Probab. Theory Relat. Fields 128, 141–160 (2004)
Digital Object Identifier (DOI) 10.1007/s00440-003-0309-8

Enzo Orsingher† · Luisa Beghin

Time-fractional telegraph equations and telegraph processes with brownian time*

Received: 5 January 2002 / Revised version: 27 September 2003 /
Published online: 12 November 2003 –

Abstract. We study the fundamental solutions to time-fractional telegraph equations of order 2α. We are able to obtain the Fourier transform of the solutions for any α and to give a representation of their inverse, in terms of stable densities. For the special case $\alpha = 1/2$, we can show that the fundamental solution is the distribution of a telegraph process with Brownian time. In a special case, this becomes the density of the iterated Brownian motion, which is therefore the fundamental solution to a fractional diffusion equation of order $1/2$ with respect to time.

1. Introduction

Fractional diffusion equations have been considered and solved by several authors such as Wyss (1986), Schneider and Wyss (1989), Fujita (1990, I-II). More recently fractional diffusion equations with random initial conditions have been analyzed by Anh and Leonenko (2000). Angulo et al. (2000) have studied diffusion equations with space-fractional derivatives (in the sense of Riesz inverse operator). Fractional equations of different type, like the Black and Scholes one (see Wyss (2000)) and the space-fractional telegraph equation (see Orsingher and Zhao (2003)) have recently been considered.

The study of fractional diffusion equations has been motivated by the analysis of thermal diffusion in fractal media by Nigmatullin (1986) and Saichev and Zaslavsky (1997). One of the aim of this paper is to show that the law of some processes (the iterated Brownian motion and the telegraph process with Brownian time) are governed by time-fractional telegraph equations.

We examine here the solutions to the time-fractional telegraph equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + 2\lambda \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{for } 0 < \alpha \le 1 \tag{1.1}$$

E. Orsingher, L. Beghin: Università di Roma “La Sapienza”, p.le Aldo Moro 5, 00185, Roma, Italy. e-mail: enzo.orsingher@uniroma1.it

* This research has been partially supported by the NATO grant No. SA (PST.CLG.976361) 5437.

Key words or phrases: Telegraph Equation – Fractional-Derivatives – Stable Laws – Fractional Heat – Wave Equations – Iterated Brownian Motion – Mittag-Leffler Function

subject, for $0 < \alpha \le 1/2$, to the initial condition

$$u(x, 0) = \delta(x) \tag{1.2a}$$

while, for $1/2 < \alpha \le 1$, besides condition (1.2a), also

$$u_t(x, 0) = 0 \tag{1.2b}$$

is assumed.

The fractional derivatives appearing in (1.1) must be understood in the sense of Dzherbashyan-Caputo (see Dzherbashyan and Nersesian (1968)), that is as

$$(D^\alpha f)(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(z)}{(t-z)^{1+\alpha-m}} dz, & \text{for } m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \text{for } \alpha = m \end{cases}, \tag{1.3}$$

where $m - 1 = \lfloor \alpha \rfloor$ ($\lfloor \alpha \rfloor$ denoting the integer part of the real number α) and $f \in C^m$ (for general reference on fractional calculus, see the encyclopedic volume by Samko et al. (1993)).

We consider solutions to equation (1.1) in the class $C^2(\mathbb{R} \times [0, \infty))$ of functions $u = u(x, t)$ such that $\lim_{|x|\to\infty} u(x, t) = \lim_{|x|\to\infty} \frac{\partial}{\partial x} u(x, t) = 0$ (see Fujita (1990, I)).

We are able to obtain the general expression of the Fourier transform of the solutions to problem (1.1)–(1.2) in terms of the Mittag-Leffler functions

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta > 0, \ x \in \mathbb{C} \tag{1.4}$$

that is

$$\begin{aligned} U_\alpha(\beta, t) &= \int_{-\infty}^{+\infty} e^{i\beta x} u_\alpha(x, t) dx \\ &= E_{\alpha,1}(\eta_1 t^\alpha) + \frac{(2\lambda + \eta_2) t^\alpha}{\eta_1 - \eta_2} [\eta_1 E_{\alpha,\alpha+1}(\eta_1 t^\alpha) - \eta_2 E_{\alpha,\alpha+1}(\eta_2 t^\alpha)] \\ &= \frac{1}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\beta^2}} \right) E_{\alpha,1}(\eta_1 t^\alpha) \right. \\ &\quad \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\beta^2}} \right) E_{\alpha,1}(\eta_2 t^\alpha) \right], \end{aligned} \tag{1.5}$$

where

$$\eta_1 = -\lambda + \sqrt{\lambda^2 - c^2\beta^2}, \qquad \eta_2 = -\lambda - \sqrt{\lambda^2 - c^2\beta^2}.$$

We prove that the inverse of (1.5) is a non-negative, symmetric (with respect to x) density function which integrates to one. This means that, for any $0 < \alpha \le 1$, the solutions (which we show to be unique) $u_\alpha = u_\alpha(x, t)$ to the fractional equations (1.1) can be viewed as probability density functions. We denote by $X_\alpha = X_\alpha(t)$,

$t > 0$ the process whose distribution, at time t, coincides with u_α. The solution to problem (1.1) can be expressed as convolution of stable laws for all $0 < \alpha \le 1$ (excluding the case $\alpha = 1/2$).

Because of the complicated structure of (1.5) the inverse Fourier transform cannot be determined for any $0 < \alpha \le 1$. However we are able to obtain the explicit distribution emerging from (1.5) for the case $\alpha = 1/2$.

The case $\alpha = 1$ is related to the well-known telegraph process, which is defined as

$$T(t) = V(0) \int_0^t (-1)^{N(s)} ds$$

where $V(0)$ is a two-valued random variable (with values $\pm c$ taken with probability $1/2$) and $N(t)$ is the number of events in $[0, t]$ of a homogeneous Poisson process, independent of $V(0)$.

The case $\alpha = 1/2$ gives the following fine expression for the distribution

$$\begin{aligned} u_{\frac{1}{2}}(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_0^\infty & e^{-\frac{w^2}{4t} - \lambda w} \{[\lambda I_0(\frac{\lambda}{c}\sqrt{c^2w^2 - x^2}) \\ & + \frac{\partial}{\partial w} I_0(\frac{\lambda}{c}\sqrt{c^2w^2 - x^2})] 1_{\{|x| < cw\}} \\ & + c[\delta(x - cw) + \delta(x + cw)]\} dw \end{aligned} \tag{1.6}$$

where

$$I_0(x) = \sum_{k=0}^\infty \frac{\left(\frac{x}{2}\right)^{2k}}{(k!)^2}$$

is the zero-order modified Bessel function of the first kind (see Tranter (1968) pag.16, formula 1.51).

