Archive of Applied Mechanics (2021) 91:1407–1422
https://doi.org/10.1007/s00419-020-01829-6

ORIGINAL

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Semi-analytical solution for the trapped orbits of satellite near the planet in ER3BP

Received: 21 August 2020 / Accepted: 20 October 2020 / Published online: 9 November 2020

Abstract In this paper, we present a new ansatz for solving equations of motion for the *trapped orbits* of the infinitesimal mass (satellite), which is locked in the space trap to be moving near the planet in case of the elliptic restricted problem of three bodies, ER3BP (with Keplerian *elliptic* trajectories of primaries Sun and planet around each other). A new type of the solving procedure is implemented here to obtain the coordinates of the infinitesimal mass (satellite) with its orbit located near the planet. The system of equations of motion was applied for obtaining of the semi-analytic and analytic solutions. It is obtained that two Cartesian coordinates (in a plane of mutual rotation of primaries Sun and planet around each other) depend on the true anomaly and a function which determines the quasi-periodic character of solution, while the third coordinate (perpendicular to the plane of rotation of primaries) is quasi-periodically varying with true anomaly.

Keywords Elliptic restricted three-body problem (ER3BP) · trapped motion · *forced oscillations*

1 Introduction

In the restricted three-body problem (R3BP), the equations of motion describe the dynamics of an infinitesimal mass m under the action of gravitational forces effected by two celestial bodies of giant masses M_{Sun} and m_{planet} ($m_{planet} < M_{Sun}$), which are rotating around their common center of mass on Keplerian trajectories. The small mass m (satellite) is supposed to be moving as first approximation inside of *restricted* region of space near the planet of mass m_{planet} or inside of so-called *Hill sphere* [1] radius:

$$r_H \cong a_p \cdot (1 - e^2) \cdot \left(\frac{m_{\text{planet}}}{3(M_{\text{Sun}} + m_{\text{planet}})} \right)^{\frac{1}{3}}$$

where a_p is semimajor axis of the planet's orbit, e is the eccentricity of its orbit.

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It is worth noting that there is a large number of previous and recent works concerning analytical development with respect to the R3BP equations which should be mentioned accordingly [2–5].

We should especially emphasize the theory of orbits, which was developed in profound work [5] by V. Szebehely for the case of the circular restricted problem of three bodies (CR3BP) (primaries M_{Sun} and m_{planet} are rotating around their common center of mass on *circular* orbits) as well as the case of the elliptic restricted problem of three bodies (ER3BP), where the primaries M_{Sun} and m_{planet} are rotating around their common center of mass on *elliptic* orbits.

$$\rho = \frac{a_p}{1 + e \cdot \cos f}$$

Unlike the CR3BP [6], the position of the primaries is not fixed in the rotating frame as they move along elliptical orbits: Their relative distance ρ is not constant in time where f is the true anomaly (the unit of distances is chosen so that $a_p = 1$).

As for the purpose of the current research, we can formulate it as follows: The main aim is to find a kind of the semi-analytical solution to the system of equations under consideration. Namely, each exact or even semi-analytical solution can clarify the structure, intrinsic code and topology of the variety of possible solutions (from mathematical point of view); here, exact or semi-analytical solution should be treated not only as analytical formulae in quadratures, but a system of ordinary differential equations (each for one appropriate variable) with well-known code for analytical or numerical resolving to be presented in their final form.

2 Mathematical model, equations of motion

According to [6,7], in the ER3BP equations of motion of the infinitesimal mass m (satellite) can be represented in the synodic co-rotating frame of a Cartesian coordinate system $\vec{r} = \{x, y, z\}$ in non-dimensional form (*at given initial conditions*):

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x}, \\ \ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y}, \\ \ddot{z} &= \frac{\partial \Omega}{\partial z}, \end{aligned} \tag{1}$$

$$\Omega = \frac{1}{1 + e \cdot \cos f}\left[\frac{1}{2}\left(x^2 + y^2 - z^2 \cdot e \cdot \cos f\right) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2}\right], \tag{2}$$

where dot indicates (d/df) in (1), Ω is the scalar function, and

$$\begin{aligned} r_1^2 &= (x-\mu)^2 + y^2 + z^2, \\ r_2^2 &= (x-\mu+1)^2 + y^2 + z^2, \end{aligned} \tag{3}$$

where r_i ($i = 1, 2$) are distances of the infinitesimal mass m from the primaries M_{Sun} and m_{planet}, respectively [7].

Now, the unit of mass is chosen in (1) so that the sum of the primary masses is equal to 1. We suppose that $M_{Sun} \cong 1 - \mu$ and $m_{planet} = \mu$, where μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 < \mu \leq 1/2$. The unit of time is chosen so that the gravitational constant is equal to 1 in (2).

We neglect the effect of variable masses of the primaries [8] as well as the effect of their *oblateness* as was considered earlier in [9]. As for the domain where the aforesaid infinitesimal mass m is supposed to be moving, let us consider the Cauchy problem in the whole space. Besides, we should note that the second terms in the left parts of Eq. (1) are associated with the components of the *Coriolis* acceleration. Finally, let us additionally note that the spatial ER3BP when $e > 0$ and $\mu > 0$ is not conservative, and no integrals of motion are known [7].

3 Reduction of the system of equations (1)

Aiming the aforementioned way of constructing the semi-analytical solution, let us present Eq. (1) in a suitable for analysis form as below by appropriately transforming the right parts with regard to partial derivatives with respect to the proper coordinates $\{x, y, z\}$ [10]

$$
\begin{aligned}
\ddot{x} - 2\dot{y} &= \frac{1}{1+e\cdot\cos f}\cdot\left[x - \frac{(1-\mu)(x-\mu)}{\left((x-\mu)^2 + y^2 + z^2\right)^{\frac{3}{2}}} - \frac{\mu(x-\mu+1)}{\left((x-\mu+1)^2 + y^2 + z^2\right)^{\frac{3}{2}}}\right],\\
\ddot{y} + 2\dot{x} &= \frac{1}{1+e\cdot\cos f}\cdot\left[y - \frac{(1-\mu)y}{\left((x-\mu)^2 + y^2 + z^2\right)^{\frac{3}{2}}} - \frac{\mu y}{\left((x-\mu+1)^2 + y^2 + z^2\right)^{\frac{3}{2}}}\right],\\
\ddot{z} &= \frac{1}{1+e\cdot\cos f}\cdot\left[-z\cdot e\cdot\cos f - \frac{(1-\mu)z}{\left((x-\mu)^2 + y^2 + z^2\right)^{\frac{3}{2}}} - \frac{\mu z}{\left((x-\mu+1)^2 + y^2 + z^2\right)^{\frac{3}{2}}}\right],
\end{aligned}
\tag{4}
$$

Let us transform system of equations (4) to the form which would be convenient for further analysis. From first and second of Eq. (4), we obtain ($y \neq 0$):

$$
\begin{aligned}
(\ddot{x} - 2\dot{y})\cdot(1+e\cdot\cos f) - x &= (x-\mu)\cdot\left\{\frac{(\ddot{y} + 2\dot{x})\cdot(1+e\cdot\cos f) - y}{y}\right\}\\
&- \frac{\mu}{\left((x-\mu+1)^2 + y^2 + z^2\right)^{\frac{3}{2}}}
\end{aligned}
\tag{5}
$$

In (5), z^2 is given as the function of $\{x, y\}$, their derivatives $\{\dot{x}, \dot{y}\}$ with respect to the f, accelerations $\{\ddot{x}, \ddot{y}\}$, and true anomaly f.

