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ORIGINAL

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Green's functions for the magneto-electro-elastic anisotropic half-space and their applications to contact and crack problems

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Abstract A general solution is obtained for a magneto-electro-elastic half-space $x_3 \geq 0$ subjected to arbitrary point forces or arbitrary point dislocations, as well as electric and magnetic influence by using two-dimensional Fourier transform. The final results are presented as single integrals over a unit circle. Using the theory of generalized functions, all basic parameters at the half-space boundary are defined in a finite form, and no computation of any integral is needed. Knowledge of Green's functions in finite form allows us to derive the governing integral equations for the normal and tangential contact and crack problems, as well as to establish certain relationship between the kernels of the relevant integral equations. We also established some interesting general properties of the determinants, which might be new.

Keywords Green's functions · Magneto-electro-elastic anisotropic half-space · Contact and crack problems

1 Introduction

Contemporary nanotechnology and intelligent material systems require further development of mathematical methods for treatment of various problems of practical interest in magneto-electro-elastic (MEE) composites. Quite a few publications appeared in the past 20 years devoted to generally anisotropic MEE materials, as well as to various particular cases, such as the case of transverse isotropy, piezoelasticity. We refer mainly to books, which compile various results in specific fields. Ding and Chen [2] published solutions to various three-dimensional problems in transversely isotropic piezoelastic bodies. The most general case of transverse isotropy was considered by Hou et al. [5], where they gave fundamental solutions for the case of thermo-magneto-electro-elastic body.

The first book on anisotropic elasticity was published by Lekhnitskii [7]. More advanced results on anisotropic elasticity can be found in Ting [12]. Efficient method of computation of Green's function and their derivatives suitable for BEM was published by Shiah et al. [11]. The residue approach to derive Green's function in the case of multiple roots was employed by Phan et al. [10]. Yet another approach to evaluation of Green's function by using advanced Stroh's formalism and Radon transform is presented in Xie et al. [13]. One of the earliest treatments of the problem of evaluation of Green's function for magneto-electro-elastic anisotropic body was achieved by Pan [8]. Evaluation of Green's function as well as it derivatives can found in Buroni and Saez [1].

Recently, Pan and Chen published their book [9], where detailed results are given for the most general case of anisotropic MEE materials, as well as for particular cases of transverse isotropy and isotropy. There seems to be no publication on relationship between contact and crack problems in MEE materials, similar to the one described below. This article may be considered as generalization of recently published results ([3] and [4]).

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2 Mathematical formulation of the problem and its solution

We consider magneto-electro-elastic half-space subjected to electric and mechanical point sources. Such half-space ($x_3 \geq 0$) is described by 5 partial differential equations of second order

$$\sum_{k=1}^{5} \sum_{t,m=1}^{3} a_{\text{ktmn}} \frac{\partial^2 u_k}{\partial x_t \partial x_m} = 0, \; n = 1, 2, 3, 4, 5. \tag{1}$$

Here a_{ktmn} are physical constants; u_1, u_2 and u_3 are the displacements of the half-space in the directions of x_1, x_2 and x_3, respectively, u_4 and u_5 are the electric and magnetic potentials. Without loss of generality, we presume that this half-space is subject to the following boundary conditions at $x_3 = 0$

$$\sum_{k=1}^{5} \sum_{m=1}^{3} b_{\text{kmn}} \frac{\partial u_k}{\partial x_m} = \alpha_n \delta(x_1, x_2), n = 1, 2, 3, 4, 5. \tag{2}$$

Here b_{kmn} are physical constants, which have certain relation with the set a_{ktmn}; α_n are given source intensities, some of which could be zero as well; $\delta(\cdot, \cdot)$ is the two-dimensional Dirac delta function.

We presume the solution in the form of two-dimensional Fourier transforms:

$$u_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{s=1}^{5} u_{ks} \exp(-\zeta_s x_3) \exp[-i(x_1\xi_1 + x_2\xi_2)] \mathrm{d}\xi_1 \mathrm{d}\xi_2, \; k = 1, \ldots 5. \tag{3}$$

Here u_{ks} and ζ_s are yet unknown functions of ξ_1 and ξ_2. Substitution of (3) in the set (1) and application of inverse Fourier transform leads to the following

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \mathcal{M}_{14} & \mathcal{M}_{15} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & \mathcal{M}_{24} & \mathcal{M}_{25} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} & \mathcal{M}_{34} & \mathcal{M}_{35} \\ \mathcal{M}_{41} & \mathcal{M}_{42} & \mathcal{M}_{43} & \mathcal{M}_{44} & \mathcal{M}_{45} \\ \mathcal{M}_{51} & \mathcal{M}_{52} & \mathcal{M}_{53} & \mathcal{M}_{54} & \mathcal{M}_{55} \end{bmatrix} \begin{bmatrix} u_{1s} \\ u_{2s} \\ u_{3s} \\ u_{4s} \\ u_{5s} \end{bmatrix} = 0. \tag{4}$$

