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On the rotating rod with variable cross section

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Summary Stability of a heavy rotating rod with a variable cross section is studied by energy method. Bifurcation points for the system of equilibrium equations are analyzed. It is shown that for the case when the rotation speed exceeds the critical one, the trivial solution ceases to be the minimizer of the potential energy, so that rod loses stability, according to the energy criteria. Also, a new estimate of the maximal rod deflection in the post-critical state is obtained.

Key words rotation, stability, energy criterion, variational analysis, functional analysis, eigenvalue problem

1 Introduction

Consider an elastic rod, fixed at the one end and free at the other end. Suppose that the rod is rotating with constant angular velocity ω about its axis. At certain angular velocity, the rod could lose its stability and deform so that its axis is bent under the action of centrifugal forces. In the simplest case, the deformed axis of the rod is a plane curve. The deformed configuration is a relative equilibrium configuration, with respect to a plane rotating with the angular velocity ω about the straight line determined by the axis of the rod in the undeformed state. The problem of determining critical rotation speed, i.e. the speed at which rod loses stability, for the case of a rod with constant cross section, has been subject of many investigations, see for example [1], [2], [3], [4] and [5]. In all these works, the classic Bernoulli-Euler theory of rods was used, and it was assumed that the rod cross section is constant, i.e. the moment of inertia of the rod and its line density are assumed to be constant. We propose, in this note, to study the rotating rod problem for the case when Bernoulli-Euler theory of rods is used, and when the rod cross section is not constant. We shall use variational method to determine the bifurcation points of the equations describing the relative equilibrium configurations of the rotating rod. The use of variational method to study the stability of a relative equilibrium configuration has the advantage over the straightforward bifurcation analysis (Liapunov-Schmidt, method for example). One can prove by it the existence of bifurcating solutions, *and* examine stability of each bifurcating branch by examining the value of the potential energy on the particular branch. For the value of rotation speed larger than the critical one for which bifurcation occurs, we shall give an estimate of the maximal deflection. The estimate is based on one of the special integral inequalities obtained in [6].

2 Model

Consider an elastic rod whose axis coincides with $\bar{x}$ axis of a rectangular Cartesian coordinate system $\bar{x} - B - \bar{y}$ lying in a plane Π. Suppose that end B of the rod is fixed, while the other end is free. Suppose further that Π is rotating about $\bar{x}$ with the constant angular velocity ω.

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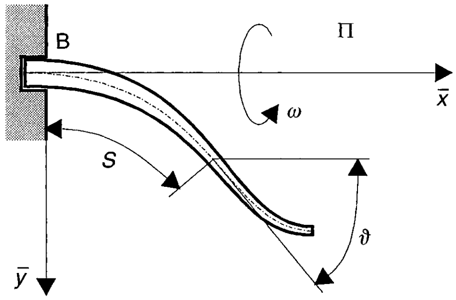

Fig. 1. Coordinate system and rod configuration in the post-critical state

At certain value of ω the rod could deform and take the configuration shown in Fig. 1. The axis of the rod, in the deformed state, is a smooth, nonintersecting curve C. The coordinates of an arbitrary point of the rod axis, in the deformed state, are

$$\bar{x} = \hat{X}(S); \quad \bar{y} = \hat{Y}(S) \ , \tag{1}$$

where S is the arc length of C, and $\hat{X}$, $\hat{Y}$ are smooth functions. We assume that the length of the rod axis is L so that $S \in [0, L]$. Since the C is nonintersecting it has well-defined tangent at each point. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit vectors along the $\bar{x}$ and $\bar{y}$ axis, respectively. Then, if $\mathbf{t}$ is the unit tangent vector at the point $(\bar{x}, \bar{y})$, the relations

$$\cos\vartheta(S) = \mathbf{t}\cdot\mathbf{e}_1, \quad \sin\vartheta(S) = \mathbf{t}\cdot\mathbf{e}_2 \ , \tag{2}$$

where $\cdot$ denotes the scalar product of two vectors, determine a unique $\vartheta(S)$. We assume that the rod is so fixed that $\vartheta(0) = 0$. With $\vartheta(S)$ determined from (2), the curvature of C is $K = \mathrm{d}\vartheta/\mathrm{d}S$. The equilibrium equations and geometrical relations expressing the fact that the rod axis is inextensible, written in the system $\bar{x} - B - \bar{y}$, are

