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Dynamics of a small planetoid in Newtonian gravity field of Lagrangian configuration of three primaries

Abstract Novel method for semi-analytical solving of equations of a *trapped dynamics* for a planetoid m_4 close to the plane of mutual motion of main bodies around each other (in case of a special type of Bi-Elliptic Restricted 4-Bodies Problem) is presented. We consider here three primaries m_1, m_2, m_3 orbiting around their center of mass on *elliptic* orbits which are permanently forming Lagrangian configuration of an equilateral triangle. Our aim is to obtain approximate coordinates of quasi-planar trajectory of the infinitesimal planetoid m_4, when the primaries have masses equal to 1/3 (not stable configuration of the Lagrange solution). Results are as follows: (1) equations for coordinates $\{\overline{x}, \overline{y}\}$ are described by system of coupled second-order ODEs with respect to true anomaly f and (2) expression for $\overline{z}$ stems from solving second-order *Riccati* ordinary differential equation that determines the *quasi-periodical* oscillations of planetoid m_4 not far from invariant plane $\{\overline{x}, \overline{y}, 0\}$.

1 Introduction

It is a well-established fact in celestial mechanics that equations of the restricted 3-bodies problem [1–3], hereafter refereed as R3BP, are non-integrable. Here we consider orbiting of infinitesimal planetoid gravitationally influenced by a duet of 2 primaries M_{Sun} and m_{planet}, both moving on Keplerian orbits, where m_{planet} is assumed to be less than M_{Sun}. This is a well-known problem: two primaries revolve on Keplerian orbits around the barycenter in a company of a third body, named 'planetoid', which moves under the gravitational attraction of the primaries, without affecting their motion. If the motion of the primaries is circular, then one obtains the

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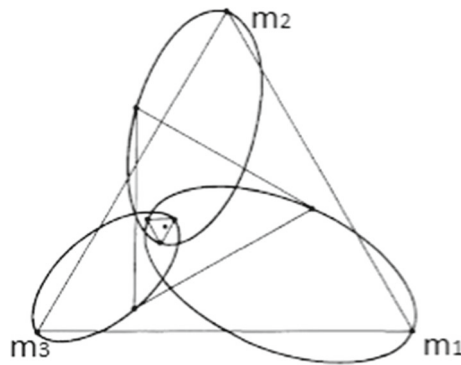
Fig. 1 Lagrange solution (equilateral triangle) to the three-body problem

classical R3BP. It is worth noting valuable contributions of famous participants of celestial mechanics community regarding analytical methods and the obtained results in R3BP [1–7], stability of solutions [8–10], input of such findings to investigation of influence of tidal phenomena on fluid-type planets fluidic envelope dynamics and their rotational dynamics along with their satellites [11, 12], variety of partial formulations and applications in celestial mechanics [13, 14] (including those of non-gravitational nature [15]), investigations of stability of the solutions [16–20] (including of those in the vicinity of libration points [21–24]), and partial application of the aforedescribed findings in investigations of various nonlinear perturbing effects and phenomena [25, 26], e.g., in investigations of the escape and collision dynamics [27]. In particular, let us mention a case of elliptic restricted 3-bodies problem [4, 5] (ER3BP, where primaries are orbiting not far from their center of masses on *elliptic* orbits) and, also, of BiER4BP, Bi-Elliptic Restricted 4-Bodies Problem [6, 7]. The problem we study may be presented as follows: consider three primaries moving on homographic elliptic orbits in a triangular configuration; analyze the dynamics of a fourth body, 'planetoid', subject to attraction of the primaries without affecting their orbits. The existence of homographic triangular motions for the three-body problem is known since Lagrange's time. The idea exploited in the paper is: the use of the triangular homographic solutions as a basis, in the same sense that homographic solutions of a two body system are the basis for the restricted problem, and then analysis of the motion of a planetoid.

Following the ansatz in [6] for BiER4BP, we will consider here 3 primaries m_1, m_2, m_3 which in our research are supposed to be moving not far from the center of masses in *elliptic* orbital mutual motions on homographic triangular orbits, Fig. 1 (in general case, homographic solutions are not restricted in size).

Let us note that we schematically imagine in Fig. 1 the motions of three primaries which are forming the equilateral triangle in their mutual motion (such Lagrange solution can be presented as non-equal masses located in the vertices of the above mentioned triangle, in general case, whereas stability of such solution should be discussed further [9]).

