Archive of Applied Mechanics 70 (2000) 673–693 © Springer-Verlag 2000

Spheroidal inhomogeneity in a transversely isotropic piezoelectric medium

V. M. Levin, Th. Michelitsch, I. Sevostianov

Summary Piezoelectric material containing an inhomogeneity with different electroelastic properties is considered. The coupled electroelastic fields within the inclusion satisfy a system of integral equations solved in a closed form in the case of an ellipsoidal inclusion. The solution is utilized to find the concentration of the electroelastic fields around an inhomogeneity, and to derive the expression for the electric enthalpy of the electroelastic medium with an ellipsoidal inclusion that is relevant for various applications. Explicit closed-form expressions are found for the electroelastic fields within a spheroidal inclusion embedded in the transversely isotropic matrix. Results are specialized for a cylinder, a flat rigid disk and a crack. For a penny-shaped crack, the quantities entering the crack propagation criterion are found explicitly.

Key words Inclusion, piezoelectric material, electro-mechanical field

1 Introduction

Solutions for spheroidal inhomogeneities in a piezoelectric material are of key importance in connection with several problems. First, they constitute the basic building block for modeling the effective electroelastic properties of piezocomposites. Second, they yield concentration factors for the electroelastic fields near inclusions. Third, in the limiting case of a crack, the results yield the quantities that enter the crack propagation criterion, [1].

The problem of a spheroidal inclusion in a piezoelectric material has been considered by several authors. The classical approach of Eshelby was extended to the piezoelectric material in [2], but results were not derived in an explicit form. In [3, 4], electroelastic fields in the case of an ellipsoidal inclusion and in the limiting case of an elliptical crack were derived; however, the results were given in the form of integrals, containing Green's function that was unknown at that time and could not, therefore, be readily used. General representation for an inhomogeneity of an arbitrary (not necessary ellipsoidal) shape was given in [5]. In [6, 7], Eshelby's tensor was considered for an ellipsoidal inhomogeneity in a transversely isotropic and an orthotropic medium, correspondingly; however, similarly to [3, 4], these results were given in an integral, nonexplicit form due to the fact that Green's function was not available in an explicit form at that time. Explicit expressions for components of Eshelby's tensor for the spheroidal inhomogeneity were obtained in [8]. Similar results were derived by a different method in [9], where the limiting case of an infinite cylinder (a "fiber") was also analyzed in detail.

Received 17 February 2000; accepted for publication 9 May 2000

V. M. Levin
Division of Mechanics, Petrozavodsk State University,
Petrozavodsk, 185640, Russia

Th. Michelitsch
Institute for Theoretical Physics, University of Stuttgart,
D-70550, Stuttgart, Germany

I. Sevostianov (✉)
Department of Mechanical Engineering, Tufts University,
Medford, MA 02155, USA
Fax: +1-617 627 3058
e-mail: isevos01@tufts.edu

The present work constitutes further progress in studies of spheroidal inclusions in piezoelectric media. The new results obtained here can be outlined as follows. First, the electroelastic compliance tensors that describe the contribution of an inhomogeneity to the overall electroelastic response (and, thus, are of direct relevance for the effective electroelastic properties of a composite with multiple inclusions) are derived in the explicit form, in terms of elementary functions. These tensors constitute a generalization of the inclusion compliance tensors in the elasticity of materials with inclusions, [10, 11]. Second, general expressions for coefficients of electromechanical fields concentrations are derived. Third, in the case of a circular crack, the quantities that enter the crack propagation criterion proposed in [1] are explicitly calculated. Finally, the important asymptotic cases of strongly oblate and strongly prolate spheroids are analyzed in detail.

2
Electric and elastic fields in a medium with an inhomogeneity

In this section, we review some general results and modify them to a form suitable for the present work. We consider a homogeneous piezoelastic material under isothermal condition. The governing equations for such a material have the form

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k, \quad D_i = e^T_{ikl}\varepsilon_{kl} + \eta_{ik}E_k \ , \tag{1}$$

where $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ are the stress and strain tensors, $\mathbf{E}$, $\mathbf{D}$ are the electric field intensity and electric induction vectors, $\mathbf{C}$ is the tensor of elastic moduli, $\boldsymbol{\eta}$ is the tensor of dielectric permeabilities and $\mathbf{e}$ is the tensor of piezoelectric constants characterizing coupled electroelastic effects (the superscript T means the transpose).

Relations (1) can be written in the following short form:

$$\mathbf{J} = \mathscr{L}\mathbf{F}, \quad \mathbf{J} = \left\| \begin{matrix} \sigma \\ \mathbf{D} \end{matrix} \right\|, \quad \mathscr{L} = \left\| \begin{matrix} \mathbf{C} & \mathbf{e} \\ \mathbf{e}^T & -\eta \end{matrix} \right\|, \quad \mathbf{F} = \left\| \begin{matrix} \varepsilon \\ -\mathbf{E} \end{matrix} \right\| \ , \tag{2}$$

where the symbolic matrix $\mathscr{L}$ must be regarded as a linear operator, which transforms the tensor–vector pair $[\boldsymbol{\sigma}, \mathbf{D}]$ into the pair $[\boldsymbol{\varepsilon}, \mathbf{E}]$.

The relations inverse to (1) have the form

$$\mathbf{F} = \mathscr{M}\mathbf{J}, \quad \mathscr{M} = \left\| \begin{matrix} \mathbf{S} & \mathbf{d} \\ \mathbf{d}^T & -\boldsymbol{\kappa} \end{matrix} \right\| \ , \tag{3}$$

where

$$\mathbf{S} = \left(\mathbf{C} + \mathbf{e}\boldsymbol{\eta}^{-1}\mathbf{e}^{\mathrm{T}}\right)^{-1}, \quad \boldsymbol{\kappa} = \left(\boldsymbol{\eta} + \mathbf{e}^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{e}\right)^{-1}, \quad \mathbf{d} = \mathbf{S}\mathbf{e}\boldsymbol{\eta}^{-1} = \mathbf{C}^{-1}\mathbf{e}\boldsymbol{\kappa} \ .$$

Let us consider now an infinite homogeneous piezoelectric body with the operator of electroelastic characteristics $\mathscr{L}^0$, containing an inclusion with different operator of electroelastic constants $\mathscr{L}$ occupying a region v. The strain $\varepsilon_{ij}(\mathbf{x})$ and electric intensity $E_i(\mathbf{x})$ fields in an arbitrary point $\mathbf{x}$ of the medium with inhomogeneity satisfy the following system of integral equations, [12]:

$$F(\mathbf{x}) = F^0(\mathbf{x}) + \int_v \mathscr{P}(\mathbf{x} - \mathbf{x}')\mathscr{L}^1 F(\mathbf{x}')\mathrm{d}\mathbf{x}' \ , \tag{4}$$

$$\mathscr{P}(\mathbf{x}) = \mathscr{D}\mathscr{G}(\mathbf{x})\mathscr{D}, \quad \mathscr{D} = \left\| \begin{matrix} \mathrm{def} & 0 \\ 0 & \mathrm{grad} \end{matrix} \right\|, \quad \mathscr{L}^1 = \mathscr{L} - \mathscr{L}^0 \ .$$

Here, $F^0(\mathbf{x})$ stands for the external elastic and electric fields which would have taken place in the homogeneous matrix (without the inclusion) under the same boundary conditions. We assume in the present work that, in the absence of the inclusion, fields $F^0(\mathbf{x})$ can be taken as constant at the length scale of v. The kernel of this equation $\mathscr{P}(\mathbf{x})$ is concentrated in the region v and expressed via the second derivatives of Green's function $\mathscr{G}(\mathbf{x})$ of the equilibrium equations of the coupled electroelasticity. This function satisfies the following equation:

$$\mathscr{T}(\nabla)\mathscr{G}(\mathbf{x})+\delta(\mathbf{x})\mathbf{1}=0, \quad \mathbf{1}=\left\|\begin{matrix}\delta_{ij} & 0\\ 0 & 1\end{matrix}\right\| , \tag{5}$$

$$\mathscr{T}(\nabla)=\left\|\begin{matrix}C_{ipjq}\partial_p\partial_q & e_{ipq}\partial_p\partial_q\\ e^T_{pkq}\partial_p\partial_q & -\eta_{pq}\partial_p\partial_q\end{matrix}\right\|, \quad \mathscr{G}(x)=\left\|\begin{matrix}G_{ik}(x) & \gamma_i(x)\\ \gamma^T_k(x) & g(x)\end{matrix}\right\| . \tag{6}$$

Relations (4) yield the equation for the pair $J=[\sigma,D]$, in terms of the external fields σ^0 and D^0

$$J(\mathbf{x})=J^0(\mathbf{x})+\int_v \mathscr{Q}(\mathbf{x}-\mathbf{x}')\mathscr{M}^1 J(\mathbf{x}')\mathrm{d}\mathbf{x}' ,$$
$$\mathscr{Q}(\mathbf{x})=-\left(\mathscr{L}^0\delta(\mathbf{x})+\mathscr{L}^0\mathscr{P}(\mathbf{x})\mathscr{L}^0\right), \quad \mathscr{M}^1=\mathscr{M}-\mathscr{M}^0 . \tag{7}$$

Let the inclusion be of an ellipsoidal shape, with semiaxes a_1,a_2,a_3. As is well known (see, for example, [13–15]) the constancy of F^0 in v implies that the fields ε_{ij} and E_i inside v are uniform as well. We will need the expressions for the fields inside the inclusion in terms of F^0. They were derived in [13–15]. However, their expressions were given in terms of Fourier integrals, and are unsuitable for our purposes. Therefore, we re-derive these expressions in a closed form.

The electroelastic Green's function can be represented in the following general form:

$$\mathscr{G}(\mathbf{x})=\frac{1}{r}\mathscr{G}^*(\mathbf{a}^r), \quad \mathbf{a}^r=\frac{\mathbf{x}}{r}, \quad r=|\mathbf{x}| . \tag{8}$$

As follows from $\mathscr{G}(\mathbf{x})$ being a quadratic function of its argument, function $\mathscr{G}^*$ is symmetric with respect to the center of the unit sphere: $\mathscr{G}^*(\mathbf{a}^\mathbf{r})=\mathscr{G}^*(-\mathbf{a}^\mathbf{r})$. Introducing the local tangential basis of the spherical system of coordinates (r,θ,φ)

$$\mathbf{a}^r=\frac{\partial\mathbf{x}}{\partial r}, \quad \mathbf{a}^\varphi=\frac{1}{r\sin\theta}\frac{\partial\mathbf{x}}{\partial\varphi}, \quad \mathbf{a}^\theta=\frac{1}{r}\frac{\partial\mathbf{x}}{\partial\theta} , \tag{9}$$

we have

$$\nabla=\mathbf{a}^r\frac{\partial}{\partial r}+\frac{1}{r}\nabla^*, \quad \nabla^*=\frac{\mathbf{a}^\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}+\mathbf{a}^\theta\frac{\partial}{\partial\theta} , \tag{10}$$

and, as follows from Eqs. (9) and (10),

$$\nabla\mathscr{G}(x)=\frac{1}{r^2}\mathscr{G}^1, \quad \mathscr{G}^1=\nabla^*\mathscr{G}^*-\mathscr{G}^*\mathbf{a}^r , \tag{11}$$

where function $\mathscr{G}^1$ is defined on the unit sphere. In contrast to $\mathscr{G}^*$, function $\mathscr{G}^1$ is antisymmetric with respect to the center of this sphere: $\mathscr{G}^1(-\mathbf{a}^r)=-\mathscr{G}^1(\mathbf{a}^r)$. This fact is essential for solving the electroelastic analogue of Eshelby's problem.

