# Negative Poisson's ratios in composites with star-shaped inclusions: a numerical homogenization approach

**P. S. Theocaris, G. E. Stavroulakis P. D. Panagiotopoulos**



**Summary** Materials with specific microstructural characteristics and composite structures are able to exhibit negative Poisson's ratio. This result has been proved for continuum materials by analytical methods in previous works of the first author, among others [1]. Furthermore, it also has been shown to be valid for certain mechanisms involving beams or rigid levers, springs or sliding collars frameworks and, in general, composites with voids having a nonconvex microstructure.Recently microstructures optimally designed by the homogenization approach have been verified. For microstructures composed of beams, it has been postulated that non-convex shapes with re-entrant corners are responsible for this effect [2]. In this paper, it is numerically shown that mainly the shape of the re-entrant corner of a non-convex, star-shaped, microstructure influences the apparent (phenomenological) Poisson's ratio. The same is valid for continua with voids or for composities with irregular shapes of inclusions, even if the individual constituents are quite usual materials. Elements of the numerical homogenization theory are reviewed and used for the numerical investigation.

**Key words** negative Poisson's ratio, mechanics and design of composites, numerical homogenization

## 1 Introduction

Composite materials usually present a certain nonhomogeneous and isotropic microstructure. Only on the macroscale it is possible to accept these materials as quasi-homogeneous and eventually isotropic for the cases considered in this paper. By using the method of optimal topology design in the numerical homogenization [1, 2], a choice of the appropriate quantities for the constituents of the microstructure could be achieved on the respective characteristic unit cell. The origin of this micro macro approach can be traced back to the modelling of elastic frameworks in continuous structures [3, 4]. By this procedure, overall elastic moduli of the anisotropic structure can be evaluated through the stiffnesses of the members composing the unit cell walls.

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P. S. Theocaris
National Academy of Athens, P.O. Box 77230, GR-17510 Athens, Greece

G. E. Stavroulakis
Technical University of Crete, Greece

P. D. Panagiotopoulos
Aristotle University of Thessaloniki, Greece

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By applying these ideas, it is further possible to explain the appearance of negative Poisson's ratios and their effects on the behaviour of the structure [3–7]. Recently, it has been shown that by an appropriate selection of the anisotropic properties of the material or the structure, and especially by varying its values of Poisson's ratios, a beneficial effect can be achieved in the strength characteristics. In particular this is due to a reduction of stress concentration factors caused by geometric discontinuities within the structure [9–12]. The advantages of the use of materials with negative Poisson's ratio have been already appreciated. They permit, among others, a reduction of the stress concentration factors, and production of layered composite panels and beams, which allow for smooth treatment by cold metal forming processes [13, 14].

In this paper, ways for designing materials with negative Poisson's ratios will be indicated, based on configurations of arrays of inclusions with polygonal shaped and re-entrant corners. These nonconvex two-dimensional cellular microstructures, where inclusions are made of a material of lower moduli than the moduli of the matrix of the structure, are convenient to create composites with negative Poisson's ratios of different values. They depend on the ratio of the moduli of the constituents of the composite, as well as on the shape of the inclusions. By applying methods of numerical analysis, it will be shown in this paper that while the choice of the material properties of the individual constituents of the composite does not influence significantly this effect, the shape of the star-shaped micro-inclusions is mainly responsible for this phenomenon.



## 2
## True bounds of Poisson's ratios in anisotropic bodies

The positiveness of the stiffness **C** and the compliance **S** tensors in anisotropic materials is imposed by thermodynamic principles based on the fact that the elastic potential should remain always a positive quantity. The positive definiteness of these two tensors for any anisotropic material implies that the following four eigenvalues of the minimum polynomial for **S** can be expressed by [15]

$$\begin{aligned}
\lambda_1 &= \frac{1+\nu_{23}}{E_{23}} = \frac{1}{2G_{23}} \quad , \\
\lambda_2 &= \frac{1}{2G_{12}} = \frac{1}{2G_{13}} \quad , \\
\lambda_3, \lambda_4 &= \frac{(1-\nu_{23})}{2E_{23}} + \frac{1}{2E_{12}} \pm \left\{ \left( \frac{1-\nu_{23}}{2E_{23}} - \frac{1}{2E_{12}} \right)^2 + \frac{2\nu_{12}^2}{2E_{12}^2} \right\}^{1/2} \quad .
\end{aligned} \tag{1}$$

The above values for the four roots of the minimum polynomial for **S** are simplified for the transversely isotropic body where the 2nd, 3rd principal directions correspond to the transverse plane of symmetry of the material, so that $E_{12} = E_{13}$ and $\nu_{12} = \nu_{13}$.

From the above relationships (1), and for positiveness of $E_{23}$ and $G_{23}$, one easily finds for the components of Poisson's ratios $\nu_{23}$ and $\nu_{12} = \nu_{13}$ the expressions [16, 17]

$$\nu_{23} < 1 \quad \text{and} \quad |\nu_{12}| = |\nu_{13}| < \left( \frac{(1-\nu_{23})E_{12}}{2E_{23}} \right)^{1/2} . \tag{2}$$

It should be pointed out and emphasized that positiveness of the elastic potential is guaranteed only when both above inequalities hold, a fact which has been sometimes overlooked in the literature and has led to inaccurate conclusions [17]. Then, for othotropic solids the following system of relations must hold [17]:

$$|\nu_{12}| < \left( \frac{E_{11}}{E_{22}} \right)^{1/2}, |\nu_{23}| < \left( \frac{E_{22}}{E_{33}} \right)^{1/2}, |\nu_{13}| < \left( \frac{E_{11}}{E_{33}} \right)^{1/2} \quad , \tag{3}$$

and

$$2\nu_{12}\nu_{23}\nu_{13} \frac{E_{33}}{E_{11}} < 1 - \nu_{12}^2 \frac{E_{22}}{E_{11}} - \nu_{23}^2 \frac{E_{33}}{E_{22}} - \nu_{13}^2 \frac{E_{33}}{E_{11}} \quad . \tag{4}$$

