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A general way of obtaining novel closed-form solutions for functionally graded columns

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Abstract In this paper, we present a general methodology for solving buckling problems for inhomogeneous columns. Columns that are treated are functionally graded in axial direction. The buckling mode is postulated as the general order polynomial function that satisfies all boundary conditions. For specificity, we concentrate on the boundary conditions of simple support, and employ the second-order ordinary differential equation that governs the buckling behavior. A quadratic polynomial is adopted for the description of the column's flexural rigidity. Satisfaction of the governing differential equation leads to a set of nonlinear algebraic equations that are solved exactly. In addition to the recovery of the solutions previously found by Duncan and Elishakoff, several new solutions are arrived at.

Keywords Functionally graded materials · Axial grading · Closed-form solutions

1 Introduction

Leonhard Euler [1] was the first investigator that derived a solution for the buckling load of inhomogeneous column. In the next, in the nineteenth century, Engesser [2] determined a closed-form solution, although with vanishing flexure rigidity values at the ends of the column. The twentieth century witnessed several other closed-form solutions. Namely, Duncan [3] derived the following closed-form solution for the beam. He studied the column with variable moment of inertia

$$I = \left(1 - \frac{3}{7}\xi^2\right) I_0 \tag{1}$$

with I_0 being the moment of inertia at the origin, and $\xi = x/L$ is the nondimensional axial coordinate, L being the column's length. Duncan postulated the mode-shape $f(\xi)$ of buckling as a fifth-order polynomial

$$f(\xi) = 7\xi - 10\xi^2 + 3\xi^5 \tag{2}$$

and arrived at the following closed-form expression for the buckling load P_{Cr}

$$P_{\mathrm{Cr}} = \frac{60}{7}\frac{EI_0}{L^2} \tag{3}$$

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where E is the modulus of elasticity. Duncan [3] did not explain how he derived the expression for the buckling mode and the moment of inertia. We note that since the governing differential equation contains the product EI, with E denoting modulus of elasticity, one can treat the Duncan's problem as that of the axially functionally graded column with variable flexural rigidity $D(\xi)$

$$D(\xi) = E(\xi)I(\xi), \tag{4}$$

since it is immaterial whether the modulus of elasticity or the moment of inertia constitutes the variable property.

Elishakoff [4,5] suggested to postulate a fourth-order polynomial as representing the buckling mode shape

$$f(\xi) = \xi - 2\xi^3 + \xi^4 \tag{5}$$

He also postulated polynomial expression for the flexural rigidity

$$D(\xi) = b_0 + b_1\xi + b_2\xi^2 \tag{6}$$

and derived the following expression for the buckling load

$$P_{\text{Cr}} = -\frac{12b_2}{L^2} \tag{7}$$

where in order to guarantee the positive value of the buckling load the coefficient b_2 must take a negative value. Numerous other solutions are summarized in the monograph by Elishakoff [6]. Recently Elishakoff et. al. [7] gave the complete derivation of the solutions by Duncan and Elishakoff, and obtained two additional solutions which are rederived hereto. These results were obtained for parabolic variation of the axially functionally graded member stiffness, and using a fifth-order polynomial function for the buckling mode shape.

In this study we generalize both the Duncan's [3], Elishakoff's [6] and Elishakoff's et al. [7] solutions in such a manner that our methodology yields novel solutions that have not been reported elsewhere. Recent studies devoted to Buckling of nonhomogeneous columns include those by Eisenberger [8,9], Ayadoğlu [10], Li [11], Maalawi [12], Singh and Li [13], Coscun [14], Darbandi et al. [15], Huang and Li [16], Huang and Luo [17], Bubilio [18].

2 Basic equations

We study buckling of a functionally graded column that is simply supported at its both ends. We resort to the governing differential equation

$$D(\xi)\frac{d^2 f(\xi)}{d\xi^2} + P_{\text{cr}}L^2 f(\xi) = 0 \tag{8}$$

Instead of postulating the fourth-order polynomial for $f(\xi)$ as was done by Elishakoff [4,5], or the fifth-order polynomial as was adopted by Duncan [3], we resort to a higher-order polynomial. Let us demonstrate how the suggested methodology is performing for ninth-order polynomial, postulating the buckling mode as follows:

$$\begin{aligned} f(\xi) = \sum\nolimits_{i=0}^{9} a_i\xi^i &= a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 \\ &+ a_5\xi^5 + a_6\xi^6 + a_7\xi^7 + a_8\xi^8 + a_9\xi^9 \end{aligned} \tag{9}$$