The general component of the matrix $\mathcal{M}$ is defined as

$$\mathcal{M}_{ij} = a_{j33i}\zeta^2 + i\zeta \left(\sum_{t=1}^{2} a_{jt3i}\xi_t + \sum_{m=1}^{2} a_{j3mi}\xi_m \right) - \sum_{t,m=1}^{2} a_{\text{jtmi}}\xi_m\xi_t. \tag{5}$$

As system of linear algebraic equation (4) is homogeneous, it can have non-trivial solutions, only if the determinant of $\mathcal{M}$ is zero. This leads to the tenth-order algebraic equation with respect to ζ

$$\sum_{s=0}^{10} i^s h_s(\xi_1, \xi_2)\zeta^{10-s} = 0 \tag{6}$$

Here $i = \sqrt{-1}$ is the imaginary unit; h_s are homogeneous polynomials of ξ_1 and ξ_2 with real coefficients of the total order of s. This means that (6) can be factorized as follows

$$h_0 \prod_{s=1}^{5} (\zeta - \zeta_s)(\zeta + \bar{\zeta}_s) = 0, \tag{7}$$

where ζ_s are the roots of (6) and the overbar indicates the complex conjugate value. The structure of (7) confirms that 5 of the roots ζ_s will have positive real parts, while the remaining 5 will have negative real parts. We take only the roots with positive real parts in order to keep the integrals in (3) convergent.

We can find the non-trivial solutions of (4) by assuming

$$u_{ks} = \mathcal{X}_{ks} u_{1s}, \quad k = 2, 3, 4, 5; \quad s = 1, 2, 3, 4, 5 \tag{8}$$

and

$$\mathcal{X}_{1s} = 1 \quad \text{for } s = 1, 2, 3, 4, 5. \tag{9}$$

The parameters $\mathcal{X}_{ks}$ for $k = 2, 3, 4, 5$ can be found from the set of linear algebraic equations

$$\begin{bmatrix} \mathcal{M}_{12}(\zeta_s) & \mathcal{M}_{13}(\zeta_s) & \mathcal{M}_{14}(\zeta_s) & \mathcal{M}_{15}(\zeta_s) \\ \mathcal{M}_{22}(\zeta_s) & \mathcal{M}_{23}(\zeta_s) & \mathcal{M}_{24}(\zeta_s) & \mathcal{M}_{25}(\zeta_s) \\ \mathcal{M}_{32}(\zeta_s) & \mathcal{M}_{33}(\zeta_s) & \mathcal{M}_{34}(\zeta_s) & \mathcal{M}_{35}(\zeta_s) \\ \mathcal{M}_{42}(\zeta_s) & \mathcal{M}_{43}(\zeta_s) & \mathcal{M}_{44}(\zeta_s) & \mathcal{M}_{45}(\zeta_s) \end{bmatrix} \begin{bmatrix} \mathcal{X}_{2s} \\ \mathcal{X}_{3s} \\ \mathcal{X}_{4s} \\ \mathcal{X}_{5s} \end{bmatrix} = \begin{bmatrix} -\mathcal{M}_{11}(\zeta_s) \\ -\mathcal{M}_{21}(\zeta_s) \\ -\mathcal{M}_{31}(\zeta_s) \\ -\mathcal{M}_{41}(\zeta_s) \end{bmatrix} \tag{10}$$

After the parameters $\mathcal{X}_{ks}$ are found, the remaining parameters u_{1s} can be defined from the boundary conditions (2), which will lead to the set of linear algebraic equations with respect to u_{1s}, namely,

$$\begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \end{bmatrix} = \begin{bmatrix} \alpha_1/2\pi \\ \alpha_2/2\pi \\ \alpha_3/2\pi \\ \alpha_4/2\pi \\ \alpha_5/2\pi \end{bmatrix}, \tag{11}$$

where

$$\mathcal{C}_{ns} = \sum_{k=1}^{5} \mathcal{X}_{ks}(-b_{k1n} i\xi_1 - b_{k2n} i\xi_2 - b_{k3n}\zeta_s). \tag{12}$$

The solution of (11) will take the form

$$u_{1s} = \frac{1}{D_c} \sum_{n=1}^{5} \frac{\alpha_n}{2\pi} (-1)^{n+s} D_c^{(n,s)}. \tag{13}$$

Here D_c is the determinant of the matrix $\mathcal{C}$ and $D_c^{(n,s)}$ is its minor, corresponding to the nth row and sth column. Now substitution of (13) into (3) gives the complete solution in the form

$$u_k(x_1, x_2, x_3) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{s=1}^{5} \mathcal{X}_{ks} \sum_{n=1}^{5} \alpha_n (-1)^{n+s} \frac{D_c^{(n,s)}}{D_c} \exp[-x_3\zeta_s - i(x_1\xi_1 + x_2\xi_2)] \mathrm{d}\xi_1 \mathrm{d}\xi_2. \tag{14}$$