But from the second and third of Eq. (4), it follows ($\{y, z\} \neq 0$, $\{\dot{x}, \dot{y}, \dot{z}\} \neq 0$):

$$
(1+e\cdot\cos f)\left(\frac{\ddot{z}}{z}+1\right) = \frac{(\ddot{y} + 2\dot{x})\cdot(1+e\cdot\cos f)}{y}, \quad\Rightarrow\quad \frac{\ddot{z}}{z} + 1 = \frac{(\ddot{y} + 2\dot{x})}{y}
\tag{6}
$$

4 Approximated solutions of Eqs. (1)–(3) for the class of *trapped motions*

Let us assume that coordinates $\vec{r} = \{x, y, z\}$ of solutions of system (1) belong to the class of *trapped motions* of the infinitesimal mass m, which is moving near the planet m_{planet} (taking into account the equality $\rho = \frac{a_p}{1+e\cdot\cos f}$):

$$
\frac{|\vec{r}_2|}{|\vec{r}_1|} << 1, \quad |\vec{r}_1| \cong \frac{a_p}{1+e\cdot\cos f} + \delta, \quad |\delta| << a_p
\tag{7}
$$

but the aforementioned infinitesimal mass m is, nevertheless, located on each step of its trajectory at a large distance from the Sun (M_{Sun}) insofar; here $|\vec{r}_2| > R_p$, whereas R_p is the radius of planet m_{planet}.

Thus, if we take into consideration the additional restriction (7) with respect to the components of solution in Eqs. (1)–(3), the aforesaid assumption should simplify the third of equations (4) accordingly (except the obvious case $\{z, \ddot{z}\} = 0$ in our further analysis):

$$
\begin{aligned}
&\ddot{z}\cdot(1+e\cdot\cos f) + z\cdot e\cdot\cos f = -\frac{z}{|\vec{r}_2|^3}\cdot\left\{(1-\mu)\cdot\frac{|\vec{r}_2|^3}{|\vec{r}_1|^3} + \mu\right\}, \quad\Rightarrow\\
&\left\{\left(\frac{|\vec{r}_2|}{|\vec{r}_1|}\right)^3 \to 0\right\} \quad\Rightarrow\quad |\vec{r}_2| \cong \frac{\mu^{\frac{1}{3}}}{\left(-\frac{\ddot{z}}{z}\cdot(1+e\cdot\cos f) - e\cdot\cos f\right)^{\frac{1}{3}}}, \quad\Rightarrow\\
&y \cong \pm\sqrt{\frac{\mu^{\frac{2}{3}}}{\left(\frac{\ddot{z}}{z}\cdot(1+e\cdot\cos f) + e\cdot\cos f\right)^{\frac{2}{3}}} - \left((x-\mu+1)^2 + z^2\right)},
\end{aligned}
\tag{8}
$$

where appropriate restriction should be valid

$$\frac{\mu^{\frac{2}{3}}}{\left(\frac{\ddot{z}}{z}\cdot(1+e\cdot\cos f)\ +\ e\cdot\cos f\right)^{\frac{2}{3}}}\ -\ z^2 \geq\ (x-\mu+1)^2\ ,$$

Thus, we have expressed in (8) the coordinate y *via* coordinates $\{x, z\}$ and second derivative of coordinate $\ddot{z}$ with respect to the true anomaly f (as a first approximation).

Now let us present Eq. (6) in a form of the *Riccati*-type ordinary differential equation [11] for coordinate z, depending on the coordinate y and on the appropriate derivatives of coordinates $\{\dot{x}\ ,\ \ddot{y}\}$ with respect to the true anomaly f

$$\ddot{z}\ +\ \left(1\ -\ \frac{(\ddot{y}\ +\ 2\dot{x})}{y}\right)\cdot z\ =\ 0 \tag{9}$$

So, Equation (9) should determine the proper *quasi-periodic* solution for coordinate z if the solutions for coordinates $\{x, y\}$ are already obtained.

Let us present further the solutions of Eq. (9)

$$\left(1\ -\ \frac{(\ddot{y}\ +\ 2\dot{x})}{y}\right)\ =\ \alpha\,(f)\quad\Rightarrow\quad \ddot{z}\ +\ \alpha\cdot z\ =\ 0 \tag{10}$$

The aforementioned presentation of solutions in a form (10) for coordinate z is obviously useful from practical point of view in celestial mechanics for the reason that such the solutions, e.g., could be presenting the *quasi-periodical* dependence of coordinate z with respect to the true anomaly f (if α is considered to be slowly varying parameter or *circa constant*, as we can see in our analysis).

The second advance of exploring the differential invariant (6) in a form (10) is that we can reduce one of two equations (second or third) of system (4), which was used at derivation of differential invariant (6). Let us choose the third equation for this aim

$$\alpha\ =\ \frac{1}{1+e\cdot\cos f}\cdot\left[e\cdot\cos f + \frac{(1-\mu)}{\left((x-\mu)^2+y^2\ +\ (z)^2\right)^{\frac{3}{2}}}\ +\ \frac{\mu}{\left((x-\mu+1)^2\ +\ y^2\ +\ (z)^2\right)^{\frac{3}{2}}}\right] \tag{11}$$

Meanwhile, equality (11) reveals the *quasi-periodic* type of the solutions for Eq. (10) (if α is considered to be slowly varying parameter or *circa constant*): Indeed, taking into account the additional restriction (7), we can make a reasonable conclusion from (11) that $\alpha\ >0$ in any case (for the motions which can be expected according to the additional assumption (7)).

So, in this case Eq. (10) yields the classical periodic type of solutions for coordinate z as presented below (e.g., if α is considered to be *circa constant*)

$$z\ =\ C_1\cos\left(f\cdot\sqrt{\alpha}\right)\ +\ C_2\sin\left(f\cdot\sqrt{\alpha}\right) \tag{12}$$

where $\{C_1, C_2\} = const.$

We should note also that Eq. (10) reveals the obvious *quasi-periodic* character of dependence of coordinate y on the derivative of coordinate x (with respect to the true anomaly f). Indeed, we obtain from (10) for coordinate y:

$$\ddot{y}\ +\ (\alpha\ -\ 1)\cdot y\ =\ -2\dot{x} \tag{13}$$

which is, in fact, the *equation of forced oscillations* ([10], example 2.36).

For the sake of simplicity, let us consider in our further analysis the *partial* case $\alpha = 1$ in formulae (10)–(13). Then, integrating both the parts of Eq. (13) with respect to the true anomaly f, we could use the result of integrating in the transformation of the left part of first equation of system (4).

Thus, we should obtain in result the *nonlinear* ordinary differential equation of the second order in regard to the coordinate $x(f)$ in case of the given function (12) for the coordinate z, $\alpha = 1$ (whereas the true anomaly f is to be slowly varying independent coordinate). Obviously, such the *nonlinear* ordinary differential equation of second order (Appendix A1) could be solved by means of numerical methods only. Similar simple case was investigated first in [1] (the well-known Clohessy–Wiltshire equations for relative motion when $e \neq 0$) but without obtaining expression for y presented by (8).

Finally, let us note that we should restrict choosing of the obtained solutions for the aforesaid *nonlinear* ODE of second order with regard to the coordinate $x(f)$ by taking into account the additional condition for the function $r_2 \to R_p$ (while the magnitude of this function should be exceeding the minimal distances within Roche-lobe's region [4,5] for the planet with mass $m_{planet} = \mu$ around which infinitesimal mass is currently rotating in its trapped motion).

As for expression for the function $\delta(f)$, we can obtain it from (7) as below

$$\begin{aligned}
&(x-\mu)^2 + y^2 + z^2 \cong \left(\frac{a_p}{1+e\cdot\cos f} + \delta\right)^2, \quad |\delta| << a_p \quad \Rightarrow \\
&\delta \cong \sqrt{(x-\mu)^2 + y^2 + z^2} - \left(\frac{a_p}{1+e\cdot\cos f}\right)
\end{aligned} \tag{14}$$

where expression for y is given in (8) (where expression for $\frac{\ddot z}{z}$ could be expressed from (10)), but expression for z is given in (12), $\alpha = 1$.