Assuming that vectors $\mathbf{x}$ and $\mathbf{x}'$ originate at the center of the ellipsoid ($\mathbf{x}\in v$) and denoting $\mathbf{R}=\mathbf{x}'-\mathbf{x}$, we have

$$\int_v \nabla\mathscr{G}(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}'=-\int_0^{R_s}\mathrm{d}R\int_{S_1}(\nabla^*\mathscr{G}^*-\mathscr{G}^*\mathbf{a}^r)\mathrm{d}S_1 , \tag{12}$$

where R_s denotes the value of $R=|\mathbf{R}|$ on the ellipsoid's surface and S_1 indicates the unit sphere with the center at point $\mathbf{x}$. Let $\mathbf{T}$ be the second rank tensor

$$\mathbf{T}=\frac{\mathbf{a}^1\otimes\mathbf{a}^1}{a_1^2}+\frac{\mathbf{a}^2\otimes\mathbf{a}^2}{a_2^2}+\frac{\mathbf{a}^3\otimes\mathbf{a}^3}{a_3^2} , \tag{13}$$

$\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3$ being unit vectors along the ellipsoid's axes so that $\mathbf{x}^s\cdot\mathbf{T}\cdot\mathbf{x}^s=1$ is the equation of ellipsoid's surface. Since $\mathbf{R}=\mathbf{x}'-\mathbf{x}$, $\mathbf{x}^s=\mathbf{R}^s+\mathbf{x}$, we have $(R_s\mathbf{a}^r+\mathbf{x})\cdot\mathbf{T}\cdot(R_s\mathbf{a}^r+\mathbf{x})=1$. This is a quadratic equation with respect to R_s having only one positive root

$$R_s = (\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{a}^r)^{-1}\left[-\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{x} + \sqrt{\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{x} - (\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{a}^r)(\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} - 1)}\right] \quad . \tag{14}$$

Since $\mathbf{x}^s \cdot \mathbf{T} \cdot \mathbf{x}^s - 1 = 0$, we have $\mathbf{x} \cdot \mathbf{T} \cdot \mathbf{x} - 1 \leq 0$ for $x \in v$. Therefore, the expression under square root in Eq. (14) is positive and $R_s \geq 0$.

To utilize (14), we rewrite Eq. (12) in the form

$$\int_v \nabla \mathscr{G}(\mathbf{x} - \mathbf{x}')\mathrm{d}\mathbf{x}' = -\int_{S_1} (\nabla^* \mathscr{G}^* - \mathscr{G}^* \mathbf{a}^r) R_s \, \mathrm{d}S_1 \quad , \tag{15}$$

We observe that representation of function $R_s(\mathbf{a}^r)$ given by Eq. (14) constitutes a decomposition of R_s into the symmetric and antisymmetric parts. Since multiplication of two anti-symmetric functions results in a symmetric function, placing R_s from Eq. (14) in the right-hand side of Eq. (15) produces a sum of symmetric and antisymmetric functions. An integral of an anti-symmetric function over a unit sphere is zero, therefore,

$$\int_v \nabla \mathscr{G}(\mathbf{x} - \mathbf{x}')\mathrm{d}\mathbf{x}' = \mathscr{H} \cdot \mathbf{x} \quad (x \in v) \quad , \tag{16}$$

where

$$\mathscr{H} = \mathbf{T} \cdot \int_v (\mathbf{a}^r \cdot \mathbf{T} \cdot \mathbf{a}^r)^{-1} \mathbf{a}^r (\nabla^* \mathscr{G}^*(\mathbf{a}^r) - \mathscr{G}^*(\mathbf{a}^r)\mathbf{a}^r)\mathrm{d}S_1 \quad , \tag{17}$$

is an operator that when written in the matrix form (similar to (2)), has constant components. Thus, fields ε^0 and E^0 are uniform in ellipsoidal domain v, Eq. (13) transforms to the algebraic one and solving for F yields

$$F = \mathscr{A} F^0, \quad \mathscr{A} = \left(\mathscr{I} - \mathscr{P}\mathscr{L}^0\right)^{-1}, \quad \mathscr{I} = \left\| \begin{matrix} I_{ijkl} & 0 \\ 0 & \delta_{ik} \end{matrix} \right\|, \quad I_{ijkl} = \delta_{i(k}\delta_{l)j} \quad . \tag{18}$$

Similarly, solving Eq. (7) yields

$$J = \mathscr{B} F^0, \quad \mathscr{B} = \left(\mathscr{I} + \mathscr{Q}\mathscr{M}^1\right)^{-1}, \quad \mathscr{Q} = \mathscr{L}^0(\mathscr{I} + \mathscr{P}\mathscr{L}^0) \quad , \tag{19}$$

where constant operator $\mathscr{P}$ can be obtained from $\mathscr{H}$ by an appropriate symmetrization, and is given for the spheroidal inclusion in Sec. 6.

3
Concentration of electroelastic fields at the inclusion

We now express the electroelastic fields outside of inclusion on its surface (fields concentration coefficients). Let the volume forces and electric dipoles, with intensities $q_{\alpha\beta}(\mathbf{x})$ and $\chi_\alpha(\mathbf{x})$, respectively, be distributed in a domain v bounded by a sufficiently smooth surface ∂v. The strain $\varepsilon(\mathbf{x})$ and the electric field $E(\mathbf{x})$ in an arbitrary point $\mathbf{x}$ of the medium are presented by the following expression:

$$F(\mathbf{x}) = \int_v \mathscr{P}(\mathbf{x} - \mathbf{x}')\mathscr{K}(\mathbf{x}')\mathrm{d}\mathbf{x}', \quad \mathscr{K}(\mathbf{x}) = \left\| \begin{matrix} q_{\alpha\beta}(\mathbf{x}) \\ \chi_\alpha(\mathbf{x}) \end{matrix} \right\| \quad . \tag{20}$$

Here, function $F(\mathbf{x})$ is continuous inside and outside of v, but is discontinuous on ∂v. The jump can be determined analogously to the purely elastic case, [16]. We present integral (20) in the following form:

$$F(\mathbf{x}) = \int_v \mathscr{P}(\mathbf{x} - \mathbf{x}')[\mathscr{K}(\mathbf{x}') - \mathscr{K}(\mathbf{x})]\mathrm{d}\mathbf{x}' + \int_v \mathscr{P}(\mathbf{x} - \mathbf{x}')\mathrm{d}\mathbf{x}' \cdot \mathscr{K}(\mathbf{x}) \quad . \tag{21}$$

For the smooth and bounded function $\mathcal{K}(\mathbf{x})$, the first term in the right-hand side of Eq. (21) is continuous on ∂v. Let us consider the limit of the second integral in (21) when point $\mathbf{x}$ tends to point $\mathbf{x}_0$ on ∂v inside and outside of domain v.

We introduce the cartesian coordinates y_1, y_2, y_3 with the origin at $\mathbf{x}_0$ and axis y_3 directed along external normal $\mathbf{n}(x_0)$ to ∂v. As a first step, we consider the limit of the integral

$$\mathcal{J}(\mathbf{y}) = \int_v \mathcal{P}(\mathbf{y} - \mathbf{y}')\mathrm{d}\mathbf{y}' \ , \tag{22}$$

when $\mathbf{y} \to 0$ and $\mathbf{y} \notin v$. Let us fix the point $\mathbf{y} = \mathbf{y}_0 \notin v$ and introduce the dimensionless variables $\zeta_i = \mathbf{y}_i/|\mathbf{y}_0| \quad (i = 1, 2, 3)$. Since function $\mathcal{P}(y)$ is homogeneous of power -3 we have

$$\mathcal{J}(\mathbf{y}) = \mathcal{J}(\zeta|\mathbf{y}_0|) = \int \mathcal{P}(\zeta - \zeta')U(\zeta')\mathrm{d}\zeta' \ , \tag{23}$$

where $U(\zeta)$ is the characteristic function of the region v. Substituting $\mathbf{y} = \mathbf{y}_0$ into the integral and letting $\mathbf{y}_0 \to 0$, vector $\boldsymbol{\zeta}_0 = \mathbf{y}_0/|\mathbf{y}_0|$ is the unit vector that determines the direction in which point $\mathbf{y}_0$ approaches the origin. In the limit $\mathbf{y}_0 \to 0$, region v is transformed to half-space $\zeta_3 < 0$, i.e. $U(\zeta_1, \zeta_2, \zeta_3) \to 1 - H(\zeta_3)$, where $H(\zeta)$ is the Heavyside's function. It follows from here

$$\lim_{y_0 \to 0} \mathcal{J}(y_0) = \int \mathcal{P}(\zeta_0 - \zeta')H_1(\zeta')\mathrm{d}\zeta' = \frac{1}{(2\pi)^3} \int \mathcal{P}^*(k)H_1^*(k)\exp(-ik \cdot \zeta_0)\mathrm{d}k \ , \tag{24}$$

where $f^*(k)$ is Fourier transformation of $f(\mathbf{x})$ and where it is denoted

$$H_1(\zeta_1, \zeta_2, \zeta_3) = 1 - H(\zeta_3) \ . \tag{25}$$

Taking into account the relation

$$H^*(k_1, k_2, k_3) = (2\pi)^2 \delta(k_1)\delta(k_2)\left[\pi\delta(k_3) + \frac{i}{k_3}\right] \ , \tag{26}$$

we obtain

$$\lim_{\mathbf{y}_0 \to 0} \mathcal{J}(\mathbf{y}_0) = \tfrac{1}{2}[\mathcal{P}^*(0) - \mathcal{P}^*(n)] = \mathcal{J}^+(0), \quad \mathbf{y}_0 \notin v \ . \tag{27}$$

The limit of $\mathcal{J}(\mathbf{y}_0)$ at $\mathbf{y}_0 \to 0$ and $\mathbf{y} \in v$ can be found analogously

$$\lim_{\mathbf{y}_0 \to 0} \mathcal{J}(\mathbf{y}_0) = \tfrac{1}{2}[\mathcal{P}^*(0) + \mathcal{P}^*(n)] = \mathcal{J}^-(0), \quad \mathbf{y}_0 \in v \ . \tag{28}$$

Hence, the jump of the integral $\mathcal{J}(\mathbf{y})$ on interface ∂v is given by

$$[\mathcal{J}(n)] = \mathcal{J}^+(0) - \mathcal{J}^-(0) = -\mathcal{P}^*(n) \ . \tag{29}$$

This implies, with the account of Eq. (20), that

$$[F(\mathbf{x}_0)] = F^+(\mathbf{x}_0) - F^-(\mathbf{x}_0) = \mathcal{P}^*(\mathbf{n}_0)\mathcal{K}(\mathbf{x}_0) \ , \tag{30}$$

where $\mathbf{n}_0 = \mathbf{n}(\mathbf{x}_0)$ is the vector of inward normal to ∂v in point $\mathbf{x}_0 \in \partial v$.

Returning to Eq. (4), we can present function $F(\mathbf{x})$ as the following sum:

$$F(x) = F^0(x) + F^1(x), \quad F^1(\mathbf{x}) = \int_v \mathcal{P}(\mathbf{x} - \mathbf{x}')\mathcal{L}^1 F(\mathbf{x}')\mathrm{d}\mathbf{x}' \ , \tag{31}$$

where $F^0(\mathbf{x})$ is assumed to be a continuous function. Therefore, the jump of the field $F(\mathbf{x})$ is determined by $F^1(\mathbf{x})$ only. This function can be interpreted as representing the electroelastic fields in the homogeneous medium with the properties $\mathcal{L}^0$ induced by dipoles with density

$$\mathscr{K}(\mathbf{x}) = \mathscr{L}^1 F(\mathbf{x}) \ , \tag{32}$$

distributed in domain v. Hence, we can write

$$F^-(\mathbf{x}_0) - F^+(\mathbf{x}_0) = -\mathscr{P}^*(\mathbf{n}_0)\mathscr{L}^1 F^+(\mathbf{x}_0) \ . \tag{33}$$

It follows that the limiting values F^+ and F^- of the electroelastic fields inside and outside of domain v are interrelated as follows:

$$F^-(\mathbf{x}_0) = \left[\mathscr{I} - \mathscr{P}^*(\mathbf{n}_0)\mathscr{L}^1\right] F^+(\mathbf{x}_0) \ . \tag{34}$$

Taking into account (18), we finally obtain, in the case of the ellipsoidal inclusion,

$$F^-(\mathbf{n}_0) = \mathscr{R}^F(\mathbf{n}_0) F^0, \quad \mathscr{R}^F(\mathbf{n}_0) = \left[\mathscr{I} - \mathscr{P}^*(\mathbf{n}_0)\mathscr{L}^1\right]\mathscr{A} \ , \tag{35}$$

where $\mathscr{R}^F(\mathbf{n}_0)$ can be interpreted as an operator of tensor coefficients of electroelastic fields concentration on the inclusion in the piezoelectric medium.

