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Green function estimate for censored stable processes

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Abstract. Sharp two-sided estimates for Green functions of censored α-stable process Y in a bounded $C^{1,1}$ open set D are obtained, where $\alpha \in (1, 2)$. It is shown that the Martin boundary and minimal Martin boundary of Y can all be identified with the Euclidean boundary ∂D of D. Sharp two-sided estimates for the Martin kernel of Y are also derived.

1. Introduction

Markov processes with discontinuous sample paths constitute an important family of stochastic processes in probability theory. It is well known that (cf. e.g., Janicki and Weron [14], Samorodnitsky and Taqqu [18]) many physical and economic systems should be and in fact have been successfully modeled by discontinuous processes, such as stable processes.

Although a lot is known about symmetric stable as well as general Markov processes and their potential theory (see, e.g., Bliedtner and Hansen [1], Blumenthal and Getoor [2], Landkof [17], and Sharpe [19]), some fine properties for symmetric stable processes, such as Green function estimates, Martin boundary, and conditional gauge theorem, have only been recently studied. See Chen [7] for a recent survey on these.

Very recently, another class of discontinuous Markov processes, namely censored stable processes, has been studied by Bogdan, Burdzy and Chen [6]. Roughly speaking, for $\alpha \in (0, 2)$, a censored α-stable process Y in an open set $D \subset \mathbf{R}^n$ is a process obtained from a symmetric α-stable Lévy process by restricting its Lévy measure to D. The censored process is repelled from the complement of the open set D because it is prohibited to make jumps outside D. In this sense, the censored stable process is analogous to the reflecting Brownian motion in a domain. The last process plays a prominent role in stochastic analysis. It was shown in Bogdan,

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Burdzy and Chen [6] that when D is a bounded Lipschitz open set, the censored α-stable process Y in D is recurrent if $\alpha \le 1$ and is transient with finite lifetime ζ when $\alpha > 1$. In the latter case, with probability one, the process Y_t approaches to a boundary point of D as $t \uparrow \zeta$.

The main objective of this paper is to obtain the following two-sided sharp estimates for the Green function of Y.

Theorem 1.1. *Let D be a bounded $C^{1,1}$ open set in $\mathbf{R}^n$ and Y be the censored α-stable process in D with $\alpha \in (1, 2)$. There is a unique positive continuous function $G(x, y)$ on $D \times D$ except along the diagonal such that*

$$\int_D G(x, y) f(y) dy = \mathbf{E}_x \left[\int_0^\zeta f(Y_s) ds \right]$$

for every Borel function $f \ge 0$ on D. Moreover there exists a constant $c = c(D, \alpha) > 1$ such that for $x, y \in D$,

$$\begin{aligned} \frac{1}{c} \min \left\{ \frac{1}{|x-y|^{n-\alpha}}, \frac{\delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1}}{|x-y|^{n-2+\alpha}} \right\} &\le G(x, y) \\ &\le c \min \left\{ \frac{1}{|x-y|^{n-\alpha}}, \frac{\delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1}}{|x-y|^{n-2+\alpha}} \right\}. \end{aligned}$$

Here $\delta_D(x)$ denotes the Euclidean distance between x and the Euclidean boundary ∂D of D. Furthermore, the constant $c = c(D, \alpha)$ can be chosen to be domain translation and dilation invariant.

The main difficulty in deriving the above two-sided estimate for Green function is its interior estimate. Note that unlike the Green functions G_D for symmetric stable processes in D, Green functions for censored stable processes do not have domain monotonicity property and no formula is known even when D is a unit ball in $\mathbf{R}^n$. So even the upper bound $G(x, y) \le c\, |x - y|^{\alpha - n}$ is not trivial at all. We get around these difficulties using a capacity argument. For this, we use the Hardy inequality, Harnack principle and boundary Harnack principle for censored stable processes recently established in Chen and Song [11] and in Bogdan, Burdzy and Chen [6] respectively.

We further identify the Martin boundary and minimal Martin boundary of Y when $\alpha > 1$. Fix $x_0 \in D$ and define

$$M(x, y) = \frac{G(x, y)}{G(x_0, y)}, \quad x, y \in D.$$

Theorem 1.2. *Let D and Y be as in Theorem 1.1. For each $z \in \partial D$, $M(x, z) := \lim_{y \to z} M(x, y)$ exists and $M(x, z)$ is jointly continuous on $D \times \partial D$. The function $M(x, z)$ is called the Martin kernel of Y. For each $z \in \partial D$, $x \mapsto M(x, z)$ is a minimal harmonic function of Y and there is a constant $c = c(x_0, D, \alpha) > 1$ such that*

$$\frac{1}{c} \frac{\delta_D(x)^{\alpha-1}}{|x - z|^{n-2+\alpha}} \le M(x, z) \le c\, \frac{\delta_D(x)^{\alpha-1}}{|x - z|^{n-2+\alpha}}.$$

This implies that the Martin boundary and the minimal Martin boundary of Y can all be identified with the Euclidean boundary ∂D of D.

The remaining of the paper is organized as follows. In section 2 we collect the basic definition and properties of censored stable processes, including Harnack principle and boundary Harnack principle and Hardy inequality. The proofs for Theorems 1.1 and 1.2 are given in sections 3 and 4 respectively.

In this paper, we use ":=" as a way of definition, which is read as "is defined to be". For functions f and g, notation "$f \approx g$" means that there exist constants $c_2 > c_1 > 0$ such that $c_1 g \le f \le c_2 g$. We will use c to denote positive constants whose values are insignificant and may change from line to line.

2. Preliminaries

We recall the definition of censored stable process and its equivalent characterizations. Let $X = \{X_t\}$ denote a symmetric α-stable process in $\mathbf{R}^n$ with $\alpha \in (0, 2)$ and $n \ge 1$, that is, let X_t be a Lévy process whose transition density $p(t, y - x)$ relative to the Lebesgue measure is given by the following Fourier transform,

$$\int_{\mathbf{R}^n} e^{ix\cdot\xi} p(t, x)dx = e^{-t|\xi|^\alpha}.$$

It is well known (cf. (1.2.20) of Blumenthal and Getoor [2] and Example 1.4.1 of Fukushima, Oshima and Takeda [13]) that the Dirichlet form $(\mathcal{C}, \mathcal{F}^{\mathbf{R}^n})$ associated with X is given by

$$\mathcal{C}(u, v) = \frac{1}{2}\mathcal{A}(n, -\alpha)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}}\, dxdy, \tag{2.1}$$

$$\mathcal{F}^{\mathbf{R}^n} = \left\{u \in L^2(\mathbf{R}^n) : \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}}\, dxdy < \infty\right\}, \tag{2.2}$$

where

$$\mathcal{A}(n, -\alpha) = \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+n}{2})}{\pi^{n/2}\Gamma(1 - \frac{\alpha}{2})}.$$

Here Γ is the Gamma function defined by $\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}dt$ for $\lambda > 0$. Every function u in $\mathcal{F}^{\mathbf{R}^n}$ has a quasi-continuous version and it is this version that will be used hereafter for $u \in \mathcal{F}^{\mathbf{R}^n}$.