$$\begin{aligned} H' &= 0, \\ V' &= -\rho\omega^2\bar{y}, \\ M' &= V\cos\vartheta - H\sin\vartheta, \\ \bar{x}' &= \cos\vartheta, \\ \bar{y}' &= \sin\vartheta \ , \end{aligned} \tag{3}$$

where the differentiation with respect to S is denoted by prime, $\mathbf{F} = H\mathbf{e}_1 + V\mathbf{e}_2$ is the contact force representing the influence of the part of the rod [0, S] on the part (S, L], M is the resultant couple, ρ is the mass of the rod per unit length of the rod axis in the undeformed state. To (3) we adjoin the following boundary conditions:

$$H(L) = 0, \quad V(L) = 0, \quad \bar{x}(0) = 0, \quad \bar{y}(0) = 0, \quad M(L) = 0, \quad \vartheta(0) = 0 \ . \tag{4}$$

In the Bernoulli-Euler rod theory, M is the only constitutive quantity, so that

$$M = -EI\vartheta' \ , \tag{5}$$

where E is the module of elasticity and I is axial moment of inertia of the cross section. Since the cross section is variable, we assume that

$$EI = EI_0 f(S), \quad \rho(S) = \rho_0\psi(S) \ , \tag{6}$$

where I_0 and ρ_0 are constants and $f(S)$ and $\psi(S)$ are known continuous functions. For $f(S)$ and $\psi(S)$, we assume that they are positive, decreasing, infinitely differentiable functions on the closed interval [0, L] and that the following inequalities hold:

$$b_1 > f(S) > c_1 > 0, \ \ b_2 > \psi(S) > c_2 > 0 \ .$$

Integrating $(3)_1$, and using the boundary condition $(4)_1$, we obtain $H \equiv 0$. Then, from (3) and (5) follows

$$[EI_0 f(S)\vartheta']' = -V\cos\vartheta, \quad \left[\frac{V'}{\rho_0\psi(S)}\right]' = -\omega^2\sin\vartheta \ . \tag{7}$$

We introduce the following nondimensional quantities:

$$W = \frac{VL^2}{EI_0}, \quad \lambda^2 = \frac{\omega^2\rho_0 L^4}{EI_0}, \quad t = \frac{S}{L}, \quad x = \frac{\bar{x}}{L}, \ y = \frac{\bar{y}}{L} \ . \tag{8}$$

and define a new dependent variable as

$$u = -W/\lambda \ .$$

Then setting $g = 1/\psi$, the system (7) transforms to

$$[g\dot{u}]^{\cdot} = \lambda\sin\vartheta, \quad [f\dot{\vartheta}]^{\cdot} = \lambda u\cos\vartheta, \quad \dot{y} = \sin\vartheta, \quad \dot{x} = \cos\vartheta \ , \tag{9}$$

where $(\dot{\cdot}) = \mathrm{d}(\cdot)/\mathrm{d}t$. Note also that $g \geq 1/b_2$. The boundary conditions corresponding to (9) are

$$\dot{u}(0) = 0, \quad u(1) = 0, \quad \vartheta(0) = 0, \quad \dot{\vartheta}(1) = 0, \quad y(0) = 0, \quad x(0) = 0 \ . \tag{10}$$

The functions $u_0 = \vartheta_0 = y_0 = 0$, $x_0 = S$ satisfy (9), (10) and represent a solution of (9), (10) valid for all values of λ. We call this solution the trivial solution. For the study of bifurcation only two first equations of the system (9) may be analyzed, since x and y could be determined by an independent integration, after u and ϑ are determined. Thus we analyze

$$[g\dot{u}] = \lambda\sin\vartheta; \quad [f\dot{\vartheta}]^{\cdot} = \lambda u\cos\vartheta \ , \tag{11}$$

subject to

$$\dot{u}(0) = 0; \quad u(1) = 0; \quad \vartheta(0) = 0; \quad \dot{\vartheta}(1) = 0 \ . \tag{12}$$

Note that, in the case of a rod with constant cross section, i.e. $g = \text{const.}$, $f = \text{const.}$, the system (11) possesses a first integral given as

$$K = g\frac{\dot{u}^2}{2} + f\frac{\dot{\vartheta}^2}{2} - \lambda u\sin\vartheta = \text{const} \ . \tag{13}$$

The problem (11), (12) with g and f constant, was treated in the papers mentioned before.

