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Universal spherically symmetric solution of nonlinear dislocation theory for incompressible isotropic elastic medium

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Abstract The equilibrium problem of a nonlinearly elastic medium with a given dislocation distribution is considered. The system of equations consists of the equilibrium equations for the stresses, the incompatibility equations for the distortion tensor, and the constitutive equations. Deformations are considered to be finite. For a special distribution of screw and edge dislocations, an exact spherically symmetric solution of these equations is found. This solution is universal in the class of isotropic incompressible elastic bodies. With the help of the obtained solution, the eigenstresses in a solid elastic sphere and in an infinite space with a spherical cavity are determined. The interaction of dislocations with an external hydrostatic load was also investigated. We have found the dislocation distribution that causes the spherically symmetric quasi-solid state of an elastic body, which is characterized by zero stresses and a nonuniform elementary volumes rotation field.

Keywords Nonlinear elasticity · Dislocation density · Eigenstresses · Large deformations · Exact solution · Quasi-solid states

1 Introduction

Universal solutions, or universal deformations, in the mechanics of a continuous medium are the solutions of equilibrium equations which are valid for any constitutive equations from a certain class of materials. The importance of universal solutions is in their convenience for the experimental determining of the constitutive equations and also in their use in numerical solution testing. An example of a universal solution for a homogeneous nonlinearly elastic medium in the absence of mass forces is an arbitrary uniform deformation at which the stresses are constant. For the nonlinear elasticity of incompressible isotropic nonlinearly elastic bodies, five families of nonuniform universal deformations are known [8,11,18,24,26–28,31,32]. The concept of universal solutions was originally introduced for static problems of the elasticity theory without taking mass forces into account, but later it was extended to static problems with mass forces, as well as to dynamic problems [25]. Universal deformations of nonlinear solid bodies with constraints that differ from the incompressibility condition have been investigated in [25,29,30]. Currently, generalized models of continua with complex physico-mechanical properties (see, e.g., [7,19,20,22], etc.) are widely used. Universal solutions for some of these models are constructed in [6,40].

In this paper, we have found and analyzed a new solution of the nonlinear elasticity theory taking into account distributed dislocations. This solution is of spherical symmetry and is universal in the class of isotropic incompressible materials. The solution supplements a small list of known exact solutions of the nonlinear continuum theory of dislocations [33–35,37,39] and describes the effect of distributed screw and edge dislocations

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on large spherically symmetric deformations of an elastic medium. While taking into account the distributed edge dislocations, a spherically symmetric solution of nonlinear elasticity for a particular model of a compressible material was found [37].

Dislocations are a common element of the solid structure. Dislocation models are useful in describing phenomena such as crystal growth, fatigue, destruction, plastic flow, and inelasticity [3,4,10,21]. If there are a lot of dislocations in the body, then it makes sense to pass from the discrete dislocation distribution to the continuous one. In this case, the continuum theory of continuously distributed dislocations is used.

2 Input relations

Statics system of equations for nonlinear elastic medium in the absence of mass forces consists [18,24,31] of the equilibrium equations for stresses

$$\mathrm{div}\mathbf{D} = 0, \tag{1}$$

the constitutive equations

$$\mathbf{D}(\mathbf{F}) = \mathrm{d}W(\mathbf{G})/\mathrm{d}\mathbf{F}, \qquad \mathbf{G} = \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}}, \tag{2}$$

and the geometric equations

$$\mathbf{F} = \mathrm{grad}\boldsymbol{R}, \qquad \boldsymbol{R} = X_k\boldsymbol{i}_k, \tag{3}$$

where $\mathbf{D}$ is the asymmetric Piola stress tensor (the first Piola–Kirchhoff stress tensor), W is the energy of deformation, $\mathbf{F}$ is the deformation gradient, $\mathbf{G}$ is the metric tensor, also called the Cauchy strain measure, X_k ($k = 1, 2, 3$) are the Cartesian coordinates of body particles in the deformed configuration, and $\boldsymbol{i}_k$ are the fixed coordinate orts. In (1) and (3) and further, we use the operators of gradient, divergence, and rotor in the reference configuration:

$$\mathrm{grad}\boldsymbol{\Psi} = \boldsymbol{r}^s\otimes\frac{\partial\boldsymbol{\Psi}}{\partial q^s}, \qquad \mathrm{div}\boldsymbol{\Psi} = \boldsymbol{r}^s\cdot\frac{\partial\boldsymbol{\Psi}}{\partial q^s}, \qquad \mathrm{rot}\boldsymbol{\Psi} = \boldsymbol{r}^s\times\frac{\partial\boldsymbol{\Psi}}{\partial q^s};$$
$$\boldsymbol{r}^s = \boldsymbol{i}_m\frac{\partial q^s}{\partial x_m}, \quad s, m = 1, 2, 3\ .$$

Here, x_m are the Cartesian coordinates of the reference configuration of the material body, q^s are some curvilinear coordinates, and $\boldsymbol{\Psi}$ is the arbitrary differentiable tensor field.

If dislocations with a given tensor density $\boldsymbol{\alpha}(q^s)$ are distributed in the body, then the coordinates of the particles in the deformed state $X_k(q^s)$ and the vector field $\boldsymbol{R}(q^s)$ do not exist, and the geometric equations (3) are replaced by the tensor incompatibility equation [9,15,23,38]:

$$\mathrm{rot}\mathbf{F} = \boldsymbol{\alpha}\ . \tag{4}$$

For $\boldsymbol{\alpha}\neq 0$, nonsingular tensor $\mathbf{F}$ is called the distortion tensor. The dislocation density tensor $\boldsymbol{\alpha}$ cannot be specified completely arbitrary, but must satisfy the solenoidality condition

$$\mathrm{div}\boldsymbol{\alpha} = 0\ . \tag{5}$$

The physical meaning of the second-rank tensor field $\boldsymbol{\alpha}$ is that the flux of this tensor through any surface is equal to the total Burgers vector of all dislocations crossing this surface [23].

In the nonlinear continual dislocation theory, there are other incompatibility equations [1,2,12,14] containing the dislocation density tensor that differ from (4). They represent a nonlinear system of second-order differential equations with respect to the metric tensor $\mathbf{G}$, and as shown in [5], they are the consequences of the first-order incompatibility equations (4). Because of the complexity of the second-order incompatibility equations, their use for solving boundary value problems of the nonlinear dislocation theory involves severe difficulties. Therefore, in the present paper, the incompatibility equations in the form (4) will be applied.