5 Final presentation of the solution

Let us present the solution $\vec r = \{x, y, z\}$ for the *trapped motion* (7) of the infinitesimal mass m (satellite), which is moving near the planet m_{planet} in the ER3BP (1)–(4)

- The key *nonlinear* ordinary differential equation of the second order in regard to the coordinate $x(f)$ in case of the given function (12) for the coordinate z ($\alpha = 1$) is obtained below:

Let us we substitute expression (8) for coordinate y, expression (12) for coordinate z, and the integrated expression for $\dot y$ directly $\to$ into the first equation of system (4) to obtain the *nonlinear* ordinary differential equation of the second order with regard to the unknown coordinate $x(f)$ ($\alpha = 1$, $x_0 = const$)

$$\begin{aligned}
&\ddot x + 4(x - x_0) = \frac{1}{1+e\cdot\cos f}\cdot\left[x - \frac{(1-\mu)(x-\mu)}{\left((x-\mu)^2 + y^2 + z^2\right)^{\frac{3}{2}}} - \frac{\mu(x-\mu+1)}{(|\vec r_2|)^3}\right], \\
&\left\{|\vec r_2| \cong \mu^{\frac{1}{3}}\right\}, \\
&\ddot x + 4(x - x_0) = -\frac{(1-\mu)}{1+e\cdot\cos f}\cdot\left[1 + \frac{(x-\mu)}{\left((x-\mu)^2 + y^2 + z^2\right)^{\frac{3}{2}}}\right], \\
&\left\{ y \cong \pm\sqrt{\mu^{\frac{2}{3}} - \left((x-\mu+1)^2 + z^2\right)}, \quad \mu^{\frac{2}{3}} - \left((x-\mu+1)^2 + z^2\right) > 0 \right\} \\
&\Rightarrow \quad \ddot x + 4(x - x_0) = -\frac{(1-\mu)}{1+e\cdot\cos f}\cdot\left[1 + \frac{(x-\mu)}{\left(\mu^{\frac{2}{3}} - 2(x-\mu) - 1\right)^{\frac{3}{2}}}\right],
\end{aligned} \tag{15}$$

- The expression (8) for coordinate y is given via coordinates $\{x, z\}$, true anomaly f, and additional parameter α in (10) ($\alpha = 1$):

$$\begin{aligned}
&y \cong \pm\sqrt{\frac{\mu^{\frac{2}{3}}}{\left(\frac{\ddot z}{z}\cdot(1+e\cdot\cos f) + e\cdot\cos f\right)^{\frac{2}{3}}} - \left((x-\mu+1)^2 + z^2\right)}, \\
&\frac{\mu^{\frac{2}{3}}}{\left(\frac{\ddot z}{z}\cdot(1+e\cdot\cos f) + e\cdot\cos f\right)^{\frac{2}{3}}} - z^2 \geq (x-\mu+1)^2,
\end{aligned}$$

– expression for z is given in (12) ($\{C_1, C_2\} = const, \alpha = 1$):

$$z \;=\; C_1 \cos\; f \;+\; C_2 \sin\; f$$

The aforementioned *nonlinear* ODE of the second order for function $x(f)$ should be solved under the optimizing condition $r_2 \to R_p$, as soon as coordinates $\{x, y, z\}$ are already calculated (while the magnitude of this function r_2 should exceed the minimal distance or Roche limit [4,5] for the planet with mass $m_{planet} = \mu$).

Let us also consider the case $\alpha(f) \neq$ const in Eqs. (10)–(11) (which are proved to be valid for the *trapped motion* (7) of infinitesimal mass m in the ER3BP (1)–(4)).

Using (10), we obtain

$$\ddot{z} \;+\; \alpha(f)\cdot z \;=\; 0\,, \tag{16}$$

where Eq. (16) could be transformed by the change of variables $\frac{\dot{z}}{z}$ to the *Riccati* ODE of the first order [10] (in case $z \neq 0$).

Then, let us simplify the second equation of system (4) by using the left part of (10)

$$\begin{gathered}(\ddot{y} \;+\; 2\,\dot{x})\cdot(1 + e\cdot\cos f) \;-\; y \;=\; -\,\frac{y}{|\vec{r}_2|^3}\cdot\left\{(1-\mu)\cdot\frac{|\vec{r}_2|^3}{|\vec{r}_1|^3} \;+\; \mu\right\}\,, \\ \left\{\left(\frac{|\vec{r}_2|}{|\vec{r}_1|}\right)^3 \to\; 0\right\} \quad\Rightarrow\quad \frac{1}{|\vec{r}_2|^3} \;\cong\; \frac{1}{\mu}\cdot(1 \;+(\alpha \;-\; 1)\cdot(1 + e\cdot\cos f))\end{gathered} \tag{17}$$

where (in case $z \neq 0$)

$$\alpha(f) \;=\; -\,\frac{\ddot{z}}{z}$$

Furthermore, based on (16)–(17), we can write out the key *nonlinear* ordinary differential equation of the second order in regard to the coordinate $x(f)$ in case of the given function (16) for the coordinate z

$$\begin{gathered}\ddot{x} \;-\; 2\,\dot{y} \;+\; (\alpha \;-\; 1)\cdot(x-\mu+1) \;=\; -\,\frac{(1-\mu)}{1+e\cdot\cos f}\cdot\left[\,1 \;+\; \frac{(x-\mu)}{\left((x-\mu)^2 \;+\; y^2 \;+\; z^2\right)^{\frac{3}{2}}}\,\right], \\ \Rightarrow\left\{\; y \;\cong\; \pm\sqrt{\frac{\mu^{\frac{2}{3}}}{(\alpha(f)\cdot(1+e\cdot\cos f) \;-\; e\cdot\cos f)^{\frac{2}{3}}} \;-\; \left((x-\mu+1)^2 \;+\; z^2\right)}\;\right\} \quad\Rightarrow \\ \ddot{x} \;-\; 2\,\dot{y} \;+\; (\alpha \;-\; 1)\cdot(x-\mu+1) \;=\; -\,\frac{(1-\mu)}{1+e\cdot\cos f}\cdot[\,1 \;+\; \frac{(x-\mu)}{\left(\frac{\mu^{\frac{2}{3}}}{(\alpha(f)\cdot(1+e\cdot\cos f)\,-\,e\cdot\cos f)^{\frac{2}{3}}} \;-\; 2(x-\mu) \;-\; 1\right)^{\frac{3}{2}}}\,]\,,\end{gathered} \tag{18}$$

where expression (8) for coordinate y in (18) is given via coordinates $\{x, z\}$, true anomaly f, and the additional parameter α in (10) or (16):

$$\begin{gathered}y \;\cong\; \pm\sqrt{\frac{\mu^{\frac{2}{3}}}{(\alpha(f)\cdot(1+e\cdot\cos f) \;-\; e\cdot\cos f)^{\frac{2}{3}}} \;-\; \left((x-\mu+1)^2 \;+\; z^2\right)}\,, \\ \frac{\mu^{\frac{2}{3}}}{(\,\alpha\cdot(1+e\cdot\cos f) \;-\; e\cdot\cos f)^{\frac{2}{3}}} \;-\; z^2 \;>\; (x-\mu+1)^2\,.\end{gathered}$$

The aforementioned *nonlinear* ODE of the second order for function $x(f)$ should be solved under the optimizing condition $r_2 \to R_p$, as soon as coordinates $\{x, y, z\}$ are already calculated (while the magnitude of this function r_2 should exceed the minimal distances within the Roche-limit [4,5] for the planet with mass m_{planet}).