Given an open set $D \subset \mathbf{R}^n$, define $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Let $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and set $X_t^D(\omega) = \partial$ if $t \ge \tau_D(\omega)$, where ∂ is a coffin state added to $\mathbf{R}^n$. The process X^D, i.e., the process X killed upon leaving D, is called the symmetric α-stable process in D. Note that X^D is irreducible even when D is disconnected. The Dirichlet form of X^D on $L^2(D, dx)$ is $(\mathcal{C}, \mathcal{F}^D)$, where

$$\mathcal{F}^D = \{f \in \mathcal{F}^{\mathbf{R}^n} : f = 0 \text{ q.e. on } D^c\}.$$

Here q.e. is the abbreviation for quasi-everywhere (cf. Fukushima, Oshima and Takeda [13]). For $u, v \in \mathcal{F}^D$, by (2.1),

$$\mathcal{C}(u, v) = \frac{1}{2}\mathcal{A}(n, -\alpha)\int_D\int_D \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\alpha}}\, dxdy + \int_D u(x)v(x)\kappa_D(x)dx,$$

where

$$\kappa_D(x) = \mathcal{A}(n, -\alpha) \int_{D^c} \frac{1}{|x-y|^{n+\alpha}} dy \tag{2.3}$$

is the density of the killing measure of X^D. We will use $C_c(D)$ ($C_c^\infty(D)$) to denote the space of continuous (smooth) functions in D with compact support. It is well known that $\mathcal{F}^D$ is the $\mathcal{C}_1$-closure of $C_c^\infty(D)$, where $\mathcal{C}_1 = \mathcal{C} + (\cdot, \cdot)_{L^2(D)}$. Note that typically, $\lim_{t\uparrow\tau_D} X_t$ exists and belongs to D.

Define a bilinear form $\mathcal{E}$ on $C_c^\infty(D)$:

$$\mathcal{E}(u, v) = \frac{1}{2}\mathcal{A}(n, -\alpha) \int_D \int_D \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}} dxdy, \qquad u, v \in C_c^\infty(D). \tag{2.4}$$

Using Fatou's lemma, it is easy to check that the bilinear form $(C_c^\infty(D), \mathcal{E})$ is closable in $L^2(D, dx)$ Let

$\mathcal{F}$ be the closure of $C_c^\infty(D)$ under the Hilbert inner product $\mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_{L^2(D)}$.

As it is noted in Bogdan, Burdzy and Chen [6], $(\mathcal{E}, \mathcal{F})$ is Markovian and hence a regular Dirichlet form on $L^2(D, dx)$ (cf. Theorem 3.1.1 of Fukushima, Oshima and Takeda [13]) and therefore there is a Hunt process Y associated with it. This process Y is called the censored α-stable process in D. There are other ways to construct a censored stable processes. The following was proved in Bogdan, Burdzy and Chen [6].

Theorem 2.1. *(Theorem 2.1 of [6]) The following processes have the same distribution.*

(1) *The symmetric Hunt process Y associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D, dx)$;*
(2) *The strong Markov process Y obtained from the symmetric α-stable process X^D in D through the Ikeda–Nagasawa–Watanabe piecing together procedure;*
(3) *The process Y obtained from X^D through the Feynman-Kac transform $e^{\int_0^t \kappa_D(X_s^D)ds}$.*

The Ikeda–Nagasawa–Watanabe piecing together procedure mentioned in (2) goes as follows. Let $Y_t(\omega) = X_t^D(\omega)$ for $t < \tau_D(\omega)$. If $X_{\tau_D-}^D(\omega) \notin D$, set $Y_t(\omega) = \partial$ for $t \geq \tau_D(\omega)$. If $X_{\tau_D-}^D(\omega) \in D$, let $Y_{\tau_D}(\omega) = X_{\tau_D-}^D(\omega)$ and glue an independent copy of X^D starting from $X_{\tau_D-}^D(\omega)$ to $Y_{\tau_D}(\omega)$. Iterating this procedure countably many times, we obtain a process on D which is a version of the strong Markov process Y; the procedure works for every starting point in D.

The following theorem is a special case of Theorem 2.9 of Bogdan, Burdzy and Chen [6].

Theorem 2.2. *Suppose that $D \subset \mathbf{R}^n$ is a bounded Lipschitz open set, i.e., ∂D lies above the graph of a Lipschitz function in a neighborhood of every boundary point.*

(1) *If* $\alpha \le 1$ *then the censored symmetric* α*-stable process* Y *in* D *is conservative and will never approach* ∂D*;*
(2) *If* $\alpha > 1$ *then the process* Y *in* D *is transient with finite lifetime* ζ*. Moreover,* $P_x(\lim_{t\uparrow\zeta} Y_t \in \partial D,\ \zeta < \infty) = 1$ *for all* $x \in D$*.*

It is well known that when D is a bounded Lipschitz open set, there is a positive continuous function $G_D(x, y)$ on $(D \times D) \setminus d$, where d denotes the diagonal, such that for any Borel function $f \ge 0$,

$$\mathbf{E}_x \left[\int_0^{\tau_D} f(X_s) ds \right] = \int_D G_D(x, y) f(y)\, dy.$$

Function $G_D(x, y)$ is called the Green function of X^D, or the Green function of X in D. When D is a bounded $C^{1,1}$-smooth open set in $\mathbf{R}^n$, sharp estimates on G_D were obtained in Chen and Song [9] and in Kulczycki [15]:

$$G_D(x, y) \approx \min \left\{ \frac{1}{|x - y|^{n-\alpha}}, \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{|x - y|^n} \right\} \quad \text{for } x, y \in D. \tag{2.5}$$

Recall that a bounded open set D in $\mathbf{R}^n$ is said to be $C^{1,1}$ if there is a localization radius $r_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there is a $C^{1,1}$-function $\phi = \phi_Q : \mathbf{R}^{n-1} \to \mathbf{R}$ satisfying $\phi(0) = 0$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \le \Lambda|x - z|$, and an orthonormal coordinate system $y = (y_1, \cdots, y_{n-1}, y_n) := (\tilde{y}, y_n)$ such that $B(Q, r_0) \cap D = B(Q, r_0) \cap \{y : y_n > \phi(\tilde{y})\}$. The pair (r_0, Λ) is called the **characteristics** of the $C^{1,1}$ open set D.