It is worth noting that motivation, method of semi-analytical investigation of the problem, and presented final results of the current research obviously differ from those presented in [18] (and thus such results should be considered as novel). In paper [18], the case of *trapped* orbiting of infinitesimal mass, which is moving *exclusively* in the vicinity of the smallest from triplet of primaries in alternative variant of BiER4BP (with respect to presented here), has been studied. In the present paper, we extend our study to the case where the planetoid is freely moving between all three primaries. Besides, in [18] all three primaries are orbiting around the center of masses in elliptic orbits with *hierarchical* assumption regarding masses of bodies, where masses of the two smallest primaries (distant planets) are much less than the mass of the main primary (Sun). The aforementioned formulation of the problem [18] differs quite a lot from the formulation of the problem presented in the current research, where three primary bodies are forming the Lagrange solution (equilateral triangle). Finally, let us outline that various approximations considered along with the mathematical development, presented in the current research, must be specified from the very beginning of the paper given that they restrict the range of applicability of the presented approach. Among these assumptions are equal masses for the primaries. Such configuration of the Lagrange solution is proved not to be stable [9]), low (and equal) eccentricities of the trajectories of all the primaries (homographic orbits are by definition similar; hence, the eccentricities of the primaries need to be equal), and small values for the coordinate $\bar{z}$, in *pulsating* system of rotating coordinates $\{\bar{x},\ \bar{y},\ \bar{z}\}$ (this assumption will be presented and discussed in the next section).

The introduction of a (nonuniformly) rotating and pulsating reference system in order to describe elliptic homographic orbits for the three-body problem is a standard tool (see, for instance, [2]).

2 Description of the model, equations of motion

Let us present in the current investigation numerical findings and a novel method for resolving equations of a *quasi-planar orbiting* of infinitesimal planetoid m_4 close to the plane of mutual orbitings of the main primaries around each other (in case of a special type of BiER4BP).

We consider three primaries m_1, m_2, m_3 which are orbiting not far from their center of mass on *elliptic* orbits geometrically forming an equilateral triangle for them (Fig. 1). Our aim for constructing the aforementioned *quasi-planar motion* is obtaining coordinates of a infinitesimal planetoid m_4 which maintain its orbit located close to plane of orbiting primaries m_1, m_2, m_3 (of equal mass m, for the sake of simplicity). It is a well-known fact that the locations of the main bodies are variable in ER3BP in view of their mutual Kepler elliptic motion, so we can conclude that relative distance ρ between them is also variable at such an elliptic motion [2]

$$\rho = \frac{a(1 - e^2)}{1 + e \cos f} \tag{1}$$

where a is a semimajor axis of an *elliptic* orbit of the chosen primary in its motion around the barycenter of all primaries (e is eccentricity) and f is the true anomaly of the chosen primary (planet) in its elliptic motion around the common center of masses of all three primaries. It is worth noting that the eccentricity e and the true anomaly f are the same for all primaries; the reference to a chosen primary is not appropriate, except for the semimajor axis a); besides, here the distance scaling results in $a = 1$. In addition, angular motion is given by

$$\frac{df}{dt} = \left(\frac{GM}{a^3 (1 - e^2)^3} \right)^{\frac{1}{2}} (1 + e \cos f)^2 \tag{2}$$

where G is the constant of Newton gravity law (in units of International System of physical measurement SI, a coherent system of units of measurement) and M is the sum of masses of all 3 primaries (the total mass of the system). The time scaling results in $G = 1$ in (2). Following the approach [6], system of equations of BiER4BP for a planetoid m_4 can be represented in *pulsating–rotating* system of Cartesian coordinates $\vec{r} = \{\bar{x}, \bar{y}, \bar{z}\}$ by using relation (1) for primaries (*with appropriate initial conditions*)

$$\ddot{\bar{x}} - 2\dot{\bar{y}} = \frac{\partial \Omega}{\partial \bar{x}},$$
$$\ddot{\bar{y}} + 2\dot{\bar{x}} = \frac{\partial \Omega}{\partial \bar{y}}, \tag{3}$$
$$\ddot{\bar{z}} = \frac{\partial \Omega}{\partial \bar{z}},$$

$$\Omega = \frac{1}{1 + e \cos f} \left[\frac{1}{2} \left(\bar{x}^2 + \bar{y}^2 - \bar{z}^2 e \cos f \right) + \frac{1}{3} \sum_{i=1}^{3} \frac{1}{\bar{r}_i} \right] \tag{4}$$

$$\bar{r}_i^2 = (\bar{x} - \bar{x}_i)^2 + (\bar{y} - \bar{y}_i)^2 + (\bar{z} - \bar{z}_i)^2, \quad i = 1, 2, 3 \tag{5}$$

