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Effect of the regular term on the stress field in a joint of dissimilar materials under remote mechanical load

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Summary For most geometries and material combinations, stresses at the intersection of outer edges and the interface of a joint of two dissimilar materials are singular for elastic response under mechanical or thermal loads. Near the singular point the stresses can be described by a sum of singular terms and one regular term, which is independent of the distance from the singular point. Earlier investigations have shown that the regular term is also important in the description of the singular field. For thermal loading, for instance, there is a homogeneous temperature change in the joint; the regular stress term is nonzero. For a remote mechanical loading, the regular stress term is always nonzero for some geometries and material combinations. One important case is a joint with an interface crack, in which the so called T-stress term is the regular term and always nonzero.

In this paper, the regular stress term will be determined for a joint under a remote mechanical load. The conditions of a joint with a nonzero regular term and the formulas to calculate the regular stress term will be presented both for an arbitrary joint geometry (θ_1, θ_2) and for some special cases. Examples will be introduced, to show the contribution of the regular term to the stress distribution near the singular point.

Key words Regular stress term, stress singularity, interface crack, stress intensity factor, asymptotic solution

1 Introduction

For most geometries and material combinations, stresses at the intersection of outer edges and the interface of a joint of two dissimilar materials are singular assuming elastic material behavior under mechanical or thermal load. In many investigations, the characteristics of the singular term were described, [1–9]. However, in many cases a term, which is independent of the distance from the singular point – called regular stress term – is also important for the description of the stress field even close to the singular point. To assess the stress field and develop failure criteria, both terms – the singular and the regular one – have to be considered, [10]. Regular stress terms for joints with free outer edges under thermal loading [11–13] and for joints with edge tractions [14] have been given in an explicit form for an arbitrary geometry (θ_1, θ_2), Fig. 1. For a joint under thermal loading or a joint with edge tractions, the regular stress term is always nonzero. For a joint under remote mechanical load, the regular stress term is zero for most joint geometries and material combinations. In this paper, the regular stress term will be studied for a joint under remote mechanical loading. The method and the explicit form to calculate the regular stress term will be given for a joint with an arbitrary geometry, Fig. 1, and simplified for several special geometries, which often appear in the engineering structures. Examples will be introduced to show the contribution of the regular stress term to the stress distribution near the singular point. The significance of applying a regular term is due to the fact that both the singular and the regular terms are necessary to describe the stress field near the singular point in a large range, which is the scale of practical interest.

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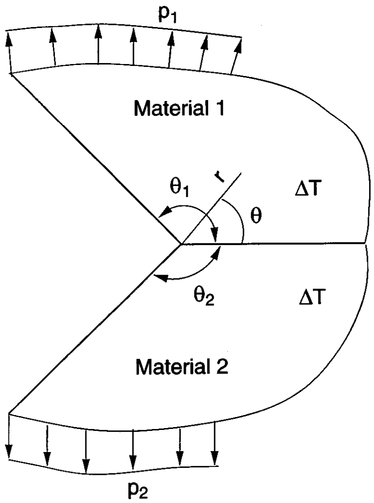

Fig. 1. General joint geometry and the coordinates

2 Basic equations

To study the regular stress term, the following Airy's stress function [11]

$$\Phi_k(r,\theta) = r^2[\mathscr{A}_{k0}\theta + \mathscr{B}_{k0} + \mathscr{C}_{k0}\sin(2\theta) + \mathscr{D}_{k0}\cos(2\theta)] \ , \tag{1}$$

is used in polar coordinates, where the coefficients $\mathscr{A}_{k0}, \mathscr{B}_{k0}, \mathscr{C}_{k0}, \mathscr{D}_{k0}$ are unknown (for the coordinates see Fig. 1). The index for material 1 is $k = 1$ and $k = 2$ for material 2. The regular stress term can be calculated as

$$\sigma_{rrk}(r,\theta) = 2[\mathscr{A}_{k0}\theta + \mathscr{B}_{k0} - \mathscr{C}_{k0}\sin(2\theta) - \mathscr{D}_{k0}\cos(2\theta)] \ , \tag{2}$$

$$\sigma_{\theta\theta k}(r,\theta) = 2[\mathscr{A}_{k0}\theta + \mathscr{B}_{k0} + \mathscr{C}_{k0}\sin(2\theta) + \mathscr{D}_{k0}\cos(2\theta)] \ , \tag{3}$$

$$\sigma_{r\theta k}(r,\theta) = -2\left[\frac{1}{2}\mathscr{A}_{k0} + \mathscr{C}_{k0}\cos(2\theta) - \mathscr{D}_{k0}\sin(2\theta)\right] \ , \tag{4}$$

which is independent of the distance r from the singular point.

Following the relations between stress, strain and displacement, the displacements for the plane stress can be determined by

$$u_k(r,\theta) = \frac{2r}{E_k}[\mathscr{A}_{k0}\theta(1-\nu_k) + \mathscr{B}_{k0}(1-\nu_k) - \mathscr{C}_{k0}(1+\nu_k)\sin(2\theta) - \mathscr{D}_{k0}(1+\nu_k)\cos(2\theta)] \ , \tag{5}$$

$$v_k(r,\theta) = \frac{2r}{E_k}[-\mathscr{C}_{k0}(1+\nu_k)\cos(2\theta) + \mathscr{D}_{k0}(1+\nu_k)\sin(2\theta)] + \mathscr{F}_{k0}r - \frac{4\mathscr{A}_{k0}}{E_k}r\ln(r) \ , \tag{6}$$

where $u = 0$ and $v = 0$ at $r = 0$. The coefficient $\mathscr{F}_{k0}$ is an unknown. To determine the coefficients $\mathscr{A}_{k0}, \mathscr{B}_{k0}, \mathscr{C}_{k0}, \mathscr{D}_{k0}$ and $\mathscr{F}_{k0}$, the boundary conditions have to be used. For a joint with free edges, the boundary conditions are:
at the interface

$$\begin{aligned} &u_1(r,0) = u_2(r,0), \quad v_1(r,0) = v_2(r,0), \\ &\sigma_{\theta\theta 1}(r,0) = \sigma_{\theta\theta 2}(r,0), \quad \sigma_{r\theta 1}(r,0) = \sigma_{r\theta 2}(r,0) \ , \end{aligned} \tag{7}$$

for the free edges

$$\begin{aligned} &\sigma_{\theta\theta 1}(r,\theta_1) = 0, \quad \sigma_{\theta\theta 2}(r,\theta_2) = 0, \\ &\sigma_{r\theta 1}(r,\theta_1) = 0, \quad \sigma_{r\theta 2}(r,\theta_2) = 0 \ , \end{aligned} \tag{8}$$

where a perfect bond at the interface is assumed.

The regular stress term should also satisfy the boundary conditions. From Eqs. (3)–(8) and due to r being arbitrary, the following equations can be obtained:

$$2g[\mathscr{B}_{10}(1-\nu_1)-\mathscr{D}_{10}(1-\nu_1)]-2[\mathscr{B}_{20}(1-\nu_2)-\mathscr{D}_{20}(1-\nu_2)]=0 \ , \tag{9}$$

$$2g[-\mathscr{C}_{10}(1+\nu_1)]-2[-\mathscr{C}_{20}(1+\nu_2)]+E_2(\mathscr{F}_1-\mathscr{F}_{20})=0 \ , \tag{10}$$

$$g\mathscr{A}_{10}-\mathscr{A}_{20}=0 \ , \tag{11}$$

$$(\mathscr{B}_{10}+\mathscr{D}_{10})-(\mathscr{B}_{20}+\mathscr{D}_{20})=0 \ , \tag{12}$$

$$(\mathscr{A}_{10}+2\mathscr{C}_{10})-(\mathscr{A}_{20}+2\mathscr{C}_{20})=0 \ , \tag{13}$$

$$\mathscr{A}_{10}\theta_1+\mathscr{B}_{10}+\mathscr{C}_{10}\sin(2\theta_1)+\mathscr{D}_{10}\cos(2\theta_1)=0 \ , \tag{14}$$

$$\mathscr{A}_{20}\theta_2+\mathscr{B}_{20}+\mathscr{C}_{20}\sin(2\theta_2)+\mathscr{D}_{20}\cos(2\theta_2)=0 \ , \tag{15}$$