Table 1 Comparison of the averaged parameters of the moons in Solar system

Masses of the planets (*Solar system*), kg	Ratio $\left(\frac{m_{\text{planet}}\ /\ m_{\text{Earth}}}{M_{\text{Sun}}\ /\ m_{\text{Earth}}}\right)=\mu$	Semimajor axis a_p of the planet, AU	Possible distance $\lvert\vec{r}_2\rvert \cong a_p \cdot \left(\frac{\mu}{3}\right)^{\frac{1}{3}}$ (*between moon-planet*), AU (10^3 km)	Distance of real moon (*between Moon-Planet*), 10^3 km
Mercury, $3.3{\cdot}10^{23}$	$\left(\frac{0.055}{332{,}946}\right)=0.165\cdot10^{-6}$	0.387 AU	0.015 AU (2,208)	
Venus, $4.87{\cdot}10^{24}$	$\left(\frac{0.815}{332{,}946}\right)=2.448\cdot10^{-6}$	0.723 AU	0.068 AU (10,124)	
Earth, $5.97{\cdot}10^{24}$ + Moon, $7.36{\cdot}10^{22}$	$\left(\frac{1.0123}{332{,}946}\right)=3.040\cdot10^{-6}$	1 AU	0.101 AU (15,040)	383.4
Mars, $6.42{\cdot}10^{24}$	$\left(\frac{0.107}{332{,}946}\right)=0.321\cdot10^{-6}$	1.523 AU	0.073 AU (10,900)	1) Phobos 9.38 2) Deimos 23.46
Jupiter, $1.9{\cdot}10^{27}$	$\left(\frac{317.8}{332{,}946}\right)=954.509\cdot10^{-6}$	5.205 AU	3.555 AU (531,781)	1) Ganymede 1,070 2) Callisto 1,883 3) Io 422 4) Europa 671
Saturn, $5.69{\cdot}10^{26}$	$\left(\frac{95.16}{332{,}946}\right)=285.812\cdot10^{-6}$	9.582 AU	4.378 AU (654,964)	1) Titan 1,222 2) Rhea 527 3) Iapetus 3,561 4) Dione 377 5) Tethys 294.6 6) Enceladus 238
Uranus, $8.69{\cdot}10^{25}$	$\left(\frac{14.37}{332{,}946}\right)=43.160\cdot10^{-6}$	19.201 AU	4.673 AU (699,048)	1) Titania 436 2) Oberon 584 3) Ariel 191 4) Umbriel 266.3 5) Miranda 129.4
Neptune, $1.02{\cdot}10^{26}$	$\left(\frac{17.15}{332{,}946}\right)=51.510\cdot10^{-6}$	30.048 AU	7.750 AU (1,159,403)	1) Triton 355 2) Proteus 118 3) Nereid 5,513
Pluto, $1.3{\cdot}10^{22}$	$\left(\frac{0.002}{332{,}946}\right)=0.006\cdot10^{-6}$	39.238 AU	0.490 AU (73,258)	Charon 20

Table 2 Results of numerical calculations for Eqs. (16)–(19) (case of Earth)

f, rad	x	y	r_1	r_2	α	δ
0	−1.01037	0	1.013419	0.014405	1000	0.030135
0.1	−0.98354	0.005301	0.986595	0.014430	995.0125	0.003228
0.2	−1.01040	0.005327	1.013451	0.014455	990.0991	0.029839
0.3	−0.98351	0.005354	0.986562	0.014480	985.3306	0.002543
0.4	−1.01043	0.005394	1.013485	0.014505	980.7715	0.028902
0.5	−0.98348	0.005438	0.986530	0.014530	976.4776	0.001230
0.6	−1.01045	0.005525	1.013503	0.014554	972.4948	0.027339
0.7	−0.98348	0.005623	0.986533	0.014577	968.8588	−0.000630
0.8	-1.01041	0.005810	1.013461	0.014599	965.5951	0.025166
0.9	-0.98357	0.006008	0.986623	0.014620	962.7207	−0.002920
1.0	-1.01025	0.006333	1.013304	0.014639	960.245	0.022406
1.1	−0.98379	0.006652	0.986856	0.014657	958.1721	−0.00549
1.2	−1.00992	0.007111	1.012986	0.014673	956.5022	0.019109
1.3	−0.98420	0.007532	0.987270	0.014687	955.2333	−0.00820
1.4	−1.00941	0.008082	1.012480	0.014700	954.3624	0.015361
1.5	-0.98481	0.008562	0.987882	0.014710	953.8868	-0.010920
1.6	-1.00871	0.009144	1.011787	0.014719	953.8051	0.011291
1.7	−0.98559	0.009632	0.988672	0.014726	954.1167	−0.013520
1.8	−1.00785	0.010189	1.010940	0.014730	954.8229	0.007063
1.9	−0.98650	0.010640	0.989593	0.014733	955.9259	−0.015930
2.0	−1.00690	0.011130	1.010000	0.014733	957.4287	0.002875
2.1	−0.98746	0.011510	0.990568	0.014730	959.3339	−0.018090
2.2	−1.00594	0.011905	1.009051	0.014726	961.6422	−0.001050
2.3	−0.98839	0.012196	0.991503	0.014718	964.3516	−0.019950
2.4	−1.00508	0.012485	1.008189	0.014709	967.4549	−0.004510
2.5	−0.98918	0.012679	0.992299	0.014696	970.9387	−0.021510
2.6	−1.00439	0.012864	1.007510	0.014682	974.781	−0.007270
2.7	−0.98974	0.012962	0.992863	0.014665	978.9504	−0.022750
2.8	−1.00398	0.013047	1.007098	0.014646	983.4055	−0.009180
2.9	−0.99000	0.013053	0.993120	0.014625	988.0943	−0.023660
3.0	−1.00389	0.013043	1.007013	0.014603	992.9557	−0.010110
3.1	−0.98991	0.012954	0.993028	0.014579	997.9213	−0.024250
3.2	−1.00416	0.012848	1.007279	0.014555	1002.918	−0.009990
3.3	−0.98946	0.012654	0.99258	0.014530	1007.871	−0.024490
3.4	−1.00477	0.012441	1.007881	0.014505	1012.708	−0.008830
3.5	−0.98870	0.012125	0.991816	0.014481	1017.361	−0.024360
3.6	−1.00565	0.011790	1.008759	0.014456	1021.772	−0.006720
3.7	−0.98771	0.011336	0.990816	0.014433	1025.889	−0.023810
3.8	−1.00672	0.010869	1.009813	0.014411	1029.673	−0.003820
3.9	−0.98661	0.010270	0.989698	0.014389	1033.094	−0.022800
4.0	−1.00783	0.009679	1.010915	0.014369	1036.132	−0.000320

As for the numerical checking of the proper solutions of Eq. (18), we have tested the cases of Earth, Mars and Venus (see their appropriate parameters at Table 1 in Appendix A1). By the way, for the Earth it means $z = 0$ in (16)–(18) where we can choose function $\alpha(f)$ absolutely arbitrary, but taking into account that it should be the slowly varying function, e.g., as below

$$\alpha(f) \;\cong\; 1000 \;-\; 100 \cdot \left(\frac{1 - \exp(\sin f)}{1 + \exp(\sin f)}\right) \tag{19}$$

Meanwhile, such the same choice of function $\alpha(f)$ (19) can also be applied for calculations in cases of Mars and Venus as well.

We should note that we have used for calculating the data (Tables 2, 3, 4) the Runge–Kutta fourth-order method with step 0.001 at initial values for Eqs. (16) and (18) as follows: (1) $x_0 = -1.0103652222598$ and $(\dot{x})_0 = 0$ (for Earth; we consider $z = 0$); (2) $x_0 = -1, (\dot{x})_0 = 0, z_0 = 0, (\dot{z})_0 = -0.1$ (for Mars); 3) $x_0 = -1, (\dot{x})_0 = 0, z_0 = 0, (\dot{z})_0 = -0.3$ (for Venus). All the results of numerical calculations for Eqs. (16)–(19) (see Tables 2, 3, 4 in Appendix, A2) we schematically imagine at Figs. 1, 2, 3, 4, and 5.

