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State–space approach on two-temperature three-phase-lag thermoelastic medium with a spherical cavity due to memory-dependent derivative

Abstract In this article, thermoelastic interaction in a two-temperature generalized thermoelastic unbounded isotropic medium having spherical cavity has been studied in the context of memory-dependent derivative (MDD). The governing coupled equations for the problem associated with kernel function and time delays are considered in the perspective of two-temperature (2 T) three-phase-lag thermoelasticity theory. The bounding surface of the spherical cavity is subjected to mechanical and thermal loading. Using Laplace transform, the problem is transformed from the space–time domain and then solved by the state–space approach method. Suitable numerical technique is used to find the inversion of Laplace transforms. Comparisons are made graphically, between the 2 T three-phase-lag model and 2 T Lord Shulman model with MDD. Also, the effects of time-delay parameter and the kernel function on the distributions of the strain component, thermodynamic temperature, conductive temperature, displacement components, radial and hoop stresses are examined and illustrated graphically. The results show that due to the influence of the three-phase-lag-effect, memory effect, two-temperature parameter, the kernel function and time-delay, all the distributions are affected extensively.

Keywords State–space approach · Three-phase-lag · Laplace transform · Memory-dependent derivative · Spherical cavity · Two-temperature generalized thermoelasticity

1 Introduction

In the past few decades, generalized thermoelasticity theory has gained more attention to the researchers due to its applications in many fields of sciences, earthquake engineering, nuclear reactor's design, soil dynamics, high energy particle accelerators, geophysics, etc. The generalized dynamic coupled theory of thermoelasticity is one of the modified versions of classical thermoelasticity. The primary theory of generalized thermoelasticity is created by Lord and Shulman [1] who got a wave-type heat condition by proposing another law of heat conduction to displace the established Fourier law. This new theory comprises the heat flux vector in addition to its time derivative. It additionally involves another parameter that goes about as relaxation time. The second hypothesis is proposed by Green and Lindsay [2], who included two constants that go about as relaxation times. This hypothesis modifies not only the law of heat conduction but also the other equations of coupled hypothesis. Several problems of each of the theories were investigated and a few existing phenomena were discovered in [3–7].

Tzou [8] proposed a new modification of the generalized scheme of thermoelasticity known as the dual-phase lag model (DPL). After DPL model, Roy Choudhuri [9] established the three-phase-lags (TPL) equation of heat conduction utilizing the extension of the DPL model where the Fourier transform law of the heat

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conduction was substituted by estimation of TPL using the heat flux component τ_q, gradient of temperature τ_T and thermal displacement τ_v. According to this TPL model,

$$\vec{q}_i\left(r, t + \tau_q\right) = -K\phi_{,i}(r, t + \tau_T) - K^*v_{,i}(r, t + \tau_v), \dot{v} = \phi \tag{1}$$

where τ_q catches the thermal wave conduct (a little-scale reaction in time) and τ_T catches the impact of phonon electron interactions (a microscopic reaction in time). The other delay time τ_v is effective since, in the TPL model, the thermal displacement gradient is considered as a constitutive variable while in the conventional thermo-elasticity theory temperature gradient is considered as a constitutive variable. TPL model is very much useful in the problems of phonon-scattering, exothermic catalytic reactions, phonon–electron interactions, nuclear boiling, etc.

In the 2nd law of thermodynamics for continuous bodies in which the entropy due to heat conduction was governed by single temperature, the heat provided by an additional temperature was suggested by Gurtin and Williams [10, 11]. In light of this idea, Chen and Gurtin [12] and Chen et al. [13, 14] developed two temperature (2 T) theory of thermoelasticity. Several problems of wave propagation on 2 T theory of thermoelasticity have been solved by Warren and Chen [15] and Boley and Tollins [16]. The 2 T thermoelasticity theory has increased much consideration of the researchers in the ongoing years. Quintanilla and Tien [17] studied the spatial behavior, structural stability, existence and convergence of 2 T thermoelasticity theory. A new model of 2 T generalized thermoelasticity theory was introduced by Youssef [18]. Sarkar and Lahiri [19] and Youssef and Al-Lehaibi [20] discussed problems on the 2 T thermoelasticity with a single relaxation time parameter. The propagation of plane waves in the context of 2 T theory was investigated by Puri and Jordan [21]. The reciprocity and uniqueness theorems for 2 T thermoelasticity theory were established by Lesan [22]. Using the state–space approach, the problem of an unbounded body having a spherical cavity and cylindrical cavity for 2 T generalized thermoelasticity has solved by Youssef and Al-Harby [23] and Kumar, Mukhopadhyay [24] and Abbas [25]. Kumar et al. [26] established the reciprocal and Variational principles for 2 T generalized thermoelasticity. The 2 T electro-magneto-thermoelastic one-dimensional problem for the perfect conducting medium has studied by Ezzat and El-Karamany [27]. The three-dimensional problem of 2 T thermoelasticity with Lord Shulman model of a semi-infinite space subjected to traction free surface and ramp-type heating was introduced by Ezzat and Youssef [28].

Enlightened by the Caputo [29] and Caputo and Mainardi [30] fractional derivative, Wang and Li [31] presented the essential idea of MDD to portray the memory effect. He expressed a function f of the first-order derivative as an integral form over a slipping interval regarding a common derivative associated with the kernel function as follows:

$$D_\omega^{(1)} f(x, t) = \frac{1}{\omega} \int\limits_{t-\omega}^{t} K(t - \xi) f'(x, \xi) d\xi \tag{2}$$

where ω is the time delay and the kernel function $K(t - \xi)$ is of the form [31].

$$K(t - \xi) = 1 - \frac{2d}{\omega}(t - \xi) + \frac{c^2(t - \xi)^2}{\omega^2} = \begin{cases} 1 & \text{if } c = d = 0, \\ 1 - \frac{(t-\xi)}{\omega} & \text{if } c = 0, d = \frac{1}{2}, \\ \left(1 - \frac{(t-\xi)}{\omega}\right)^2 & \text{if } c = d = 1. \end{cases}$$

The pioneering contribution was dedicated to explore different theoretical and practical perspectives in continuum mechanics with memory-dependent heat transfer. Yu et al. [32] presented MDD rather than fractional calculus within Lord Shulman [1] universal thermoelasticity. Ezzat et al. [33, 34] proposed a novel MDD model of thermo-viscoelasticity in the presence of time-dependent thermal shock. Ezzat et al. [35] developed a new magneto-thermoelastic theory involving MDD for a perfect electrically conducting semi-infinite space, which was subjected to thermal loading. Lofty and Sarkar [36] had studied 2 T generalized thermoelasticity problem with MDD for photothermal semiconducting medium. Ezzat and El-Bary [37] solved the 1D problem of functionally graded magneto-thermoelasticity in the perspective of MDD heat transfer. Sur and Kanoria [38] studied the transient phenomena and three-phase lag effect on generalized thermoelasticity under MDD for a fiber-reinforced plate having a heat source. The propagation of magneto-thermoelastic interaction on 2 T generalized thermoelastic perfectly conducting medium with MDD was investigated by Sarkar et al. [39]. Sarkar and Othman [40] considered a 3D problem of generalized thermoelasticity with MDD subjected to thermal shock. The effect of MDD on two-dimensional thermoelastic rotating medium in the context of Lord-Shulman model was dealt with by Othman and Mondal [41]. Mondal [42] studied memory response, nonlocal

and moving heat source effect on a magneto-thermoelastic rod in the framework of the DPL model. Later on, presenting this methodology of MDD, a few examinations were done by [43–51].