$$\mathscr{A}_{10}+2\mathscr{C}_{10}\cos(2\theta_1)-2\mathscr{D}_{10}\sin(2\theta_1)=0 \ , \tag{16}$$

$$\mathscr{A}_{20}+2\mathscr{C}_{20}\cos(2\theta_2)-2\mathscr{D}_{20}\sin(2\theta_2)=0 \ , \tag{17}$$

where $g=E_2/E_1$. By solving Eqs. (9, 11–17), coefficients $\mathscr{A}_{k0},\mathscr{B}_{k0},\mathscr{C}_{k0},\mathscr{D}_{k0}$ can be determined. From Eq. (10), the difference $\mathscr{F}_{10}-\mathscr{F}_{20}$ can be obtained. The absolute values of $\mathscr{F}_{10}$ and $\mathscr{F}_{20}$ depend on the overall geometry of the component, and have to be determined numerically. For the regular stress term, only the coefficients $\mathscr{A}_{k0},\mathscr{B}_{k0},\mathscr{C}_{k0},\mathscr{D}_{k0}$ are interesting. Equations (9, 11–17) can be rewritten in a matrix form as

$$[A_0]_{8\times8}\{X_0\}_{8\times1}=\{0\}_{8\times1} \ , \tag{18}$$

where $\{X_0\}_{8\times1}=\{\mathscr{A}_{10},\mathscr{B}_{10},\mathscr{C}_{10},\mathscr{D}_{10},\mathscr{A}_{20},\mathscr{B}_{20},\mathscr{C}_{20},\mathscr{D}_{20}\}^T$ is unknown. Let $[A_0]_{8\times8}$ be the coefficient matrix, in which material properties (E_k,ν_k) and geometry angles (θ_1,θ_2) are included. Equation (18) is a homogeneous system. The condition of Eq. (18) having a nonzero solution applies if and only if

$$\mathrm{Det}([A_0]_{8\times8})=0 \ , \tag{19}$$

is satisfied. This means that if $\mathrm{Det}([A_0]_{8\times8})\neq0$, the regular stress term is always zero. In the following the explicit form of $\mathrm{Det}([A_0]_{8\times8})$ will be given for an arbitrary geometry and for several special geometries, which often appear in the engineering structures, e.g. (a) $\theta_1=-\theta_2$; (b) $\theta_1-\theta_2=\pi$; (c) $\theta_1-\theta_2=2\pi$; (d) $\theta_1=\pi$ and θ_2 is arbitrary. For the case of $\mathrm{Det}([A_0]_{8\times8})=0$, the nonzero coefficients $\mathscr{A}_{k0},\mathscr{B}_{k0},\mathscr{C}_{k0},\mathscr{D}_{k0}$ can be determined with an arbitrary constant, which will be given in the next Section.

3 Determination of the regular stress term

In this Section, the conditions of the regular stress term being nonzero and the relations to determine the corresponding nonzero regular stress term will be presented for an arbitrary joint geometry and some special geometries.

3.1 A joint with an arbitrary geometry

For a joint with an arbitrary geometry θ_1,θ_2, the determinant of $[A_0]_{8\times8}$ is

$$\begin{aligned}\mathrm{Det}([A_0]_{8\times8})=\frac{32}{(1+\alpha)^2}\{&1-\cos[2(\theta_1-\theta_2)]+(\theta_2-\theta_1)\sin[2(\theta_1-\theta_2)]\\&+\alpha^2[-1+2\cos(2\theta_1)-\cos[2(\theta_1-\theta_2)]+2\cos(2\theta_2)\\&-2\cos[2(\theta_1+\theta_2)]-(\theta_1+\theta_2)\sin[2(\theta_1+\theta_2)]]\\&+\alpha\beta[4-4\cos(2\theta_1)+2\cos[2(\theta_1-\theta_2)]-4\cos(2\theta_2)\\&+2\cos[2(\theta_1+\theta_2)]-2\theta_1\sin(2\theta_1)+(\theta_1-\theta_2)\sin[2(\theta_1-\theta_2)]\\&-2\theta_2\sin(2\theta_2)+(\theta_1+\theta_2)\sin[2(\theta_1+\theta_2)]]\end{aligned}$$

$$+ \alpha[-2\cos(2\theta_1) + 2\cos(2\theta_2) - (\theta_1 + \theta_2)\sin[2(\theta_1 - \theta_2)]$$
$$+ (-\theta_1 + \theta_2)\sin[2(\theta_1 + \theta_2)]]$$
$$+ \beta[-2\theta_1 \sin(2\theta_1) + (\theta_1 + \theta_2)\sin[2(\theta_1 - \theta_2)]$$
$$+ 2\theta_2 \sin(2\theta_2) + (\theta_1 - \theta_2)\sin[2(\theta_1 + \theta_2)]]\} \ , \quad (20)$$

where α and β are the Dundurs' parameters, which are defined as

$$\alpha = \frac{m_2 - km_1}{m_2 + km_1}, \qquad \beta = \frac{(m_2 - 2) - k(m_1 - 2)}{m_2 + km_1} \ ,$$

with

$$m = \begin{cases} \frac{4}{(1+\nu)} & \text{for plane stress} \\ 4(1-\nu) & \text{for plane strain} \end{cases} \ ,$$

and $k = G_2/G_1$, where G is the shear modulus. From Eq. (20) we can see that, in general, $\mathrm{Det}([A_0]_{8\times 8})$ is nonzero, i.e. the regular stress term is zero. For a given geometry with θ_1 and θ_2, however, material combinations with α and β exist, which lead to $\mathrm{Det}([A_0]_{8\times 8}) = 0$. This can be seen if Eq. (20) is rewritten as

$$\mathrm{Det}([A_0]) = f_0(\theta_1, \theta_2) + \alpha^2 f_1(\theta_1, \theta_2) + \alpha f_2(\theta_1, \theta_2) + \alpha\beta f_3(\theta_1, \theta_2) + \beta f_4(\theta_1, \theta_2) \ . \quad (21)$$

For given θ_1 and θ_2, the solution of $\mathrm{Det}([A_0]) = 0$ is

$$\beta = -\frac{f_0(\theta_1, \theta_2) + \alpha^2 f_1(\theta_1, \theta_2) + \alpha f_2(\theta_1, \theta_2)}{\alpha f_3(\theta_1, \theta_2) + f_4(\theta_1, \theta_2)} \ . \quad (22)$$

In the Dundurs' diagram, in which α is plotted versus β, this represent a curve, which shall be seen clearly in the next Sections for some special joint geometries. In fact, for a given geometry with θ_1 and θ_2, there are infinite material combinations with E_1, E_2, ν_1, ν_2, which satisfy Eq. (22), so that the regular stress term is nonzero.

On the other hand, for a given material combination with α and β, one or more geometries with θ_1 and θ_2 exist, which lead to $\mathrm{Det}([A_0]_{8\times 8}) = 0$.

For the case of $\mathrm{Det}([A_0]_{8\times 8}) = 0$, the nonzero coefficients $\mathscr{A}_{k0}, \mathscr{B}_{k0}, \mathscr{C}_{k0}, \mathscr{C}_{k0}$ are not unique. They are normalized as

$$\mathscr{A}_{k0} = K_0 \mathscr{A}^*_{k0}, \quad \mathscr{B}_{k0} = K_0 \mathscr{B}^*_{k0}, \quad \mathscr{C}_{k0} = K_0 \mathscr{C}^*_{k0}, \quad \mathscr{D}_{k0} = K_0 \mathscr{D}^*_{k0} \ , \quad (23)$$

where K_0 is an unknown constant and $\mathscr{A}^*_{k0}, \mathscr{B}^*_{k0}, \mathscr{C}^*_{k0}, \mathscr{D}^*_{k0}$ can be determined from

$$\mathscr{A}^*_{10} = 2(1+\alpha)\{1 - \alpha + 2\beta - 2\beta\cos(2\theta_1) + (-1+\beta)\cos[2(\theta_1 - \theta_2)] + 2(\alpha - \beta)\cos(2\theta_2)$$
$$+ (-\alpha + \beta)\cos[2(\theta_1 + \theta_2)]\} \ , \quad (24)$$