The construction of the censored α-stable process Y in D via Ikeda–Nagasawa–Watanabe piecing together procedure in Theorem 2.1, Theorem 2.2(2) above and Theorem 1 in Kunita and Watanabe [16] imply that for $\alpha \in (1, 2)$ there is a unique Borel measurable function $G(x, y)$ on $(D \times D) \setminus d$ such that (i) $G(x, y) = G(y, x)$ for $x, y \in D$, (ii) $x \mapsto G(x, y)$ is an excessive function of Y for each fixed $y \in D$, (iii) for any Borel $f \ge 0$,

$$\mathbf{E}_x \left[\int_0^{\zeta} f(Y_s) ds \right] = \int_D G(x, y) f(y)\, dy.$$

Function $G(x, y)$ is called the Green function of Y.

Remark 2.3. It follows from Theorem 2.1 that for any $r > 0$, $\{r^{-1} Y_{r^\alpha t}, \mathbf{P}_x\}$ has the same distribution as the censored α-stable process in the open set $r^{-1}D$ starting from $r^{-1}x$. Consequently, function $(x, y) \mapsto r^{n-\alpha}\, G(rx, ry)$ is the Green function for censored α-stable process in open set $r^{-1}D$.

One of the main results of this paper is a two-sided Green function estimate for Y when D is a bounded $C^{1,1}$ open set. For this, we need the Hardy inequality, Harnack principle and boundary Harnack principle for censored stable processes recently established in Chen and Song [11] and in Bogdan, Burdzy and Chen [6] respectively.

Theorem 2.4. *(Corollary 2.4 of Chen and Song [11]) Suppose that D is a bounded Lipschitz open set in $\mathbf{R}^n$ and $\alpha \neq 1$. Then there is a constant $c(D, \alpha) > 0$ such that*

$$\int_D \frac{u(x)^2}{\delta_D(x)^\alpha} dx \leq c(D, \alpha) \int_D \int_D \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} dx dy \quad \text{for } u \in \mathcal{F}. \tag{2.6}$$

The above Hardy inequality implies that the domain $\mathcal{F}$ of the Dirichlet form $(\mathcal{F}, \mathcal{E})$ for the censored α-stable process Y in D is the same as $\mathcal{F}^D$ for the killed symmetric α-stable process X^D in D for bounded Lipschitz open set D and for $\alpha \neq 1$.

To state Harnack principle and boundary Harnack principle for Y, we need the following definition.

Definition 2.5. *Let O be an open subset of D. An integrable Borel function f defined on D taking values in $(-\infty, \infty]$ is said to be*

1) harmonic in O with respect to Y if f is continuous in O and for each $x \in O$ and each ball $B(x, r)$ with $\overline{B(x, r)} \subset O$,

$$f(x) = \mathbf{E}_x[f(Y_{\tau_{B(x,r)}}); \tau_{B(x,r)} < \zeta];$$

2) superharmonic in O with respect to Y if f is lower semicontinuous in O and for each $x \in O$ and each ball $B(x, r)$ with $\overline{B(x, r)} \subset O$,

$$f(x) \geq \mathbf{E}_x[f(Y_{\tau_{B(x,r)}}); \tau_{B(x,r)} < \zeta].$$

Remark 2.6. (1) If we define $f = 0$ on D^c, clearly

$$\mathbf{E}_x[f(Y_{\tau_{B(x,r)}}); \tau_{B(x,r)} < \zeta] = \mathbf{E}_x[f(Y_{\tau_{B(x,r)}})]$$

and so the definition of harmonicity or superharmonicity can be rephrased in terms of the relation between $f(x)$ and $\mathbf{E}_x[f(Y_{\tau_{B(x,r)}})]$. The latter are consistent with the definitions given in Landkof [17] for symmetric α-stable processes. It is shown in Bogdan, Burdzy and Chen [6] that such a continuous function f in O is harmonic with respect to Y if and only if $(-(-\Delta)^{\alpha/2} + \kappa_D) f = 0$ in O, where κ_D is given by (2.3).

(2) For any C^2-smooth open set U with $\overline{U} \subset D$, let G_U^Y be the Green function of Y in U. From Theorem 2.1(3), we see that G_U^Y is the Green function for the process obtained from the symmetric α-stable process killed upon leaving U through the Feynman-Kac transform $e^{\int_0^t \kappa_D(X_s^U) ds}$. As the potential $1_U \kappa_D$ is bounded, it is in the Kato class of X^U. By Theorem 3.3 and Lemma 3.5 in Chen [8], we have

$$G_U^Y(x, y) \approx G_U(x, y) \approx \min\left\{ \frac{1}{|x - y|^{n-\alpha}}, \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x - y|^n} \right\} \quad \text{for } x, y \in U, \tag{2.7}$$

where $\delta_U(y) = \operatorname{dist}(y, \partial U)$ is the Euclidean distance from point y to the set ∂U. It can be shown in a similar way as those in Chung and Zhao [12] that the Green function $(x, y) \mapsto G_U^Y(x, y)$ is continuous off the diagonal. Using the Lévy system for Y, we conclude that

$$\mathbf{E}_x[\phi(Y_{\tau_U})] = \int_{D\setminus U} \phi(z) K_U(x, z) dz$$

for any bounded Borel measurable function ϕ in D, where

$$K_U(x, z) = \mathcal{A}(n, -\alpha) \int_U \frac{G_U^Y(x, y)}{|y - z|^{n+\alpha}} dy \quad \text{for } x \in U \text{ and } z \in D \setminus U. \tag{2.8}$$

It follows from (2.7) and the calculations done in Chen [7] and Chen and Song [9] that

$$K_U(x, z) \approx \frac{\delta_U(x)^{\alpha/2}}{\delta_U(z)^{\alpha/2} |x - z|^n} \quad \text{for } x \in U \text{ and } z \in D \setminus U. \tag{2.9}$$

(3) If f is a lower semicontinuous function defined on O taking values in $(-\infty, \infty]$, then f is bounded from below on any open subset whose closure is contained in O. Thus for such kind of function f that is integrable on D, it follows from (2.9) that $\mathbf{E}_x[f^-(Y_{\tau_{B(x,r)}})] < \infty$ for any ball $B(x, r)$ with $\overline{B(x, r)} \subset D$. Therefore the expectations in Definition 2.5 are well defined.

(4) Using a similar argument as that for Theorems 2.1 and 2.2 in Chen and Song [10], it can be shown by using Theorem 2.1(3) that an integrable Borel function f on D that is lower semicontinuous in O is superharmonic (harmonic, respectively) in O with respect to Y if and only if for any relatively compact open subset U of O, $f(x) \ge \mathbf{E}_x[f(Y_{\tau_U}); \tau_U < \zeta]$ $(f(x) = \mathbf{E}_x[f(Y_{\tau_U}); \tau_U < \zeta]$, respectively) for every $x \in U$.