3
Variational formulation for Eqs. (11), (12)

Let R denote the set of real numbers, and let $H^1((0,1), R^2)$ be the space of square integrable vector functions $\mathbf{w} = (u, \vartheta)$ mapping the interval (0,1) into R^2 and having square integrable first derivative in the sense of distributions. Suppose that the components of $\mathbf{w}$ satisfy the boundary conditions $(10)_{2,3}$, that is

$$H^1 = \left\{\mathbf{w} = (u, \vartheta) : \int_0^1 \mathbf{w}\mathbf{w}^T \ \mathrm{d}t < \infty; \int_0^1 \dot{\mathbf{w}}\dot{\mathbf{w}}^T \ \mathrm{d}t < \infty; u(1) = \vartheta(0) = 0\right\} \ , \tag{14}$$

where w^T denotes the transpose of $\mathbf{w}$. The norm on H^1 is taken as

$$\|\mathbf{w}\|_1 = \left(\int_0^1 [\dot{u}^2 + u^2 + \dot{\vartheta}^2 + \vartheta^2]\mathrm{d}t\right)^{1/2} \ . \tag{15}$$

The space H^1 is separable reflexive Banach space. The Sobolev imbedding theorem implies that $H^1 \subset C([0,1]), R^2)$, where $C([0,1]), R^2)$ is the space of continuous functions mapping the interval [0, 1] into R^2. Therefore, $u(1)$ and $\vartheta(0)$ in (14) make sense. The space H^1 is a real Hilbert space with the inner product $< \mathbf{w}_1, \mathbf{w}_2 >$ given by

$$< \mathbf{w}_1, \mathbf{w}_2 >= \int_0^1 (\dot{u}_1\dot{u}_2 + \dot{\vartheta}_1\dot{\vartheta}_2 + u_1u_2 + \vartheta_1\vartheta_2)\mathrm{d}t \quad . \tag{16}$$

Consider the functional $F: H^1 \to R$, defined by

$$F = \int_0^1 \left(g\frac{\dot{u}^2}{2} + f\frac{\dot{\vartheta}^2}{2} + \lambda u \sin\vartheta \right)\mathrm{d}t \quad . \tag{17}$$

The functional F represents the nondimensional total potential energy of outer loads and inner forces. We shall show that F attains a minimum on H^1, and that the minimizing element is a solution to the boundary value problem (11), (12). Thus, we have the following:
Theorem: For each λ there exist $\overline{\mathbf{w}} = (\overline{u}, \overline{\vartheta}) \in H^1$ such that $F(\overline{u}, \overline{\vartheta}) \leq F(u, \vartheta)$ for all $\mathbf{w} = (u, \vartheta) \in H^1$. Moreover, since f and g are infinitely differentiable functions, then $\overline{\mathbf{w}}$ is an infinitely differentiable vector function from the interval (0,1) into R^2, i.e., $\overline{\mathbf{w}} = (\overline{u}, \overline{\vartheta}) \in C^\infty([(0,1)], R^2)$ and satisfies (11), (12).
Proof. We prove first the existence of a minimizer of F in H^1. To do this, we have to show that the functional (17) is sequentially weakly lower semicontinuous and coercive (see [7]p. 45). Let the L_2 norm of a function be denoted as

$$\|u_0\| = \left(\int_0^1 u^2 \mathrm{d}t \right)^{1/2} \quad .$$

Then, by using Cauchy inequality, F could be estimated as

$$F \geq \frac{1}{2}\frac{1}{b_2}\|\dot{u}\|_0^2 + \frac{1}{2}c_2\|\dot{\vartheta}\|_0^2 - \lambda\|u\|_0 \geq \frac{1}{2}\alpha\left[\|\dot{u}\|_0^2 + \|\dot{\vartheta}\|_0^2\right] - \lambda\|u\|_0 \quad , \tag{18}$$

where $\alpha = \min\{1/b_2, c_2\}$. However, from the boundary conditions $u(1) = 0$ and $\vartheta(0) = 0$, it follows that

$$\begin{aligned} u(t) &= -\int_t^1 \dot{u}(m)\mathrm{d}m \leq \left(\int_t^1 1\mathrm{d}m \right)^{1/2} \left(\int_0^t \dot{u}^2(m)\mathrm{d}m \right)^{1/2} \leq (1-t)^{1/2}\|\dot{u}\|_0 \quad , \\ \vartheta(t) &= \int_0^t \dot{\vartheta}(m)\mathrm{d}m \leq \left(\int_0^t 1\mathrm{d}m \right)^{1/2} \left(\int_0^t \dot{\vartheta}^2(m)\mathrm{d}m \right)^{1/2} \leq t^{1/2}\|\dot{\vartheta}\|_0 \quad , \end{aligned} \tag{19}$$

where, again, we have used Cauchy inequality. From (18), (19) we obtain

$$F \geq \frac{1}{4}\alpha[\|u\|_0^2\|\dot{u}\|_0^2 + \|\vartheta\|_0^2 + \|\dot{\vartheta}\|_0^2] - \lambda\|u\|_0 \geq \frac{1}{4}\alpha\|w\|_1^2 - \lambda\|w\|_0 \quad , \tag{20}$$