Analogously, one can find

$$J^-(\mathbf{n}_0) = \mathscr{R}^J(\mathbf{n}_0) J^0, \quad \mathscr{R}^J(\mathbf{n}_0) = \left[\mathscr{I} - \mathscr{Q}^*(\mathbf{n}_0)\mathscr{M}^1\right]\mathscr{B} \ , \tag{36}$$

where $\mathscr{R}^J(\mathbf{n}_0)$ can be regarded as an operator of concentration coefficients of fields σ and $\mathbf{D}$.

4
Electric enthalpy of a solid with an inhomogeneity

We consider the electric enthalpy that was defined for the piezoelectric solid in [1] as follows:

$$W = \tfrac{1}{2}(\sigma_{ij}\varepsilon_{ij} - E_i D_i) = \tfrac{1}{2} J \cdot F \ , \tag{37}$$

so that the stress and the electric induction are given by

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, \quad D_i = -\frac{\partial W}{\partial E_i} \ . \tag{38}$$

The density of electric enthalpy in a certain volume V containing an inhomogeneity is given by the expression

$$W = W^0 + \frac{1}{2V}\int_V (J \cdot F - J^0 \cdot F^0)\mathrm{d}V \ , \tag{39}$$

where $W^0 = \frac{1}{2}\int_V J^0 \cdot F^0 \, \mathrm{d}V$ is the electric enthalpy for the homogeneous matrix material under the same boundary conditions. Using equations of the elastic and electric equilibrium: $\partial_j \sigma_{ij} = 0, \partial_i D_i = 0$, and applying Gauss' theorem, we obtain

$$W = W^0 + \Delta W, \quad \Delta W = \frac{1}{2V}\int_{\partial V} \Sigma^0 \cdot (U - U^0)\mathrm{d}s \ , \tag{40}$$

where

$$U = \left\| \begin{matrix} u_i \\ \varphi \end{matrix} \right\|, \quad \Sigma = \left\| \begin{matrix} \sigma_{ij} n_j \\ D_i n_i \end{matrix} \right\|, \quad \Sigma \cdot U = (\sigma_{ij} u_i + D_j \varphi) n_j \ , \tag{41}$$

and where u_i is the elastic displacement, φ is the electric potential and n_i is the unit outward normal to ∂V.

The expression for ΔW in (40) can be transformed to the quadratic form in J^0 the same way that was used in [17] for the uncoupled elasticity. Representing fields U, F and D in the form

$$U = U^0 + U', \quad F = F^0 + F', \quad D = D^0 + D' \ , \tag{42}$$

where the primed quantities denote perturbations due to the inclusion, we have, taking into account that $U' = 0$ on ∂V,

$$\Delta W = \frac{1}{2V}\int\limits_{\partial V} \Sigma^0 \cdot U' \, \mathrm{d}s = \frac{1}{2V}\int\limits_{V} J^0 \cdot F' \, \mathrm{d}v = \frac{1}{2V}\left(\int\limits_{v} J^0 \cdot F' \, \mathrm{d}v + \int\limits_{V-v} J^0 \cdot F' \, \mathrm{d}v\right) \ . \tag{43}$$

Since

$$J^0 \cdot F' = J^0 \cdot (\mathscr{M}^0 J') = (\mathscr{M}^0 J^0) \cdot J' = F^0 \cdot J' \ , \tag{44}$$

in the domain $V - v$, expression (43) can be rewritten as

$$\Delta W = \frac{1}{2V}\left(\int\limits_{\partial v} \Sigma^0 \cdot U' \, \mathrm{d}s - \int\limits_{\partial v} U^0 \cdot \Sigma' \, \mathrm{d}s + \int\limits_{\partial V} U^0 \cdot \Sigma' \, \mathrm{d}s\right) \ , \tag{45}$$

and, since $J' = 0$ on ∂V, expression (45) with the account of (42) takes the form

$$\Delta W = \frac{1}{2V}\int\limits_{\partial v} \left(U \cdot \Sigma^0 - U^0 \cdot \Sigma\right) \mathrm{d}s = \frac{1}{2V}\int\limits_{v} J^0 \mathscr{M}^1 J \, \mathrm{d}v \ , \tag{46}$$

if the boundary conditions are given in terms of stress and electric induction D. Similarly, it can be shown that

$$\Delta W = \frac{1}{2V}\int\limits_{\partial v} \left(U^0 \cdot \Sigma - U \cdot \Sigma^0\right) \mathrm{d}s = \frac{1}{2V}\int\limits_{v} F^0 \mathscr{L}^1 F \, \mathrm{d}v \ . \tag{47}$$

Formulae (46) and (47) are full analogues of Eshelby's results for the elastic energy of a medium with an inhomogeneity.

Let us consider the case when the external fields σ^0 and D^0 (or ε^0 and E^0) are uniform in the ellipsoidal domain v. Substituting the expressions for F and J from (18) and (19) in (46) and (47), we finally obtain

$$W = W^0 + \frac{v}{2V} J^0 \mathscr{M}^B J^0, \quad \mathscr{M}^B = \mathscr{M}^1 \mathscr{B} \ , \tag{48}$$

if the external fields σ^0 and D^0 are given on ∂V, and

$$W = W^0 + \frac{v}{2V} F^0 \mathscr{L}^A F^0, \quad \mathscr{L}^A = \mathscr{L}^1 \mathscr{A} \ , \tag{49}$$

if ε^0 and E^0 are given on ∂V. In these expressions, $v = \frac{4}{3}\pi a_1 a_2 a_3$ is the ellipsoid's volume.

The obtained formulae allow one to find the expressions for the effective electroelastic characteristics of the piezoelectric materials containing a random set of inhomogeneities in the case of dilute concentration, and may be useful in various self-consistent schemes.

5
Green's function for the transversely isotropic piezoelectric medium

The general formulae (18), (19), (49) and (4.14) are valid for the arbitrary anisotropic materials of the inclusion and of the matrix. We consider now the special case when the matrix has hexagonal (transversely isotropic) symmetry. Such materials are characterized by five independent elastic moduli (written in standard Voigt's two-indices notation): $\mathbf{C} = \{C_{11}, C_{13}, C_{33}, C_{44}, C_{66} = (C_{11} - C_{12})/2\}$, three piezoelectric constants $\mathbf{e} = \{e_{31}, e_{15}, e_{33}\}$ and two permeability coefficients $\boldsymbol{\eta} = \{\eta_{11}, \eta_{33}\}$. The electroelastic Green's function was constructed earlier in [18–21]. The most compact and closed form with the convenient separation on φ and θ dependence of this function was given in [21]. Following this approach, we utilize the Fourier transform of operator $\mathscr{T}(\nabla)$ in Eq. (6) in the form

$$\mathscr{T}(\mathbf{k}) = k^2 \left\| \begin{matrix} T_{ik}(\mathbf{n}) & t_i(\mathbf{n}) \\ t_k^T(\mathbf{n}) & \tau(\mathbf{n}) \end{matrix} \right\|, \quad \mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|} \ , \tag{50}$$

where $\mathbf{k}$ is the wavevector of the three-dimensional Fourier transform and where it is denoted

$$\begin{aligned} T_{ik}(\mathbf{n}) &= T_{b\perp} e_i^2 e_k^2 + T_b e_i^1 e_k^1 + T_{bc}(e_i^1 e_k^3 + e_i^3 e_k^1) + T_c e_i^3 e_k^3, \\ t_i(\mathbf{n}) &= t_b e_i^1 + t_c e_i^3 \ . \end{aligned} \tag{51}$$

Here, unit vectors $\mathbf{e}^1, \mathbf{e}^2$ and $\mathbf{e}^3$ are elements of the following vector basis:.5

$$\mathbf{e}^1 = \frac{1}{n_b}(n_1, n_2, 0), \quad \mathbf{e}^2 = \frac{1}{n_b}(-n_2, n_1, 0), \quad \mathbf{e}^3 = (0, 0, 1) \ , \tag{52}$$

$\mathbf{e}^3$ being in the direction of the axis of symmetry of the transversely isotropic medium and $n_b = \sqrt{n_1^2 + n_2^2}$. Scalar quantities $T_{b\perp}, T_b, T_{bc}, T_c, t_b, t_c$ and τ are as follows:

$$\begin{aligned} &T_{b\perp} = C_{66}^0 n_b^2 + C_{44}^0 n_3^2, \quad T_b = C_{11}^0 n_b^2 + C_{44}^0 n_3^2, \quad T_{bc} = (C_{13}^0 + C_{44}^0) n_b n_3, \\ &T_c = C_{44}^0 n_b^2 + C_{33}^0 n_3^2, \quad t_b = (e_{31}^0 + e_{15}^0) n_b n_3, \quad t_c = e_{15}^0 n_b^2 + e_{33}^0 n_3^2, \\ &\tau = -(\eta_{11}^0 n_b^2 + \eta_{33}^0 n_3^2) \ . \end{aligned} \tag{53}$$

Matrix $\tilde{\mathscr{G}}(\mathbf{k}) = \mathscr{T}^{-1}(\mathbf{k})$ can now be written in the form

$$\tilde{\mathscr{G}}(\mathbf{k}) = \frac{1}{k^2} \tilde{\mathscr{G}}(\mathbf{n}), \quad \tilde{\mathscr{G}}(\mathbf{n}) = \frac{1}{f(\mathbf{n})} \left\| \begin{matrix} G_{ik}(\mathbf{n}) & \gamma_i(\mathbf{n}) \\ \gamma_k(\mathbf{n}) & g(\mathbf{n}) \end{matrix} \right\| \ , \tag{54}$$

$$G_{ik}(\mathbf{n}) = G_{b\perp} e_i^2 e_k^2 + G_b e_i^1 e_k^1 + G_{bc}(e_i^1 e_k^3 + e_i^3 e_k^1) + G_c e_i^3 e_k^3 \ ,$$

$$\gamma_i(\mathbf{n}) = \gamma_b e_i^1 + \gamma_c e_i^3 \ ,$$

Here, the following scalar quantities are introduced:

$$\begin{aligned} &G_{b\perp} = \tau(T_b T_c - T_{bc}^2) - (t_c^2 T_b - 2 t_b t_c T_{bc} + t_b^2 T_c), \quad G_b = T_{b\perp}(T_c \tau - t_c^2), \\ &G_{bc} = -T_{b\perp}(T_{bc}\tau - t_b t_c), \quad G_c = T_{b\perp}(T_b \tau - t_b^2), \quad \gamma_b = T_{b\perp}(T_{bc} t_c - T_c t_b), \\ &\gamma_c = -T_{b\perp}(T_b t_c - T_{bc} t_b), \quad g = T_{b\perp}(T_b T_c - T_{bc}^2) \ . \end{aligned} \tag{55}$$