Theorem 2.7. *(Theorem 1.2 of Bogdan, Burdzy and Chen [6]) Let D be a bounded $C^{1,1}$ open set in $\mathbf{R}^n$ with characteristics $r_0 \le 1$ and Λ. Let Y be the censored stable process in D with index of stability $\alpha \in (1, 2)$. Let $Q \in \partial D$ and $r \in (0, r_0)$. Assume that u is a nonnegative function on D which vanishes continuously on $\partial D \cap B(Q, r)$ and is harmonic in $D \cap B(Q, r)$ for Y. Then there is a constant $K = K(n, \alpha, \Lambda) > 1$ such that*

$$\frac{u(x)}{u(y)} \le K \frac{\delta_D(x)^{\alpha-1}}{\delta_D(y)^{\alpha-1}} \qquad \textit{for } x, y \in D \cap B(Q, r/2). \tag{2.10}$$

We will also need the following scale-invariant version of Harnack inequality for nonnegative harmonic functions of the censored process Y from Bogdan, Burdzy and Chen [6].

Theorem 2.8. *(Theorem 3.2 in [6]) Let $D \subset \mathbf{R}^n$ be an open set and let Y be the censored process on D. Let $x_1, x_2 \in D$, $r > 0$ with $B(x_1, r) \cup B(x_2, r) \subset D$ and $k \in \{1, 2, \dots\}$, such that $|x_1 - x_2| < 2^k r$. If $u \ge 0$ is harmonic for Y in $B(x_1, r) \cup B(x_2, r)$, then there exists a constant J depending only on n and α, such that*

$$J^{-1} 2^{-k(n+\alpha)} u(x_2) \le u(x_1) \le J 2^{k(n+\alpha)} u(x_2)\,. \tag{2.11}$$

3. Green function estimate

Throughout this section D is a bounded $C^{1,1}$ open set and $\alpha \in (1, 2)$. The key to establish the Green function estimate in Theorem 1.1 is the following interior estimate, which is obtained through a capacity argument. We point out that our capacity argument is different from Stampacchia's approach in [20] for the Green function-capacity comparison result, since the sample paths of the censored stable process Y are discontinuous and therefore the first exit distribution from a smooth domain, unlike that of diffusion processes, does not concentrate on the boundary of the domain.

Theorem 3.1. *The Green function $G(x, y)$ is continuous on $D \times D$ except along the diagonal. For $k > 0$, there is a constant $c = c(D, \alpha, k) > 1$ such that*

$$c^{-1} |x - y|^{\alpha-n} \leq G(x, y) \leq c\,|x - y|^{\alpha-n} \tag{3.1}$$

for $x, y \in D$ satisfying $|x - y| \leq k \min\{\delta_D(x),\ \delta_D(y)\}$.

Proof. By the scaling property of symmetric α-stable process X and the $3G$-estimate in Chen and Song [9],

$$\frac{G_B(x, y)G_B(y, z)}{G_B(x, z)} \leq c(n, \alpha)(|x - y|^{\alpha-n} + |y - z|^{\alpha-n}), \quad x, y, z \in B,$$

for any ball $B \subset D$, where G_B is the Green function of X in B. As $\kappa_D(x) \approx \delta_D(x)^{-\alpha}$, there is a constant $\lambda = \lambda(D, \alpha) \in (0,\ 1/2)$ such that for any $w \in D$ and ball $B = B(w,\ \lambda\delta_D(w))$,

$$\sup_{x,z\in D} \int_B \frac{G_B(x, y)G_B(y, z)}{G_B(x, z)} \kappa_D(y)\, dy < 1/2. \tag{3.2}$$

For any ball $B \subset D$, we will use G_B^Y to denote the Green function of Y in B; that is,

$$\int_B G_B^Y(x, y) f(y)dy = \mathbf{E}_x \left[\int_0^{\tau_B} f(Y_t)dt \right]$$

for Borel $f \geq 0$. Here $\tau_B := \inf\{t > 0 : Y_t \notin B\}$ is the exit time of Y from B. It follows from Theorem 2.1, (3.2) above, Khasminskii's inequality (cf. [12]) and Lemma 3.5 in Chen [8] that

$$\frac{1}{2} G_B(x, y) \leq G_B^Y(x, y) \leq 2G_B(x, y) \tag{3.3}$$

for any $B = B(w, \lambda\delta_D(w))$ and $x, y \in B$. It can be shown in a similar way as those in Chung and Zhao [12] that the Green function $(x, y) \mapsto G_B^Y(x, y)$ is continuous off the diagonal.

We now show that the theorem holds for $k = \lambda/2$ and for $x_0, y_0 \in D$ satisfying $|x_0 - y_0| \leq k \min\{\delta_D(x_0),\ \delta_D(y_0)\}$. In this case there is a ball $B := B(w, \lambda\delta_D(w))$

for some $w \in D$ such that $x_0, y_0 \in B_{1/2} := B(w, \lambda\delta_D(w)/2)$. By the strong Markov property of Y, we have

$$G(x, y) = G_B^Y(x, y) + \mathbf{E}_x \left[G(Y_{\tau_B}, y)\right] \quad \text{for } x, y \in B. \tag{3.4}$$

By (3.3), (2.5) and the scaling property of X, there is a constant $c_1 = c_1(D, \alpha) > 1$ such that

$$c_1^{-1} |x - y|^{\alpha-n} \leq G_B^Y(x, y) \leq c_1 |x - y|^{\alpha-n} \quad \text{for } x, y \in B_{1/2}. \tag{3.5}$$

Note that $h(x, y) := \mathbf{E}_x \left[G(Y_{\tau_B}, y)\right]$ has the property that for each fixed $x \in B$, $y \mapsto h(x, y)$ is harmonic in $B_{2/3} := B(w, 2\lambda\delta_D(w)/3)$ with respect to process Y and for each fixed $y \in B$, $x \mapsto h(x, y)$ is harmonic in $B_{2/3}$ with respect to process Y. So it follows from Harnack inequality, h is bounded and is continuous on $B_{2/3} \times B_{2/3}$ (cf. (3.25) of Bogdan, Burdzy and Chen [6]) and this implies that Green function $G(x, y)$ is continuous on $D \times D$ except along the diagonal. By Theorem 2.8 there is a universal constant $c > 1$ such that

$$c^{-1}h(x_1, y_1) \leq h(x, y) \leq ch(x_1, y_1)$$

for any $x_1, y_1, x, y \in B_{2/3}$ and so

$$\max_{x,y \in B_{2/3}} h(x, y) \leq c \min_{x,y \in B_{2/3}} h(x, y) \leq c \min_{x,y \in B_{2/3}} G(x, y). \tag{3.6}$$