Since

$$\|\mathbf{w}\|_1 \geq \|\mathbf{w}\|_0$$

the inequality (20) could be written as

$$F \geq \left[\frac{1}{2}\sqrt{\alpha}\|\mathbf{w}\|_1 - \frac{\lambda}{\sqrt{\alpha}} \right]^2 - \frac{\lambda^2}{\alpha} \quad . \tag{21}$$

Therefore, $F \to \infty$ as $\|\mathbf{w}\|_1 \to \infty$, i.e. F is coercive over H^1. Further, let

$$F = \int_0^1 L(\mathbf{w}, \dot{\mathbf{w}})\mathrm{d}t.$$

Then from (17) it follows that $L(\mathbf{w}, \dot{\mathbf{w}})$ is given as

$$L = g\frac{\dot{u}^2}{2} + f\frac{\dot{\vartheta}^2}{2} + \lambda u \sin\vartheta \ . \tag{22}$$

Note that $L(\mathbf{w},\cdot)$ is a strictly convex function (see [8] p. 117). Thus, we conclude (see [7] p. 75) that F is a weakly lower semicontinuous functional. Since F is coercive over H^1, and H^1 is a reflexive Banach space, F attains a minimum on H^1. Let $\overline{\mathbf{w}} = (\overline{u}, \overline{\vartheta}) \in H^1$ be the minimizer of F. Then the Fréchet derivative of F at the point $(\overline{\mathbf{w}}, \lambda) \in H^1 \times R$ calculated in the direction $\mathbf{w} = (v_1, v_2)$ satisfies

$$\mathbf{D}F(\overline{\mathbf{w}}, \lambda)[\mathbf{w}] = \int_0^1 \{g\dot{\overline{u}}\dot{v}_1 + f\dot{\overline{\vartheta}}\dot{v}_2 + \lambda[v_1 \sin\overline{\vartheta} + \overline{u}\cos\overline{\vartheta} v_2]\}\mathrm{d}t = 0 \ , \tag{23}$$

for all $\mathbf{w} = (v_1, v_2) \in H^1$. The functional (23) is just the first variation of (17). From (23), it follows that $\overline{\mathbf{w}} = (\overline{u}, \overline{\vartheta})$ is a weak solution to (9), (10). However, since f and g are infinitely differentiable and the system (11) is analytic, it follows that $(\overline{u}, \overline{\vartheta}) \in C_0^\infty((0,1), R^2)$. This proves the theorem.

Remark 1. From (11), (12) it follows that if $(\overline{u}, \overline{\vartheta})$ is a solution, then $(-\overline{u}, -\hat{\vartheta})$ is also a solution. Also, since $F(\overline{u}, \overline{\vartheta}) \leq F(u, \vartheta)$ for *all* $(u, \vartheta) \in H^1$, we conclude that $(\overline{u}, \overline{\vartheta})$, and $(-\overline{u}, -\overline{\vartheta})$ also, are global minimizers of F.

From (23), it follows that the system (11), (12) could be written as

$$\mathbf{D}F(\overline{\mathbf{w}}, \lambda) = 0 \ . \tag{24}$$

In (24), $\mathbf{D}F(\overline{\mathbf{w}}, \lambda)$ is the Fréchet derivative of F at the point $(\overline{\mathbf{w}}, \lambda) \in H^1 \times R$. It represents a linear mapping between H^1 and R. As such, it is an element of the dual space of H^1. However, since H^1 is a Hilbert space, we can identify the dual of H^1 with H^1. Equation (24) has a trivial solution $\mathbf{w}_0 = \mathbf{0}$ for all values of λ. At certain values of λ it may, however, have a nontrivial solution. We characterize those values in the following:

Proposition 1. The points $(\mathbf{0}, \lambda_n) \in H^1 \times R$, where λ_n are eigenvalues of the following linear boundary value problem

$$[g\dot{u}] = \lambda\vartheta, \quad [f\dot{\vartheta}] = \lambda u \ , \tag{25}$$

and

$$\dot{u}(0) = 0, \quad u(1) = 0, \quad \vartheta(0) = 0, \ \ \dot{\vartheta}(1) = 0 \ , \tag{26}$$

are bifurcation points for the system (24).