For the transversely isotropic body, these relations reduce to the simpler ones

$$|\nu_{12}| = |\nu_{13}| < \left(\frac{E_{11}}{E_{22}}\right)^{1/2}, |\nu_{23}| < 1 \quad , \tag{5}$$

and

$$\nu_{12}^2 \nu_{23} < (1 - \nu_{23}^2) \frac{E_{11}}{2E_{22}} - \nu_{12}^2 \quad . \tag{6}$$



It can readily be derived from these relations that the inequalities (4) or (6) are more restrictive and severe than the respective inequalities (3) or (5). Therefore, they are the relationships which should be considered for evaluating limits of variation of Poisson's ratios in composites. Application of these relationships may then protect the researcher from admitting excessive bounds for this important mechanical property (see, for example, the excessive value for $\nu_{23} = 1.97$ given in Ref. [17] for the transverse Poisson's ratio of some particular composite).

For isotropic elastic materials, the bounds of Poisson's ratio values are reduced to the well-known limits varying between (−1.0 and +0.5). The right-side limit corresponds to incompressible materials, with rubbery materials and, especially, polymers approaching this limit. The negative values for Poisson's ratio appear in special substances and, in particular, in those possessing low values of the bulk modulus and high values of the respective shear modulus. The lower limit of the negative units is an extreme value, which may be achieved only in very special structures of substances. In all other cases, the possibility of the appearance of a negative Poisson's ratio, at least in one direction of loading is not excluded from the theory of general anisotropic elasticity. Instances of this effect will be reviewed in this section [18, 19].

Thus, single crystals with a polygonal structure at the atomic level are reported to have negative Poisson's ratio along some directions of loading. Such materials are reported to be cadmium [20], single crystals of pyrite [21] and lattice-structured pyrolytic graphite [22]. On the other hand, thermomechanically treated low-density open-cell thermoplastic polymeric foams are materials which eventually exhibit negative Poisson's ratios. It is of interest to remark that such materials are usually porous, have a spongy nature, many voids, and a complicated microstructure. From the microstructural picture of the latter materials exhibiting nonconvex cells and containing cells with re-entrant corners, a number of microstructures and mechanisms have been proposed as an explanation of this effect [13, 23–25]. These examples are not actually materials which can be found in nature. However, as manufacturing technology and micromechanics attain a higher level of development, the possibility of constructing materials with these microstructures as prototypes grows continuously. On the other hand, it should be remarked that almost all structures in living creatures are practically composed of a combination of such materials.

Cellular microstructures composed of beams have been used with success for modelling of linear and nonlinear elastic properties of two-dimensional and three-dimensional cellular materials or honeycombs; the results correlate well with experimental measurements [3, 4, 7]. It should be noted that experience gathered up to now indicates that all usual materials and composites with positive values for Poisson's ratio should be formed from units containing exclusively or predominantly convex cells, whereas, foamy materials with very high porosity containing nonconvex cells constituting re-entrant corners are convenient to create substances with negative Poisson's ratio [3, 14, 23].

The importance of creating materials with negative Poisson's ratio has been recognized with respect to modern structural analysis applications, especially in the aerospace industry. It was recognized that these materials should normally have a very high shear modulus relative to their respective bulk modulus. This is appreciated if the material is used in a sheet or beam form, as it is actually the case in most structural applications where materials having a high shear modulus rather than a high bulk modulus are beneficial [14]. Moreover, the deformation patterns of elastic structures made of these kind of materials generally differ from those made of classical materials (see ref. [13] for a detailed description). The latter effect requires a new way of thought for the design of structural elements of structures, but at the same time opens new possibilities for applications. For example, a sandwich panel or beam with a core made of this new material will exhibit a dome-like double curvature on flexure. This fact allows an improved cold metal-forming treatment for the production of shells from initially plane panels, thus reducing the stress concentration factors which, in turn , enhance the crack and fatigue strengths of structures.

## 3
## A numerical homogenization method for adapting negative effective Poisson's ratios

From a series of experimental results on foams with re-entrant corner cells, e.g.[23], and from the relevant results by applying the numerical homogenization theory, e.g.[2], it can be shown that we may construct microstructures with an adjustable mechanical behaviour exhibiting positive or negative Poisson's ratio. For the study of the overall mechanical properties of these materials, we assume that they are periodic, i.e. the same microstructural pattern is repeated for the whole area of a structure. We assume, moreover, that the overall mechanical behaviour of the material can be described by classical elasticity relations. In this framework, the homogenization problem is posed as follows:

find the elasticity constants of the continuous model which lead to the same mechanical behaviour as the one of the material with the periodic microstructure.



To this end, a detailed analysis of a representative material cell is performed and the best-fit method is followed. This will be shown in the numerical examples later in this paper. The possibility to adjust the overall mechanical properties by changing either the geometric or the material properties of the microstructure constitutes the inverse (optimal) design problem:

find a microstructure for which the material has a given (or optimal in some sense) mechanical behaviour.

Let us assume a representative unit cell of the periodic structure, which for simiplicity is considered to be two-dimensional, (Fig. 1). Let the unit cell be orthogonal, with dimensions equal to $l_1$ and $l_2$ along the two coordinate axes. Let it occupy the area $\Omega$ with boundary $\Gamma$. The boundary is composed of the complementary and nonoverlapping parts $\Gamma_1, \Gamma_2, \Gamma'_1$ and $\Gamma'_2$, i.e.

$$\Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma'_1 \bigcup \Gamma'_2 = \Gamma_2 \Gamma_1 \bigcap \Gamma_2 = \phi \ .$$

A unit cell of the real structure (case *II* in Fig. 1) and a unit cell with the same dimensions of the sought homogeneous structure (case *I* in Fig. 1) are considered. The cells *I* and *II* are subjected to three types of unit prestresses

