Arch Appl Mech (2019) 89:545–559
https://doi.org/10.1007/s00419-018-1488-1

ORIGINAL

Michael Grinfeld · Pavel Grinfeld

Finite kinetics effects in the problems of oscillations of two-phase liquid heterogeneous systems

To 80th birthday of Konstantin Lurie with admiration

Received: 29 January 2018 / Accepted: 5 November 2018 / Published online: 10 December 2018

Abstract For centuries, statics and dynamics of two-phase heterogeneous systems were and remain to be in the focus of multiple academic and engineering disciplines. Phase transformations phenomena include both strong reversible and irreversible effects. Many of the irreversible effects, such as friction, viscosity, heat conduction, etc., are the same as in the single-phase systems. In addition to those, the two-phase systems are known for one more irreversible effect. It is the effect entailed by finite rate of kinetics of phase transformation. Below, we explore this irreversible phenomenon for two two-phase heterogeneous systems: (i) the layered system of incompressible phases in the external gravity field and (ii) the two-phase, two-layered self-gravitating planet.

Keywords Phase transformations · Kinetics · Heterogeneous systems · Self-gravitating bodies

1 Introduction

We consider heterogeneous one-component liquid systems, undergoing phase transformations. The heterogeneity is understood in the sense of Gibbs [1]. In other words, we deal with systems in which macroscopic phases are separated by macroscopic boundaries; the boundaries are modeled as mathematical surfaces of vanishing thickness. The interfaces under consideration can be either contact or phase. The contact interfaces separate chemically different non-reacting liquid substances; these interfaces "propagate with particles" not permitting particles to move from one side of the interface to another. In other words, our contact interfaces are the classical interfaces, separating immiscible liquids. In the case of phase interfaces, the particles of one phase can penetrate through the interface to join the other phase: this is the *kinematical* essence of the process of phase transformation. This, kinematic, definition of phase transformation interfaces is not sufficient to model phase transformation. It is equally important to amend this definition with the relevant model of the substance. For the sake of simplicity and transparency of the final relationships, we assume that the phases are both incompressible and "cold." The coldness is typically treated by physicists as the low absolute temperature approximation. For us, in this paper, the interpretation of coldness is somewhat different: it means that we just ignore all the thermal aspects of the phenomena. Whenever possible, we consider the simplest models aiming the qualitative rather than quantitative results and conclusions.

Although both phases are assumed incompressible, the densities of the 2 phases are different. Also, different are the internal energy densities and chemical potentials of the phases. The incompressibility of the phases

M. Grinfeld (✉)
The ARL, Adelphi, USA
E-mail: michael.greenfield4.civ@mail.mil

P. Grinfeld
Drexel University, Philadelphia, USA

causes some secondary technical inconvenience (for instance, introduction of functional indefinite multipliers) as compared with the more general compressible case. These inconveniences, however, are fully compensated by transparency of final results: in the compressible case, analogous results would be practically intractable by means of explicit analytical techniques.

When considering each of the systems, we first remind the already known facts relating to the reversible case of instantaneous kinetics of phase transformation, and, then, discuss the changes caused by the assumption of finite kinetics.

2 The main formulas for phase transformation waves in incompressible liquid systems with instantaneous kinetics

The central result, around which thousands of papers in different physical, engineering, and mathematical disciplines revolve, concerns the surface waves atop a pressure-free heavy, incompressible, non-viscous liquid half-space placed in uniform field of gravity field with the constant free-fall acceleration g. The key result here connects g with the time frequency ω and wave number k of the propagating sinusoidal surface wave

$$\omega^2 = gk \tag{1}$$

(for the relevant history see, for instance, the classical treatise of Lamb [2]).

Consider the simplest generalization of this classical result relating to self-oscillations of a layered system consisting of two non-viscous incompressible heavy liquids (Fig. 1). The case, when the two liquid are chemically different non-reacting substances without intermixing, was analyzed in many publications in the last two centuries. The simplest results, related to infinitesimal waves in such a system, and the historical information can be found again in Lamb [2].

In the absence of surface tension the dispersion equation, connecting the frequency ω_c and horizontal wave number k, for infinitesimal waves reads

$$\frac{\omega_c^4}{g^2k^2}\left(\tanh kh_o + \gamma \tanh^{-1} kh_i\right) - \frac{\omega_c^2}{gk}\gamma\left(1 + \tanh kh_o \tanh^{-1} kh_i\right) + (\gamma - 1)\tanh kh_o = 0 \tag{2}$$

where, h_o and ρ_0 are the thickness and density of the upper layer, h_i and ρ_2 are the thickness and density of the lower layer; $\gamma \equiv \rho_i/\rho_o$.

For the analysis of the short wavelengths, we ought to assume the inequalities $kh_o >> 1$ and $kh_i >> 1$. Then, Eq. (2) has two simple roots

$$\text{(a)}\frac{\omega_c^2}{gk} = 1\ , \quad \text{(b)}\frac{\omega_c^2}{gk} = \frac{\gamma - 1}{\gamma + 1}, \tag{3}$$

first of which corresponds to the surface wave concentrated near the upper interface and the second corresponds to the wave concentrated near the internal (contact) interface. The second of relationships (3) implies that, for the statically stable stratification $\gamma > 1$, the all the eigen frequencies ω_c are real, and we indeed, deal with non-decaying oscillations.

These elementary results have been generalized in various directions by taking into account viscosity, thermal conductivity, nonlinearity, etc. The contact interface waves still generate a lot of interest of researchers and a lot of physical, mathematical, and numerical problems, demanding further considerations. However, our knowledge of contact interfaces is incomparably better than the knowledge of phase interfaces. By definition, a phase interface is the interface separating two different physical states of the same substance, for instance, water and its vapor. In this case, the mass exchange between the two phases can play a crucial role in the physics of the system.