In the theory of finite elastic deformations, along with the Piola stress tensor $\mathbf{D}$, we use the symmetric Cauchy stress tensor [18,24,31]

$$\mathbf{T} = (\mathrm{det}\mathbf{F})^{-1}\mathbf{F}^{\mathrm{T}}\cdot\mathbf{D}, \tag{6}$$

and the Kirchhoff stress tensor, also called the second Piola–Kirchhoff stress tensor

$$\mathbf{P} = \mathbf{D} \cdot \mathbf{F}^{-1} \, . \tag{7}$$

In the absence of dislocations, that is, for $\boldsymbol{\alpha} = 0$, the equilibrium equations (1) can be rewritten in the form [18]:

$$\operatorname{Div} \mathbf{T} = 0, \tag{8}$$

where Div is a divergence operator in the coordinates of the deformed state. For a medium with distributed dislocations, such coordinates do not exist and Eq. (8) does not make sense. At the same time, the concepts of the Cauchy stress tensor and the Kirchhoff stress tensor as characteristics of the stressed state of the body do not become invalid in the presence of distributed dislocations too.

We also note that the Piola identity known in the mechanics of finite deformations of the continuum [18]

$$\operatorname{div}[(\det \mathbf{F}) \mathbf{F}^{-\mathrm{T}}] = 0, \qquad \mathbf{F}^{-\mathrm{T}} = \left(\mathbf{F}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{F}^{-1}\right)^{\mathrm{T}} \tag{9}$$

in the presence of distributed dislocations, generally speaking, is not satisfied. It can be shown that the generalization of the identity (9) to the case $\boldsymbol{\alpha} \neq 0$ is the relation

$$\operatorname{div}[(\det \mathbf{F}) \mathbf{F}^{-\mathrm{T}}] = \boldsymbol{r}_s \cdot \left(\boldsymbol{\alpha} \times \mathbf{F}^{\mathrm{T}}\right) \cdot \boldsymbol{r}^s,$$

where $\boldsymbol{r}_s$ is a vector basis mutual to the basis $\boldsymbol{r}^s$, and the third-rank tensor $\mathbf{B} \times \mathbf{C}$ is defined as follows:

$$\left(B_{mn} \boldsymbol{r}^m \otimes \boldsymbol{r}^n\right) \times \left(C_{pq} \boldsymbol{r}^p \otimes \boldsymbol{r}^q\right) = B_{mn} C_{pq} \boldsymbol{r}^m \otimes \left(\boldsymbol{r}^n \times \boldsymbol{r}^p\right) \otimes \boldsymbol{r}^q \, .$$

3 Spherically symmetric state

Let us introduce the spherical coordinates r, φ, θ by the following formulas:

$$x_1 = r \cos\varphi \cos\theta, \qquad x_2 = r \sin\varphi \cos\theta, \qquad x_3 = r \sin\theta \, .$$

On a spherical surface $r = \text{const}$, the parameters φ and θ are the geographical coordinates: longitude and latitude. We denote the unit vectors tangent to the coordinate lines by $\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$, $\boldsymbol{e}_\theta$.

Suppose that the dislocation density tensor is given in the form

$$\begin{aligned} \boldsymbol{\alpha} &= \alpha_1(r)\mathbf{g} + \alpha_2(r)\mathbf{d} + \alpha_3(r)\mathbf{e}_r \otimes \mathbf{e}_r, \\ \mathbf{g} &= \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta, \quad \mathbf{d} = \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta - \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi \, . \end{aligned} \tag{10}$$

The tensor field (10) is of spherical symmetry [39]. The first term in (10) describes the distribution of screw dislocations, which lines coincide with the parallels and meridians, and the last term describes the distribution of screw dislocations with a radial axis. The middle term in (10) corresponds to the distribution of edge dislocations.

It was established in [39] that in the case of an isotropic material, the system of Eqs. (1), (2), (4), (10) has the following solution:

$$\mathbf{F} = F_1(r)\mathbf{g} + F_2(r)\mathbf{d} + F_3(r)\mathbf{e}_r \otimes \mathbf{e}_r, \tag{11}$$

$$\mathbf{D} = D_1(r)\mathbf{g} + D_2(r)\mathbf{d} + D_3(r)\mathbf{e}_r \otimes \mathbf{e}_r, \tag{12}$$

and the vector equilibrium equation (1) reduces to a single scalar equation:

$$\frac{\mathrm{d}D_3}{\mathrm{d}r} + \frac{2(D_3 - D_1)}{r} = 0 \, . \tag{13}$$

By virtue of (11)–(13), the equilibrium problem for an elastic isotropic body with a spherically symmetric dislocation distribution reduces to the nonlinear ordinary differential equations [39].

We consider a special case of the representation (10):

$$\boldsymbol{\alpha} = \frac{1}{r} \left(\gamma_0 \mathbf{g} + \beta_0 \mathbf{d} + 2\gamma_0 \boldsymbol{e}_r \otimes \boldsymbol{e}_r\right) \, . \tag{14}$$

Here β_0, γ_0 are the dimensionless constants. Expression (14) satisfies the solenoidality condition (5).

It can be verified that the incompatibility equation (4) with the dislocation density (10) has the following solution:

$$\mathbf{F} = C_1\mathbf{g} + \gamma_0\mathbf{d} + (C_1 + \beta_0)\boldsymbol{e}_r \otimes \boldsymbol{e}_r, \qquad C_1 = \text{const} \, . \tag{15}$$

According to (15), the distortion tensor has constant components in the basis of the spherical coordinates $\boldsymbol{e}_\varphi$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_r$. This does not mean that the body experiences a uniform deformation, since grad$\mathbf{F} \neq 0$.

Furthermore, we consider an incompressible elastic medium. According to (15), the incompressibility condition

$$\det\mathbf{F} = 1 \tag{16}$$

leads to the following restriction on the distortion tensor field:

$$(C_1 + \beta_0)\left(C_1^2 + \gamma_0^2\right) = 1 \, . \tag{17}$$

From (17) we obtain:

$$C_1 + \beta_0 > 0 \, . \tag{18}$$

Using (15), (17), (18), we find a metric tensor $\mathbf{G}$, a positive defined stretch tensor $\mathbf{U}$, a properly orthogonal rotation tensor $\mathbf{A}$, and an inverse distortion tensor $\mathbf{F}^{-1}$:

$$\mathbf{G} = \mathbf{F}\cdot\mathbf{F}^{\mathrm{T}} = \left(C_1^2 + \gamma_0^2\right)\boldsymbol{g} + (C_1 + \beta_0)^2\,\boldsymbol{e}_r \otimes \boldsymbol{e}_r, \tag{19}$$

$$\mathbf{U} = \mathbf{G}^{1/2} = \sqrt{C_1^2 + \gamma_0^2}\,\mathbf{g} + (C_1 + \beta_0)\boldsymbol{e}_r \otimes \boldsymbol{e}_r, \tag{20}$$

$$\mathbf{A} = \mathbf{U}^{-1}\cdot\mathbf{F} = \frac{1}{\sqrt{C_1^2 + \gamma_0^2}}\left(C_1\mathbf{g} + \gamma_0\mathbf{d}\right) + \boldsymbol{e}_r \otimes \boldsymbol{e}_r, \tag{21}$$

$$\mathbf{F}^{-1} = \frac{C_1}{C_1^2 + \gamma_0^2}\mathbf{g} - \frac{\gamma_0}{C_1^2 + \gamma_0^2}\mathbf{d} + \frac{1}{C_1 + \beta_0}\boldsymbol{e}_r \otimes \boldsymbol{e}_r \, . \tag{22}$$

Introducing the notation

$$\cos\psi = \frac{C_1}{\sqrt{C_1^2 + \gamma_0^2}}, \quad \sin\psi = \frac{\gamma_0}{\sqrt{C_1^2 + \gamma_0^2}},$$

we can see that the orthogonal tensor (21) describes a rotation through an angle ψ about the vector $\boldsymbol{e}_r$.