where the dot denotes the derivative with respect to the independent variable f in (1); Ω is the proper real (non-vector) function; besides, $\{\bar{r}_i\}$ is a set of appropriate space distances of planetoid m_4 from primaries $\{m_i\}$, respectively. In such the pulsating rotating coordinate system, primaries are to be in the stationary positions; this fact allows us to use the appropriate space-scaled expressions for coordinates in (5) as presented in (5.1) (another type of scaling was used earlier in [6]); besides, we choose $\bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$ in (5)

$$\bar{x}_1 = \frac{a_1}{a}, \quad \bar{y}_1 = 0$$
$$\bar{x}_2 = \frac{a_2 e}{a(1 - e^2)}, \quad \bar{y}_2 = \frac{a_2}{a} \tag{5.1}$$
$$\bar{x}_3 = \frac{a_3 e}{a(1 - e^2)}, \quad \bar{y}_3 = -\frac{a_3}{a}$$

where $\{a_1, a_2, a_3\}$ are semimajor axes of elliptic orbits of primaries $\{m_1, m_2, m_3\}$ around the common center of masses, respectively. Then further, expressions (5.1) can be simplified in case of equal masses m of all primaries as considered in [6]. In addition, we can choose in the current research in (5.1) as follows: $2a_2 = 2a_3 = a$, $a_1 = a\left(\frac{\sqrt{3}}{2} + \frac{e}{2(1-e^2)}\right)$, such type of scaling keeps size of the appropriate side a of equilateral triangle that equals to 1, whereas its bisector equals to $\left(\frac{\sqrt{3}}{2}\right)$ as $(x_1 - x_2) = (x_1 - x_3) = \left(\frac{\sqrt{3}}{2}\right)$. Here, Eq. (6) has been chosen to be differed from those presented in [6] earlier:

$$
\begin{aligned}
\overline{x}_1 &= \left(\frac{\sqrt{3}}{2} + \frac{e}{2(1-e^2)}\right), \quad \overline{y}_1 = 0 \\
\overline{x}_2 &= \frac{e}{2(1-e^2)}, \qquad \overline{y}_2 = 1/2 \\
\overline{x}_3 &= \frac{e}{2(1-e^2)}, \qquad \overline{y}_3 = -1/2
\end{aligned}
\tag{6}
$$

It is worth noting that a different sign was used by mistake in expression for $\overline{y}_3$ in Eq. (11) in [6]. It seems to be simply a typo (all other fundamental results in [6] are obviously correct). First, we should note that if we choose $a_2 = a_3$ in Eq. (11) in [6], this would mean that two primaries are to be at the same fixed position in a space (which is senseless). Moreover, if we calculate the size of the side of equilateral triangle based on various expressions for coordinates of vertices given in Eq. (11) in [6], we will come to conclusion that it is not an equilateral triangle. Thus, we will use (here, in Eqs. (5.1)-(6)) only the correct expressions as above.

3 Resolving the system of Eq. (3)

Let us aim to present Eq. (3) (with the help of (4) and (6)) in a convenient form for further analysis. By properly rewriting Eqs. (3) with respect to the partial derivatives which relate to coordinates $\{\overline{x}, \ \overline{y}, \ \overline{z}\}$, this yields:

$$
\ddot{\overline{x}} - 2\dot{\overline{y}} = \frac{1}{1+e\cos f}\left[\overline{x} - \frac{1}{3}\left(\frac{\overline{x} - \left(\frac{\sqrt{3}}{2} + \frac{e}{2(1-e^2)}\right)}{\left((\overline{x} - \frac{\sqrt{3}}{2} - \frac{e}{2(1-e^2)})^2 + \overline{y}^2 + \overline{z}^2\right)^{\frac{3}{2}}} + \frac{\left(\overline{x} - \frac{e}{2(1-e^2)}\right)}{\left(\left(\overline{x} - \frac{e}{2(1-e^2)}\right)^2 + (\overline{y} - \frac{1}{2})^2 + \overline{z}^2\right)^{\frac{3}{2}}} + \frac{\left(\overline{x} - \frac{e}{2(1-e^2)}\right)}{\left(\left(\overline{x} - \frac{e}{2(1-e^2)}\right)^2 + (\overline{y} + \frac{1}{2})^2 + \overline{z}^2\right)^{\frac{3}{2}}}\right)\right],
$$

$$
\ddot{\overline{y}} + 2\dot{\overline{x}} = \frac{1}{1+e\cos f}\left[\overline{y} - \frac{1}{3}\left(\frac{\overline{y}}{\left((\overline{x} - \frac{\sqrt{3}}{2} - \frac{e}{2(1-e^2)})^2 + \overline{y}^2 + \overline{z}^2\right)^{\frac{3}{2}}} + \frac{(\overline{y} - \frac{1}{2})}{\left(\left(\overline{x} - \frac{e}{2(1-e^2)}\right)^2 + (\overline{y} - \frac{1}{2})^2 + \overline{z}^2\right)^{\frac{3}{2}}} + \frac{(\overline{y} + \frac{1}{2})}{\left(\left(\overline{x} - \frac{e}{2(1-e^2)}\right)^2 + (\overline{y} + \frac{1}{2})^2 + \overline{z}^2\right)^{\frac{3}{2}}}\right)\right],
$$