The importance of the state–space approach is familiar in fields where the time behavior of any physical procedure is important. This approach is more general compared to the Fourier transform and Laplace theories. Several examinations were carried out using the state–space approach in the following literatures [52–58].

Inspired by the above research works, in our present article, we analyze in details the TPL effect and memory response on a 2 T generalized thermoelastic isotropic infinite medium having a spherical cavity. The bounding surface of the spherical cavity is subjected to a ramp type heating thermal shock. The governing equations involving 2 T thermoelasticity theory with MDD are exposed in the Laplace transform domain and solved by utilizing the state–space approach. The numerical inversions of the Laplace transform have been found utilizing Honig Hirdes [59]. The analytical solutions of the thermophysical quantities are obtained numerically for copper material. These complete solutions are presented graphically for various forms of kernel functions and for the various time delay. Also, to estimate the effect of 2 T parameter, some comparisons (2 T TPL model and 2 T LS model) have been made.

2 Derivation of two-temperature TPL heat conduction equation model with MDD

The 2 T thermoelasticity is depended based on the classical conjecture of heat conductivity. The classical Fourier's law is given by

$$\vec{q}_i = -K\phi_{,i}. \tag{3}$$

The energy equation is [54]

$$\rho C_e \dot{T} + \gamma T_0 \dot{e} = -\vec{q}_{i,i} + \rho Q \tag{4}$$

where ρ is the mass density, T_0 is uniform reference temperature, q_i is the heat flux component, $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is coefficient of linear thermal expansion, λ, μ are Lame's constant, C_e is specific heat and Q is the heat source.

The 2 T relation is as follows [23]

$$\phi - T = b\phi_{,ii}, \quad b \geq 0 \tag{5}$$

where T is the thermodynamic temperature, ϕ is the conductive temperature and $b(> 0)$ is the 2 T parameter, known as temperature discrepancy [12–14].

Using Taylor's series expansion and MDD in (1) we obtain

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\vec{q}_i = -K\phi_{,i}(1 + \tau_T D_\omega) - K^* v_{,i}(1 + \tau_v D_\omega), \tag{6}$$

where $\dot{v} = \phi$, K^* is a constant material and K is the thermal conductivity, D_ω is the memory-dependent derivative operator of a differentiable function $f(t)$.

Taking divergence of both sides of Eq. (6), we get

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\vec{q}_{i,i} = -K\phi_{,ii}(1 + \tau_T D_\omega) - K^* v_{,ii}(1 + \tau_v D_\omega). \tag{7}$$

Eliminating $\vec{q}_{i,i}$ from Eqs. (4) and (7) the heat equation takes the form

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)(\rho C_e \dot{T} + \gamma T_0 \dot{e} - \rho Q) = K(1 + \tau_T D_\omega)\phi_{,ii} + K^*(1 + \tau_v D_\omega)v_{,ii}. \tag{8}$$

or,

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)(\rho C_e \ddot{T} + \gamma T_0 \ddot{e} - \rho \dot{Q}) = K(1 + \tau_T D_\omega)\dot{\phi}_{,ii} + K^*(1 + \tau_v D_\omega)\phi_{,ii}. \tag{9}$$

Equation (9) represents the 2 T hyperbolic thermoelasticity heat transfer equation with MDD comprising TPL effects.

3 Formulation of the problem:

We consider a homogeneous, isotropic, thermoelastic solid having a spherical cavity with radius 'a'. We shall use a system of spherical polar system of co-ordinates (r, ψ, χ) with the origin at the center of the spherical cavity. Due to spherical symmetry of the present problem, every function is treated as functions of r (radial distance) and time t only. Therefore,

$$\overrightarrow{u} = (u(r, t), 0, 0), \quad T = T(r, t), \quad \phi = \phi(r, t). \tag{10}$$

The strain components are as follows

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\psi\psi} = e_{\chi\chi} = \frac{u}{r}. \tag{11}$$

The cubic dilatation e is given by

$$\varepsilon_{kk} = e = e_{rr} + e_{\psi\psi} + e_{\chi\chi} = \frac{\partial u}{\partial r} + 2\frac{u}{r} = \frac{1}{r^2}\frac{\partial \left(r^2 u\right)}{\partial r}. \tag{12}$$

The stress–strain–temperature relations are given by

$$\sigma_{rr} = 2\mu\frac{\partial u}{\partial r} + \lambda e - \gamma T, \tag{13}$$

$$\sigma_{\psi\psi} = \sigma_{\chi\chi} = 2\mu\frac{u}{r} + \lambda e - \gamma T. \tag{14}$$

The equation of motion and the TPL heat conduction in the nonappearance of the external forces and internal heat sources under 2 T thermoelasticity based on MDD are as follows

$$(\lambda + 2\mu)\frac{\partial e}{\partial r} - \gamma\frac{\partial T}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{15}$$

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\left(\rho C_e \frac{\partial^2 T}{\partial t^2} + \gamma T_0 \frac{\partial^2 e}{\partial t^2}\right) = K(1 + \tau_T D_\omega)\nabla^2\dot{\phi} + K^*(1 + \tau_v D_\omega)\nabla^2\phi \tag{16}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{17}$$

For numerical simulation, we introduce the following non-dimensional variables

$$r' = c_0\eta r, \ u' = c_0\eta u, \ t' = c_0^2\eta t, \ T' = \frac{(T - T_0)}{T_0}, \ \sigma_{ij}' = \frac{\sigma_{ij}}{(\lambda + 2\mu)}, \ \tau_q' = c_0^2\eta\tau_q$$

$\tau_T' = c_0^2\eta\tau_T, \ \tau_v' = c_0^2\eta\tau_v, \ c_0^2 = \frac{(\lambda + 2\mu)}{\rho}, \ \phi' = \frac{\phi}{T_0}, \ \eta = \frac{\rho C_e}{k}$. For simplicity omitting primes, Eqs. (13)–(16) and (5) take the non-dimensional form as

$$\sigma_{rr} = \left(1 - \beta^2\right)\frac{\partial u}{\partial r} + \beta^2 e - \alpha_2 T, \tag{18}$$

$$\sigma_{\psi\psi} = \sigma_{\chi\chi} = \left(1 - \beta^2\right)\frac{u}{r} + \beta^2 e - \alpha_2 T, \tag{19}$$

$$\frac{\partial e}{\partial r} - \alpha_2\frac{\partial T}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \tag{20}$$

$$\left(1 + \tau_q D_\omega + \frac{\tau_q^2}{2} D_\omega^2\right)\left(\frac{\partial^2 T}{\partial t^2} + \varepsilon\frac{\partial^2 e}{\partial t^2}\right) = (1 + \tau_T D_\omega)\nabla^2\dot{\phi} + \varepsilon'(1 + \tau_v D_\omega)\nabla^2\phi, \tag{21}$$

$$T = \phi - \omega'\nabla^2\phi, \tag{22}$$

where $\alpha_2 = \frac{T_0\gamma}{(\lambda + 2\mu)}, \varepsilon' = \frac{k^*}{C_e(\lambda + 2\mu)}, \varepsilon = \frac{T_0\gamma^2}{\rho C_e(\lambda + 2\mu)}, \beta^2 = \frac{\lambda}{(\lambda + 2\mu)}, \omega' = bc_0^2\eta^2.$

Applying the divergence operator on Eq. (20) we get

$$\nabla^2 e - \alpha_2\nabla^2 T = \frac{\partial^2 e}{\partial t^2}. \tag{23}$$

4 Special cases

(1) For $\tau_v = \tau_T = \tau_q^2 = 0$, $K^* = 0$ and $\tau_q \neq 0$, in Eq. (16), the present problem reduces to the two temperature Lord–Shulman theory with MDD

(2) For $\tau_v = \tau_T = \tau_q = 0$ in Eq. (16), the present problem reduces to the two temperature Green–Naghdi type III theory with MDD.