Although many issues related to the thermodynamics of phase equilibrium became transparent after publications of the classical papers of Gibbs [1] and of van der Waals [3], the analogy of the dispersion relation (2) for the phase interface was found years later (see, Grinfeld [4]). For the approximation of instantaneous kinetics of phase transformation, discussed below, the dispersion equation for the two layer system reads

$$\frac{\omega_p^4}{g^2k^2}\tanh kh_o - \frac{\omega_p^2}{gk}\gamma\left(1 + \tanh kh_o \tanh kh_i\right) + \tanh kh_o + \gamma \tanh kh_i = 0 \tag{4}$$

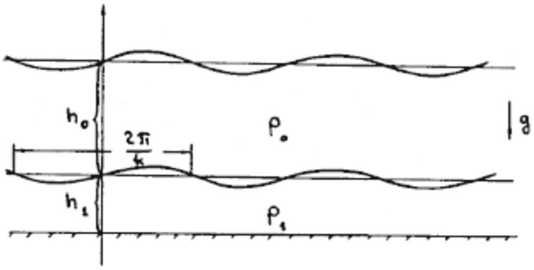

Fig. 1 A layered liquid system in gravity field

The short wavelength analogy of Eq. (4) leads to the following roots:

$$(\mathrm{a})\frac{\omega_p^2}{gk} = 1, \quad (\mathrm{b})\ \frac{\omega_p^2}{gk} = \frac{\gamma + 1}{\gamma - 1} \quad . \tag{5}$$

One can see that the wave concentrated near the upper interface has the same dispersion relation as the similar wave in the case of contact interface. This is a physically expected result: in the short wavelength asymptotics the wave does not "feel" the nature of the interface, and it has the same relationship $\omega^2 = gk$ as the surface wave propagating at an infinite liquid half-space. The second root shows that the behavior of the phase interface wave is qualitatively different as compared with the contact interface situation. For instance, at $\gamma \to 1$ the "contact" mode ω_c approaches 0 whereas for the "phase" root ω_p approaches infinity.

Although the physics of the two systems is different and the key formula for them are, in certain sense opposite, from the purely mathematical point of view, there is still a remarkable similarity in the dispersion Eqs. 1 and 3. This similarity, presumably, reflects the *mathematical* similarity of these systems. Therefore, a considerable portion of mathematical results and techniques, related to the systems with contact interfaces, can be fruitfully transferred to the systems with phase interfaces. Such a research effort may result in the discovery of new explicit formulas, the mathematical structure of which will be similar to classical formulas of hydrodynamics with contact boundaries, but the physics will be totally different. Interestingly, this venue was not explored for several decades.

The considerations of the interfacial waves concentrated nearby phase interface was stimulated by the problems of stability and free oscillations of self-gravitating liquid celestial bodies. The starting point of this study was the remarkable paper of Ramsay [5] in which he analyzed radial stability of a two-phase liquid planet with incompressible phases. Such a planet is shown schematically in Fig. 2.

Ramsay's approach to radial stability was purely static: it is based on the "astrophysical" criterion of stability of self-gravitating liquid spheres. According this criterion, a gravitating barotropic liquid body is stable if the equilibrium outer radius r_0 of the planet grows with the growth of the central pressure inside the body.

Although Ramsay's analysis does not mention any variational principle, the "astrophysical" criterion has an immediate relation to the Jacoby condition for a minimum in the classical one-dimensional problem of the calculus of variations Shnol [6]. Based on the aforementioned static criterion, Ramsay demonstrated in [5] that the planet is unstable if corresponding value of the function $F(\gamma, R) \equiv (\gamma - 1)^2 R^4 + 4(\gamma - 1)R + 3 - 2\gamma$ is negative; in this formula we use the notation: $R \equiv r_o/r_i$ and $\gamma \equiv \rho_i/\rho_o$. For instance, the function $F(\gamma, R)$ is negative for the planet with a small inner core (i.e., when $R << 1$) if the "core" phase is sufficiently essentially more dense than the "mantle" phase (more specifically, if $\gamma > 3/2$). The Ramsay result found various applications in Earth and planetary physics (Jeffryes [7], Grinfeld [8], Grinfeld and Wisdom [9]) and astrophysics (Blinnikov et al. [10], Haensel et al. [11]), and it was further studied in more mathematical papers of Lighthill [12] and of Shnol [6].

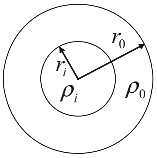

Fig. 2 Two-phase Ramsay planet

Interest in two-phase celestial bodies raised the problem of free oscillations of such bodies when mass transfer due to the phase transformation is taken into account. This problem was explored in the papers of Grinfeld [4,8]. The following formula was given in Grinfeld [4] for the purely radial free oscillations

$$\omega_p^2 = \frac{4\pi G\rho_o}{3} \frac{4(\gamma-1)R + (\gamma-1)^2 R^4 + 3 - 2\gamma}{(1-R)(\gamma-1)}, \tag{6}$$

where G is the universal gravitational constant. We will not dwell here on the wide spectrum of non-radial harmonics explored in Grinfeld [8]: for the higher-order harmonics the spectrum approaches the values given by formulas (4).

As always, the appearance of negative values for ω^2 means the loss of stability of the system. Obviously, the dynamic criterion (6) is consistent with the Ramsay static stability criterion. Formula (6) found various applications in helioseismology and astrophysics [10,11]. To the best of the author's knowledge, it still remains unknown whether the phase transformation oscillations of celestial bodies is a planetological/astrophysical reality or just a theoretical fiction. To get ultimate evidence about interior structure of celestial bodies is always a difficult task.

All the results related to phase transformation waves discussed above are based on the assumption of "instantaneous kinetics." In the context of phase transformation waves, this assumption means that the conditions of phase equilibrium across the phase interface are maintained even when dealing with dynamic problems. Physicists would say that this approximation is valid when considering physical phenomena with the typical timescales of establishing phase equilibrium at the phase interface which are much shorter than the periods of free oscillations of the system in question. Such an assumption can be realistic for celestial bodies, in which low frequencies of free oscillations are of order of hours and the phase transformations zones appear at high temperatures. Usage of the instantaneous kinetics approximation in dynamics was discussed in detail in Grinfeld [14] and references therein. Possible corrections, associated with dynamics, are immaterial when dealing with small oscillations. Interested readers are referred also to the monograph [14]. Below we give a more detailed presentation of the above-mentioned results taking into account the finite rate of phase transformation.