$$
\ddot{\overline{z}} = \frac{1}{1+e\cos f}\left[-\overline{z}\,e\cos f - \frac{1}{3}\left(\frac{\overline{z}}{\left((\overline{x} - \frac{\sqrt{3}}{2} - \frac{e}{2(1-e^2)})^2 + \overline{y}^2 + \overline{z}^2\right)^{\frac{3}{2}}} + \frac{\overline{z}}{\left(\left(\overline{x} - \frac{e}{2(1-e^2)}\right)^2 + (\overline{y} - \frac{1}{2})^2 + \overline{z}^2\right)^{\frac{3}{2}}} + \frac{\overline{z}}{\left(\left(\overline{x} - \frac{e}{2(1-e^2)}\right)^2 + (\overline{y} + \frac{1}{2})^2 + \overline{z}^2\right)^{\frac{3}{2}}}\right)\right],
\tag{7}
$$

$$
\Omega = \frac{1}{1+e\cos f}\left[\frac{1}{2}\left(\overline{x}^2 + \overline{y}^2 - \overline{z}^2 e\cos f\right) + \frac{1}{3}\left(\frac{1}{\overline{r}_1} + \frac{1}{\overline{r}_2} + \frac{1}{\overline{r}_3}\right)\right],
$$

$$
\overline{r}_1 = \sqrt{(\overline{x} - \frac{\sqrt{3}}{2} - \frac{e}{2(1-e^2)})^2 + \overline{y}^2 + \overline{z}^2}, \quad \overline{r}_2 = \sqrt{\left(\overline{x} - \frac{e}{2(1-e^2)}\right)^2 + (\overline{y} - \frac{1}{2})^2 + \overline{z}^2}, \text{where}
$$

$$
\overline{r}_3 = \sqrt{\left(\overline{x} - \frac{e}{2(1-e^2)}\right)^2 + (\overline{y} + \frac{1}{2})^2 + \overline{z}^2}\,.
$$

we use expressions (4) and (6).

Let us further assume that $\overline{z}$-coordinate for system (7) belongs to the sub-variety of *quasi-planar orbits* for infinitesimal planetoid m_4, $\overline{z} \to 0$ (i.e., planetoid m_4 is moving in close vicinity of invariant plane $\{\overline{x}, \overline{y}, 0\}$). Then afterward, we should not take into account all terms less than the second order of smallness in (7); thus, third Eqns. of system (7) would led us to conclusion as follows ($\overline{x} \neq \left(\frac{\sqrt{3}}{2} + \frac{e}{2(1-e^2)}\right)$, $\overline{x} \neq \frac{e}{2(1-e^2)}$, $\overline{y} \neq \{0, \pm\frac{1}{2}\}$, $\overline{z} \neq 0$)

$$
\ddot{\overline{z}} + \xi\,\overline{z} = 0,
\tag{8}
$$

$$\xi = \frac{1}{1+e\cos f}\left[e\cos f + \frac{1}{3}\left(\frac{1}{\left((\overline{x}-\frac{\sqrt{3}}{2}-\frac{e}{2(1-e^2)})^2+\overline{y}^2\right)^{\frac{3}{2}}} + \frac{1}{\left(\left(\overline{x}-\frac{e}{2(1-e^2)}\right)^2+(\overline{y}-\frac{1}{2})^2\right)^{\frac{3}{2}}} + \frac{1}{\left(\left(\overline{x}-\frac{e}{2(1-e^2)}\right)^2+(\overline{y}+\frac{1}{2})^2\right)^{\frac{3}{2}}}\right)\right] \tag{9}$$

Thus, Eq. (8) for resolving the coordinate $\overline{z}(f)$ stems from equations of system (7) and can be recognized as being of the *Riccati* ODEs class [11, 12].