(3) For $\tau_v = \tau_T = \tau_q = 0$ and $K = 0$, in Eq. (16), the present problem reduces to the two temperature Green–Naghdi type II theory with MDD.

5 Initial and boundary conditions

The initial conditions are

$$u(r, 0) = T(r, 0) = \phi(r, 0) = 0 \text{ at } t = 0 \text{ for } r \geq a,$$

$$\frac{\partial u(r, 0)}{\partial t} = \frac{\partial T(x, 0)}{\partial t} = \frac{\partial \phi(x, 0)}{\partial t} = 0 \text{ at } t = 0 \text{ for } r \geq a.$$

The boundary conditions of the problem are as follows:

The surface $r = a$ is subjected to thermal and mechanical loading given by

(i) Thermal boundary condition:

$$\phi(a, t) = h(t), \tag{24}$$

where $h(t) = \begin{cases} 0, & t \leq 0 \\ \frac{h_0}{t_0}t, & 0 < t \leq t_0 \\ h_0, & t > t_0 \end{cases}$.

(ii) Mechanical boundary condition:

$$\sigma_{rr}(a, t) = 0. \tag{25}$$

6 Laplace transform domain

The Laplace transform with parameter p is defined as

$$\overline{f}(x, p) = L[f(x, t)] = \int_0^\infty f(x, t)e^{-pt} dt, \quad Re(p) > 0. \tag{26}$$

Using the Laplace transform in (18) and (19) and (21)–(25) we obtain

$$\overline{\sigma}_{rr} = (1 - \beta^2)\frac{\partial \overline{u}}{\partial r} + \beta^2 \overline{e} - \alpha_2 \overline{T}, \tag{27}$$

$$\overline{\sigma}_{\psi\psi} = \overline{\sigma}_{\chi\chi} = (1 - \beta^2)\frac{\overline{u}}{r} + \beta^2 \overline{e} - \alpha_2 \overline{T}, \tag{28}$$

$$\nabla^2 \overline{e} - \alpha_2 \nabla^2 \overline{T} = p^2 \overline{e}, \tag{29}$$

$$\left(1 + \tau_q G(p) + \frac{\tau_q^2}{2}G(p)\right)\left(p^2 \overline{T} + \varepsilon p^2 \overline{e}\right) = p(1 + \tau_T G(p))\nabla^2 \overline{\phi} + \varepsilon'(1 + \tau_v G(p))\nabla^2 \overline{\phi}, \tag{30}$$

$$\overline{T} = \overline{\phi} - \omega' \nabla^2 \overline{\phi}, \tag{31}$$

$$\overline{\phi}(a, p) = \overline{h}(p) = \frac{h_0(1 - e^{-pt_0})}{t_0 p^2}, \tag{32}$$

$$\overline{e}(a, p) = 0. \tag{33}$$

Here the following rule has been used

$$L\{D_\omega f(t)\} = \frac{\overline{f}(p)G(p)}{\omega}$$

where $G(p) = \left(1 - \frac{2d}{\omega p} + \frac{2c^2}{\omega^2 p^2}\right) - e^{-p\omega}\left[\left(1 - 2d + c^2\right) + \frac{2(c^2 - d)}{\omega p} + \frac{2c^2}{\omega^2 p^2}\right]$.

Substituting Eq. (31) in (30), we get

$$\overline{T} = \left(1 - \omega' N\right)\overline{\phi} - \varepsilon\omega' N\overline{e} \tag{34}$$

where $N = \dfrac{p^2\left(1 + \tau_q G(p) + \frac{\tau_q^2}{2}G(p)\right)}{p + p\tau_T G(p) + \varepsilon' + \varepsilon'\tau_v G(p) + \omega' p^2\left(1 + \tau_q G(p) + \frac{\tau_q^2}{2}G(p)\right)}$.

Substituting the value of $\overline{T}$ from Eq. (34) into (30), we get

$$\nabla^2\overline{\phi} = N\overline{\phi} + \varepsilon N\overline{e}. \tag{35}$$

By using Eqs. (34) and (35) with Eq. (29), we obtain

$$\nabla^2\overline{e} = L_1\overline{e} + L_2\overline{\phi}, \tag{36}$$

where $L_1 = \dfrac{p^2 + (1 - \omega' N)\alpha_2\varepsilon N}{1 + \omega'\alpha_2\varepsilon N}$, $L_2 = \dfrac{(1 - \omega' N)\alpha_2 N}{1 + \omega'\alpha_2\varepsilon N}$.

7 State–space approach

Choosing the strain dilatation $\overline{e}$ and the conductive temperature $\overline{\phi}$ in the r direction as state variables, Eqs. (35) and (36), can be written explicitly in vector–matrix differential equation form as:

$$\nabla^2\overline{V}(r, p) = A(p)\overline{V}(r, p), \tag{37}$$

where $\overline{V}(r, p) = \begin{pmatrix} \overline{\phi}(r, p) \\ \overline{e}(r, p) \end{pmatrix}$, $A(p) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $a_{11} = N$, $a_{12} = \varepsilon N$, $a_{21} = L_2$, $a_{22} = L_1$.

The formal solution of Eq. (37) can be expressed as

$$\overline{V}(r, p) = \frac{a}{r}e^{-\sqrt{A(p)}(r-a)}\overline{V}(a, p) \tag{38}$$

where $\overline{V}(a, p) = \begin{pmatrix} \overline{\phi}(a, p) \\ \overline{e}(a, p) \end{pmatrix} = \begin{pmatrix} \overline{h}(p) \\ 0 \end{pmatrix}$.

The characteristic equation of the matrix $A(p)$ is given by

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{39}$$

The roots of Eq. (39), namely λ_1 and λ_2, satisfy the relations:

$$\lambda_1 + \lambda_2 = a_{11} + a_{22} = N + L_1, \ \ \lambda_1\lambda_2 = (a_{11}a_{22} - a_{12}a_{21}) = NL_1 - \varepsilon NL_2. \tag{40}$$

Now, we can write the spectral decomposition of $A(p)$ given by [55] as:

$$A(p) = \lambda_1 E_1 + \lambda_2 E_2, \tag{41}$$

where E_1 and E_2 satisfy the following conditions [60]:

$$E_1 + E_2 = I, \ \ E_1 E_2 = 0, \ \ E_i^2 = E_i \ \text{for} \ i = 1, 2 \tag{42}$$

where E_1 and E_2 are called the projectors of $A(p)$, I is the identity matrix and 0 is the zero matrix.