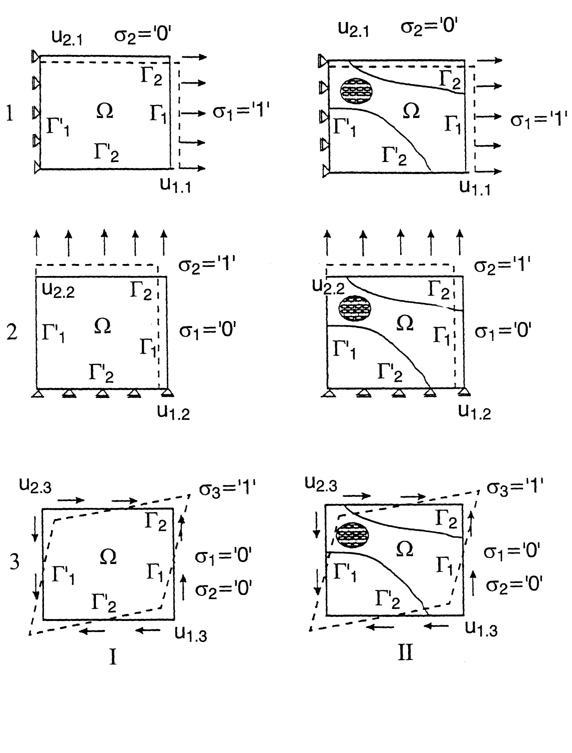


**Fig. 1.** Elements of the numerical homogenization technique for a unit-cell. Case (*I*) the homogeneous cell, case (*II*) the real structure cell. Problems (1), (2) and (3) indicate loading modes of the respective cells

$$\begin{aligned}
&\text{Problem}(1): \ \sigma_1 = 1,\ \sigma_2 = 0,\ \sigma_3 = \tau_{12} = \tau_{21} = 0 \ , \\
&\text{Problem}(2): \ \sigma_1 = 0,\ \sigma_2 = 1,\ \sigma_3 = \tau_{12} = \tau_{21} = 0 \ , \\
&\text{Problem}(3): \ \sigma_1 = 0,\ \sigma_2 = 0,\ \sigma_3 = \tau_{12} = \tau_{21} = 1 \ ,
\end{aligned} \tag{7}$$

as it is shown in Fig. 1.

The solution of cell *I* for these load problems can be based on simple engineering mechanics relations, due to the assumption that the dimensions of the periodic cell are small with respect to the dimensions of the structure.



For the cell *II*, the finite element method is employed for the solution of the above static analysis problems. Moreover, the following periodicity restraints, which result from technical mechanics consideration, are taken into account as multipoint constraints in the above described problems:

- for problems 1 and 2, displacements on boundaries $\Gamma_1, \Gamma_1'$ along the horizontal direction 1 are the same;
- for problems 1 and 2, displacements on boundaries $\Gamma_2, \Gamma_2'$ along the vertical direction 2 are the same; and
- for problem 3, boundaries $\Gamma_1, \Gamma_1' \Gamma_2$ and$\Gamma_2'$ remain straight lines after deformation.

The essence of the energy-based numerical homogenization method is that the parameters of the homogeneous cell *I* need to be appropriately chosen, in order to obtain the same deformation energy as in the cell *II* of the real structure. Both are subjected to the same deformation patterns, which should respect the periodicity assumptions, i.e. they are periodic for the whole structure.

If the parameters defining the mechanical behaviour of the cell *I* (e.g. the elasticity constants) are gathered up in the design vector $\boldsymbol{\alpha}$, the numerical homogenization method can be described by the following identification problem:

find $\boldsymbol{\alpha}$ as a solution of the optimization problem

$$\min_{\alpha \in A_{ad}} \frac{1}{2} \sum_{i=1}^{3} w_i \left\{ \prod_{\text{in}}^{I(i)} (e^{(i)}, \boldsymbol{\alpha}) - \prod_{\text{in}}^{II(i)} (e^{(i)}) \right\}^2 \ . \tag{8}$$

Here, $A_{ad}$ is the admissible set for the material parameters of the homogenized cell, index $i$ runs over all independent periodic deformation patterns $e^{(i)}$ considered, $w_i$ are appropriate weights which transform the multi-objective optimization problem into a classical one with a cost function as in (8), superscripts *I* or *II* stand for the quantities of cells *I* or *II*, respectively, and $\prod_{\text{in}}$ is the internal energy of the considered structure.

The identification problem (8) can be solved either by classical numerical optimization techniques, or by neural-network based methods as presented in [26]. Here, we use a simple procedure based on the optimality criteria method for the solution of a certain class of problems (8). The method avoids formulation and solution of large scale optimization problems, and, if it can be used, is considered to be suitable for structural analysis applications [1].

Let us assume for simplicity that all $w_i$ are equal to one. We assume that the homogenized unit cell I obeys the classical isotropic elasticity relations, i.e. we have [17]

$$\mathbf{e} = \left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{array} \right\} = \left\{ \begin{array}{ccc} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1-\nu)}{E} \end{array} \right\} \left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{array} \right\} = \mathbf{K}_0 \boldsymbol{\sigma} \ . \tag{9}$$

The design vector $\boldsymbol{\alpha}$ is chosen as $\boldsymbol{\alpha} = [\alpha_1,\ \alpha_2]^{\mathrm{T}} = [1/E,\ -\nu/E]^{\mathrm{T}}$.

The internal energy is expressed by

$$\prod_{\text{in}}^{i(j)} = \int_{\Omega} \mathbf{o}^{i(j)\mathrm{T}} \mathbf{e}^{i(j)} \mathrm{d}\Omega \text{ for all } i = I, II, j = 1, 2, 3,$$

where $\Omega$ is the area of the considered cell. For simplicity, we assume that $A_{ad} = R^2$.

For the assumed unit stresses (7) and the elasticity relations (9), we get for the unit cell *I*

$$
\begin{aligned}
&e_1^{I(1)} = \alpha_1 \sigma_1^{I(1)} = \alpha_1 \ , \qquad e_2^{I(1)} = \alpha_2 \sigma_1^{I(1)} = \alpha_2 \ , \\
&e_1^{I(2)} = \alpha_2 \sigma_2^{I(2)} = \alpha_2 \ , \qquad e_2^{I(2)} = \alpha_1 \sigma_2^{I(2)} = \alpha_1 \ , \\
&e_3^{I(3)} = 2(\alpha_1 - \alpha_2)\sigma_3^{I(3)} = 2(\alpha_1 - \alpha_2) \ ,
\end{aligned} \tag{10}
$$

with all other components equal to zero.