$$\mathscr{B}^*_{10} = \{-2(1+\alpha)(1 - \alpha + 2\beta)\theta_1 + 2(1-\alpha)(1+\beta)\theta_2 \cos[2(\theta_1 - \theta_2)]$$
$$+ 4(1+\alpha)(\beta - \alpha)\theta_1 \cos(2\theta_2) + 2(1-\alpha)(\alpha - \beta)\theta_2 \cos[2(\theta_1 + \theta_2)]$$
$$+ [4\alpha(1-\alpha) - 2(1-3\alpha)\beta]\sin(2\theta_1) + [1 - \alpha + 2\alpha^2 + (1-3\alpha)\beta]\sin[2(\theta_1 - \theta_2)]$$
$$+ (1-3\alpha)(\beta - \alpha)\sin[2(\theta_1 + \theta_2)]\} \ , \quad (25)$$

$$\mathscr{C}^*_{10} = \{4\alpha(\alpha - 1) + 2(1-3\alpha)\beta + [1 - \alpha^2 - 2(1-3\alpha)\beta]\cos(2\theta_1)$$
$$+ [\alpha(1+\alpha) + (1-3\alpha)\beta]\cos[2(\theta_1 - \theta_2)]$$
$$+ [-1 + 2\alpha - 5\alpha^2 - 2(1-3\alpha)\beta]\cos(2\theta_2)$$
$$+ [\alpha(1+\alpha) + (1-3\alpha)\beta]\cos[2(\theta_1 + \theta_2)] - 2(1-\alpha)\beta\theta_2 \sin[2(\theta_1 - \theta_2)]$$
$$+ 2(1-\alpha)[(1+\alpha)\theta_1 - (1 - \alpha + 2\beta)\theta_2]\sin(2\theta_2)$$
$$+ 2(1-\alpha)\beta\theta_2 \sin[2(\theta_1 + \theta_2)]\} \ , \quad (26)$$

$$\mathscr{D}^*_{10} = \{4(1+\alpha)\beta\theta_1 - 2(1-\alpha)\beta\theta_2 \cos[2(\theta_1-\theta_2)] + 2(1+\alpha)[(1+\alpha-2\beta)\theta_1 - (1-\alpha)\theta_2]\cos(2\theta_2) + 2(1-\alpha)\beta\theta_2\cos[2(\theta_1+\theta_2)] + [-1+\alpha^2+2(1-3\alpha)\beta]\sin(2\theta_1) + [-\alpha(1+\alpha)+(-1+3\alpha)\beta]\sin[2(\theta_1-\theta_2)] + (1-\alpha^2)\sin(2\theta_2) + [-\alpha(1+\alpha)+(-1+3\alpha)\beta]\sin[2(\theta_1+\theta_2)]\} \;, \tag{27}$$

$$\mathscr{A}^*_{20} = 2(\alpha-1)\{-1+\alpha-2\beta+2\beta\cos(2\theta_1)+(1-\beta)\cos[2(\theta_1-\theta_2)] + 2(-\alpha+\beta)\cos(2\theta_2)+(\alpha-\beta)\cos[2(\theta_1+\theta_2)]\} \;, \tag{28}$$

$$\mathscr{B}^*_{20} = (1-\alpha)\{-2(1+\alpha)\theta_1 + 2(1-\beta)\theta_2\cos[2(\theta_1-\theta_2)] + 4(-\alpha+\beta)\theta_2\cos(2\theta_2) + 2(\alpha-\beta)\theta_2\cos[2(\theta_1+\theta_2)] + 2\alpha\sin(2\theta_1) + (1-\beta)\sin[2(\theta_1-\theta_2)] + 2(\alpha-\beta)\sin(2\theta_2) + (-\alpha+\beta)\sin[2(\theta_1+\theta_2)]\} \;, \tag{29}$$

$$\mathscr{C}^*_{20} = (1-\alpha)\{-2\alpha+2\beta+(1+\alpha-2\beta)\cos(2\theta_1)+(-\alpha+\beta)\cos[2(\theta_1-\theta_2)] + (-1+\alpha-2\beta)\cos(2\theta_2)+(\alpha+\beta)\cos[2(\theta_1+\theta_2)] - 2\beta\theta_2\sin[2(\theta_1-\theta_2)] + [2(1+\alpha)\theta_1 + 2(-1+\alpha-2\beta)\theta_2]\sin(2\theta_2) + 2\beta\theta_2\sin[2(\theta_1+\theta_2)]\} \;, \tag{30}$$

$$\mathscr{D}^*_{20} = (\alpha-1)\{-2\beta\theta_2\cos[2(\theta_1-\theta_2)] + [-2(1+\alpha)\theta_1 + 2(1-\alpha+2\beta)\theta_2]\cos(2\theta_2) - 2\beta\theta_2\cos[2(\theta_1+\theta_2)] + (1-\alpha)\sin(2\theta_1) + (\alpha-\beta)\sin[2(\theta_1-\theta_2)] + (-1+\alpha-2\beta)\sin(2\theta_2) + (\alpha+\beta)\sin[2(\theta_1+\theta_2)]\} \;. \tag{31}$$

Then, the regular stress term in polar coordinates can be calculated from

$$\sigma_{rrk0}(r,\theta) = K_0[\mathscr{A}^*_{k0}\theta + \mathscr{B}^*_{k0} - \mathscr{C}^*_{k0}\sin(2\theta) - \mathscr{D}^*_{k0}\cos(2\theta)] \;, \tag{32}$$

$$\sigma_{\theta\theta k0}(r,\theta) = K_0[\mathscr{A}^*_{k0}\theta + \mathscr{B}^*_{k0} + \mathscr{C}^*_{k0}\sin(2\theta) + \mathscr{D}^*_{k0}\cos(2\theta)] \;, \tag{33}$$

$$\sigma_{r\theta k0}(r,\theta) = -K_0\left[\frac{1}{2}\mathscr{A}^*_{k0} + \mathscr{C}^*_{k0}\cos(2\theta) - \mathscr{D}^*_{k0}\sin(2\theta)\right] \;, \tag{34}$$

where K_0 has to be determined from the stress analysis of the total joint, e.g. using the Finite Element Method (FEM).

For convenience in the engineering applications, simplified equations (also in cartesian coordinates) for some special geometries will be presented in the following.

3.2 A joint with $\theta_1 = -\theta_2$

For a joint geometry with $\theta_1 = -\theta_2$, the determinant of $[A_0]_{8\times 8}$ is

$$\mathrm{Det}([A_0]_{8\times 8}) = \frac{64}{(1+\alpha)^2}\sin(\theta_1)\{[-2\theta_1\cos(\theta_1) - 2\theta_1\cos(3\theta_1) + \sin(\theta_1) + \sin(3\theta_1)] + \alpha^2[-3\sin(\theta_1) + \sin(3\theta_1)] + 2\alpha\beta[-\theta_1\cos(\theta_1) + \theta_1\cos(3\theta_1) + 3\sin(\theta_1) - \sin(3\theta_1)]\} \;. \tag{35}$$

For $\theta_1 \neq \pi$, if

$$\beta = \frac{2\theta_1\cos(\theta_1) + 2\theta_1\cos(3\theta_1) - \sin(\theta_1) - \sin(3\theta_1) + \alpha^2(3\sin(\theta_1) - \sin(3\theta_1))}{2\alpha[-\theta_1\cos(\theta_1) + \theta_1\cos(3\theta_1) + 3\sin(\theta_1) - \sin(3\theta_1)]} \;, \tag{36}$$

one gets $\mathrm{Det}([A_0]_{8\times 8}) = 0$, i.e. the regular stress term is nonzero. The nonzero coefficients $\mathscr{A}_{k0}, \mathscr{B}_{k0}, \mathscr{C}_{k0}, \mathscr{D}_{k0}$ can be determined from Eq. (23) with

$$\mathscr{A}^*_{10} = 8(1+\alpha)\sin^2(\theta_1)\{1-\alpha+\beta+(1-\beta)\cos(2\theta_1)\} \;, \tag{37}$$

$$\mathscr{B}^*_{10} = \{2[-1-\alpha+2\alpha^2-(1+3\alpha)\beta]\theta_1 + 4(1+\alpha)(\beta-\alpha)\theta_1\cos(2\theta_1) - 2(1-\alpha)(1+\beta)\theta_1\cos(4\theta_1) + 2[2\alpha(1-\alpha)+(-1+3\alpha)\beta]\sin(2\theta_1) + [1-\alpha+2\alpha^2+(1-3\alpha)\beta]\sin(4\theta_1)\} \ , \tag{38}$$