Then, we have

$$\sqrt{A(p)} = \sqrt{\lambda_1} E_1 + \sqrt{\lambda_2} E_2, \tag{43}$$

where

Table 1 Physical parameters of copper material

Symbol	Numerical value	Units
λ	7.76×10^{10}	$\mathrm{N\,m^{-2}}$
ρ	8954	$\mathrm{kg\,m^{-3}}$
c_e	383.1	$\mathrm{J\,kg^{-1}\,deg^{-1}}$
μ	3.86×10^{10}	$\mathrm{N\,m^{-2}}$
T_0	293	K
α_t	0.0104	$\mathrm{(K)^{-1}}$
K	386	$\mathrm{W\,m^{-1}\,K^{-1}}$
ε	0.0168	

$$E_1 = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} N - \lambda_2 & \varepsilon N \\ -\frac{(\lambda_1-N)(\lambda_2-N)}{\varepsilon N} & \lambda_1 - N \end{pmatrix}, \quad E_2 = \frac{-1}{\lambda_1 - \lambda_2}\begin{pmatrix} N - \lambda_1 & \varepsilon N \\ -\frac{(\lambda_1-N)(\lambda_2-N)}{\varepsilon N} & \lambda_2 - N \end{pmatrix}.$$

Thus, we get

$$\sqrt{A(p)} = \frac{1}{\sqrt{\lambda_1} + \sqrt{\lambda_2}}\begin{pmatrix} N + \sqrt{\lambda_1\lambda_2} & \varepsilon N \\ L_2 & L_1 + \sqrt{\lambda_1\lambda_2} \end{pmatrix}. \tag{44}$$

To find the matrix $\mathrm{e}^{-\sqrt{A(p)}(r-a)}$, we use Cayley–Hamilton theorem. The characteristic equation of the matrix $\sqrt{A(p)}$ is given by

$$m^2 - \left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right)m + \sqrt{\lambda_1\lambda_2} = 0. \tag{45}$$

The roots of Eq. (45), namely m_1 and m_2, take the forms

$$m_1 = \sqrt{\lambda_1}, \quad m_2 = \sqrt{\lambda_2}. \tag{46}$$

The Taylor's series expansion for the matrix exponential in Eq. (38) is as follows

$$\mathrm{e}^{-\sqrt{A(p)}(r-a)} = \sum_{n=0}^{\infty} \frac{\left(-\sqrt{A(p)}(r-a)\right)^n}{n!}. \tag{47}$$

We can express A^2 and higher powers of the matrix A in terms of A and I, where I the identity matrix is of second-order [61].

Thus, $\mathrm{e}^{-\sqrt{A(p)}(r-a)}$ in (47) can be reduced to

$$\mathrm{e}^{-\sqrt{A(p)}(r-a)} = a_0(r, p)I + a_1(r, p)\sqrt{A(p)} \tag{48}$$

where a_0 and a_1 are coefficients depending on r and p.

By Cayley–Hamilton's theorem, the characteristic roots m_1 and m_2 of the matrix $\sqrt{A(p)}$ must satisfy Eq. (48), thus

$$\mathrm{e}^{-m_1(r-a)} = a_0 I + a_1 m_1, \tag{49}$$

$$\mathrm{e}^{-m_2(r-a)} = a_0 I + a_1 m_2. \tag{50}$$

The solution of Eqs. (49) and (50) is given by

$$a_0 = \frac{m_1 \mathrm{e}^{-m_2(r-a)} - m_2 \mathrm{e}^{-m_1(r-a)}}{m_1 - m_2}, \tag{51}$$

$$a_1 = \frac{\mathrm{e}^{-m_1(r-a)} - \mathrm{e}^{-m_2(r-a)}}{m_1 - m_2}. \tag{52}$$

Now, Eq. (48) can be written as

$$\mathrm{e}^{-\sqrt{A(p)}(r-a)} = L(r, p) = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \tag{53}$$
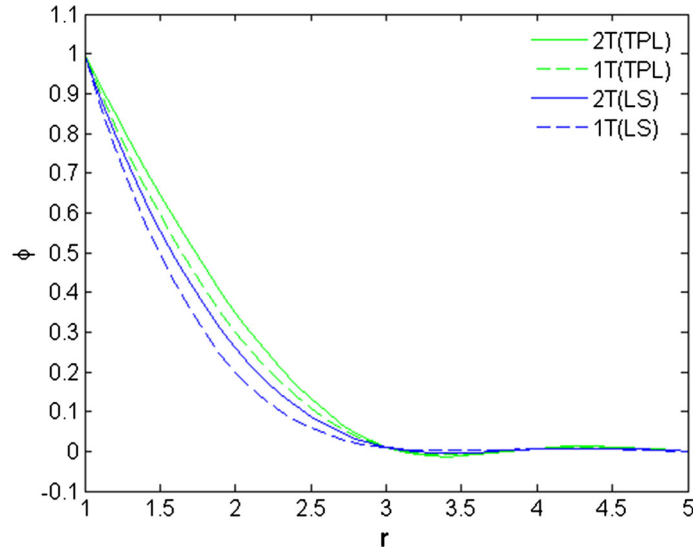
Fig. 1 Distribution of ϕ against r

where

$$\left. \begin{array}{l} l_{11} = \frac{(m_1^2 - N)e^{-m_2(r-a)} - (m_2^2 - N)e^{-m_1(r-a)}}{m_1^2 - m_2^2} \\[2mm] l_{12} = \frac{\varepsilon N e^{-m_1(r-a)} - \varepsilon N e^{-m_2(r-a)}}{m_1^2 - m_2^2} \\[2mm] l_{21} = \frac{L_2 e^{-m_1(r-a)} - L_2 e^{-m_2(r-a)}}{m_1^2 - m_2^2} \\[2mm] l_{22} = \frac{(m_1^2 - L_1)e^{-m_2(r-a)} - (m_2^2 - L_1)e^{-m_1(r-a)}}{m_1^2 - m_2^2} \end{array} \right\}. \tag{54}$$

Therefore, considering (53) and (38), we get the solution of (37) in the following form

$$\overline{V}(r, p) = \frac{a}{r} \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \overline{V}(a, p). \tag{55}$$

Thus, the solutions for $\overline{\phi}(a, p)$ and $\overline{e}(a, p)$ can be obtained from Eq. (55), (32) and (33) as follows:

$$\overline{\phi}(r, p) = \frac{a\overline{h}(p)}{r(m_1^2 - m_2^2)} \left\{ (m_1^2 - N)e^{-m_2(r-a)} - (m_2^2 - N)e^{-m_1(r-a)} \right\}, \tag{56}$$

$$\overline{e}(r, p) = \frac{L_2 a\overline{h}(p)}{r(m_1^2 - m_2^2)} \left\{ e^{-m_1(r-a)} - e^{-m_2(r-a)} \right\}. \tag{57}$$

Using Eqs. (56) and (57) into (34) we obtain expressions of $\overline{T}$

$$\overline{T}(r, p) = \frac{a\overline{h}(p)}{r(m_1^2 - m_2^2)} \left[\left\{ (m_1^2 - N)(1 - \omega' N) - L_2 \omega' N \varepsilon \right\} e^{-m_2(r-a)} \right]$$
$$- \frac{a\overline{h}(p)}{r(m_1^2 - m_2^2)} \left[\left\{ (m_2^2 - N)(1 - \omega' N) - L_2 \omega' N \varepsilon \right\} e^{-m_1(r-a)} \right]. \tag{58}$$