3 Phase transformation waves in a layered liquid system with incompressible liquid phases (finite kinetics of transformation)

Let us come back to the two-layered system shown in Fig. 1. Because of the symmetry, we can limit ourselves with the 2D case: the in-plane axis x is directed along the interface, the axis z coincide with the direction of the external gravity field. Let $V_x(x,z,t)$ and $V_z(x,z,t)$ be the corresponding components of the particles' velocities, whereas $p(x,z,t)$ is the pressure distribution.

3.1 The master system

Bulk equations for ideal heavy phases in the Eulerian description are well-known [Lamb] and include the momentum equations

$$\begin{aligned} \rho\left(\frac{\partial V_x}{\partial t} + V_x\partial_x V_x + V_z\partial_z V_x\right) &= -\partial_x p \\ \rho\left(\frac{\partial V_z}{\partial t} + V_x\partial_x V_z + V_z\partial_z V_z\right) &= -\partial_z p - \rho g \end{aligned} \tag{7}$$

and the incompressibility equation

$$\partial_x V_x + \partial_z V_z = 0 \tag{8}$$

Obviously, we need two sets of Eqs. (8, 9): one for the upper layer and another for the lower layer.

In the following, we use the marks "−" and "+" for the lower and upper layers, respectively.

The boundary conditions at the flat horizontal bottom $z = -h_-$ and the pressure-free top are, respectively,

$$V_z|_{z=-h_d} = 0, \quad p|_{\text{top}} = 0 \tag{9}$$

The equilibrium conditions across the phase interface are the following:

$$[p]_-^+ = 0 \tag{10}$$

and

$$[\mu]_-^+ = 0, \tag{11}$$

where $\mu_\pm$ are the chemical potentials of the phases. In the case of a "cold" incompressible liquid, the chemical potential is given by the formula

$$\mu \equiv e + \frac{p}{\rho}, \tag{12}$$

where e is the internal energy density and ρ is the density. Contrary to the *constants* e and ρ, the quantities p and μ appear to be *functions*: $p = p\,(x, z, t)\,, \quad \mu = \mu\,(x, z, t)\,.$

Phase transformation leads to mass transfer across the interface that separates the phases. Contrary to the case of a contact interface, there is a mass transfer across the phase interface, and the normal component of the velocity field is not continuous $\left[V^i\right]_-^+ N_i \neq 0$. Instead, the mass conservation equation should be satisfied across the interface

$$C\,[\rho]_-^+ - \left[\rho V^i\right]_-^+ N_i = 0, \tag{13}$$

where C is the spatial velocity of the interface. Equation (13) leads to the equality

$$\left(C - V_+^i N_i\right)\rho_+ = \left(C - V_-^i N_i\right)\rho_- = J \tag{14}$$

where J is the mass flux through the interface.

Gibbs' variational principles [1] imply equilibrium conditions (10–12), if the phase transformation is possible. We assume that the same conditions should be satisfied in quasi-static processes which can be treated as a *sequence* of equilibrium configurations. In quasi-static processes, the time t is just a scalar parameter and the inertia term is ignored in the bulk equation. In the quasi-static situation, the bulk equations of hydrostatics, the continuity Eq. (14), and the boundary conditions of phase equilibrium (10–12) imply the conservation of the total (static) energy of the system

$$\frac{\mathrm{d}E_{\text{total}}}{\mathrm{d}t} = 0, \tag{15}$$

where the total static energy includes the total internal energy of the system and the total gravitational energy.

Equations (7–13) comprise a self-consistent system. The system remains mathematically consistent if the "chemical" condition of phase equilibrium (11) is replaced with the following kinetic condition:

$$J = -K\,[\mu]_-^+\,, \tag{16}$$

where the positive constant K is called the kinetic constant. Obviously, at K approaching infinity the postulated condition (16) implies the "chemical" condition of phase equilibrium. Our choice of the kinetic master equation (16) is motivated by the demand of temporal decay of the total energy

$$\frac{\mathrm{d}E_{\text{total}}}{\mathrm{d}t} \leq 0 \tag{17}$$

At K approaching zero, the mass flux J due to phase transformation vanishes and the system behaves as if there were a standard contact interface instead of a phase interface.

As we have seen earlier, the kinetic equation (16) is motivated by consideration of quasi-static processes. From the standpoint of physics, however, quasi-statics is the approximation for a sufficiently slow dynamic process. In other words, we should expect to deal with the self-consistent and practically useful system (7–13) if (a) the equations of hydrostatics are replaced with the dynamic bulk Eqs. (7) and (b) the dynamic processes under consideration are sufficiently slow. Of course, the dynamic bulk equations (7) differ qualitatively from the system of hydrostatics: for instance, they require different sets of initial conditions.

The system of Eqs. (7) and (8) is called the dynamic system for two-phase liquid system with instantaneous kinetics of phase transformations. The same system, but with the boundary condition (16) instead of (11), is the dynamic system for the case of finite kinetics.

3.2 The layers’ thicknesses in equilibrium configuration

When all thermal effects are ignored, the internal energy density should be replaced with the function of the density ρ alone. If we consider incompressible phases with fixed densities $\rho_\pm$, we ought to accept that the internal energy densities of the two phases are certain fixed constants $e_\pm$ characterizing the two phases. The chemical potential, however, changes due to changes in the interface pressure p.

According to (10–12), at equilibrium, the interface pressure is equal to

$$p^\circ = -\frac{[e]_-^+}{\left[\rho^{-1}\right]_-^+}. \tag{18}$$

Hence, the equilibrium thickness h_+ of the upper layer is fixed and equal to

$$h_+ = \frac{p^\circ}{g\rho_+} = \frac{\rho_-}{g}\frac{e_+ - e_-}{\rho_+ - \rho_-}. \tag{19}$$

If we pour more liquid to the upper layer of system, this action would not change the equilibrium thickness of the upper layer as it would happen when dealing with contact interface. Instead, this will result in a phase transformation at the internal interface, and the lower layer (not the upper one!) become thicker. This is one of the key physical differences in the two systems, implied by the physical nature of the interface.

