Archive of Applied Mechanics (2023) 93:2081–2095
https://doi.org/10.1007/s00419-023-02373-9

**ORIGINAL**

**Peyman Gholami**

# Fuzzy fatigue life prediction of fiber-reinforced laminated composites by continuum damage mechanics

Received: 29 September 2022 / Accepted: 31 January 2023 / Published online: 27 February 2023


**Abstract** Due to the uncertainty sources in fiber-reinforced plastic, deterministic analysis of damage of laminates under specific loadings, such as fatigue modes is not given much attention. This study's purpose is to define a fuzzy fatigue damage model (FFDM) for the prediction of composite laminate under cyclic conditions based on the continuum damage mechanics (CDM) model. At first, a damaged material's evolution and constitutive equations are formulated. Therefore, to estimate the unknown parameters of the CDM-based damage model the experimental data of damage behavior in fiber, matrix, and shear direction at ply scale are expressed by fuzzy sets and are fitted utilizing a fuzzy linear regression. Data scattering and uncertainty in the empirical failure model are reflected in the definition of membership functions. After estimating unknown parameters of the FFDM according to obtained experimental data of damage mechanisms associated with the composite laminate under cyclic loading, a finite element model was executed to estimate the fatigue life of multidirectional fiber-reinforced plastic laminates. The prediction results have been compared with the experimental data in the literature and shown that by considering the uncertainty sources in stiffness reduction, the proposed model will be able to estimate the lower and upper bounds of a fatigue life of composite laminate under cyclic loading conditions. Overall, the results show that the use of fuzzy theory allows us to consider many more parameters than deterministic or classical statistical methods.

**Keywords** Fatigue damage · Continuum damage mechanics · Fuzzy sets · Fuzzy regression

## 1 Introduction

Fiber-reinforced laminated composites reduce the weight of the structure due to the lack of strength reduction, which makes them used in many engineering structures in the aerospace field, automotive industry, wind power generation, and civil infrastructure. To study the behavior of composite materials, it is necessary to consider several uncertainties such as variation of the fiber volume fraction, load, boundary conditions, geometry and structure of composite material. These uncertainties cause considerable variation in the mechanical behavior of composite material, which leads to the alternative analysis being often used instead of the conventional deterministic method.

One of the most important changes in science and mathematics in this century is the gradual transition from the traditional to the modern perspective in the field of uncertainty. In general, one of the most important works done in the evolution of the modern concept of uncertainty is the publication of a basic article by Zadeh [1] in 1965. In this article, Zadeh introduced the fuzzy set theory whose members are not sets with precise boundaries. Membership in a fuzzy set is not approved or denied, but a degree. Research on the fuzzy set theory has been growing steadily since the theory began in the mid-1960s. The set of concepts and implications of

P. Gholami (✉)
Aerospace Research Institute (Ministry of Science, Research and Technology), Tehran, Iran
e-mail: p.gholami@sun.ari.ac.ir

this theory is now quite impressive. Research on a wide range of engineering applications has also been very active and has produced results that may be even more impressive.

To describe the uncertainty and randomness of composite materials based on the experimental data Muc et al. [2–4] proposed a fuzzy set methodology and expressed a broad description of the possible implementation of the fuzzy set theory to predict damage mechanisms of the composite structures. On the other hand, Kędziora and Muc studied the fuzzy fatigue damage behavior of composite plates containing a central circular hole subjected to cyclic tensile and shear loading. In this study, a one-dimensional damage parameter relying on the classical Tsai-Wu static failure criterion is defined and then the input and output parameters in the proposed model were fuzzy parameters. Song et al. [5] by using the stress strength interference, proposed the reliability method coupling with fuzzy failure domain for carbon fiber composites. Dey et al. [6] studied the uncertainty propagation in the dynamic behavior of laminated composite plates utilizing fuzzy set theory. Considering uncertainty propagation in a composite structure, a non-intrusive Gram–Schmidt polynomial chaos expansion (GPCE) approach is applied and showed that the suggested method is more efficient compared to the conventional global optimization method in terms of computational time and cost.

Naskar et al. [7] suggested a non-probabilistic fuzzy relying on multi-scale uncertainty propagation to describe the dynamic and stability behavior of cross-ply composite materials with spatially varying system properties. Hence, in this study, a new fuzzy representative volume element (FRVE) is developed and showed that neglecting the spatial variation in material properties over-estimates the fuzzy response bounds such as natural frequencies and buckling loads. By using the fuzzy set approach coupled with the finite element model (FEM) Muc [8, 9] investigated the static and fatigue strength of composite structures with a circular hole. In this study, fatigue life of composite estimated by Paris, Collipriest, and Forman formula and compared with each other in view of fuzzy set theory. Muc showed that the fuzzy set theory lets us to take into consider much more parameters than deterministic or statistical approaches.

The composites used in the structures can experience fatigue and degradation due to cyclic loads. Compared to static loading, less attention has been paid to fatigue conditions. The response of structures made of composites exposed to fatigue loading has been studied by several researchers, which has led to the development of several fatigue damage models such as fatigue life concepts [10], phenomenological models [11], strength degradation models [12], micromechanics models [13], and continuum damage mechanics (CDM) models [14].

The CDM approaches are based on the thermodynamic potential theory to represent material constitutive equations. In the composite structure context, the CDM model firstly has been proposed by Ladevèze et al. [15, 16] for static loading. After the Ladevèze article, several articles were published in this field, relying on the development of theory and practical examples to obtain the damage behavior of composite material subjected to fatigue loading. Movaghghar and Lvov [17] by considering plane stress and single scalar damage variable presented an energy-based model in the framework of irreversible thermodynamics to estimate fatigue number of cycles and assess the damage. Salimi-Majd and co-workers [18] by defining three damage variables developed a fatigue damage model based on the energy to estimate the intralaminar fatigue life of the composite. To estimate the fatigue life of composite material, Mohammadi et al. [19] by considering three damage variables in the fiber, matrix, and shear direction have proposed a CDM method relying on effective average local stresses so that it can estimate the fatigue damage of multidirectional composites with allowable accuracy. In this model, the characterization of the model is according to $S$–$N$ and material stiffness reduction diagrams in different directions. Mahmoudi et al. [20] by using micromechanics combined with the CDM method suggested a fatigue damage approach to predict the life of carbon/epoxy laminate, which considers both damages because of static and fatigue loading. Hohe et al. [21] by considering that the damage of composite is based on the microplastic work, proposed the CDM brittle damage approach that characterizes the degradation behavior of the laminate composite structure subjected to harmonic cyclic loading.