We know from Theorem 2.4 there is a constant $c_3 = c_3(D, \alpha) > 1$ such that

$$c_3^{-1}\mathcal{C}(u, u) \leq \mathcal{E}(u, u) \leq c_3 \mathcal{C}(u, u) \quad \text{for } u \in \mathcal{F}^D.$$

Therefore

$$c_3^{-1}\mathrm{Cap}_{X^D}(A) \leq \mathrm{Cap}_Y(A) \leq c_3\mathrm{Cap}_{X^D}(A) \quad \text{for open set } A. \tag{3.7}$$

Here $\mathrm{Cap}_Y(A)$ is the capacity of the set A with respect to censored stable process Y in D; that is, for open set A,

$$\mathrm{Cap}_Y(A) = \inf \left\{\mathcal{E}(u, u) : \ u \in \mathcal{F}^D \text{ and } u \geq 1 \ m\text{-a.e. on } A\right\}$$

(cf. Fukushima, Oshima and Takeda [13]). $\mathrm{Cap}_{X^D}(A)$ is defined similarly in terms of the Dirichlet form $(\mathcal{C}, \mathcal{F}^D)$ of the killed symmetric α-stable process X^D in D. Let Cap_X denote the capacity for the symmetric α-stable process X in $\mathbf{R}^n$; that is, for open set A:

$$\mathrm{Cap}_X(A) = \inf \left\{\mathcal{C}(u, u) : \ u \in \mathcal{F}^{\mathbf{R}^n} \text{ and } u \geq 1 \ m\text{-a.e. on } A\right\}.$$

Clearly for any ball $B = B(w_0, r) \subset D$, by scaling

$$\mathrm{Cap}_{X^D}(B) \geq \mathrm{Cap}_X(B) = c_1(n, \alpha)r^{n-\alpha}. \tag{3.8}$$

On the other hand, for ball $B = B(w_0, r)$ with $B_2 = B(w_0, 2r) \subset D$, again by scaling we have

$$\mathrm{Cap}_{X^D}(B) \le \mathrm{Cap}_{X^{B_2}}(B) = c_2(n, \alpha) r^{n-\alpha}. \tag{3.9}$$

By (3.7)–(3.9), there is a constant $c_4 = c_4(D, \alpha) > 1$ such that

$$c_4^{-1} r^{n-\alpha} \le \mathrm{Cap}_Y(B) \le c_4\, r^{n-\alpha} \tag{3.10}$$

for any ball $B = B(w_0, r)$ with $B(w_0, 2r) \subset D$. For such ball B, by Theorem 2.1.5 and (2.2.13) of Fukushima, Oshima and Takeda [13] there is an equilibrium measure μ_B supported on $\overline{B}$ such that $G\mu_B \in \mathcal{F}^D$, $G\mu_B = 1$ on B and

$$\mathrm{Cap}_Y(B) = \int_{\overline{B}} G\mu_B(x)\, \mu_B(dx) = \mu_B(\overline{B})$$

Now applying above to ball $B_{1/2} := B(w, \lambda\delta_D(w)/2)$, we have

$$\begin{aligned}\mathrm{Cap}_Y(B_{1/2}) &= \int_D G\mu_{B_{1/2}}(x)\mu_{B_{1/2}}(dx)\\ &\ge \min_{x,y\in \overline{B}_{1/2}} G(x,y) \left(\mu_{B_{1/2}}(\overline{B}_{1/2})\right)^2 = \min_{x,y\in \overline{B}_{1/2}} G(x,y)\, \mathrm{Cap}_Y(B_{1/2})^2.\end{aligned}$$

Hence

$$\min_{x,y\in \overline{B}_{1/2}} G(x,y) \le \mathrm{Cap}_Y(B_{1/2})^{-1} \le c_4(\lambda\delta_D(w)/2)^{\alpha-n}. \tag{3.11}$$

Putting (3.4)–(3.6) and (3.11) together yields that there is a constant $c_5 = c_5(D, \alpha) > 1$ such that

$$c_5^{-1}|x_0 - y_0|^{\alpha-n} \le G(x_0, y_0) \le c_5|x_0 - y_0|^{\alpha-n} \tag{3.12}$$

for any $x_0, y_0 \in D$ satisfying $|x_0 - y_0| \le k_0 \min\{\delta_D(x_0),\ \delta_D(y_0)\}$ with $k_0 := \lambda/2$.

Now for general $k > k_0$ and $x, y \in D$ satisfying

$$k_0 \min\{\delta_D(x),\ \delta_D(y)\} < |x - y| \le k \min\{\delta_D(x),\ \delta_D(y)\},$$

without loss of generality, assume that $\delta_D(x) \ge \delta_D(y)$ and thus $|x - y| \ge k_0\delta_D(y)$. Choose $y_0 \in D$ such that

$$k_0\, |x - y|/(2k) \le |x - y_0| \le k_0 \min\{\delta_D(x),\ \delta_D(y_0)\}.$$

Then by Theorem 2.8, there is a constant $c_6 = c_6(D, \alpha, k) > 1$ such that

$$c_6^{-1}\, G(x, y_0) \le G(x, y) \le c_6\, G(x, y_0).$$

Thus by (3.12) there is a constant $c = c(D, k)$ such that (3.1) holds. □

Lemma 3.2. *For each fixed $y \in D$, there is a constant $c = c(D, \alpha, y) > 1$,*

$$c^{-1}\, \delta_D(x)^{\alpha-1} \le G(x, y) \le c\, \delta_D(x)^{\alpha-1} \quad \text{for } x \in D \setminus B(y, \delta_D(y)/2).$$

In particular, $x \mapsto G(x, y)$ vanishes continuously on ∂D.

Proof. Let $U := D \setminus \overline{B(y, \frac{1}{4}\delta_D(y))}$. For $x \in U$, clearly

$$G(x, y) = \mathbf{E}_x \left[G(Y_{\tau_U}, y) \right].$$

The conclusion of this theorem now follows directly from Theorem 1.2 and Remark 6.2 in Bogdan, Burdzy and Chen [6]. □

Proof of Theorem 1.1. Note that

$$\min\left\{\frac{1}{|x-y|^{n-\alpha}}, \frac{\delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1}}{|x-y|^{n-2+\alpha}}\right\} = \frac{1}{|x-y|^{n-\alpha}} \left(\min\left\{1, \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right\}\right)^{\alpha-1}. \tag{3.13}$$

As $\delta_D(x) \le \delta_D(y) + |x-y|$ for any $x, y \in D$, it is easy to see that

$$\frac{\delta_D(x)\delta_D(y)}{(\max\{\delta_D(x), \delta_D(y), |x-y|\})^2} \approx \min\left\{1, \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right\} \tag{3.14}$$