The determinant $f(\mathbf{n})$ of $\mathscr{T}(\mathbf{n})$ in (54) then takes the form

$$f(\mathbf{n}) = T_{b\perp}(\mathbf{n}) G_{b\perp}(\mathbf{n}) \ . \tag{56}$$

Introducing Eqs. (53)–(55) in Eq. (56) shows that only those terms that are proportional to $n_b^{2s} n_3^{8-2s}$ $(s = 0, 1, ..., 4)$ appear. Thus $G_{b\perp}$ is a polynomial of third degree in $a = n_b^2/n_3^2$. Therefore,

$$G_{b\perp} = (Aa^3 + Ba^2 + Ca + D) n_3^6 \ , \tag{57}$$

where the coefficients of the third degree polynomial are given by

$$\begin{aligned} A &= -(\eta_{11}^0 C_{11}^0 C_{44}^0 + C_{11}^0 (e_{15}^0)^2), \\ B &= -\eta_{33}^0 C_{11}^0 C_{44}^0 - \eta_{11}^0 (C_{11}^0 C_{33}^0 - 2 C_{13}^0 C_{44}^0 - (C_{13}^0)^2) - C_{44}^0 (e_{15}^0)^2 - 2 C_{11}^0 e_{15}^0 e_{33}^0 \\ &\quad + 2(C_{13}^0 + C_{44}^0) e_{15}^0 (e_{31}^0 + e_{15}^0) - C_{44}^0 (e_{31}^0 + e_{15}^0)^2, \end{aligned}$$

$$C = -\eta^0_{33}(C^0_{11}C^0_{33} - 2C^0_{13}C^0_{44} - (C^0_{13})^2) - \eta^0_{11}C^0_{33}C^0_{44} - 2C^0_{44}e^0_{15}e^0_{33}$$
$$- (e^0_{33})^2C^0_{11} + 2e^0_{33}(e^0_{31} + e^0_{15})(C^0_{13} + C^0_{44}) - C^0_{33}(e^0_{31} + e^0_{15})^2, \quad (58)$$
$$D = -(\eta^0_{33}C^0_{33}C^0_{44} + (e^0_{33})^2C^0_{44}) \ ,$$

so that

$$f(\mathbf{n}) = n_3^8 C^0_{66} A(a + A_1)(a + A_2)(a + A_3)(a + A_4) \ , \quad (59)$$

with $T_{b\perp}(\mathbf{n}) = C^0_{66}(a + A_1)n_3^2$, $A_1 = C^0_{44}/C^0_{66}$, and A_2, A_3, A_4 being the roots of the equation

$$Aa^3 - Ba^2 + Ca - D = 0 \ , \quad (60)$$

the coefficients of which (58) are expressed in terms of the components of tensors $\mathbf{C}^0$, $\mathbf{e}^0$ and $\boldsymbol{\eta}^0$. Furthermore, the subdeterminants (55) yield

$$G_{b\perp}(a) = Aa^3 + Ba^2 + Ca + D = A(a + A_2)(a + A_3)(a + A_4),$$
$$G_b(a) = -(C^0_{66}a + C^0_{44})\left[(\eta^0_{11}a + \eta^0_{33})(C^0_{44}a + C^0_{33}) + (e^0_{15}a + e^0_{33})^2\right],$$
$$G_{bc}(a) = \sqrt{a}(C^0_{66}a + C^0_{44})\left[(e^0_{31} + e^0_{15})(e^0_{15}a + e^0_{33}) + (\eta^0_{11}a + \eta^0_{33})(C^0_{13} + C^0_{44})\right],$$
$$G_c(a) = -(C_{66}a + C^0_{44})\left[(\eta^0_{11}a + \eta^0_{33})(C^0_{11}a + C^0_{44}) + a(e^0_{31} + e^0_{15})^2\right], \quad (61)$$
$$\gamma_b(a) = \sqrt{a}(C^0_{66}a + C^0_{44})\left[(C^0_{13} + C^0_{44})(e^0_{15}a + e^0_{33}) - (C^0_{44}a + C^0_{33})(e^0_{31} + e^0_{15})\right],$$
$$\gamma_c(a) = -(C^0_{66}a + C^0_{44})\left[(C^0_{11}a + C^0_{44})(e^0_{15}a + e^0_{33}) - a(C^0_{13} + C^0_{44})(e^0_{31} + e^0_{15})\right],$$
$$g(a) = (C^0_{66}a + C^0_{44})\left[a^2C^0_{11}C^0_{44} + a(C^0_{11}C^0_{33} - 2C^0_{13}C^0_{44} - (C^0_{13})^2) + C^0_{33}C^0_{44}\right] \ .$$

The dependence on a is obtained by setting $n_b = \sqrt{a}$ and $n_3 = 1$ in Eqs. (53) and (56), respectively.

The k-representation of Green's function (54) yields operator $\mathscr{P}^*(\mathbf{n})$

$$\mathscr{P}^*(\mathbf{n}) = \frac{1}{f(\mathbf{n})}\left\| \begin{matrix} P^*_{ijkl}(\mathbf{n}) & p^*_{ijk}(\mathbf{n}) \\ p^{*T}_{ikl}(\mathbf{n}) & \pi^*_{ik}(\mathbf{n}) \end{matrix} \right\| \ , \quad (62)$$
$$P^*_{ijkl}(\mathbf{n}) = n_{(j}G_{i)(k}(\mathbf{n})n_{l)}, \quad p^*_{ijk}(\mathbf{n}) = n_{(i}\gamma_{j)}(\mathbf{n})n_k, \quad \pi^*_{ik}(\mathbf{n}) = n_i n_k g(\mathbf{n}) \ .$$

The $\mathbf{r}$-representation of the Green's function $\mathscr{G}(\mathbf{r})$ can be obtained by residue calculation. Omitting details of the derivation, we present the results.

The electroelastic Green's function for the medium with hexagonal symmetry can be written in the form

$$\mathscr{G}(\mathbf{r}) = \frac{1}{r}\mathscr{G}(\theta, \varphi), \quad \mathscr{G}(\theta, \varphi) = \left\| \begin{matrix} G_{ik}(\theta, \varphi) & \gamma_i(\theta, \varphi) \\ \gamma_k(\theta, \varphi) & g(\theta, \varphi) \end{matrix} \right\| \ ,$$
$$G_{ik}(\theta, \varphi) = G_{\varphi\varphi}(\theta)\mathbf{e}^\varphi_i\mathbf{e}^\varphi_k + G_{\rho\rho}(\theta)\mathbf{e}^\rho_i\mathbf{e}^\rho_k + G_{\rho z}(\theta)(\mathbf{e}^\rho_i\mathbf{e}^z_k + \mathbf{e}^z_i\mathbf{e}^\rho_k) + G_{zz}(\theta)\mathbf{e}^z_i\mathbf{e}^z_k, \quad (63)$$
$$\gamma_i(\theta, \varphi) = \gamma_\rho(\theta)\mathbf{e}^\rho_i + \gamma_z(\theta)\mathbf{e}^z_i \ ,$$

where r, φ, θ are the spherical coordinates, and where the following quantities are introduced:

$$G_{\varphi\varphi}(\theta)=\frac{1}{4\pi}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l\Delta_l(\theta)}\left[G_b(-A_l)+\Gamma_b(-A_l)\cot^2\theta\right],$$

$$G_{\rho\rho}(\theta)=\frac{1}{4\pi}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l\Delta_l(\theta)}\left[G_{b\perp}(-A_l)-\Gamma_b(-A_l)\cot^2\theta\right],$$

$$G_{\rho z}(\theta)=-\frac{1}{4\pi}\sum_{l=1}^{4}\frac{\Gamma_{bc}(-A_l)}{\mathscr{E}_l\Delta_l(\theta)}\cot\theta,\quad G_{zz}(\theta)=\frac{1}{4\pi}\sum_{l=1}^{4}\frac{G_c(-A_l)}{\mathscr{E}_l\Delta_l(\theta)},\tag{64}$$

$$\gamma_\rho(\theta)=-\frac{1}{4\pi}\sum_{l=1}^{4}\frac{g_b(-A_l)}{\mathscr{E}_l\Delta_l(\theta)}\cot\theta,\quad \gamma_z(\theta)=\frac{1}{4\pi}\sum_{l=1}^{4}\frac{\gamma_c(-A_l)}{\mathscr{E}_l\Delta_l(\theta)},$$

$$g(\theta)=\frac{1}{4\pi}\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l\Delta_l(\theta)},\quad \Delta_l(\theta)=\sqrt{A_l\sin^2\theta+\cos^2\theta},\quad \mathscr{E}_l=AC_{66}^0\prod_{j=1(j\neq l)}^{4}(A_j-A_l)\ .$$

In these expressions, $\mathbf{e}^\rho,\mathbf{e}^\varphi$ and $\mathbf{e}^z$ are the basis vectors

$$\mathbf{e}^\rho=(\cos\varphi,\sin\varphi,0),\quad \mathbf{e}^\varphi=(-\sin\varphi,\cos\varphi,0),\quad \mathbf{e}^z=(0,0,1)\ ,\tag{65}$$

vector $\mathbf{e}^z$ coincides with the symmetry axis of the transversely isotropic medium. It is also denoted

$$G_{b\perp}(a)-G_b(a)=a\Gamma_b(a),\quad \Gamma_{bc}(a)=\frac{1}{\sqrt{a}}G_{bc}(a),\quad g_b(a)=\frac{1}{\sqrt{a}}\gamma_b(a)\ .\tag{66}$$

6
Operators $\mathscr{P}$ and $\mathscr{A}$ for a spheroidal inhomogeneity in a transversely isotropic piezoelectric medium

We consider a spheroidal inclusion ($a_1=a_2=a,a_3$) in the transversely isotropic piezoelectric medium, with the axis $2a_3$ parallel to the symmetry axis x_3 (also coinciding with z-axis of the spherical coordinate system). Then,

$$\begin{aligned}
&\mathbf{e}^r=\mathbf{e}^\rho\sin\theta+\mathbf{e}^z\cos\theta,\quad \mathbf{e}^\theta=\mathbf{e}^\rho\cos\theta-\mathbf{e}^z\sin\theta,\\
&\nabla^*=\mathbf{e}^\varphi\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}+(\mathbf{e}^\rho\cos\theta-\mathbf{e}^z\sin\theta)\frac{\partial}{\partial\theta},\\
&T_{ij}=\frac{1}{a^2}\left(\theta_{ij}+\xi^2e_i^ze_j^z\right),\quad \theta_{ij}=\delta_{ij}-e_i^ze_j^z,\quad \xi=\frac{a}{a_3},\\
&\mathbf{e}^r\cdot\mathbf{T}\cdot\mathbf{e}^r=\frac{1}{a^2}(\sin^2\theta+\xi^2\cos^2\theta)\ .
\end{aligned}\tag{67}$$

The result of calculation of operator $\mathscr{P}$ for Green's function (50) and spheroidal inclusion can be presented in the form

$$\mathscr{P}=\left\|\begin{matrix}P_{ijkl} & p_{ijk}\\ p_{ikl}^T & \pi_{ik}\end{matrix}\right\|\ .\tag{68}$$

To calculate tensor π_{ik}, we note that, in accordance with general formula (17),

$$\pi_{ik}=a^2T_{i\alpha}\int_0^\pi\left(\sin^2\theta+\xi^2\cos^2\theta\right)^{-1}\left[\int_0^{2\pi}e_\alpha^r\left(\nabla_k^*g(\theta)-e_k^rg(\theta)\right)\mathrm{d}\varphi\right]\sin\theta\ \mathrm{d}\theta\ ,\tag{69}$$

where $g(\theta)$ is given by (63). Integration with respect to φ yields

$$\int_0^{2\pi}e_\alpha^r\left(\nabla_k^*g(\theta)-e_k^rg(\theta)\right)\mathrm{d}\varphi=2\pi\left[\frac{1}{2}f_1(\theta)\theta_{\alpha k}+f_2(\theta)m_\alpha m_k\right],\quad m_\alpha=e_\alpha^z\ ,\tag{70}$$

where

$$f_1(\theta) = \frac{\partial g(\theta)}{\partial \theta}\sin\,\theta\cos\,\theta - g(\theta)\sin^2\,\theta = -\frac{1}{4\pi}\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l}\frac{A_l\sin^2\,\theta}{\Delta_l^3(\theta)},$$
$$f_2(\theta) = -\frac{\partial g(\theta)}{\partial \theta}\sin\theta\cos\theta - g(\theta)\cos^2\theta = -\frac{1}{4\pi}\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l}\frac{A_l^2\cos^2\theta}{\Delta_l^3(\theta)}, \quad (71)$$
$$\Delta_l(\theta) = \left(A_l\sin^2\theta + \cos^2\theta\right)^{\frac{1}{2}}\ .$$

Introducing these expressions into (53) and performing integration with respect to θ we obtain

$$\pi_{ik} = \pi_1\theta_{ik} + \pi_2 m_i m_k,$$
$$\pi_1 = -\frac{1}{4}\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l}J_1^{(l)}, \quad \pi_2 = -\frac{1}{2}\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l}\xi^2 J_2^{(l)}\ , \quad (72)$$

where it is denoted ($u = \cos\theta$)

$$J_1^{(l)} = A_l\int_{-1}^{1}\frac{(1-u^2)\mathrm{d}u}{\left[1+(\xi^2-1)u^2\right]\left[A_l+(1-A_l)u^2\right]^{\frac{3}{2}}} = 2\lambda_l^2\left[1-\frac{1}{2}\xi^2A_l\lambda_l\ln\left(\frac{\lambda_l+1}{\lambda_l-1}\right)\right],$$
$$J_2^{(l)} = \int_{-1}^{1}\frac{u^2\,\mathrm{d}u}{\left[1+(\xi^2-1)u^2\right]\left[A_l+(1-A_l)u^2\right]^{\frac{3}{2}}} = 2\lambda_l^2\left[\frac{1}{2}\lambda_l\ln\left(\frac{\lambda_l+1}{\lambda_l-1}\right)-1\right], \quad (73)$$
$$\lambda_l = \left(1-A_l\xi^2\right)^{-\frac{1}{2}}\ ,$$

Expressions (73) remains valid when λ_l is complex. These integrals, being functions of the aspect ratio of the inclusion, represent shape factors.