Nadjafi and Gholami [22] proposed a probabilistic method based on the CDM theory to assess the failure probability of the anisotropic damage. Gholami et al. [23] extended a CDM model coupling with micromechanics that assumed three variables for damage of fiber, matrix, and shear direction. In most of the fatigue models in the framework of the CDM-based models, the multi-stage process for damage caused by cyclic loading is not explicitly considered in the formulation of the damage accumulation law. In this study, a modified model is proposed, applicable to multi-stage damage due to cyclic loading, which leads to more accurate estimations about fatigue life. Stochastic analysis was considered using MCS by Gholami et al. [24]. They estimated the fatigue life of composite material taking into account mechanical properties as random parameters, and a CDM method coupled with elastic–plastic theory is utilized to predict the mechanical response of laminated composite. Also, Nadjafi and Gholami [25–27] estimated the failure probability of laminated composite using the finite element method (FEM) and CDM method coupled with FORM, SORM and, MCS.

Since there are several possibilities in the combination of components to form a composite, many factors can affect the mechanical properties of composite materials, their behavior under different boundary and loading conditions, and their ultimate failure. To describe it, the fuzzy set theory is suggested. Fuzzy theory is a method of variable processing that enables multiple possible truth values to be processed through the same variable. Fuzzy logic attempts to solve problems with an open and imprecise range of data and heuristics that make it possible to obtain a set of precise results. In this article, a CDM-based fatigue damage model coupled with fuzzy set theory is developed to estimate the stiffness reduction and fatigue life of fiber-reinforced laminated composites based on data scattering and uncertainty of experimental data. Hence, the rest of this paper is organized as follows. Section 2 gives the CDM approach, damage mechanism, and damage evolution for fiber-reinforced laminated composites. Section 3 introduces fuzzy sets, fuzzy operations, and fuzzy linear regression to determine the unknown parameters of the CDM approach. Section 4 explains how to apply the fuzzy approach and fuzzy linear regression in the CDM-based fatigue damage model and proposed fuzzy fatigue damage model. Extensive experimental results on AS4/3501–6 composites subjected to longitudinal, transverse, and shear loading are used to show the validity of the developed method in Sect. 5, followed by concluding remarks in Sect. 6.

## 2 Continuum damage mechanics

In the CDM theory, thermodynamic laws are utilized to formulate the constitutive equations of the damaged materials. Hence, at first, the free energy and dissipation function are defined. Free energy expresses the relationship between the variables of the internal variable and the dissipation functions demonstrates the development of internal variables.

The CDM approach considered in this work is at a ply scale, in which the damage is assumed to be in the form of fiber breakage, matrix cracking, and fiber/matrix debonding. In this model damage of material is characterized by stiffness degradation. Due to the brittle behavior of composite material, the plastic strain has been disregarded. In addition, all the developed equations in the system are the principal coordinates of the materials, therefore 1 and 2 display the fiber direction and perpendicular direction of the fibers, respectively.

### 2.1 Damage mechanism

The $E_{11}^0$ and $E_{22}^0$ are the Young modulus of the undamaged fiber and matrix, and $G_{12}^0$ is shear modulus. On the other hand, $\nu_{12}$ is the Poisson's ratio. Elastic modulus degradation of fiber and matrix are considered as fiber breakage, matrix cracking, respectively. Also, shear modulus degradation is assumed fiber/matrix interface debonding. Thus, if the damage variable is represented by $D_1$,$D_2$ and $D_{12}$ of $E_{11}^0$, $E_{22}^0$ and $G_{12}^0$, then the Gibbs free energy for the damaged ply is written as follows [28]:

$$\rho\Gamma = \frac{1}{2}\left[\frac{\sigma_{11}^2}{E_{11}^0(1-D_1)} - \frac{2\nu_{12}\sigma_{11}\sigma_{22}}{E_{11}^0} + \frac{\langle\sigma_{22}\rangle_+^2}{E_{22}^0(1-D_2)} + \frac{\langle\sigma_{22}\rangle_-^2}{E_{22}^0} + \frac{\tau_{12}^2}{G_{12}^0(1-D_{12})}\right] \tag{1}$$

where:

$$\begin{aligned} \langle a\rangle_+ &= a \ if \ a \geq 0;\ \text{otherwise} \langle a\rangle_+ = 0 \\ \langle a\rangle_- &= a \ if \ a \leq 0;\ \text{otherwise} \langle a\rangle_- = 0 \end{aligned} \tag{2}$$

The constitutive equation is defined as follows:

$$\varepsilon_{11}^e = \rho\frac{\partial\Gamma}{\partial\sigma_{11}} = \frac{\sigma_{11}}{E_{11}^0(1-D_1)} - \frac{\nu_{12}}{E_{11}^0}\sigma_{22} \tag{3-a}$$

$$\varepsilon_{22}^e = \rho\frac{\partial\Gamma}{\partial\sigma_{22}} = \frac{\langle\sigma_{22}\rangle_+}{E_{22}^0(1-D_2)} + \frac{\langle\sigma_{22}\rangle_-}{E_{22}^0} - \frac{\nu_{12}}{E_{11}^0}\sigma_{11} \tag{3-b}$$

$$\gamma_{12}^e = \rho\frac{\partial\Gamma}{\partial\tau_{12}} = \frac{\tau_{12}}{G_{12}^0(1-D_{12})} \tag{3-c}$$

The damage process is controlled by three energy density release rates $Y_1$, $Y_2$ and $Y_{12}$ obtained from the partial derivatives of the Gibbs free energy, respectively, relative to $D_1$, $D_2$ and, $D_{12}$:

$$Y_1 = \rho \frac{\partial \Gamma}{\partial D_1} = \frac{\sigma_{11}^2}{2E_{11}^0(1 - D_1)^2} \tag{4-a}$$

$$Y_2 = \rho \frac{\partial \Gamma}{\partial D_2} = \frac{\sigma_{22}^2}{2E_{22}^0(1 - D_2)^2} \tag{4-b}$$

$$Y_{12} = \rho \frac{\partial \Gamma}{\partial D_{12}} = \frac{\tau_{12}^2}{2G_{12}^0(1 - D_{12})^2} \tag{4-c}$$

## 2.2 Damage evolution

Based on the stiffness degradation, the damage variable of fiber and matrix directions is expressed as follows:

$$D_k = \frac{E_k^0 - E_k}{E_k^0} \tag{5}$$

where $E_k^0$ and $E_k$ are the undamaged and damaged stiffness, respectively, such that $k = f, m$ is fiber and matrix. Also, damage variable of shear direction is written as:

$$D_{12} = \frac{G_k^0 - G_k}{G_k^0} \tag{6}$$

where $G_k^0$ and $G_k$ are the undamaged and damaged shear modulus, respectively.