Applying Laplace transform to Eq. (12) and using Eq. (57), we obtain expressions of $\overline{u}$ as follows:

$$\overline{u}(r, p) = \frac{L_2 a\overline{h}(p)}{r(m_1^2 - m_2^2)m_1^2 m_2^2} \left\{ m_2^2(m_1 r + 1)e^{-m_1(r-a)} - m_1^2(m_2 r + 1)e^{-m_2(r-a)} \right\}. \tag{59}$$
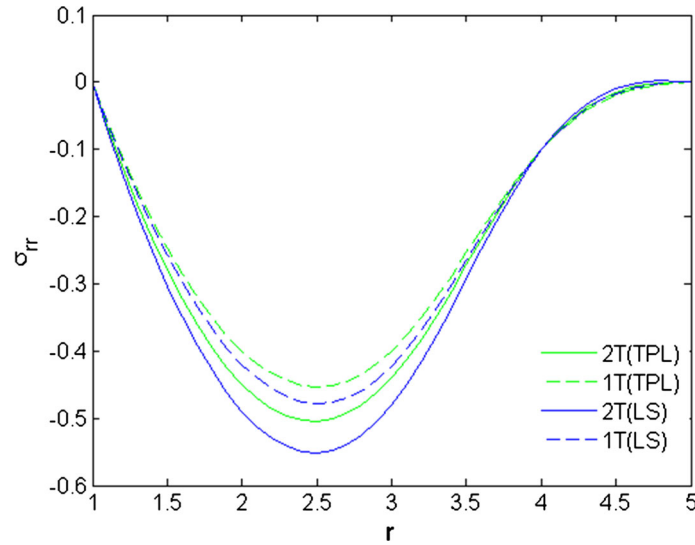
Fig. 2 Distribution of σ_{rr} against r

Using (57)–(59) and (27) the stress components $\overline{\sigma}_{rr}$ and $\overline{\sigma}_{\chi\chi}$ are obtained as:

$$\overline{\sigma}_{rr} = \frac{a\overline{h}(p)}{r^3(m_1^2 - m_2^2)m_1^2 m_2^2}\left[m_2^2 \mathrm{e}^{-m_1(r-a)}\{m_1 r^2(L_2 + \alpha_2 T_2)\} + 2L_2(1 - \beta^2)(m_1 r + 1)\right]$$

$$- \frac{a\overline{h}(p)}{r^3(m_1^2 - m_2^2)m_1^2 m_2^2}\left[m_1^2 \mathrm{e}^{-m_2(r-a)}\{m_2 r^2(L_2 + \alpha_2 T_1)\} + 2L_2(1 - \beta^2)(m_2 r + 1)\right], \quad (60)$$

where $T_1 = (m_1^2 - N)(1 - \omega' N) - L_2 \omega' N \varepsilon$, $T_2 = (m_2^2 - N)(1 - \omega' N) - L_2 \omega' N \varepsilon$,

$$\tilde{\sigma}_{\chi\chi} = \frac{a\overline{h}(p)}{r^2(m_1^2 - m_2^2)m_1^2 m_2^2}\left[m_2^2 e^{-m_1(r-a)}\left\{L_2(1 - \beta^2)(m_1 r + 1) + m_1^2 r(\beta^2 L_2 - \alpha_2 T_1)\right\}\right.$$

$$\left. + m_1^2 e^{-m_2(r-a)}\left\{L_2(1 - \beta^2)(m_2 r + 1) + m_2^2 r(\beta^2 L_2 - \alpha_2 T_2)\right\}\right]. \quad (61)$$

8 Inversion of Laplace transform

For the final solution of the thermodynamic temperature, conductive temperature, stress and displacement in the space–time domain, we have used a numerical technique [59] where the inversion of Laplace transform formula can be written as

$$f(t) \approx f_N(t) = \frac{\mathrm{e}^{rt} f(r)}{2t_1} + \sum_{k=1}^{N} d_k, \quad \text{for } 0 \le t \le 2t_1 \quad (62)$$

where

$$d_k = \frac{\mathrm{e}^{rt}}{t_1}\mathrm{Re}\left[\mathrm{e}^{\frac{ik\pi t}{t_1}}\overline{f}(r + ik\pi/t_1)\right], \quad (63)$$

N is a large integer and N is picked in such a manner that $d_k \le \varepsilon$ (ε is a small, nonnegative quantity), which corresponds to the achievement of the degree of accuracy. The best possible choices of t_1 and r are made according to the criteria outlined in [59].
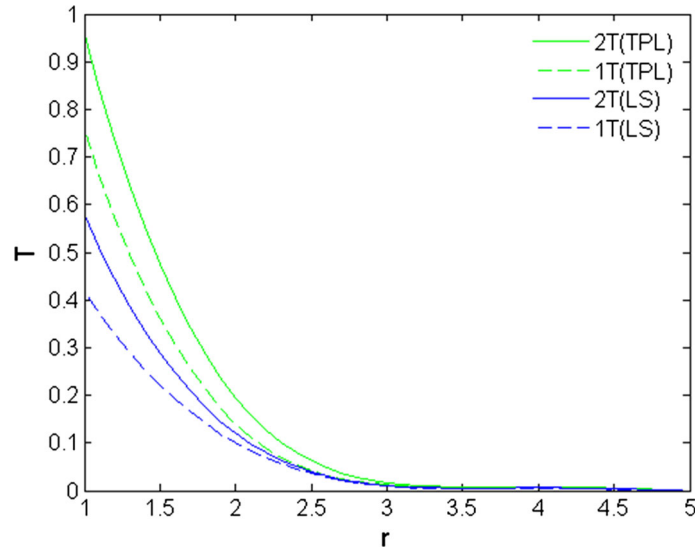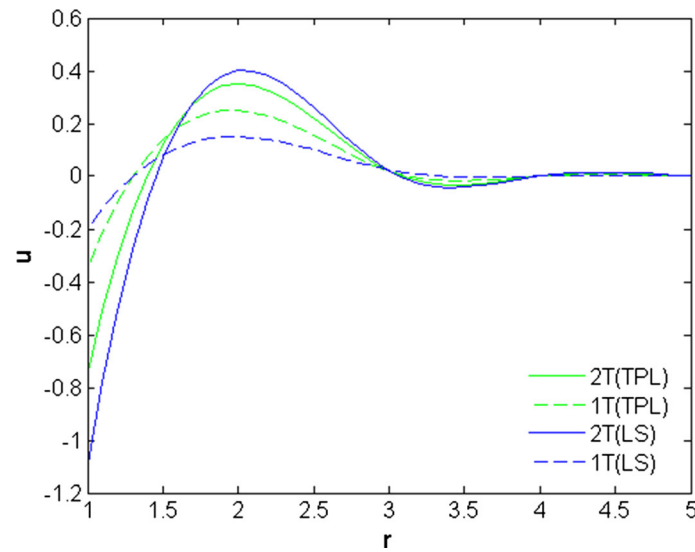
Fig. 3 Distribution of T against r

Fig. 4 Distribution of u against r

9 Particular cases

(i) For the case of two-temperature thermoelasticity with MDD problem, taking $\tau_T = \tau_v = K^* = 0$ and $\tau_q = \tau$ in Eq. (8) this problem agrees with Sarkar and Mondal [46].

(ii) If we put $\tau_v = \tau_T = K^* = \tau_q^2 = 0$ and $\tau_q = \omega$, this problem reduces to two-temperature thermoelasticity with MDD and agrees with the work of Ezzat et al. [43].