$$\mathscr{C}^*_{10} = 2\sin(\theta_1)\{-2(1-\alpha)(2+\beta)\theta_1\cos(\theta_1) + 2(1-\alpha)\beta\theta_1\cos(3\theta_1) + [\alpha(-3+5\alpha)+3(1-3\alpha)\beta]\sin(\theta_1) + [-\alpha(1+\alpha)+(-1+3\alpha)\beta]\sin(3\theta_1)\} \ , \tag{39}$$

$$\mathscr{D}^*_{10} = \{2(1+3\alpha)\beta\theta_1 + 4(1+\alpha)(1-\beta)\theta_1\cos(2\theta_1) + 2(1-\alpha)\beta\theta_1\cos(4\theta_1) + 2[-1+\alpha^2+(1-3\alpha)\beta]\sin(2\theta_1) + [-\alpha(1+\alpha)+(-1+3\alpha)\beta]\sin(4\theta_1)\} \ , \tag{40}$$

$$\mathscr{A}^*_{20} = -8(1-\alpha)\sin^2(\theta_1)\{-1+\alpha-\beta+(-1+\beta)\cos(2\theta_1)\} \ , \tag{41}$$

$$\mathscr{B}^*_{20} = (1-\alpha)\{2(-1-2\alpha+\beta)\theta_1 + 4(\alpha-\beta)\theta_1\cos(2\theta_1) + 2(-1+\beta)\theta_1\cos(4\theta_1) + 2\beta\sin(2\theta_1) + (1-\beta)\sin(4\theta_1)\} \ , \tag{42}$$

$$\mathscr{C}^*_{20} = 2(1-\alpha)\sin(\theta_1)\{-2(2+\beta)\theta_1\cos(\theta_1) + 2\beta\theta_1\cos(3\theta_1) + (-\alpha+3\beta)\sin(\theta_1) + (\alpha-\beta)\sin(3\theta_1)\} \ , \tag{43}$$

$$\mathscr{D}^*_{20} = -(1-\alpha)\{2\beta\theta_1 - 4(1+\beta)\theta_1\cos(2\theta_1) + 2\beta\theta_1\cos(4\theta_1) + 2(1-\alpha+\beta)\sin(2\theta_1) + (\alpha-\beta)\sin(4\theta_1)\} \ . \tag{44}$$

For the special case with $\theta_1 = -\theta_2 = \pi/2$,

$$\mathrm{Det}([A_0]_{8\times 8}) = \frac{256\alpha}{(1+\alpha)^2}(2\beta-\alpha) \ . \tag{45}$$

This means that if $\alpha = 2\beta$ or $\alpha = 0$ with an arbitrary β, the regular stress is nonzero. The nonzero coefficients $\mathscr{A}_{k0}, \mathscr{B}_{k0}, \mathscr{C}_{k0}, \mathscr{D}_{k0}$ are

$$\mathscr{A}_{10} = \mathscr{C}_{10} = \mathscr{A}_{20} = \mathscr{C}_{20} = 0, \quad \mathscr{B}_{10} = \mathscr{D}_{10} = \mathscr{B}_{20} = \mathscr{D}_{20} = K_0 \ . \tag{46}$$

The regular stress term in cartesian coordinates is

$$\sigma_{yy10} = \sigma_{yy20} = 4K_0, \quad \sigma_{xx10} = \sigma_{xx20} = \sigma_{xy10} = \sigma_{xy20} = 0 \ . \tag{47}$$

This means that only σ_{yy0}, which is perpendicular to the interface, is nonzero.

3.3
A joint with $\theta_1 - \theta_2 = \pi$

For a joint geometry with $\theta_1 - \theta_2 = \pi$, the determinant of $[A_0]_{8\times 8}$ is

$$\mathrm{Det}([A_0]_{8\times 8}) = \frac{128}{(1+\alpha)^2}\sin(\theta_1)[\beta+\alpha\cos(2\theta_1)-\beta\cos(2\theta_1)] \times [-\pi\cos(\theta_1)+\alpha\pi\cos(\theta_1)-2\alpha\theta_1\cos(\theta_1)+2\alpha\sin(\theta_1)] \ . \tag{48}$$

The determinant $\mathrm{Det}([A_0]_{8\times 8}) = 0$ if

$$\alpha = \frac{\beta[\cos(2\theta_1)-1]}{\cos(2\theta_1)}, \ \text{for} \ \cos(2\theta_1) \neq 0 \tag{49}$$

or

$$\alpha = \frac{\pi\cos(\theta_1)}{\pi\cos(\theta_1) - 2\theta_1\cos(\theta_1) + 2\sin(\theta_1)} \quad , \tag{50}$$

with an arbitrary β, which represents two curves in the Dundurs' diagram. This means that there are infinite material combinations, in which the regular stress is nonzero. For these combinations of α and β, the regular stress term can be simplified. After transformation of the regular stress term from polar to cartesian coordinates by

$$\begin{aligned}
\sigma_{xx} &= \sigma_{rr}\cos^2(\theta) + \sigma_{\theta\theta}\sin^2(\theta) - 2\sigma_{r\theta}\sin(\theta)\cos(\theta), \\
\sigma_{yy} &= \sigma_{rr}\sin^2(\theta) + \sigma_{\theta\theta}\cos^2(\theta) + 2\sigma_{r\theta}\sin(\theta)\cos(\theta), \\
\sigma_{xy} &= (\sigma_{rr} - \sigma_{\theta\theta})\cos(\theta)\sin(\theta) + \sigma_{r\theta}(\cos^2(\theta) - \sin^2(\theta)) \quad ,
\end{aligned} \tag{51}$$

the regular stress term can be obtained in cartesian coordinates. For α given by Eq. (49), there is

$$\begin{aligned}
\sigma_{xx10} = \sigma_{xx20} &= \frac{64\cos^2(\theta_1)[\beta + \cos(2\theta_1) - \beta\cos(2\theta_1)]}{[\cos(2\theta_1) + \beta\cos(2\theta_1) - \beta]^2} K_0 \\
&\times \{\beta\theta_1 - \pi\cos(2\theta_1)(1+\beta) + \beta\cos(4\theta_1)(\pi - \theta_1) - 2\beta\sin(2\theta_1) + \beta\sin(4\theta_1)\} \quad ,
\end{aligned} \tag{52}$$

$$\sigma_{yy10} = \sigma_{yy20} = \frac{\sin^2(\theta_1)}{\cos^2(\theta_1)}\sigma_{xx10}, \quad \tau_{xy10} = \tau_{xy20} = \frac{\sin(\theta_1)}{\cos(\theta_1)}\sigma_{xx10} \quad . \tag{53}$$

For α given by Eq. (50), there is

$$\begin{aligned}
\sigma_{xx10} = &-\frac{16\sin^2(\theta_1)}{\pi\cos(\theta_1) - \theta_1\cos(\theta_1) + \sin(\theta_1)} K_0 \\
&\times \{2[2\beta + (1-2\beta)\pi\theta_1 + 4\beta\theta_1^2]\cos(\theta_1) + 2[-2\beta + (1-2\beta)\pi\theta_1 + 4\beta\theta_1^2]\cos(3\theta_1) \\
&+ 2\beta(\pi + 2\theta_1)\sin(\theta_1) + 2\beta(\pi - 6\theta_1)\sin(3\theta_1) \\
&+ \sin(2\theta)[[-(1+\beta)\pi + 2\beta\theta_1]\cos(\theta_1) + [(-1+\beta)\pi - 2\beta\theta_1]\cos(3\theta_1) \\
&- 6\beta\sin(\theta_1) + 2\beta\sin(3\theta_1)] \\
&+ 2\theta[[-(1+\beta)\pi + 2\beta\theta_1]\cos(\theta_1) + [(-1+\beta)\pi - 2\beta\theta_1]\cos(3\theta_1) \\
&- 6\beta\sin(\theta_1) + 2\beta\sin(3\theta_1)]\} \quad ,
\end{aligned} \tag{54}$$