The micromechanical concept is utilized to calculate the average stress distributions and damage evolution in each matrix and fiber direction. So, the stress–strain relationship is written by [29]:

$$\left[\sigma_i^m\right] = \left[A_{ij}\right]\left[\sigma_j^f\right] \tag{7}$$

and $\left[A_{ij}\right]$ is defined as:

$$\left[A_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \tag{8}$$

with:

$$a_{11} = \frac{E_m}{E_f} \tag{9-a}$$

$$a_{22} = a_{33} = a_{44} = \frac{1}{2} + \frac{E_m}{2E_f} \tag{9-b}$$

$$a_{55} = a_{66} = \frac{1}{2} + \frac{G_m}{2G_f} \tag{9-c}$$

$$a_{12} = a_{13} = (a_{11} - a_{22})\left[\left(\frac{V_m}{E_m} - \frac{V_f}{E_f}\right) \Big/ \left(\frac{1}{E_f} - \frac{1}{E_m}\right)\right] \tag{9-d}$$

$E_f$ and $E_m$ are the Young's modulus of fiber and matrix, while $V_f$ and $V_m$ represent the volume fractions of fiber and matrix, respectively. Therefore, the stress matrix in damaged fiber and matrix is written as [29]:

$$\left[\sigma_i^f\right] = \left[B_{ij}\right]\left[\sigma_j\right] \tag{10-a}$$

$$\left[\sigma_i^m\right] = \left[A_{ij}\right]\left[B_{ij}\right]\left[\sigma_j\right] \tag{10-b}$$

where $\left[\sigma_j\right]$ is the stress of lamina, and $\left[B_{ij}\right] = \left(V_f[I] + V_m\left[A_{ij}\right]\right)^{-1}$, while $[I]$ represents a unit matrix.

The undamaged elastic modulus $E_{11}^0$ in the fiber direction and $E_{22}^0$ in the matrix direction is written as:

$$E_{11}^0 = V_f E_f^0 + V_m E_m^0 \tag{11-a}$$

$$E_{22}^0 = \frac{\left(V_f + V_m a_{11}\right)\left(V_f + V_m a_{22}\right)}{\left(V_f + V_m a_{11}\right)\left(\frac{V_f}{E_f^0} + a_{22}\frac{V_m}{E_m^0}\right) + V_f V_m \left(\frac{V_f}{E_f^0} - \frac{V_m}{E_m^0}\right) a_{12}} \tag{11-b}$$

Therefore, the damaged elastic modulus can be written as:

$$E_{11} = V_f\left(1 - D_f\right)E_f^0 + V_f(1 - D_m)E_m^0 \tag{12-a}$$

$$E_{22} = \frac{\left(V_f + V_m a_{11}\right)\left(V_f + V_m a_{22}\right)}{\left(V_f + V_m a_{11}\right)\left(\frac{V_f}{(1-D_f)E_f^0} + a_{22}\frac{V_m}{(1-D_m)E_m^0}\right) + V_f V_m \left(\frac{V_f}{(1-D_f)E_f^0} - \frac{V_m}{(1-D_m)E_m^0}\right) a_{12}} \tag{12-b}$$

$$G_{12} = G_{12}^0(1 - D_{12}) \tag{12-c}$$

So, the damage variable in fiber and matrix directions can be defined as:

$$D_1 = \frac{E_{11}^0 - E_{11}}{E_{11}^0} \tag{13-a}$$

$$D_2 = \frac{E_{22}^0 - E_{22}}{E_{22}^0} \tag{13-b}$$

### 2.3 Damage evolution equation

According to the CDM approach, there has been much effort to develop damage evolution models under fatigue loading conditions in recent years. In this work, to estimate the damage evolution of composite laminate under cyclic loading, the following model is used [23, 24]:

$$\frac{\mathrm{d}D_k}{\mathrm{d}N} = \frac{A_k Y_k^{B_k}}{(1 - D_k)^{C_k}} \tag{14}$$

where $k$ indicates the fiber, matrix, and shear direction, while $A_k$, $B_k$, and $C_k$ are the material constants. The associated forces are:

$$Y_k = \frac{\sigma_k^2}{2E_k(1 - D_k)^2} \tag{15}$$

The model's unknown parameters, such as $A_k$, $B_k$, $C_k$ should be characterized. Since most of the fatigue models by which composite materials have been analyzed are based on experimental observations, the parameters used in these models have many possibilities and uncertainties. Therefore, in this model, fuzzy theory is used to estimate the damage parameters in fatigue damage model. Therefore, the fuzzy theory will be discussed in the following**.**

## 3 Fuzzy set theory

Members and non-members of a crisp set are specified with a value of 1 or 0 using the characteristic function. This function can be extended so that the values allocated to the elements of a set are specified with a membership degree and are within a certain range. This generalized function is known as the membership function, and the set described by it is a fuzzy set. The common interval used by membership functions is the unit interval [0, 1]. So, the membership function of a fuzzy set $A$ with $\mu_A$ is described as follows: [1, 30]:

$$\mu_A : X \to [0, 1] \tag{16}$$

Each fuzzy set is completely and uniquely expressed by a specific membership function. In the fuzzy set theory literature, several membership functions are introduced as standard and have many applications in practice, including the triangular membership function, trapezoid membership function, exponential membership function, and the sigmoidal membership function. In the present study, the triangular membership function and sigmoidal membership function are used. As well, the main operations of the fuzzy number, because of the use of them in the desired fuzzy fatigue approach, the mathematical expression is given.

The addition of two fuzzy numbers $(A + B)$ is written as follows [1, 30]:

$$\mu_{A(+)B}(z) = \bigvee_{z=x+y} (\mu_A(x) \wedge \mu_B(y)) \tag{17}$$

also, the subtraction of two fuzzy numbers $(A - B)$ is written as follows [1, 30]:

$$\mu_{A(-)B}(z) = \bigvee_{z=x-y} (\mu_A(x) \wedge \mu_B(y)) \tag{18}$$

and, the multiplication of two fuzzy numbers $(A \cdot B)$ is written as follows [1, 30]:

$$\mu_{A(\cdot)B}(z) = \bigvee_{z=x\cdot y} (\mu_A(x) \wedge \mu_B(y)) \tag{19}$$

Finally, the division of two fuzzy numbers $(A/B)$ is written as follows [1, 30]:

$$\mu_{A(/)B}(z) = \bigvee_{z=x/y} (\mu_A(x) \wedge \mu_B(y)) \tag{20}$$

where "$\wedge$" and "$\vee$" operators are minimum and maximum, respectively. So, the maximum and minimum on fuzzy numbers are expressed as follows respectively [1, 30]:

$$\mu_{A(\wedge)B}(z) = \bigvee_{z=x\wedge y} (\mu_A(x) \wedge \mu_B(y)) \tag{21-a}$$

$$\mu_{A(\vee)B}(z) = \bigvee_{z=x\vee y} (\mu_A(x) \wedge \mu_B(y)) \tag{21-b}$$

### 3.1 Fuzzy event

In classical probability theory, an event has its exact boundary, but when a boundary of the event is not sharp, it can be thought of as a fuzzy set, a fuzzy event. In this case, the probability of this event is specified as a crisp probability as in the following method.