Table 2 continued

f, rad	x	y	r_1	r_2	α	δ
4.1	−0.98552	0.008957	0.988602	0.014351	1038.774	−0.021270
4.2	−1.00886	0.008281	1.011926	0.014333	1041.015	0.003522
4.3	-0.98461	0.007506	0.987672	0.014318	1042.852	−0.019190
4.4	−1.00965	0.006839	1.012714	0.014303	1044.287	0.007462
4.5	−0.98397	0.006161	0.987031	0.014291	1045.324	−0.016570
4.6	−1.01012	0.005681	1.013175	0.014280	1045.963	0.011264
4.7	−0.98371	0.005347	0.986765	0.014271	1046.209	−0.013450
4.8	−1.01019	0.005278	1.013248	0.014263	1046.061	0.014733
4.9	−0.98385	0.005474	0.986907	0.014258	1045.519	−0.009930
5.0	−1.00987	0.005847	1.012927	0.014254	1044.581	0.017726
5.1	−0.98437	0.006440	0.987436	0.014253	1043.245	−0.006180
5.2	−1.00919	0.007043	1.012259	0.014254	1041.508	0.020161
5.3	−0.98520	0.007771	0.988272	0.014256	1039.367	−0.002390
5.4	−1.00826	0.008407	1.011339	0.014262	1036.824	0.022013
5.5	−0.98622	0.009101	0.989299	0.014269	1033.883	0.001203
5.6	−1.00721	0.009663	1.010294	0.014279	1030.555	0.023307
5.7	−0.98728	0.010244	0.990377	0.014292	1026.859	0.004369
5.8	−1.00617	0.010688	1.009266	0.014306	1022.821	0.024096
5.9	−0.98826	0.011129	0.991365	0.014323	1018.479	0.006888
6.0	−1.00528	0.011443	1.008386	0.014342	1013.881	0.024446
6.1	−0.98903	0.011746	0.992142	0.014363	1009.083	0.008583
6.2	−1.00465	0.011935	1.007760	0.014386	1004.152	0.024419
6.3	−0.98950	0.012113	0.992621	0.014409	999.1593	0.009334
6.4	−1.00434	0.012189	1.007456	0.014434	994.1791	0.024060
6.5	−0.98964	0.012255	0.992755	0.014459	989.2853	0.009086
6.6	−1.00438	0.012224	1.007497	0.014484	984.5477	0.023394
6.7	−0.98943	0.012185	0.992546	0.014509	980.0295	0.007853
6.8	−1.00475	0.012048	1.007863	0.014534	975.7850	0.022427
6.9	−0.98892	0.011907	0.992032	0.014558	971.8583	0.005710

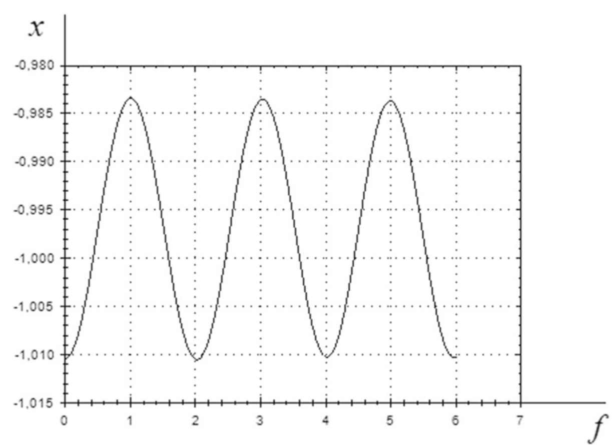

Fig. 1 Results of numerical calculations of the coordinate x by Eq. (18) for Earth

6 Discussion

As we can see from the derivation above, equations of motion (4) even for the case of *trapped motion* $\vec{r} = \{x, y, z\}$ (in the sense of additional assumption (7) for the infinitesimal mass m (satellite), which is moving near the planet m_{planet}) are proved to be very hard to solve analytically.

Nevertheless, at first step we have succeeded in obtaining the elegant expression for the differential invariant (6), which interconnects the second and third equations of system (4). The aforesaid invariant (6) yields Equation (9) of a *Riccati*-type for the coordinate z which should determine the proper solution (for coordinate z) if the solutions for coordinates $\{x, y\}$ are already obtained. Then, we suggest a kind of reduction in a form (10) for Eq. (9) which let us obtain the classical analytical (*quasi-periodic*) solution (12) for coordinate z (if additional

Table 3 Results of numerical calculations for Eqs. (16)–(19) (case of Mars)

f, rad	x	y	z	r_1	r_2	α	δ
0	−1	0	0	1.000343	0.006645	1000	0.086266
0.1	−0.99936	0.006649	0.000053	0.999701	0.006657	995.0125	0.085231
0.2	−1	0.006663	−8.1E−05	1.000344	0.006671	990.0991	0.084699
0.3	−0.99936	0.006679	0.000085	0.9997	0.006687	985.3306	0.082102
0.4	−1	0.006696	−6.5E−05	1.000345	0.006704	980.7715	0.080026
0.5	−0.99936	0.006713	0.000023	0.9997	0.006722	976.4776	0.075906
0.6	−1	0.006732	0.000041	1.000346	0.006741	972.4948	0.072342
0.7	−0.99936	0.00675	−0.00012	0.9997	0.006761	968.8588	0.066773
0.8	−1	0.006767	0.000225	1.000345	0.006783	965.5951	0.06181
0.9	−0.99936	0.006783	−0.00034	0.999703	0.006804	962.7207	0.054909
1	−1	0.006795	0.000471	1.000341	0.006827	960.245	0.048675
1.1	−0.99937	0.006805	−0.00061	0.99971	0.006849	958.1721	0.040604
1.2	−0.99999	0.006812	0.00076	1.000334	0.006872	956.5022	0.033274
1.3	−0.99938	0.006814	−0.00092	0.99972	0.006895	955.2333	0.024249
1.4	−0.99998	0.006813	0.001072	1.000322	0.006918	954.3624	0.016048
1.5	−0.99939	0.006807	−0.00123	0.999736	0.00694	953.8868	0.006341
1.6	−0.99996	0.006799	0.001385	1.000306	0.006962	953.8051	−0.00245
1.7	−0.99941	0.006787	−0.00154	0.999755	0.006983	954.1167	−0.01251
1.8	−0.99994	0.006773	0.00168	1.000285	0.007004	954.8229	−0.02154
1.9	−0.99943	0.006757	−0.00182	0.999778	0.007022	955.9259	−0.03156
2	−0.99992	0.00674	0.001941	1.000263	0.00704	957.4287	−0.04045
2.1	−0.99946	0.006723	−0.00205	0.999801	0.007056	959.3339	−0.05002
2.2	−0.99989	0.006706	0.002155	1.00024	0.00707	961.6422	−0.05832
2.3	−0.99948	0.00669	−0.00224	0.999824	0.007081	964.3516	−0.06699
2.4	−0.99987	0.006675	0.002316	1.000219	0.007091	967.4549	−0.07426
2.5	−0.9995	0.006662	−0.00238	0.999843	0.007097	970.9387	−0.0816
2.6	−0.99986	0.006651	0.00242	1.000202	0.007101	974.781	−0.0874
2.7	−0.99951	0.006641	−0.00245	0.999857	0.007103	978.9504	−0.09302
2.8	−0.99985	0.006634	0.002467	1.000192	0.007101	983.4055	−0.09698
2.9	−0.99952	0.006629	−0.00247	0.999863	0.007097	988.0943	−0.10057
3	−0.99984	0.006627	0.002453	1.00019	0.00709	992.9557	−0.10242
3.1	−0.99952	0.006628	−0.00242	0.999861	0.007081	997.9213	−0.10379
3.2	−0.99985	0.006632	0.002376	1.000196	0.007069	1002.918	−0.10336
3.3	−0.9995	0.006639	−0.00231	0.99985	0.007054	1007.871	−0.10247
3.4	−0.99987	0.00665	0.002228	1.00021	0.007038	1012.708	−0.09975
3.5	−0.99949	0.006665	−0.00212	0.999832	0.00702	1017.361	−0.09669
3.6	−0.99989	0.006683	0.001999	1.000231	0.007	1021.772	−0.09182
3.7	−0.99946	0.006703	−0.00185	0.999808	0.006979	1025.889	−0.08682
3.8	−0.99991	0.006725	0.001684	1.000257	0.006957	1029.673	−0.08007
3.9	−0.99944	0.006748	−0.00149	0.99978	0.006934	1033.094	−0.07346
4	−0.99994	0.006769	0.001282	1.000283	0.006911	1036.132	−0.06518
4.1	−0.99941	0.006786	−0.00105	0.999753	0.006887	1038.774	−0.05737
4.2	−0.99996	0.006798	0.000801	1.000307	0.006863	1041.015	−0.048
4.3	−0.99939	0.006803	−0.00054	0.99973	0.006839	1042.852	−0.03942
4.4	−0.99998	0.006799	0.000261	1.000326	0.006815	1044.287	−0.02942
4.5	−0.99937	0.006785	0.000023	0.999713	0.006792	1045.324	−0.0205
4.6	−0.99999	0.006759	−0.00031	1.000337	0.006769	1045.963	−0.01032
4.7	−0.99936	0.006722	0.000598	0.999705	0.006747	1046.209	−0.00146
4.8	−1	0.006674	−0.00088	1.000339	0.006727	1046.061	0.008497
4.9	−0.99936	0.006616	0.001153	0.999707	0.006707	1045.519	0.016937
5	−0.99999	0.00655	−0.00141	1.000331	0.006689	1044.581	0.026303
5.1	−0.99938	0.006478	0.001657	0.999718	0.006673	1043.245	0.034029
5.2	−0.99997	0.006404	−0.00188	1.000315	0.006659	1041.508	0.042498
5.3	−0.99939	0.006328	0.002087	0.999737	0.006646	1039.367	0.049267
5.4	−0.99995	0.006255	−0.00227	1.000293	0.006636	1036.824	0.056595
5.5	−0.99942	0.006186	0.002429	0.999761	0.006628	1033.883	0.062215
5.6	−0.99993	0.006124	−0.00257	1.000268	0.006622	1030.555	0.068218
5.7	−0.99944	0.006069	0.002682	0.999786	0.006618	1026.859	0.07254
5.8	−0.9999	0.006024	−0.00278	1.000244	0.006617	1022.821	0.077086
5.9	−0.99947	0.005988	0.002855	0.999809	0.006619	1018.479	0.080001
6	−0.99988	0.005961	−0.00292	1.000223	0.006622	1013.881	0.083007
6.1	−0.99949	0.005943	0.002964	0.999828	0.006628	1009.083	0.084435
6.2	−0.99987	0.005933	−0.003	1.000208	0.006637	1004.152	0.08586