In what follows, we assume that the equilibrium phase interface coincides with the axis $z = 0$. Then, the equilibrium position of the upper interface will be $z = h_+$.

3.2.1 Linearized equations of evolution

Linearizing the master bulk system (7) in the vicinity of equilibrium configuration, we get

$$\rho\frac{\partial \bar{V}_x}{\partial t} = -\partial_x \bar{p}, \tag{20}$$

$$\rho\frac{\partial \bar{V}_z}{\partial t} = -\partial_z \bar{p} \tag{21}$$

The linearized incompressibility condition (8) reads

$$\partial_x \bar{V}_x + \partial_z \bar{V}_z = 0. \tag{22}$$

Linearization of the boundary condition at the flat horizontal bottom $z = -h_d$ implies

$$\bar{V}_z\big|_{z=-h_d} = 0. \tag{23}$$

This linearization is straightforward since the bottom surface is assumed immobile.

Linearization of boundary condition at the pressure-free upper boundary implies

$$\left(\frac{\partial \bar{p}}{\partial t} - g\rho_+ \bar{V}_z\right)\bigg|_{z=h_+} = 0. \tag{24}$$

The second term in (24) is entailed by the fact that the upper surface “moves with particles.” This is why the term $g\rho_+\bar{V}_z$ is called convective.

Linearization of the quasi-static boundary conditions 10, 11, 13 at the phase interface implies 3 relationships:

$$\left(\frac{\partial \bar{p}}{\partial t}_+ - \frac{\partial \bar{p}}{\partial t}_-\right)_{z=0} + \bar{C}g\left(\rho_+ - \rho_-\right)_{z=0} = 0, \tag{25}$$

$$\left(\bar{C} + \bar{V}_+^z\right)\rho_+ = \left(\bar{C} + \bar{V}_-^z\right)\rho_- \tag{26}$$

$$\left(\frac{\partial \bar{C}}{\partial t} + \frac{\partial \bar{V}_+^z}{\partial t}\right)\rho_+ = -K\left(\frac{1}{\rho_+}\frac{\partial \bar{p}}{\partial t}_+ - \frac{1}{\rho_-}\frac{\partial \bar{p}}{\partial t}_-\right), \tag{27}$$

where $\bar{V}_\pm^x$, $\bar{V}_\pm^z$, $\bar{C}$, and $\bar{p}$ are the corresponding small perturbations.

3.3 Small vertical oscillations

Let us consider small vertical oscillations of heterogeneous systems with heavy liquid layers. In the case of a contact interface, such oscillations are impossible due to the conditions of incompressibility. At the same time, in the two-phase layered system vertical oscillations become possible due to mass transfer across the phase interface.

For these oscillations, we are looking for solutions of the linearized boundary value problem of the form

$$\bar{V}^x_\pm \equiv 0,\ \ \bar{V}^z_\pm = W_\pm(z)e^{i\omega t},\ \ \bar{p}_\pm = P_\pm(z)e^{i\omega t},\ \ \bar{C} = Se^{i\omega t} \tag{28}$$

Substituting (28) in the bulk Eqs. (21, 22), we get, respectively:

$$i\omega\rho_+ W_+ = -\frac{\mathrm{d}P_+(z)}{\mathrm{d}z}, \tag{29}$$

$$i\omega\rho_- W_- = -\frac{\mathrm{d}P_-(z)}{\mathrm{d}z} \tag{30}$$

The incompressibility bulk Eq. (22) implies

$$\frac{\mathrm{d}W_+(z)}{\mathrm{d}z} = 0, \tag{31}$$

and

$$\frac{\mathrm{d}W_-(z)}{\mathrm{d}z} = 0. \tag{32}$$

Combining (32) with the linearized boundary condition (33), we get

$$W_- = const = 0 \tag{33}$$

The upper boundary condition (24) reads

$$(i\omega P_+ - g\rho_+ W_+)|_{z=h_+} = 0 \tag{34}$$

The linearized boundary conditions (25–27) at the phase boundary imply

$$i\omega\,(P_+ - P_-)_{z=0} + Sg\,(\rho_+ - \rho_-) = 0, \tag{35}$$

$$(S + W_+)\,\rho_+ = (S + W_-)\,\rho_-, \tag{36}$$

$$i\omega\,(S + W_+)\,\rho_+ = -i\omega K\left(\frac{1}{\rho_+}P_+ - \frac{1}{\rho_-}P_-\right), \tag{37}$$

respectively.

Equations (33) imply the relationships

$$W_+\,(z) = A_+ = const \tag{38}$$

With the help of (38), Eq. (29) implies

$$P_- = \Pi_- = const,\quad P_+ = -i\omega\rho_+ A_+ z + \Pi_+, \tag{39}$$

where $\Pi_\pm$ are certain constants.

Equations (34–37) now imply

$$i\omega\,(-i\omega\rho_+ A_+ h_+ + \Pi_+) - g\rho_+ A_+ = 0 \tag{40}$$

$$i\omega\,(\Pi_+ - \Pi_-) + Sg\,(\rho_+ - \rho_-) = 0 \tag{41}$$

$$(S + A_+)\,\rho_+ = S\rho_- \tag{42}$$

$$i\omega\,(S + A_+)\,\rho_+ = -i\omega K\left(\frac{1}{\rho_+}\Pi_+ - \frac{1}{\rho_-}\Pi_-\right) \tag{43}$$

We arrive at the linear uniform system of 4 equations with respect to 4 unknowns constants $A_{+,}\quad \Pi_{\pm},\quad S$. The secular equation of system (40–43) reads

$$\omega^2 - i\omega\frac{1}{Kh_+}\frac{\rho_-^2\rho_+}{(\rho_- - \rho_+)^2} - \frac{g}{h_+}\frac{\rho_-}{\rho_- - \rho_+} = 0 \tag{44}$$

Equation (44) shows that the phase interface waves always decay, provided $K \neq 0$.