If crisp event $A$ is defined in space $\Re^n$ and all events in space $\Re^n$ are mutually exclusive, then the probability of each event is expressed as follows [31]:

$$P(A) = \int_A \mathrm{d}P \tag{22}$$

also, if $\mu_A(x)$ is the membership function of the set $A$ and $E_P(\mu_A)$ is its expectation, then the following relation will be established [31]:

$$P(A) = \int_A \mu_A \mathrm{d}P = E_P(\mu_A) \tag{23}$$

therefore, if event $A$ is a fuzzy set that is assumed in space $\Re^n$ [31]:

$$A = \left\{ (x, \mu_A(x)) | x \in \Re^n \right\} \tag{24}$$

In this case, the probability of this fuzzy set is expressed as [31]:

$$P(A) = \int_A \mu_A \mathrm{d}P = E_P(\mu_A) \tag{25}$$

## 3.2 Fuzzy linear regression

Although classical linear regression models were originally developed for estimating dependent variables via independent variables, much effort is still being devoted to developing new models. Despite some successful applications of classical linear regression models, for practical applications because of the inadequate number of observations, the improper definition of the distribution function, ambiguity in the relationship between dependent and independent variables, ambiguity in the occurrence or degree of occurrence of events, inaccuracy, and error, it is difficult to define the proper model to fit the experimental data and may lead to erroneous results.

Fuzzy linear regression models were produced to create an alternative approach towards modeling the relationship between a dependent variable and one or more independent variables based on the fuzzy sets theory or possibilistic approach proposed by Zadeh [1, 32, 33]. Fuzzy linear regression was first proposed by Tanaka et al. [34] in 1982. According to the basic concept of fuzzy theory and fuzzy regression, the error term is not generated from the residuals between the estimated values and the original values or observations, but it took account of vagueness or the uncertainty parameters and the possibility of distribution. A fuzzy linear regression approach is generally as follows [34]:

$$\begin{aligned} Y &= A_0 + A_1x_1 + A_2x_2 + \cdots + A_nx_n \\ &= A_jx_j \\ j &= 1, 2, ..., n \end{aligned} \tag{26}$$

where $x$ is the vector of the independent variables, $n$ is the number of variables and $A_j$ is the fuzzy set representing the $i$th parameter of the model. These fuzzy numbers are used as triangular fuzzy numbers according to the following equation [34]:

$$\mu_{A_j}(x_j) = \begin{cases} 1 - \frac{|a_j - x_j|}{c_j} & a_j - c_j \le x_j \le a_j + c_j \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

where $\mu_{A_i}(x_i)$ represents the fuzzy set membership function expressing the parameters $A_j$, $a_j$ and $c_j$ is the center and limit values of interval, respectively, so that $A = (a, c)$. The membership function of the regression output variable $y$ can be expressed as follows [34]:

$$\mu_Y(y) = \begin{cases} 1 - \frac{|y - x^t a|}{c^t|x|} & x \ne 0 \\ 1 & x = 0, y = 0 \\ 0 & x = 0, y \ne 0 \end{cases} \tag{28}$$

where $|x| = (|x_1|, \cdots |x_n|)^t$. The problem of finding fuzzy regression parameters by Tanaka is formulated as linear programming. In general, the model uses the minimization of total vagueness, which equal to the sum of the individual limit values of each of the fuzzy parameters of the model [34]:

$$\text{minimize } S = \sum_{i=1}^{N} c^t|x_i| \tag{29}$$

subject to

$$\begin{aligned} a^t x_i + (1+h)\sum_j c_j\left|x_{ij}\right| &\ge y_i \\ -a^t x_i + (1+h)\sum_j c_j\left|x_{ij}\right| &\ge -y_i \end{aligned} \tag{30}$$

where $y_i$ and $x_{ij}$ are observation and independent variable for the $i$th sample, while $N$ is the number of observations.

## 4 Fuzzy Fatigue approach

Most of the fatigue damage models utilize a deterministic approach to predict fatigue life but they cannot to cover all uncertainties in the mechanical behavior. In the present study, in order to predict the damage propagation under fatigue loading conditions, the Fuzzy logic is used. In this model the fiber breakage, matrix cracking, and fiber/matrix debonding is considered and the delamination is ignored. Also, plastic damage is neglected due to the brittle behavior of composite materials. In order to overcome the uncertainties in the model and the composite material, triangular fuzzy number has been used to estimate the parameters of the model. According to Eqs. (14) and (15) the damage evolution law is written as follows:

$$\frac{\mathrm{d}D_k}{\mathrm{d}N} = \frac{A_k}{\left(2E_k^0\right)^{B_k}} \frac{\sigma_{\max\ k}^{2B_k}}{(1 - D_k)^{2B_k + c_k}} \tag{31}$$

where $\sigma_{\max k}$ represents the maximum applied stress in each direction. By integration of Eq. (31) from $D = 0$ to $D = D_{cr}$, this relation is derived as follows:

$$\sigma_{\max k}^{2B_k} \cdot N = \frac{(2E_k)^{B_k}}{A_k(2B_k + C_k + 1)}\left(1 - \left(1 - D_{cr,k}\right)^{2B_k + C_k + 1}\right) \tag{32}$$

and $D_{cr,k}$ is the critical damage parameter in each direction and is obtained experimentally. The logarithm of Eq. (32) is expressed as:

$$N_L^1 = \mathrm{A}_0^1 + \mathrm{A}_1^1 \cdot S_L \tag{33}$$

where

$$N_L^1 = \log N_k \tag{34-a}$$

$$\mathrm{A}_0^1 = \log\left[\frac{(2E_k)^{B_k}}{A_k(2B_k + C_k + 1)}.\left(1 - \left(1 - D_{c,cr}\right)^{2B_k + C_k + 1}\right)\right] \tag{34-b}$$

$$\mathrm{A}_1^1 = -2B_k \tag{34-c}$$

$$S_L = \log \sigma_{\max,k} \tag{34-d}$$

To estimate the unknown parameter of Eq. (33), fuzzy linear regression is used. Hence, according to this approach, each regression coefficient is supposed to be the triangular shape of the membership function of the feature $x$. A triangular fuzzy number can be expressed as:

$$\mu_A(x) = \begin{cases} 0 & x \le a_m - c_L \\ \frac{x - a_m}{c_L} + 1 & a_m - c_L \le x \le a_m \\ \frac{a_m - x}{c_L} + 1 & a_m \le x \le a_m + c_R \\ 0 & x > a_m + c_R \end{cases} \tag{35}$$

It is assumed that, in each level of applied stress, $n_t$ tests are performed. According to triangular fuzzy number, for each $S_L$, the parameter $N_L^1$ can be expressed as:

$$N_{Lt}^1 = \left(N_{Lt\min}^1, \overline{N}_{Lt}^1, N_{Lt\max}^1\right) \tag{36}$$

where

$$\overline{N}_{Lt}^1 = \frac{1}{n_t} \sum_{i}^{n_t} N_{Lti}^1 \tag{37}$$

and the experimental interval is assumed:

$$\left(N_{Lt\min}^1; N_{Lt\max}^1\right) \forall t \tag{38}$$

Since the unknown parameters of Eq. (33) is assumed fuzzy number, $\mathrm{A}_0^1$ and $\mathrm{A}_1^1$ are defined as:

$$\mathrm{A}_0^1 = \left(c_{L0}^1, a_0^1, c_{L0}^1\right) \tag{39-a}$$

$$\mathrm{A}_1^1 = \left(c_{L1}^1, a_1^1, c_{R1}^1\right) \tag{39-b}$$

and the estimated interval is written as follows:

$$\left[\left(a_0^1 - c_{L0}^1\right) + \left(a_1^1 + c_{L1}^1\right)S_{Lt}; \left(a_0^1 + c_{R0}^1\right) + \left(a_1^1 + c_{R1}^1\right)S_{Rt}\right] \forall t \tag{40}$$