Define $r(x, y) = \max\{\delta_D(x), \delta_D(y), |x-y|\}$. Choose $z_0 \in D$ and $0 < R < r_0$ so that $B(z_0, R) \in D$ and $B(z_0, R)$ is disjoint from any interior tangential ball in D with radius less than R. Fix $x_1 \in \partial B(z_0, R/2)$ and define

$$A(x, y) = \begin{cases} B(x_0 + r(x, y)\mathbf{n}_{x_0}, r(x, y)/2) \cup B(y_0 + r(x, y)\mathbf{n}_{y_0}, r(x, y)/2), & \text{if } r(x, y) < R/5 \\ \{x_1\}, & \text{otherwise.} \end{cases}$$

where $\mathbf{n}_Q$ is unit inward normal vector at $Q \in \partial D$, x_0 and y_0 are points in ∂D such that $\delta_D(x) = |x_0 - x|$ and $\delta_D(y) = |y_0 - y|$. It is clear that

$$\frac{r(x, y)}{2} \le \delta_D(z) \le \frac{3r(x, y)}{2} \tag{3.15}$$

for every $z \in A(x, y)$. On the other hand, by Lemma 3.2,

$$\phi(x) := \min\left\{G(z_0, x), \max_{y \in D:\, |y-z_0| \ge R} G(z_0, y)\right\} \approx \delta_D(x)^{\alpha-1}. \tag{3.16}$$

In particular, by (3.15), for $z \in A(x, y)$,

$$\phi(z) \approx r(x, y)^{\alpha-1} = (\max\{\delta_D(x), \delta_D(y), |x-y|\})^{\alpha-1} \quad \text{for } z \in A(x, y). \tag{3.17}$$

In view of (3.13)–(3.17), it suffices to show that

$$G(x, y) \approx \frac{\phi(x)\phi(y)}{\phi^2(z)} |x-y|^{\alpha-n}, \qquad \text{where } z \in A(x, y). \tag{3.18}$$

Thanks to Theorems 2.7, 2.8 and 3.1, the proof of (3.18) is almost the same as the proof of Proposition 6 and Theorem 2 in Bogdan [5]. In fact here the proof can be simplified quite a bit due to the fact that D is a $C^{1,1}$ open set and the stronger version of boundary Harnack principle and Harnack principle in Theorems 2.7 and 2.8. Clearly the Green function of Y is domain translation invariant. By Remark 2.3, the constant $c = c(D, \alpha)$ can be chosen to be domain dilation invariant. □

The following is a direct consequence of Theorem 1.1.

Corollary 3.3.

$$\lim_{x \to \partial D} G(x, y) = 0$$

uniformly on $D_r = \{y \in D : \delta_D(y) \ge r\}$.

4. Martin boundary and Martin kernel estimates

Throughout this section D is a bounded $C^{1,1}$ open set in $\mathbf{R}^n$, $\alpha \in (1, 2)$, and Y is the censored α-stable process in D. We will show in this section that the Martin boundary and the minimal Martin boundary of Y can all be identified with the Euclidean boundary ∂D of D. Our approach in this section is similar to the one used in Chen and Song [10] for symmetric α-stable processes.

Fix $x_0 \in D$ and define

$$M(x, y) = \frac{G(x, y)}{G(x_0, y)} \quad x, y \in D.$$

Recall that r_0 is the characteristic radius of D defined at the end of the paragraph following Theorem 2.2.

Theorem 4.1. *For every $x \in D$ and $z \in \partial D$, $M(x, z) := \lim_{y \to z} M(x, y)$ exists, which is called the Martin kernel of Y. The Martin kernel $M(x, z)$ is a continuous function on $D \times \partial D$. Furthermore there exists $c := c(x_0, D, \alpha)$ such that*

$$\frac{1}{c} \frac{\delta_D(x)^{\alpha-1}}{|x - z|^{n-2+\alpha}} \le M(x, z) \le c \frac{\delta_D(x)^{\alpha-1}}{|x - z|^{n-2+\alpha}}.$$

Proof. For every $0 < \varepsilon < (r_0 \wedge \delta_D(x_0))/4$, define $D_\varepsilon := \{y \in D : \delta_D(y) < \varepsilon\}$ and $U_\varepsilon := \{x \in D : \delta_D(x) > 2\varepsilon\}$. By Theorem 1.1, for $x \in U_\varepsilon$, both function $y \mapsto G(x, y)$ and function $y \mapsto G(x_0, y)$ are bounded and harmonic in D_ε with respect to Y, and vanish continuously on ∂D. So by Remark 6.4 in Bogdan, Burdzy and Chen [6], $M(x, z) := \lim_{y \to z} M(x, y)$ exists for every $z \in \partial D$ and $x \in U_\varepsilon$, and there are positive constants $\sigma = \sigma(D, \alpha, x_0, \varepsilon)$ and $c = c(D, \alpha, x_0, \varepsilon)$ such that

$$|M(x, y) - M(x, z)| \le c\, |y - z|^\sigma \tag{4.1}$$

for every $x \in U_\varepsilon$, $z \in \partial D$ and $y \in D \cap B(z, \varepsilon)$. This implies that $M(x, z)$ is jointly continuous on $U_\varepsilon \times \partial D$, and hence on $D \times \partial D$ after letting $\varepsilon \downarrow 0$. The Martin kernel estimate is an immediate consequence of the Green function estimate in Theorem 1.1. □

Theorem 4.1 in particular implies that $M(\cdot, z_1)$ differs from $M(\cdot, z_2)$ if z_1 and z_2 are two different points on ∂D.

Theorem 4.2. *For every $z \in \partial D$, function $x \mapsto M(\cdot, z)$ is harmonic in D with respect to Y.*

Proof. According to Chen and Song [10], it suffices to show that for every $x \in D$ there exists $R_0 = R_0(x) < \delta_D(x)$ such that

$$M(x, z) = \mathbf{E}_x[M(Y_{\tau_{B(x,r)}}, z)] \quad \text{for every } 0 < r < R_0. \tag{4.2}$$

Let $R_0 := \lambda\delta_D(x)$, where $\lambda < 1/2$ is the constant in (3.2), and for $r \in (0, R_0)$, let $r_1 := \frac{\delta_D(x)-r}{3}$. For $m \geq 1$, set $z_m := z + \frac{r_1}{m}\mathbf{n}_z$. As D is bounded and $C^{1,1}$ smooth with characteristics (r_0, Λ), there is an integer $m_0 > 1$ such that

$$B(z_m, r_1/(2m)) \subset B(z, 3r_1/(2m)) \cap D \subset B(z, 3r_1/m) \cap D \not\subseteq\subset B(z, r_0) \cap D \\ \subset D \setminus B(x, r)$$

for all $m \geq m_0$. Since $M(\cdot, z_m)$ is harmonic in $D \setminus \{z_m\}$ with respect to Y for every $m \geq m_0$, we have

$$M(x, z_m) = \mathbf{E}_x[M(Y_{\tau_{B(x,r)}}, z_m)].$$

By Fatou's lemma,

$$\mathbf{E}_x[M(Y_{\tau_{B(x,r)}}, z)] = \mathbf{E}_x\left[\lim_{m\to\infty} M(Y_{\tau_{B(x,r)}}, z_m)\right] \leq \liminf_{m\to\infty} M(x, z_m) \\ = M(x, z) < \infty.$$

Hence $M(Y_{\tau_{B(x,r)}}, z)$ is $\mathbf{P}_x$-integrable.