4 Approximate forms of Eq. (7) with their solutions belong to *quasi-planar orbits*.

Exploiting the main assumption above (assumption of boundedness), we should further suggest solutions {$\overline{x}$, $\overline{y}$, $\overline{z}$} of Eq. (7) that are corresponding to sub-variety of quasi-planar *orbits* for planetoid m_4 which are close to plane of orbiting of primaries m_1, m_2, m_3. This transforms the two first Eq. of (7) accordingly:

$$\ddot{\overline{x}} - 2\dot{\overline{y}} = \frac{1}{1+e\cos f}\left[\overline{x} - \frac{1}{3}\left(\frac{(\overline{x}-\frac{\sqrt{3}}{2})}{\left((\overline{x}-\frac{\sqrt{3}}{2}-\frac{e}{2(1-e^2)})^2+\overline{y}^2\right)^{\frac{3}{2}}} + \frac{\left(\overline{x}-\frac{e}{2(1-e^2)}\right)}{\left(\left(\overline{x}-\frac{e}{2(1-e^2)}\right)^2+(\overline{y}-\frac{1}{2})^2\right)^{\frac{3}{2}}} + \frac{\left(\overline{x}-\frac{e}{2(1-e^2)}\right)}{\left(\left(\overline{x}-\frac{e}{2(1-e^2)}\right)^2+(\overline{y}+\frac{1}{2})^2\right)^{\frac{3}{2}}}\right)\right],$$

$$\ddot{\overline{y}} + 2\dot{\overline{x}} = \frac{1}{1+e\cos f}\left[\overline{y} - \frac{1}{3}\left(\frac{\overline{y}}{\left((\overline{x}-\frac{\sqrt{3}}{2}-\frac{e}{2(1-e^2)})^2+\overline{y}^2\right)^{\frac{3}{2}}} + \frac{\overline{y}-\frac{1}{2}}{\left(\left(\overline{x}-\frac{e}{2(1-e^2)}\right)^2+(\overline{y}-\frac{1}{2})^2\right)^{\frac{3}{2}}} + \frac{\overline{y}+\frac{1}{2}}{\left(\left(\overline{x}-\frac{e}{2(1-e^2)}\right)^2+(\overline{y}+\frac{1}{2})^2\right)^{\frac{3}{2}}}\right)\right]. \tag{10}$$

Let us consider further the case $e<<1$ (which corresponds to the case of low (and equal) eccentricities of the trajectories of all the primaries):

$$\ddot{\overline{x}} - 2\dot{\overline{y}} = \frac{1}{1+e\cos f}\left[\overline{x} - \frac{1}{3}\left(\frac{\left(\overline{x}-\frac{\sqrt{3}}{2}-\frac{e}{2}\right)}{\left((\overline{x}-\frac{\sqrt{3}}{2})^2-e(\overline{x}-\frac{\sqrt{3}}{2})+\overline{y}^2\right)^{\frac{3}{2}}} + \frac{\left(\overline{x}-\frac{e}{2}\right)}{\left(\overline{x}^2-e\overline{x}+(\overline{y}-\frac{1}{2})^2\right)^{\frac{3}{2}}} + \frac{\left(\overline{x}-\frac{e}{2}\right)}{\left(\overline{x}^2-e\overline{x}+(\overline{y}+\frac{1}{2})^2\right)^{\frac{3}{2}}}\right)\right],$$

$$\ddot{\overline{y}} + 2\dot{\overline{x}} = \frac{1}{1+e\cos f}\left[\overline{y} - \frac{1}{3}\left(\frac{\overline{y}}{\left((\overline{x}-\frac{\sqrt{3}}{2})^2-e(\overline{x}-\frac{\sqrt{3}}{2})+\overline{y}^2\right)^{\frac{3}{2}}} + \frac{\overline{y}-\frac{1}{2}}{\left(\overline{x}^2-e\overline{x}+(\overline{y}-\frac{1}{2})^2\right)^{\frac{3}{2}}} + \frac{\overline{y}+\frac{1}{2}}{\left(\overline{x}^2-e\overline{x}+(\overline{y}+\frac{1}{2})^2\right)^{\frac{3}{2}}}\right)\right]. \tag{11}$$

5 Discussion

We consider in the current research the dynamics of an infinitesimal planetoid governed by gravitational field of three primaries situated on the vertices of an equilateral triangle. But we should make a reasonable conclusion that Eq. (3) (presented by equations of system (7)) for a planetoid m_4, orbiting close to the plane of mutual motion of primaries m_1, m_2, m_3 (of equal mass m), is hard to be solved analytically for sub-class of approximated *quasi-planar orbits* $\overline{z} \cong 0$ for Eq. (7) insofar as presented herein, whereas it should be particularly noted that such configuration of the Lagrange solution is not stable [9]. Nevertheless, we have obtained from the third equation of (7) the *Riccati*-type Eq. (8) for coordinate $\overline{z}$ (in case of already known solutions for coordinates {$\overline{x}$, $\overline{y}$}). Thus, solution $\overline{z}$ is to be oscillating quasi-periodically (close to plane {$\overline{x}$, $\overline{y}$, 0}) with an appropriate variable frequency (9), depending on coordinates {$\overline{x}$, $\overline{y}$}. Then afterward, we suggest special type of approximations (10)-(11) for the two first equations of system (7) with aim to obtain the *quasi-stable* numerical solutions for coordinates {$\overline{x}$, $\overline{y}$}(at least, up to the restricted value of true anomaly $f = 50$).