As a result, based on the fuzzy linear regression algorithm, the problem is obtaining unknown fuzzy parameters, which:

$$\min\left[\max\left(c_{R0}^1, c_{L0}^1, c_{R_0}^1, c_{L0}^1\right)\right] \tag{41}$$

subject to:

$$\begin{aligned} &\left(a_0^1 + c_{R0}^1\right) + \left(a_1^1 + c_{R1}^1\right)S_{Lt} \geq N_{Lt\,\max} \,\forall t \\ &\left(a_0^1 - c_{L0}^1\right) + \left(a_1^1 - c_{L1}^1\right)S_{Lt} \geq N_{Lt\,\min} \,\forall t \\ &c_{R0}^1 \geq 0,\; c_{L0}^1 \geq 0,\; c_{R1}^1 \geq 0,\; c_{L1}^1 \geq 0, \end{aligned} \tag{42}$$

where $a_0^1$, $a_1^1$, $c_{L0}^1$, $c_{L1}^1$, $c_{R0}^1$, and $c_{R1}^1$ are variables that should be calculated. On the other hand, by using the logarithm of Eq. (31) is obtained as:

$$N_L^2 = \mathrm{A}_0^2 + \mathrm{A}_1^2 \cdot D_L \tag{43}$$

where

$$N_L^2 = \log \frac{\mathrm{d}D_k}{\mathrm{d}N} \tag{44-a}$$

$$\mathrm{A}_0^2 = \log\left(\frac{A_k}{(2E_k)^{B_k}} \sigma_{\max k}^{2B_k}\right) \tag{44-b}$$

$$\mathrm{A}_1^2 = (2B_k + c_k) \tag{44-c}$$

$$D_L = \log(1 - D_k)^{-1} \tag{44-d}$$

The parameter $N_L^2$ is triangular fuzzy number and can is written as:

$$N_{Lt}^1 = \left(N_{Lt\,\min}^1, \overline{N}_{Lt}^1, N_{Lt\,\max}^1\right) \tag{45}$$

where

$$\overline{N}_{Lt}^2 = \frac{1}{n_t} \sum_{i}^{n_t} N_{Lti}^2 \tag{46}$$

and the experimental interval is written as follows:

$$\left(N_{Lt\,\min}^1; N_{Lt\,\max}^2\right) \forall t \tag{47}$$

and $\mathrm{A}_0^2$ and $\mathrm{A}_1^2$ are assumed such as triangular fuzzy number:

$$\mathrm{A}_0^2 = \left(c_{L0}^2, a_0^2, c_{L0}^2\right) \tag{48-a}$$

$$\mathrm{A}_1^2 = \left(c_{L1}^2, a_1^2, c_{R1}^2\right) \tag{48-b}$$

and the estimated interval is written as follows:

$$\left[\left(a_0^1 - c_{L0}^1\right) + \left(a_1^1 + c_{L1}^1\right)S_{Lt}; \left(a_0^1 + c_{R0}^1\right) + \left(a_1^1 + c_{R1}^1\right)S_{Rt}\right] \forall t \tag{49}$$

So, the target function is expressed as:

$$\min\left[\max\left(c_{R0}^2, c_{L0}^2, c_{R_0}^2, c_{L0}^2\right)\right] \tag{50}$$

subject to:

$$\begin{aligned} &\left(a_0^2 + c_{R0}^2\right) + \left(a_1^2 + c_{R1}^2\right) S_{Lt} \geq N_{Lt\max}^2 \quad \forall t \\ &\left(a_0^2 - c_{L0}^2\right) + \left(a_1^2 - c_{L1}^2\right) S_{Lt} \geq N_{Lt\min}^2 \quad \forall t \\ &c_{R0}^2 \geq 0,\ c_{L0}^2 \geq 0,\ c_{R1}^2 \geq 0,\ c_{L1}^2 \geq 0, \end{aligned} \tag{51}$$

where $a_0^2$, $a_1^2$, $c_{L0}^2$, $c_{L1}^2$, $c_{R0}^2$, and $c_{R1}^2$ should be obtained.

According to the fuzzy failure probability approach, if strength R is defined as a fuzzy event, then the probability of damage $P_f$ is written as:

$$P_f(S_L) = P(R < S_L) \tag{52}$$

and membership function of damage event is assumed sigmoidal membership function as follows:

$$\mu_f(S_L) = \mu(R < S_L) = \begin{cases} 1 & S_L > S_{L\max} \\ \frac{1}{2}\left[1 + erf\left(\frac{S_L - m}{\sigma\sqrt{2}}\right)\right] & S_{L\min} \leq S_L \leq S_{L\min} \\ 0 & S_L < S_{L\min} \end{cases} \tag{53}$$

where $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-z^2\right) dz$, $m = \frac{S_{L\min} + S_{L\max}}{2}$, and $\sigma = \frac{S_{L\max} - S_{L\min}}{6}$, while $S_{L\min}$ and $S_{L\max}$ represent maximum and minimum applied stress, respectively. So, according Eq. (25), the probability of damage event can be defined as:

$$P_f(S_L) = \int_A \mu_f(S_L) \mathrm{d}P \tag{54}$$

Since then $dP(S_L) = f(S_L) dS_L$, Eq. (54) is rewritten as follows:

$$P_f(S_L) = \int_{-\infty}^{S_L} \mu_f(S_L) \cdot f(S_L) \mathrm{d}S_L \tag{55}$$

where $f(S_L)$ is the probability distribution function of $S_L$.

The above method for characterization of material constants is applied for fiber, matrix, and shear directions, and model constants are obtained in these directions. Therefore, according to the fatigue experimental data in each direction, the fuzzy coefficients of linear regression are determined.

## 5 Results and discussion

To show the ability of the proposed model, the FFDM described in Sect. 4 is applied to the experimental data obtained by Shokrieh and Lessard [35]. So, the constants of the CDM approach are assumed triangular fuzzy numbers, and then these fuzzy coefficients of linear regression are obtained based on the observed damage for AS4/3501-6 composites exposed to longitudinal, transverse, and shear loading. Finally, the sigmoidal membership function for the damage event is used to calculate the probability of fracture in each direction. Figure 1 shows the overall flowchart of the approach to estimating the fatigue life of the composite laminates.

As a result, by applying the linear fuzzy regression method on experimental data, the unknown parameters in fiber, matrix, and shear direction are obtained and shown in Figs. 2, 3, 4. As mentioned earlier, the input and output unknown parameters are defined in the form of the triangular membership function. So, Figs. 1, 2, 3 present the membership function of the unknown constants $A_k$, $B_k$, and $C_k$ in fiber, matrix, and shear directions, respectively.