Relations (7) and (10) written for the cell $I$ are introduced in (8) and yield

$$
\min_{\boldsymbol{\alpha}\in R^2} \left\langle \frac{1}{2} \int_\Omega \left\{ \left[\sigma^{I(1)\mathrm{T}}(\boldsymbol{\alpha}) e^{I(1)}(\alpha) - \sigma^{II(1)T} e^{II(1)}\right]^2 + \left[\sigma^{I(2)T}(\boldsymbol{\alpha}) e^{I(2)}(\boldsymbol{\alpha}) - \sigma^{II(2)\mathrm{T}} e^{II(2)}\right]^2 \right.\right.
$$

$$
\left.\left. + \left[\sigma^{I(3)\mathrm{T}}(\boldsymbol{\alpha}) e^{I(3)}(\boldsymbol{\alpha}) - \sigma^{II(3)\mathrm{T}} e^{II(3)}\right]^2 \right\} \mathrm{d}\Omega \right\rangle \ . \tag{11}
$$



Moreover, the virtual work equality for the cell $II$ reads

$$
\int_\Omega \boldsymbol{\sigma}^{II(j)\mathrm{T}} \mathbf{e}^{II(j)} \ \mathrm{d}\Omega = \int_\Gamma S^{II(j)\mathrm{T}} u^{II(j)} \ \mathrm{d}\Gamma, \quad j = 1, 2, 3 \ , \tag{12}
$$

for all given unit stresses of (7), i.e.

$S^{II(1)} = 1$ on $\Gamma_1, S^{II(1)} = 0$ on $\Gamma_2, \ \Gamma_2'$.

Finally, the optimality conditions for (11) are written by means of (10) in the form of an equation for $\alpha_1, \alpha_2$

$$
\int_\Omega \left(\alpha_1 - \sigma^{II(2)\mathrm{T}} e^{II(2)}\right) \mathrm{d}\Omega \frac{\partial \alpha_1}{\partial \alpha_1} + \int_\Omega \left(2(\alpha_1 - \alpha_2) - \sigma^{II(3)\mathrm{T}} e^{II(3)}\right) \mathrm{d}\Omega \frac{\partial\{2(\alpha_1 - \alpha_2)\}}{\partial \alpha_1} = 0 \ , \tag{13}
$$

$$
\int_\Omega \left(2(\alpha_1 - \alpha_2) - \sigma^{II(3)\mathrm{T}} e^{II(3)}\right) \mathrm{d}\Omega \frac{\partial\{2(\alpha_1 - \alpha_2)\}}{\partial \alpha_2} = 0 \ . \tag{14}
$$

By using (12), the surface integrals are transformed into line integrals. Thus, we get the following optimality conditions:
find $\alpha_1, \ \alpha_2$ such that

$$
\left(\alpha_1 l_1 l_2 - u_1^{II(1)} l_2\right) \ 1 + \left(\alpha_1 l_1 l_2 - u_2^{II(2)} l_1\right) \ 1 + \left(2(\alpha_1 - \alpha_2) l_1 l_2 - \int_\Omega \sigma^{II(3)\mathrm{T}} e^{II(3)} \mathrm{d}\Omega\right) \ 2 = 0 \ , \tag{15}
$$

$$
\left(2(\alpha_1 - \alpha_2) l_1 l_2 - \int_\Omega \sigma^{II(3)\mathrm{T}} e^{II(3)} \mathrm{d}\Omega\right) \ (-2) = 0 \ . \tag{16}
$$

The variable $\alpha_1$ (the elastic modulus $E$) results from (14) and (15)

$$
\alpha_1 = \frac{u_1^{II(1)} l_2 + u_2^{II(2)} l_1}{2 l_1 l_2} \ . \tag{17}
$$

The variable $\alpha_2$ (the Poisson ratio $\nu$) may now be calculated either from (16), or from the elasticity relations (9), which have been assumed to hold true.

Analogous relations can be extracted for the more general case, where the homogeneous model $I$ is assumed to obey orthotropic elasticity relations or general anisotropic elasticity relations [2].

## 4
## The inverse problem of defining a material with given homogenized elastic constants

The aim of this chapter is to formulate and implement a procedure to define linear elastic materials with prescribed constitutive parameters, possessing a periodic microstructure such as fiber composites. Such materials are prone to be defined for their macroscopic behaviour by effective average elastic constants through an analysis of the microstructure represented by representative unit cells. Then, the inverse homogenization problem can be formulated analogously to the direct problem of the previous section. We make here the same assumptions and consider again the cells *I* and *II* and the unit load problems (1), (2), (3) of Fig. 1.



Now the "homogeneous" cell *I* is given, i.e. relation (9) is valid; the elastic constants are known and they constitute the goal of the optimal design problem. On the other hand, the real cell *II* may now be modified by means of a certain number of design parameters which are summed up in the design vector $\boldsymbol{\beta}$. For instance, either elasticity constants of various constituents in a composite structure, the shape of the inclusions in a reinforced composite or the type and shape of the microstructure may be considered as design variables by an appropriate choice of the elements of vector $\boldsymbol{\beta}$.

By an analogous reasoning to the one used in the previous section, the optimal design problem reads like Eq.(8)

find $\boldsymbol{\beta}$ as a solution of the optimization problem

$$\min_{\boldsymbol{\beta}\in B_{ad}} \frac{1}{2}\sum_{i=1}^{3} w_i \left\{ \prod_{\text{in}}^{I(i)}(e^{(i)}) - \prod_{\text{in}}^{II(i)}(e^{(i)},\boldsymbol{\beta}) \right\}^2 . \tag{18}$$

Here, $B_{ad}$ is the admissible set for the design variables $\boldsymbol{\beta}$, and all other quantities are defined after problem (8).