Besides, it is worth mentioning the qualitative behavior of solutions of *Riccati*-ODEs, being their instability or *jumping* of the absolute values of the solutions [11, 12]. So, we conclude that any solutions of Eq. (7) of motion (3) may obviously be unstable even during a restricted period of an orbiting process of planetoid m_4 close to the plane of mutual motion of the three primaries m_1, m_2, m_3 (including the ones which can be found for the libration points [21–24]). But we should especially note that the chosen assumption $\overline{z} \to 0$ (or a kind

of simplification for system of Eq. (7) for motion of a planetoid), based on which we then obtain our semi-analytical solutions, stems from Eq. (8) of *Riccati*-type [11, 12]. The latter is known to have *quasi-periodic* (and thus, bounded in magnitude) solutions determined almost everywhere in the range of changing of true anomaly f.

Then hereafter, we have obtained the graphical plots of *numerical* solutions of Eqs. (11) for coordinates $\{\overline{x}, \overline{y}\}$. We have considered the current case with an assumption of low eccentricity $e << 1$.

At the end of discussion, we should note that the numerical calculating for semi-analytical solutions of the first and second Eq. of (11) is provided in Appendix. The Runge–Kutta scheme of the fourth order (step 0.001 proceeding from initial values) has been used for calculating the data. Graphical results of numerical calculating are depicted in Figs. 2, 3, 4, 5 and 6.

6 Conclusions

The given paper proposes a novel method for semi-analytical solving of equations of a *trapped dynamics* for a planetoid m_4 close to the plane of mutual motion of main bodies around each other (in case of a special type of Bi-Elliptic Restricted 4-Bodies Problem). Thus, the paper analyzes the dynamics of a small body under the action of three masses (called primaries) orbiting in a triangular, elliptic homographic configuration. The problem may be considered as an extension of the classical restricted three-body problem to the case of four bodies. It is restricted in a sense that the motion of the three primaries is an exact homographic solution of the three-body problem, and the object of study is the motion of the fourth point. We restrict our study to the particular case of primaries with equal masses (solution of such configuration is not stable [9]) and to orbits of the fourth body which are close to the plane of the primaries. As stated above, we have found that the dynamical equations for the fourth body may be approximated through a *Riccati* equation, thus describing a family of *quasi-periodic* oscillations.

As a general conclusion, we should also say that the problem proposed in the current research belongs to a wide class of problems of academic interest, aimed at describing particular aspects of the dynamics of bodies under the Newtonian interaction. As such, it is a classical problem, and the method of approximation proposed in this paper is an valuable one. In terms of the perspectives of future development of the suggested approach, it would be interesting in the future to consider researching another particular case of equilateral triangle of primaries with two of them of almost equal masses but negligible with respect to the main primary (considered as host star in celestial system). Solution of such configuration is stable [9] and can be considered to be used not only in theoretical aims as in the current research but also in the practical applications for the real celestial mechanics problem (beyond the aims of our study). Namely, as mentioned in [9], we can consider the exo-planetary system with the star TRAPPIST-1, having two of its planets with the mass of the two planets being not equal but approximately similar (circa 1.3105 ± 0.0453 or 1.3238 ± 0.0171 times the mass of Earth, respectively).

Also, it is worth mentioning the profound works which have also tackled the problem under the current investigation presented herein [28, 29] (besides, it is worth noting that non-classical effects like [30, 31] have been ignored in mathematical formulation of the problem here). It is also worth noting that in [29] (where BiCR3BP was investigated in a proper way, i.e., *bi-circular* restricted problem of *four* bodies) another meaning of solutions $\overline{x}_1, \overline{x}_2, \overline{x}_3$ from Eq. (6) for positions of fixed vertices of equilateral triangle was used (in pulsating system of rotating coordinates). They, nevertheless, keep the size of the appropriate side of the equilateral triangle that equals to 1, whereas its bisector equals to $\left(\frac{\sqrt{3}}{2}\right)$ (according to the definition of equilateral triangle). Thus, such an alternative choice of coordinates for the positions of fixed vertices of equilateral triangle does not change the geometry and general dynamical properties of motion of an infinitesimal planetoid in the problem under consideration.