The other components of operator $\mathscr{P}$ can be obtained by performing analogous but lengthier calculations. For the explicit presentation of these tensors it is convenient to use the following tensorial basis, formed by unit vector $\mathbf{m} \equiv \mathbf{e}^z$ and tensor $\theta_{ij} = \delta_{ij} - m_i m_j$:

$$T^1_{ijkl} = \theta_{i(k}\theta_{l)j}, \quad T^2_{ijkl} = \theta_{ij}\theta_{kl}, \quad T^3_{ijkl} = \theta_{ij}m_k m_l, \quad T^4_{ijkl} = m_i m_j\theta_{kl},$$
$$T^5_{ijkl} = \theta_{i)(k}m_{l)}m_{(j}, \quad T^6_{ijkl} = m_i m_j m_k m_l, \quad U^1_{ijk} = \theta_{ij}m_k, \quad (74)$$
$$U^2_{ijk} = 2m_{(i}\theta_{j)k}, \quad U^3_{ijk} = m_i m_j m_k, \quad t^1_{ij} = \theta_{ij}, \quad t^2_{ij} = m_i m_j\ .$$

This tensorial basis is convenient for the following reasons: the contraction of the T-basis tensors over two indices gives tensors of the same basis; the contraction of the U-basis tensors over one index gives tensors of the T-basis, and over two indices – tensors of the t-basis. As for as the t-basis is concerned, it is orthogonal with respect to contraction over one index: $t^r_{i\alpha}t^s_{\alpha j} = \delta_{rs}t^r_{ij}$ (no summation over r). Tensorial operations in this basis are discussed in Appendix 1.

Thus, we have

$$\mathbf{P} = P_1\mathbf{T}^2 + P_2\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + P_3\left(\mathbf{T}^3 + \mathbf{T}^4\right) + P_5\mathbf{T}^5 + P_6\mathbf{T}^6\ , \quad (75)$$

$$P_1 = -\frac{1}{8}\sum_{l=1}^{4}\frac{G_b(-A_l)}{\mathscr{E}_l}J_1^{(l)}, \quad P_2 = -\frac{1}{8}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l}[G_b(-A_l) + G_{b\perp}(-A_l)]J_1^{(l)}\ ,$$

$$P_3 = -\frac{1}{8}\sum_{l=1}^{4}\frac{\Gamma_{bc}(-A_l)}{\mathscr{E}_l}\left(J_1^{(l)} - \xi^2 A_l J_2^{(l)}\right),$$

$$P_5 = -\frac{1}{4}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l}\left\{[G_b(-A_l) + G_{b\perp}(-A_l)]\xi^2 J_2^{(l)} + \Gamma_{bc}(-A_l)\left(J_1^{(l)} - \xi^2 A_l J_2^{(l)}\right) + G_c(-A_l)J_1^{(l)}\right\},$$

$$P_6 = -\frac{1}{2}\sum_{l=1}^{4}\frac{G_c(-A_l)}{\mathscr{E}_l}\xi^2 J_2^{(l)} \quad ,$$

$$\mathbf{p} = p_1\mathbf{U}^1 + p_2\mathbf{U}^2 + p_3\mathbf{U}^3 \quad , \tag{76}$$

$$p_1 = -\frac{1}{8}\sum_{l=1}^{4}\frac{\Gamma_b(-A_l)}{\mathscr{E}_l}\left(J_1^{(l)} - \xi^2 A_l J_2^{(l)}\right),$$

$$p_2 = \frac{1}{16}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l}\left[\Gamma_b(-A_l)\left(J_1^{(l)} - \xi^2 A_l J_2^{(l)}\right) + 2\gamma_c(-A_l)J_1^{(l)}\right],$$

$$p_3 = -\frac{1}{2}\sum_{l=1}^{4}\frac{\gamma_c(-A_l)}{\mathscr{E}_l}\xi^2 J_2^{(l)} \quad ,$$

$$\pi = \pi_1\mathbf{t}^1 + \pi_2\mathbf{t}^2 \quad , \tag{77}$$

where π_1 and π_1 are given by (72).

In accordance with (18), operator $\mathscr{A}$ for the spheroidal inhomogeneity can be written in the form

$$\mathscr{A} = \left\| \begin{matrix} \mathbf{A} & \mathbf{a} \\ \mathbf{a}'^T & \alpha \end{matrix} \right\| \quad , \tag{78}$$

$$\mathbf{A} = A_1\mathbf{T}^2 + A_2\left(\mathbf{T}^2 - \frac{1}{2}\mathbf{T}^2\right) + A_3\mathbf{T}^3 + A_4\mathbf{T}^4 + A_5\mathbf{T}^5 + A_6\mathbf{T}^6,$$

$$\mathbf{a} = a_1\mathbf{U}^1 + a_2\mathbf{U}^2 + a_3\mathbf{U}^3, \quad \mathbf{a}' = a_1'\mathbf{U}^1 + a_2'\mathbf{U}^2 + a_3'\mathbf{U}^3, \quad \alpha = \alpha_1\mathbf{t}^1 + \alpha_2\mathbf{t}^2 \ .$$

Operations in the above-mentioned tensorial basis yield

$$A_1 = \frac{1}{2\Delta_A}\left(B_6 - \frac{q_3 Q_3}{b_2}\right), \quad A_2 = \frac{1}{B_2}, \quad A_3 = -\frac{1}{\Delta_A}\left(B_3 - \frac{q_3 Q_1}{b_2}\right),$$

$$A_4 = -\frac{1}{\Delta_A}\left(B_4 - \frac{q_1 Q_3}{b_2}\right), \quad A_5 = 4\left(B_5 - \frac{q_2 Q_2}{b_1}\right)^{-1}, \quad A_6 = \frac{2}{\Delta_A}\left(B_1 - \frac{q_1 Q_1}{b_2}\right),$$

$$\Delta_A = 2\left[\left(B_1 - \frac{q_1 Q_1}{b_2}\right)\left(B_6 - \frac{q_3 Q_3}{b_2}\right) - \left(B_3 - \frac{q_3 Q_1}{b_2}\right)\left(B_4 - \frac{q_1 Q_3}{b_2}\right)\right] \ , \tag{79}$$

$$a_1 = -\frac{\alpha_2}{\Delta_B}(B_6 Q_1 - B_3 Q_3), \quad a_2 = -\frac{2Q_2\alpha_1}{B_5}, \quad a_3 = -\frac{2\alpha_2}{\Delta_B}(B_1 Q_3 - B_4 Q_1),$$

$$a_1' = \frac{1}{b_2}(2q_1 A_1 + q_3 A_1), \quad a_2' = -\frac{q_2 A_5}{2b_1}, \quad a_3' = -\frac{1}{b_2}(2q_1 A_3 + q_3 A_6),$$

$$\alpha_1 = \left(b_1 - \frac{4q_2 Q_2}{B_5}\right)^{-1}, \quad \Delta_B = 2(B_1 B_6 - B_3 B_4),$$

$$\alpha_2 = \left\{b_2 - \frac{2}{\Delta_B}[(q_1 B_6 - q_3 B_4)Q_1 + (q_3 B_1 - q_1 B_3)Q_3]\right\}^{-1}$$

where it is denoted

$$B_1 = \frac{1}{2} - P_1\left(C^1_{11} + C^1_{12}\right) - P_3 C^1_{13} - p_1 e^1_{31}, \quad B_2 = 1 - 2P_2 C^1_{66},$$

$$B_3 = -2P_1 C^1_{13} - P_3 C^1_{33} - p_1 e^1_{33}, \quad B_4 = -P_3\left(C^1_{11} + C^1_{12}\right) - P_6 C^1_{13} - p_3 e^1_{31},$$

$$B_5 = 2\left(1 - P_5 C^1_{44} - 2p_2 e^1_{15}\right), \quad B_6 = 1 - P_6 C^1_{33} - 2P_3 C^1_{13} - p_3 e^1_{33},$$

$$Q_1 = -\left(2P_1 e^1_{31} + P_3 e^1_{33} - p_1 \eta^1_{33}\right), \quad Q_2 = -\left(\frac{1}{2} P_5 e^1_{15} - p_2 \eta^1_{11}\right), \tag{80}$$

$$Q_3 = -\left(2P_3 e^1_{31} + P_6 e^1_{33} - p_3 \eta^1_{33}\right),$$

$$q_1 = -\left[p_1\left(C^1_{11} + C^1_{12}\right) + p_3 C^1_{13} + \pi_2 e^1_{31}\right], \quad q_2 = -\left(2p_2 C^1_{44} + \pi_1 e^1_{15}\right),$$

$$q_3 = -\left(2p_1 C^1_{13} + p_3 C^1_{33} + \pi_2 e^1_{33}\right),$$

$$b_1 = 1 - 2p_2 e^1_{15} + \pi_1 \eta^1_{11}, \quad b_2 = 1 - 2p_1 e^1_{31} - p_3 e^1_{33} + \pi_2 \eta^1_{33} \ .$$

These formulae yield operator $\mathscr{L}^A$ in (33) in the form

$$\mathscr{L}^A = \begin{Vmatrix} \mathbf{C}^A & \mathbf{e}^A \\ \mathbf{e}^{AT} & -\eta^A \end{Vmatrix} , \tag{81}$$

$$\mathbf{C}^A = \tfrac{1}{2}(C^A_{11} + C^A_{12})\mathbf{T}^2 + 2C^A_{66}\left(\mathbf{T}^1 - \tfrac{1}{2}\mathbf{T}^2\right) + C^A_{13}(\mathbf{T}^3 + \mathbf{T}^4) + 4C^A_{44}\mathbf{T}^5 + C^A_{33}\mathbf{T}^6,$$

$$\mathbf{e}^A = e^A_{31}\mathbf{U}^1 + e^A_{15}\mathbf{U}^2 + e^A_{33}\mathbf{U}^3, \quad \eta^A = \eta^A_{11}\mathbf{t}^1 + \eta^A_{33}\mathbf{t}^3 \ ,$$

where

$$\tfrac{1}{2}\left(C^A_{11} + C^A_{12}\right) = \left(C^1_{11} + C^1_{12}\right)A_1 + C^1_{13}A_4 + e^1_{31}a'_1, \quad C^A_{66} = C^1_{66}A_2,$$

$$C^A_{13} = 2C^1_{13}A_1 + C^1_{33}A_4 + e^1_{33}a'_1, \quad C^A_{44} = \tfrac{1}{2}C^1_{44}A_5 + e^1_{15}a'_2,$$

$$C^A_{33} = C^1_{33}A_6 + 2C^1_{13}A_3 + e^1_{33}a'_3, \tag{82}$$

$$e^A_{31} = \left(C^1_{11} + C^1_{12}\right)a_1 + C^1_{13}a_3 + e^1_{31}\alpha_2, \quad e^A_{15} = 2C^1_{44}a_2 + e^1_{15}\alpha_1,$$

$$e^A_{33} = 2C^1_{13}a_1 + C^1_{33}a_3 + e^1_{33}\alpha_2,$$

$$\eta^A_{11} = -2e^1_{15}a_2 + \eta^1_{11}\alpha_1, \quad \eta^A_{33} = -2e^1_{31}a_1 - e^1_{33}a_3 + \eta^1_{33}\alpha_2 \ .$$

Quantities marked by superscript “1” stand for the difference between the inclusion and matrix electroelastic constants

$$C^1 = C - C^0, \quad e^1 = e - e^0, \quad \eta^1 = \eta - \eta^0 \ . \tag{83}$$

In the next section, we specialize these results to several special cases of the spheroidal geometry.