In the following, based on the proposed fatigue damage model and parameter identification, for simulation of fatigue life of laminated composite, the procedure is implemented in Abaqus software by the subroutine. So, the results of experiments on the [30]$_{16}$ [36] and [90/45/-45/0]$_s$ [37] laminates made of AS4/351-6 under fatigue loading conditions with stress ratio equal to 0.1 and different maximum stresses, are used and compared with the results of the simulation.

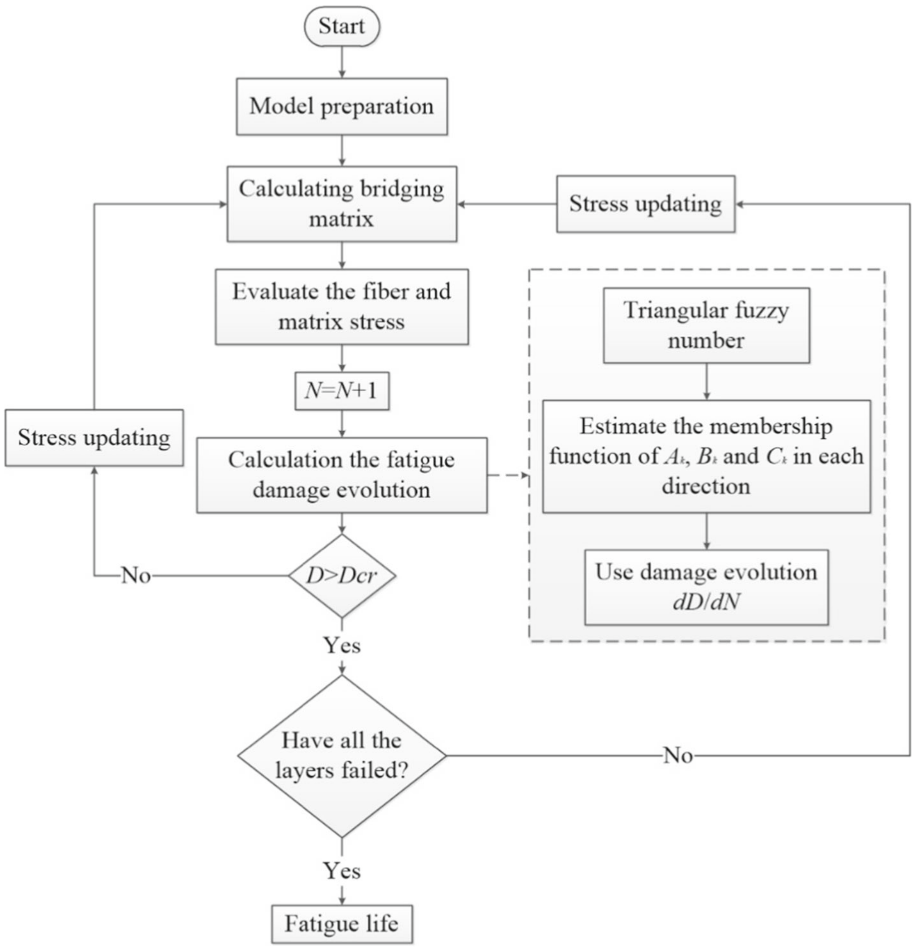


**Fig. 1** Flowchart of fuzzy fatigue damage model

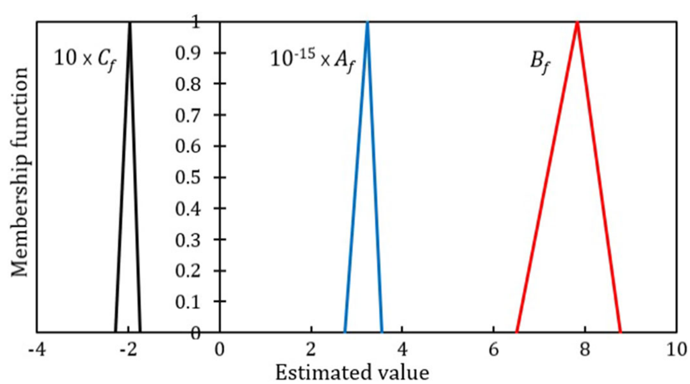


**Fig. 2** The estimated membership functions for the parameters of the fiber direction

In this study, the predicted results by the fuzzy fatigue damage model are evaluated in terms of *S*–*N* curves with lines at 5%, 50%, and 95% damage probability, and compared with the deterministic fatigue damage model proposed by Mohammadi et al. [19]. It is shown in Figs. 5 and 6 that the predicted life of the fuzzy fatigue model is similar to the deterministic fatigue damage model. It is worth noting that the uncertainty and randomness of parameters describing the laminated composite behaviors and fatigue life are large, hence the predicted fatigue life of the deterministic fatigue damage model is not exact. As can be seen, the range of life between the specified probability values includes most of the experimental data. But on the other