Taking into consideration that $\mathcal{X}_{ks}$ is homogeneous with respect to ξ_1 and ξ_2 of the order zero, ζ_s is homogeneous of the order 1; $D_c^{(n,s)}$ and D_c are homogeneous of the orders 4 and 5, respectively, the introduction of polar coordinates

$$\xi_1 = \rho \cos\vartheta, \qquad \xi_2 = \rho \sin\vartheta \tag{15}$$

allows us to compute the integral with respect to ρ and simplify (14) as follows

$$u_k(x_1, x_2, x_3) = \frac{1}{4\pi^2} \int_0^{2\pi} \sum_{s=1}^{5} \sum_{n=1}^{5} \frac{\alpha_n (-1)^{n+s} \mathcal{X}_{ks}(\vartheta) D_c^{(n,s)}(\vartheta)}{D_c(\vartheta)[x_3\zeta_s(\vartheta) + i(x_1\cos\vartheta + x_2\sin\vartheta)]} \mathrm{d}\vartheta. \tag{16}$$

The solution may be considered in principle as finished, since all the parameters of interest can now be computed by simple differentiation of (16). We should note though that (16) is not valid at $x_3 = 0$. A different procedure needs to be followed. The integral (14) will take the form

$$u_k(x_1, x_2, 0) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} \sum_{s=1}^{5} \mathcal{X}_{ks}(\vartheta) \sum_{n=1}^{5} \alpha_n (-1)^{n+s} \frac{D_c^{(n,s)}(\vartheta)}{D_c(\vartheta)} \times \exp[-i\rho(x_1\cos\vartheta + x_2\sin\vartheta)] \mathrm{d}\rho \mathrm{d}\vartheta. \tag{17}$$

Since ρ enters only the exponential, we can use the theory of generalized functions to get

$$\int_0^\infty \exp[-i\rho(x_1\cos\vartheta + x_2\sin\vartheta)]\mathrm{d}\rho = \pi\delta(x_1\cos\vartheta + x_2\sin\vartheta). \tag{18}$$

Here $\delta(\cdot)$ is the Dirac delta function. The following property of delta functions can be found in [6]

$$\delta[f(\vartheta)] = \sum_n \frac{\delta(\vartheta - \vartheta_n)}{|f'(\vartheta_n)|}, \tag{19}$$

where ϑ_n are all the roots of equation $f(\vartheta) = 0$. In our case, $f(\vartheta) = x_1\cos\vartheta + x_2\sin\vartheta$ and we have 2 roots in the interval $0 \le \vartheta < 2\pi$

$$\vartheta_1 = \pi - \tan^{-1}(x_1/x_2), \qquad \vartheta_2 = 2\pi - \tan^{-1}(x_1/x_2) \tag{20}$$

with

$$|f'(\vartheta_1)| = |f'(\vartheta_2)| = \sqrt{x_1^2 + x_2^2}. \tag{21}$$

Now the final result will take the form

$$u_k(x_1, x_2, 0) = \frac{1}{2\pi}\Re\left[\sum_{s=1}^{5}\mathcal{X}_{ks}(-x_2, x_1)\sum_{n=1}^{5}\alpha_n(-1)^{n+s}\frac{D_c^{(n,s)}(-x_2, x_1)}{D_c(-x_2, x_1)}\right]. \tag{22}$$

Here \Re stands for the real part of the expression to follow and each parameter with the arguments $(-x_2, x_1)$ is understood as similar parameter in the article with ξ_1 formally replaced by $-x_2$ and ξ_2 is replaced by x_1. In order to better visualize (22), we may deduce that

$$\sum_{s=1}^{5}\mathcal{X}_{ks}(-x_2, x_1)(-1)^{n+s}D_c^{(n,s)}(-x_2, x_1) = D_{\mathrm{cnk}}, \tag{23}$$

which is the determinant of the matrix $\mathcal{C}$(see 11) with the nth row replaced by $\mathcal{X}_{ks}$. Now (22) can be rewritten as

$$u_k(x_1, x_2, 0) = \frac{1}{2\pi}\Re\left[\sum_{n=1}^{5}\alpha_n\frac{D_{\mathrm{cnk}}(-x_2, x_1)}{D_c(-x_2, x_1)}\right]. \tag{24}$$

Now we can proceed to application of the results of this section to contact and crack problems.

3 Normal contact and crack problems and their relationship

Presume that we have a magneto-electro-elastic generally anisotropic half-space $x_3 \ge 0$, which is indented by a rigid punch, whose surface may be described in the system of coordinates (x_1, x_2, x_3) as

$$x = w(x_1, x_2). \tag{25}$$

In the general case, the domain of contact S should be considered unknown. The governing integral equation can be presented in the form

$$\int_S\int K_c(x_1 - x_{10}, x_2 - x_{20})\sigma_3(x_{10}, x_{20})\mathrm{d}x_{10}\mathrm{d}x_{20} = \delta - w(x_1, x_2). \tag{26}$$

Here δ denotes yet unknown (in general case) maximum penetration of the punch. The kernel K_c in (26) represents the normal displacement of the point $(x_1, x_2, 0)$ of our half-space due to the action of a unit normal force applied at the point $(x_{10}, x_{20}, 0)$. The result is readily available from (24) by taking all $\alpha_n = 0$, except $\alpha_3 = -1$. This means that

$$K_c(x_1, x_2) = -\frac{1}{2\pi}\Re\left[\frac{D_{c33}(-x_2, x_1)}{D_c(-x_2, x_1)}\right]. \tag{27}$$