Satisfaction of the boundary condition $f(0) = 0$ leads to $a_0 = 0$. The demand that the bending moment vanishes at $\xi = 0$, results in $a_2 = 0$. Enforcement of the condition that the displacement vanishes at the end $\xi = 1$ yields the condition

$$a_1 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 0 \tag{10}$$

Likewise, the demand that the bending moment vanishes at $\xi = 1$ leads to

$$6a_3 + 12a_4 + 20a_5 + 30a_6 + 42a_7 + 56a_8 + 72a_9 = 0 \tag{11}$$

The second term in Eq. (8), namely $P_{\text{Cr}}L^2 f(\xi)$ represents a ninth-order polynomial. In the first term, $f^{''}(\xi)$ constitutes a seventh-order polynomial. In order that the first term $D(\xi)f^{''}$ to represent the ninth-order polynomial, as is the second term, the flexural rigidity must be a quadratic polynomial equation:

$$D(\xi) = b_0 + b_1\xi + b_2\xi^2 \tag{12}$$

where D_0 is the value of the flexural rigidity at the origin of the coordinate system. We substitute Eqs. (9) and (12) into Eq. (8) subject to constraints given in Eqs. (10) and (11). The result is the ninth-order polynomial,

$$\begin{aligned} R(\xi) = r_0 + r_1\xi + r_2\xi^2 + r_3\xi^3 + r_4\xi^4 + r_5\xi^5 + r_6\xi^6 + r_7\xi^7 \\ + r_8\xi^8 + r_9\xi^9 = 0 \end{aligned} \tag{13}$$

In order Eq. (8) to be fulfilled, it is necessary and sufficient all coefficients r_i vanish, i.e.

$$r_0 = 0 \tag{14}$$
$$r_1 = 6D_0a_3 + P_{\text{cr}}L^2a_1 = 0 \tag{15}$$
$$r_2 = 6a_3b_1 + 12a_4 = 0 \tag{16}$$
$$r_3 = 12D_0a_4b_1 + 6D_0a_3b_2 + 20D_0a_5 + P_{\text{cr}}L^2a_3 = 0 \tag{17}$$
$$r_4 = 20D_0a_5b_1 + 12D_0a_4b_2 + 30D_0a_6 + P_{\text{cr}}L^2a_4 = 0 \tag{18}$$
$$r_5 = 30D_0a_6b_1 + 20D_0a_5b_2 + 42D_0a_7 + P_{\text{cr}}L^2a_5 = 0 \tag{19}$$
$$r_6 = 42D_0a_7b_1 + 30D_0a_6b_2 + 56D_0a_8 + P_{\text{cr}}L^2a_6 = 0 \tag{20}$$
$$r_7 = 56D_0a_8b_1 + 42D_0a_7b_2 + 72D_0a_9 + P_{\text{cr}}L^2a_7 = 0 \tag{21}$$
$$r_8 = 72D_0a_9b_1 + 56D_0a_8b_2 + P_{\text{cr}}L^2a_8 = 0 \tag{22}$$
$$r_9 = 72D_0a_9b_1 + P_{\text{cr}}L^2a_9 = 0 \tag{23}$$

We obtain 11 equations (10–11, 15–23) with 11 unknowns: $a_1, a_3, a_4, a_5, a_6, a_7, a_8, a_9, b_1, b_2$, and P_{Cr}.

Table 1 Results of the coupled equations for the mode and stiffness variation (14 solutions)