10 Numerical results and discussions

In this part, the numerical results of the non-dimensional displacement component, stress and temperature distribution are demonstrated. Numerical computations are conducted for a copper material [62] whose physical information is given below (Table 1).
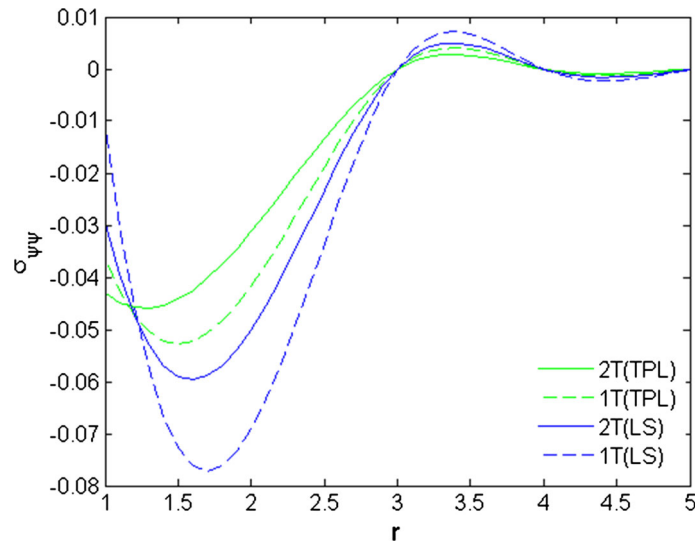
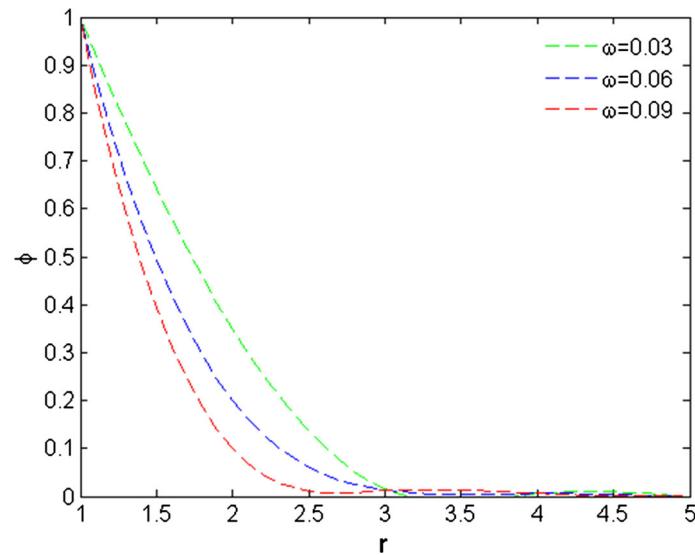**Fig. 5** Distribution of $\sigma_{\psi\psi}$ against r

Fig. 6 Distribution of ϕ against r for various time delays

We take the other parameters for computational purposes $\omega' = 0$, $\phi_0 = 1$, $a = 1$ m. Moreover, take the other parameters for TPL theory $\tau_q = 0.001$ s, $\tau_T = 0.05$ s, $\tau_v = 0.05$ s, $K^* = 200$ according to the Quintanilla and Racke [63] stability condition $K^* < K + K^*\tau_v < \frac{2K\tau_T}{\tau_q}$.

In view of the above physical constants, the field quantities are computed numerically by utilizing MATLAB programming and are depicted graphically. The simulations were performed for three various cases.

10.1 Case I

In case I, the distributions of the dimensionless conductive temperature ϕ, radial stresses σ_{rr}, thermodynamic temperature T, displacement u and hoop stresses $\sigma_{\psi\psi}$ with respect to distance r are examined for two-temperature three-phase-lag model (2 T-TPL)($\omega' > 0$), two-temperature Lord Shulman model (2 T-LS) ($\omega' > 0$), one-temperature three-phase-lag model (1 T-TPL)$\left(\omega' = 0\right)$ and one-temperature Lord Shulman model (1 T-LS) $\left(\omega' = 0\right)$ in Figs. 1, 2, 3, 4 and 5.
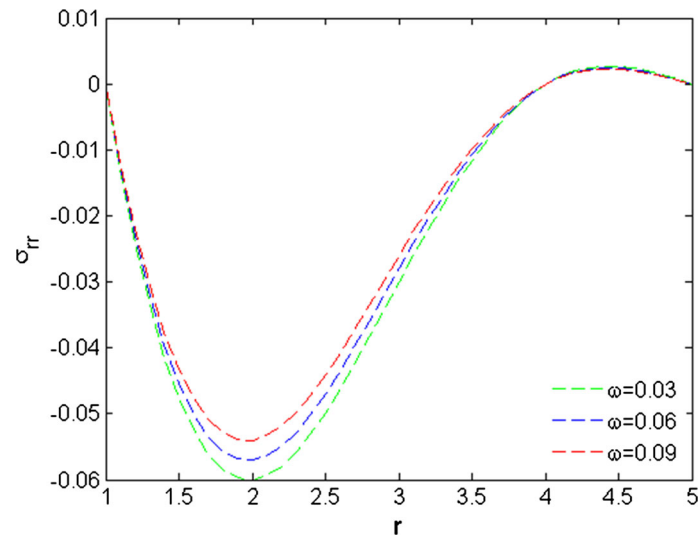
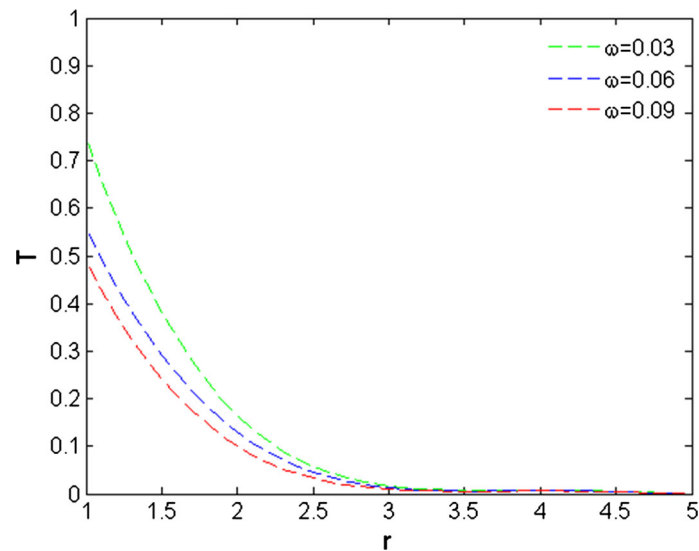**Fig. 7** Distribution of σ_{rr} against r for various time delays

Fig. 8 Distribution of T against r for various time delays

In Figs. 1 and 3 it is clear that the magnitudes of the dimensionless conductive temperature ϕ and thermodynamic temperature T for 2 T-LS $(\omega' > 0)$ and 1 T-LS $(\omega' = 0)$ are smaller than that of 2 T-TPL $(\omega' > 0)$ and 1 T-TPL $(\omega' = 0)$ models. Also,ϕ and T converge towards zero with the increase in distance (r). However, we notice that conductive temperature ϕ agrees with the physical boundary condition. It is observed from Figs. 2 and 4 that the values of radial stress σ_{rr} and u are larger in the case of 2 T-LS $(\omega' > 0)$ model compared to the 2 T-TPL $(\omega' > 0)$, 1 T-TPL $(\omega' = 0)$ and 1 T-LS $(\omega' = 0)$ models. It is clear from Fig. 5 that the absolute value of the hoop stress $\sigma_{\psi\psi}$ for 2 T-LS $(\omega' > 0)$ model is larger than that of 2 T-TPL $(\omega' > 0)$ model and also 1 T-LS $(\omega' = 0)$ is larger than that of 1 T-TPL $(\omega' = 0)$ model. Moreover, $\sigma_{\psi\psi}$ reaches the peak value around $r = 1.8$ and converges towards zero after $r = 3.5$.