Declarations

Appendix

We have provided here in the current research the numerical calculating for appropriate semi-analytical solutions of the first and second Eq. of system (11). The Runge–Kutta scheme of the fourth order (step 0.001 proceeding from initial values) has been used for calculating the data. Also, eccentricity $e = 0.0167$ has been chosen for calculations (e.g., as in "Sun–Earth" system) for modeling the same binary mutual motions for various pairs of primaries m_1, m_2, m_3. Graphical results of numerical calculation are depicted in Figs. 2, 3, 4, 5 and 6, with the initial data presented below:

    1) $\bar{x}_0 = 0.001$, $(\dot{\bar{x}})_0 = -0.4$, $\bar{y}_0 = -0.2$, $(\dot{\bar{y}})_0 = -0.3$

Meanwhile, it was numerically obtained for the dynamics of infinitesimal planetoid m_4 (see Figs. 2, 3, 4, 5 and 6) that this planetoid should be moving not far from primaries m_1, m_2, m_3 ($\{\bar{r}_1, \bar{r}_2, \bar{r}_3\} < 1.3$) up to the meaning of true anomaly $f \cong 14.5$ or more than 2 full turns of the first primary around the common center of masses. It is worth noting that dynamics of components of the numerical solution is checked to be *quasi-stable* (at least, up to the value of true anomaly $f = 50$).

We should note that additional numerical experiments regarding solving Eq. (8) (with already known numerical solutions for coordinates $\{\bar{x}, \bar{y}\}$) have brought reasonable results which can indeed be regarded as *quasi-periodical* oscillations of a planetoid in close vicinity of plane $\{\bar{x}, \bar{y}, 0\}$, see Fig. 7
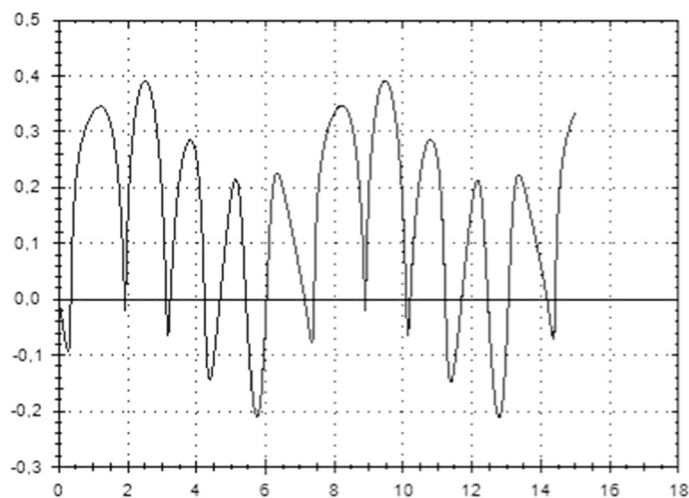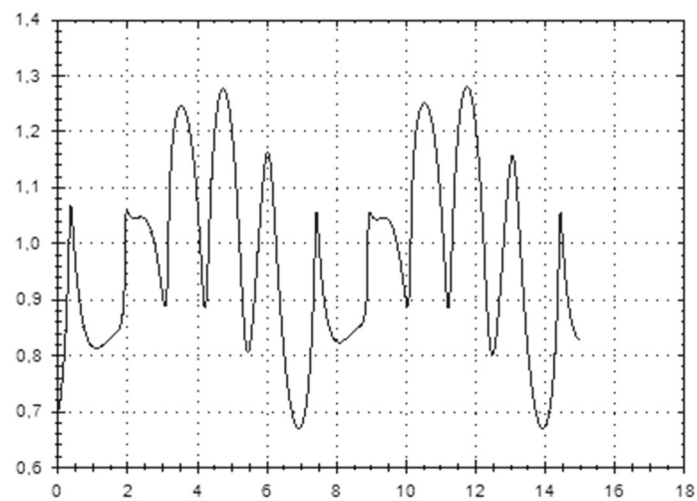