Table 4 Results of numerical calculations for Eqs. (16)–(19) (case of Venus)

f, rad	x	y	z	r_1	r_2	α	δ
0	−1	0.013221	0	1.002535	0.013446	1000	0.009487
0.1	−0.9951	0.013243	0.000159	0.997637	0.013469	995.0125	0.004554
0.2	−1.00001	0.013264	−0.00024	1.002541	0.013491	990.0991	0.009355
0.3	−0.9951	0.013286	0.000255	0.997631	0.013514	985.3306	0.004274
0.4	−1.00001	0.013309	−0.0002	1.002548	0.013536	980.7715	0.008954
0.5	−0.99509	0.013331	0.000068	0.997627	0.013557	976.4776	0.003733
0.6	−1.00002	0.013351	0.000122	1.002552	0.013577	972.4948	0.008296
0.7	−0.99509	0.013367	−0.00037	0.997631	0.013596	968.8588	0.002956
0.8	−1.00001	0.013373	0.000674	1.002544	0.013613	965.5951	0.007397
0.9	−0.99511	0.01337	−0.00102	0.997651	0.013629	962.7207	0.001983
1	−0.99998	0.013352	0.001413	1.002515	0.013644	960.245	0.006283
1.1	−0.99516	0.013319	−0.00184	0.997698	0.013656	958.1721	0.000863
1.2	−0.99992	0.013266	0.002281	1.002456	0.013667	956.5022	0.004987
1.3	−0.99524	0.013197	−0.00275	0.997777	0.013676	955.2333	−0.00035
1.4	−0.99982	0.013104	0.003217	1.002363	0.013683	954.3624	0.003551
1.5	−0.99535	0.012997	−0.00369	0.997892	0.013689	953.8868	−0.00161
1.6	−0.9997	0.012869	0.004156	1.002234	0.013692	953.8051	0.00203
1.7	−0.9955	0.012731	−0.00461	0.998039	0.013694	954.1167	−0.00286
1.8	−0.99954	0.012577	0.005041	1.002077	0.013694	954.8229	0.000484
1.9	−0.99567	0.01242	−0.00545	0.998209	0.013692	955.9259	−0.00406
2	−0.99936	0.012254	0.005822	1.001903	0.013687	957.4287	−0.00102
2.1	−0.99585	0.012095	−0.00616	0.998387	0.013681	959.3339	−0.00516
2.2	−0.99919	0.011936	0.006465	1.001727	0.013673	961.6422	−0.00241
2.3	−0.99602	0.011792	−0.00673	0.998558	0.013663	964.3516	−0.00613
2.4	−0.99903	0.011657	0.006948	1.001567	0.01365	967.4549	−0.00362
2.5	−0.99616	0.011543	−0.00713	0.998703	0.013636	970.9387	−0.00694
2.6	−0.9989	0.011444	0.007261	1.001441	0.01362	974.781	−0.00459
2.7	−0.99627	0.011371	−0.00735	0.998805	0.013602	978.9504	−0.00756
2.8	−0.99883	0.011319	0.0074	1.001364	0.013583	983.4055	−0.00528
2.9	−0.99631	0.011296	−0.0074	0.998851	0.013562	988.0943	−0.00799
3	−0.99881	0.011296	0.00736	1.001348	0.01354	992.9557	−0.00563
3.1	−0.9963	0.011328	−0.00727	0.998833	0.013518	997.9213	−0.00821
3.2	−0.99886	0.011385	0.007128	1.001397	0.013496	1002.918	−0.00564
3.3	−0.99621	0.011474	−0.00693	0.998751	0.013473	1007.871	−0.00821
3.4	−0.99897	0.011587	0.006683	1.001508	0.013451	1012.708	−0.00531
3.5	−0.99607	0.011729	−0.00637	0.998611	0.01343	1017.361	−0.00799
3.6	−0.99913	0.011888	0.005998	1.001671	0.013409	1021.772	−0.00465
3.7	−0.99589	0.012067	−0.00556	0.998427	0.01339	1025.889	−0.00755
3.8	−0.99933	0.012252	0.005053	1.001865	0.013371	1029.673	−0.0037
3.9	−0.99569	0.012441	−0.00448	0.998221	0.013354	1033.094	−0.00689
4	−0.99953	0.012618	0.003846	1.002069	0.013339	1036.132	−0.00253
4.1	−0.99548	0.012781	−0.00315	0.998018	0.013325	1038.774	−0.00602
4.2	−0.99972	0.012914	0.002403	1.002255	0.013313	1041.015	−0.00119
4.3	−0.99531	0.013013	−0.00161	0.997843	0.013303	1042.852	−0.00497
4.4	−0.99987	0.013067	0.000782	1.0024	0.013294	1044.287	0.000244
4.5	−0.99519	0.013074	0.00007	0.99772	0.013286	1045.324	−0.00376
4.6	−0.99995	0.013029	−0.00093	1.002485	0.01328	1045.963	0.001699
4.7	−0.99513	0.01293	0.001793	0.997664	0.013276	1046.209	−0.00242
4.8	−0.99997	0.012783	−0.00264	1.002498	0.013274	1046.061	0.00311
4.9	−0.99515	0.012587	0.003458	0.997683	0.013273	1045.519	−0.00101
5	−0.99991	0.012357	−0.00424	1.002439	0.013274	1044.581	0.004421
5.1	−0.99524	0.012091	0.004972	0.997772	0.013277	1043.245	0.000411
5.2	−0.99978	0.011812	−0.00565	1.002317	0.013281	1041.508	0.005586
5.3	−0.99538	0.011518	0.006261	0.997919	0.013288	1039.367	0.001784
5.4	−0.99962	0.011233	−0.00681	1.00215	0.013296	1036.824	0.006573
5.5	−0.99557	0.010956	0.007286	0.998102	0.013307	1033.883	0.003038
5.6	−0.99943	0.010706	−0.0077	1.00196	0.013319	1030.555	0.00736
5.7	−0.99576	0.01048	0.008045	0.998296	0.013333	1026.859	0.004105
5.8	−0.99924	0.010292	−0.00833	1.001774	0.013349	1022.821	0.007935
5.9	−0.99594	0.010134	0.008565	0.998474	0.013366	1018.479	0.004925
6	−0.99908	0.010015	−0.00875	1.001615	0.013386	1013.881	0.008291
6.1	−0.99608	0.009924	0.008892	0.998616	0.013406	1009.083	0.005452
6.2	−0.99897	0.009864	−0.009	1.001502	0.013428	1004.152	0.00843