In the case of an isotropic incompressible elastic material, the specific energy W is given as a function of the first and the second invariants (I_1 and I_2, respectively) of the Cauchy strain measure [18]. The third invariant of the tensor $\mathbf{G}$ is equal to unity because of the incompressibility condition. Now the constitutive equation (2) takes a more specific form

$$\begin{gathered}\mathbf{D} = \mathbf{D}^* - p\mathbf{F}^{-\mathrm{T}}, \qquad \mathbf{D}^* = (\tau_1 + I_1\tau_2)\mathbf{F} - \tau_2\mathbf{G}\cdot\mathbf{F};\\ \tau_1 = 2\frac{\partial W(I_1, I_2)}{\partial I_1}, \quad \tau_2 = 2\frac{\partial W(I_1, I_2)}{\partial I_2}, \quad I_1 = \mathrm{tr}\mathbf{G}, \quad I_2 = \frac{1}{2}\left(\mathrm{tr}^2\mathbf{G} - \mathrm{tr}\mathbf{G}^2\right) \, .\end{gathered} \tag{23}$$

Here, p is a pressure in an incompressible body, not expressed in terms of strain. In the problem we consider, according to (19) the invariants I_1, I_2 are expressed as follows:

$$\begin{aligned}I_1 &= 2\left(C_1^2 + \gamma_0^2\right) + (C_1 + \beta_0)^2,\\ I_2 &= \left(C_1^2 + \gamma_0^2\right)^2 + 2(C_1 + \beta_0) \, .\end{aligned} \tag{24}$$

Using (15), (19), (22), (23), we find the components of the Piola stress tensor in the decomposition (12):

$$\begin{aligned}
D_1 &= D_1^* - p(r)\frac{C_1}{C_1^2+\gamma_0^2},\\
D_2 &= D_2^* - p(r)\frac{\gamma_0}{C_1^2+\gamma_0^2},\\
D_3 &= D_3^* - p(r)\frac{1}{C_1+\beta_0},\\
D_1^* &= \tau_1 C_1 + \tau_2 C_1[C_1^2+\gamma_0^2+(C_1+\beta_0)^2],\\
D_2^* &= \gamma_0\left(\tau_1+\tau_2\left[C_1^2+\gamma_0^2+(C_1+\beta_0)^2\right]\right),\\
D_3^* &= \tau_1(C_1+\beta_0)+2\tau_2\,.
\end{aligned} \tag{25}$$

The quantities D_1^*, D_2^*, D_3^* are constant, that is, they do not depend on the variable r.

The incompressibility condition (17) is a cubic equation with respect to C_1. This equation has one real root, which can be found using the formulas given in [41]. We have

$$\begin{aligned}
C_1 &= -\frac{\beta_0}{3}+\frac{9\sqrt[3]{K^2}+\beta_0^2-3\gamma_0^2}{9\sqrt[3]{K}},\\
K &= \frac{27-2\beta_0^3-18\gamma_0^2\beta_0}{54}+\sqrt{\left(\frac{27-2\beta_0^3-18\gamma_0^2\beta_0}{54}\right)^2-\left(\frac{\beta_0^2-3\gamma_0^2}{9}\right)^3}\,.
\end{aligned} \tag{26}$$

According to (12), (15), (26), the distortion tensor components F_1, F_2, F_3 are completely expressed in terms of the scalar dislocation densities β_0, γ_0, while the quantities F_1, F_3 are the even functions of the parameter γ_0. From this, it follows that the strain measure **G** and stresses D_1, D_3 do not depend on the sign of γ_0. This suggests that a given screw dislocation distribution creates nonlinear elastic effects that cannot be detected in the context of the linear elasticity theory.

On the basis of (12), (23), (25), the equilibrium equation (13) is reduced to an ordinary differential equation with respect to the function $p(r)$:

$$\begin{aligned}
&r\frac{\mathrm{d}p}{\mathrm{d}r}+2\left[1-C_1(C_1+\beta_0)^2\right]p\\
&-2(C_1+\beta_0)\left\{\tau_1\beta_0+\tau_2\left[\gamma_0^2C_1+\beta_0\left(2\left(C_1^2+\gamma_0^2\right)-C_1\left(2C_1+\beta_0\right)\right)\right]\right\}=0\,.
\end{aligned} \tag{27}$$

Recall that the quantities τ_1 and τ_2 in (27) are the known functions of the invariants I_1, I_2 expressed in terms of the parameters β_0, γ_0 with the help of (24), (26). Equation (27) has the following solution:

$$p = A_1 r^{\lambda}+\frac{\tau_1\beta_0+\tau_2\left[\beta_0(2\gamma_0^2-C_1\beta_0)+\gamma_0^2C_1\right]}{\gamma_0^2-\beta_0C_1},\qquad A_1=\text{const}, \tag{28}$$

$$\lambda = \frac{2\left(C_1\beta_0-\gamma_0^2\right)}{C_1^2+\gamma_0^2}\,. \tag{29}$$

Note that according to (29), the power function exponent λ does not depend on the material properties, but it is completely expressed through the dislocation parameters.

Since the equilibrium equations are satisfied, the tensor distortion field (15), (26) and the stress field (12), (25), (28) represent the nonlinear dislocation theory solution that is universal in the class of isotropic incompressible elastic bodies.

The dislocation distribution given by the two-parameter family of tensor functions (14) is noteworthy in that it allows one to construct an exact solution that reflects the nonlinear interaction of edge and screw dislocations and is universal in the class of isotropic incompressible elastic bodies. Of course, formula (14) does not describe many possible cases of dislocation distribution, in particular, the case when the dislocation density is a strongly oscillating function of coordinates. We note that if the dislocation tensor field is taken in the form $\boldsymbol{\alpha}=\beta(r)\left(\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\theta-\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\varphi\right)$, then the solenoidality condition (5) imposes no restrictions on the function $\beta(r)$. The function $\beta(r)$ can be any function, including a strongly oscillating function and the Dirac delta function. The solution of the spherically symmetric problem of the nonlinear dislocation theory for one specific elastic material model with the given tensor dislocation density is found in [37].