On the other hand, by the boundary Harnack principle in Theorem 2.7 with $B(z, 3r_1/m)$ in place of $B(Q, r)$ there, there is a constant $c_1 > 0$ such that for every $w \in D \setminus B(z, 3r_1/m)$ and $y \in D \cap B(z, 3r_1/(2m))$,

$$M(w, z_m) = \frac{G(w, z_m)}{G(x_0, z_m)} \leq c_1 \frac{G(w, y)}{G(x_0, y)} = c_1 M(w, y), \quad m \geq m_0.$$

Letting $y \to z \in \partial D$ yields

$$M(w, z_m) \leq c_1 M(w, z) \quad m \geq m_0, \tag{4.3}$$

for every $w \in D \setminus B(z, 3r_1/m)$.

To prove (4.2), it suffices to show that $\{M(Y_{\tau_{B(x,r)}}, z_m) : m \geq m_0\}$ is $\mathbf{P}_x$-uniformly integrable. Since $M(Y_{\tau_{B(x,r)}}, z)$ is $\mathbf{P}_x$-integrable, for any $\varepsilon > 0$, there is an $N_0 > 1$ such that

$$\mathbf{E}_x\left[M(Y_{\tau_{B(x,r)}}, z); M(Y_{\tau_{B(x,r)}}, z) > N_0/c_1\right] < \varepsilon/4c_1. \tag{4.4}$$

Note that by (4.3) and (4.4)

$$\mathbf{E}_x[M(Y_{\tau_{B(x,r)}}, z_m);\ M(Y_{\tau_{B(x,r)}}, z_m) > N_0 \text{ and } Y_{\tau_{B(x,r)}} \in D \setminus B(z, 3r_1/m)] \\ \leq c_1\mathbf{E}_x[M(Y_{\tau_{B(x,r)}}, z);\ c_1 M(Y_{\tau_{B(x,r)}}, z) > N_0] \\ < c_1\varepsilon/4c_1 = \varepsilon/4.$$

It follows from the Lévy system for process Y, that the distribution of $Y_{\tau_{B(x,r)}}$ under $\mathbf{P}_y$ is absolutely continuous with respect to the Lebesgue measure on $D \setminus B(x, r)$ and its density function $K^Y_{B(x,r)}(y, z)$ is

$$K^Y_{B(x,r)}(y, z) = \int_{B(x,r)} \frac{G^Y_{B(x,r)}(y, w)}{|w - z|^{n+\alpha}} dw, \quad y \in B(x, r),\ z \in D \setminus B(x, r), \tag{4.5}$$

where $G^Y_{B(x,r)}$ is the Green function of Y in $B(x,r)$. Similarly, the density $K_{B(x,r)}(y,z)$ for the exit distribution $X^D_{\tau_{B(x,r)}}$ for symmetric α-stable process X^D in D starting from $y \in B(x,r)$ is given by

$$K_{B(x,r)}(y,z) = \int_{B(x,r)} \frac{G_{B(x,r)}(y,w)}{|w-z|^{n+\alpha}} dw, \quad z \in D \setminus B(x,r), \tag{4.6}$$

It follows from Chen and Song [9] and Chen [7] that

$$K_{B(x,r)}(y,z) \le c \frac{\delta_{B(x,r)}(y)^{\alpha/2}}{\delta_{B(x,r)}(z)^{\alpha/2}(1+\delta_{B(x,r)}(z)/r)^{\alpha/2}|y-z|^n} \tag{4.7}$$

for $y \in B(x,r)$ and $z \in D \setminus \overline{B(x,r)}$. Note that $r < \lambda\, \delta_D(x)$. So by (3.3), (4.5)–(4.7), for $m \ge 2$,

$$\begin{aligned}
\mathbf{E}_x & [M(Y_{\tau_{B(x,r)}}, z_m);\ Y_{\tau_{B(x,r)}} \in D \cap B(z, 3r_1/m)] \\
&\le 2\mathbf{E}_x[M(X^D_{\tau_{B(x,r)}}, z_m) 1_{D \cap B(z,3r_1/m)}(X^D_{\tau_{B(x,r)}})] \\
&\le c \int_{B(z,3r_1/m)} M(w, z_m) dw \\
&= c\, G(x_0, z_m)^{-1} \int_{B(z,3r_1/m)} G(w, z_m) dw \\
&\le c\, (r_1/m)^{1-\alpha} \int_{B(z,3r_1/m)} |w - z_m|^{\alpha - n} dw \\
&\le c\, r_1/m.
\end{aligned}$$

In the second to last inequality we used the Green function estimate in Theorem 1.1; in particular its lower bound estimate implies that

$$G(x_0, z_m) \ge c^{-1} \min\left\{ \frac{1}{|x_0 - z_m|^{n-\alpha}}, \frac{\delta_D(x_0)^{\alpha-1}\delta_D(z_m)^{\alpha-1}}{|x_0 - z_m|^{n-2+\alpha}} \right\} \ge c^{-1}\, (r_1/m)^{\alpha-1}$$

Therefore by taking N large enough we have for $m \ge N$,

$$\begin{aligned}
& \mathbf{E}_x[M(Y_{\tau_{B(x,r)}}, z_m);\ M(Y_{\tau_{B(x,r)}}, z_m) > N] \\
&\quad \le \mathbf{E}_x[M(Y_{\tau_{B(x,r)}}, z_m);\ Y_{\tau_{B(x,r)}} \in D \cap B(z, 3r_1/m)] \\
&\qquad + \mathbf{E}_x[M(Y_{\tau_{B(x,r)}}, z_m);\ M(Y_{\tau_{B(x,r)}}, z_m) > N \text{ and } Y_{\tau_{B(x,r)}} \in D \setminus B(z, 3r_1/m)] \\
&\quad < c\, r_1/m + \varepsilon/4 \\
&\quad < \varepsilon.
\end{aligned}$$

As each $M(Y_{\tau_{B(x,r)}}, z_m)$ is $\mathbf{P}_x$-integrable, we conclude that $\{M(Y_{\tau_{B(x,r)}}, z_m) : m \ge 1\}$ is uniformly integrable under $\mathbf{P}_x$. □

Lemma 4.3. *If h is positive harmonic function with respect to Y and continuous on $\overline{D}$, then* $\sup_{x \in \overline{D}} h(x) = \sup_{x \in \partial D} h(x)$.