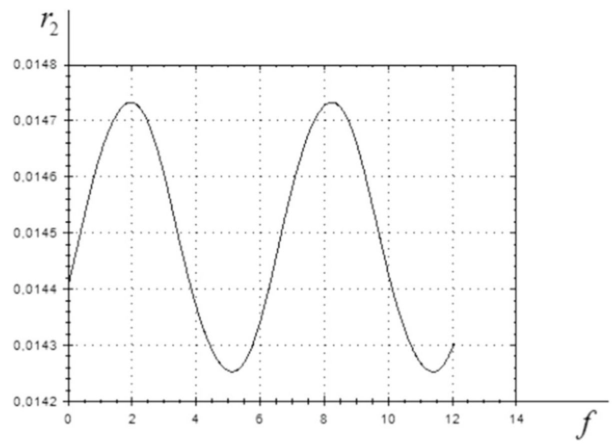

Fig. 2 Results of numerical calculations of r_2 by using (17) and (19) for Earth

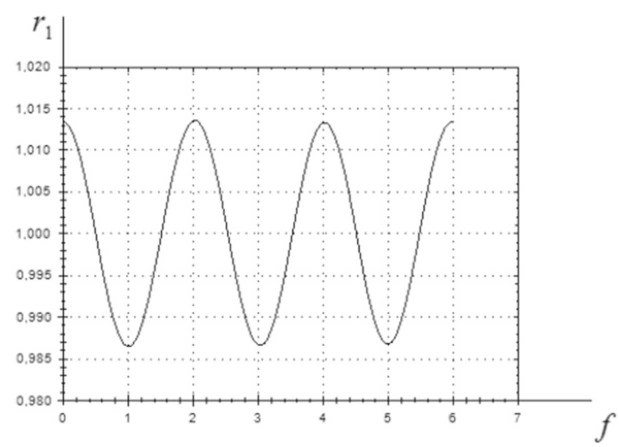

Fig. 3 Results of numerical calculations of r_1 by Eqs. (17)–(19) for Earth

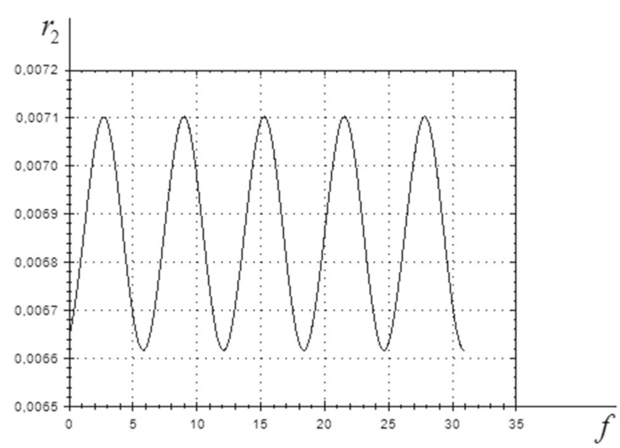

Fig. 4 Results of numerical calculations of r_2 by Eqs. (16)–(19) for Mars

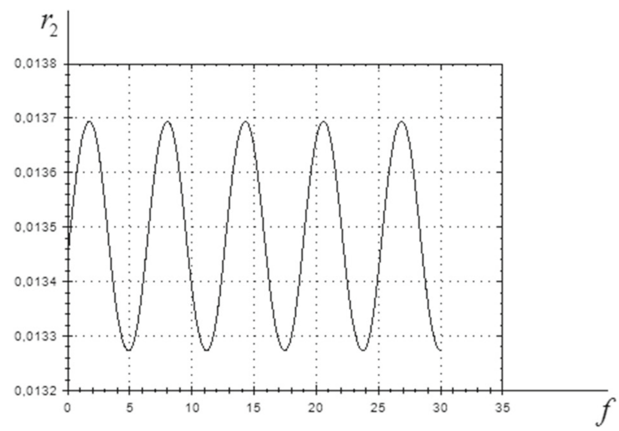

Fig. 5 Results of numerical calculations of r_2 by Eqs. (16)–(19) for Venus

function α is considered to be slowly varying parameter or *circa constant*), as well as it let us obtain in Equation (8) the *approximated* solution for coordinate y, which was derived under assumption (7). Hereafter, we have considered the case $\alpha = 1$ for the sake of simplicity. The key *nonlinear* ODE in regard to the coordinate $x(f)$ in case of the given function (12) for the coordinate z, $\alpha = 1$ (whereas true anomaly f is to be slowly varying independent coordinate) is obtained to be presented in (15).

Meanwhile, the key *nonlinear* ODE in regard to coordinate $x(f)$ in case of the given function for coordinate z via (10) is obtained to be presented in a form (18) for $\alpha \neq const$. This result outlines the novelty of the paper which should be discussed accordingly. Indeed, results (16)–(18) and (20) in Appendix A1 are obtained as generalization of the simple case $\alpha = 1$ which was investigated first in work [1] by F.Cabral (the well-known Clohessy–Wiltshire equations for relative motion when $e \neq 0$) but without obtaining expression for y presented by (8) here.

According to our knowledge, semi-analytical solution, obtained by F.Cabral in [1] for simple case $\alpha = 1$, is most close to our general case of solutions, presented by formulae (16)–(18) for $\alpha \neq const$; moreover, such nontrivial type of solution has not been suggested until the current research and the only way to get some kind of information about the intrinsic properties and behavior of even the particular dynamical system (1) or (4) was to calculate approximated solutions by using various numerical methods. So, any new theoretical method or semi-analytical approach for even the particular solving of such the system of equations would be useful on the level of practical applications.

We also determine the *analytical* expression which interrelates the coordinate x, depending on the given coordinate z and true anomaly f, relative to the additional function $\delta(f)$ in (14), which determines deviation of distance of infinitesimal mass m (to M_{Sun}) from variable relative distance $\rho = \frac{a_p}{1+e\cdot\cos f}$ between primaries M_{Sun} and m_{planet}.

Ending discussion, let us note also that natural restriction $r_2 \to R_p$ should be valid for the *trapped motion* $\vec{r} = \{x, y, z\}$ with respect to the aforementioned function r_2 in case of the given function for coordinate z in (10) (R_p is the radius of planet m_{planet}). It means that we should choose among the solutions of *nonlinear* ODE of second order in regard to the coordinate $x(f)$ only the optimized solutions, for which the condition $r_2 \to R_p$ is valid accordingly (while its magnitude should exceed the level of minimal distances outside the Roche-lobe's region [4,5] for the planet with mass m_{planet}).