As with problem (8), problem (18) can be solved by means of various methods. A detailed presentation of the solution of this homogenization problem is not undertaken in this paper, since the method is well established and known. The reader may consult [2], [27] and [28] among others, for recent studies.

The inverse problem is to construct materials with designated properties. It is expected that a number of differently composed bodies may exhibit the same mechanical behaviour. Then, one can choose from a practical standpoint the goal to construct the simplest material with the given parameters, thus solving an optimization problem, whose cost function must be minimized. If the cost function should be the weight of the structure, then the constraints are expressed by the constitutive parameters to be satisfied, and the design variables should define the composition and the topology of the body.

Since the composite materials are periodic structures, they are described by a representative unit cell, which constitutes the smallest repetitive unit of material. A calculation of the effective moduli of the substance can be obtained by analyzing only the unit cell. Considering that the typical composite is a complicated microstructure, an analytic approach for the determination of the properties of the material is rather imposssible and, therefore, a finite-element based numerical method is better suited, due to its simplicity. Here, we are using the homogenization procedure in terms of element mutual energies which renders the inverse problem better suited for optimization. Then, the optimization problem is formulated as a multiple load minimum weight problem, and solved by a modified version of the optimality criterion method proposed in [29].

In fiber-reinforced materials, we are concerned with the general constitutive laws in two-dimensional linear elasticity. We consider here a case of a particular type of microstructure consisting of a star-shaped inclusion with re-entrant corners, as it is indicated in Fig. 2. It is related with materials of a specific microstructure, which can be modelled by means of a truss-like cell. The principal analogue of this example comes from a foamed porous material. Indeed, the truss structure may be a continuum with holes, with the provision for an analytic solution of the problem that none of the holes intersect the cell boundaries.

However, this constraint may be relaxed for the case of a solution based on numerical analysis, provided that the appropriate boundary conditions of the examined cases were conveniently defined. Then, the homogenization relationships can be solved by a finite element approach. Here, the individual bars in the truss-like cell are considered as continuum elements with two modes disposing only of a certain longitudinal stiffness and zero-shear stiffness. In this way, the same software, which is used in finding the homogenized coefficients for the truss-

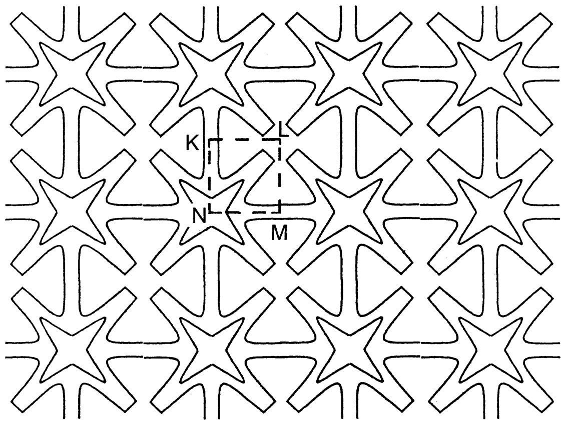


**Fig. 2.** A periodic fiber-reinforced composite with star-shaped encapsulated inclusions

like structure, yields also the continuum-like material. Figure 3 presents a periodic composite material with star-shaped inclusions convenient for developing negative Poisson's ratios.

The described asymptotic homogenization procedure provides rigorous convergence estimates for the displacements of the real structure and those derived by using the homogenized coefficients. Concerning the effective material properties, the method tallies with the approach based on energy principles that employ average stress and strain theorems [30]. This technique presents the advantage to effectively use methods and solutions existing for trusses and similar structures. In this method, three tests of the representative unit cell were considered: two simple tension tests along either of the principal directions of the unit-cell, and a test where the unit cell is deformed under simple shear, as described in the previous section.

According to the standard homogenization procedure, the displacement fields developed on the boundaries of the unit-cell under these three modes of loading are expanded into an asymptotic series, involving functions which depend on the global macroscopic variable and a local microscopic one. The series are truncated to the desired order for each problem. They are used to express the global properties of the material, as indicated in the previous section, taking also into consideration the periodic boundary conditions imposed on the unit cell during each simple mode of loading.

In order to explore the possibility of introducing a convenient shape of cross sections of the inclusions in a fiber composite, which would contribute to the creation of a negative value for the transverse Poisson's ratio of the composite, we examine the case of the truss-like structure in the form of a convex star created by a number of beams and rods, whose principal analogy derives from open foam and porous materials. It is indeed anticipated that in order for a porous material to present a negative Poisson's ratio, its porosity should be rather high and the material should be classified among the open-foam materials. We start our investigation with the convex shaped-beam cell of Fig. 2.

The microstructure of the material produced by this cell is schematically shown in Fig. 3. Using the numerical homogenization concepts of Sec. 4, we model the unit cell of Fig. 2 by

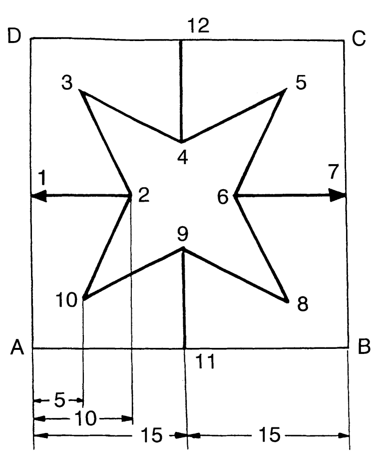


**Fig. 3.** A star-shaped, two-dimensional beam-like cell with re-entrant corners simulating the unit cell of Fig. 2. Finite element discretization and mode numbering

means of two-dimensional beam finite elements. Fixed-end boundary conditions (support) are considered at point 1, and a unit load in the horizontal direction applied at point 7. For the cell described above, with geometric dimensions as in Fig. 2, we assume that the beams have a cross section equal to unity, a moment of inertia equal to 1000, and they are made of an elastic material with elastic modulus equal to $E = 1000$. For a shear factor equal to 0.3 (resp. to 0.9), the phenomenological elastic modulus $E$ and Poisson's ratio $\nu$ calculated by the numerical homogenization theory, are plotted in Figs. 4a and 5a respectively, for low values of the shear modulus $G$ of the structure varying between $G = 10^2$ and $G = 10^3$. For higher values of the shear modulus $G$, varying between $G = 10^3$ and $G = 10^4$, the variation of $E$ and $\nu$ is plotted in Figs. 4b and 5b respectively.