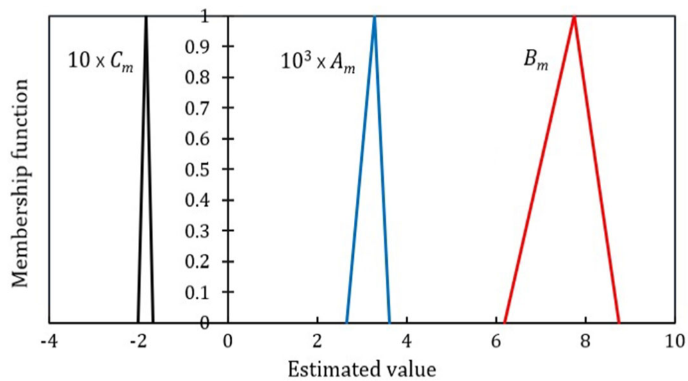


**Fig. 3** The estimated membership functions for the parameters of the matrix direction

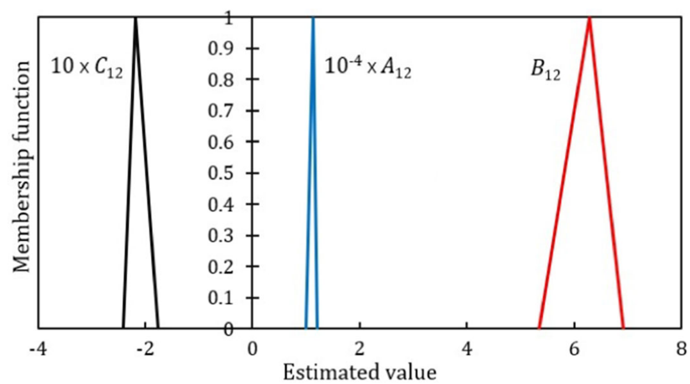


**Fig. 4** The estimated membership functions for the parameters of the shear direction

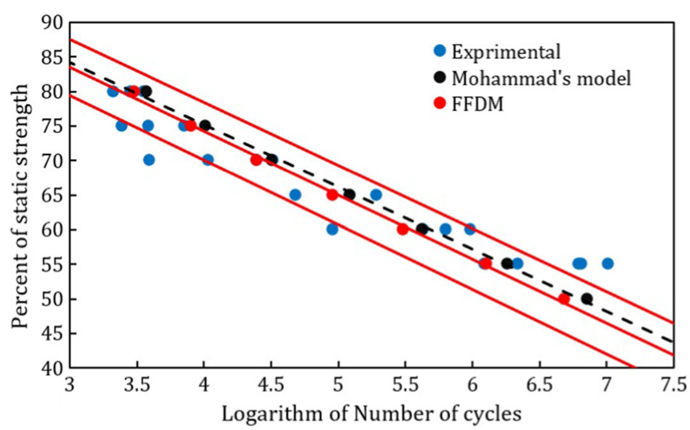


**Fig. 5** Fuzzy fatigue life simulation and experimental results of $[30]_{16}$ laminate

hand, the proposed FFDM provides a new approach to predict the fatigue life of laminated composites due to the scattering of mechanical properties of laminated composite. The differences between the two models are because of considering the uncertainty of composite material's mechanical properties. Thus, the results obtained from the FFDM are rational and satisfactory, from the engineering point of view.

On the other hand, based on the fuzzy failure probability approach, the probability distributions versus the number of cycles for the $[30]_{16}$ and [90/45/-45/0] laminates were obtained and presented in Figs. 7 and 8, respectively.

It can be concluded that considering the experimental data scattering in laminated composites, more accurate results can be achieved. It is worth noting that the results obtained are only for AS4/3501-6 composites subjected

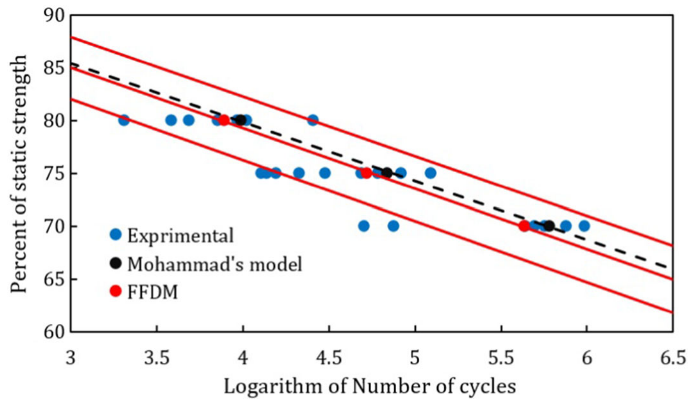


**Fig. 6** Fuzzy fatigue life simulation and experimental results of [90/45/-45/0]$_s$ laminate

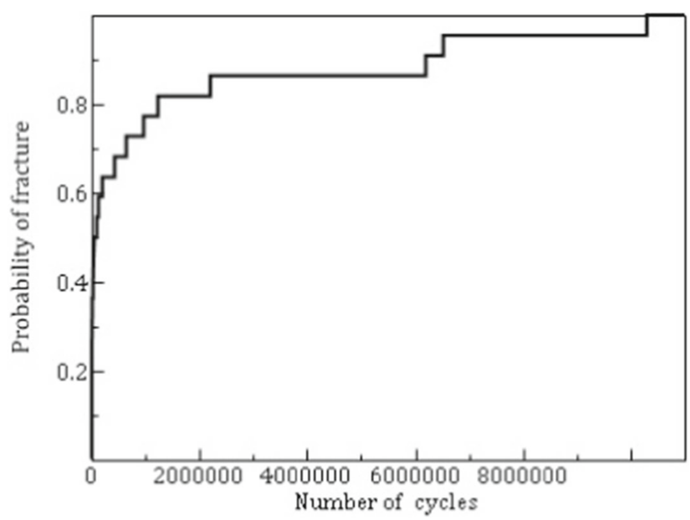


**Fig. 7** Fuzzy probability of fracture for [30]$_{16}$ laminate

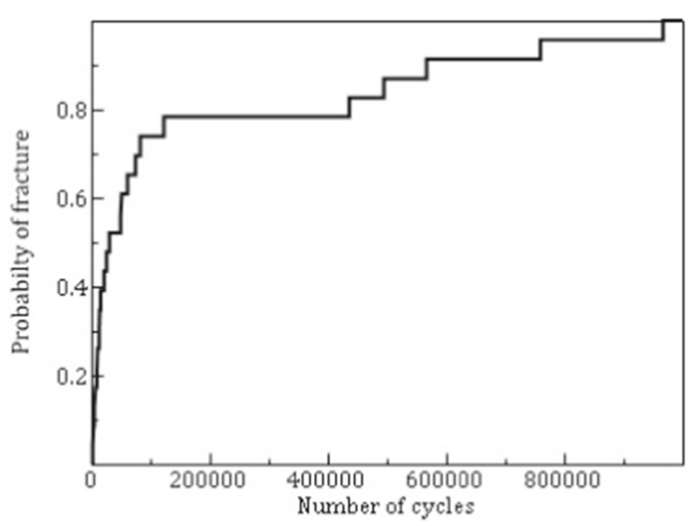


**Fig. 8** Fuzzy probability of fracture for [90/45/-45/0]$_s$ laminate

to a specific load, and these results will be different for other materials and load and geometry conditions. Investigating the fatigue life of complex geometry with different stacking in addition to plastic strain and hardening processes using the fuzzy fatigue-based CDM approach is also recommended for future studies.

## 6 Conclusion

According to the experimental failure description, uncertainty sources in stiffness reduction due to cyclic loading led to significant random variations in the mechanical behavior of composite material. Hence, the need for a useful and supple approach is essential relying on possibility theory for the analysis of damage models for a structure system under fatigue loading. In this article, a fatigue damage model based on the continuum damage mechanics (CDM) combined with fuzzy theory has been presented to predict the fatigue life of composite laminate. The fuzzy sets theory and fuzzy linear regression are applied to the experimental data of the stiffness reduction in the fiber, matrix, and shear direction to estimate unknown parameters of fatigue damage, where vagueness in the input data is represented as membership functions. This fuzzy fatigue damage model (FFDM) is implemented into the analysis of the fatigue life of the [30]16 and [90/45/-45/0]s laminates. It was shown that the FFDM approach is rational and satisfactory and can be considered a novel method to obtain the *S–N* curve of composite materials. In the fatigue damage simulation, the knowledge of a great number of several factors is needed and the validity of a pure deterministic analysis is always not perfect due to the scatter of experimental data. In general, the present study shows that the fuzzy theory lets us take into account much more parameters than classical deterministic or statistical methods. Also, the fatigue life investigation for composite materials with complex geometry such as glass fiber-reinforced polymer (GFRP) composite pipes under different loads, using the current approach is suggested.

**Acknowledgements** Not applicable

**Authors' contribution** The author confirms sole responsibility for the following: study conception, analysis and interpretation of results, and manuscript preparation.

**Funding** Also, no funding was received for this work.

**Availability of data and materials** All data generated or analyzed during this study are included in this published article.