Because the stress tensors of Cauchy and Kirchhoff are symmetric, in a spherically symmetric state of an isotropic medium they have representations:

$$\mathbf{T} = T_1(r)\mathbf{g} + T_3(r)\boldsymbol{e}_r \otimes \boldsymbol{e}_r, \qquad \mathbf{P} = P_1(r)\mathbf{g} + P_3(r)\boldsymbol{e}_r \otimes \boldsymbol{e}_r\ .$$

In accordance with (6), (7), the components of these tensors are expressed in terms of the Piola stress tensor components by the formulas:

$$\begin{aligned} T_1 &= C_1 D_1 + \gamma_0 D_2 = C_1^{-1}\left(C_1^2 + \gamma_0^2\right) D_1, \qquad T_3 = (C_1 + \beta_0) D_3, \\ P_1 &= \frac{C_1 D_1 + \gamma_0 D_2}{\left(C_1^2 + \gamma_0^2\right)} = C_1^{-1} D_1, \qquad P_3 = \frac{D_3}{C_1 + \beta_0}\ . \end{aligned} \tag{30}$$

Here, we have taken into account the relation followed from (11), (23):

$$C_1 D_2 = \gamma_0 D_1\ .$$

Using (25), (26), (30), we write stresses in the form (31):

$$\begin{aligned} T_1 &= \tau_1(C_1^2 + \gamma_0^2) + \tau_2[(C_1^2 + \gamma_0^2)^2 + C_1 + \beta_0] - p, \qquad T_3 = (\tau_1 + 2\tau_2)(C_1 + \beta_0)^2 - p, \\ P_1 &= \tau_1 + \tau_2\left[C_1^2 + \gamma_0^2 + (C_1 + \beta_0)^2\right] - (C_1 + \beta_0)p, \qquad P_3 = \tau_1 + 2\tau_2 - \frac{p}{(C_1 + \beta_0)^2}, \end{aligned} \tag{31}$$

where p is given by (28).

Note that the stress deviator

$$\operatorname{dev}\mathbf{T} = \mathbf{T} - \frac{1}{3}\mathbf{I}\operatorname{tr}\mathbf{T}$$

has constant components in the basis $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_\theta$. Here, $\mathbf{I}$ is a second-order unit tensor.

Next, we consider the application of the obtained universal solution for determining a stress state in some equilibrium problems for elastic bodies containing dislocations.

4 Eigenstresses in infinite space

When an elastic medium fills all the infinite space, then it is not necessary to satisfy any boundary conditions. If $A_1 \neq 0$, then, according to (25), (28), the stresses increase unboundedly for $r \to 0$ or $r \to \infty$. In the infinite space, a solution with bounded stresses exists only for $A_1 = 0$. In this case, the stress components in the basis of spherical coordinates are constant. This does not mean that the stress state is uniform.

From the equilibrium equation (13) for $D_1 = \text{const}$, it follows that $D_1 = D_3$ at each point of the medium. Also this equation results from the formulas (25), (28) for $A_1 = 0$. At the same time, the coincidence of different components of the stress tensors of Cauchy and Kirchhoff is not the case. Indeed, from (30) for $D_1 = D_3$ we obtain that $T_1 \neq T_3$, $P_1 \neq P_3$.

Stresses are determined by (31). In this case, p is computed according to the formula (28) under the assumption that $A_1 = 0$.

5 Eigenstresses in a solid sphere

Consider the eigenstress problem for a solid sphere of radius r_0. The boundary condition

$$T_3(r_0) = 0$$

means that the surface $r = r_0$ is not loaded. Then, the integration constant A_1 is determined by the formula (32):

$$A_1 = r_0^{-\lambda}\left\{(C_1 + \beta_0)^2\tau_1 + 2(C_1 + \beta_0)\tau_2 - \frac{\tau_1\beta_0 + \tau_2\left[\beta_0\left(2\gamma_0^2 - C_1\beta_0\right) + C_1\gamma_0^2\right]}{\gamma_0^2 - \beta_0 C_1}\right\}\ . \tag{32}$$

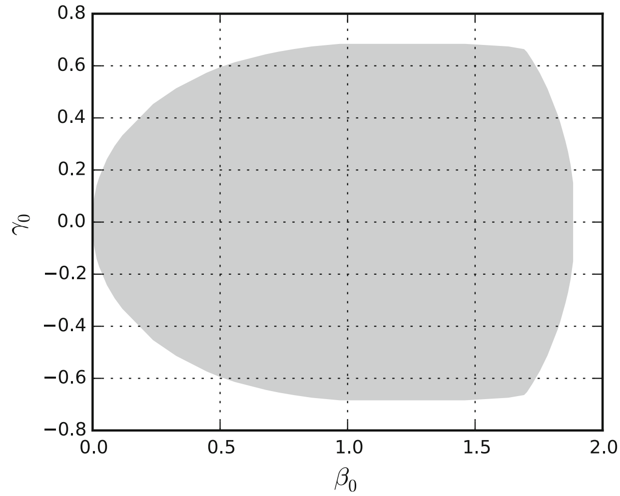

Fig. 1 Domain of allowed values of the parameters β_0 and γ_0 for a sphere

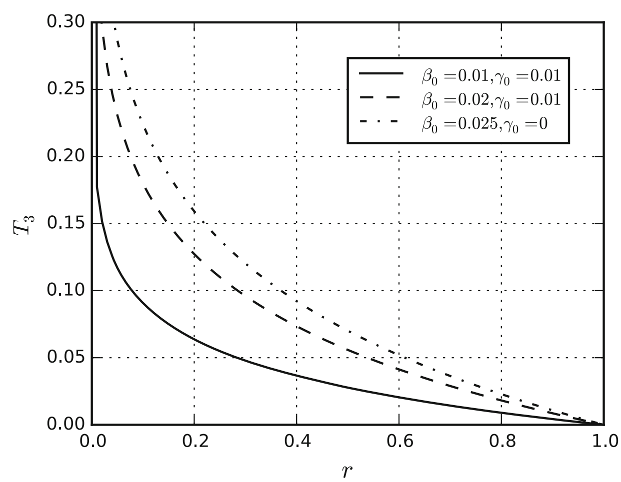

Fig. 2 Radial eigenstress T_3

We require that in the center of the sphere, that is, when $r = 0$, the stresses are limited. This requirement is equivalent to the fact that expression (29) must be positive. Taking into account this restriction, as well as the requirement (18) and the fact that C_1 is real (suppose that $|C_1| \leq 2$), we plot the domain of allowed values of the parameters β_0 and γ_0. It is shown in Fig. 1 in gray. From Eqs. (25), (28), it follows that in the center of the sphere $D_1 = D_3$. At the same time, $T_1 \neq T_3$, $P_1 \neq P_3$ in the presence of dislocations.

The numerical results are plotted for the neo-Hookean material [18]. Therefore, we use $\tau_1 = \mu$, $\tau_2 = 0$. Since all the stresses are related to the shear modulus μ, we use $\mu = 1$. In the calculations, it is assumed that $r_0 = 1$. Figures 2 and 3 show that at sufficiently low dislocation densities, the stresses are distributed not uniformly with the maximum stress at the center of the sphere. For higher densities, the maximum stress is also observed at the center of the sphere. Besides, as shown in Fig. 3, in a solid sphere there is a spherical surface, on which the circumferential stress does not depend on the dislocation density.