3.4 Asymptotics of instantaneous kinetics K

At $K = \infty$ Eq. (44) implies

$$\omega_{inst}^2 = \frac{g}{h_+}\frac{\rho_-}{\rho_- - \rho_+} \tag{45}$$

A more accurate calculations gives us the following formula at K approaching infinity implies

$$\omega_{1,2} = \pm\sqrt{\frac{g}{h_+}\frac{\rho_-}{\rho_- - \rho_+}} + i\omega\frac{1}{2Kh_+}\frac{\rho_-^2\rho_+}{(\rho_- - \rho_+)^2} \tag{46}$$

Equations (45) show that the system under consideration allows phase transformation oscillations with the frequency ω_{inst}. According to Eq. (46), these oscillations are dissipative with decrement η

$$\eta = \frac{\omega_{inst}}{2Kh_+}\frac{\rho_-^2\rho_+}{(\rho_- - \rho_+)^2} \tag{47}$$

Relationship (47) shows that at very large kinetic constant K, the waves experience rather small dissipation.

3.5 Asymptotics of small kinetics constant K approaching zero

At $K \to 0$ the two roots of Eq. (44) are approximately equal

$$\omega_1 \approx i\frac{1}{Kh_+}\frac{\rho_-^2\rho_+}{(\rho_- - \rho_+)^2}, \quad \omega_2 \approx i\frac{K(\rho_- - \rho_+)}{\rho_-\rho_+}g \tag{48}$$

Both modes of the motion decay exponentially and cannot be treated as oscillations.

3.6 Finite kinetics

In general, the discriminant D of (44) is equal to

$$D \equiv \frac{g}{h_+}\frac{\rho_-}{\rho_- - \rho_+} - \frac{1}{4K^2h_+^2}\frac{\rho_-^4\rho_+^2}{(\rho_- - \rho_+)^4} \tag{49}$$

If the inequality

$$K < \frac{h_+\rho_+\omega_{inst}^3}{2g^2} \tag{50}$$

holds, we get an exponentially decaying regime of vertical evolution. If K is not so small, there are some oscillations superimposed on exponential decay.

3.7 Infinitesimal interface waves with spatial oscillations in the tangential direction

Let us consider now sinusoidal solutions of the master system (29–37), i.e., solutions of the following form:

$$\begin{aligned}&\bar{V}^x_\pm \equiv U_\pm(z)e^{i(kx+\omega t)},\ \bar{V}^z_\pm = W_\pm(z)e^{i(kx+\omega t)},\\&\bar{p}_\pm = P_\pm(z)e^{i(kx+\omega t)},\ \bar{C} = Se^{i(kx+\omega t)}\end{aligned} \tag{51}$$

Inserting (51) in the bulk Eqs. (20–22), we get

$$i\omega\rho U = -ikP,\ i\omega\rho W = -\frac{\mathrm{d}P}{\mathrm{d}z} \tag{52}$$

System (52) should be amended with the incompressibility condition

$$ikU + \frac{\mathrm{d}W}{\mathrm{d}z} = 0. \tag{53}$$

Looking for the solution of (51–53) in the form

$$\begin{bmatrix} U \\ V \\ P \end{bmatrix} = \begin{bmatrix} A_U \\ A_V \\ A_P \end{bmatrix} e^{pkz}, \tag{54}$$

we get the following general solution

$$\begin{bmatrix} U \\ V \\ P \end{bmatrix} = A_+ \begin{bmatrix} 1 \\ -i \\ -\frac{\omega\rho}{k} \end{bmatrix} e^{kz} + A_- \begin{bmatrix} 1 \\ i \\ -\frac{\omega\rho}{k} \end{bmatrix} e^{-kz} \tag{55}$$

For the sake brevity, let us consider short wavelength oscillations concentrated near the phase interface. In this case, we can treat the liquid layers as half-space and ignore boundary conditions (9). Instead, we demand exponential decay of the fields at $z \to \pm\infty$. Then, the disturbances at the upper and lower half-spaces are given by

$$\begin{bmatrix} U_+ \\ V_+ \\ P_+ \end{bmatrix} = R \begin{bmatrix} 1 \\ i \\ -\frac{\omega\rho_+}{k} \end{bmatrix} e^{-kz}\ ,\quad \begin{bmatrix} U_- \\ V_- \\ P_- \end{bmatrix} = Q \begin{bmatrix} 1 \\ -i \\ -\frac{\omega\rho_-}{k} \end{bmatrix} e^{kz} \tag{56}$$

Equations (25–27) now read

$$\begin{aligned}&i\frac{\omega^2}{gk\,(\rho_+ - \rho_-) - \sigma k^2}\,(\rho_- Q - \rho_+ R) + S = 0,\\&(S + iR)\,\rho_+ = (S - iQ)\,\rho_-,\\&S + iR = K\frac{\omega}{\rho_+ k}\,(R - Q)\end{aligned} \tag{57}$$

The secular equation of the algebraic linear uniform system (57) reads

$$\frac{\omega^2}{gk}\frac{\rho_- + \rho_+}{\rho_- - \rho_+} + iK\frac{\omega}{\rho_+\rho_- k}\left[\frac{\omega^2}{gk}\,(\rho_- - \rho_+) - (\rho_- + \rho_+)\right] - 1 = 0 \tag{58}$$

Thus, we arrive at the cubic equation with respect to the complex frequency ω.

3.8 Asymptotic of instantaneous kinetics

In this asymptotic case, K approaches infinity and Eq. (58) implies

$$\frac{\omega_{inst}^2}{gk} = \frac{\rho_- + \rho_+}{\rho_- - \rho_+}, \tag{59}$$

which is coherent with the earlier established Eq. (5).