Proof. Take an increasing sequence of smooth domain $\{D_m\}_{m\geq 1}$ such that $\overline{D_m} \subset D_{m+1}$ and $\cup_{m=1}^{\infty} D_m = D$. By bounded convergence theorem, we have

$$h(x) = \lim_{m\to\infty} \mathbf{E}_x \left[h(Y_{\tau_{D_m}}) \right] = \mathbf{E}_x \left[h\big(\lim_{t\uparrow\zeta} Y_t \big) \right].$$

Therefore, $\sup_{x\in\overline{D}} h(x) = \sup_{x\in\partial D} h(x)$. □

As process Y satisfies Hypothesis (B) in Kunita and Watanabe [16], the process Y has a Martin boundary. A consequence of Theorem 4.1 is that the Martin boundary of Y can be identified with the Euclidean boundary ∂D of D. We know from the general theory in Kunita and Watanabe [16] that non-negative superharmonic functions with respect to Y admit a Martin representation. That is, for every superharmonic function $u \geq 0$ with respect to Y, there is a unique Radon measure μ in D and a finite measure ν on ∂D such that

$$u(x) = \int_D G(x, y)\mu(dy) + \int_{\partial D} M(x, z)\nu(dz). \tag{4.8}$$

Furthermore, u is harmonic if and only if the measure $\mu = 0$. The above can also be proved directly by adapting the proofs in Bass [3] for the Martin representation of the classical superharmonic functions for Brownian motions.

Theorem 4.4. *For each $z \in \partial D$, $M(x, z)$ is minimal harmonic. That is, if h is a harmonic function with respect to Y and $h(x) \leq M(x, z)$, then $h(x) = cM(x, z)$ for some constant $c \leq 1$.*

Proof. Suppose that Martin kernel $x \mapsto M(x, z_0)$ is not minimal for some $z_0 \in \partial D$. Then there is a non-trivial harmonic function $h \geq 0$ of Y in D such that $h(x) \leq M(x, z_0)$ but h is not a constant multiple of $M(x, z_0)$. By Martin representation (4.8), there is a finite measure μ on ∂D which is not concentrated at z_0 such that

$$h(x) = \int_{\partial D} M(x, w)\mu(dw) \quad \text{for } x \in D.$$

Thus there exists $\varepsilon > 0$ such that $\mu_\varepsilon := \mu|_{\partial D \setminus B(z_0,\varepsilon)}$ is non-trivial. By Theorem 4.2 and Fubini Theorem, $\int M(x, w)\mu_\varepsilon(dw)$ is a harmonic function of Y that is bounded by $h(x)$ and hence by $M(x, z_0)$. By the Martin kernel estimate in Theorem 4.1,

$$\lim_{x\to z} \int M(x, w)\mu_\varepsilon(dw) = 0 \quad \text{for } z \in \partial D \cap B(z_0, \varepsilon/2).$$

On the other hand, for $z \in \partial D \setminus B(z_0, \varepsilon/2)$,

$$\begin{aligned}\lim_{x\to z} \int M(x, w)\mu_\varepsilon(dw) &\leq \lim_{x\to z} \int M(x, w)\mu(dw) = \lim_{x\to z} h(x) \\ &\leq \lim_{x\to z} M(x, z_0) = 0.\end{aligned}$$

Thus the harmonic function $x \mapsto \int_{\partial D} M(x, w)\mu_\varepsilon(dw)$ vanishes continuously on ∂D so by Lemma 4.3,

$$\int_{\partial D} M(x, w)\mu_\varepsilon(dw) = 0 \quad \text{for } x \in D.$$

This is impossible as μ_ε is non-trivial and $M(x, w) > 0$. So $x \mapsto M(x, z_0)$ has to be a minimal harmonic function of Y in D. $\square$

The above theorem implies that every boundary point $z \in \partial D$ is a minimal Martin boundary point of Y, which completes the proof of Theorem 1.2.

References

1. Bliedtner, J., Hansen, W.: *Potential Theory: An Analytic and Probabilistic Approach to Balayage*. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1986
2. Blumenthal, R.M., Getoor, R.K.: *Markov Processes and Potential Theory*. Academic Press, 1968
3. Bass, R.F.: *Probabilistic Techniques in Analysis*. Springer-Verlag, 1995
4. Bogdan, K.: The boundary Harnack principle for the fractional Laplacian. *Studia Math.* **123**, 43–80 (1997)
5. Bogdan, K.: Sharp estimates for the Green function in Lipschitz domains. *J. Math. Anal. Appl.* **243**, 326–337 (2000)
6. Bogdan, K., Burdzy, K., Chen, Z.-Q.: Censored stable processes. Preprint 2001
7. Chen, Z.-Q.: Multidimensional symmetric stable processes. *Korean J. Comput. Appl. Math.* **6**, 227–266 (1999)
8. Chen, Z.-Q.: Gaugeability and conditional gaugeability. Preprint, 2001. To appear in *Trans. Amer. Math. Soc.*
9. Chen, Z.-Q., Song, R.: Estimates on Green functions and Poisson kernels of symmetric stable processes. *Math. Ann.* **312**, 465–601 (1998)
10. Chen, Z.-Q., Song, R.: Martin boundary and integral representation for harmonic functions of symmetric stable processes. *J. Funct. Anal.* **159**, 267–294 (1998)
11. Chen, Z.-Q., Song, R.: Hardy inequality for censored stable processes. Preprint 2001 To appear in Tohoku Math. J.
12. Chung, K.L., Zhao, Z.: *From Brownian motion to Schrödinger's equation*. Berlin: Springer, 1995
13. Fukushima, M., Oshima, Y., Takeda, M.: *Dirichlet Forms and Symmetric Markov Processes*. Walter de Gruyter, Berlin, 1994
14. Janicki, A., Weron, A.: *Simulation and Chaotic Behavior of α-Stable Processes*. Dekker, 1994
15. Kulczycki, T.: Properties of Green function of symmetric stable processes. *Probab. Math. Stat.* **17**, 339–364 (1997)
16. Kunita, H., Watanabe, T.: Markov processes and Martin boundaries. *Ill. J. Math.* **9**, 485–526 (1965)
17. Landkof, N.S.: *Foundations of Modern Potential Theory*. Springer-Verlag, Berlin-Heidelberg-New York, 1972
18. Samorodnitsky, G., Taqqu, M.S.: *Stable Non-Gaussian Random Processes*. Chapman & Hall, New York-London, 1994
19. Sharpe, M.: *General theory of Markov processes*. Academic Press, 1988
20. Stampacchia, G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier, Grenoble*, **15**, 189–258 (1965)