Figures 4 and 5 present the Kirchhoff stresses. The radial Kirchhoff stress is lower than the radial Cauchy stress, the Kirchhoff circumferential stress is slightly higher than the corresponding Cauchy stress.

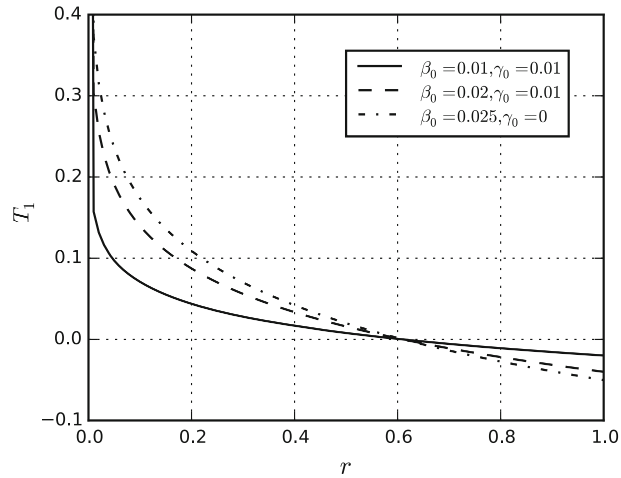

Fig. 3 Circumferential eigenstress T_1

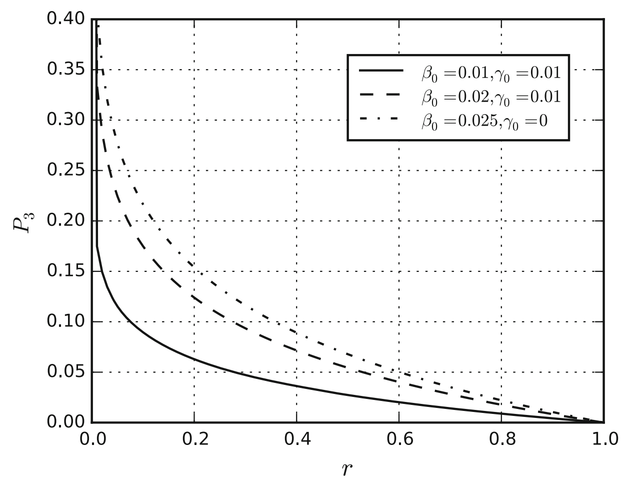

Fig. 4 Radial eigenstress P_3

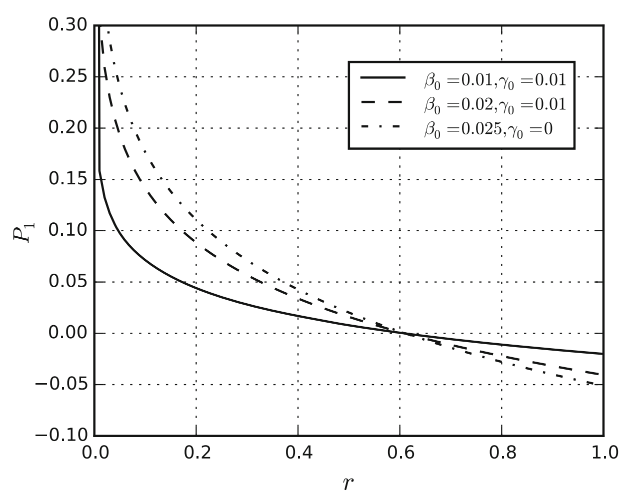

Fig. 5 Circumferential eigenstress P_1

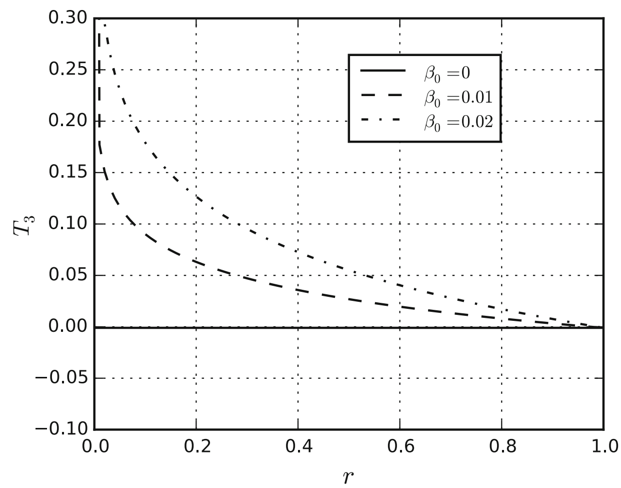

Fig. 6 Radial stress T_3 in a sphere loaded with pressure $q_0 = 0.001\mu$

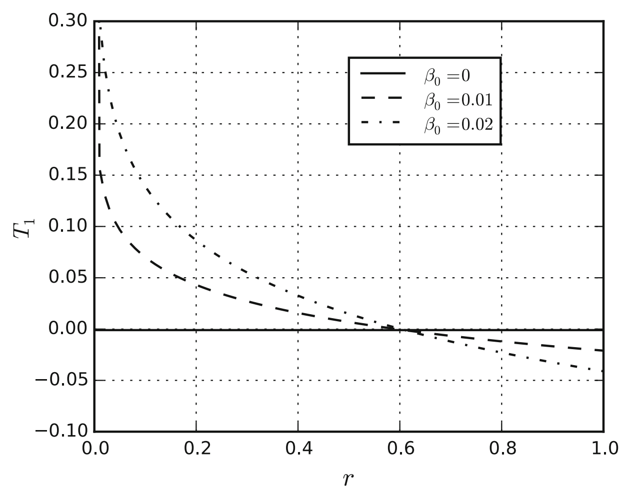

Fig. 7 Circumferential stress T_1 in a sphere loaded with pressure $q_0 = 0.001\mu$

6 Sphere loading with external pressure

The sphere is supposed to be loaded by a constant pressure q_0. Then, the boundary condition for Eq. (27) takes the form (33):

$$T_3(r_0) = -q_0 \ . \tag{33}$$

Hence, the constant A_1 is defined by the formula:

$$A_1 = \tilde{A}_1 + r_0^{-\lambda} q_0,$$

where $\tilde{A}_1$ is the integration constant (32) found in the eigenstress problem.

According to Figs. 6, 7, 8, and 9 in the case of $\gamma_0 = 0$ at the same pressure, the higher the dislocation density, the higher the stress. For the stress T_1, this fails only in the negligible neighborhood of the center of the sphere.

We determine the effect of the applied load and dislocations on a stressed state of an elastic body. With respect to the presence of load and dislocations, we consider three cases listed in Table 1, where the stress (radial / circumferential stress) is denoted as f_i $(i = 1, 2, 3)$, "+" denotes the presence of load / dislocations, and "−" means the absence of those.