$$\begin{aligned}
\sigma_{yy10} = &\frac{16\sin^2(\theta_1)}{\pi\cos(\theta_1) - \theta_1\cos(\theta_1) + \sin(\theta_1)} K_0 \\
&\times \{-2\pi\theta_1\cos(\theta_1) - 2\pi\theta_1\cos(3\theta_1) - 2\pi\sin(\theta_1) + 2\pi\sin(3\theta_1) \\
&+ 2\theta[[(1+\beta)\pi - 2\beta\theta_1]\cos(\theta_1) + [(1-\beta)\pi + 2\beta\theta_1]\cos(3\theta_1) \\
&+ 6\beta\sin(\theta_1) - 2\beta\sin(3\theta_1)] \\
&+ \sin(2\theta)[[-(1+\beta)\pi + 2\beta\theta_1]\cos(\theta_1) + [(-1+\beta)\pi - 2\beta\theta_1]\cos(3\theta_1) \\
&- 6\beta\sin(\theta_1) + 2\beta\sin(3\theta_1)]\} \quad ,
\end{aligned} \tag{55}$$

$$\begin{aligned}
\sigma_{xy10} = &\frac{16\sin^2(\theta_1)}{\pi\cos(\theta_1) - \theta_1\cos(\theta_1) + \sin(\theta_1)} K_0 \\
&\times \{[(1-\beta)\pi + 6\beta\theta_1]\cos(\theta_1) + [(1+\beta)\pi - 6\beta\theta_1]\cos(3\theta_1) \\
&+ 2(-3\beta + \beta\pi\theta_1 - 2\beta\theta_1^2)\sin(\theta_1) + 2\beta(1 + \pi\theta_1 - 2\theta_1^2)\sin(3\theta_1) \\
&+ \cos(2\theta)[[-(1+\beta)\pi + 2\beta\theta_1]\cos(\theta_1) + [(-1+\beta)\pi - 2\beta\theta_1]\cos(3\theta_1) \\
&- 6\beta\sin(\theta_1) + 2\beta\sin(3\theta_1)]\} \quad ,
\end{aligned} \tag{56}$$

$$\begin{aligned}
\sigma_{xx20} = &\frac{16\sin^2(\theta_1)[\theta_1\cos(\theta_1) - \sin(\theta_1)]}{[\pi\cos(\theta_1) - \theta_1\cos(\theta_1) + \sin(\theta_1)]^2} K_0 \\
&\times \{2(2\beta - \pi^2 + 2\beta\pi^2 + \pi\theta_1 - 6\beta\pi\theta_1 + 4\beta\theta_1^2)\cos(\theta_1) \\
&+ 2(-2\beta - \pi^2 + 2\beta\pi^2 + \pi\theta_1 - 6\beta\pi\theta_1 + 4\beta\theta_1^2)\cos(3\theta_1)
\end{aligned}$$

$$+ 2\beta(-3\pi + 2\theta_1)\sin(\theta_1) + 2\beta(5\pi - 6\theta_1)\sin(3\theta_1)$$
$$+ \sin(2\theta)[(-\pi - \beta\pi + 2\beta\theta_1)\cos(\theta_1) + (-\pi + \beta\pi - 2\beta\theta_1)\cos(3\theta_1)$$
$$- 6\beta\sin(\theta_1) + 2\beta\sin(3\theta_1)]$$
$$+ 2\theta[(-\pi - \beta\pi + 2\beta\theta_1)\cos(\theta_1) + (-\pi + \beta\pi - 2\beta\theta_1)\cos(3\theta_1)$$
$$- 6\beta\sin(\theta_1) + 2\beta\sin(3\theta_1)]\} \ , \tag{57}$$

$$\sigma_{yy20} = \frac{16\sin^2(\theta_1)[\theta_1\cos(\theta_1) - \sin(\theta_1)]}{[\pi\cos(\theta_1) - \theta_1\cos(\theta_1) + \sin(\theta_1)]^2} K_0$$
$$\times \{2\pi(-\pi + \theta_1)\cos(\theta_1) + 2\pi(-\pi + \theta_1)\cos(3\theta_1) + 2\pi\sin(\theta_1) - 2\pi\sin(3\theta_1)$$
$$+ \sin(2\theta)[(\pi + \beta\pi - 2\beta\theta_1)\cos(\theta_1) + (\pi - \beta\pi + 2\beta\theta_1)\cos(3\theta_1)$$
$$+ 6\beta\sin(\theta_1) - 2\beta\sin(3\theta_1)]$$
$$+ 2\theta[(-\pi - \beta\pi + 2\beta\theta_1)\cos(\theta_1) + (-\pi + \beta\pi - 2\beta\theta_1)\cos(3\theta_1)$$
$$- 6\beta\sin(\theta_1) + 2\beta\sin(3\theta_1)]\} \ , \tag{58}$$

$$\tau_{xy20} = \frac{16\sin^2(\theta_1)[\theta_1\cos(\theta_1) - \sin(\theta_1)]}{[\pi\cos(\theta_1) - \theta_1\cos(\theta_1) + \sin(\theta_1)]^2} K_0$$
$$\times \{(-\pi + 5\beta\pi - 6\beta\theta_1)\cos(\theta_1) + (-\pi - 5\beta\pi + 6\beta\theta_1)\cos(3\theta_1)$$
$$+ 2\beta(3 + \pi^2 - 3\pi\theta_1 + 2\theta_1^2)\sin(\theta_1) + 2\beta(-1 + \pi^2 - 3\pi\theta_1 + 2\theta_1^2)\sin(3\theta_1)$$
$$+ \cos(2\theta)[(\pi + \beta\pi - 2\beta\theta_1)\cos(\theta_1) + (\pi - \beta\pi + 2\beta\theta_1)\cos(3\theta_1)$$
$$+ 6\beta\sin(\theta_1) - 2\beta\sin(3\theta_1)]\} \ . \tag{59}$$

3.4
A joint with $\theta_1 - \theta_2 = 2\pi$

For a joint with $\theta_1 - \theta_2 = 2\pi$, which is the case of a joint with a crack having the tip at the end of the interface, all equations given in Sec. 3.3 (Eqs. (48)–(59)) are valid by replacing π with 2π. For example, the determinant of $[A_0]_{8\times 8}$ is

$$\mathrm{Det}([A_0]_{8\times 8}) = \frac{256}{(1+\alpha)^2}\sin(\theta_1)[\beta + \alpha\cos(2\theta_1) - \beta\cos(2\theta_1)][-\pi\cos(\theta_1)$$
$$+ \alpha\pi\cos(\theta_1) - \alpha\theta_1\cos(\theta_1) + \alpha\sin(\theta_1)] \ , \tag{60}$$

Eq. (50) is replaced by

$$\alpha = \frac{\pi\cos(\theta_1)}{\pi\cos(\theta_1) - \theta_1\cos(\theta_1) + \sin(\theta_1)} \ , \tag{61}$$

and Eq. (52) is replaced by

$$\sigma_{xx10} = \sigma_{xx20} = \frac{64\cos^2(\theta_1)[\beta + \cos(2\theta_1) - \beta\cos(2\theta_1)]}{[\cos(2\theta_1) + \beta\cos(2\theta_1) - \beta]^2} K_0\{\beta\theta_1$$
$$- 2\pi\cos(2\theta_1)(1+\beta) + \beta\cos(4\theta_1)(2\pi - \theta_1)$$
$$- 2\beta\sin(2\theta_1) + \beta\sin(4\theta_1)\} \ . \tag{62}$$

3.5
A joint with $\theta_1 = \pi$ and arbitrary θ_2

For a joint geometry with $\theta_1 = \pi$ and an arbitrary θ_2, the determinant of $[A_0]_{8\times 8}$ is

$$\mathrm{Det}([A_0]_{8\times 8}) = \frac{64}{(1+\alpha)^2}(1-\alpha)\sin(\theta_2)\{\pi\cos(\theta_2) + \alpha\pi\cos(\theta_2) - \theta_2\cos(\theta_2)$$
$$+ \alpha\theta_2\cos(\theta_2) + \sin(\theta_2) - \alpha\sin(\theta_2)\} \ . \tag{63}$$