SN PD	1 9	2 7	3 5	4 8	5 5	6 5	7 6	8 4	9 6	10 6	11 7	12 7	13 7	14 9
a_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_1	$\frac{34157}{23516}$	$\frac{44286}{26963}$	$\frac{7}{3}$	$\frac{13254}{10157}$	$\frac{8}{3}$	$-\frac{1}{6}$	$\frac{64769}{44449}$	1	$\frac{5125}{40412}$	$\frac{7808}{6343}$	$\frac{488}{6343}$	$\frac{2048}{98059}$	$\frac{5231}{72232}$	$\frac{10462}{9029}$
a_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_3	$-\frac{55679}{24227}$	$\frac{57335}{22686}$	$-\frac{10}{3}$	$-\frac{149729}{67087}$	$\frac{20}{3}$	$\frac{5}{3}$	$-\frac{52577}{20371}$	-2	$-\frac{22825}{29977}$	$-\frac{102480}{42211}$	$-\frac{25620}{42211}$	$-\frac{22147}{70033}$	$-\frac{20292}{38435}$	$-\frac{77882}{36879}$
a_4	0	0	0	$\frac{14179}{69336}$	-5	$-\frac{5}{2}$	$\frac{20320}{44449}$	1	$\frac{52041}{209587}$	$\frac{20496}{29963}$	$\frac{10248}{29963}$	$\frac{30706}{124377}$	$\frac{10973}{68422}$	$\frac{10973}{34211}$
a_5	1	1	1	1	1	1	1	0	1	1	1	1	1	1
a_6	0	0	0	$-\frac{12405}{70033}$	0	0	$-\frac{1}{3}$	0	$-\frac{29977}{48846}$	$-\frac{46612}{82005}$	$-\frac{40517}{35641}$	$-\frac{28947}{18380}$	$-\frac{35140}{58579}$	$\frac{17570}{58579}$
a_7	$-\frac{20805}{127444}$	$-\frac{9271}{80517}$	0	$-\frac{8289}{61910}$	0	0	0	0	0	$\frac{9271}{114174}$	$\frac{15623}{48100}$	$\frac{17776}{28515}$	$-\frac{12837}{26099}$	$-\frac{12837}{104396}$
a_8	0	0	0	$\frac{7951}{237542}$	0	0	0	0	0	0	0	0	$\frac{25433}{51128}$	$\frac{5521}{88791}$
a_9	$\frac{5319}{593060}$	0	0	0	0	0	0	0	0	0	0	0	$-\frac{12641}{114355}$	$-\frac{1599}{231442}$
d_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
d_1	0	0	0	$\frac{7795}{42537}$	$\frac{3}{2}$	3	$\frac{20065}{56641}$	1	$\frac{19447}{29817}$	$\frac{15124}{26839}$	$\frac{30248}{26839}$	$\frac{44650}{28597}$	$\frac{19369}{31882}$	$\frac{16647}{54803}$
d_2	$-\frac{18008}{136575}$	$-\frac{7562}{34401}$	$-\frac{3}{7}$	$-\frac{7795}{42537}$	$-\frac{3}{4}$	-3	$-\frac{20065}{56641}$	-1	$-\frac{25960}{21619}$	$-\frac{27087}{96137}$	$\frac{30248}{26839}$	$-\frac{33227}{15361}$	$\frac{19369}{31882}$	$-\frac{16647}{109606}$
P_{cr}	9.4935	9.2324	8.5714	10.2621	15	60	10.6275	12	36.0239	11.8337	47.3347	90.8492	43.7415	10.9354

3 Derived solutions

Using Maple we obtain 14 distinct solution for the 11 unknowns. These are given in Table 1 ordered in the sequence of derivation. The order of the polynomial in each solution is given too. One can see that the Duncan's [3], Elishakoff's [6], and Elishakoff's et. al. [7] solutions are listed as solution number 3, 8, 5 and 6, respectively. One can also identify that there are solutions with significantly higher buckling loads, and these are related to the second and even third buckling modes. In Table 2 the buckling mode shapes $f(\xi)$, and the stiffness variation $D(\xi)$, are plotted schematically indicating symmetric, anti-symmetric or general behavior of the mode shape and/or grading variations. Additionally, the solutions numbers are indicated for the solutions that have the above general shapes. In parentheses the polynomial order is given. It can be seen

Table 2 Buckling modes and stiffness variations

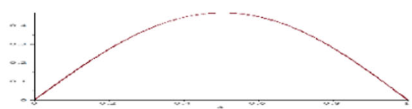
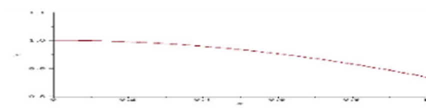
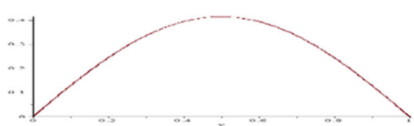
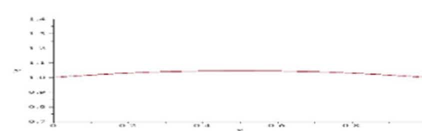
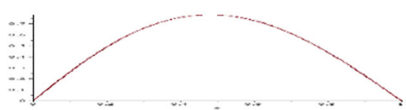
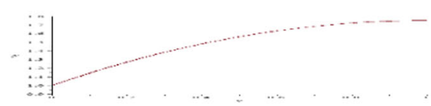
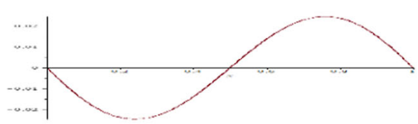
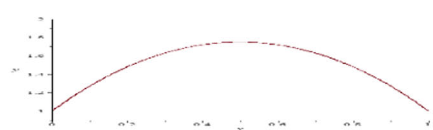
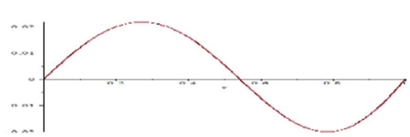
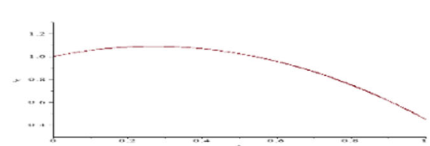
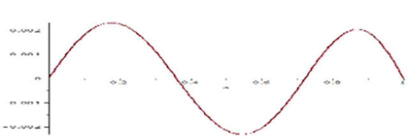
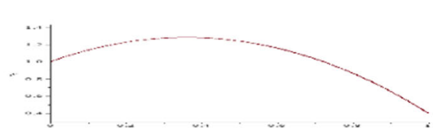
Type	*F(x)*	*D(x)*	Solutions
A	 Symmetric	 Descending	1(9) 2(7) 3(5)
B	 Symmetric	 Symmetric	4(8) 7(6) 8(4)
C	 Symmetric	 Ascending	5(5) 10(7) 14(9)
D	 Anti-symmetric	 Symmetric	6(5) 11(7) 13(9)
E	 General	 General	9(6)
F	 General	 General	12(7)