The law (1.6) coincides with that of the composition of the telegraph process $T = T(t)$, $t > 0$ with a reflecting Brownian motion $|B| = |B(t)|$, $t > 0$ (independent of T). This means that $u_{\frac{1}{2}}(x, t)$ coincides with the distribution of the telegraph process with a Brownian time, that is

$$W(t) = T(|B(t)|), \qquad t > 0. \tag{1.7}$$

The process W can be thought of as the random motion of a particle moving with alternating velocities $\pm c$ (changing at Poisson times) during an interval of length $|B(t)|$. In other words the particle is located at time t in the random space interval $(-c|B(t)|, c|B(t)|)$. This shows that the distribution related to equation

$$\frac{\partial u}{\partial t} + 2\lambda \frac{\partial^{1/2} u}{\partial t^{1/2}} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.8}$$

covers the whole real line and differs substantially from the case of the telegraph process, where the distribution is concentrated on a finite interval (spreading as time passes) because of the finite velocity of motion. We recall that, analogously, the law of the process

$$\overline{W}(t) = \begin{cases} B(T(t)), & \text{when } T(t) > 0 \\ iB(-T(t)), & \text{when } T(t) < 0 \end{cases} . \tag{1.9}$$

satisfies the fourth-order equation

$$\frac{\partial^2 u}{\partial t^2} + 2\lambda \frac{\partial u}{\partial t} = c^2 \frac{\partial^4 u}{\partial x^4} , \tag{1.10}$$

as proved by Hochberg and Orsingher (1996).

As a consequence of our analysis we also show that the law of the iterated Brownian motion

$$I(t) = B_1(|B_2(t)|), \qquad t > 0$$

is a solution to the fractional equation

$$\frac{\partial^{1/2} u}{\partial t^{1/2}} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} , \tag{1.11}$$

where B_1 and B_2 are independent Brownian motions. Equation (1.11) can be obtained from (1.1) either for $\lambda = 0$, $c^2 = 1/2$, $\alpha = 1/4$ or for $\lambda, c \to \infty$, in such a way that $c^2/\lambda \to 1$ and $\alpha = 1/2$.

In Funaki (1979) it is proved that the process related to $I(t)$, namely

$$\overline{I}(t) = \begin{cases} B_1(B_2(t)) & \text{when } B_2(t) > 0 \\ iB_1(-B_2(t)) & \text{when } B_2(t) < 0 \end{cases} ,$$

has a law satisfying the fourth-order equation

$$\frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4}.$$

The fractional telegraph equation in the case $\alpha = 1/2$ can be interpreted as a heat equation subject to a damping effect, represented by the 1/2-order time-derivative, involving the values of u in the whole interval $[0, t]$.

We are also able to obtain from (1.5) the general expression of the distributions related to the heat-wave fractional equations (as particular case when $\lambda = 0$). These laws can be expressed either in terms of the Wright functions

$$W_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\Gamma(\alpha k + \beta)} ,$$

or by means of integrals on Hankel paths (see Mainardi (1996)).

Moreover we give a representation of the solution to the heat-wave fractional equation as fractional integrals of space-symmetric stable distributions (see formula (3.5)).

Finally the general expression of the variance of the processes $X_\alpha = X_\alpha(t), t > 0$ with characteristic function (1.5) is

$$EX_\alpha^2(t) = 2c^2 t^{2\alpha} E_{\alpha,2\alpha+1}(-2\lambda t^\alpha). \tag{1.12}$$

We show that, for the special cases $\alpha = 1$, $\alpha = 1/2$, the asymptotic behavior of the variance is

$$\begin{aligned} EX_1^2(t) &\sim \frac{c^2}{\lambda} t \qquad \text{as } t \to \infty \\ EX_{1/2}^2(t) &\sim \frac{2c^2}{\lambda\sqrt{\pi}} \sqrt{t}. \end{aligned}$$

A general information on the behavior of $EX_\alpha^2(t)$, as $t \to \infty$, for any α seems not possible.

2. The Fourier transform of the solutions

In order to find the solution to equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + 2\lambda \frac{\partial^\alpha u}{\partial t^\alpha} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < \alpha \le 1 \tag{2.1a}$$

with initial conditions

$$\begin{cases} u(x,0) = \delta(x) \\ u_t(x,0) = 0 \end{cases}, \qquad \text{for } \frac{1}{2} < \alpha \le 1 \tag{2.1b}$$

and

$$u(x,0) = \delta(x), \qquad \text{for } 0 < \alpha \le \frac{1}{2}, \tag{2.1c}$$

we consider the Fourier transform

$$U(\beta,t) = \int_{-\infty}^{+\infty} e^{i\beta x} u(x,t)\,dx \tag{2.2}$$

which satisfies

$$\frac{\partial^{2\alpha} U}{\partial t^{2\alpha}} + 2\lambda \frac{\partial^\alpha U}{\partial t^\alpha} + c^2\beta^2 U = 0 \tag{2.3a}$$

with initial conditions

$$\begin{cases} U(\beta,0) = 1 \\ U_t(\beta,0) = 0. \end{cases}, \qquad \text{for } \frac{1}{2} < \alpha \le 1 \tag{2.3b}$$

and

$$U(\beta,0) = 1, \qquad \text{for } 0 < \alpha \le \frac{1}{2}. \tag{2.3c}$$

The integration of (2.3a) with conditions (2.3b) or (2.3c) can be performed by resorting to the Laplace transform

$$\mathcal{L}U(\beta,t)(s)=\int_0^\infty e^{-st}U(\beta,t)dt. \tag{2.4}$$

We remark here that the assumption that the fractional derivatives are of the form (1.3) permits us to impose initial conditions in terms of integer-order derivatives. This clearly emerges from the following formula for the Laplace transform of the α-order derivatives:

$$\mathcal{L}D^\alpha U(\beta,t)(s)=s^\alpha\mathcal{L}U(\beta,t)(s)-\sum_{k=0}^{m-1}s^{\alpha-1-k}D^kU(\beta,t)\Big|_{t=0} \tag{2.5}$$

where $\lfloor\alpha\rfloor=m-1$.

Formula (2.5) can be obtained by considering the definition of Dzherbashyan-Caputo derivative as follows

$$\begin{aligned}\mathcal{L}D^\alpha U(\beta,t)(s)&=\frac{1}{\Gamma(m-\alpha)}\int_0^\infty e^{-st}\left\{\int_0^t\frac{\frac{\partial^m}{\partial z^m}U(\beta,z)}{(t-z)^{1+\alpha-m}}dz\right\}dt\\&=\frac{1}{\Gamma(m-\alpha)}\int_0^\infty\frac{\partial^m}{\partial z^m}U(\beta,z)\left\{\int_z^\infty\frac{e^{-st}}{(t-z)^{1+\alpha-m}}dt\right\}dz\\&=s^{\alpha-m}\int_0^\infty e^{-sz}\frac{\partial^m}{\partial z^m}U(\beta,z)dz.\end{aligned}$$

By inserting now the well-known formula for the Laplace transform of integer derivatives

$$\begin{aligned}&\int_0^\infty e^{-sz}\frac{\partial^m}{\partial z^m}U(\beta,z)dz\\&=s^m\mathcal{L}U(\beta,t)(s)-\sum_{k=0}^{m-1}s^{m-1-k}\frac{\partial^k}{\partial t^k}U(\beta,t)\Big|_{t=0},\end{aligned}$$

we obtain (2.5).

Formula (2.5) shows that, since $U_t(\beta,0)=0$, there is no difference between the cases where $0<\alpha\le 1/2$ and $1/2<\alpha\le 1$.