10.2 Case II

In case II, the effects of the various time delay parameters $\omega = 0.03$ s, 0.06 s, 0.09 s on the variations of the dimensionless conductive temperature ϕ, radial stress σ_{rr}, thermodynamic temperature T, displacement u
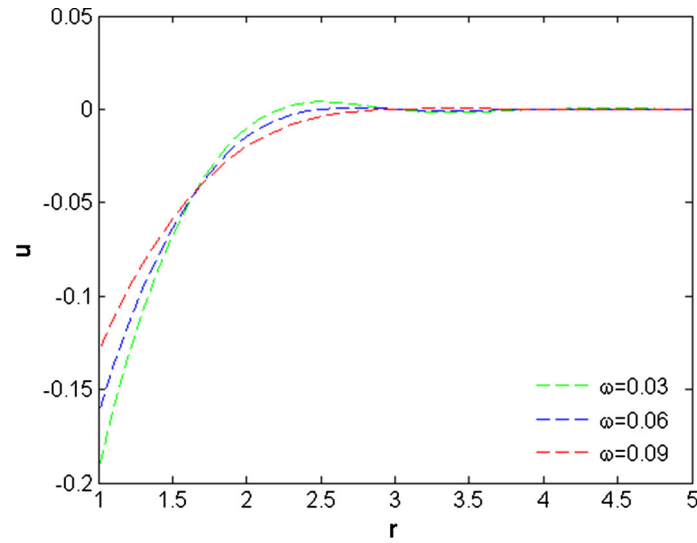
Fig. 9 Distribution of u against r for various time delays
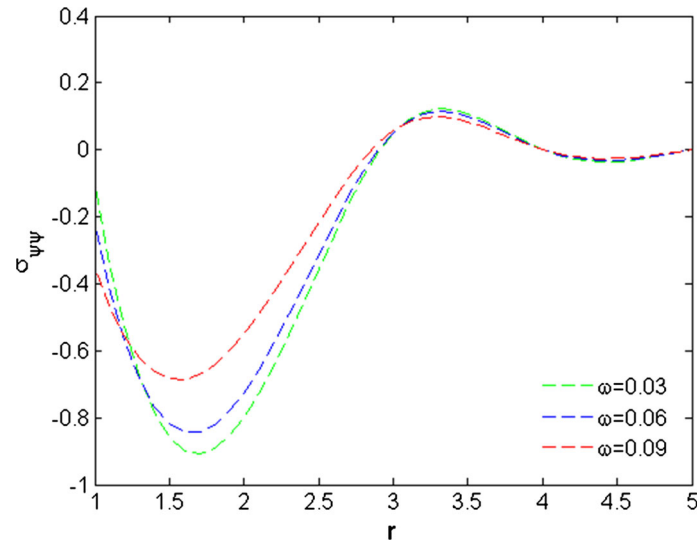

Fig. 10 Distribution of $\sigma_{\psi\psi}$ against r for various time delays

and hoop stress $\sigma_{\psi\psi}$ for suitable kernel function $K(t - \xi) = \left(1 - \frac{(t-\xi)}{\omega}\right)^2$, $t = 0.01$ with respect to radial distance r are examined. The obtained results are displayed graphically in Figs. 6, 7, 8, 9 and 10.

From Figs. 5, 6, 7, 8, 9 and 10, we see that the absolute values of all the thermo-physical quantities have been affected by the time delay parameter ω in the perspective of MDD. We observed that the time delay parameter ω has a significant effect on all the field quantities. However, we also noticed that the conductive temperature ϕ and thermodynamic temperature T have been affected by the time-delay ω, where the rising of the value of the parameter ω causes diminishing effect in temperature fields. The thermal waves are smooth, continuous functions and reach to steady state depending on the value ω, which means that the particles transport the heat to the other particles easily and this makes the diminishing rate of the temperature greater than the other ones. Also, the magnitudes of the non-dimensional conductive temperature ϕ and radial stresses σ_{rr} satisfy the boundary conditions and validate the numerical scheme incorporated for the solution of the problem.
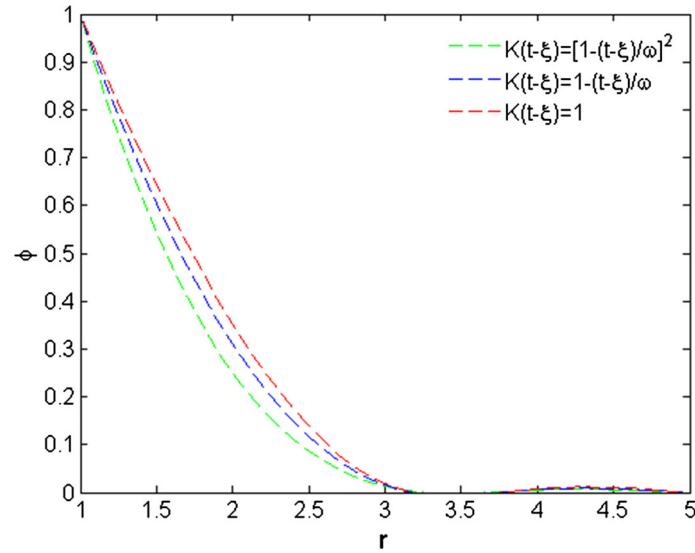
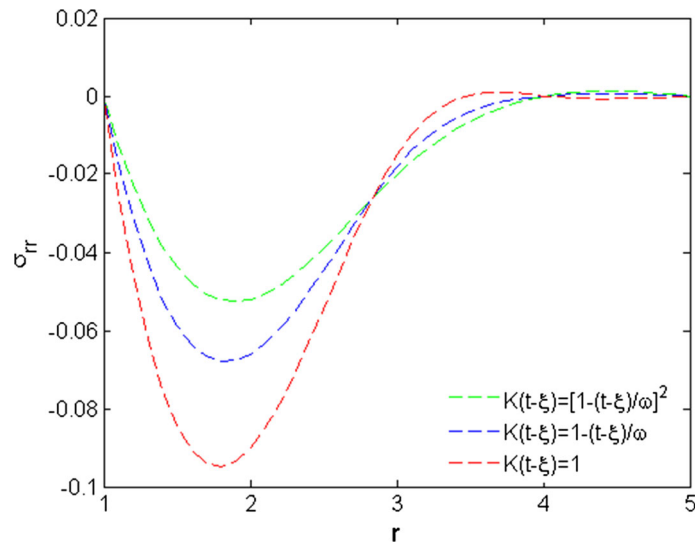**Fig. 11** Distribution of ϕ against r for various $K(t - \xi)$

Fig. 12 Distribution of σ_{rr} against r for various $K(t - \xi)$

10.3 Case III

In Figs. 11, 12, 13, 14 and 15, we study the variations of the dimensionless conductive temperature ϕ, radial stresses σ_{rr}, thermodynamic temperature T, displacement u and hoop stress $\sigma_{\psi\psi}$ for various forms of kernel functions $K(t - \xi) = \left(1 - \frac{(t-\xi)}{\omega}\right)^n$ ($n = 0, 1, 2$) and for suitable time delay parameters and time $(t, \omega) = (0.02, 0.05)$, $(0.1, 0.02)$, $(0.03, 0.006)$, $(0.06, 0.05)$, $(0.004, 0.07)$ and $(0.004, 0.08)$, respectively.