From the above results, a negative Poisson's ratio effect is clearly demonstrated. One should nevertheless underline here that the above parametric investigation is extrapolated outside the range of mechanically admissible values for the material constants in order to give a better visualization of the sought dependence between Poisson's ratio and structural constants for a given cell geometry. In fact, a value of $G = 333.30$ corresponds to a beam material with $\nu = 0.5$, which leads to a Poisson's ratio for the microstructure equal to $-0.2815$ for a material with shear factor equal to 0.3. Whereas, for a material with shear factor equal to 0.9, the respective value for Poisson's ratio is equal to $-0.1538$ for the low range of variation of $G$ ($100 \leq G \leq 1000$). However, a value $G = 1000$ corresponds to a beam material with $\nu = -0.5$ , which leads to a Poisson's ratio for the microstructure equal to $-0.1524$ for a low shear factor 0.3, and equal to $\nu = -0.0120$ for a high shear factor. These results indicate that the shape and the geometry of the microstructure, and not the material constants of its elements, are mainly responsible for a negative Poisson's ratio.

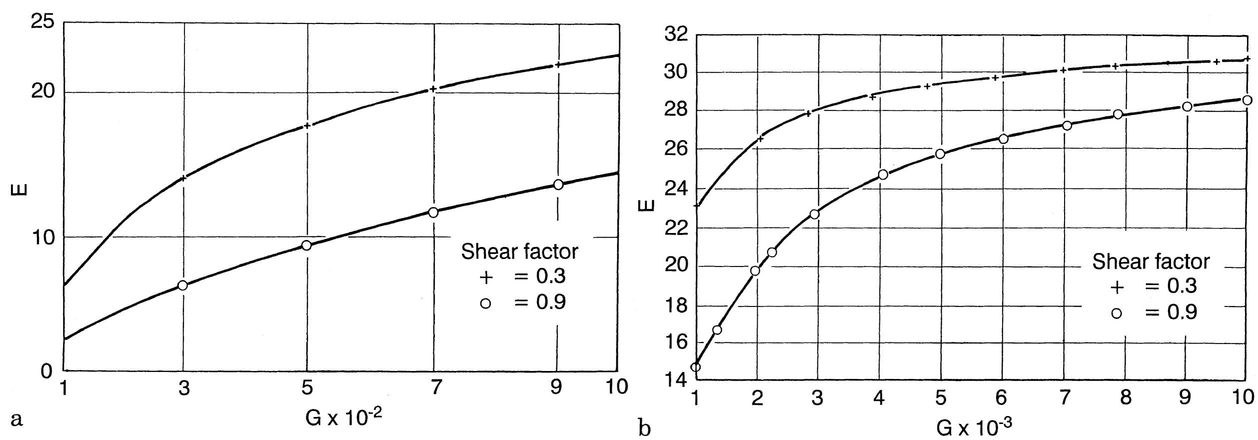


**Fig. 4a,b.** The variation of the elastic modulus $E$ of the composite versus its shear modulus $G$ for $10^2$ <G< $10^3$ **a** and for $10^3 < G < 10^4$ **b**

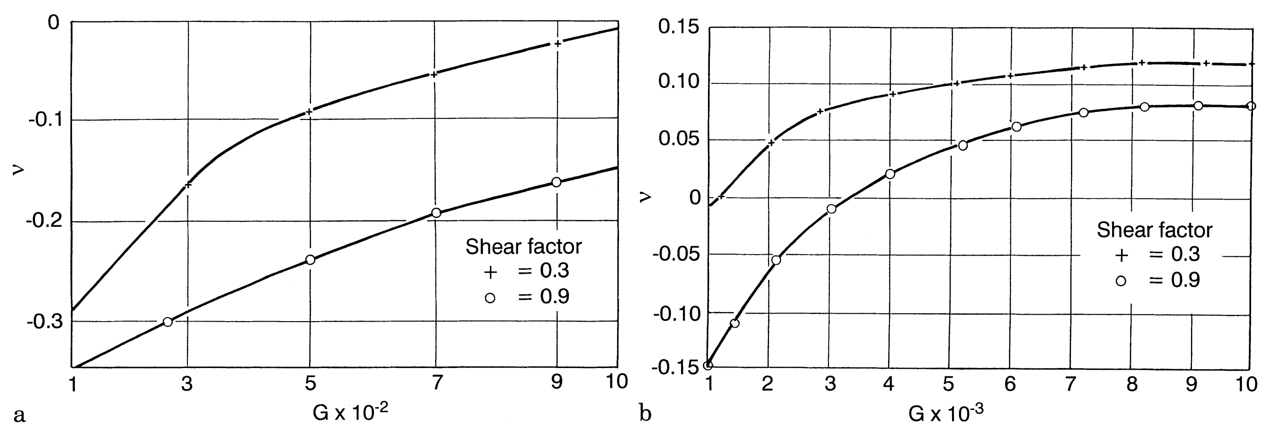


**Fig. 5a,b.** The variation of Poisson's ratio $\nu$ of the composite versus its shear modlus $G$ for $10^2 < G < 10^3$ **a** and for $10^3 < G < 10^4$ **b**