Fig. 2 Numerical solution for $\bar{x}(f)$, depicted on ordinate axis
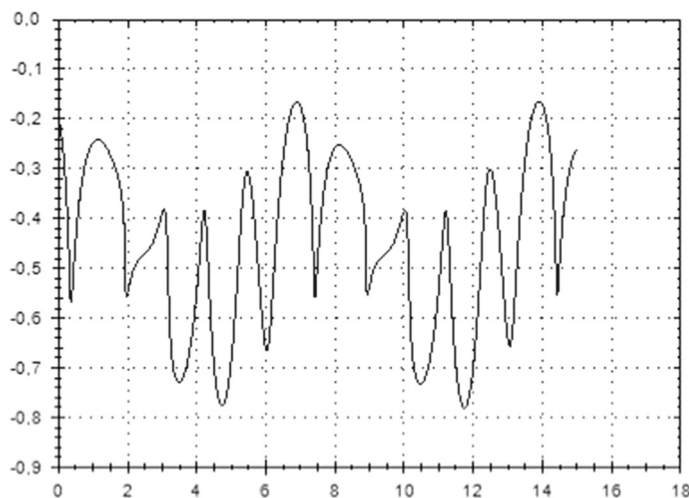
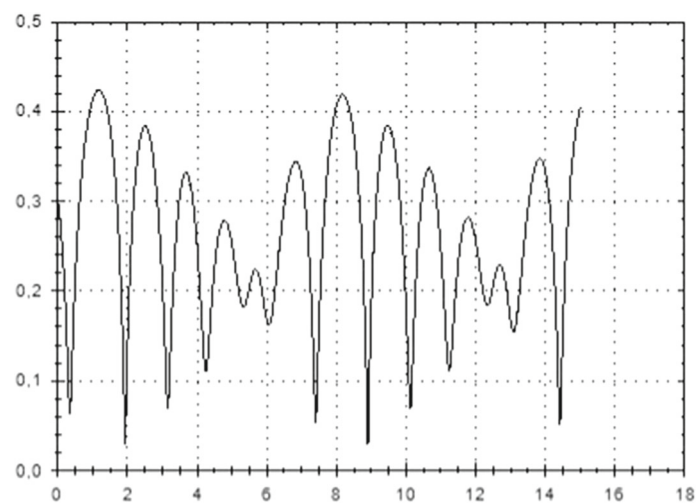**Fig. 3** Numerical solution for $\bar{y}(f)$, depicted on ordinate axis
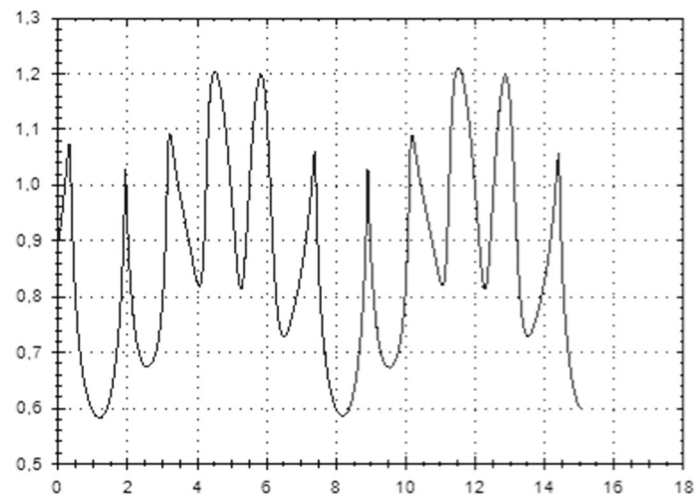
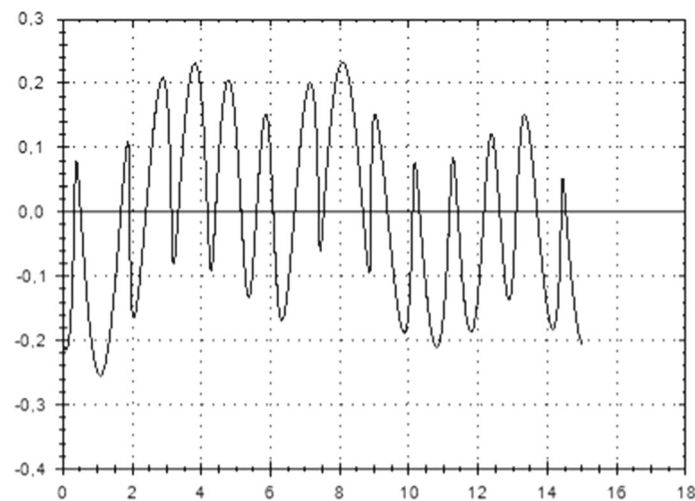**Fig. 4** Numerical solution for distance $\bar{r}_1(f)$, depicted on ordinate axis

Fig. 5 Numerical solution for distance $\bar{r}_2(f)$, depicted on ordinate axis

Fig. 6 Numerical solution for distance $\bar{r}_3(f)$, depicted on ordinate axis

Fig. 7 Numerical solution for $\bar{z}(f)$, depicted on ordinate axis

Thus, trajectories have the *quasi-stable* dynamics (for the chosen initial conditions, including those $\{\bar{z}_0 = -0.2, \ (\dot{\bar{z}})_0 = -0.3\}$ for coordinate $(\bar{z})$) without sudden jumping of the solutions.

Also, works [32–53] should be mentioned as a part of novel methods used in celestial mechanics applications related to the current research.

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