In Figs. 11 and 12, it is observed that the magnitudes of the conductive temperature ϕ and radial stresses σ_{rr} have a significant consequence on the occurrence of various kernel functions. Also, ϕ and σ_{rr} acquire maximum values for $n = 0$ in $K(t - \xi) = \left(1 - \frac{(t-\xi)}{\omega}\right)^n$ and minimum value for the exponent $n = 2$ of $K(t - \xi) = \left(1 - \frac{(t-\xi)}{\omega}\right)^n$. On the other hand, conductive temperature ϕ and radial stresses σ_{rr} satisfy Eqs. (24) and (25) at $r = a = 1$ and then they converge towards zero with increasing values of r. It is observed from Figs. 13, 14 and 15 that the thermodynamic temperature T, displacement u and hoop stress $\sigma_{\psi\psi}$ are affected
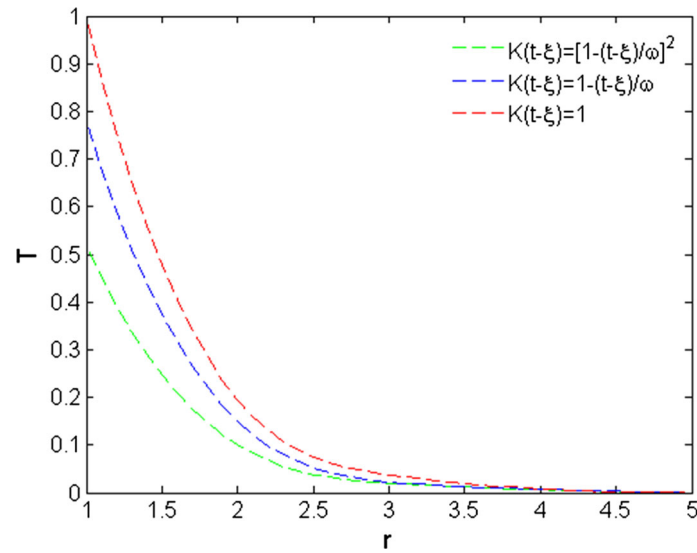
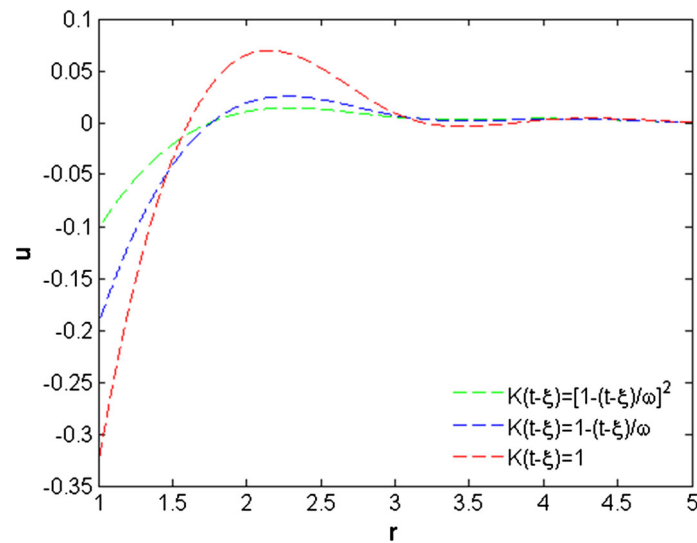**Fig. 13** Distribution of T against r for various $K(t - \xi)$

Fig. 14 Distribution of u against r for various $K(t - \xi)$

by various forms of $K(t - \xi)$. Increment the exponent n from 0 to 1 and 2 in $K(t - \xi) = \left(1 - \frac{(t-\xi)}{\omega}\right)^n$ causes decrement in all fields.

11 Conclusions

The behavior of strain component, conductive temperature, thermodynamic temperature, radial and hoop stresses and displacement in a homogeneous, isotropic, two-temperature generalized thermoelastic medium with a spherical cavity has been investigated in the framework of memory-dependent TPL model by utilizing state–space approach. The theoretical developments and the numerical simulation expose that all the considered parameters, namely time delay, kernel functions, ramp type parameter, and 2 T parameters, have considerable effects on the field variables. As per the above investigation, the following concluding remarkable points can be drawn:

1. According to the results, we can see the presence of the time delay parameter with its various values and the kernel function with its various forms can play a significant role to increase or decrease the wave
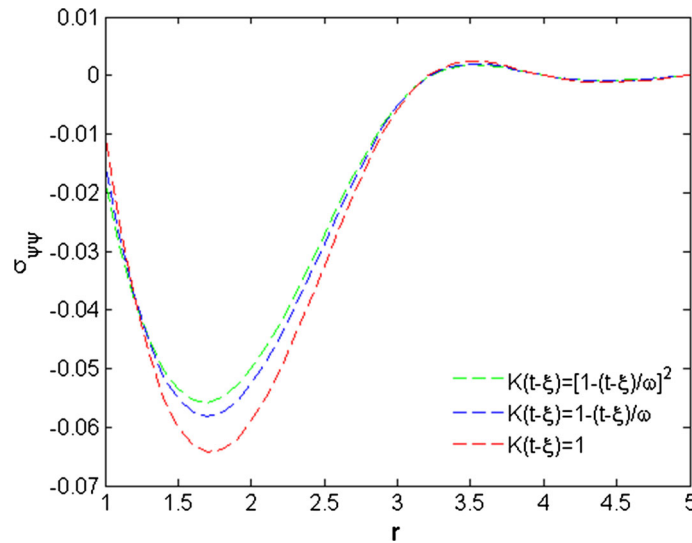
Fig. 15 Distribution of $\sigma_{\phi\phi}$ against r for various $K(t - \xi)$

propagation speed of all the considered distributions. To examine the memory impact, it is progressively beneficial to deal with diverse kernel functions.

2. The generalized thermoelasticity with MDD has more capacity in depicting the memory impact than fractional type.
3. According to this present work, some fundamental theories on two-temperature generalized thermoelasticity with MDD can be easily obtained, such as the LS model, GN-III, II models and DPL heat conduction model.
4. MDD is more flexible than the fractional ones. The time-delay and Kernel function of MDD can be selected randomly according to the need of applications.
5. For various kernel functions significant differences are observed for all the thermophysical quantities.
6. According to the figures of this work, we have noticed that larger value of time delay parameter ω causes the smaller magnitude of all the field quantities, from which we can conclude that ω has become an indicator of its ability to conduct heat in a conducting medium.
7. The conductive temperature ϕ and thermodynamic temperature T plays a significant role for various time delay and different kernel functions.
8. Also, it may be concluded that 2 T thermoelasticity theory with MDD is more realistic than the 1 T thermoelasticity theory with MDD.
9. The vital phenomena are observed in all figures that the variations of the dimensionless thermodynamic field variables in the present generalized thermoelasticity theory with MDD are restricted on the boundary of the cavity and its neighboring region. In the distant region, the variation of the all the thermodynamic variable vanishes identically. This physically implies that the domain of influence for each thermodynamic variable is finite for any kernel function and time delay, and hence present problem with MDD shows the behavior of finite speed propagation of thermal wave [47] and also all results are in agreement with the generalized theory of two temperature thermoelasticity.

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