The determinant $\mathrm{Det}([A_0]_{8\times 8}) = 0$ if $\alpha = 1$ or

$$\alpha = -\frac{(\pi - \theta_2)\cos(\theta_2) + \sin(\theta_2)}{(\pi + \theta_2)\cos(\theta_2) - \sin(\theta_2)} \quad , \tag{64}$$

with an arbitrary β. Therefore, the regular stress term is nonzero. For the α given in Eq. (64), the regular stress terms in cartesian coordinates are

$$\begin{aligned}\sigma_{xx10} = {} & \frac{32\pi \sin^2(\theta_2) K_0}{\theta_2 \cos(\theta_2) - \sin(\theta_2)} \{2\theta \cos(\theta_2) + \cos(\theta_2)\sin(2\theta) \\ & - 2(1 - 2\beta)\sin(\theta_2) - 2(\pi + 2\beta\pi + 2\beta\theta_2)\cos(\theta_2)\} \quad ,\end{aligned} \tag{65}$$

$$\sigma_{yy10} = \frac{32\pi \cos(\theta_2)\sin^2(\theta_2) K_0}{\theta_2 \cos(\theta_2) - \sin(\theta_2)} \{-2\pi + 2\theta - \sin(2\theta)\} \quad , \tag{66}$$

$$\tau_{xy10} = \frac{64\pi \cos(\theta_2)\sin^2(\theta_2) K_0}{\theta_2 \cos(\theta_2) - \sin(\theta_2)} \sin^2(\theta) \quad , \tag{67}$$

$$\sigma_{xx20} = \frac{8\pi^2 \sin^2(\theta_2) K_0}{[\theta_2 \cos(\theta_2) - \sin(\theta_2)]^2} \{-2\theta_2 + 2\theta + \sin(2\theta)\} \quad , \tag{68}$$

$$\begin{aligned}\sigma_{yy20} = {} & \frac{32\pi^2 \cos(\theta_2)\sin^2(\theta_2) K_0}{[\theta_2 \cos(\theta_2) - \sin(\theta_2)]^2} \{2\theta \cos(\theta_2) - 2\theta_2 \cos(\theta_2) \\ & - \cos(\theta_2)\sin(2\theta) + 2\sin(\theta_2)\} \quad ,\end{aligned} \tag{69}$$

$$\tau_{xy20} = \frac{16\pi^2 \sin^2(2\theta_2) K_0}{[\theta_2 \cos(\theta_2) - \sin(\theta_2)]^2} \sin^2(\theta) \quad , \tag{70}$$

where $\theta_2 \cos(\theta_2) - \sin(\theta_2) \neq 0$, i.e. $\alpha \neq -1$. For $\alpha = 1$

$$\begin{aligned}&\sigma_{xx10} = 64\pi K_0[-\beta - \cos(2\theta_2) + \beta\cos(2\theta_2)], \\ &\sigma_{xx20} = 0, \quad \sigma_{yy10} = \sigma_{yy20} = \sigma_{xy10} = \sigma_{xy20} = 0 \quad .\end{aligned} \tag{71}$$

The special case is $\theta_2 = -\pi$, which is the case of a joint with an interface crack. For this joint

$$\mathrm{Det}([A_0]_{8\times 8}) \equiv 0 \quad . \tag{72}$$

This means that for any material combination the regular stress is always nonzero. The nonzero coefficients $\mathscr{A}_{k0}, \mathscr{B}_{k0}, \mathscr{C}_{k0}, \mathscr{D}_{k0}$ are

$$\mathscr{A}_{10} = \mathscr{C}_{10} = \mathscr{A}_{20} = \mathscr{C}_{20} = 0 \quad , \tag{73}$$

$$\mathscr{B}_{10} = -\mathscr{D}_{10} = K_0 \frac{1+\alpha}{\alpha - 1}, \quad \mathscr{B}_{20} = -\mathscr{D}_{20} = -K_0 \quad . \tag{74}$$

The regular stress term in cartesian coordinates is

$$\begin{aligned}&\sigma_{xx10} = 4\mathscr{B}_{10} = \frac{4K_0(1+\alpha)}{\alpha - 1}, \quad \sigma_{xx20} = 4\mathscr{B}_{20} = -4K_0, \\ &\sigma_{yy10} = \sigma_{yy20} = \sigma_{xy10} = \sigma_{xy20} = 0 \quad ,\end{aligned} \tag{75}$$

for $\alpha \neq 1$. From Eq. (75) it can be seen that for a joint with an interface crack only the component σ_{xx0}, which is parallel to the crack and called T-stress, is nonzero.

4
Asymptotic description of a singular stress field

In general, the stress field near the singular point in a joint of dissimilar materials can be described by

$$\sigma_{ij}(r,\theta) = \sum_{n=1}^{N} \frac{K_n}{(r/L)^{\omega_n}} f_{ijn}(\theta) + K_0 f_{ijo}(\theta) \ , \tag{76}$$

when the eigenvalue is real [15], and by

$$\sigma_{ij}(r,\theta) = \sum_{n=1}^{N} \frac{K_n}{(r/L)^{\omega_n}} \{\cos[p_n \ln(r/L)] f^c_{ijn}(\theta) + \sin[p_n \ln(r/L)] f^s_{ijn}(\theta)\} + K_0 f_{ijo}(\theta) \ , \tag{77}$$

when the eigenvalue is complex $(\omega_n \pm \mathrm{i}p_n)$, [16]. Here, ω_n are the singular stress exponents, $f_{ijn}(\theta)$, $f^c_{ijn}(\theta)$ and $f^s_{ijn}(\theta)$ are the angular functions, K_n are the stress intensity factors and $K_0 f_{ijo}(\theta)$ is the regular stress term. For an arbitrary joint geometry with θ_1 and θ_2, the quantities ω_n, p_n, $f_{ijn}(\theta)$ and $f_{ijo}(\theta)$ can be calculated analytically. The factors K_n should be determined by applying a numerical method, e.g. FEM. For a joint under thermal loading or with edge tractions, K_0 can also be calculated analytically [11–14]. However, for a joint under a remote mechanical loading, K_0 has to be determined like K_n by using a numerical method. In Eqs. (76) and (77), the distance r is normalized by L, which is a characteristic length of the joint, such that the factors K_n have the unit of stress. For an arbitrary geometry and an arbitrary material combination, there may be more than one singular term. Equations (76) and (77) include all singular terms in the sum, i.e. $\omega_n > 0$ in Eqs. (76) and (77).

The method to determine factors K_0 and K_n for a finite joint, which is given in [15], is based on the least squares method and the stress analysis of the overall joint.

5
Examples and discussions

In this Section, three examples will be presented to show the effect of the regular stress term on the stress field near the singular point. For the examples, different geometries and material combinations are chosen, which lead to different singular stress exponents.

The results given in the following are for plane strain. The loading is a remote tensile stress perpendicular to the interface of the joint, Fig. 1. For the FEM calculation, the ABAQUS code was used with a 8 nodes' standard element. The mesh near the singular point is fine. The smallest length in the element is about $10^{-6}L$.

5.1
Joints with $|\theta_1|+|\theta_2|<360°$

The geometry of Example 1 is $\theta_1 = 135°, \theta_2 = -45°$, which is the case with $\theta_1 - \theta_2 = \pi$, Fig. 2. For this geometry, it is known from Eq. (50) that $\mathrm{Det}([A_0]_{8\times 8}) = 0$ if $\alpha = -0.879802$ and β is arbitrary. This means that the regular stress term doesn't vanish. Material data $E_1 = 100$ GPa, $\nu_1 = 0.425041$, $E_2 = 1832.41$ GPa, $\nu_2 = 0.2$ are chosen for Example 1, which gives $\beta = -0.1$. The singular stress exponent is $\omega = 0.04827$, i.e. there is only one singular term. According to Eq. (76), if the regular stress term could be neglected, the plot of $\log_{10}(\sigma_{ij})$ vs $\log_{10}(r/L)$ should provide straight lines with the slope of $-\omega$. The stresses obtained by the FEM along $\theta = 90°$ are plotted in Fig. 3 in a double-logarithmic scale (here, for τ_{xy} the absolute value is presented). It is evident that the three curves are not parallel straight lines, also for small r/L values. This fact shows that, in the range of $r/L > 10^{-6}$, the singular stress field cannot be described by using the singular stress term exclusively. Therefore, the effect of the regular stress term should be considered. The factors $K_n(n = 0, 1)$ in Eq. (76) were determined by using the method given in [15] and $K_0 = 3.442$ MPa, $K_1 = 2.115$ MPa. Using the K-factors as determined, at arbitrary points stresses can be calculated analytically from Eq. (76). Considering the regular stress term, a comparison of the stresses obtained from FEM (points) and from Eq. (76) (curves) along $\theta = 0$ is shown in Fig. 4, here, σ_x is missing due to a jump at the interface. The results demonstrate that stresses calculated by FEM and from Eq. (76) are in good agreement with each other in the range near the singular point $(r/L < 10^{-2})$ when the regular stress term is taken into account.