Proof: We write (24) as

$$\mathbf{D}F(\mathbf{w}, \lambda) = \mathbf{A}(\lambda)\mathbf{w} + \mathbf{N}(\lambda, \mathbf{w}) \ , \tag{27}$$

where

$$\mathbf{A}\mathbf{w} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} g & \dot{u} \\ f & \dot{\vartheta} \end{bmatrix} - \lambda\begin{bmatrix} \vartheta \\ u \end{bmatrix}; \quad \mathbf{N}(\mathbf{w}, \lambda) = -\lambda\begin{bmatrix} \sin\vartheta - \vartheta \\ u(\cos\vartheta - 1) \end{bmatrix} \ . \tag{28}$$

Note that $\mathbf{N}(\mathbf{w}, \lambda) = o(\|\mathbf{w}\|_1, \lambda)$ uniformly in λ. The linear boundary value problem (25), (26) is equivalent to

$$\mathbf{A}(\lambda)\mathbf{w} = \mathbf{0} \ . \tag{29}$$

Let $\dot{y} = \vartheta$. Then, (29) reduces to

$$[f\,\ddot{y}]^{\cdot\cdot} = \lambda^2 \frac{1}{g} y \ , \tag{30}$$

with the boundary conditions

$$y(0)=0, \quad \dot{y}(0)=0, \quad \ddot{y}(1)=0, \quad \dot{f}(1)\ddot{y}(1)+f(1)\dddot{y}(1)=0 \ . \tag{31}$$

The selfadjoint, positive boundary value problem (30), (31) has countable many *positive* simple eigenvalues. To show that λ in (25), (26) is positive, we may use different approach. Namely, $f(S) > c_1 > 0$, $g \geq 1/b_2$ so that we may apply the result of [3] p. 444, directly to (25), (26) to conclude that the smallest λ is positive. Therefore, there exist a countable increasing sequence of eigenvalues $\lambda_n^2, n = 1, 2, 3, \ldots$ with the corresponding functions $y_n(t)$. If $y_n(t)$ is the solution of (30), then the eigenfunctions of (29) are $\mathbf{w}_n = (u_n, \vartheta_n)$, where

$$\vartheta_n = \dot{y}_n; \quad u_n = -\lambda_n \int_t^1 \frac{y_n(p)}{g(p)} \mathrm{d}p \ . \tag{32}$$

For Eq. (24), all conditions needed to apply the bifurcation theorem presented in [9] are satisfied, so that $(0, \lambda_n) \in H^1 \times R$ are bifurcation points of (24). This proves Proposition 1.

Note that in the first mode, that corresponds to smallest eigenvalue denoted by λ_1, we have (see [3] p. 444 Eq. (4.5))

$$\vartheta_1 \geq 0; \quad u_1 \leq 0 \ . \tag{33}$$

Therefore, from $(9)_3, (10)_5$ follows $y_1 > 0, \dot{y}_1 > 0$. Also, the smallest eigenvalue of (30), (31) is characterized as [10]

$$\lambda_1^2 = \min_{y \in Y} \frac{\int_0^1 f \ddot{y}^2 \mathrm{d}t}{\int_0^1 \frac{1}{g} y^2 \ \mathrm{d}t} \ , \tag{34}$$

where Y is the set of admissible trial functions, i.e.

$$Y = \{y : y \in C^4((0,1), R) y(0) = \dot{y}(0) = \ddot{y}(1) = \dot{f}(1)\ddot{y}(1) + f(1)\dddot{y}(1) = 0\} \ . \tag{35}$$

We show now that the rod loses stability when $\lambda > \lambda_1$. We shall do this by showing that for $\lambda > \lambda_1$ the functional (17) does not attain the minimum on the trivial solution $\mathbf{w}_0 = (u_0, \vartheta_0) = \mathbf{0}$. Therefore, according to the energy criterion of stability, the trivial solution $\mathbf{w}_0$ is *not stable*. We state this as:

Proposition 2. If $\lambda > \lambda_1$, where λ_1 is the smallest eigenvalue of the linear boundary value problem (25), (26), then the minimizer $\overline{\mathbf{w}} = (\overline{u}, \overline{\vartheta})$ of F is nontrivial, i.e. $\overline{\mathbf{w}} = (\overline{u}, \overline{\vartheta}) \neq \mathbf{w}_0 = (u_0, \vartheta_0)$ where $(u_0, \vartheta_0) = (0, 0)$.