7 Special cases

We now specialize the general results obtained above to several special cases of spheroid’s geometry and of the contrast between the matrix and inclusion properties.

7.1
Infinite circular cylinder

In the case of a cylinder (continuous cylindrical fiber), $a_3 \to \infty$, whereas a remains fixed. Expressions (75)–(77) take the form

$$
\begin{aligned}
&P_1 = -\frac{1}{4}\sum_{l=1}^{4}\frac{G_b(-A_l)}{\mathscr{E}_l}, \quad P_2 = -\frac{1}{4}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l}[G_b(-A_l) + G_{b\perp}(-A_l)],\\
&P_3 = -\frac{1}{4}\sum_{l=1}^{4}\frac{\Gamma_{bc}(-A_l)}{\mathscr{E}_l}, \quad P_5 = -\frac{1}{2}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l}[G_c(-A_l) + \Gamma_{bc}(-A_l)],\\
&P_6 = 0, \quad p_1 = -\frac{1}{4}\sum_{l=1}^{4}\frac{\Gamma_b(-A_l)}{\mathscr{E}_l}, \quad p_2 = \frac{1}{2}\left(p_1 - \sum_{l=1}^{4}\frac{\gamma_c(-A_l)}{\mathscr{E}_l}\right),\\
&p_3 = 0, \quad \pi_1 = -\frac{1}{2}\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l}, \quad \pi_2 = 0 \ .
\end{aligned}
\tag{84}
$$

Taking into account the following relations (Appendix 2):

$$
\begin{aligned}
&\sum_{l=1}^{4}\frac{G_b(-A_l)}{\mathscr{E}_l} = \frac{1}{C_{11}^0}, \quad \sum_{l=1}^{4}\frac{G_{b\perp}(-A_l)}{\mathscr{E}_l} = \frac{1}{C_{66}^0},\\
&\sum_{l=1}^{4}\frac{\Gamma_{bc}(-A_l)}{\mathscr{E}_l} = 0, \quad \sum_{l=1}^{4}\frac{G_c(-A_l)}{\mathscr{E}_l} = \left(C_{44}^0 + \frac{(e_{15}^0)^2}{\eta_{11}^0}\right)^{-1},\\
&\sum_{l=1}^{4}\frac{\Gamma_b(-A_l)}{\mathscr{E}_l} = 0, \quad \sum_{l=1}^{4}\frac{\gamma_c(-A_l)}{\mathscr{E}_l} = e_{15}^0\left(C_{44}^0\eta_{11}^0 + (e_{15}^0)^2\right)^{-1},\\
&\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l} = -\left(\eta_{11}^0 + \frac{(e_{15}^0)^2}{C_{44}^0}\right)^{-1},
\end{aligned}
\tag{85}
$$

one obtains

$$
\begin{aligned}
&\mathbf{P} = -\frac{1}{4C_{11}^0}\mathbf{T}^2 - \frac{1}{4}\left(\frac{1}{C_{11}^0} + \frac{1}{C_{66}^0}\right)\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) - \frac{\eta_{11}^0}{2\Delta_0}\mathbf{T}^5,\\
&\mathbf{p} = -\frac{e_{15}^0}{4\Delta_0}\mathbf{U}^2, \quad \pi = \frac{C_{44}^0}{2\Delta_0}\mathbf{t}^1, \quad \Delta_0 = \eta_{11}^0 C_{44}^0 + (e_{15}^0)^2 \ .
\end{aligned}
\tag{86}
$$

These expressions, together with general formulae (81), (82), yield

$$
\begin{aligned}
&\frac{1}{2}\left(C_{11}^A + C_{12}^A\right) = \frac{C_{11}^1 + C_{12}^1}{2}\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1}, \quad C_{66}^A = C_{66}^1\left[1 + \frac{C_{66}^1}{2}\left(\frac{1}{C_{11}^0} + \frac{1}{C_{66}^0}\right)\right]^{-1},\\
&C_{13}^A = C_{13}^1\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1}, \quad C_{44}^A = \frac{1}{\Delta_f}\left[C_{44}^1 + \frac{C_{44}^0}{2\Delta_0}\left(C_{44}^1\eta_{11}^1 + (e_{15}^1)^2\right)\right],\\
&C_{33}^A = C_{33}^1 - \frac{(C_{13}^1)^2}{C_{11}^0}\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1},\\
&e_{31}^A = e_{31}^1\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1}, \quad e_{15}^A = \frac{1}{\Delta_f}\left[e_{15}^1 + \frac{e_{15}^0}{2\Delta_0}\left(C_{44}^1\eta_{11}^1 + (e_{15}^1)^2\right)\right],
\end{aligned}
\tag{87}
$$

$$e^A_{33} = e^1_{33} - \frac{C^1_{13}e^1_{31}}{C^0_{11}}\left(1 + \frac{C^1_{11} + C^1_{12}}{2C^0_{11}}\right)^{-1},$$

$$\eta^A_{11} = \frac{1}{\Delta_f}\left[\eta^1_{11} + \frac{\eta^0_{11}}{2\Delta_0}\left(C^1_{44}\eta^1_{11} + (e^1_{15})^2\right)\right], \quad \eta^A_{33} = \left[\eta^1_{33} + \frac{(e^1_{31})^2}{C^0_{11}}\left(1 + \frac{C^1_{11} + C^1_{12}}{2C^0_{11}}\right)^{-1}\right],$$

$$\Delta_f = \left[1 + \frac{1}{2\Delta_0}\left(e^0_{15}e^1_{15} + C^0_{44}\eta^1_{11}\right)\right]\left[1 + \frac{1}{2\Delta_0}\left(e^0_{15}e^1_{15} + C^1_{44}\eta^0_{11}\right)\right]$$
$$- \frac{1}{4\Delta_0^2}\left(C^1_{44}e^0_{15} - C^0_{44}e^1_{15}\right)\left(\eta^0_{11}e^1_{15} - \eta^1_{11}e^0_{15}\right) .$$

These results are in agreement with the ones obtained earlier in [9] and [22]. We note that the multiplier $v/2V$ in (48) is transformed for the continuous fiber to $\pi a^2/S$, where S is the reference area in the plane perpendicular to the fiber axis.

7.2
Strongly oblate spheroid

We now consider the limit of a strongly oblate, disk-like inclusion, $\xi \to \infty$. In the expansion of shape factors $J^{(l)}_1, J^{(l)}_2$ in powers of ξ^{-1}, we retain terms up to the second order for the purpose of inversion. Thus,

$$J^{(l)}_1 = \frac{\pi}{\xi\sqrt{A_l}}, \quad \xi^2 J^{(l)}_2 = \frac{2}{A_l}\left(1 - \frac{\pi}{2\xi\sqrt{A_l}}\right), \qquad \xi \gg 1 . \tag{88}$$

So that

$$\mathscr{P} = \left\|\begin{matrix} P^0 & p^0 \\ p^{0T} & \pi^0 \end{matrix}\right\| + \frac{\pi}{\xi}\left\|\begin{matrix} P^1 & p^1 \\ p^{1T} & \pi^1 \end{matrix}\right\| + O\left(\frac{1}{\xi^2}\right) , \tag{89}$$

where tensors P^0, p^0 and π^0 can be presented in the tensorial basis (74) as follows:

$$\begin{aligned} &P^0_1 = 0, \quad P^0_2 = 0, \quad P^0_3 = 0, \quad P^0_5 = -\frac{1}{2}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l A_l}[G_b(-A_l) + G_{b\perp}(A_l)], \\ &P^0_6 = -\sum_{l=1}^{4}\frac{G_c(-A_l)}{\mathscr{E}_l A_l}, \quad p^0_1 = 0, \quad p^0_2 = 0, \quad p^0_3 = -\sum_{l=1}^{4}\frac{\gamma_c(-A_l)}{\mathscr{E}_l A_l}, \\ &\pi^0_1 = 0, \quad \pi^0_2 = -\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l A_l} . \end{aligned} \tag{90}$$

Using the results of summation presented in Appendix 2, we obtain

$$P^0_1 = -\frac{1}{C^0_{44}}, \quad P^0_6 = -\frac{\eta^0_{33}}{\Delta_c}, \quad p^0_3 = -\frac{e^0_{33}}{\Delta_c}, \quad \pi^0_2 = \frac{C^0_{33}}{\Delta_c}, \quad \Delta_c = C^0_{33}\eta^0_{33} + (e^0_{33})^2 . \tag{91}$$

Tensors $\mathbf{P}^1, \mathbf{p}^1$ and $\boldsymbol{\pi}^1$ can be presented in the same tensorial basis as follows:

$$P^1_1 = -\frac{1}{8}\sum_{l=1}^{4}\frac{G_b(-A_l)}{\mathscr{E}_l\sqrt{A_l}}, \quad P^1_2 = -\frac{1}{8}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l\sqrt{A_l}}[G_b(-A_l) + G_{b\perp}(A_l)],$$

$$P^1_3 = -\frac{1}{4}\sum_{l=1}^{4}\frac{\Gamma_{bc}(-A_l)}{\mathscr{E}_l\sqrt{A_l}},$$

$$P^1_5 = -\frac{1}{4}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l\sqrt{A_l}}\left\{2\Gamma_{bc}(-A_l) + G_c(-A_l) - \frac{1}{A_l}[G_b(-A_l) + G_{b\perp}(A_l)]\right\},$$

$$P_6^1 = -\frac{1}{2}\sum_{l=1}^{4}\frac{G_c(-A_l)}{\mathscr{E}_l\sqrt{A_l}} \ , \tag{92}$$

$$p_1^1 = -\frac{1}{4}\sum_{l=1}^{4}\frac{\Gamma_b(-A_l)}{\mathscr{E}_l\sqrt{A_l}}, \quad p_2^1 = -\frac{1}{8}\sum_{l=1}^{4}\frac{1}{\mathscr{E}_l\sqrt{A_l}}[\Gamma_b(-A_l)+\gamma_c(-A_l)],$$

$$p_3^1 = \frac{1}{2}\sum_{l=1}^{4}\frac{\gamma_c(-A_l)}{\mathscr{E}_l\sqrt{A_l}}, \quad \pi_1^1 = -\frac{1}{4}\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l\sqrt{A_l}}, \quad \pi_2^1 = -\frac{1}{4}\sum_{l=1}^{4}\frac{g(-A_l)}{\mathscr{E}_l A_4\sqrt{A_l}} \ .$$

Operators $\mathscr{M}^B$ and $\mathscr{L}^A$ in (49) and (48) do not contribute to the expression for the electric enthalpy W (multiplier v/V having the order of $1/\xi$), except for the two special cases considered below.

The problem needs more attention in the cases of rigid and absolutely permeable disc or an impermeable crack-like pore. These cases are analyzed below.