The more precise asymptotics at K approaching infinity reads

$$\omega_{inst} = \pm\sqrt{\frac{\rho_- + \rho_+}{\rho_- - \rho_+} gk} + 2i\frac{k}{K}\frac{\rho_-^2 \rho_+^2}{(\rho_- - \rho_+)^2 (\rho_- + \rho_+)} \tag{60}$$

Thus, these waves have the decrement η

$$\eta = 2\frac{k}{K}\frac{\rho_-^2 \rho_+^2}{(\rho_- - \rho_+)^2 (\rho_- + \rho_+)} \tag{61}$$

approaching zero at K approaching infinity.

3.9 Asymptotics of infinitely slow kinetics

In this case K approaches zero, and Eq. (58) implies the dispersion relationship

$$\frac{\omega_{cont}^2}{gk} = \frac{\rho_- - \rho_+}{\rho_- + \rho_+}, \tag{62}$$

which is coherent with the classical dispersion Eq. (3).

The more precise asymptotics at K approaching zero reads

$$\omega_{inst} = \pm\sqrt{\frac{\rho_- - \rho_+}{\rho_- + \rho_+} gk} + 2i\frac{K}{k}\frac{\rho_- - \rho_+}{(\rho_- + \rho_+)^2} \tag{63}$$

According to (63), we arrive at the following decrement η

$$\eta = 2\frac{K}{k}\frac{\rho_- - \rho_+}{(\rho_- + \rho_+)^2} \tag{64}$$

According to (60, 63), in both extreme situations the waves decay very slowly and we get the real oscillatory regimes.

3.10 The finite kinetics regimes

Equation (58) can be rewritten as

$$\frac{\omega^2}{\omega_{cont}^2} - 1 + iK\frac{\omega(\rho_- + \rho_+)}{\rho_+\rho_- k}\left(\frac{\omega^2}{\omega_{inst}^2} - 1\right) = 0 \tag{65}$$

At $\rho_-/\rho_+ >> 1$ we get $\omega_{cont}^2 \approx \omega_{inst}^2 \approx gk$ and Eq. (65) gives us

$$\left(\frac{\omega^2}{gk} - 1\right)\left(1 + iK\frac{\omega}{\rho_+ k}\right) = 0 \tag{66}$$

The last equation shows that in addition to an almost dissipation-free mode

$$\frac{\omega_\pm^2}{gk} = 1 \tag{67}$$

there is an additional mode with $\omega = \omega_\Delta$:

$$\frac{\omega_\Delta}{\rho_+ k} = iK^{-1} \tag{68}$$

The last root corresponds to the purely dissipative mode that would be interesting to detect. In the following, we will be using the notation Δ for

$$\Delta \equiv \rho_+ k K^{-1} \tag{69}$$

We can now rewrite (65) in the following transparent form

$$\frac{\omega^2}{\omega_{cont}^2} - 1 + i\frac{\omega}{\Delta}\left(1 + \frac{1}{\gamma}\right)\left(\frac{\omega^2}{\omega_{inst}^2} - 1\right) = 0 \tag{70}$$

This third-order equation can be explored numerically.

3.11 The quasi-static dynamics

3.11.1 The vertical evolution

In the quasi-static theory, we ignore the inertia in the bulk equations. The only difference in the equations of small oscillations will be in equations (39, 40)

$$P_- = \Pi_- = const, \quad P_+ = \Pi_+ = const \tag{71}$$

$$i\omega\Pi_+ - g\rho_+ A_+ = 0 \tag{72}$$

Equations (41–43) stay the same.

Skipping routine calculations, we arrive at the formula

$$\omega = iK\frac{\rho_- - \rho_+}{\rho_+\rho_-}g, \tag{73}$$

which is obviously equivalent to the second of formulas (48).

4 Dynamics of a two-phase self-gravitating planet with incompressible liquid phases

4.1 The exact nonlinear master system

The bulk equations of motion of a two-phase, self-gravitating, liquid planet with incompressible phases are the following

$$\rho\left(\frac{\partial V^i}{\partial t} + V^j \nabla_j V^i\right) = -\nabla^i p + \rho\nabla^i \chi \tag{74}$$

$$\nabla_i V^i = 0 \tag{75}$$

$$\nabla_i \nabla^i \chi = -4\pi G\rho \tag{76}$$

where χ is the Newtonian potential of the gravitating body and G is the gravitational constant (we note that many authors prefer to define the Newtonian potential with the function $-\chi$). In vacuum, the Laplace equation should be used instead of the Poisson equation (76).

At the upper (free) surface of the planet, we have to impose the condition of vanishing of the pressure

$$p|_{out} = 0 \tag{77}$$

and of continuity of the Newtonian potential and its first derivatives

$$[\chi]^+_{-out} = 0, \quad [\nabla_i \chi]^+_{-out} N^i = 0 \tag{78}$$

where N^i is the unit normal to the interface. Obviously, the "gravitational sub-system" should be supplemented with the condition of vanishing of the potential at infinity $\chi(\infty, t) = 0$.

At the internal phase interface, the continuity conditions for the potential χ still should be satisfied. In addition, we assume that the quasi-static boundary condition of pressure continuity are still valid

$$[p]^{+}_{-\mathrm{int}} = 0\,. \tag{79}$$

Because the mass flux due to phase transformation cannot be neglected, we use the condition of continuity of the mass flux J

$$J = \rho_{+}(C - V^i_{+} N_i) = \rho_{-}(C - V^i_{-} N_i) \tag{80}$$

instead of the continuity of the normal components of the velocity vectors $V^i_{\pm} N_i$.

We also accept the kinetics master equation 8 for the mass flow

$$\rho_{+}(C - V^i_{+} N_i)\Big|_{\mathrm{int}} = -K \left[e + \frac{p}{\rho} \right]^{+}_{-\mathrm{int}} \tag{81}$$

where $K > 0$ is a corresponding kinetic constant.