For the purpose of clarity, we write below explicitly that D_{c33} is the determinant of the following matrix

$$D_{c33} = \begin{Vmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \\ \mathcal{X}_{31} & \mathcal{X}_{32} & \mathcal{X}_{33} & \mathcal{X}_{34} & \mathcal{X}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix}. \tag{28}$$

Now we turn to the investigation of the normal crack problem. We have a flat crack of shape S, located in the plane $x_3 = 0$ of a magneto-electro-elastic generally anisotropic space. This crack is being opened by a normal stress

$$\sigma_{33} = -\sigma_{330}(x_1, x_2) \text{ for } (x_1, x_2) \subseteq S. \tag{29}$$

Due to the symmetry of the problem, it can be reduced to the one for the half-space, subjected to a unit normal dislocation with the boundary conditions, leading to the set of linear algebraic equations

$$\begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \\ \mathcal{X}_{31} & \mathcal{X}_{32} & \mathcal{X}_{33} & \mathcal{X}_{34} & \mathcal{X}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/2\pi \\ 0 \\ 0 \end{bmatrix}, \tag{30}$$

Its solution will have the form

$$u_{1s} = \frac{(-1)^{3+s} D_{c33}^{(3,s)}}{2\pi D_{c33}}, \tag{31}$$

where $\mathrm{D}_{c33}^{(3,s)}$ is the minor of D_{c33}, with the third row and sth column deleted. Taking into consideration that

$$\sigma_{33}(x_1, x_2, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{s=1}^{5} \mathcal{C}_{3s} u_{1s} \exp[-i(x_1\xi_1 + x_2\xi_2)] d\xi_1 d\xi_2. \tag{32}$$

and that

$$\sum_{s=1}^{5} \mathcal{C}_{3s} (-1)^{3+s} D_{c33}^{(3,s)} = D_c, \tag{33}$$

we may conclude that the normal stress at the surface of the half-space due to a unit normal dislocation will be defined as

$$\sigma_{33}(x_1, x_2, 0) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{D_c}{D_{c33}} \exp[-i(x_1\xi_1 + x_2\xi_2)] d\xi_1 d\xi_2. \tag{34}$$

Utilization of (14) allows us to write the following expression for the normal displacements on the boundary of half-space due to a unit normal concentrated force as

$$u_3(x_1, x_2, 0) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{D_{c33}}{D_c} \exp[-i(x_1\xi_1 + x_2\xi_2)] d\xi_1 d\xi_2. \tag{35}$$

Since the kernels of governing integral equations of the normal crack and contact problems come directly from (34) and (35), respectively, we may conclude that the integrands of Fourier transforms of both kernels are inverse to each other.

Recalling that D_{c33} and D_c are homogeneous with respect to ξ_1 and ξ_2 of the order 4 and 5, respectively, we may conclude that the integration in (34) is divergent. In order to regularize (34), we rewrite it as follows:

$$\sigma_{33}(x_1, x_2, 0) = -\frac{1}{4\pi^2} \Delta_{12} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{D_c}{(\xi_1^2 + \xi_2^2) D_{c33}} \exp[-i(x_1\xi_1 + x_2\xi_2)] d\xi_1 d\xi_2. \tag{36}$$

Here

$$\Delta_{12} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}. \tag{37}$$

Now the integration in (36) is convergent and may be performed in exactly the same manner, as it was done in (15–21). The final result is

$$\sigma_{33}(x_1, x_2, 0) = -\frac{1}{2\pi}\backslash Re\left[\Delta_{12}\left(\frac{D_c(-x_2, x_1)}{\sqrt{x_1^2 + x_2^2}D_{c33}(-x_2, x_1)}\right)\right]. \tag{38}$$

This result allows us to formulate the governing integral equation of the normal crack problem for a flat crack of the shape S, located in the plane $x_3 = 0$ of the magneto-electro-elastic anisotropic space and subjected to normal pressure σ_{330} as follows:

$$\int_S\int K_{cr}(x_1 - x_{10}, x_2 - x_{20})u_3(x_{10}, x_{20})dx_{10}dx_{20} = \sigma_{330}(x_1, x_2)$$

with

$$K_{cr}(x_1, x_2) = \frac{1}{2\pi}\backslash Re\left[\Delta_{12}\left(\frac{D_c(-x_2, x_1)}{\sqrt{x_1^2 + x_2^2}D_{c33}(-x_2, x_1)}\right)\right]. \tag{39}$$

4 Tangential contact problems

We consider a magneto-electro-elastic anisotropic half-space $x_3 \geq 0$. We presume that inside an arbitrary domain S in the plane $x_3 = 0$ some arbitrary tangential displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in the directions of the axes Ox_1 and Ox_2, respectively, are prescribed, while the rest of the boundary is free of tangential stresses, and the normal stress vanishes all over the boundary $x_3 = 0$. There is also no outside electrical or magnetic interference. We need to derive governing integral equations, relating the tangential displacements with tangential stresses.