After some calculations based on (2.5) we see that the Laplace transform of the solution to (2.3a), equipped with the initial conditions, reads

$$\mathcal{L}U_\alpha(\beta,t)(s)=\frac{s^{2\alpha-1}+2\lambda s^{\alpha-1}}{s^{2\alpha}+2\lambda s^\alpha+c^2\beta^2}=F(\beta,s), \tag{2.6}$$

for all $0<\alpha\le 1$.

We remark that, if the derivatives appearing in equations (2.1a) and (2.3a) were meant in the sense of Riemann-Liouville, initial conditions should be considered in the fractional form and the expression of the Fourier-Laplace transform of the solutions would be consequently different from (2.6).

The task of finding the inverse Laplace transform of (2.6) is carried out in the next theorem.

Theorem 2.1. *The Fourier transform of the solutions of problems (2.1a)–(2.1b) and (2.1a)–(2.1c) can be written in the following equivalent forms:*

$$\begin{aligned} U_\alpha(\beta,t) &= E_{\alpha,1}(\eta_1 t^\alpha) + \frac{(2\lambda+\eta_2)t^\alpha}{\eta_1-\eta_2}[\eta_1 E_{\alpha,\alpha+1}(\eta_1 t^\alpha) - \eta_2 E_{\alpha,\alpha+1}(\eta_2 t^\alpha)] \\ &= \frac{1}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\beta^2}}\right)E_{\alpha,1}(\eta_1 t^\alpha)\right. \\ &\quad \left. + \left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\beta^2}}\right)E_{\alpha,1}(\eta_2 t^\alpha)\right], \quad t>0, \end{aligned} \tag{2.7}$$

where $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined in (1.4) and

$$\eta_1 = -\lambda + \sqrt{\lambda^2-c^2\beta^2}\,, \qquad \eta_2 = -\lambda - \sqrt{\lambda^2-c^2\beta^2}\,. \tag{2.8}$$

Proof. It is convenient to write (2.6) as follows

$$\mathcal{L}U_\alpha(\beta,t)(s) = s\frac{s^{\alpha-1}}{s^\alpha-\eta_1}\frac{s^{\alpha-1}}{s^\alpha-\eta_2} + 2\lambda\frac{s^{\alpha-1}}{s^\alpha-\eta_1}\frac{1}{s^\alpha-\eta_2} \tag{2.9}$$

and then apply the following relationships

$$\int_0^\infty e^{-st}E_{\alpha,1}(\eta_j t^\alpha)dt = \frac{s^{\alpha-1}}{s^\alpha-\eta_j} \qquad j=1,2 \tag{2.10}$$

$$\int_0^\infty e^{-st}t^{\alpha-1}E_{\alpha,\alpha}(\eta_j t^\alpha)dt = \frac{1}{s^\alpha-\eta_j}, \qquad j=1,2 \tag{2.11}$$

valid for $s > \eta_j^{1/\alpha}$.

The reader can easily check result (2.10) in the following manner

$$\begin{aligned} \int_0^\infty e^{-st}E_{\alpha,1}(\eta_j t^\alpha)dt &= \sum_{k=0}^\infty \frac{\eta_j^k}{\Gamma(\alpha k+1)}\int_0^\infty e^{-st}t^{\alpha k}dt \\ &= \frac{1}{s}\sum_{k=0}^\infty\left(\frac{\eta_j}{s^\alpha}\right)^k = \frac{s^{\alpha-1}}{s^\alpha-\eta_j}\,. \end{aligned}$$

From the last step it is clear why formulas (2.10) and (2.11) are valid for $s > \eta_j^{1/\alpha}$. Similar calculations yield (2.11).

In order to invert the first term in (2.9) we write

$$\begin{aligned} & s\frac{s^{\alpha-1}}{s^\alpha-\eta_1}\frac{s^{\alpha-1}}{s^\alpha-\eta_2} = \int_0^\infty se^{-st}\left[\int_0^t E_{\alpha,1}(\eta_1 z^\alpha)E_{\alpha,1}(\eta_2(t-z)^\alpha)dz\right]dt \\ &= -e^{-st}\int_0^t E_{\alpha,1}(\eta_1 z^\alpha)E_{\alpha,1}(\eta_2(t-z)^\alpha)dz\Big|_{t=0}^{t=\infty} \\ &+ \int_0^\infty e^{-st}E_{\alpha,1}(\eta_1 t^\alpha)dt + \int_0^\infty e^{-st}dt \\ &\times \int_0^t E_{\alpha,1}(\eta_1 z^\alpha)\frac{d}{dt}E_{\alpha,1}(\eta_2(t-z)^\alpha)dz \end{aligned} \tag{2.12}$$

To realize that the second line of (2.12) is equal to zero it is necessary to take into account the following formula (see Podlubny (1999), page 26, formula (1.108)):

$$\begin{aligned}
&\int_0^t z^{\gamma-1}E_{\alpha,\gamma}(yz^\alpha)(t-z)^{\beta-1}E_{\alpha,\beta}(w(t-z)^\alpha)dz \\
&\quad = \frac{t^{\beta+\gamma-1}}{y-w}\sum_{k=0}^\infty \frac{t^{\alpha k}(y^{k+1}-w^{k+1})}{\Gamma(\alpha k+\beta+\gamma)} \\
&\quad = \frac{t^{\beta+\gamma-1}}{y-w}[yE_{\alpha,\beta+\gamma}(yt^\alpha) - wE_{\alpha,\beta+\gamma}(wt^\alpha)], \qquad (2.13)
\end{aligned}$$

for $\beta, \gamma > 0$ and $y \neq w$, which for $\gamma = \beta = 1$, $y = \eta_1$, $w = \eta_2$ yields

$$\begin{aligned}
&\int_0^t E_{\alpha,1}(\eta_1 z^\alpha)E_{\alpha,1}(\eta_2(t-z)^\alpha)dz \\
&\quad = \frac{t}{\eta_1-\eta_2}[\eta_1 E_{\alpha,2}(\eta_1 t^\alpha) - \eta_2 E_{\alpha,2}(\eta_2 t^\alpha)].
\end{aligned}$$

By considering now the asymptotic expansion of the Mittag-Leffler function (see Podlubny (1999), Theorem 1.4, page 33), it is straightforward that the second step in (2.12) holds.

Since

$$\begin{aligned}
\frac{d}{dt}E_{\alpha,1}(\eta_2(t-z)^\alpha) &= \sum_{k=0}^\infty \frac{(\eta_2(t-z)^\alpha)^{k-1}}{\Gamma(\alpha k+1)}\alpha k\eta_2(t-z)^{\alpha-1} \\
&= \sum_{k=1}^\infty \frac{(\eta_2(t-z)^\alpha)^{k-1}}{\Gamma(\alpha k)}\eta_2(t-z)^{\alpha-1} \\
&= \sum_{k=0}^\infty \frac{(\eta_2(t-z)^\alpha)^{k}}{\Gamma(\alpha k+\alpha)}\eta_2(t-z)^{\alpha-1} \\
&= \eta_2(t-z)^{\alpha-1}E_{\alpha,\alpha}(\eta_2(t-z)^\alpha).
\end{aligned}$$

we have that

$$\begin{aligned}
s\frac{s^{\alpha-1}}{s^\alpha-\eta_1}\frac{s^{\alpha-1}}{s^\alpha-\eta_2} &= \int_0^\infty e^{-st}E_{\alpha,1}(\eta_1 t^\alpha)dt + \eta_2\int_0^\infty e^{-st}dt \\
&\quad \times \int_0^t (t-z)^{\alpha-1}E_{\alpha,1}(\eta_1 z^\alpha)E_{\alpha,\alpha}(\eta_2(t-z)^\alpha)dz. \qquad (2.14)
\end{aligned}$$