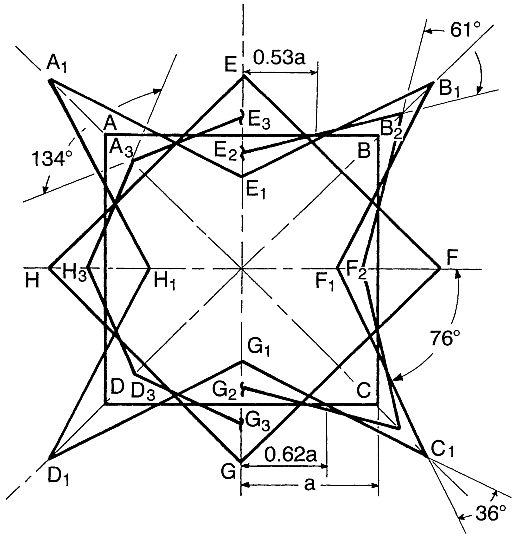


**Fig. 6.** Parametric investigation of five types of a star-shaped two-dimensional unit cell expressed as a truss-like structure

The influence of the shape of the inclusions will be studied subsequently. For this purpose we consider five different shapes and orientations of inclusions, whose forms and orientation are indicated in Fig. 6. Indeed, from the shape of square cell type of Fig. 3 with four re-entrant sides, whose angles at its corners are equal to $\theta = 36°$, we create the four successive forms by increasing progressively the angles $\theta$ to be : $\theta_b = 61°$, $\theta_c = 90°$, $\theta_d = 134°$ and $\theta_e = 180°$. The microstructures of the three materials produced by the cells a, c and e are schematically shown in Fig. 7. The deformation modes of these three types of cells under horizontal tensile unit-loads applied to the respective frames are shown in Fig. 8. It is schematically shown in this figure that for case (a) the Poisson's ratio should be negative, whereas for the two other modes, either Poisson's ratio is insignificant (case (c)), or it takes large values (case (e)). For the beam elements, we consider the following constants: cross area 50, moment of inertia 416.66, shear factor 0.9, $E = 10^6$, $G = 333.30$, that is a material with $\nu = 0.5$. The previously outlined numerical homogenization method is applied. Numerical analysis and application of the homogenization method indicated that these three types of frames exhibit Poisson's ratios equal to −0.2715, +0.02928 and +0.40134, respectively. Examples of the variation of the elastic modulus $E$, and the Poisson's ratio for the types of materials with such microstructures are shown in Figs. 9 and 10, respectively, as found by applying the homogenization and the numerical analysis technique.

It can be derived from the above results given in Figs. 9 and 10, that a negative Poisson's ratio effect is clearly developing with such a type of composite, where the inclusions have a star-like shaped cross section with sides containing re-entrant corners. Furthermore, the same figures yield the conclusion that, as the ratio of $G/E$ of the material is reduced and its shear factor is also reduced absolutely, higher negative values for the Poisson ratio may be attained.

The following conclusion may be derived from the previous results. It can be stated that mainly the shape and the geometry of microstructure and, secondarily, the particular mechanical properties of the composite are responsible for creating composite materials with negative values of Poisson's ratio.

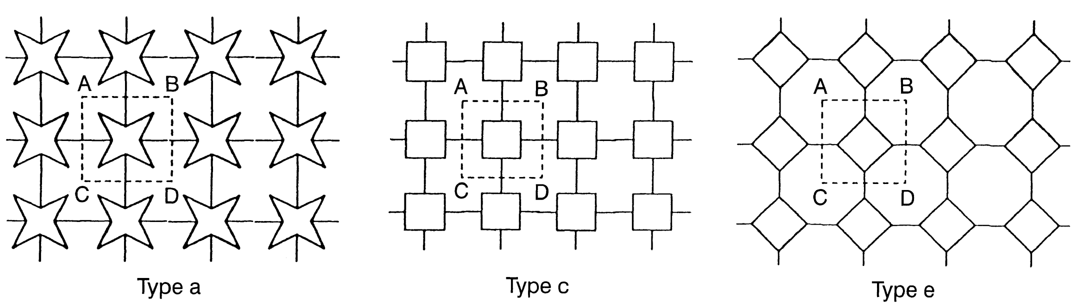


**Fig. 7.** Three types of microstructures produced by the unit-cells (a), (c) and (e) of Fig. 6

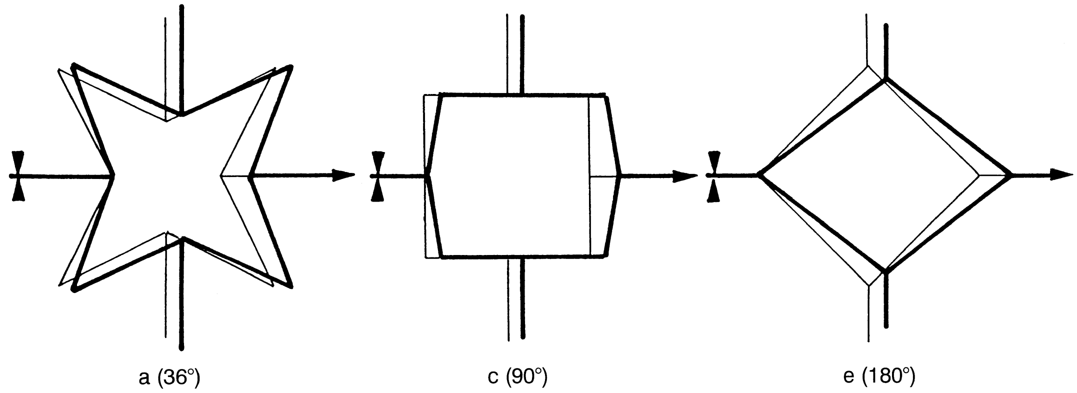


**Fig. 8.** Initial (thin lines) and deformed (solid lines) configurations for the cells of Fig. 7. The effect of negative, near zero, and positive Poisson's ratio are shown