The set of governing integral equations can be written in the following form

$$\begin{aligned} u_1 = &\int_S\int K_{11}(x_1 - x_{10}, x_2 - x_{20})\tau_{31}(x_{10}, x_{20})\mathrm{d}x_{10}\mathrm{d}x_{20} \\ &+ \int_S\int K_{12}(x_1 - x_{10}, x_2 - x_{20})\tau_{23}(x_{10}, x_{20})\mathrm{d}x_{10}\mathrm{d}x_{20}, \end{aligned} \tag{40}$$

$$\begin{aligned} u_2 = &\int_S\int K_{21}(x_1 - x_{10}, x_2 - x_{20})\tau_{31}(x_{10}, x_{20})\mathrm{d}x_{10}\mathrm{d}x_{20} \\ &+ \int_S\int K_{22}(x_1 - x_{10}, x_2 - x_{20})\tau_{23}(x_{10}, x_{20})\mathrm{d}x_{10}\mathrm{d}x_{20}. \end{aligned} \tag{41}$$

Here K_{ij} is the displacement in the i direction of the point $(x_1, x_2, 0)$ due to a unit force P_j applied at the point $(x_{10}, x_{20}, 0)$. We can find K_{ij} as a solution of a particular case of (11) with $\alpha_1 = -P_1$, $\alpha_2 = -P_2$, $\alpha_3 = \alpha_4 = \alpha_5 = 0$. Here P_1 and P_2 are unit concentrated forces applied at the coordinates origin in the directions Ox_1 and Ox_2, respectively. According to (13), the solution will take the form

$$u_{1s} = \frac{(-1)^s}{2\pi D_c}\left(P_1 D_c^{(1,s)} - P_2 D_c^{(2,s)}\right) \quad \text{for } s = 1, 2, 3, 4, 5. \tag{42}$$

Taking into consideration that

$$\sum_{s=1}^{5}(-1)^s D_c^{(1,s)} = -\begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} = D_{11}, \tag{43}$$

$$-\sum_{s=1}^{5}(-1)^s D_c^{(2,s)} = -\begin{Vmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} \\ 1 & 1 & 1 & 1 & 1 \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} = D_{12}, \tag{44}$$

$$\sum_{s=1}^{5}(-1)^s \mathcal{X}_{2s} D_c^{(1,s)} = -\begin{Vmatrix} X_{21} & \mathcal{X}_{22} & \mathcal{X}_{23} & X_{24} & \mathcal{X}_{25} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} = D_{21}, \tag{45}$$

$$-\sum_{s=1}^{5}(-1)^s \mathcal{X}_{2s} D_c^{(2,s)} = -\begin{Vmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} \\ X_{21} & \mathcal{X}_{22} & \mathcal{X}_{23} & X_{24} & \mathcal{X}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} = D_{22}, \tag{46}$$

we may conclude that

$$K_{11}(x_1, x_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{D_{11}}{D_c}\exp[-i(x_1\xi_1 + x_2\xi_2)]\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{47}$$

$$K_{12}(x_1, x_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{D_{12}}{D_c}\exp[-i(x_1\xi_1 + x_2\xi_2)]\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{48}$$

$$K_{21}(x_1, x_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{D_{21}}{D_c}\exp[-i(x_1\xi_1 + x_2\xi_2)]\mathrm{d}\xi_1\mathrm{d}\xi_2, \tag{49}$$

$$K_{22}(x_1, x_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{D_{22}}{D_c}\exp[-i(x_1\xi_1 + x_2\xi_2)]\mathrm{d}\xi_1\mathrm{d}\xi_2. \tag{50}$$

Since D_{11}, D_{12}, D_{21} and D_{22} are homogeneous in ξ_1 and ξ_2 of the order 4 and D_c is homogeneous of the order 5, the integrals in (47–50) can be computed in the same way as it was done (15–21). The final results are

$$K_{11}(x_1, x_2) = \frac{1}{4\pi}\left(\frac{D_{11}(-x_2, x_1)}{D_c(-x_2, x_1)} + \frac{D_{11}(x_2, -x_1)}{D_c(x_2, -x_1)}\right), \tag{51}$$

$$K_{12}(x_1, x_2) = \frac{1}{4\pi}\left(\frac{D_{12}(-x_2, x_1)}{D_c(-x_2, x_1)} + \frac{D_{12}(x_2, -x_1)}{D_c(x_2, -x_1)}\right), \tag{52}$$

$$K_{21}(x_1, x_2) = \frac{1}{4\pi}\left(\frac{D_{21}(-x_2, x_1)}{D_c(-x_2, x_1)} + \frac{D_{21}(x_2, -x_1)}{D_c(x_2, -x_1)}\right), \tag{53}$$

$$K_{22}(x_1, x_2) = \frac{1}{4\pi}\left(\frac{D_{22}(-x_2, x_1)}{D_c(-x_2, x_1)} + \frac{D_{22}(x_2, -x_1)}{D_c(x_2, -x_1)}\right). \tag{54}$$

Since the first term in (51–54) is complex conjugate to the second, the final results there will be real, as they should. The basic derivation for the governing integral equations of the tangential contact problem for a magneto-electro-elastic anisotropic half-space may be considered finished.