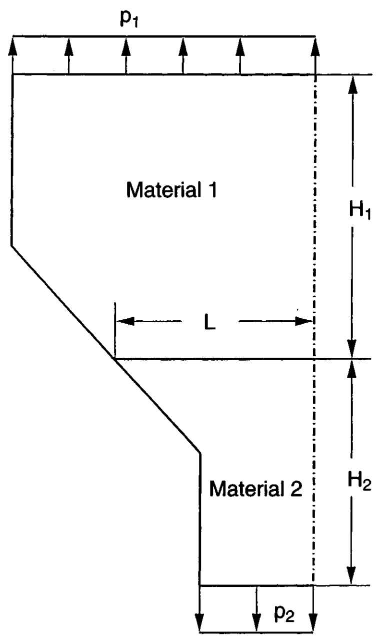

Fig. 2. Joint geometry and load; Example 1 with H_1/L = 1.695, H_2/L =1, p_1=1 MPa and p_2 = 3.4284 MPa

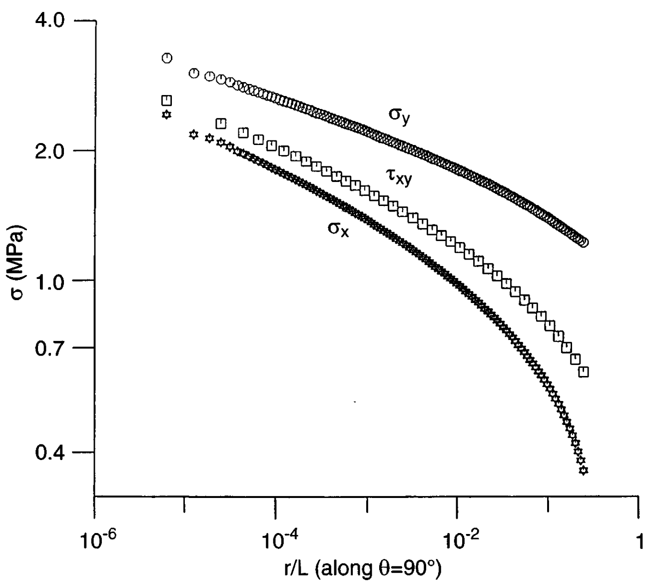

Fig. 3. Dependence of $\log_{10}(\sigma_{ij})$ vs. $\log_{10}(r/L)$ along θ = 90°; Example 1

The geometry of Example 2 is $\theta_1 = 180°, \theta_2 = -60°$, which is shown in Fig. 5 with the overall geometry and load. For this geometry, from Eq. (64) it is known that $\mathrm{Det}([A_0]_{8\times 8}) = 0$ if $\alpha = -0.642042$ or $\alpha = 1$ and β is arbitrary. Therefore, the regular stress term doesn't vanish. Material data $E_1 = 100$ GPa, $\nu_1 = 0.258878$, $E_2 = 472.009$ GPa, $\nu_2 = 0.2$ are chosen for Example 2, which gives $\alpha = -0.642042$ and $\beta = -0.2$. The singular stress exponent is $\omega = 0.48588$. Although the singular stress exponent is very large, the effect of the regular stress term will be shown to be obvious.

The obtained K-factors are $K_1 = 1.204$ MPa, if only the singular term is considered, and $K_0 = 3.557$ MPa, $K_1 = 1.186$ MPa, when two terms are used. Applying the K factors as determined, stresses have been calculated from Eq. (76) using either one term or two terms. Comparison of stresses obtained from FEM and from Eq. (76) along $\theta = -45°$ is shown in Fig. 6. It can be seen that if only the singular term is used, the agreement is good only in the range of $r/L < 10^{-3}$ or smaller, Fig. 6a. However, if the regular stress term is considered, agreement is good in a very large range near the singular point for $r/L < 10^{-1}$, Fig. 6b. To obtain a good description of the stress field near the singular point over a large range, the regular stress term should be considered, also for the joint with a large singular stress exponent.

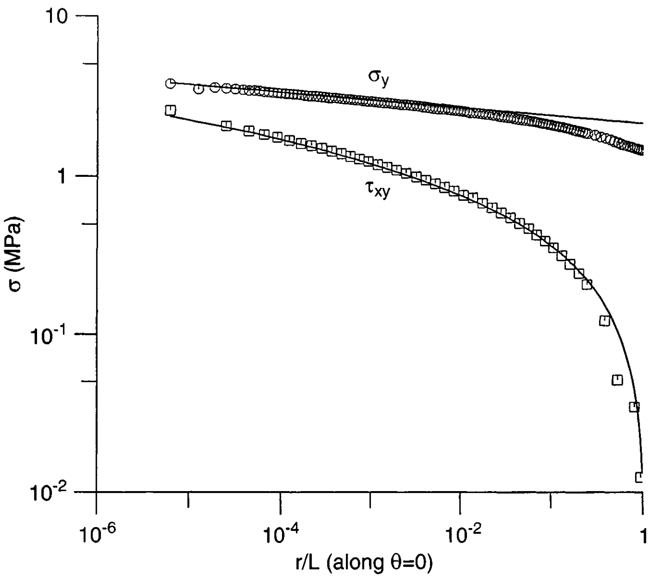

Fig. 4. Comparison of the stresses along $\theta = 0°$ calculated by FEM (points) and from the asymptotic solution (lines); Example 1

5.2
A joint with an interface crack

The geometry of Example 3 is $\theta_1 = 180°$, $\theta_2 = -180°$, which is shown in Fig. 7 with the overall geometry and loading. For this geometry the regular stress term is always nonzero.

Material data $E_1 = 400\,\text{GPa}$, $\nu_1 = 0.3$, $E_2 = 100\,\text{GPa}$, $\nu_2 = 0.2$ are chosen for Example 3, which gives $\alpha = 0.61684$ and $\beta = 0.24842$.

For this joint, the eigenvalue is complex ($\lambda = \omega + ip$). The singular stress exponent is $\omega = 0.5$, $p = \pm\frac{1}{2\pi}\ln\left[\frac{1+\beta}{1-\beta}\right]$. To calculate the stresses analytically, Eq. (77) should be used. In Eq. (77), the angular functions are normalized as follows: $f_\theta^c(\theta = 0) = 1, f_\theta^s(\theta = 0) = 0$ for K_1, and $f_\theta^c(\theta = 0) = 0, f_\theta^s(\theta = 0) = -1$ for K_2, which are the same as that defined in the fracture mechanics. Following this normalization both eigenvalues $\omega + ip$ and $\omega - ip$ give the same information on the stresses.