7 Conclusion

In this paper, we present a new ansatz for solving equations of motion for the *trapped orbits* of the infinitesimal mass m (satellite), which is locked in the space trap to be moving near the planet m_{planet} in case of the elliptic restricted problem of three bodies, ER3BP (with Keplerian *elliptic* trajectories of the primaries M_{Sun} and m_{planet} around each other): a new type of the solving procedure is implemented here to obtain the coordinates $\vec{r} = \{x, y, z\}$ of the infinitesimal mass m with its orbit located near the planet m_{planet}. Meanwhile, the system of equations of motion has been successfully explored with respect to the existence of analytical or semi-analytical (approximated) way for presentation of the solution.

We obtain as follows: (1) Equation for coordinate x is given via coordinate y, true anomaly f, and the additional function α, which determines the *quasi-periodic* (a *Riccati*-type) character of solution for coordinate z, (2) expression for coordinate y is given via coordinate x, true anomaly f, and the aforementioned parameter α, (3) coordinate z is to be *quasi-periodically* varying with respect to the true anomaly f.

We have pointed out the optimizing procedure for the *nonlinear* ordinary differential equation of second order in regard to the aforementioned coordinate $x(f)$ in case of the given function for coordinate z (which is valid only for the optimized solutions $r_2 \to R_p$, while the magnitude of function r_2 should exceed the level of minimal distances outside the Roche-limit for the planet).

The suggested approach can be used in future researches for optimizing the maneuvers of spacecraft which is moving near the planet $m_{planet} = \mu$ in case of the elliptic restricted problem of three bodies (ER3BP).

So, in this case we also could suggest a scheme for r_2-optimizing for the maneuvers of spacecraft which is moving near the planet m_{planet}.

Also, some remarkable articles should be cited, which concern the problem under consideration, [12–24] and [25–34]; the results of the most remarkable and comprehensive works (in the sense of algorithms of obtaining the families of solutions in ER3BP) should be outlined and commented additionally hereto. In [17] Dr. J.Singh and A.Umar describe the motion around the collinear libration points in the elliptic R3BP with a bigger triaxial primary. In work [19] E.I.Abouelmagd and M.A.Sharaf classified trajectories of test particle around the libration points in the restricted three-body problem with the effect of radiation and oblateness. In [21], Kushvah, et al. explored the stability (as authors say, of nonlinear character) in the generalized photogravitational restricted three body problem with additional influence of Poynting–Robertson drag. In [28], Wiegert et al. investigated the problem of stability of quasi-satellites in the outer solar system. In the profound work of Wiesel [29], a lot of theoretical and numerical findings regarding stable orbits around of the martian moons was established thoroughly (he, e.g., found that stable retro-grade orbits exist about both moons, staying in the moon vicinity for at least 25 days, and quite probably longer).

Acknowledgements Sergey Ershkov appreciates advice of Dr. V.V. Sidorenko during process of meaningful navigation through special literature on celestial mechanics regarding the subject of research (especially, results reported in [35]). Authors are thankful to unknown esteemed Reviewers with respect to their valuable efforts and advices which have improved structure of the article significantly.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Remark regarding contributions of authors as below In this research, Dr. Sergey Ershkov is responsible for the general ansatz and the solving procedure, simple algebra manipulations, calculations, results of the article in Sects. 1–5, Appendix A1 and also is responsible for the search of approximated solutions. Dr. Alla Rachinskaya is responsible for approximated solving the *nonlinear* ordinary differential equation of second order (15) and (18) by means of advanced numerical methods as well as is responsible for numerical data of calculations and graphical plots of numerical solutions. Both authors agreed with results and conclusions of each other in Sects. 1–7.

Appendix, A1 (estimation of possible orbits according to (15))

Let us we estimate possible orbits for moons in Solar system [15] which are in agreement with condition reported in (15), $|\vec{r}_2| \;\cong\; \mu^{\frac{1}{3}}$:

Data and results, shown in Table 1, can be compared with previous research, ref. [15] (see Table 1).

As we can see from Table 1, most realistic data (for the problem under consideration) appears to be associated with cases of Mercury or Venus: these planets might have had the moons, which were rotating on circular orbits around their planets. But numerical solving procedure for Eq. (15) reveals that the ER3BP (in case of Mercury or Venus) allows the existing of the solution on only the limited range of true anomaly f. It means that numerical modeling by means of the ER3BP forbids or exclude the existing of the free-gravitating moon near the Mercury (or Venus), taking into account interactions between “Sun-Planet”-system and the aforementioned moon.

As for the case of Venus, we know that during its closest approach to Earth, mutual distance is ~38,000$\cdot$ 10^3 km (which equals to 0.254 AU); so, we obtain from the appropriate data (see Table 1):

$$0.254\mathrm{AU} - |\boldsymbol{r}_{2,\,Venus}| - |\boldsymbol{r}_{2,\,Earth}| \cong 0.085\mathrm{AU}$$

but if we take into consideration the mass of possible moon of Venus, the aforementioned difference would tend to zero. So, such (possible) moon of Venus was to interact to the Hill sphere of Earth at their closest approaches to each other, that's why its motion was obviously not to be stable. Meanwhile, we could estimate aforesaid minimal mass of possible moon of Venus, for which the (closest) relative distance 0.085 AU would disappear at all.

Let us also we estimate parameter $\alpha(f)$ in expression (11), taking into account the approximation (7):

$$\begin{aligned}
\alpha \; = \; & \frac{1}{1 + e \cdot \cos f} \cdot \Bigg[e \cdot \cos f \\
& + \frac{(1-\mu)}{\left((x-\mu)^2 + y^2 + (z)^2\right)^{\frac{3}{2}}} + \frac{\mu}{\left((x-\mu+1)^2 + y^2 + (z)^2\right)^{\frac{3}{2}}} \Bigg] \; \Rightarrow \\
\alpha \; = \; & \frac{1}{1 + e \cdot \cos f} \cdot \Bigg[e \cdot \cos f \\
& + \frac{1}{\left((x-\mu+1)^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \left\{ (1-\mu) \cdot \left(\frac{\left((x-\mu+1)^2 + y^2 + z^2\right)^{\frac{1}{2}}}{\left((x-\mu)^2 + y^2 + z^2\right)^{\frac{1}{2}}} \right)^3 + \mu \right\} \Bigg] \; \Rightarrow \\
\alpha \; \cong \; & \frac{1}{1 + e \cdot \cos f} \cdot \left[e \cdot \cos f \; + \; \frac{1}{r_2^3} \cdot \{\mu\} \right]
\end{aligned} \tag{20}$$

Such the result (20) for estimating the parameter $\alpha(f)$ in expression (11) can be compared with the result of F.Cabral for the simple case $\alpha = 1$ which was investigated first in work [1].

Obviously, the case of function α to be the slowly varying parameter or *circa constant* corresponds to the condition $r_2 \cong$ const $<< r_1$ (where $(r_2)^3 \cong \mu$, but $r_1 \cong 1$). For example, if even we choose $r_2 = 0.05$, $\mu = 0.001$, $e = 0.015$, estimations according (20) should yield as follows:

$$\alpha \; \cong \; \frac{1}{1 + (0.015) \cdot \cos f} \cdot [(0.015) \cdot \cos f \; + \; 8] \quad \Rightarrow \quad \alpha_{\min} = 7.867\,, \quad \alpha_{\max} = 8.137 \quad (8 \, \pm \, 1.7\%)$$

Appendix, A2 (results of numerical calculations for Eqs. (16)–(19))

Let us present all the results of numerical calculations for Eqs. (16)–(19) below:

We should note that we have used for calculating the data (Tables 2, 3, 4) the Runge–Kutta fourth-order method with step 0.001 at initial values for Eqs. (16) and (18) as follows: (1) $x_0 = -1.0103652222598$ and $(\dot{x})_0 = 0$ (for Earth; we consider $z = 0$); (2) $x_0 = -1$, $(\dot{x})_0 = 0$, $z_0 = 0$, $(\dot{z})_0 = -0.1$ (for Mars); (3) $x_0 = -1$, $(\dot{x})_0 = 0$, $z_0 = 0$, $(\dot{z})_0 = -0.3$ (for Venus).

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