5 Tangential crack problems

Now we start the derivation of the governing integral equations for the tangential crack problem. We have a flat crack of shape S, located in the plane $x_3 = 0$. Crack faces are subjected to the tangential stresses $\tau_{310}(x_1, x_2)$ and $\tau_{230}(x_1, x_2)$, acting in opposite directions. Due to symmetry, the problem can be reduced to that for a half-space with the following boundary conditions at $x_3 = 0$

$$\tau_{31} = -\tau_{310}(x_1, x_2), \tau_{23} = -\tau_{230}(x_1, x_2) \quad \text{for } (x_1, x_2) \subseteq S, \tag{55}$$

$$u_1 = u_2 = 0 \quad \text{for } (x_1, x_2) \notin S; \sigma_{33} = 0 \,\text{for} - \infty < (x_1, x_2) < \infty. \tag{56}$$

The conditions in (56) become evident from the fact that tangential stresses applied to the crack faces are anti-symmetric with respect to the plane $x_3 = 0$, as well as from the presumption that electrical displacement and magnetic induction are presumed zero all over the plane $x_3 = 0$. The governing integral equations may be written in the form

$$\begin{aligned}&\int_S\int K^0_{11}(x_1 - x_{10}, x_2 - x_{20})u_{10}(x_{10}, x_{20})\mathrm{d}x_{10}\mathrm{d}x_{20} + \\ &\quad + \int_S\int K^0_{12}(x_1 - x_{10}, x_2 - x_{20})u_{20}(x_{10}, x_{20})\mathrm{d}x_{10}dx_{20} = -\tau_{310},\end{aligned} \tag{57}$$

$$\begin{aligned}&\int_S\int K^0_{21}(x_1 - x_{10}, x_2 - x_{20})u_{10}(x_{10}, x_{20})\mathrm{d}x_{10}\mathrm{d}x_{20} + \\ &\quad + \int_S\int K^0_{22}(x_1 - x_{10}, x_2 - x_{20})u_{20}(x_{10}, x_{20})\mathrm{d}x_{10}\mathrm{d}x_{20} = -\tau_{320}.\end{aligned} \tag{58}$$

The physical meaning of K^0_{nk} is the tangential stress in the direction x_n at the point $(x_1, x_2, 0)$ due to a unit dislocation in the direction x_k at the point $(x_{10}, x_{20}, 0)$. The auxiliary problem to be solved has the boundary conditions at $x_3 = 0$

$$\sigma_{33} = 0, \qquad u_1 = u_{10}\delta(x_1 - 0, x_2 - 0), \qquad u_2 = u_{20}\delta(x_1 - 0, x_2 - 0). \tag{59}$$

Instead of (11) we shall have the following set of equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ \mathcal{X}_{21} & \mathcal{X}_{22} & \mathcal{X}_{23} & \mathcal{X}_{24} & \mathcal{X}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \end{bmatrix} = \begin{bmatrix} u_{10}/2\pi \\ u_{20}/2\pi \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{60}$$

The solution of (60) will have the form

$$u_{1s} = \frac{1}{2\pi D_{\text{crt}}}\left((-1)^{1+s} D^{(1,s)}_{\text{crt}} u_{10} + (-1)^{2+s} D^{(2,s)}_{\text{crt}} u_{20}\right) \quad \text{for} \;\; s = 1, 2, 3, 4, 5. \tag{61}$$

Here

$$D_{\text{crt}} = \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \\ X_{21} & \mathcal{X}_{22} & \mathcal{X}_{23} & X_{24} & \mathcal{X}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} \tag{62}$$

and $D^{(n,s)}_{\text{crt}}$ are the minors of the first order, corresponding to nth row and sth column.

$$K^0_{11}(x_1, x_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{D^0_{11}}{D_{\text{crt}}}\exp[-i(x_1\xi + x_2\eta)]\mathrm{d}\xi\mathrm{d}\eta, \tag{63}$$

$$K^0_{12}(x_1, x_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{D^0_{12}}{D_{\text{crt}}}\exp[-i(x_1\xi + x_2\eta)]\mathrm{d}\xi\mathrm{d}\eta, \tag{64}$$

$$K^0_{21}(x_1,x_2)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{D^0_{21}}{D_{\text{crt}}}\exp[-i(x_1\xi+x_2\eta)]\mathrm{d}\xi\,\mathrm{d}\eta, \tag{65}$$

$$K^0_{22}(x_1,x_2)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{D^0_{22}}{D_{\text{crt}}}\exp[-i(x_1\xi+x_2\eta)]\mathrm{d}\xi\,\mathrm{d}\eta, \tag{66}$$

with

$$D^0_{11}=\sum_{s=1}^{5}(-1)^{1+s}\mathcal{C}_{1s}D^{(1,s)}_{\text{crt}}=\begin{Vmatrix}\mathcal{C}_{11}&\mathcal{C}_{12}&\mathcal{C}_{13}&\mathcal{C}_{14}&\mathcal{C}_{15}\\ X_{21}&\mathcal{X}_{22}&\mathcal{X}_{23}&X_{24}&\mathcal{X}_{25}\\ \mathcal{C}_{31}&\mathcal{C}_{32}&\mathcal{C}_{33}&\mathcal{C}_{34}&\mathcal{C}_{35}\\ \mathcal{C}_{41}&\mathcal{C}_{42}&\mathcal{C}_{43}&\mathcal{C}_{44}&\mathcal{C}_{45}\\ \mathcal{C}_{51}&\mathcal{C}_{52}&\mathcal{C}_{53}&\mathcal{C}_{54}&\mathcal{C}_{55}\end{Vmatrix}=-D_{22} \tag{67}$$