From Figs. 10 and 11, it follows that the superposition of the solutions f_1 and f_2 is not equal to the solution f_3. This indicates an essential nonlinearity of the problem.

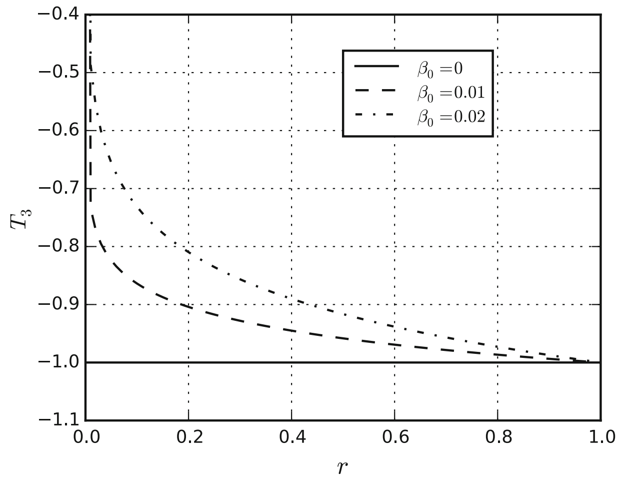

Fig. 8 Radial stress P_3 in a sphere loaded with pressure $q_0 = \mu$

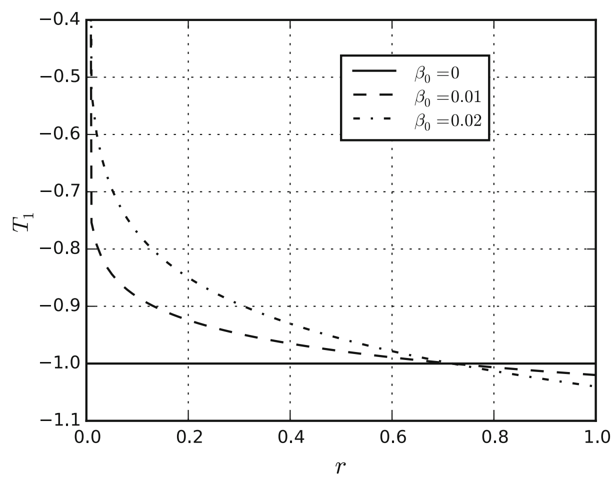

Fig. 9 Circumferential stress P_1 in a sphere loaded with pressure $q_0 = \mu$

Table 1 Superposition principle (Is $f_1 + f_2$ equal to f_3?)

Stress	Load	Dislocations
f_1	−	+
f_2	+	−
f_3	+	+

7 Eigenstresses in infinite space with a cavity

Let us consider infinite space with a cavity of radius r_1. The boundary condition for this case is written as

$$T_3(r_1) = 0 \,.$$

From this, we find the integration constant (34):

$$A_1 = r_1^{-\lambda} \left\{ (C_1 + \beta_0)^2 \tau_1 + 2(C_1 + \beta_0)\tau_2 - \frac{\tau_1 \beta_0 + \tau_2 \left[\beta_0 \left(2\gamma_0^2 - C_1 \beta_0 \right) + C_1 \gamma_0^2 \right]}{\gamma_0^2 - \beta_0 C_1} \right\} \,. \tag{34}$$

Similar to the problem for the sphere (see Sect. 5), we plot the domain of allowed values of the parameters β_0 and γ_0. It follows from Fig. 12 (in gray) that the range of possible values of the parameter γ_0 is much wider than the corresponding range for β_0.

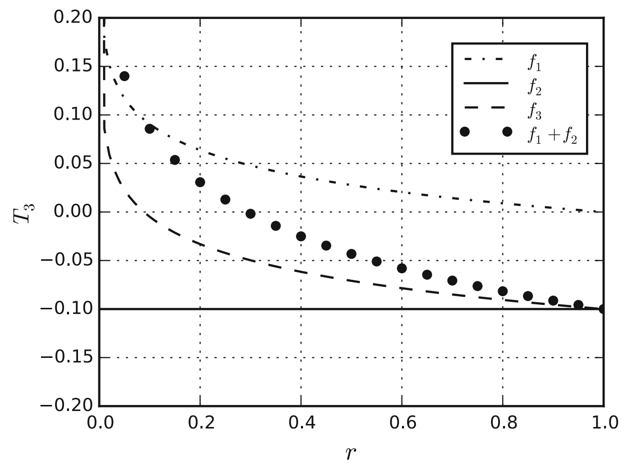

Fig. 10 Radial stress T_3 in a sphere loaded with pressure $q_0 = 0.1\mu$; $\beta_0 = \gamma_0 = 0.01$

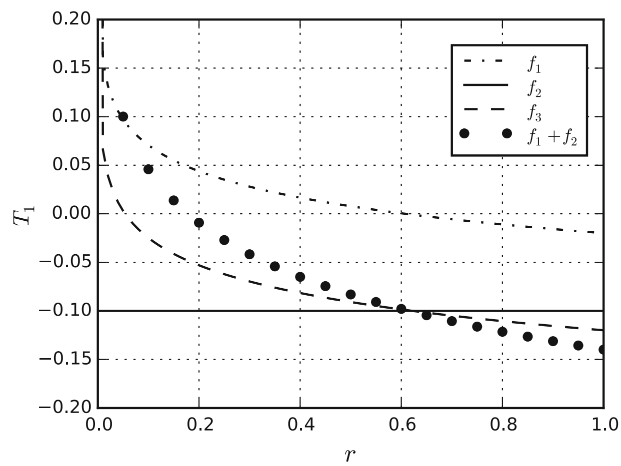

Fig. 11 Circumferential stress T_1 in a sphere loaded with pressure $q_0 = 0.1\mu$; $\beta_0 = \gamma_0 = 0.01$