Therefore the inverse Laplace transform of (2.9) reads

$$\begin{aligned}
U_\alpha(\beta,t) &= E_{\alpha,1}(\eta_1 t^\alpha) + (2\lambda+\eta_2)\int_0^t (t-z)^{\alpha-1}E_{\alpha,1}(\eta_1 z^\alpha)E_{\alpha,\alpha}(\eta_2(t-z)^\alpha)dz \\
&= E_{\alpha,1}(\eta_1 t^\alpha) + \frac{(2\lambda+\eta_2)t^\alpha}{\eta_1-\eta_2}\left[\eta_1 E_{\alpha,\alpha+1}(\eta_1 t^\alpha) - \eta_2 E_{\alpha,\alpha+1}(\eta_2 t^\alpha)\right]. \qquad (2.15)
\end{aligned}$$

In the last step we have applied formula (2.13) for $\gamma = 1$, $\beta = \alpha$, $y = \eta_1$ and $w = \eta_2$.

Taking into account that

$$E_{\alpha,\alpha+1}(x) = \frac{1}{x}\left[E_{\alpha,1}(x) - 1\right]$$

we obtain the second expression in (2.7). □

Remark 2.1. For $\alpha = 1$, considering that $E_{1,1}(x) = e^x$ and $E_{1,2}(x) = \frac{1}{x}(e^x - 1)$, from (2.7) we obtain

$$\begin{aligned} U_1(\beta, t) =& \frac{2\lambda + \eta_1}{\eta_1 - \eta_2} e^{\eta_1 t} - \frac{2\lambda + \eta_2}{\eta_1 - \eta_2} e^{\eta_2 t} \\ =& \frac{e^{-\lambda t}}{2} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - c^2\beta^2}} \right) e^{t\sqrt{\lambda^2 - c^2\beta^2}} \right. \\ & \left. + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - c^2\beta^2}} \right) e^{-t\sqrt{\lambda^2 - c^2\beta^2}} \right]. \end{aligned} \tag{2.16}$$

Formula (2.16) represents the well-known characteristic function of the telegraph process.

Remark 2.2. For $\lambda = 0$, formula (2.7) coincides with the Fourier transform of the solution to the fractional heat-wave equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < \alpha \le 1 \tag{2.17}$$

with the suitable initial conditions.

Indeed, in this case, $\eta_1 = ic\beta$, $\eta_2 = -ic\beta$ and thus the Fourier transform (2.7), denoted by V_α, becomes

$$\begin{aligned} V_\alpha(\beta, t) &= E_{\alpha,1}(ic\beta t^\alpha) - \frac{ic\beta t^\alpha}{2}[E_{\alpha,\alpha+1}(ic\beta t^\alpha) + E_{\alpha,\alpha+1}(-ic\beta t^\alpha)] \\ &= \sum_{k=0}^{\infty} \frac{(ic\beta t^\alpha)^k}{\Gamma(\alpha k + 1)} - ic\beta t^\alpha \sum_{k=0}^{\infty} \frac{(ic\beta t^\alpha)^{2k}}{\Gamma(2\alpha k + \alpha + 1)} \\ &= \sum_{k=0}^{\infty} \frac{(-c^2\beta^2 t^{2\alpha})^k}{\Gamma(2\alpha k + 1)} = E_{2\alpha,1}(-c^2\beta^2 t^{2\alpha}). \end{aligned} \tag{2.18}$$

We note that, for $\alpha = 1/2$, (2.18) gives

$$V_{\frac{1}{2}}(\beta, t) = e^{-c^2\beta^2 t}$$

which is the characteristic function of a Gaussian distribution with variance $2c^2t$. If $\alpha = 1$ we obtain from (2.16) (for $\lambda = 0$) and alternatively from the first line of (2.18)

$$V_1(\beta, t) = \frac{1}{2}(e^{ic\beta t} + e^{-ic\beta t})$$

which is the Fourier transform of

$$v_1(x,t) = \frac{1}{2}\{\delta(x-ct)+\delta(x+ct)\}, \tag{2.19}$$

that is the D'Alembert solution to the wave equation.

3. First properties of the solutions

By taking first the inverse Fourier transform of (2.6) we immediately obtain that

$$\int_0^\infty e^{-st} u_\alpha(x,t)dt = \frac{\sqrt{s^{2\alpha}+2\lambda s^\alpha}}{2sc} e^{-\frac{|x|}{c}\sqrt{s^{2\alpha}+2\lambda s^\alpha}}. \tag{3.1}$$

The Laplace transform (3.1) can be further inverted in a simple way only when $\lambda = 0$.

It is well known (see Samorodnitsky and Taqqu (1994), page 15) that for a stable random variable $X \sim S(\sigma, 1, 0)$ we have that

$$Ee^{-\gamma X} = e^{-\frac{\sigma^\alpha}{\cos(\pi\alpha/2)}\gamma^\alpha} \qquad 0 < \alpha \leq 2,\ \alpha \neq 1, \tag{3.2}$$

where $\sigma, \gamma > 0$. Therefore (3.1) can be written (for $\lambda = 0$) as

$$\int_0^\infty e^{-st} v_\alpha(x,t)dt = \frac{s^{\alpha-1}}{2c} e^{-\frac{|x|s^\alpha}{c}} \qquad \text{for } 0 < \alpha < 1 \tag{3.3}$$

where

$$e^{-\frac{|x|s^\alpha}{c}} = Ee^{-sX((\frac{|x|}{c}\cos\frac{\pi\alpha}{2})^{\frac{1}{\alpha}},1,0)}. \tag{3.4}$$

Denoting by $p_\alpha(|x|,t)$ the probability law of $X((\frac{|x|}{c}\cos\frac{\pi\alpha}{2})^{\frac{1}{\alpha}}, 1, 0)$, we can write the inverse Laplace transform of (3.3) as

$$v_\alpha(x,t) = \frac{1}{2c\Gamma(1-\alpha)} \int_0^t \frac{p_\alpha(|x|,s)}{(t-s)^\alpha} ds \qquad \text{for } 0 < \alpha < 1. \tag{3.5}$$

Clearly (3.5) shows that the solution to (2.17) is non-negative.

For $\alpha = 1/2$ we can invert (3.3) by means of the relationship

$$e^{-\frac{|x|s^{1/2}}{c}} = \int_0^\infty e^{-st} \frac{|x|}{\sqrt{2}c} \frac{1}{\sqrt{2\pi t^3}} e^{-\frac{|x|^2}{4c^2 t}} dt. \tag{3.6}$$

Therefore the Laplace inverse transform of (3.3) is

$$\begin{aligned} v_{\frac{1}{2}}(x,t) &= \frac{1}{2c}\int_0^t \frac{1}{\sqrt{\pi(t-s)}} \frac{|x|}{\sqrt{2}c} \frac{1}{\sqrt{2\pi s^3}} e^{-\frac{x^2}{4c^2 s}} ds \\ &= \frac{1}{\sqrt{4\pi c^2 t}} e^{-\frac{x^2}{4c^2 t}}. \end{aligned} \tag{3.7}$$

The last step in (3.7) is due to the following result

$$\frac{1}{\pi}\int_0^t \frac{e^{-\frac{x^2}{2s}}}{\sqrt{s^3(t-s)}}ds = \frac{2}{|x|\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\ , \tag{3.8}$$

which can be easily checked.

