in \mathbb{R}^{Md} \,:\, x^{(k)} = x^{(j)}\} \right)$$

and let the functions ψ, $\bar{\psi}$, and b be defined on Q by

$$\psi(t,x) := \sum_{1 \le k < j \le M} V(x^{(k)} - x^{(j)}), \quad \bar{\psi}(t,x) := \sum_{1 \le k < j \le M} \bar{V}(x^{(k)} - x^{(j)}),$$

$$b = (b^{(1)}, ..., b^{(M)}), \quad b^{(k)}(t,x) = 4K \sum_{j=1, j \ne k}^{M} (x^{(k)} - x^{(j)}).$$

Obviously, $2\Delta U \le N e^{\varepsilon U}$ for an appropriate constant N. Also obviously ψ and $\bar{\psi}$ are continuously differentiable in Q. Furthermore, in the sense of distributions on Q (cf. (9.6))

$$\begin{aligned} \Delta \bar{V} &\le K + K(e^{\varepsilon(V+U)} - 1) + N + N(e^{\varepsilon U} - 1) \le N + N(e^{\varepsilon \bar{V}} - 1), \\ \Delta \bar{\psi}(t,x) &= 2 \sum_{1 \le k < j \le M} \Delta \bar{V}(x_k - x_j) \le N + N \sum_{1 \le k < j \le M} (e^{\varepsilon \bar{V}(x_k - x_j)} - 1) \\ &\le N + N(e^{\varepsilon \bar{\psi}(t,x)} - 1) \le N e^{\varepsilon \bar{\psi}(t,x)}. \end{aligned}$$

It follows that all conditions on $\bar{\psi}$ in Theorem 2.7 are fulfilled and therefore by Remark 8.2 the corresponding stochastic equation for a process $x_t = (x_t^{(1)}, ..., x_t^{(M)})$ has a unique strong solution defined for all times whenever for the initial condition x we have $(0, x) \in Q$. The equation in question is the following system

$$\begin{aligned} x_t^{(i)} &= x^{(i)} + w_t^{(i)} - \int_0^t \sum_{j=1, j \ne i}^{M} \bar{V}_x\big((x_s^{(i)} - x_s^{(j)}) \operatorname{sign}(j-i)\big) \operatorname{sign}(j-i)\, ds \\ &\quad + \int_0^t b^{(i)}(s, x_s)\, ds, \quad i = 1, ..., M. \end{aligned}$$

Simple arithmetics shows that this system is rewritten as the following one with $i = 1, ..., M$

$$x_t^{(i)} = x^{(i)} + w_t^{(i)} - \int_0^t \sum_{j=1, j \ne i}^{M} V_x\big((x_s^{(i)} - x_s^{(j)}) \operatorname{sign}(j-i)\big) \operatorname{sign}(j-i)\, ds, \tag{9.8}$$

which thus has a unique strong solution defined for all times whenever $(0, x^{(1)}, ..., x^{(M)}) \in Q$. Finally, notice that if V is symmetric: $V(x) = V(-x)$, equation (9.8) becomes

$$x_t^{(i)} = x^{(i)} + w_t^{(i)} - \int_0^t \sum_{j=1, j \ne i}^{M} V_x(x_s^{(i)} - x_s^{(j)})\, ds.$$

Remark 9.2. (i) The results of this subsection substantially improve those in [4] (see also [16]) where under stronger assumptions on V merely the existence of weak solutions to (9.8) was proved. The latter results themselves generalized earlier results in [24], [25]. We also improve the recent result on strong solutions in [26, Lemma 1], where as in [24], [25] much stronger conditions than ours were imposed on the behavior of V at zero (cf. [26, Condition (P)]).
(ii) By the same arguments as in the previous subsection one can investigate the hard core case where the particles are balls of a fixed radius rather than points in $\mathbb{R}^d$.

10. Appendix

Take a constant $T \in (0, \infty)$.

Lemma 10.1. *Let $p, q \in (1, \infty)$, $2/q < \beta \le 2$, and $u \in H_p^{2,q}(T)$. Then for any $s, t \in [0, T]$ and $a > 0$*

$$\|u(t) - u(s)\|_{H_p^{2-\beta}} \le N|t-s|^{\beta/2-1/q} a^{\beta-1} (a\|u\|_{\mathbb{H}_p^{2,q}(T)} + a^{-1}\|D_t u\|_{\mathbb{L}_p^q(T)}), \tag{10.1}$$

where N depends only on p, q, β, which upon minimizing with respect to $a > 0$ yields

$$\|u(t) - u(s)\|_{H_p^{2-\beta}} \le N|t-s|^{\beta/2-1/q} \|u\|_{\mathbb{H}_p^{2,q}(T)}^{1-\beta/2} \|D_t u\|_{\mathbb{L}_p^q(T)}^{\beta/2}. \tag{10.2}$$

This lemma is a particular case of Theorem 7.3 of [19].