The obtained K-factors are $K_1 = 9.290\,\text{MPa}$ and $K_2 = 0.6591\,\text{MPa}$, when only two terms are used, and $K_0 = 0.2348\,\text{MPa}$, $K_1 = 9.336\,\text{MPa}$, $K_2 = 0.6999\,\text{MPa}$ when the regular stress term is considered. Using the K-factors as determined, stresses have been calculated with Eq. (77) for the use of two terms only, as well as for three terms. Comparisons of stresses obtained from

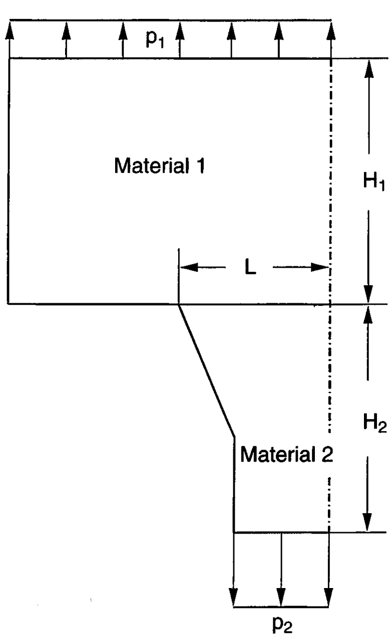

Fig. 5. Joint geometry and load Example 2 with $H_1/L = H_2/L = 2$, $p_1 = 4.226$ MPa and $p_2 = 20$ MPa

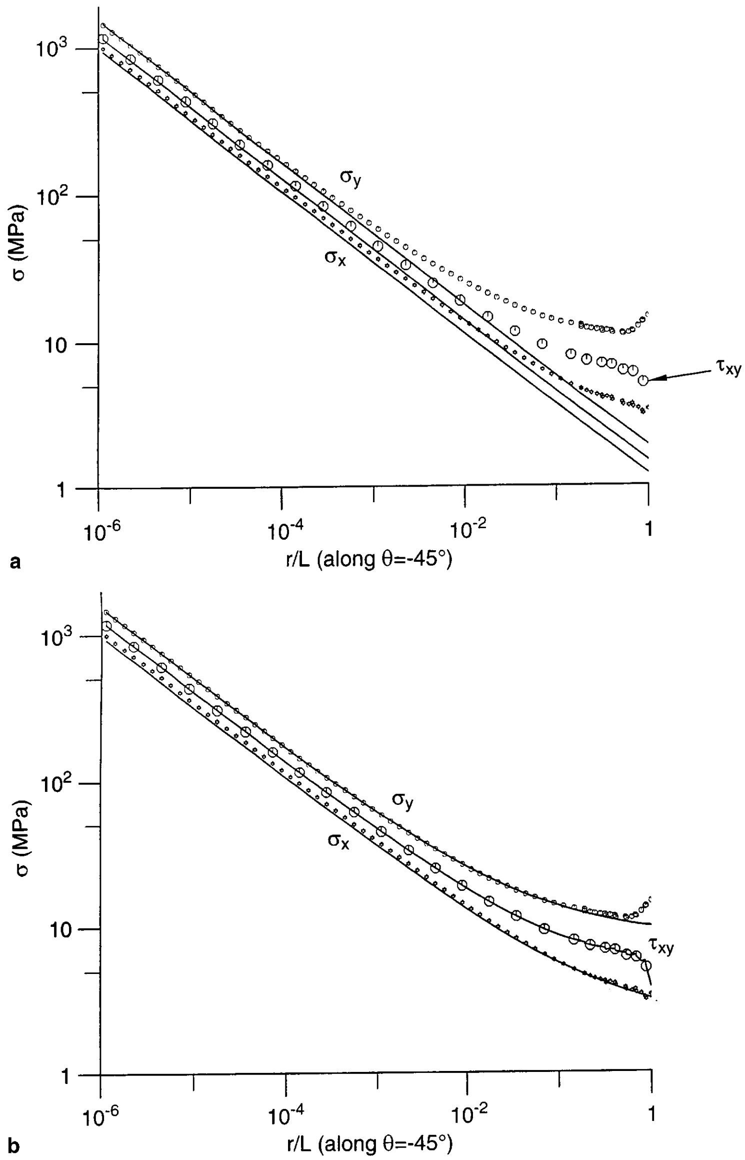

Fig. 6. Comparison of the stresses along $\theta = -45°$ calculated by FEM and from the asymptotic solution for Example 2, **a** only the singular term is used (1 term), **b** the regular stress term is considered also (2 terms)

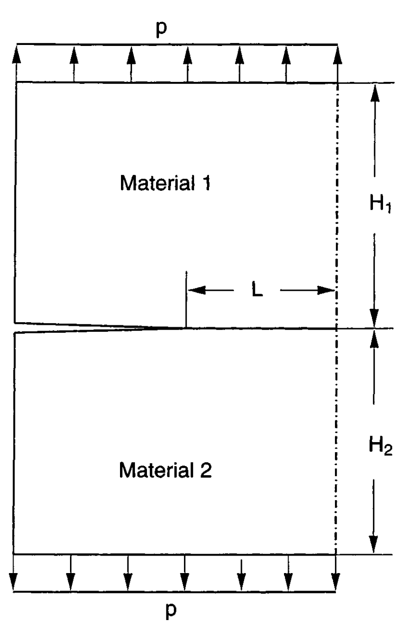

Fig. 7. Joint geometry and load; Example 3 with $H_1/L = H_2/L = 2$ and $p = 1$ MPa

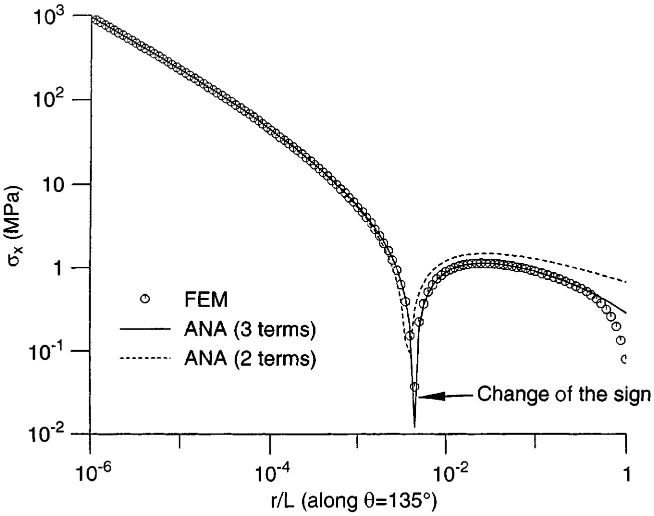

Fig. 8. Comparison of the stresses along $\theta = 135°$ calculated by FEM and from the asymptotic solution for Example 3

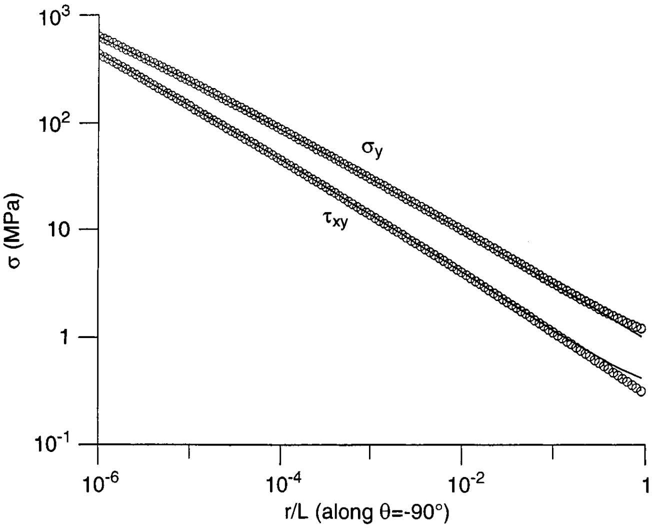

Fig. 9. Comparison of the stresses along $\theta = -90°$ calculated by FEM and from the asymptotic solution (3 terms) for Example 3

FEM and from Eq. (77) along $\theta = -90°$ and $\theta = 135°$ are shown in Figs. 8–9. It can be seen that if only the singular terms are used, they are in good agreement only in the range of $r/L < 10^{-3}$ or smaller, Fig. 8; it should be noted that the stresses are here in a logarithmic scale. However, if the regular stress term is considered, they are in good agreement in a very large range near the singular point for $r/L < 10^{-1}$, Figs. 8 and 9.

6
Conclusions

For a joint under remote mechanical loading, the regular stress term is zero for most joint geometries and material combinations. However, for a given joint geometry there are infinite material combinations corresponding to one or two curves in the Dundurs' diagram, in which the regular stress term is nonvanishing.

The conditions for a joint with a nonzero regular stress term have been given. Explicit forms needed to calculate the regular stress term have been presented for a joint with an arbitrary geometry, and simplified for several special geometries.

The examples have shown that by using the singular terms only, the asymptotic solution can describe the stress field near the singular point over a small range, even for the case of a joint with an interface crack.

The significance of applying a regular stress term is due to the fact that the singular and the regular stress term are both necessary to describe the stress field near the singular point over a large range, which is the scale of practical interest.

If one material is ceramic, failure starts from flaws in the order of $10 \sim 100\,\mu m$. Therefore, stresses in the range of $0.0001 < r < 0.1$ mm are important, which is corresponding to $0.00001 < r/L < 0.01$ for $L = 10$ mm. As can be seen from the results, the stresses have to be described including the regular stress term.

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