In the case where $\lambda \neq 0$, $0 < \alpha \leq 1$, bearing in mind the identity

$$\frac{a}{\pi(a^2+x^2)} = \int_0^\infty \frac{a}{\sqrt{2\pi s^3}}e^{-\frac{a^2}{2s}}\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}ds, \tag{3.9}$$

the Fourier-Laplace transform (2.6) can be written as

$$F(\beta, s) = (s^{2\alpha-1}+2\lambda s^{\alpha-1})\frac{1}{2c^2}\int_0^\infty \frac{1}{w^2}e^{-\frac{s^{2\alpha}+2\lambda s^\alpha}{2c^2 w}}e^{-\frac{\beta^2}{2w}}dw. \tag{3.10}$$

In view of (3.2) we have that

$$e^{-\frac{s^{2\alpha}}{2c^2w}} = Ee^{-sX\left(\left(\frac{\cos\pi\alpha}{2c^2w}\right)^{\frac{1}{2\alpha}},1,0\right)} \tag{3.11}$$

$$e^{-\frac{\lambda s^\alpha}{c^2w}} = Ee^{-sX\left(\left(\frac{\lambda\cos\pi\alpha/2}{c^2w}\right)^{\frac{1}{\alpha}},1,0\right)}, \tag{3.12}$$

for any $0 < \alpha < 1$, $\alpha \neq 1/2$. The distribution connected with (3.11) will be denoted by $q_{2\alpha}(w, t)$ and that related to (3.12) will be indicated by $q_\alpha(w, t)$.

By considering that

$$\frac{1}{\Gamma(1-2\alpha)}\int_0^\infty e^{-st}\frac{dt}{t^{2\alpha}} = s^{2\alpha-1} \qquad \text{for } \alpha < 1/2, \tag{3.13a}$$

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\infty e^{-st}\frac{dt}{t^{\alpha}} = s^{\alpha-1} \qquad \text{for } \alpha < 1, \tag{3.13b}$$

the Fourier-Laplace inverse transform of (3.10) reads

$$\begin{aligned} u_\alpha(x,t) = {} & \frac{1}{2c^2\Gamma(1-2\alpha)}\int_0^\infty \frac{e^{-\frac{wx^2}{2}}}{\sqrt{2\pi w^3}}dw\int_0^t \frac{ds}{(t-s)^{2\alpha}} \\ & \times \int_0^s q_{2\alpha}(w,z)q_\alpha(w,s-z)dz + \frac{\lambda}{c^2\Gamma(1-\alpha)}\int_0^\infty \frac{e^{-\frac{wx^2}{2}}}{\sqrt{2\pi w^3}}dw \\ & \times \int_0^t \frac{ds}{(t-s)^\alpha}\int_0^s q_{2\alpha}(w,z)q_\alpha(w,s-z)dz. \end{aligned} \tag{3.14}$$

Clearly the second term of (3.14) is non-negative for any $0 < \alpha < 1$. The first term of (3.14) is non-negative for $0 < \alpha < 1/2$; by a different argument we show that it is non-negative also for $1/2 < \alpha < 1$.

If we consider the first term of (3.10) and use the mean-value theorem we have:

$$\begin{aligned}&\frac{s^{2\alpha-1}}{2c^2}\int_0^\infty \frac{1}{w^2}e^{-\frac{s^{2\alpha}+2\lambda s^\alpha}{2c^2 w}}e^{-\frac{\beta^2}{2w}}dw\\ &\quad= e^{-\frac{\lambda s^\alpha}{c^2\overline{w}}}\left\{\frac{s^{2\alpha-1}}{2c^2}\int_0^\infty \frac{1}{w^2}e^{-\frac{s^{2\alpha}}{2c^2 w}}e^{-\frac{\beta^2}{2w}}dw\right\}\end{aligned} \tag{3.15}$$

for a suitable $\overline{w} \in (0, \infty)$. While the first factor of (3.15) coincides with (3.12) and refers to a stable distribution of order α, the second one represents the Fourier-Laplace transform of the solution to the fractional heat-wave equation (2.17), which we proved to be non-negative by means of the representation (3.5).

From (2.7) we can argue that $U_\alpha(0,t) = 1$, for any α, since for $\beta = 0$ we get $\eta_1 = 0$, $\eta_2 = -2\lambda$ and $E_{\alpha,1}(0) = 1$.

In conclusion the solutions to the initial value problem of the fractional telegraph equation (2.1a) can be interpreted as true probability distributions for any $0 < \alpha \leq 1$.

Remark 3.1. The uniqueness of the solution to the initial-value problems considered here can be proved by means of arguments similar to those appearing in Fujita (1990, I).

Let $u_1,\ u_2$ be two different solutions to (2.1a) with initial condition (2.1c) $(0 < \alpha \leq 1/2,\ m = 1)$; therefore $w = u_1 - u_2$ satisfies equation (2.1a) with $w(x,0) = 0$.

We note that for the Riemann-Liouville integral of order 2α $(0 < \alpha < 1)$, denoted by $I^{2\alpha}$ (see Samko et al. (1993), definition 2.1, page 33), and the Dzherbashyan-Caputo derivative of order α we have, for the Fourier transform W of w, that

$$\begin{aligned}&I^{2\alpha}\frac{\partial^\alpha}{\partial t^\alpha}W(\beta,t)\\ &= \frac{1}{\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}\frac{ds}{\Gamma(1-\alpha)}\int_0^s \frac{\partial}{\partial z}W(\beta,z)\frac{dz}{(s-z)^\alpha}\\ &= \frac{1}{\Gamma(2\alpha)\Gamma(1-\alpha)}\int_0^t \frac{\partial}{\partial z}W(\beta,z)\,dz\int_z^t (t-s)^{2\alpha-1}\frac{ds}{(s-z)^\alpha}\\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^t \frac{\partial}{\partial z}W(\beta,z)(t-z)^\alpha dz\\ &= \frac{1}{\Gamma(\alpha)}\int_0^t W(\beta,z)(t-z)^{\alpha-1}dz\\ &= I^\alpha W(\beta,t),\end{aligned}$$

since $W(\beta,0) = 0$.