$$D^0_{12}=\sum_{s=1}^{5}(-1)^{2+s}\mathcal{C}_{1s}D^{(2,s)}_{\text{crt}}=-\begin{Vmatrix}\mathcal{C}_{11}&\mathcal{C}_{12}&\mathcal{C}_{13}&\mathcal{C}_{14}&\mathcal{C}_{15}\\ 1&1&1&1&1\\ \mathcal{C}_{31}&\mathcal{C}_{32}&\mathcal{C}_{33}&\mathcal{C}_{34}&\mathcal{C}_{35}\\ \mathcal{C}_{41}&\mathcal{C}_{42}&\mathcal{C}_{43}&\mathcal{C}_{44}&\mathcal{C}_{45}\\ \mathcal{C}_{51}&\mathcal{C}_{52}&\mathcal{C}_{53}&\mathcal{C}_{54}&\mathcal{C}_{55}\end{Vmatrix}=D_{12} \tag{68}$$

$$D^0_{21}=\sum_{s=1}^{5}(-1)^{1+s}\mathcal{C}_{2s}D^{(1,s)}_{\text{crt}}=-\begin{Vmatrix}X_{21}&\mathcal{X}_{22}&\mathcal{X}_{23}&X_{24}&\mathcal{X}_{25}\\ \mathcal{C}_{21}&\mathcal{C}_{22}&\mathcal{C}_{23}&\mathcal{C}_{24}&\mathcal{C}_{25}\\ \mathcal{C}_{31}&\mathcal{C}_{32}&\mathcal{C}_{33}&\mathcal{C}_{34}&\mathcal{C}_{35}\\ \mathcal{C}_{41}&\mathcal{C}_{42}&\mathcal{C}_{43}&\mathcal{C}_{44}&\mathcal{C}_{45}\\ \mathcal{C}_{51}&\mathcal{C}_{52}&\mathcal{C}_{53}&\mathcal{C}_{54}&\mathcal{C}_{55}\end{Vmatrix}=D_{21} \tag{69}$$

Here we used the property of determinant to change its sign, when two rows interchange places.

$$D^0_{22}=\sum_{s=1}^{5}(-1)^{2+s}\mathcal{C}_{2s}D^{(2,s)}_{\text{crt}}=\begin{Vmatrix}1&1&1&1&1\\ \mathcal{C}_{21}&\mathcal{C}_{22}&\mathcal{C}_{23}&\mathcal{C}_{24}&\mathcal{C}_{25}\\ \mathcal{C}_{31}&\mathcal{C}_{32}&\mathcal{C}_{33}&\mathcal{C}_{34}&\mathcal{C}_{35}\\ \mathcal{C}_{41}&\mathcal{C}_{42}&\mathcal{C}_{43}&\mathcal{C}_{44}&\mathcal{C}_{45}\\ \mathcal{C}_{51}&\mathcal{C}_{52}&\mathcal{C}_{53}&\mathcal{C}_{54}&\mathcal{C}_{55}\end{Vmatrix}=-D_{11} \tag{70}$$

6 Additional relationships between contact and crack problems

Equations (67–70) provide clear connections between tangential contact and crack problems, and the relationship does not end there. In order to make an educated guess, we may present Eqs. (40–41) in a schematic form as follows

$$u_1=\frac{D_{11}}{D_c}\tau_{31}+\frac{D_{12}}{D_c}\tau_{23},\qquad u_2=\frac{D_{21}}{D_c}\tau_{31}+\frac{D_{22}}{D_c}\tau_{23} \tag{71}$$

The solution of the algebraic equations (71) will take the form

$$\tau_{31}=\frac{D_{22}u_1-D_{12}u_2}{D_{11}D_{22}-D_{12}D_{21}}D_c\qquad \tau_{23}=\frac{D_{11}u_2-D_{21}u_1}{D_{11}D_{22}-D_{12}D_{21}}D_c \tag{72}$$

Using the same schematic, we can rewrite Eqs. (57–58) as follows

$$-\tau_{31}=\frac{D^0_{11}}{D_{\text{crt}}}u_1+\frac{D^0_{12}}{D_{\text{crt}}}u_2,\qquad -\tau_{23}=\frac{D^0_{21}}{D_{\text{crt}}}u_1+\frac{D^0_{22}}{D_{\text{crt}}}u_2. \tag{73}$$