**Declarations**

**Conflict of interest** The authors declare that they have no competing interests.

## References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
2. Muc, A., Kedziora, P.: A fuzzy set analysis for a fatigue damage response of composite materials. In: ICCST/3- International Conference on Composite Science and Technology, 3 rd, Durban, South Africa, pp. 417–422 (2000)
3. Muc, A., Kędziora, P.: A fuzzy set analysis for a fracture and fatigue damage response of composite materials. Compos. Struct. **54**, 283–287 (2001)
4. Muc, A., Kędziora, P.: Application of fuzzy set theory in mechanics of composite materials. In: Soft Computing in Textile Sciences, ed: Springer, pp. 16–50 (2003)
5. Song, Y., Yao, J., Yang, J., Zhou, J.: Fuzzy reliability research on carbon fibre composites under random vibration based on fuzzy failure domain. Mater. Res. Innov. **19**, S5–47-S5–52 (2015)
6. Dey, S., Mukhopadhyay, T., Khodaparast, H.H., Adhikari, S.: Fuzzy uncertainty propagation in composites using Gram-Schmidt polynomial chaos expansion. Appl. Math. Model. **40**, 4412–4428 (2016)
7. Naskar, S., Mukhopadhyay, T., Sriramula, S.: Spatially varying fuzzy multi-scale uncertainty propagation in unidirectional fibre reinforced composites. Compos. Struct. **209**, 940–967 (2019)
8. Muc, A.: Fuzzy approach to fatigue problems in composite materials and structures. In: 2017 13th International Conference on Natural Computation, Fuzzy Systems and Knowledge Discovery (ICNC-FSKD), pp. 1247–1253 (2017)
9. Muc, A.: Fuzzy approach in modeling static and fatigue strength of composite materials and structures. Neurocomputing **393**, 156–164 (2020)
10. Vassilopoulos, A.P.: Fatigue life prediction of composites and composite structures: Woodhead publishing (2019)

11. Elenchezhian, M.R.P., Das, P.P., Rahman, M., Vadlamudi, V., Raihan, R., Reifsnider, K.: Stiffness degradation in fatigue life of composites using dielectric state variables. Compos. Struct. **273**, 114272 (2021)
12. Gao, J., Zhu, P., Yuan, Y., Wu, Z., Xu, R.: Strength and stiffness degradation modeling and fatigue life prediction of composite materials based on a unified fatigue damage model. Eng. Fail. Anal. **137**, 106290 (2022)
13. Davidson, P., Waas, A.M.: A micromechanics-driven model for compressive fatigue of fiber-reinforced composites. J. Reinf. Plast. Compos. **41**, 661–669 (2022)
14. Llobet, J., Maimí, P., Essa, Y., De La Escalera, F.M.: A continuum damage model for composite laminates: Part III-Fatigue. Mech. Mater. **153**, 103659 (2021)
15. Ladevèze, P.: A damage computational method for composite structures. Comput. Struct. **44**, 79–87 (1992)
16. Allix, O., Ladevèze, P.: Interlaminar interface modelling for the prediction of delamination. Compos. Struct. **22**, 235–242 (1992)
17. Movaghghar, A., Lvov, G.: An energy model for fatigue life prediction of composite materials using continuum damage mechanics. Appl. Mech. Mater. **110**, 1353–1360 (2012)
18. Salimi-Majd, D., Helmi, M., Mohammadi, B.: Damage growth prediction of unidirectional layered composites under cyclic loading using an energy based model. Modares Mech. Eng. **15**, 173–180 (2015)
19. Mohammadi, B., Fazlali, B., Salimi-Majd, D.: Development of a continuum damage model for fatigue life prediction of laminated composites. Compos. Part A Appl. Sci. Manuf. **93**, 163–176 (2017)
20. Mahmoudi, A., Mohammadi, B., Hosseini-Toudeshky, H.: Damage behaviour of laminated composites during fatigue loading. Fatigue Fract. Eng. Mater. Struct. **43**, 698–710 (2019)
21. Hohe, J., Gall, M., Fliegener, S., Hamid, Z.M.A.: A continuum damage mechanics model for fatigue and degradation of fiber reinforced materials. J. Compos. Mater. **54**, 2837–2852 (2020)
22. Nadjafi, M., Gholami, P.: Reliability analysis of notched plates under anisotropic damage based on uniaxial loading using continuum damage mechanics approach. Int. J. Eng. **34**, 253–262 (2021)
23. Gholami, P., Kouchakzadeh, M.A., Farsi, M.: A continuum damage mechanics-based piecewise fatigue damage model for fatigue life prediction of fiber-reinforced laminated composites. Int. J. Eng. **34**, 1512–1522 (2021)
24. Gholami, P., Farsi, M.A., Kouchakzadeh, M.A.: Stochastic fatigue life prediction of fiber-reinforced laminated composites by continuum damage mechanics-based damage plastic model. Int. J. Fatigue **152**, 106456 (2021)
25. Nadjafi, M., Gholami, P.: Reliability study of notched composite laminates under uniaxial loading based on continuum damage mechanics approach. Iran. J. Sci. Technol. Trans. Mech. Eng. **46**, 1–15 (2021)
26. Gholami, P., Kouchakzadeh, M.A., Farsi, M.A.: A continuum damage mechanics approach for reliability analysis of composite laminates containing a central circular hole. Int. J. Reliab. Risk Saf. Theory Appl. **4**, 23–30 (2021)
27. Nadjafi, M., Gholami, P.: Probability fatigue life prediction of pin-loaded laminated composites by continuum damage mechanics-based Monte Carlo simulation. Compos. Commun. **32**, 101161 (2022)
28. Ladevèze, P., LeDantec, E.: Damage modelling of the elementary ply for laminated composites. Compos. Sci. Technol. **43**, 257–267 (1992)
29. Huang, Z.-M.: Simulation of the mechanical properties of fibrous composites by the bridging micromechanics model. Compos. A Appl. Sci. Manuf. **32**, 143–172 (2001)
30. Zimmermann, H.-J.: Fuzzy set theory—and its applications. Springer Science & Business Media (2011)
31. Bordossy, A., Bogardi, I.: Fuzzy fatigue life prediction. Struct. Saf. **6**, 25–38 (1989)
32. Zadeh, L.A.: Probability measures of fuzzy events. J. Math. Anal. Appl. **23**, 421–427 (1968)
33. Zadeh, L.A.: Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets Syst. **1**, 3–28 (1978)
34. Tanaka, H., Uejima, S., Asai, Linear regression analysis with fuzzy model, pp. 903–907 (1982)
35. Shokrieh, M.M., Lessard, L.B.: Progressive fatigue damage modeling of composite materials, Part II: material characterization and model verification. J. Compos. Mater. **34**, 1081–1116 (2000)
36. Shokrieh, M.M., Lessard, L.B.: Multiaxial fatigue behaviour of unidirectional plies based on uniaxial fatigue experiments—II. Experimental evaluation. Int. J. Fatigue **19**, 209–217 (1997)
37. Yang, S.: Stiffness degradation of composite laminates. George Washington Univ., Washington, DC (USA) (1987)