Analogously, after some calculations, it is easy to show that

$$I^{2\alpha}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}W(\beta,t) = W(\beta,t).$$

By performing now the Riemann-Liouville integral of order 2α with respect to t in equation (2.3a), we get

$$\begin{aligned} W(\beta, t) &+ \frac{2\lambda}{\Gamma(\alpha)} \int_0^t W(\beta, s)(t-s)^{\alpha-1} ds \\ &= \frac{c^2\beta^2}{\Gamma(2\alpha)} \int_0^t W(\beta, s)(t-s)^{2\alpha-1} ds. \end{aligned} \tag{3.16}$$

From (3.16) it turns out that

$$\begin{aligned} |W(\beta, t)| &\le \frac{2\lambda}{\Gamma(\alpha)} \int_0^t |W(\beta, s)|\, |t-s|^{\alpha-1}\, ds \\ &\quad + \frac{c^2\beta^2}{\Gamma(2\alpha)} \int_0^t |W(\beta, s)|\, |t-s|^{2\alpha-1}\, ds \\ &\le \left\{ \frac{2\lambda t^{\alpha-1}}{\Gamma(\alpha)} + \frac{c^2\beta^2 t^{2\alpha-1}}{\Gamma(2\alpha)} \right\} \int_0^t |W(\beta, s)|\, ds \\ &\le \left\{ \frac{2\lambda T^{\alpha-1}}{\Gamma(\alpha)} + \frac{c^2\beta^2 T^{2\alpha-1}}{\Gamma(2\alpha)} \right\} \int_0^t |W(\beta, s)|\, ds, \end{aligned}$$

for $t \in [0, T]$, $T > 0$. By the Gronwall's inequality we conclude that $W(\beta, t) = 0$ and thus $w = u_1 - u_2 = 0$. The same arguments extend to equation (2.1a) with initial conditions (2.1b).

4. The explicit solution for $\alpha = 1/2$ as the law of a telegraph process with Brownian time

We consider here the case $\alpha = 1/2$ for which it is possible to obtain the explicit form of the solution to (2.1a) with initial condition (2.1c); we start by writing the Fourier transform (2.7) in a more suitable form.

Theorem 4.1. *For $\alpha = 1/2$, we obtain the following Fourier transform of the solution*

$$\begin{aligned} U_{\frac{1}{2}}(\beta, t) =& \frac{\lambda}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{z^2}{4t} - \lambda z} \left\{ \frac{e^{z\sqrt{\lambda^2 - c^2\beta^2}} - e^{-z\sqrt{\lambda^2 - c^2\beta^2}}}{\sqrt{\lambda^2 - c^2\beta^2}} \right\} dz \\ &+ \frac{1}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{z^2}{4t} - \lambda z} \left\{ e^{z\sqrt{\lambda^2 - c^2\beta^2}} + e^{-z\sqrt{\lambda^2 - c^2\beta^2}} \right\} dz. \end{aligned} \tag{4.1}$$

Proof. In order to pass from the first form of (2.7) to formula (4.1) we need explicit expressions for the Mittag-Leffler functions $E_{\frac{1}{2}, \frac{3}{2}}(x)$ and $E_{\frac{1}{2}, 1}(x)$. We have

$$
\begin{aligned}
E_{\frac{1}{2},\frac{3}{2}}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma\left(\frac{k+3}{2}\right)} = \sum_{k=0}^{\infty} \frac{x^k}{\frac{k+1}{2}\Gamma\left(\frac{k+1}{2}\right)} \\
&= \text{(by the duplication formula of the Gamma function)} \\
&= \sum_{k=0}^{\infty} \frac{x^k 2^{k+1}\Gamma\left(\frac{k}{2}+1\right)}{\sqrt{\pi}(k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{x^k 2^{k+1}}{\sqrt{\pi}(k+1)!} \int_0^\infty e^{-w} w^{\frac{k}{2}} dw \\
&= \frac{2}{\sqrt{\pi}x} \int_0^\infty e^{-w^2}(e^{2wx}-1)dw \qquad (4.2)
\end{aligned}
$$

and analogously

$$
\begin{aligned}
E_{\frac{1}{2},1}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma\left(\frac{k}{2}+1\right)} = \sum_{k=0}^{\infty} \frac{x^k 2^k \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}k!} \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-w^2+2xw} dw. \qquad (4.3)
\end{aligned}
$$

With this at hand we readily have

$$
\begin{aligned}
&U_{\frac{1}{2}}(\beta,t) \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-w^2-2\lambda\sqrt{t}w+2\sqrt{t}w\sqrt{\lambda^2-c^2\beta^2}} dw + \\
&+ \frac{\lambda}{\sqrt{\pi}} \int_0^\infty e^{-w^2-2\lambda\sqrt{t}w} \left\{ \frac{e^{2w\sqrt{t}\sqrt{\lambda^2-c^2\beta^2}} - e^{-2w\sqrt{t}\sqrt{\lambda^2-c^2\beta^2}}}{\sqrt{\lambda^2-c^2\beta^2}} \right\} dw \\
&- \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-w^2-2\lambda\sqrt{t}w} \left\{ e^{2w\sqrt{t}\sqrt{\lambda^2-c^2\beta^2}} - e^{-2w\sqrt{t}\sqrt{\lambda^2-c^2\beta^2}} \right\} dw, \qquad (4.4)
\end{aligned}
$$

which coincides with (4.1), after having introduced the change of variable $2\sqrt{t}w = z$.

Result (4.1) can also be obtained by working on the second expression of (2.7) and applying formula (4.3). □

Theorem 4.2. *The distribution obtained by inverting the Fourier transform (4.1) is*

$$
\begin{aligned}
u_{\frac{1}{2}}(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_0^\infty e^{-\frac{w^2}{4t}-\lambda w} \{ &[\lambda I_0(\frac{\lambda}{c}\sqrt{c^2w^2-x^2}) \\
&+ \frac{\partial}{\partial w} I_0(\frac{\lambda}{c}\sqrt{c^2w^2-x^2})] 1_{\{|x|<cw\}} \\
&+ c[\delta(x-cw)+\delta(x+cw)] \} dw. \qquad (4.5)
\end{aligned}
$$

Proof. If we write (4.1) as

$$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\frac{z^2}{4t}}\left\{\frac{e^{-\lambda z}}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\beta^2}}\right)e^{z\sqrt{\lambda^2-c^2\beta^2}}\right.\right.$$
$$\left.\left.+\left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\beta^2}}\right)e^{-z\sqrt{\lambda^2-c^2\beta^2}}\right]\right\}dz, \tag{4.6}$$

we see that, inside the integral (4.6), the characteristic function (2.16) of the telegraph process appears.

The distribution related to (2.16) is known to be

$$p(x,w) = \frac{e^{-\lambda w}}{2c}[\lambda I_0(\frac{\lambda}{c}\sqrt{c^2w^2-x^2})+\frac{\partial}{\partial w}I_0(\frac{\lambda}{c}\sqrt{c^2w^2-x^2})]1_{\{|x|<cw\}}$$
$$+\frac{e^{-\lambda w}}{2}[\delta(x-cw)+\delta(x+cw)]. \tag{4.7}$$

□

Remark 4.1. We can verify that $\int_{-\infty}^{+\infty}u_{\frac{1}{2}}(x,t)dx = 1$. Since

$$\int_{-cw}^{cw} I_0(\frac{\lambda}{c}\sqrt{c^2w^2-x^2})dx = \frac{c}{\lambda}(e^{\lambda w}-e^{-\lambda w})$$
$$\int_{-cw}^{cw}\frac{\partial I_0}{\partial w}(\frac{\lambda}{c}\sqrt{c^2w^2-x^2})dx = c(e^{\lambda w}+e^{-\lambda w}-2)$$

we have that

$$\begin{aligned}\int_{-\infty}^{+\infty}u_{\frac{1}{2}}(x,t)dx &= \frac{1}{2c\sqrt{\pi t}}\int_0^\infty e^{-\frac{w^2}{4t}-\lambda w}\\ &\quad\times\left[c(e^{\lambda w}-e^{-\lambda w})+c(e^{\lambda w}+e^{-\lambda w}-2)+2c\right]dw\\ &= \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\frac{w^2}{4t}}dw = 1.\end{aligned}$$

Remark 4.2. The probability density (4.5) coincides with the distribution of the telegraph process $T = T(t), t > 0$ with a Brownian time, that is

$$W(t) = T(|B(t)|). \tag{4.8}$$

This means that the fundamental solution to the fractional equation

$$\frac{\partial u}{\partial t}+2\lambda\frac{\partial^{1/2}u}{\partial t^{1/2}} = c^2\frac{\partial^2 u}{\partial x^2} \tag{4.9}$$

can be interpreted as the distribution of a particle moving back and forth on the real line with velocities $\pm c$ (switching at Poisson-paced times) for a random time interval of length $|B(t)|$. Clearly T and B are assumed independent of each other.