7.3
Rigid ideally permeable disc ($C = \infty$, $\eta = \infty$)

In this case,

$$\mathscr{L}^A = -\left(\begin{Vmatrix} \mathbf{P} & \mathbf{p} \\ \mathbf{p}^T & \pi \end{Vmatrix}\right)^{-1} = -\begin{Vmatrix} \mathbf{C}^A & \mathbf{e}^A \\ \mathbf{e}^{AT} & -\eta^A \end{Vmatrix} \tag{93}$$

$$\mathbf{C}^A = \left(\mathbf{P} - \mathbf{p}\pi^{-1}\mathbf{p}^T\right)^{-1}, \quad \mathbf{e}^A = \mathbf{P}^{-1}\mathbf{p}\eta^A, \quad \eta^A = \left(\pi - \mathbf{p}^T\mathbf{P}^{-1}\mathbf{p}\right)^{-1} \ .$$

Retaining only the leading terms of the order of ξ in these tensors yields

$$\mathscr{L}^A = \frac{\xi}{\pi}\begin{Vmatrix} \mathbf{C}^A & 0 \\ 0 & -\eta^A \end{Vmatrix} + O(1) \ , \tag{94}$$

$$\mathbf{C}^A = \frac{1}{4P_1^1}\mathbf{T}^2 + \frac{1}{P_2^1}\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right), \quad \eta^A = \frac{1}{\pi_1^1}\mathbf{t}^1 \ .$$

In the *uncoupled* case ($\mathbf{e}^0 = 0$), the expressions for P_1^1, P_2^1 and π_1^1 are simplified as follows:

$$P_1^1 = \frac{2C_{11}^0C_{44}^0(\sqrt{u_2}+\sqrt{u_3})}{C_{44}^0 + \sqrt{C_{11}^0C_{33}^0}}, \quad \pi_1^1 = \frac{1}{4\sqrt{\eta_{11}^0\eta_{33}^0}},$$

$$P_2^1 = 8C_{44}^0\left[\frac{C_{44}^0+\sqrt{C_{11}^0C_{33}^0}}{C_{11}^0(\sqrt{u_2}+\sqrt{u_3})} + \sqrt{\frac{C_{44}^0}{C_{66}^0}}\right]^{-1} \ , \tag{95}$$

where u_2 and u_3 are the roots of the quadratic equation

$$C_{11}^0C_{44}^0u^2 + \left((C_{13}^0)^2 + 2C_{13}^0C_{44}^0 - C_{11}^0C_{33}^0\right)u + C_{33}^0C_{44}^0 = 0 \ . \tag{96}$$

7.4
Strongly oblate impermeable pore ($C = 0$, $\eta = 0$)

Elliptical and penny-shaped cracks in piezoelectric media where considered in many publications (see, for example, [3]). Here, we present the explicit expressions for components of operator $\mathscr{M}^B$ in the case of the penny-shaped crack in the plane of isotropy of the transversely isotropic electroelastic solid. It allows, based on the Griffith's fracture criterion, to find the critical electroelastic loads for a crack.

Operator $\mathscr{M}^B$ in (49) takes the form

$$\mathscr{M}^B = \left(\mathscr{L}^0 + \mathscr{L}^0\mathscr{P}\mathscr{L}^0\right)^{-1} \ . \tag{97}$$

We introduce the matrix notation for the inverse operator

$$\left(\mathscr{M}^B\right)^{-1} = \left\| \begin{matrix} \mathbf{D} & \mathbf{d} \\ \mathbf{d}^T & \delta \end{matrix} \right\| , \tag{98}$$

where tensors $\mathbf{D}, \mathbf{d}$ and δ are expressed in the tensorial basis (74) as follows:

$$\begin{aligned} \mathbf{D} &= D_1\mathbf{T}^2 + D_2\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + \frac{4\pi}{\xi}\mathbf{T}^5 + \frac{\pi}{\xi}\mathbf{T}^6, \\ \mathbf{d} &= \frac{\pi}{\xi}d_3\mathbf{U}^3, \quad \delta = \delta_1\mathbf{t}^1 + \frac{\pi}{\xi}\delta_2\mathbf{t}^2, \end{aligned} \tag{99}$$

$$\begin{aligned} D_1 &= \frac{1}{2}\left(C^0_{11} + C^0_{12}\right) - \frac{1}{\Delta_c}\left(e^0_{33}C^0_{13} + e^0_{31}C^0_{33}\right), \quad D_2 = 2C^0_{66}, \\ D_5 &= P^1_5(C^0_{44})^2 + 4p^1_2C^0_{44}e^0_{15} + \pi^1_1(e^0_{15})^2, \\ D_6 &= C^0_{33}\left(P^1_6C^0_{33} + 2P^0_3C^0_{13} + p^1_3e^0_{33}\right) + 2C^0_{13}\left(2P^1_1C^0_{13} + P^1_3C^0_{33} + p^1_1e^0_{33}\right) \\ &\quad + e^0_{33}\left(2p^1_1C^0_{13} + p^1_3C^0_{33} + \pi^1_2e^0_{33}\right), \\ d_3 &= 2e^0_{31}\left(2P^1_1C^0_{13} + P^1_3C^0_{33} + p^1_1e^0_{33}\right) + e^0_{33}\left(P^1_6C^0_{33} + 2P^1_3C^0_{13} + p^1_3e^0_{33}\right) \\ &\quad - \eta^0_{33}\left(2p^1_1C^0_{13} + p^1_3C^0_{33} + \pi^1_2e^0_{33}\right), \\ \delta_1 &= -\frac{\Delta_0}{C^0_{44}}, \delta_2 = 4P^1_1(e^0_{31})^2 + 4P^1_3e^0_{31}e^0_{33} + P^1_6(e^0_{33})^2 - 4p^1_1\eta^0_{33}e^0_{31} - 2p^1_3e^0_{33}\eta^0_{33} + \pi^1_2(\eta^0_{33})^2 , \end{aligned}$$

where Δ_0 and Δ_c are given by (86) and (91).

The components of tensors $\mathbf{D}, \mathbf{d}$ and $\boldsymbol{\delta}$ that are not given in (99) do not contribute to the singular (at $\xi \to \infty$) components of operator $\mathscr{M}^B$. Determination of the inverse operator in (94) with the account of the expressions

$$\mathscr{M}^B_c = \left\| \begin{matrix} \mathbf{S}^B & \mathbf{s}^B \\ \mathbf{s}^{BT} & -\sigma^B \end{matrix} \right\| , \tag{100}$$

$$\mathbf{S}^B = \left(\mathbf{D} - \mathbf{d}\delta^{-1}\mathbf{d}^T\right)^{-1}, \quad \mathbf{s}^B = -\mathbf{D}^{-1}\mathbf{d}\sigma^B, \quad \sigma^B = \left(\delta - \mathbf{d}^T\mathbf{D}^{-1}\mathbf{d}\right)^{-1} ,$$

and formulae of Appendix 1 yields

$$\begin{aligned} \mathbf{S}^B &= \frac{\xi}{\pi}\left(\frac{1}{D_s}\mathbf{T}^5 + \frac{\delta_2}{\Delta_s}\mathbf{T}^6\right), \quad \mathbf{s}^B = -\frac{\xi}{\pi}\frac{d_3}{\Delta_s}\mathbf{U}^3, \\ \sigma^B &= \frac{\xi}{\pi}\frac{D_6}{\Delta_s}\mathbf{t}^2, \quad \Delta_s = D_6\delta_2 - d_3^2 , \end{aligned} \tag{101}$$

For the *uncoupled* electroelasticity ($\mathbf{e}^0 = 0$), the results (101), take the well-known form

$$\mathbf{S}^B = \frac{\xi}{\pi}\left(S^B_5\mathbf{T}^5 + S^B_6\mathbf{T}^6\right), \quad \sigma^B = \frac{\xi}{\pi}\sigma^B_{33}\mathbf{t}^2 , \tag{102}$$

$$\begin{aligned} S^B_5 &= \frac{4}{C^0_{44}}\left[\frac{C^0_{11}C^0_{33} - (C^0_{13})^2}{C^0_{44}\sqrt{C^0_{11}C^0_{33}}(\sqrt{u_2} + \sqrt{u_3})} + \sqrt{\frac{C^0_{66}}{C^0_{44}}}\right]^{-1}, \\ S^B_6 &= \frac{2C^0_{11}(\sqrt{u_2} + \sqrt{u_3})}{C^0_{11}C^0_{33} - (C^0_{13})^2}, \quad \sigma^B_{33} = \frac{2}{\sqrt{\eta^0_{11}\eta^0_{33}}} , \end{aligned}$$

where u_2 and u_3 are the roots of Eq. (96).

The critical stresses $(\sigma_{ij}^0)^c$ and electric displacements $(D_i^0)^c$, at which the crack propagation starts, can be determined from Griffith's criterion

$$\frac{\partial}{\partial a}\left(2\pi a^2\gamma - \Delta W\right) = 0 \ , \tag{103}$$

where γ denotes the surface energy density of the piezoelectric material and

$$\Delta W = \frac{2\pi}{3}\frac{a^3}{\xi} J^0 \mathscr{M}_c^B J^0 \ . \tag{104}$$

For example, in the simplest case of uniaxial tension $\mathbf{J}^0 = [\sigma_{33}^0, 0]$, we find the following critical stress $(\sigma_{33}^0)^c$:

$$(\sigma_{33}^0)^c = \sqrt{\frac{2\pi\gamma\Delta_s}{a\delta_2}} \tag{105}$$

If the piezoelectric coupling is ignored and the material is elastically isotropic, Eq. (105) implies the well-known result

$$(\sigma_{33}^0)^c = \sqrt{\frac{\pi\gamma\mu_0}{a(1-v_0)}} \ , \tag{106}$$

where μ_0 is the shear modulus and v_0 is the Poisson's ratio of the matrix.

References

1. **Suo, Z.; Kuo, C.-M.; Barnett, D.M.; Willis, J.R.:** Fracture mechanics for piezoelectric ceramics. J Mech Phys Solids 40(4) (1992) 739–765
2. **Deeg, W.F.:** The analysis of dislocation, crack and inclusion problems in piezoelectric solids. PhD thesis, Stanford University (1980)
3. **Wang, B.:** Three-dimensional analysis of a flat elliptical crack in a piezoelectric material. Int J Eng Sci 30(6) (1992) 781–791
4. **Wang, B.:** Effective behaviour of piezoelectric composites. In: Ostoja-Starzevski, M.; Jasiuk, I. (eds.) Micromechanics of random media. Appl Mech Rev 47 (1994) 112–121
5. **Benveniste, Y.:** The determination of the elastic and electric fields in a piezoelectric inhomogeneity. J Appl Phys 72 (1992) 1086–1095
6. **Huang, J.H.; Yu, J.S.:** Electroelastic Eshelby's tensors for an ellipsoidal piezoelectric inclusion. Composite Eng 4 (1994) 1169–1182
7. **Huang, J.H.:** An ellipsoidal inclusion or crack in orthotropic piezoelectric media. J Appl Phys 78(11) (1995) 6491–6503
8. **Dunn, M.L.; Wienecke, H.A.:** Inclusions and inhomogeneities in transversely isotropic piezoelectric solids. Int J Solids Struct 34(27) (1997) 3571–3582
9. **Michelitsch, T.; Levin, V.M.:** Inclusions and inhomogeneities in electroelastic media with hexagonal symmetry. The European Physical J B (in press)
10. **Kachanov, M.; Tsukrov, I.; Shafiro, B.:** Effective moduli of solids with cavities of various shapes. Appl Mech Rev 47(1) (1994) S151–S174
11. **Sevostianov, I.; Kachanov, M.:** Compliance tensor of ellipsoidal inclusions. Int J Fracture 96 (1999) L3–L7
12. **Levin, V.M.:** The overall properties of piezoactive matrix composite materials. In: Markov, K.Z. (ed.) Continuum models and discret systems. Proc CMDS8 pp. 225–232, World Scientific Publ Comp, Singapore (1996)
13. **Benveniste, Y.:** Universal relations in piezoelectric composites with eigenstress and polarization fields. Part I. Binary media: local fields and effective behaviour. J Appl Mech 60 (1993) 265
14. **Dunn, M.L.; Taya, M.:** Micromechanics predictions of effective electroelastic moduli of piezoelectric composites. Int J Solids Struct 30 (1993a) 161–175
15. **Dunn, M.L.; Taya, M.:** An analysis of piezoelectric composite materials containing ellipsoidal inhomogeneities. Proc R Soc London A 443 (1993b) 265–287
16. **Kanaun, S.K.; Levin, V.M.:** Effective field method in the mechanics of composite materials (in Russian). Petrozavodsk, State University (1993)
17. **Christensen, R.M.:** Mechanics of composite materials. J Wiley and Sons (1979)
18. **Chen, T.:** Green's function and the non-uniform transformation problem in a piezoelectric media. Mech Res Commun 20 (1993) 271–278

19. **Dunn, M.L.:** Electroelastic Green's functions for transversely isotropic media and their application to the solution of inclusion and inhomogeneity problems. Int J Eng Sci 32 (1994) 119–131
20. **Dunn, M.L.; Wienecke, H.A.:** Green's functions for transversely isotropic solids. Int J Solids Struct 33 (30) 4571–4581
21. **Michelitsch, Th.:** Calculation of the electroelastic Green's function of the hexagonal infinite medium. Z Phys B 104 (1997) 497–503
22. **Chen, T.:** Micromechanical estimates of the overall thermoelectroelastic moduli of multiphase fibrous composites. Int J Solids Struct 31(22) (1994) 3099–3111

Appendix 1

The tensorial basis (74) used throughout the present work is discussed here. It allows one to substantially simplify and standardize tensorial operations.