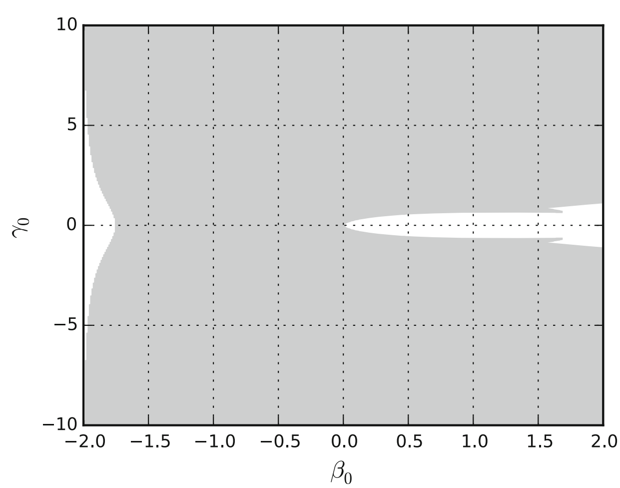

Fig. 12 Domain of allowed values of the parameters β_0 and γ_0 for a space

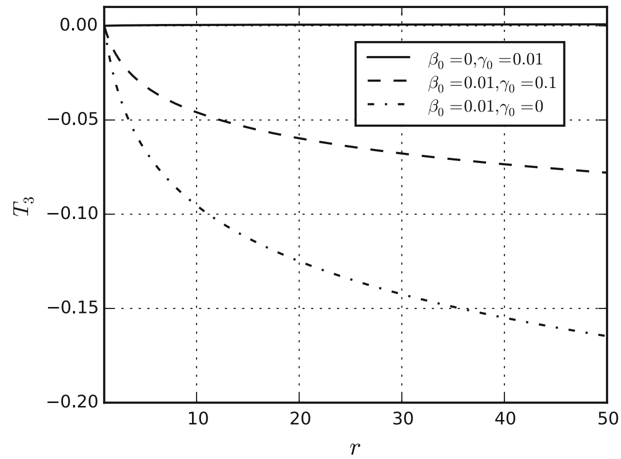

Fig. 13 Radial eigenstress T_3, the case of an infinite space with a cavity

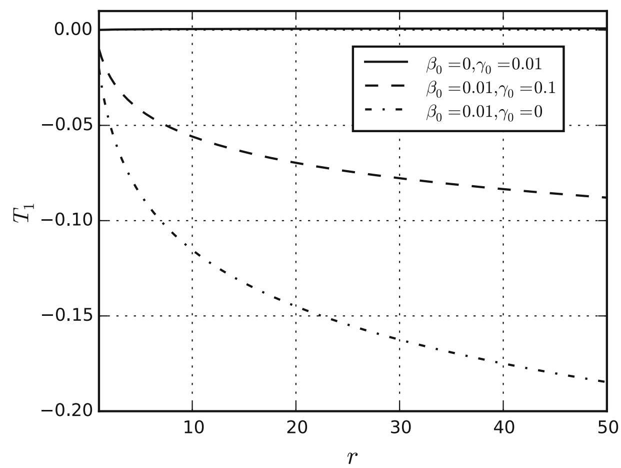

Fig. 14 Circumferential eigenstress T_1, the case of an infinite space with a cavity

Considering a point at infinity, we obtain the stresses in the form (35):

$$
\begin{aligned}
T_3 &= (\tau_1 + 2\tau_2)(C_1 + \beta_0)^2 - \frac{\tau_1\beta_0 + \tau_2\left[\beta_0(2\gamma_0^2 - C_1\beta_0) + \gamma_0^2 C_1\right]}{\gamma_0^2 - \beta_0 C_1}, \\
T_1 &= \tau_1(C_1^2 + \gamma_0^2) + \tau_2[(C_1^2 + \gamma_0^2)^2 + C_1 + \beta_0] - \frac{\tau_1\beta_0 + \tau_2\left[\beta_0(2\gamma_0^2 - C_1\beta_0) + \gamma_0^2 C_1\right]}{\gamma_0^2 - \beta_0 C_1}\,.
\end{aligned}
\tag{35}
$$

Hence, the stresses at the point at infinity are not equal. It was found that at the center of the sphere, the radial and circumferential stresses are equal to the corresponding stresses at the point at infinity. These stresses are written in the form (35).

For the neo-Hookean material, the Cauchy stresses are presented by Figs. 13 and 14. It is assumed that $r_1 = 1$. A comparison of the Cauchy stress and the Kirchhoff stress shows that the radial Kirchhoff stress is less than the radial Cauchy stress, and the Kirchhoff circumferential stress is slightly larger than the corresponding Cauchy stress. It has been established that for the same dislocation density, in the case of the Mooney material [18] the stresses are in absolute value larger than the corresponding stresses for the neo-Hookean material.

Note that for $r \to \infty$, the stresses converge to the values that are the solution of the infinite space problem (see Sect. 4).

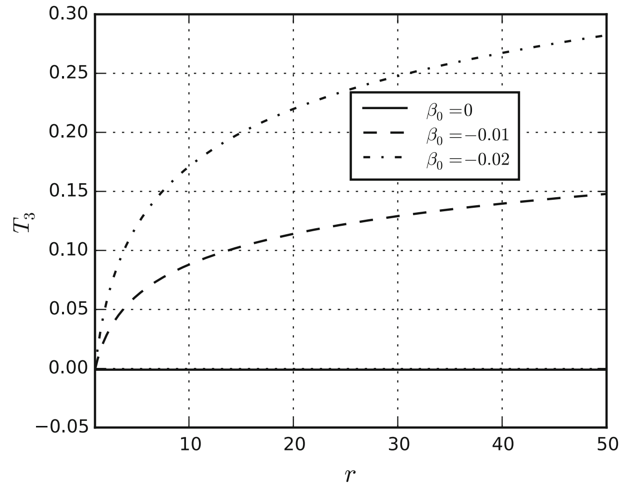

Fig. 15 Radial stress T_3 in a space with a cavity loaded with pressure $q_0 = 0.001\mu$

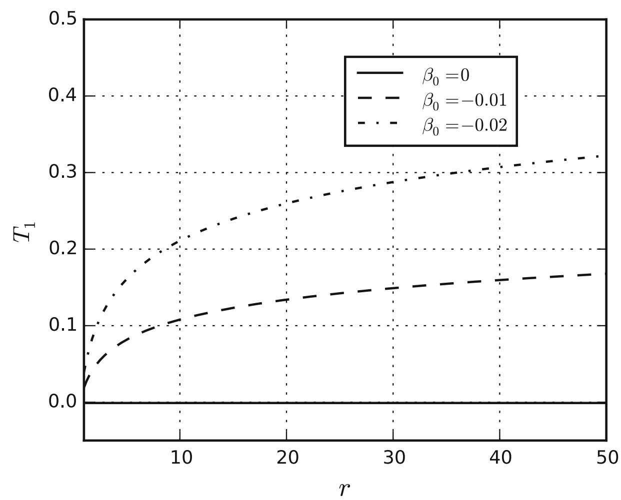

Fig. 16 Circumferential stress T_1 in a space with a cavity loaded with pressure $q_0 = 0.001\mu$

8 Loading an infinite space cavity

If a constant pressure q_0 is acting on the space cavity, then the boundary condition for the Eq. (27) is

$$T_3(r_1) = -q_0 \ .$$

The integration constant A_1 is defined by (36)

$$A_1 = \tilde{A_1} + r_0^{-\lambda} q_0, \tag{36}$$

where $\tilde{A_1}$ is the constant (34) found in the eigenstress problem.

As shown in Figs. 15, 16, 17, and 18, in the case of $\gamma_0 = 0$ at the same pressure, the higher the dislocation density, the higher the stresses. In the case of $\beta_0 = 0$ at the same pressure, the radial stress decreases with increasing dislocation density, and the circumferential stress increases.