Comparing (72) with (73) and taking into consideration (67–70), we may conclude that they would be identical, if an additional relationship held, namely,

$$D_{11}D_{22}-D_{12}D_{21}=D_{\text{crt}}D_c. \tag{74}$$

In matrix form, Eq. (74) will take the form

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} \begin{Vmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} \\ X_{21} & \mathcal{X}_{22} & \mathcal{X}_{23} & X_{24} & \mathcal{X}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} - \begin{Vmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} \\ 1 & 1 & 1 & 1 & 1 \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} \begin{Vmatrix} X_{21} & \mathcal{X}_{22} & \mathcal{X}_{23} & X_{24} & \mathcal{X}_{25} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} =$$

$$= \begin{Vmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix} \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \\ X_{21} & \mathcal{X}_{22} & \mathcal{X}_{23} & X_{24} & \mathcal{X}_{25} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} \\ \mathcal{C}_{41} & \mathcal{C}_{42} & \mathcal{C}_{43} & \mathcal{C}_{44} & \mathcal{C}_{45} \\ \mathcal{C}_{51} & \mathcal{C}_{52} & \mathcal{C}_{53} & \mathcal{C}_{54} & \mathcal{C}_{55} \end{Vmatrix}. \tag{75}$$

One can verify directly that (75) is indeed an identity for arbitrary values of $\mathcal{C}_{mk}, \mathcal{X}_k$ and that we can replace the row of '1' by arbitrary values of $\mathcal{Y}_k$, and the identity will still hold. One can also verify that such relationship will be valid for matrices of arbitrary order. Thus, we have proven that the integrands in the Fourier integral transform of the kernels of the governing integral equations for tangential contact and crack problems are related to one another as if they were in linear algebraic equations.

Taking into consideration that D^0_{nk} in (63–66) are homogeneous with respect to ξ_1 and ξ_2 of the order 4 and D_{crt} is homogeneous of the order 3, we may conclude that the integrals (63–66) are divergent. They can be regularized and computed, as it was done in (36–38) and the final result is

$$K_{nk}(x_1, x_2) = -\frac{1}{2\pi}\backslash Re\left[\Delta_{12}\left(\frac{D^0_{nk}(-x_2, x_1)}{\sqrt{x_1^2 + x_2^2}D_{\text{crt}}(-x_2, x_1)}\right)\right] \text{ for } (n, k) = 1, 2.$$

We remind that Δ_{12} is defined in (37).

7 Discussion

By method of trial and error, we established several properties of determinants, which are valid for determinants of arbitrary order and might be new. We show the properties on determinants of the third order, but generalization to an arbitrary order would be clear. We have two determinants

$$D_a = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}, \qquad D_b = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{Vmatrix}. \tag{76}$$

Let us combine them as follows

$$d_{11} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}, \qquad d_{12} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ b_{11} & b_{12} & b_{13} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}, d_{13} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{11} & b_{12} & b_{13} \end{Vmatrix} \tag{77}$$

$$d_{21} = \begin{Vmatrix} b_{21} & b_{22} & b_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}, \qquad d_{22} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}, d_{23} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{21} & b_{22} & b_{23} \end{Vmatrix} \tag{78}$$

$$d_{31} = \begin{Vmatrix} b_{31} & b_{32} & b_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}, \qquad d_{32} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ b_{31} & b_{32} & b_{33} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}, d_{33} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{Vmatrix} \tag{79}$$

The interesting result is obtained, if we compute the determinant

$$D_d = \begin{Vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{Vmatrix} = D_a^2 D_b. \tag{80}$$

If we take matrices $\{a_{ik}\}$ and $\{b_{ik}\}$ of rank n and create the matrix $\{d_{ik}\}$ by mixing them in the same manner, as in (77–79), then the determinant of the matrix $\{d_{ik}\}$ will be equal to $D_a^{n-1} D_b$.

Yet another interesting property can be demonstrated on the following example. We have three linear algebraic equations, which can be written in matrix form as

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{81}$$

The solution will have the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{D_a D_b} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{82}$$

Here

$$g_{11} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{Vmatrix}, \quad g_{12} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ a_{11} & a_{12} & a_{13} \\ b_{31} & b_{32} & b_{33} \end{Vmatrix}, \quad g_{13} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{11} & a_{12} & a_{13} \end{Vmatrix}, \tag{83}$$

$$g_{21} = \begin{Vmatrix} a_{21} & a_{22} & a_{23} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{Vmatrix}, \quad g_{22} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{Vmatrix}, \quad g_{23} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{21} & a_{22} & a_{23} \end{Vmatrix}, \tag{84}$$

$$g_{31} = \begin{Vmatrix} a_{31} & a_{32} & a_{33} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{Vmatrix}, \quad g_{32} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ a_{31} & a_{32} & a_{33} \\ b_{31} & b_{32} & b_{33} \end{Vmatrix}, \quad g_{33} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}. \tag{85}$$

The rule of creation of the inverse matrix can be deduced just by observation of (83–85). If we call creation of the matrix $\{d_{ik}\}$ as mosaic of matrix $\{b_{ik}\}$ into matrix $\{a_{ik}\}$, then creation of the inverse matrix $\{g_{ik}\}$ we call mosaic of the matrix $\{a_{ik}\}$ into matrix $\{b_{ik}\}$ and divided by the product of D_a and D_b. It is obvious that similar rule will be true for the matrices of arbitrary rank n. We do not know whether these properties are new in the theory of matrices and determinants, but they are certainly beautiful.

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