Table 3 Efficiency calculations for the 14 solutions

Solution number	Solution type	Polynomial degree	D_{ave}	P_{cr}	E_{f}
First mode					
8	B	4	1.167	12	1.0422
7	B	6	1.059	10.6275	1.0168
3	A	5	0.857	8.5714	1.0132
5	C	5	1.5	15	1.0132
2	A	7	0.927	9.2324	1.0094
10	C	7	1.188	11.8337	1.0094
4	B	8	1.031	10.2621	1.009
1	A	9	0.956	9.4935	1.0061
14	C	9	1.101	10.9354	1.0061
Second mode					
6	D	5	1.5	60	1.0132
11	D	7	1.188	47.3347	1.0094
13	D	9	1.101	43.7415	1.0061
9	E	6	0.926	36.0239	0.9856
Third mode					
12	F	7	1.06	90.8492	0.9652

that types A,B, and C are for the first mode, D and E are for the second, and F is for the third mode. All the first order modes have symmetric mode shapes; moreover, they differ with respect to the stiffness variation. Case D is for symmetric stiffness variation and anti-symmetric buckling mode, and for the remaining cases E and F there are no indications of symmetry, these constituting general cases.

In Table 3 additional values are given for the solution. First, an "equivalent uniform stiffness", D_{ave}, for each case is given. It is defined as

$$D_{\text{ave}} = \int_0^1 D(\xi)d\xi \tag{24}$$

and it will allow us to compare the efficiency of each solution. Another measure of effectiveness of the solution is also given—the nondimensional ratio of buckling loads

$$E_{\text{f}} = \frac{P_{\text{cr}}L}{\pi^2 D_{\text{ave}}} \tag{25}$$

constituting the buckling load of the graded column divided by the normalized buckling load for a constant cross section member. Then, observation of the first 9 solutions listed in Table 3 for the first mode buckling loads, enables one to deduce that there are solutions for all the polynomial degrees between 4 and 9. For the odd values, 5, 7, and 9, there are two solutions which are found to be identical, if one reverses the direction of the variation, and this is not clear as we normalize $D(\xi)$ with respect to the value at $\xi = 0$. The values in Table 3 for E_{f} are descending, and it can be seen that the best solution is the fourth-order variation (Elishakoff [4,5]), and then the sixth (Elishakoff [7]), fifth (Duncan [3]), seventh-, eighth- and ninth-order solution. We have to stress that over all the differences are small.
Here, we also like to comment that if one tries to use higher-order functions for the mode shapes in Eq. (9), the set of the nonlinear equations that is obtained does not yield additional new buckling modes.

4 Summary

In this study we present a novel method to find functionally graded beams, with axial stiffness variation, to obtain simple polynomial buckling modes. It was found that in addition to four previously known such modes one can obtain many other solutions for the first buckling mode, and also for higher-order buckling modes. It was shown that for the fourth-order polynomial variation (Elishakoff, [4,5]), one will obtain the best effective solution, with over 4% increase in overall performance. The pertinent question arises on the importance of the found closed-form solutions. As Shan and Chen [19] write "the mechanical instability of materials, especially biological materials, is often associated with material inhomogeneity and nonlinearity. In particular, exploring the role of mechanical instability in the morphogenesis of living tissues and organs represents new challenges to engineers. For example, not only does mechanical buckling have significant implications in addressing cytoskeletal mechanics, but also it plays an important role in the morphogenesis of tortuous veins often observed in a variety of diseases…" In light of these observations obtaining closed-form solutions that can serve as benchmark problems appear important. Another question that might interest the reader appears to be why we limited ourselves by the ninth-order polynomials. The previous studies dealt with fourth-order [6] and fifth-order [3] polynomials. Our goal in this work was to devote a study to higher-order polynomial solutions. Naturally, the higher the order of the polynomial more closed-form solutions presumably could be determined. Only the complexity of the derivations limits the order of the treated polynomials. We envision that in the future arbitrary order of polynomials could be treated by some artificial intelligence techniques.

The study on buckling of functionally graded columns with boundary constrains is underway and will be reported elsewhere, because of its importance [20] to implications in snake locomotion and buckling of microtubules of cell cytoskeleton.

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