Proof: We determine the second Fréchet derivative of F at the point $(\mathbf{w}_0, \lambda = \lambda_1 + \Delta\lambda)$, where $\Delta\lambda \ll 1$, in the direction of the first eigenfunction $\mathbf{w}_1 = (u_1, \vartheta_1)$ of the linearized problem (25), (26). The result is

$$\mathbf{D}^2 F(\mathbf{w}_0, \lambda)[\mathbf{w}_1, \mathbf{w}_1] = \int_0^1 \{g\dot{u}_1^2 + f\dot{\vartheta}_2^2 + 2(\lambda_1 + \Delta\lambda) u_1 \vartheta_1\} \mathrm{d}t \ . \tag{36}$$

However, from (25) follows

$$\int_0^1 \{g\dot{u}_1^2 + f\dot{\vartheta}_2^2 + 2\lambda_1 u_1 \vartheta_1\} \mathrm{d}t = 0 \ , \tag{37}$$

so that (36) becomes

$$\mathbf{D}^2 F(\mathbf{w}_0, \lambda)[\mathbf{w}_1, \mathbf{w}_1] = 2\Delta\lambda \int_0^1 u_1 \vartheta_1 \mathrm{d}t \ . \tag{38}$$

Now, if $\lambda > \lambda_1$ then $\Delta\lambda > 0$ so that (33) implies

$$\mathbf{D}^2F(\mathbf{w}_0,\lambda)[\mathbf{w}_1,\mathbf{w}_1] = 2\Delta\lambda\int_0^1 u_1\vartheta_1 \mathrm{d}t < 0 \ . \tag{39}$$

Therefore, the necessary condition for a minimum (the second Fréchet derivative to be positive) is violated at $\mathbf{w}_0$ and $\mathbf{w}_0 = (u_0, \vartheta_0)$ is *not* a minimizer of F. Since F attains a minimum of H^1 for all values of λ (see *Theorem*) this minimizer is nontrivial, i.e. it is on the branch bifurcating from $(0, \lambda_1)$.

4
An estimate of the maximal deflection in post-critical state

In this section, we derive an a priori estimate of the maximal deflection of the rod $y(1)$. Our method is similar to the procedure we used in [11]. It is based on certain integral inequalities, applied to the function $y(S)$. From (9) and (10), we obtain the following boundary value problem

$$[g\dot{u}]^{\cdot} = \lambda\sin\vartheta, \quad [f\dot{\vartheta}]^{\cdot} = \lambda u\cos\vartheta, \quad \dot{y} = \sin\vartheta \ , \tag{40}$$

subject to

$$\dot{u}(0) = 0, \quad u(1) = 0, \quad \vartheta(0) = 0, \quad \dot{\vartheta}(1) = 0, \quad y(0) = 0 \ . \tag{41}$$

Let λ_1 be the smallest eigenvalue of the problem (25), (26). Our main result in this section is the following:

Proposition 3. If, in addition to the restriction $b_1 > f(t) > c_1 > 0$, the function $f(t)$ is concave and satisfies $\dot{f}(0) \le 0$, then for $\lambda \ge \lambda_1$ the maximal deflection $\sup_{t\in[0,1]} y(t) = y(1)$ satisfies the inequality

$$\sup_{t\in[0,1]} y^2(t) \le \frac{32}{\pi^2}\frac{\int_0^1 \frac{1}{g}\mathrm{d}t}{\int_0^1 f\ \mathrm{d}t}[\lambda^2 - \lambda_1^2] \ . \tag{42}$$

Proof: From $(40)_{1,2}$ and $(41)_1$ we obtain $g\dot{u} = \lambda y$, so that (40) could be written as

$$g\dot{u} = \lambda y, \quad \dot{m} = -\lambda u\cos\vartheta, \quad f\dot{\vartheta} = -m, \quad \dot{y} = \sin\vartheta \ , \tag{43}$$

subject to

$$u(1) = 0, \quad m(1) = 0, \quad \vartheta(0) = 0, \quad y(0) = 0 \ . \tag{44}$$

From (43) and (44) follows

$$u = -\lambda\int_0^1 \frac{1}{g}y(p)\mathrm{d}p, \qquad m = -\lambda^2\int_t^1\left[\int_\xi^1 \frac{1}{g(p)}y(p)\ \mathrm{d}p\right]\sqrt{1-\dot{y}^2(\xi)}\ \mathrm{d}\xi \ ,$$
$$\dot{\vartheta} = \frac{\ddot{y}}{\sqrt{1-\dot{y}^2}} \ . \tag{45}$$