Lemma 10.2. *Let $p, q \in (1, \infty)$ and $u \in H_p^{2,q}(T)$. We assert the following.*

(i) If $d/p + 2/q < 2$, then $u(t, x)$ is a bounded Hölder continuous function on $[0, T] \times \mathbb{R}^d$. More precisely, for any $\varepsilon, \delta \in (0, 1]$ satisfying

$$\varepsilon + d/p + 2/q < 2, \quad 2\delta + d/p + 2/q < 2$$

there exists a constant N, depending only on p, q, ε, and δ, such that for all $s, t \in [0, T]$ and $x, y \in \mathbb{R}^d$ satisfying $x \neq y$ we have

$$|u(t,x) - u(s,x)| \le N|t-s|^{\delta} \|u\|_{\mathbb{H}_p^{2,q}(T)}^{1-1/q-\delta} \|D_t u\|_{\mathbb{L}_p^q(T)}^{1/q+\delta}. \tag{10.3}$$

$$|u(t,x)| + \frac{|u(t,x) - u(t,y)|}{|x-y|^{\varepsilon}} \le NT^{-1/q}(\|u\|_{\mathbb{H}_p^{2,q}(T)} + T\|D_t u\|_{\mathbb{L}_p^q(T)}) \tag{10.4}$$

(ii) If $d/p + 2/q < 1$, then $u_x(t, x)$ is Hölder continuous in $[0, T] \times \mathbb{R}^d$, namely for any $\varepsilon \in (0, 1)$ satisfying

$$\varepsilon + d/p + 2/q < 1,$$

there exists a constant N, depending only on p, q and ε, such that for all $s, t \in [0, T]$ and $x, y \in \mathbb{R}^d$ satisfying $x \neq y$, equations (10.3) and (10.4) hold with u_x in place of u and $\varepsilon/2$ in place of δ.

Proof. (i) First take $\beta = 2\delta + 2/q$ and notice that $2/q < \beta \le 2$ and $2-\beta > d/p$. Then we get (10.3) from (10.2) by the Sobolev embedding theorems after simple arithmetical manipulations showing, for instance, that $\beta/2 - 1/q = \delta$.

To prove (10.4) observe that, as is easy to see, this estimate is invariant with respect to dilations of the time axis. Therefore, we may concentrate on the case $T = 1$. Consider (10.1) with $a = 1$ and u replaced by the product of u and an infinitely differentiable function depending only on t and equal to zero either at 0 or at 1. Then from Lemma 10.1 by taking s to be 0 or 1 one obtains that for any $t \in [0, 1]$ and β satisfying $2/q < \beta \le 2$

$$\|u(t)\|_{H_p^{2-\beta}} \le N(\|u\|_{\mathbb{H}_p^{2,q}(1)} + \|D_t u\|_{\mathbb{L}_p^q(1)}) \tag{10.5}$$

Take here $\beta = \varepsilon' + 2/q$, where $0 < \varepsilon' < 2 - (\varepsilon + d/p + 2/q)$. Then $2/q < \beta \le 2$ and (10.4) follows from (10.5) and the Sobolev embedding theorems due to the fact that $2 - \beta - d/p > \varepsilon$.

(ii) Here with $\delta = \varepsilon/2$ and the same β's as above we have $2 - \beta > 1 + d/p$ and $2 - \beta - d/p > 1 + \varepsilon$, respectively, and again everything is a straightforward consequence of the Sobolev embedding theorems. The lemma is proved. □

We are now in the position to prove an existence theorem.

Assumption 10.1. We are given numbers p and q satisfying (2.1) and a Borel $\mathbb{R}^d$-valued function $b = b(t, x)$ defined on $\mathbb{R}^{d+1}$ such that $b \in L_q\text{-}L_p$.

Theorem 10.3. *Let Assumption 10.1 be satisfied. Take $\varepsilon > 0$, $f \in \mathbb{L}_p^q(T)$ and $\phi \in H_p^{2-2/q+\varepsilon}$. Then in $H_p^{2,q}(T)$ there is a unique solution of the equation*

$$D_t u + (1/2)\Delta u + b^i u_{x^i} + f = 0 \tag{10.6}$$

with boundary condition $u(T, x) = \phi(x)$. For this solution

$$\|D_t u\|_{\mathbb{L}_p^q(T)} + \|u\|_{\mathbb{H}_p^{2,q}(T)} \le N(\|f\|_{\mathbb{L}_p^q(T)} + \|\phi\|_{H_p^{2-2/q+\varepsilon}}), \tag{10.7}$$

where $N = N(d, q, p, \varepsilon, T, \|b\|_{L_q\text{-}L_p})$.

Proof. This theorem for $b = 0$ is a particular case of Theorem 1.2 of [18] where $p, q \in (1, \infty)$ is the only restriction on p, q. Actually, in that theorem the left-hand side of (10.7) contains only $\|u_{xx}\|_{\mathbb{L}_p^q(T)}$. However, then one gets an estimate of $\|D_t u\|_{\mathbb{L}_p^q(T)}$ from the equation itself and after that the only missing norm is $\|u\|_{\mathbb{L}_p^q(T)}$, which one estimates by using

$$\|u(t)\|_{L_p} \le \|\phi\|_{L_p} + \int_t^T \|D_s u(s)\|_{L_p}\, ds$$

and $\|\phi\|_{L_p} \le \|\phi\|_{H_p^{2-2/q+\varepsilon}}$. Therefore the method of continuity is applicable and to prove the theorem it suffices to prove (10.7) assuming that the solution already exists.