Equation (4.9) is a heat equation with a damping term depending on all values of u in $[0, t]$ and assigning an overwhelming weight to those close to t (because of definition (1.3)). The damping effect of $\partial^{1/2}u/\partial t^{1/2}$ reverberates on the distribution (4.5), where the governing term (solution to the heat equation) is weighted by the telegraph distribution (representing the impact of the fractional derivative).

Remark 4.3. If $\lambda = 0$, formula (4.5) reduces to

$$
\begin{aligned}
v_{\frac{1}{2}}(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{w^2}{4t}} \left[\delta(x-cw)+\delta(x+cw)\right] dw \\
&= \frac{1}{2c\sqrt{\pi t}} \int_0^\infty e^{-\frac{w^2}{4c^2t}} \left[\delta(x-w)+\delta(x+w)\right] dw \\
&= \frac{e^{-\frac{x^2}{4c^2t}}}{2c\sqrt{\pi t}} \qquad \text{for } x \in \mathbb{R}. \qquad (4.10)
\end{aligned}
$$

The Gaussian density is clearly the fundamental solution to the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

Remark 4.4. It is very interesting to point out that the density (4.5) converges to

$$
\overline{v}_{\frac{1}{2}}(x,t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{w^2}{4t}} \frac{e^{-\frac{x^2}{2w}}}{\sqrt{2\pi w}} dw \qquad (4.11)
$$

as $\lambda, c \to \infty$ in such a way that $(c^2/\lambda) \to 1$. This can be established considering that the distribution (4.7) of the telegraph process converges to the transition function of Brownian motion (as can be checked by means of the asymptotic formulas of Bessel function, see Tranter (1968), formula 3.30, p.50, and Orsingher (1990)). This can be proved in a different way by starting from the relationship (3.10). Indeed from (3.10) as $\lambda, c \to \infty$ (in such a way that $(c^2/\lambda) \to 1$) we obtain (when $\alpha = 1/2$) that

$$
\overline{F}(\beta, s) = s^{-\frac{1}{2}} \int_0^\infty \frac{1}{w^2} e^{-\frac{\sqrt{s}}{w}} e^{-\frac{\beta^2}{2w}} dw. \qquad (4.12)
$$

The inverse Laplace-Fourier transform of (4.12) reads

$$
\begin{aligned}
\overline{v}_{\frac{1}{2}}(x,t) &= \int_0^\infty \frac{dw}{w^2} \int_0^t \frac{1}{\sqrt{\pi}\sqrt{t-s}} \frac{e^{-\frac{1}{4sw^2}}}{\sqrt{2}w\sqrt{2\pi s^3}} \frac{e^{-\frac{x^2 w}{2}}}{\sqrt{2\pi}} \sqrt{w} ds \\
&= \text{(on the basis of (3.8))} \\
&= \int_0^\infty \frac{1}{w^2} \frac{1}{2w} \sqrt{w} \frac{e^{-\frac{x^2 w}{2}}}{\sqrt{2\pi}} \frac{2\sqrt{2}w}{\sqrt{2\pi t}} e^{-\frac{1}{4tw^2}} dw \\
&= \int_0^\infty \frac{e^{-\frac{x^2 w}{2}}}{\sqrt{2\pi w^3}} \frac{e^{-\frac{1}{4tw^2}}}{\sqrt{\pi t}} dw \\
&= \left(\text{by means of } \frac{1}{w} = z\right) \\
&= \int_0^\infty \frac{e^{-\frac{x^2}{2z}}}{\sqrt{2\pi z}} \frac{e^{-\frac{z^2}{4t}}}{\sqrt{\pi t}} dz. \qquad (4.13)
\end{aligned}
$$

We note that (4.11) is the law of the iterated Brownian motion

$$I(t) = B_1(|B_2(t)|) \tag{4.14}$$

where B_1 and B_2 are independent Brownian motions and B_2 possesses variance parameter equal to 2.

This shows that the well-known process $I(t)$ is a fractional diffusion governed by the equation

$$\frac{\partial^{1/2} u}{\partial t^{1/2}} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} .$$

5. About the variance of the processes $\mathbf{X}_\alpha$

By means of the Fourier transform (2.7) it is straightforward to ascertain that the mean value of processes $X_\alpha(t), t > 0$ related to the fractional telegraph equation (1.1) is zero, for any $0 < \alpha \leq 1$.

In order to obtain the explicit expression of the variance $EX_\alpha^2(t)$ we prefer a different approach, based on (2.6).

Since

$$\frac{\partial^2}{\partial \beta^2} \mathcal{L}U_\alpha(\beta, t)(s)\bigg|_{\beta=0} = -\int_0^\infty e^{-st} EX_\alpha^2(t)dt, \tag{5.1}$$

from (2.6) we obtain

$$\frac{\partial^2}{\partial \beta^2} \mathcal{L}U_\alpha(\beta, t)(s)\bigg|_{\beta=0} = -\frac{2c^2}{s^{\alpha+1}(s^\alpha + 2\lambda)} . \tag{5.2}$$

By taking profit of formula (2.11) and of the fact that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty e^{-st} t^\alpha dt = \frac{1}{s^{\alpha+1}} ,$$

we have that

$$\begin{aligned} EX_\alpha^2(t) &= \frac{2c^2}{\Gamma(1+\alpha)}\int_0^t s^{\alpha-1}(t-s)^\alpha E_{\alpha,\alpha}(-2\lambda s^\alpha)ds \\ &= \frac{2c^2}{\Gamma(1+\alpha)}\sum_{k=0}^\infty \frac{(-2\lambda)^k}{\Gamma(\alpha k+\alpha)}\int_0^t s^{\alpha-1}(t-s)^\alpha s^{\alpha k} ds \\ &= 2c^2 t^{2\alpha} E_{\alpha,2\alpha+1}(-2\lambda t^\alpha). \end{aligned} \tag{5.3}$$

Remark 5.1. We examine now some special cases of (5.3).

When $\alpha = 1$, $\lambda = 0$, it is clearly $E_{1,3}(0) = 1/2$ and thus (5.3) yields

$$EX_1^2(t) = c^2 t^2$$

as can be also inferred directly from (2.19).