4.2 The linearized evolution master system for small radial perturbations

The hydrostatic equations of radially symmetric equilibrium can be easily integrated. They give the equilibrium distribution of pressure p_0 within the planet interior. The boundary values of the derivatives $\frac{\mathrm{d}p_0}{dr}$ at the interfaces are given by the following formulas:

$$\begin{gathered}
\left.\frac{\mathrm{d}p_0}{dr}\right|_{r=r_i-0} = -\frac{4\pi G \rho_i^2 r_i}{3} = -\frac{4\pi G \rho_o^2 r_o R \gamma^2}{3}, \quad \left.\frac{\mathrm{d}p_0}{dr}\right|_{r=r_i+0} = -\frac{4\pi G \rho_o^2 r_o R \gamma}{3}, \\
\left.\frac{\mathrm{d}p_0}{dr}\right|_{r=r_o} = -\frac{4\pi G \rho_o \left[\rho_o \left(r_o^3 - r_i^3\right) + \rho_1 r_i^3\right]}{3 r_0^2} = -\frac{4\pi G \rho_o^2 r_o \left[1 - (1-\gamma) R^3\right]}{3}
\end{gathered} \tag{82}$$

Let us consider small radial disturbances of the equilibrium configuration. The bulk Eqs. (74–76) imply the following linearized equations for small disturbances of the radial velocity $V_r(r,t)$, of the pressure $p(r,t)$, and of the Newtonian potential $\bar{\chi}(r,t)$, respectively:

$$\rho \frac{\partial \bar{V}_r}{\partial t} = -\frac{\partial \bar{p}}{\partial r} + \rho \frac{\partial \bar{\chi}}{\partial r} \tag{83}$$

$$\frac{\partial \left(r^2 \bar{V}_r\right)}{\partial r} = 0 \tag{84}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \bar{\chi}}{\partial r}\right) = 0. \tag{85}$$

In linearized form, the outer boundary conditions (77, 78) for small perturbations read

$$\left.\left(\frac{\partial p}{\partial t} + \bar{V}_r \frac{\mathrm{d}p_0}{dr}\right)\right|_{r=r_o} = 0, \tag{86}$$

$$\left.\left[\frac{\partial \bar{\chi}}{\partial t}\right]^{+}_{-}\right|_{r=r_o} = 0, \quad \left.\left[\frac{\partial^2 \bar{\chi}}{\partial t \partial r} - 4\pi G \rho \bar{V}_r\right]^{+}_{-}\right|_{r=r_o} = 0. \tag{87}$$

Equations (87) for the potential perturbations should be satisfied at the (internal) phase interface as well. The linearized version of condition (79), expressing the pressure continuity, at the external boundary reads

$$\left.\left[\frac{\partial \bar{p}}{\partial t} + \bar{V}_r \frac{\mathrm{d}p_0}{dr}\right]^{+}_{-}\right|_{r=r_i} = 0 \tag{88}$$

As in (86), an extra (convective) term appears due to the boundary mobility.

The linearized flux continuity condition reads

$$\rho_+(\bar{C} + \bar{V}_{r+}) = \rho_-(\bar{C} + \bar{V}_{r-}) \tag{89}$$

At last, the linearized boundary condition (81) of the kinetics of mass flux reads

$$\rho_+(\bar{C} + \bar{V}_{r+})\big|_{r=r_i} = -K\left[\frac{1}{\rho}\bar{p}\right]^+_{-r=r_i} \tag{90}$$

Let us consider solutions of (83–90) in the form of periodic oscillations

$$\begin{gathered}\bar{V}_r(r,t) = U(r)e^{i\omega t},\ \ \bar{p}(r,t) = P(r)e^{i\omega t},\\ \bar{\chi}(r,t) = W(r)e^{i\omega t},\ \ C = Se^{i\omega t}\end{gathered} \tag{91}$$

The bulk equations (83–85) can be rewritten as

$$\begin{gathered}i\omega\rho U = -\frac{\mathrm{d}P}{dr} + \rho\frac{dW}{dr} \text{ where } \rho = \begin{cases}\rho_o \text{ when } r \in (r_i, r_o)\\ \rho_1 \text{ when } r \in (0, r_i)\end{cases}\\ \frac{d\left(r^2U\right)}{dr} = 0,\quad \frac{d}{dr}\left(r^2\frac{dW}{dr}\right) = 0\end{gathered} \tag{92}$$

The boundary conditions (86, 87) across the outer interface read

$$\left(i\omega P - \frac{4\pi G\rho_o^2 r_o\left[1 + (\gamma - 1)\,R^3\right]}{3}U\right)\Bigg|_{r=r_o} = 0, \tag{93}$$

$$[W]^+_{-r=r_o} = 0,\quad i\omega\left[\frac{dW}{dr}\right]^+_{-r=r_o} - 4\pi G\rho_o U\Bigg|_{r=r_o} = 0 \tag{94}$$

The boundary conditions (88) of the pressure continuity and the conditions of the Newtonian potential continuity at the inner interface read

$$\left(i\omega P + U\frac{\mathrm{d}p_0}{dr}\right)\Bigg|_{r=r_i+0} - \left(i\omega P + U\frac{\mathrm{d}p_0}{dr}\right)\Bigg|_{r=r_i-0} = 0 \tag{95}$$

or

$$\begin{gathered}i\omega\left(P|_{r=r_i+0} - P|_{r=r_i-0}\right) - \frac{4\pi G\rho_o^2 r_o R\gamma}{3}\left(U|_{r=r_i+0} - \gamma\ U|_{r=r_i-0}\right) = 0\\ [W]^{+}_{-}\big|_{r=r_i} = 0,\quad \left[i\omega\frac{dW}{dr} - 4\pi G\rho U\right]^+_-\Bigg|_{r=r_i} = 0\end{gathered} \tag{96}$$

The linearized flux continuity condition reads

$$S\,(1-\gamma) + U|_{r=r_i+0} - \gamma\ U_-|_{r=r_i-0} = 0 \tag{97}$$

At last, the linearized boundary condition (81) of the mass flux reads

$$S + U|_{r=r_i+0} = -\frac{K}{\gamma\rho_o^2}\left(\gamma\ P|_{r=r_i+0} - P|_{r=r_i-0}\right) \tag{98}$$