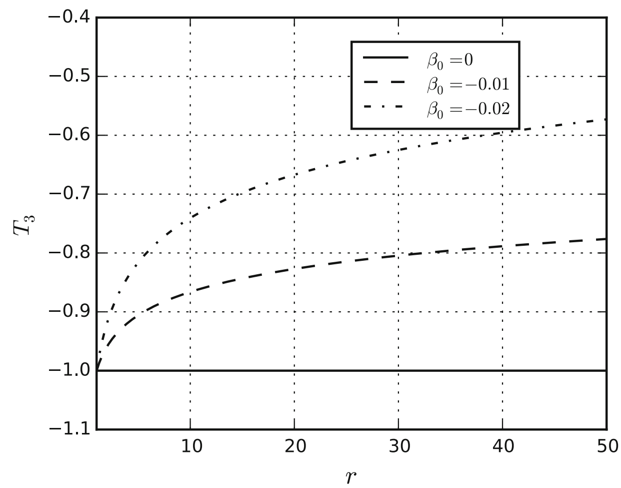

Fig. 17 Radial stress T_3 in a space with a cavity loaded with pressure $q_0 = \mu$

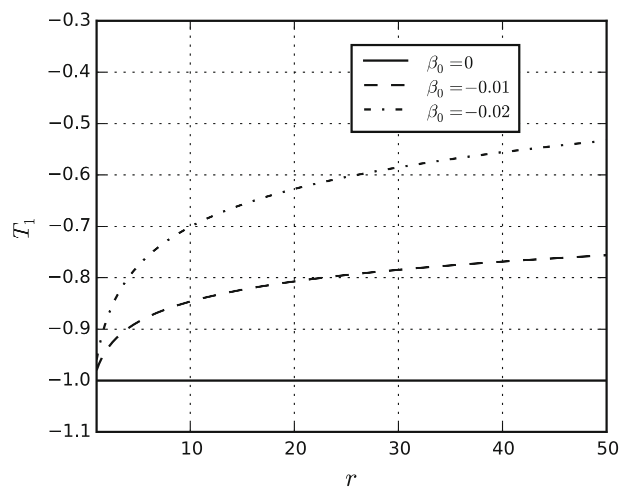

Fig. 18 Circumferential stress T_1 in a space with a cavity loaded with pressure $q_0 = \mu$

9 Quasi-solid spherically symmetric state

Let us consider a special case of dislocation distribution (14), in which $\beta_0 = 1 - \cos\omega$, $\gamma_0 = \sin\omega$, where ω is any real value. Then, according to (16), we have $C_1 = \cos\omega$, and the distortion tensor (15) takes the form:

$$\mathbf{F} = \cos\omega\mathbf{g} + \sin\omega\mathbf{d} + \mathbf{e}_r \otimes \mathbf{e}_r \ . \tag{37}$$

Distortion (37) is a proper orthogonal tensor describing the rotation by a constant angle ω around the vector $\mathbf{e}_r$. On the basis of (37), using formulas (19)–(22), we obtain $\mathbf{G} = \mathbf{I}$, $\mathbf{U} = \mathbf{I}$, $\mathbf{F} = \mathbf{A}$, $\psi = \omega$. Distortion (37) is an example of a quasi-solid state [34] of an elastic body, in which each elementary volume moves like a perfectly rigid body, and the rotation field is nonuniform. Other examples of quasi-solid states are given in [34–36].

For a compressible elastic body, the reference configuration of which coincides with the natural unstressed state, in the absence of external loads, the stresses in the quasi-solid state are identically equal to zero, since the elongations of the material fibers and the shearing strains are zero at each point of the body. In this case, the body can occupy an arbitrary region.

Quasi-solid states can exist only in the presence of distributed dislocations, since the incompatibility equation (4) for $\boldsymbol{\alpha} = 0$ has only constant solutions in the class of proper orthogonal tensors.

According to (26), (28), (31), (32), at a dislocation density determined by the parameters $\beta_0 = 1 - \cos\omega$, $\gamma_0 = \sin\omega$, in the incompressible body the stresses are identically equal to zero. Thus, dislocations, expressed through the rotation angle ω, do not create stresses in themselves, but their influence can be detected by loading

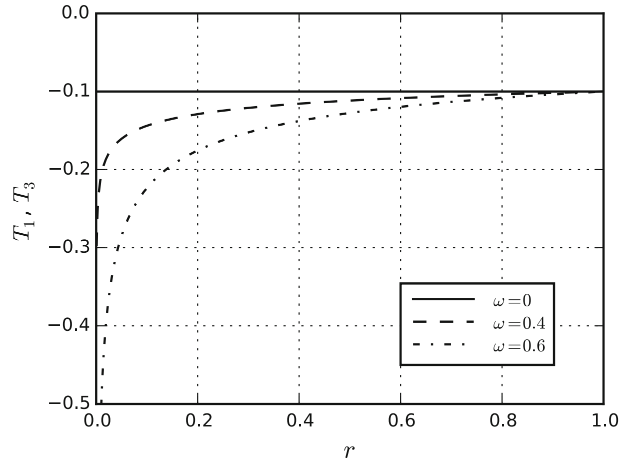

Fig. 19 Stresses in a sphere loaded with pressure $q_0 = 0.1\mu$ at a dislocation density determined by the parameters $\beta_0 = 1 - \cos\omega$, $\gamma_0 = \sin\omega$

the body, as shown in Fig. 19. Moreover, for any isotropic incompressible material, the circumferential and radial stresses are identically equal, that is, the dislocation density creates a locally hydrostatic state, while the internal pressure is not equal to the applied load.

10 Conclusion

In the present paper, we have found a new exact solution of the nonlinear continuum theory of dislocations. This solution describes a spherically symmetric stress state of elastic medium caused by the presence of continuously distributed screw and edge dislocations. The constructed solution is valid for an arbitrary isotropic incompressible material, that is, it is universal in this class of elastic bodies. For some models of materials, we have presented the calculations of the eigenstresses in a solid sphere and in an infinite space with a spherical cavity. We have also investigated the nonlinear effects caused by the interaction of dislocations with an external load in the form of hydrostatic pressure. We have shown that there is a dislocation distribution which, in the absence of an external load, does not create stresses, but affects the stress field caused by the action of external forces.

In this paper, following Kröner [14], we have investigated the problem of eigenstresses caused by distributed dislocations within the elastic body model framework without using the concept of plastic deformation. The same approach, based on the elasticity theory, is used to determine the stresses caused by isolated (singular) dislocations [9,15,38]. Singular dislocations are the limiting case of distributed dislocations, in which the dislocation density is a generalized function concentrated on some line. The general theory of continuously distributed dislocations based on the multiplicative decomposition of the distortion tensor on the elastic and plastic components is developed in [13,16,17]. Boundary value problems on the determination of stresses in the framework of models [13,16,17] are much more complicated than problems for the system of Eqs. (1), (2), (4) and can be the subject of future research. One can assume that the methods presented in this paper can be helpful in solving spherically symmetric and other problems of the general dislocation theory constructed in [13,16,17].

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