By Theorem 1.2 of [18] we find that for $S \in [0, T]$

$$
\begin{aligned}
I(S) &:= \|D_t u\|^q_{\mathbb{L}^q_p((S,T))} + \|u\|^q_{\mathbb{H}^q_{2,p}((S,T))} \\
&\le N\big(\|f\|^q_{\mathbb{L}^q_p((S,T))} + \|\phi\|^q_{H_p^{2-2/q+\varepsilon}} + \|b^i u_{x^i}\|^q_{\mathbb{L}^q_p((S,T))}\big).
\end{aligned} \tag{10.8}
$$

By Lemma 10.2 (ii) and by the Sobolev embedding theorems for $t \in (S, T)$

$$
|u_x(t,x)| \le |u_x(t,x) - \phi_x(x)| + |\phi_x| \le N I^{1/q}(t) + N\|\phi\|_{H_p^{2-2/q+\varepsilon}}.
$$

Furthermore,

$$
\|b^i u_{x^i}\|^q_{\mathbb{L}^q_p(S,T)} \le \int_S^T \sup_x |u_x(t,x)|^q \|b(t,\cdot)\|^q_{L_p}\, dt.
$$

It follows that

$$
I(S) \le N\big(\|f\|^q_{\mathbb{L}^q_p(T)} + \|\phi\|^q_{H_p^{2-2/q+\varepsilon}}\big) + N\int_S^T I(t)\|b(t,\cdot)\|^q_{L_p}\, dt.
$$

Finally, by using Gronwall's inequality we estimate $I(0)$ and arrive at (10.7). The theorem is proved. □

Remark 10.4. The term Δu in (10.6) can be replaced with $a^{ij}(t,x)u_{x^i x^j}$, if the matrix a is symmetric, uniformly nondegenerate, bounded and Borel in (t, x), and uniformly continuous in $x \in \mathbb{R}^d$ uniformly with respect to t. This is proved by using standard techniques on the basis of Theorem 1.2 of [18] which, in particular, states that the result is true for $b = 0$ and a independent of x.

Remark 10.5. The reader may have noticed that in the main part of the article, in fact, we only used the solvability of (10.6) on $[T - \varepsilon, T]$ for sufficiently small ε. In connection with this observe that the constant N in (10.7) can be chosen to increase with respect to T and thus be bounded for T small. This follows from the fact that solutions on larger time intervals also solve the equation on smaller ones.

The following result is of main importance in what concerns the needs of the present article.

Lemma 10.6. *Let Assumption 10.1 be satisfied and let B_2 be a ball in $\mathbb{R}^d$ of radius 2. Take functions $\phi^1, ..., \phi^d \in C_0^\infty(\mathbb{R}^d)$ so that $\phi^i = x^i$ in B_2. By using Theorem 10.3 introduce u^i, $i = 1, ..., d$, as the solutions of (10.6) with $f = 0$ and boundary condition $u^i(T, x) = \phi^i(x)$. Then there exists an $\varepsilon \in (0, 1)$ independent of T such that for $t \in [(T - \varepsilon)_+, T]$ and $x, y \in B_2$ we have*

$$
(1/2)|x - y|^2 \le \sum_i [u^i(t,x) - u^i(t,y)]^2 \le 2|x - y|^2.
$$

Proof. It suffices to concentrate on $T \in (0, 1]$. Indeed, for larger T one can just shift the origin of the time axis. Next, observe that

$$u^i(t, x) - u^i(t, y) = A^{ij}(t, x, y)(x^j - y^j),$$

$$\sum_i [u^i(t, x) - u^i(t, y)]^2 = G^{ij}(t, x, y)(x^i - y^i)(x^j - y^j),$$

where

$$A^{ij}(t, x, y) = \int_0^1 u^i_{x^j}(t, rx + (1 - r)y)\, dr, \quad G^{ij} = A^{ki} A^{kj}.$$

By Lemma 10.2 (ii) and Remark 10.5, if ε is small enough, then for all $t \in [(T - \varepsilon)_+, T]$ the matrices $A(t, x, y)$ and $G(t, x, y)$ are close to $A(T, x, y)$ and $G(T, x, y)$, respectively, uniformly in $x, y \in \mathbb{R}^d$. If $x, y \in B_2$, then obviously $A(T, x, y) = G(T, x, y) = I$, where I is the unit $d \times d$ matrix. It follows that the eigenvalues of $G(t, x, y)$ can be made as close to 1 as we wish if $x, y \in B_2$ on the account of choosing sufficiently small ε. This definitely implies the assertion of the lemma, which is thereby proved. □

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