If a certain tensor **A** is expressed in the **T**-basis

$$\mathbf{A} = A_1\mathbf{T}^2 + A_2\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + A_3\mathbf{T}^3 + A_4\mathbf{T}^4 + A_5\mathbf{T}^5 + A_6\mathbf{T}^6 \quad , \tag{A1.1}$$

then the inverse tensor $\mathbf{A}^{-1}$ is determined by the expression

$$\mathbf{A}^{-1} = \frac{A_6}{2\Delta}\mathbf{T}^2 + \frac{1}{A_2}\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) - \frac{A_3}{\Delta}\mathbf{T}^3 - \frac{A_4}{\Delta}\mathbf{T}^4 + \frac{4}{A_5}\mathbf{T}^5 + \frac{2A_1}{\Delta}\mathbf{T}^6 \quad , \tag{A1.2}$$

$$\Delta = 2(A_1A_6 - A_3A_4) \quad .$$

If two tensors **A** and **B** are given in the **T**-basis, the contraction of these tensors over two indices is

$$\begin{aligned} A_{ijkl}B_{klmn} = {} & (2A_1B_1 + A_3B_4)T^2_{ijmn} + A_2B_2\left(T^1_{ijmn} - \tfrac{1}{2}T^2_{ijmn}\right) + (2A_1B_3 + A_3B_6)T^3_{ijmn} \\ & + (2A_4B_1 + A_6B_4)T^4_{ijmn} + \tfrac{1}{2}A_5B_5T^5_{ijmn} + (A_6B_6 + 2A_4B_3)T^6_{ijmn} \quad . \end{aligned} \tag{A1.3}$$

We consider now two tensors **C** and **D** presented in the **U**-basis

$$C_{ijk} = \sum_{r=1}^{3} C_rU^r_{ijk}, \quad D_{ijk} = \sum_{s=1}^{3} D_sU^s_{ijk} \quad , \tag{A1.4}$$

the contraction of these tensors over one index gives tensor of the **T**-basis

$$C_{ijm}D^T_{mkl} = C_1D_1T^2_{ijkl} + C_1D_3T^3_{ijkl} + C_3D_1T^4_{ijkl} + 4C_2D_2T^5_{ijkl} + C_3D_3T^6_{ijkl} \quad , \tag{A1.5}$$

The contraction of tensors **C** and **D** over two indices gives a tensor that is presented in the **t**-basis as follows:

$$C^T_{ikl}D_{klj} = 2C_2D_2t^1_{ij} + (2C_1D_1 + C_3D_3)t^2_{ij} \quad , \tag{A1.6}$$

The **t**-basis can be shown to be orthogonal, i.e. if

$$\alpha_{ij} = \alpha_1t^1_{ij} + \alpha_2t^2_{ij}, \quad \beta_{ij} = \beta_1t^1_{ij} + \beta_2t^2_{ij} \quad , \tag{A1.7}$$

then

$$\alpha_{ik}\beta_{kj} = \alpha_1\beta_1t^1_{ij} + \alpha_2\beta_2t^2_{ij} \quad , \tag{A1.8}$$

and

$$\alpha^{-1}_{ij} = \frac{1}{\alpha_1}t^1_{ij} + \frac{1}{\alpha_2}t^2_{ij} \quad . \tag{A1.9}$$

The following results are also useful:

$$
\begin{aligned}
&A_{ijmn}C_{mnk} = (2A_1C_1 + A_3C_3)U^1_{ijk} + \tfrac{1}{2}A_5C_2U^2_{ijk} + (2A_4C_1 + A_6C_3)U^3_{ijk}, \\
&C^T_{imn}A_{mnkl} = (2C_1A_1 + C_3A_4)U^{1T}_{ijk} + \tfrac{1}{2}C_2A_5U^{2T}_{ijk} + (2C_1A_3 + C_3A_6)U^{3T}_{ijk}, \\
&\alpha_{im}C^T_{mkl} = \alpha_2C_1U^{1T}_{ikl} + \alpha_1C_2U^{2T}_{ikl} + \alpha_2C_3U^{3T}_{ikl}, \\
&C_{ijm}\alpha_{mk} = C_1\alpha_2U^1_{ijk} + C_2\alpha_1U^2_{ijk} + C_3\alpha_2U^3_{ijk} \ .
\end{aligned}
\tag{A1.10}
$$

Appendix 2

We evaluate the sums entering Eqs. (85). To this end, we consider expressions of the form

$$\alpha S_3 + \beta S_2 + \gamma S_1 + \delta S_0 = \sum_{l=1}^{4} \frac{p(-A_l)}{\mathscr{E}_l A_l} \ , \tag{A2.1}$$

Where $p(-A_l)$ is a polynomial of third order in $-A_l$

$$p(a) = \alpha a^3 + \beta a^2 + \gamma a + \delta \ . \tag{A2.2}$$

Consider now the following function:

$$h(a) = \frac{p(a)}{f(a)} = \frac{1}{AC^0_{66}} \cdot \frac{p(a)}{(a + A_1)(a + A_2)(a + A_3)(a + A_4)} \ . \tag{A2.3}$$

Since the numerator $p(a)$ is a third-order polynomial and the denominator $f(a)$ a fourth-order polynomial, we can reduce $h(a)$ to the form

$$h(a) = \sum_{l=1}^{4} \frac{h_l}{a + A_l} \ , \tag{A2.4}$$

where coefficients h_l are given by

$$h_l = (a + A_l)h(a)|_{a=-A_l} \ . \tag{A2.5}$$

From this equation, one obtains

$$h_l = \frac{p(-A_l)}{\mathscr{E}_l} \ , \tag{A2.6}$$

where the definition (64) of $\mathscr{E}_l$ is used

$$\mathscr{E}_l = C^0_{66}A \cdot \prod_{j=1, j\neq l}^{4} (A_j - A_l) \ . \tag{A2.7}$$

Thus, we can write

$$h(a) = \sum_{l=1}^{4} \frac{p(-A_l)}{\mathscr{E}_l(a + A_l)} = \frac{1}{AC^0_{66}} \cdot \frac{p(a)}{(a + A_1)(a + A_2)(a + A_3)(a + A_4)} \ . \tag{A2.8}$$

Equation (A2.8) holds if $p(a)$ is a polynomial of, at most, the third degree. Setting $a = 0$ in this equation, one obtains

$$h(a = 0) = \sum_{l=1}^{4} \frac{p(-A_l)}{\mathscr{E}_l A_l} = \frac{1}{AC^0_{66}} \cdot \frac{\delta}{A_1A_2A_3A_4} = \frac{\delta}{C^0_{44}D} \ , \tag{A2.9}$$

where $\delta = p(a = 0)$, see Eq. (A2.2). In this case, only the zeroth order term S_0 contributes to the sum. Terms S_n corresponding to the powers A_l^n $(n = 1, 2, 3)$ yield vanishing contributions. From Eq. (A2.9), we obtain therefore

$$\sum_{l=1}^{4} \frac{\alpha A_l^2 + \beta A_l + \gamma}{\mathscr{E}_l} = 0 \ . \tag{A2.10}$$

It follows from Eq. (A2.10) that the terms containing quadratic functions of A_l in their numerators are vanishing. It has the following implications for Eqs. (85):

$$\sum_{l=1}^{4} \frac{\Gamma_b(-A_l)}{\mathscr{E}_l} = 0, \quad \sum_{l=1}^{4} \frac{\Gamma_{bc}(-A_l)}{\mathscr{E}_l} = 0, \quad \sum_{l=1}^{4} \frac{g_b(-A_l)}{\mathscr{E}_l} = 0 \ . \tag{A2.11}$$

To evaluate the remaining sums entering Eqs. (85), with cubic functions of A_l in the numerators, we have to evaluate the following sum:

$$\mathscr{S} = -\sum_{l=1}^{4} \frac{A_l^3}{\mathscr{E}_l} \ . \tag{A2.12}$$

Because of (A2.10), $\mathscr{S}$ can be written in the form

$$\mathscr{S} = \sum_{l=1}^{4} \frac{(A_2 - A_l)(A_3 - A_l)(A_4 - A_l)}{\mathscr{E}_l} \ , \tag{A2.13}$$

that can be evaluated in a straightforward manner

$$\mathscr{S} = \frac{(A_2 - A_1)(A_3 - A_1)(A_4 - A_1)}{\mathscr{E}_1} = \frac{1}{C_{66}A} \ , \tag{A2.14}$$

so that

$$\mathscr{S} = -\sum_{l=1}^{4} \frac{A_l^3}{\mathscr{E}_l} = \frac{1}{C_{66}^0 A} \ . \tag{A2.15}$$

It has been seen that terms with $l = 2, 3, 4$ vanish in (A2.13) and the only term that remains is the one with $l = 1$. Utilizing (A2.15), together with (A2.10) we arrive at

$$\mathscr{S} = -\sum_{l=1}^{4} \frac{\alpha A_l^3 + \beta A_l^2 + \gamma A_l + \delta}{\mathscr{E}_l} = \frac{\alpha}{C_{66}A} \ , \tag{A2.16}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants. Thus, only the powers A_l^3 contribute to (A2.16). Using (A2.16), we calculate the remaining sum

$$\sum_{l=1}^{4} \frac{G_b(-A_l)}{\mathscr{E}_l} = \frac{1}{C_{11}^0}, \quad \sum_{l=1}^{4} \frac{G_{b\perp}(-A_l)}{\mathscr{E}_l} = \frac{1}{C_{66}^0}, \quad \sum_{l=1}^{4} \frac{G_c(-A_l)}{\mathscr{E}_l} = \left(C_{44}^0 + \frac{(e_{15}^0)^2}{\eta_{11}^0} \right)^{-1} \ , \tag{A2.17}$$

$$\sum_{l=1}^{4} \frac{\gamma_c(-A_l)}{\mathscr{E}_l} = e_{15}^0 \left(C_{44}^0 \eta_{11}^0 + (e_{15}^0)^2 \right)^{-1}, \quad \sum_{l=1}^{4} \frac{g(-A_l)}{\mathscr{E}_l} = -\left(\eta_{11}^0 + \frac{(e_{15}^0)^2}{C_{44}^0} \right)^{-1} \ . \tag{A2.18}$$