When $\alpha = 1$, $\lambda \neq 0$, since

$$E_{1,3}(x) = \frac{1}{x^2}(e^x - 1 - x), \qquad x \neq 0 \tag{5.4}$$

we obtain that

$$EX_1^2(t) = \frac{c^2}{\lambda}\left(t + \frac{e^{-2\lambda t} - 1}{2\lambda}\right) = ET^2(t), \tag{5.5}$$

which coincides with formula (28) of Orsingher (1990).

When considering the case $\alpha = 1/2$ we need to evaluate $E_{1/2,2}(x)$. Some calculations show that

$$\frac{1}{\Gamma(\frac{k}{2}+2)} = \frac{2^{k+1}(k+1)\Gamma(\frac{k+1}{2})}{\sqrt{\pi}(k+2)!} \tag{5.6}$$

so that

$$\begin{aligned}
E_{\frac{1}{2},2}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\frac{k}{2}+2)} \\
&= \sum_{k=0}^{\infty} \frac{x^k 2^{k+1}(k+1)}{\sqrt{\pi}(k+2)!} \int_0^{\infty} e^{-w} w^{\frac{k+1}{2}-1} dw \\
&= \sum_{k=0}^{\infty} \frac{x^k 2^{k+2}}{\sqrt{\pi}(k+2)!} \int_0^{\infty} e^{-w} w^{\frac{k+1}{2}} dw \\
&= \frac{1}{x^2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-w}}{\sqrt{w}} \sum_{k=0}^{\infty} \frac{x^{k+2} 2^{k+2} w^{\frac{k+2}{2}}}{(k+2)!} dw \\
&= \frac{1}{x^2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-w}}{\sqrt{w}} (e^{2\sqrt{w}x} - 1 - 2x\sqrt{w}) dw \\
&= \frac{2e^{x^2}}{x^2\sqrt{\pi}} \int_{-x}^{\infty} e^{-w^2} dw - \frac{1}{x^2} - \frac{2}{x\sqrt{\pi}}, \qquad x \neq 0.
\end{aligned} \tag{5.7}$$

It can be easily checked that formula (5.7) (as well as (5.4)) holds, by continuity, also for $x = 0$. By applying (5.7) to (5.3) we obtain

$$EX_{\frac{1}{2}}^2(t) = \frac{c^2}{\lambda^2\sqrt{\pi}}\left(e^{4\lambda^2 t} \int_{2\lambda\sqrt{t}}^{\infty} e^{-w^2} dw - \frac{\sqrt{\pi}}{2} + 2\lambda\sqrt{t}\right). \tag{5.8}$$

Result (5.8) can also be derived by taking profit of the representation (4.8) as follows

$$\begin{aligned} EX^2_{\frac{1}{2}}(t) &= ET^2(|B(t)|) \\ &= E\left\{ ET^2(|B(t)|) \Big| \, |B(t)| \right\} \\ &= \int_0^\infty \frac{e^{-\frac{w^2}{4t}}}{\sqrt{\pi t}} ET^2(w)dw \\ &= \int_0^\infty \frac{e^{-\frac{w^2}{4t}}}{\sqrt{\pi t}} \frac{c^2}{\lambda}\left(w + \frac{e^{-2\lambda w}-1}{2\lambda} \right) dw, \end{aligned} \tag{5.9}$$

where, in the last step, we have applied (5.5). Formula (5.9) coincides with (5.8) after some calculations.

We can see from (5.8) that, for large values of t,

$$EX^2_{\frac{1}{2}}(t) \sim \frac{2c^2}{\lambda\sqrt{\pi}}\sqrt{t}. \tag{5.10}$$

By taking the limit of (5.10) for $c, \lambda \to \infty$ ($c^2/\lambda \to 1$) we obtain, as expected, the variance of the iterated Brownian motion, that is

$$EI^2(t) = EB_1^2(|B_2(t)|) = \frac{2}{\sqrt{\pi}}\sqrt{t}.$$

Result (5.10) must be compared with

$$EX_1^2(t) \sim \frac{c^2}{\lambda}t, \qquad \text{for large } t. \tag{5.11}$$

An intuitive explanation of the fact that $EX^2_{\frac{1}{2}}(t)$ increases more slowly than $EX_1^2(t)$ is that the distribution of $X_{\frac{1}{2}}$ can be looked at as the distribution of X_1 in a space interval whose length takes small values with large probability.

In principle, it is possible to evaluate $EX^2_\alpha(t)$, for $\alpha = 1/k$, $k \in \mathbb{N}$, when $k > 2$, by successively applying the multiplication formula of Gamma function. However, in general, it is not possible to obtain fine, explicit expressions like (5.5) and (5.8).

Acknowledgement. We thank the referee for his accurate reading of the first draft of the paper which led to many improvements and corrections.

References

Anh, V.V., Leonenko, N.N.: Scaling laws for fractional diffusion-wave equation with singular data. Statistics and Probability Letters **48**, 239–252 (2000)

Angulo, J.M., Ruiz-Medina, M.D., Anh V.V., Grecksch, W.: Fractional diffusion and fractional heat equation. Adv. Appl. Probability **32** (4), 1077–1099 (2000)

Dzherbashyan, M.M., Nersesian, A.B.: Fractional derivatives and the Cauchy problem for differential equations of fractional order. Izv. Acad. Nauk Armjanskoy SSR **3** (1), 3–29 (1968) (in Russian)

Fujita, Y.: Integrodifferential equation which interpolates the heat equation and the wave equation (I). Osaka J. Math. **27**, 309–321 (1990)

Fujita, Y.: Integrodifferential equation which interpolates the heat equation and the wave equation (II). Osaka J. Math. **27**, 797–804 (1990)

Funaki, T.: Probabilistic construction of the solution of some higher-order parabolic equations. Proc. Japan Acad. **55**, 176–179 (1979)

Hochberg, K., Orsingher, E.: Composition of stochastic processes governed by high-order parabolic and hyperbolic equations. J. Theor. Probability **9** (2), 511–532 (1996)

Mainardi, F.: Fractional relaxation-oscillation and fractional diffusion-wave phenomena. Chaos, Solitons and Fractals. **7**, 1461–1477 (1996)

Nigmatullin, R.R.: The realization of the generalized transfer equation in a medium with fractal geometry. Phys. Stat. Sol. (b) **133**, 425–430 (1986)

Orsingher, E.: Probability law, flow function, maximum distribution of wave-governed random motions and their connection with Kirchoff's law. Stochastic Processes and their Applications. **34**, 49–66 (1990)

Orsingher, E., Zhao, X.: The space-fractional telegraph equation and the related fractional telegraph process. Chinese Ann. Math. (24) **B** (1), 1–12 (2003)

Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego, 1999

Saichev, A.I., Zaslavsky, G.M.: Fractional kinetic equations: solutions and applications. Chaos **7** (4), 753–764 (1997)

Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, 1993

Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes. Chapman and Hall, New York, 1994

Schneider, W.R., Wyss, W.: Fractional diffusion and wave equations. J. Math. Phys. **30**, 134–144 (1989)

Tranter, C.J.: Bessel Functions with Some Physical Applications. The English University Press LTD, London, 1968

Wyss, W.: Fractional diffusion equation. J. Math. Phys. **27**, 2782–2785 (1986)

Wyss, W.: The fractional Black-Scholes equation. Fractional Calculus and Appl. Anal. **3** (1), 51–61 (2000)