The bulk equations (92) and the condition for the Newtonian potential at infinity imply the following general solution:

(a) at $r > r_o$

$$W(r) = \frac{A_\infty}{r} \tag{99}$$

(b) at $r_i < r < r_o$

$$W(r) = \frac{A_o}{r} + B_o,\ U = \frac{C_o}{r^2},\ P = \frac{\rho_o}{r} A_o + \frac{i\omega\rho_o}{r} C_o + D_o$$

(c) at $r < r_i$

$$W(r) = B_i,\ U = 0,\ P = D_i$$

For the 8 constants $A_\infty,\ A_o,\ B_o,\ C_o,\ D_o,\ B_i,\ D_i$, and S we have 8 linear uniform Eqs. (95–98). Skipping quite cumbersome calculation, we arrive at the following secular equation

$$\omega^2 - i\omega \frac{\gamma^2 \rho_o}{K R r_o (1-R)(\gamma-1)^2} - \frac{4\pi G\rho_o}{3} \frac{4(\gamma-1)R + (\gamma-1)^2 R^4 + 3 - 2\gamma}{(1-R)(\gamma-1)} = 0 \tag{100}$$

At $K = \infty$, we get the formula 5 for the small radial oscillations of the two-phase planet. At $K \to \infty$, the first two terms of the expansion in K^{-1} read

$$\omega = \sqrt{\frac{4\pi G\rho_o}{3} \frac{4(\gamma-1)R + (\gamma-1)^2 R^4 + 3 - 2\gamma}{(1-R)(\gamma-1)}} + i \frac{\gamma^2 \rho_o}{2K R r_o (1-R)(\gamma-1)^2} \tag{101}$$

The last term in (101) defines the decrement η of the radial oscillations

$$\eta = \frac{\gamma^2 \rho_o}{2K R r_o (1-R)(\gamma-1)^2}\ , \tag{102}$$

and it vanishes at $K \to \infty$.

5 Conclusion

We presented a phenomenological approach to phase transformations in heterogeneous systems. For the sake of simplicity and brevity, we limited our analysis with the phase transformations liquid/liquid or liquid vapor. The approach permits immediate generalization for the solid/solid and solid/melt transformations. The suggested approach is a symbiosis of the Gibbs model of heterogeneous systems, on the one hand, and the Onsager approach of linear fluxes to irreversible system, on the other hand. Basically, we assume that the rate of phase transformations across phase boundary is proportional to the jump of the local chemical potentials of the phases. To generalize this approach to the solid/solid or solid/melt transformations, we only need to find consistent generalizations of the scalar chemical potential of liquids and gases. Possible generalizations have been suggested in multiple publication of the 1980s and summarized in the monograph [14]. The finite kinetics approach was also used for two-phase system in the monograph [15] and papers [16–18].

It was demonstrated above that in heterogeneous two-phase systems oscillatory regimes are possible when the kinetics constant is very large or very small. In the intermediate situations, dynamic processes appear to be highly dissipative. We establish explicit analytical formulas for the decrements of the oscillations.

References

1. Gibbs, J.W.: On the equilibrium of heterogeneous substances. Trans. Connect. Acad. Sci. **3**(108–248), 343–524 (1876, 1878)
2. Lamb, H.: Hydrodynamics. Dover Publications, New York (1991)
3. van der Waals, J.D.: The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density. J. Stat. Phys. **20**(2), 197–244 (1979) (Translation by J. S. Rowlinson)
4. Grinfeld, M.A.: Gravitating bodies of the Ramsay type. Doklady Akademii Nauk SSSR **262**, 1339–1344 (1982)
5. Ramsay, W.H.: On the instability of small planetary cores. Mon. Not. R. Astron. Soc. **110**(4), 325–338 (1950)
6. Shnol’, E.E.: Degeneracy in the simplest problem of variational calculus. Math. Not. **24**, 877–882 (1978)
7. Jeffreys, H.: The Earth: Its Origin, History, and Physical Constitution. Cambridge University Press, Cambridge (1976)
8. Grinfeld, M.A.: Gravitating bodies with internal phase transition surfaces. Phys. Solid Earth **24**, 540–550 (1988)
9. Grinfeld, P., Wisdom, J.: Motion of the mantle in the translational modes of the Earth and Mercury. Phys. Earth Planet. Inter. **15**, 77–87 (2005)

10. Bisnovatyi-Kogan, G.S., Blinnikov, A.I., Shnol', E.E.: Stability of a star if there is a phase-transition. Astronomicheskii Zurn. **52**, 920–929 (1975)
11. Haensel, P., Zdunik, J.L., Schaeffer, R.: Phase-transitions in dense matter and radial pulsations of neutral stars. Astron. Astrophys. **217**(1–2), 137–144 (1989)
12. Lighthill, M.J.: On the instability of small planetary cores. Mon. Not. R. Astron. Soc. **110**, 339–342 (1950)
13. Grinfeld, M.A.: Effect of surface tension on heterogeneous equilibria. Doklady Akademii Nauk SSSR **283**, 1139–1143 (1985)
14. Grinfeld, M.A.: Thermodynamic Methods in the Theory of Heterogeneous Systems. Longman, Harlow (1991)
15. Abeyaratne, R., Knowles, J.: Evolution of Phase Transitions: A Continuum Theory. Camb. Univ. Press, Cambridge (2011)
16. Eremeev, V.A., Freidin, A.B., Sharipova, L.L.: Nonuniqueness and stability in problems of equilibrium of elastic two-phase bodies. Dokl. Phys. **48**(7), 359–363 (2003)
17. Eremeev, V.A., Freidin, A.B., Sharipova, L.L.: The stability of the equilibrium of two-phase elastic solids. J. Appl. Math. Mech. **71**, 61–84 (2007)
18. Fu, Y., Freidin, A.: Characterization and stability of two-phase piece-wise homogeneous deformations. Proc. R. Soc. Lond. **A460**, 3065–3094 (2004)