Combining (45) and (43) and assuming $\cos\vartheta = +\sqrt{1-\sin^2\vartheta}$, we get

$$\frac{f(t)\ddot{y}(t)}{\sqrt{1-\dot{y}^2(t)}} = \lambda^2\int_t^1\left[\int_\xi^1 \frac{1}{g(p)}y(p)\mathrm{d}p\right]\sqrt{1-\dot{y}^2(\xi)}\ \mathrm{d}\xi \ . \tag{46}$$

The solution to the boundary value problem (43), (44) is reduced to a single integro-differential equation (46) with $y \in Y$, where Y is given by (35). Multiplying (46) by $\ddot{y}$ and integrating (after partial integration and use of boundary conditions), we obtain

$$\int_0^1 f(t)\frac{\ddot{y}^2(t)\mathrm{d}t}{\sqrt{1-\dot{y}^2(t)}} = \lambda^2\int_0^1\left[\dot{y}(t)\sqrt{1-\dot{y}^2(t)}\int_t^1\frac{1}{g(k)}y(k)\mathrm{d}k\right]\mathrm{d}t \ . \tag{47}$$

Integral relation (47) is of the central importance in the analysis that follows. Consider the inequalities

$$\frac{1}{\sqrt{1-\dot{y}^2}} \geq 1+\frac{1}{2}\dot{y}^2; \quad \dot{y}^2 \leq 1 \ , \tag{48}$$

First of (48) is given in [12] and the second follows from $(40)_3$. Using (48) in (47) we have

$$\int_0^1 f\ddot{y}^2\mathrm{d}t + \frac{1}{2}\int_0^1 f\dot{y}^2\ddot{y}^2\mathrm{d}t \leq \lambda^2\int_0^1\frac{1}{g}y^2\mathrm{d}t \ , \tag{49}$$

or

$$\frac{\int_0^1 f\ddot{y}^2\mathrm{d}t}{\int_0^1\frac{1}{g}y^2\mathrm{d}t} + \frac{1}{2}\frac{\int_0^1 f\dot{y}^2\ddot{y}^2\mathrm{d}t}{\int_0^1\frac{1}{g}y^2\mathrm{d}t} \leq \lambda^2 \ . \tag{50}$$

The first term on the left-hand side of (50) could be estimated as (see (34))

$$\lambda_1^2 \leq \frac{\int_0^1 f\ddot{y}^2\mathrm{d}t}{\int_0^1\frac{1}{g}y^2\mathrm{d}t} \ . \tag{51}$$

To estimate the second term, we introduce function Φ by

$$\Phi = \frac{\dot{y}^2}{2} \ . \tag{52}$$

Then, from (44) follows

$$\Phi(0)=0; \quad \dot{\Phi}(1)=0; \quad \dot{\Phi}(0)=0 \ . \tag{53}$$

For two functions $f(t)$ and $h(t)$ such that $f(t)$ is piecewise smooth and satisfies $f(0)=0$, and $h(t)$ is positive, concave and satisfies $\dot{h}(0)\leq 0$, the following inequality has been proved in [6]:

$$\frac{\int_0^1 h\dot{f}^2\mathrm{d}t}{\int_0^1 h\mathrm{d}t\int_0^1 f^2\mathrm{d}t} \geq \frac{\pi^2}{4} \ . \tag{54}$$

Using (54) with $h=f$ and $f=\Phi$, we obtain

$$\frac{\int_0^1 f(t)\dot{\Phi}^2(t)\mathrm{d}t}{\left[\int_0^1 f(t)\mathrm{d}t\right]\left[\int_0^1 \Phi^2(t)\mathrm{d}t\right]} \geq \frac{\pi^2}{4} \ . \tag{55}$$

Note that $y_1>0, \dot{y}_1>0$, see comment after (33). In [3] p. 450, it is shown that for $\lambda\geq\lambda_1$ there exists a unique *positive* solution for ϑ, so that boundary condition $(10)_5$ implies that $y(t)$ is positive and increasing. Also, since $1/g = \psi$ is decreasing, by Tchebyschef inequality [12] we obtain

$$\int_0^1 \frac{1}{g} y^2 \mathrm{d}t \le \left[\int_0^1 \frac{1}{g}\mathrm{d}t\right]\left[\int_0^1 y^2 \mathrm{d}t\right] \quad . \tag{56}$$