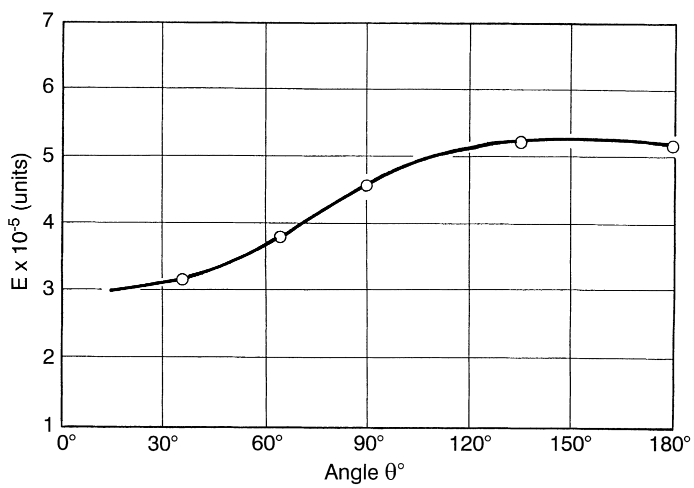


**Fig. 9.** The variation of the elastic modulus $E$ of the unit cell versus the angle of the corners of the star-shaped inclusions

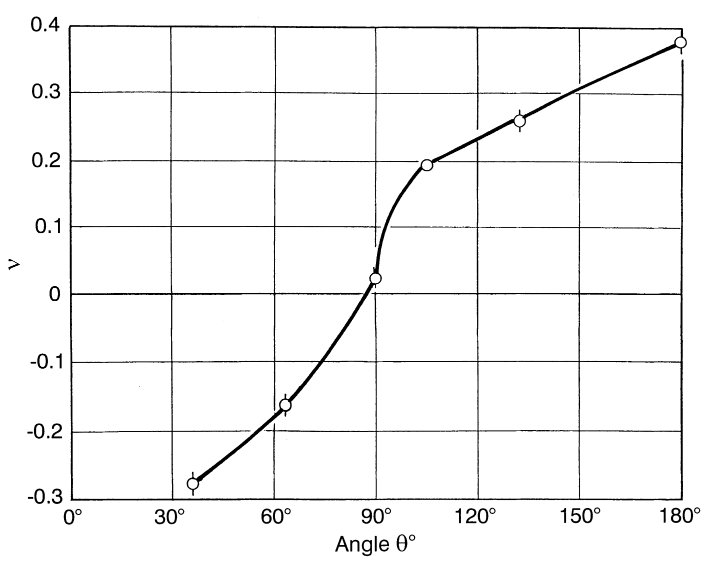


**Fig. 10.** The variation of Poisson's ratio $\nu$ of the unit cell versus the angle of the corners of the star-shaped inclusions

Let us now examine a particular microstructure of a fiber composite consisting of arrays of star-like inclusions with re-entrant corners which are encapsulated by layers of interfaces, as those indicated in Fig. 2, where the interface layer is strongly exaggerated. The microstructure of the composite may be considered as consisting of unit cells corresponding to the squares KLMN of Fig. 2, whose finite-element discretization is shown in Fig. 11.

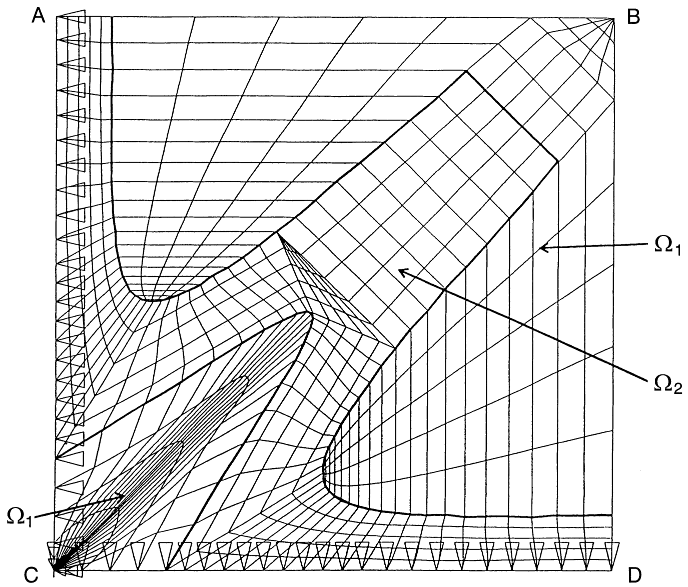


**Fig. 11.** Finite element model for the analyzed unit cell of Fig. 2

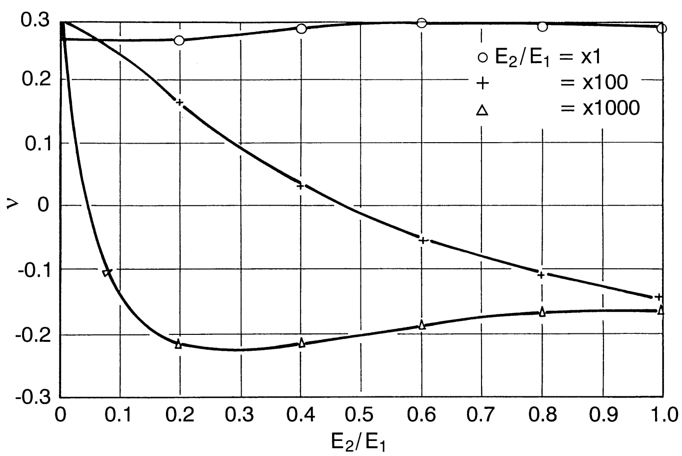


**Fig. 12.** The variation of the elastic modulus $E$ of the composite indicated by Fig. 2 and represented by the unit-cell of Fig. 11 versus the ratio $E_1/E_2$ of the elastic modulus of the phases

For the respective isotropic material, which occupies the region $\Omega_1$ (matrix) of the composite, we consider an elastic modulus $E_1 = 100$ and Poisson's ratio $\nu_1 = 0.3$. For the material of the region $\Omega_2$, that is the reinforcement of the composite, we consider $\nu_2 = 0.30$ and several values for $E_2$, from a weak material with $E_2 = 10$ to a very strong one with $E_2 = 10^5$. The dependence of Poisson's ratio of the composite on the ratio $E_2/E_1$ is presented in Fig. 12. It is clear that for low values of the ratio $E_2/E_1$ (lower than $E_2/E_1 = 46$) the Poisson ratio of the structure remains positive. Above this limit this material parameter becomes negative.

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