From (55), (56) follows

$$\frac{\int_0^1 f \dot{y}^2 \ddot{y}^2 \mathrm{d}t}{\int_0^1 \frac{1}{g} y^2 \mathrm{d}t} \ge \frac{1}{4}\frac{\pi^2}{4}\frac{\left[\int_0^1 f \mathrm{d}t\right]\int_0^1 \dot{y}^4 \mathrm{d}t}{\left[\int_0^1 \frac{1}{g}\mathrm{d}t\right]\left[\int_0^1 y^2 \mathrm{d}t\right]} \ge \frac{1}{4}\frac{\pi^2}{4}\frac{\left[\int_0^1 f \mathrm{d}t\right]\|\dot{y}\|_0^4}{\left[\int_0^1 \frac{1}{g}\mathrm{d}t\right]\|y\|_0^2} \quad , \tag{57}$$

where in the last step we used Cauchy inequality. Also, from the inequality (54) applied to the function $y(h = 1, f = y$ in $(54))$, we have

$$\int_0^1 \dot{y}^2 \mathrm{d}t \ge \frac{\pi^2}{4}\int_0^1 y^2 \mathrm{d}t \quad . \tag{58}$$

Using (58), the inequality (57) could be transformed to

$$\frac{\int_0^1 \dot{y}^2 \ddot{y}^2 \mathrm{d}t}{\int_0^1 y^2 \mathrm{d}t} \ge \frac{1}{4}\left(\frac{\pi}{2}\right)^4 \frac{\left[\int_0^1 f \ \mathrm{d}t\right]}{\left[\int_0^1 \frac{1}{g}\mathrm{d}t\right]}\|\dot{y}\|_0^2 \quad . \tag{59}$$

Finally, from the boundary condition $(44)_4$, we obtain

$$y(t) = \int_0^t \dot{y}(p)\mathrm{d}p \le \left[\int_0^t \mathrm{d}p\right]^{1/2}\left[\int_0^t \dot{y}^2(p)\mathrm{d}p\right]^{1/2} \le \|\dot{y}\|_0 \quad . \tag{60}$$

Combining (50), (51), (59) and (60), follows (42). This proves Proposition 3.
Remark 2. With $\lambda = \lambda_{cr} + \Delta\lambda$, and $\Delta\lambda \ll 1$, the expression (42) leads to the conclusion that the maximal deflection is proportional to the square root of $\Delta\lambda$, i.e.

$$\sup_{t\in[0,1]} |y(t)| \le \sqrt{\frac{64}{\pi^2}\lambda_{cr}\frac{\int_0^1 \frac{1}{g}\mathrm{d}t}{\int_0^1 f \ \mathrm{d}t}\Delta\lambda} \quad . \tag{61}$$

The result (61) is, qualitatively, in agreement with the maximal deflection obtained from the Liapunov-Schmidt procedure, in the case of a rod with constant cross section, see [4].

Conclusions

In this paper, we have analyzed the problem of determining critical value of the angular velocity for the rotating rod with variable cross section. Our main results are:

1. For the smallest value of nondimensional angular velocity λ_1 that is determined from the *linearized* equilibrium equations (25), (26), the rod loses stability. We have basically shown this in two ways. First, by *Proposition 1* we have shown that for λ_1 we have a bifurcation point of the nonlinear equilibrium equations (24). Therefore, according to the Euler stability criterion, the rod is not stable for $\lambda > \lambda_1$.
2. We have shown in the *Theorem* that the total potential energy of the system always has a minimizer, and that this minimizer is the classic solution of the equilibrium equations. Also, we have shown by *Proposition 2* that, for the case when $\lambda > \lambda_1$, the trivial solution of the

equilibrium equations in which rod axis remains straight is *not* a minimizer. Since a minimizer always exists, it follows that for $\lambda > \lambda_1$ the rod has a minimizer for which the rod axis is not straight. This implies that, according to the energy stability criterion, the rod is not stable for $\lambda > \lambda_1$.

3. In Sec. 4 we have derived an estimate of the maximal deflection for the case when the rod is rotating with the angular velocity higher than λ_1. The estimate has been derived by using several integral inequalities. The same method was applied earlier in [11] and [13]. In particular, our estimate (61) is a generalization to the case of variable cross section of the estimate presented in [13].
4. An interesting generalization of the problem treated here consists in obtaining the estimates similar to (61) for the case when the influence of shear stresses in the constitutive equations of the rod is not neglected. One such model was used in [14].

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